**MA362 — 复分析**

*Assignment 9*

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## 习题　六 (一)/1

求下列函数 $f(z)$ 在指定点的留数.

(1) $\dfrac{z}{(z-1)(z+1)^2}$ 在 $z=\pm 1, \infty$;

(3) $\dfrac{1-e^{2z}}{z^4}$ 在 $z=0, \infty$;

(6) $\dfrac{e^z}{z^2-1}$ 在 $z=\pm 1, \infty$.

### (1)

1 为一阶极点, -1 为二阶极点.

$$\operatorname*{Res}_{z=1} f(z) = \left.\frac{z}{(z+1)^2}\right|_{z=1} = \frac{1}{4},$$

$$\operatorname*{Res}_{z=-1} f(z) = \left.\left(\frac{z}{z-1}\right)'\right|_{z=-1} = -\left.\frac{1}{(z-1)^2}\right|_{z=-1} = -\frac{1}{4},$$

$$\operatorname*{Res}_{z=\infty} f(z) = -\left[\operatorname*{Res}_{z=1} f(z) + \operatorname*{Res}_{z=-1} f(z)\right] = 0.$$

### (3)

0 为四阶极点.

$$\operatorname*{Res}_{z=0} f(z) = \frac{1}{3!}\left.(1-e^{2z})'''\right|_{z=0} = -\left.\frac{4}{3}e^{2z}\right|_{z=0} = -\frac{4}{3},$$

$$\operatorname*{Res}_{z=\infty} f(z) = -\operatorname*{Res}_{z=0} f(z) = \frac{4}{3}.$$

### (6)

1 为一阶极点, -1 为一阶极点.

$$\operatorname*{Res}_{z=1} f(z) = \left.\frac{e^z}{z+1}\right|_{z=1} = \frac{e}{2},$$

$$\operatorname*{Res}_{z=-1} f(z) = \left.\frac{e^z}{z-1}\right|_{z=-1} = -\frac{1}{2e},$$

$$\operatorname*{Res}_{z=\infty} f(z) = -\left[\operatorname*{Res}_{z=1} f(z) + \operatorname*{Res}_{z=-1} f(z)\right] = \frac{1}{2e} - \frac{e}{2}.$$

## 习题　六 (一)/2

球下列函数 $f(z)$ 在其孤立奇点 (包括无穷远点) 处的留数 ($m$ 是正整数).

(3) $\dfrac{1}{(z-\alpha)^m(z-\beta)} \quad (\alpha \neq \beta)$;

(4) $\dfrac{e^z}{z^2(z-\pi i)^4}$;

### (3)

$\alpha$ 为 $m$ 阶极点, $\beta$ 为一阶极点.

$$\operatorname*{Res}_{z=\alpha} f(z) = \frac{1}{(m-1)!}\left(\frac{1}{z-\beta}\right)^{(m-1)}\Bigg|_{z=\alpha} = \frac{1}{(m-1)!}\cdot\frac{(-1)^{m-1}(m-1)!}{(z-\beta)^m}\Bigg|_{z=\alpha} = \frac{(-1)^{m-1}}{(\alpha-\beta)^m} = -\frac{1}{(\beta-\alpha)^m},$$

$$\operatorname*{Res}_{z=\beta} f(z) = \frac{1}{(z-\alpha)^m}\Bigg|_{z=\beta} = \frac{1}{(\beta-\alpha)^m},$$

$$\operatorname*{Res}_{z=\infty} f(z) = -\left[\operatorname*{Res}_{z=\alpha} f(z) + \operatorname*{Res}_{z=\beta} f(z)\right] = 0.$$

### (4)

0 为二阶极点, $\pi i$ 为四阶极点.

$$\operatorname*{Res}_{z=0} f(z) = \left[\frac{e^z}{(z-\pi i)^4}\right]'\Bigg|_{z=0} = \frac{e^z(z-4-\pi i)}{(z-\pi i)^5}\Bigg|_{z=0} = \frac{\pi-4i}{\pi^5},$$

$$\operatorname*{Res}_{z=\pi i} f(z) = \left(\frac{e^z}{z^2}\right)'''\Bigg|_{z=\pi i} = \frac{e^z(z^3-6z^2+18z-24)}{z^5}\Bigg|_{z=\pi i} = \frac{\pi^3+6\pi^2 i-18\pi-24i}{\pi^5},$$

$$\operatorname*{Res}_{z=\infty} f(z) = -\left[\operatorname*{Res}_{z=0} f(z) + \operatorname*{Res}_{z=\pi i} f(z)\right] = \frac{-\pi^3-6\pi^2 i+17\pi+28i}{\pi^5}.$$

## 习题　六 (一)/3

计算下列各积分

(1) $\displaystyle\int_{|z|=1} \frac{dz}{z\sin z}$;

(2) $\displaystyle\frac{1}{2\pi i}\int_{|z|=2} \frac{e^{zi}}{1+z^2}dz$;

(3) $\displaystyle\int_C \frac{dz}{(z-1)^2(z^2+1)}$, $C: x^2+y^2=2(x+y)$.

**(1)**

在 $|z|=1$ 内只有 0 为奇点.

$$\frac{1}{z\sin z}=\frac{1}{z}\left[\frac{1}{z}+\frac{z}{6}+o(z)\right]=\frac{1}{z^2}+6+o(1).$$

故洛朗展式中 $\frac{1}{z}$ 项的值为 0,

$$\operatorname*{Res}_{z=0} f(z)=0.$$

由留数定理得

$$\int_{|z|=1}\frac{dz}{z\sin z}=0.$$

**(2)**

在 $|z|=1$ 内 $i$ 为一阶极点, $-i$ 为一阶极点.

$$\operatorname*{Res}_{z=i}\frac{e^{zi}}{1+z^2}=\left.\frac{e^{zi}}{z+i}\right|_{z=i}=-\frac{i}{2e},$$

$$\operatorname*{Res}_{z=-i}\frac{e^{zi}}{1+z^2}=\left.\frac{e^{zi}}{z-i}\right|_{z=-i}=\frac{ei}{2}.$$

由留数定理得

$$\frac{1}{2\pi i}\int_{|z|=2}\frac{e^{zi}}{1+z^2}dz=-\frac{i}{2e}+\frac{ei}{2}=\frac{i}{2}\left(e-\frac{1}{e}\right)=i\sinh 1.$$

**(3)**

$$x^2+y^2=2(x+y)\Longrightarrow(x-1)^2+(y-1)^2=2.$$

在 $C$ 内 1 为二阶极点, $i$ 为一阶极点, $-i$ 为一阶极点.

$$\operatorname*{Res}_{z=1}\frac{1}{(z-1)^2(z^2+1)}=\left.\left(\frac{1}{z^2+1}\right)'\right|_{z=1}=-\left.\frac{2z}{(z^2+1)^2}\right|_{z=1}=-\frac{1}{2}.$$

$$\operatorname*{Res}_{z=i}\frac{1}{(z-1)^2(z^2+1)}=\left.\frac{1}{(z-1)^2(z+i)}\right|_{z=i}=\frac{1}{4},$$

$$\operatorname*{Res}_{z=-i}\frac{1}{(z-1)^2(z^2+1)}=\left.\frac{1}{(z-1)^2(z-i)}\right|_{z=-i}=\frac{1}{4}.$$

由留数定理得

$$\int_C\frac{dz}{(z-1)^2(z^2+1)}=2\pi i\left(-\frac{1}{2}+\frac{1}{4}+\frac{1}{4}\right)=0.$$