MA362 — 复分析

Assignment 1

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习题 2.1/1

研究下列函数的可微性:

(i) $f(z) = |z|$;

(ii) $f(z) = |z|^2$;

(iii) $f(z) = \operatorname{Re} z$;

(v) $f(z)$为常数

(i)

对于任意 $z \in \mathbf{C}$ 有

$$\frac{f(z+h)-f(z)}{h} = \frac{|z+h|-|z|}{h} = \frac{|z+h|^2-|z|^2}{h(|z+h|+|z|)} = \frac{z\bar{h}+\bar{z}h+|h|^2}{h(|z+h|+|z|)}.$$

当 $z \neq 0$ 时

$$\lim_{h\to 0}\frac{f(z+h)-f(z)}{h} = \lim_{h\to 0}\frac{z\bar{h}+\bar{z}h}{2|z|h} = \lim_{h\to 0}\frac{z\bar{h}/h+\bar{z}}{2|z|}.$$

如果让 h 取实数, 则上述极限为 $\dfrac{z+\bar{z}}{2|z|}$; 如果让 h 取纯虚数, 则上述极限为 $\dfrac{-z+\bar{z}}{2|z|}$.

当 $z = 0$ 时

$$\lim_{h\to 0}\frac{f(z+h)-f(z)}{h} = \lim_{h\to 0}\frac{|h|^2}{h|h|} = \lim_{h\to 0}\frac{|h|}{h}.$$

如果让 h 取实数, 则上述极限为 1; 如果让 h 取纯虚数, 则上述极限为 -1.

因此, 当 $h \to 0$ 时上述极限不存在, 因而在 $\mathbf{C}$ 中处处不可微.

(ii)

对于任意 $z \in \mathbf{C}$ 有

$$\frac{f(z+h)-f(z)}{h} = \frac{|z+h|^2-|z|^2}{h} = \frac{z\bar{h}+\bar{z}h+|h|^2}{h}.$$

当 $z \neq 0$ 时

$$\lim_{h\to 0}\frac{f(z+h)-f(z)}{h} = \lim_{h\to 0}\frac{z\bar{h}+\bar{z}h}{h} = \lim_{h\to 0}(z\bar{h}/h+\bar{z}).$$

如果让 h 取实数, 则上述极限为 $z+\bar{z}$; 如果让 h 取纯虚数, 则上述极限为 $-z+\bar{z}$.

当 $z=0$ 时

$$\lim_{h\to 0}\frac{f(z+h)-f(z)}{h}=\lim_{h\to 0}\frac{|h|^2}{h}=0.$$

因此, 当 $z\neq 0, h\to 0$ 时上述极限不存在, 因而仅在 $z=0$ 处可微.

(iii)

对于任意 $z\in\mathbf{C}$ 有

$$\lim_{h\to 0}\frac{f(z+h)-f(z)}{h}=\lim_{h\to 0}\frac{\mathrm{Re}(z+h)-\mathrm{Re}(z)}{h}=\lim_{h\to 0}\frac{\mathrm{Re}(h)}{h}.$$

如果让 h 取实数, 则上述极限为 1; 如果让 h 取纯虚数, 则上述极限为 0.
因此, 当 $h\to 0$ 时上述极限不存在, 因而在 $\mathbf{C}$ 中处处不可微.

(v)

设 $f(z)=z_0$, 对于任意 $z\in\mathbf{C}$ 有

$$\lim_{h\to 0}\frac{f(z+h)-f(z)}{h}=\lim_{h\to 0}\frac{z_0-z_0}{h}=0.$$

因此, 当 $h\to 0$ 时上述极限存在, 因而在 $\mathbf{C}$ 中处处可微.

习题　2.1/2

设 f 和 g 都在 z_0 处可微, 且 $f(z_0)=g(z_0)=0$, $g'(z_0)\neq 0$, 证明

$$\lim_{z\to z_0}\frac{f(z)}{g(z)}=\frac{f'(z_0)}{g'(z_0)}.$$

由 f 和 g 都在 z_0 处可微有

$$f'(z_0)=\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0},$$

$$g'(z_0)=\lim_{z\to z_0}\frac{g(z)-g(z_0)}{z-z_0}.$$

由 $f(z_0)=g(z_0)=0$, $g'(z_0)\neq 0$ 可得

$$\frac{f'(z_0)}{g'(z_0)}=\lim_{z\to z_0}\frac{f(z)-f(z_0)}{g(z)-g(z_0)}=\lim_{z\to z_0}\frac{f(z)}{g(z)}.$$

习题　2.1/4

设域 G 和域 D 关于实轴对称. 证明: 如果 $f(z)$ 是 D 上的全纯函数, 那么 $\overline{f(\bar{z})}$ 是 G 上的全纯函数.

对于任意 $z \in \mathbf{G}$ 有 $\bar{z} \in \mathbf{D}$, 设 $z+h \in \mathbf{G}$, 则 $\overline{z+h} \in \mathbf{D}$. 由 f 是 $\mathbf{D}$ 上的全纯函数可得

$$f'(\bar{z}) = \lim_{\bar{h}\to 0} \frac{f(\overline{z+h}) - f(\bar{z})}{\bar{h}},$$

$$\overline{f'(\bar{z})} = \overline{\lim_{\bar{h}\to 0} \frac{f(\overline{z+h}) - f(\bar{z})}{\bar{h}}} = \lim_{h\to 0} \frac{\overline{f(\overline{z+h}) - f(\bar{z})}}{h} = \lim_{h\to 0} \frac{\overline{f(\overline{z+h})} - \overline{f(\bar{z})}}{h}.$$

因此 $\overline{f(\bar{z})}$ 是 $\mathbf{G}$ 上的全纯函数.

习题 2.2/1

设 D 是 $\mathbf{C}$ 中的域, $f \in H(D)$. 如果对每一个 $z \in D$, 都有 $f'(z) = 0$, 证明 f 是一常数.

设 $f(z) = u(x,y) + iv(x,y)$, 由 $f \in H(D)$ 可知

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} = 0.$$

故

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = \frac{\partial u}{\partial y} = 0.$$

因此 $u(x,y)$ 和 $v(x,y)$ 都为常数, 因而 f 是一常数.

习题 2.2/2

设 $f \in H(D)$, 并且满足下列条件之一:

(i) $\operatorname{Re} f(z)$ 是常数;
(ii) $\operatorname{Im} f(z)$ 是常数;
(iii) $|f(z)|$ 是常数;
(iv) $\arg f(z)$ 是常数;

设 $f(z) = u(x,y) + iv(x,y)$, 由 $f \in H(D)$ 可知

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}, \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

(i)

由 $\operatorname{Re} f(z)$ 是常数可得

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0.$$

故

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = \frac{\partial u}{\partial y} = 0.$$

因此 $u(x,y)$ 和 $v(x,y)$ 都为常数, 因而 f 是一常数.

(ii)

由 $\operatorname{Im} f(z)$ 是常数可得
$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0.$$
故
$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = \frac{\partial u}{\partial y} = 0.$$
因此 $u(x, y)$ 和 $v(x, y)$ 都为常数, 因而 f 是一常数.

(iii)

由 $|f(z)|$ 是常数 $k \in \mathbf{R}$ 可得
$$u^2(x, y) + v^2(x, y) = k^2.$$
若 $k = 0$ 易知 $u(x, y) = v(x, y) = 0, f(z) = 0$.
若 $k = 0$ 且存在, 分别对 x, y 求偏导可得
$$2u(x, y)\frac{\partial u}{\partial x} + 2v(x, y)\frac{\partial v}{\partial x} = u(x, y)\frac{\partial u}{\partial x} - v(x, y)\frac{\partial u}{\partial y} = 0,$$
$$2u(x, y)\frac{\partial u}{\partial y} + 2v(x, y)\frac{\partial v}{\partial y} = u(x, y)\frac{\partial u}{\partial y} + v(x, y)\frac{\partial u}{\partial x} = 0.$$
解方程组可得
$$\left[u^2(x, y) + v^2(x, y)\right]\frac{\partial u}{\partial x} = \left[u^2(x, y) + v^2(x, y)\right]\frac{\partial u}{\partial y} = 0,$$
$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0.$$
易知 $u(x, y)$ 是常数, 由 (i) 可得 f 为常数. 因而 f 是一常数.

(iv)

由 $\arg f(z)$ 是常数 $k \in \mathbf{R}$ 可得
$$\frac{v(x, y)}{u(x, y)} = \tan k.$$
若 $\tan k = 0$ 或无穷易知 $u(x, y)$ 或 $v(x, y)$ 是常数, 由 (i)(ii) 可得 f 为常数.
若 $\tan k \neq 0$ 且存在, 分别对 x, y 求偏导可得
$$\frac{\partial v}{\partial x} = \tan k\frac{\partial u}{\partial x} = -\frac{\partial u}{\partial y}, \quad \frac{\partial v}{\partial y} = \tan k\frac{\partial u}{\partial y} = \frac{\partial u}{\partial x}.$$
解方程组可得
$$(1 + \tan^2 k)\frac{\partial u}{\partial x} = (1 + \tan^2 k)\frac{\partial u}{\partial y} = 0,$$
$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0.$$
易知 $u(x, y)$ 是常数, 由 (i) 可得 f 为常数. 因而 f 是一常数.

习题 2.2/3

设 $z = x + iy$, 证明 $f(z) = \sqrt{xy}$ 在 $z = 0$ 处满足 Cauchy-Riemann 方程, 但 f 在 $z = 0$ 处不可微.

设 $f(z) = u(x, y) + iv(x, y) = \sqrt{xy}$, $u(x, y) = \sqrt{xy}$, $v(x, y) = 0$.

$$\frac{\partial u}{\partial x} = \lim_{h \to 0} \frac{u(h, 0) - u(0, 0)}{h} = 0,$$

$$\frac{\partial u}{\partial y} = \lim_{h \to 0} \frac{u(0, h) - u(0, 0)}{h} = 0,$$

$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0,$$

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = 0.$$

故 $f(z)$ 在 $z = 0$ 处满足 Cauchy-Riemann 方程.
设 $h = x + xi$

$$\lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{x \to 0} \frac{|x|}{x + xi}.$$

如果让 x 取正数, 则上述极限为 $\frac{1}{1+i}$; 如果让 h 取负数, 则上述极限为 $-\frac{1}{1+i}$.
因此, 当 $h \to 0$ 时上述极限不存在, 因而 f 在 $z = 0$ 中处不可微.

习题 2.2/4

设 $z = r(\cos\theta + i\sin\theta)$, $f(z) = u(r, \theta) + iv(r, \theta)$, 证明 Cauchy-Riemann 方程为

$$\begin{cases} \dfrac{\partial u}{\partial r} = \dfrac{1}{r}\dfrac{\partial v}{\partial \theta}, \\ \dfrac{\partial v}{\partial r} = -\dfrac{1}{r}\dfrac{\partial u}{\partial \theta}. \end{cases}$$

由 $z = x + yi = r(\cos\theta + i\sin\theta)$ 得

$$x(r, \theta) = r\cos\theta, \quad y(r, \theta) = r\sin\theta,$$

$$\frac{\partial x}{\partial r} = \cos\theta, \quad \frac{\partial x}{\partial \theta} = -r\sin\theta,$$

$$\frac{\partial y}{\partial r} = \sin\theta, \quad \frac{\partial y}{\partial \theta} = r\cos\theta.$$

$f(z) = u(x, y) + iv(x, y) = u(r, \theta) + iv(r, \theta)$, 由链式求导法则可得

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta,$$

$$\frac{1}{r}\frac{\partial v}{\partial \theta} = \frac{1}{r}\left(\frac{\partial v}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial \theta}\right) = -\frac{\partial v}{\partial x}\sin\theta + \frac{\partial v}{\partial y}\cos\theta,$$

$$\frac{\partial v}{\partial r}=\frac{\partial v}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial v}{\partial y}\frac{\partial y}{\partial r}=\frac{\partial v}{\partial x}\cos\theta+\frac{\partial v}{\partial y}\sin\theta,$$

$$-\frac{1}{r}\frac{\partial u}{\partial\theta}=\frac{1}{r}\left(\frac{\partial u}{\partial x}\frac{\partial x}{\partial\theta}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial\theta}\right)=\frac{\partial u}{\partial x}\sin\theta-\frac{\partial u}{\partial y}\cos\theta.$$

代入 Cauchy-Riemann 方程

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y},\quad \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}.$$

可知

$$\frac{\partial u}{\partial r}=\frac{1}{r}\frac{\partial v}{\partial\theta},\quad \frac{\partial v}{\partial r}=-\frac{1}{r}\frac{\partial u}{\partial\theta}.$$

习题 2.2/5

设 $z=r(\cos\theta+i\sin\theta)$, 证明:

$$\frac{\partial f}{\partial\bar{z}}=\frac{1}{e^{i\theta}}\left(\frac{\partial f}{\partial r}+\frac{i}{r}\frac{\partial f}{\partial\theta}\right),$$
$$\frac{\partial f}{\partial z}=\frac{1}{e^{-i\theta}}\left(\frac{\partial f}{\partial r}-\frac{i}{r}\frac{\partial f}{\partial\theta}\right).$$

同上题, 由链式求导法则可得

$$\frac{\partial f}{\partial r}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial r}=\frac{\partial f}{\partial x}\cos\theta+\frac{\partial f}{\partial y}\sin\theta,$$

$$\frac{i}{r}\frac{\partial f}{\partial\theta}=\frac{i}{r}=\left(\frac{\partial f}{\partial x}\frac{\partial x}{\partial\theta}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial\theta}\right)=-\frac{\partial f}{\partial x}i\sin\theta+\frac{\partial f}{\partial y}i\cos\theta.$$

代入可知

$$\frac{1}{2}e^{i\theta}\left(\frac{\partial f}{\partial r}+\frac{i}{r}\frac{\partial f}{\partial\theta}\right)=\frac{1}{2}e^{i\theta}\left[\frac{\partial f}{\partial x}(\cos\theta-i\sin\theta)+\frac{\partial f}{\partial y}(\sin\theta+i\cos\theta)\right]=\frac{1}{2}\left(\frac{\partial f}{\partial x}+i\frac{\partial f}{\partial y}\right)=\frac{\partial f}{\partial\bar{z}},$$

$$\frac{1}{2}e^{-i\theta}\left(\frac{\partial f}{\partial r}-\frac{i}{r}\frac{\partial f}{\partial\theta}\right)=\frac{1}{2}e^{-i\theta}\left[\frac{\partial f}{\partial x}(\cos\theta+i\sin\theta)+\frac{\partial f}{\partial y}(\sin\theta-i\cos\theta)\right]=\frac{1}{2}\left(\frac{\partial f}{\partial x}-i\frac{\partial f}{\partial y}\right)=\frac{\partial f}{\partial z}.$$

习题 2.2/8

设 D 是 $\mathbf{C}$ 中的域, $f\in H(D)$, f 在 D 中不取零值. 证明: 对任意 $p>0$, 有

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^p=p^2|f(z)|^{p-2}|f'(x)|^2.$$

由 $f\in H(D)$ 可知

$$\frac{\partial f}{\partial z}=f'(z),\quad \frac{\partial f}{\partial\bar{z}}=0.$$

由 $\overline{f(\bar{z})}\in H(D)$ 由此可知

$$\overline{f(\overline{z_0+\Delta z})}-\overline{f(\overline{z_0})}=\frac{\partial\bar{f}}{\partial z}\Delta z+\frac{\partial\bar{f}}{\partial\bar{z}}\overline{\Delta z}+o(|\Delta z|).$$

$$\frac{\partial \bar{f}}{\partial \bar{z}} = \overline{f'(z)}, \quad \frac{\partial \bar{f}}{\partial z} = 0.$$

故

$$\begin{aligned}
&\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^p \\
=&\Delta|f(z)\overline{f(z)}|^{p/2} \\
=&4\frac{\partial^2}{\partial z \partial \bar{z}}|f(z)\overline{f(z)}|^{p/2} \\
=&4\frac{\partial}{\partial \bar{z}}\frac{\partial}{\partial z}|f(z)\overline{f(z)}|^{p/2} \\
=&4\frac{\partial}{\partial \bar{z}}\frac{p}{2}|f(z)|^{p/2-1}|f'(z)||\overline{f(z)}|^{p/2} \\
=&4 \cdot \frac{p}{2}|f(z)|^{p/2-1}|f'(z)| \cdot \frac{p}{2}|\overline{f(z)}|^{p/2-1}|\overline{f'(z)}| \\
=&p^2|f(z)\overline{f(z)}|^{p/2-1}|f'(x)\overline{f'(x)}| \\
=&p^2|f(z)|^{p-2}|f'(x)|^2.
\end{aligned}$$