MA362 — 复分析

Assignment 2

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习题 2.4/1

设 $z = x + yi$

$$\overline{e^z} = \overline{e^{x+yi}} = \overline{e^x(\cos y + i\sin y)} = e^x\overline{\cos y + i\sin y} = e^x(\cos y - i\sin y) = e^{x-yi} = e^{\bar{z}}.$$

习题 2.4/3

设 $z = x + yi$

$$e^z = e^{x+yi} = e^x(\cos y + i\sin y) = 1,$$

$$e^x = |e^z| = 1 \Longrightarrow x = 0,$$

$$\cos y + i\sin y = \frac{e^z}{e^x} = 1 \Longrightarrow y = 2k\pi, \quad k \in Z.$$

故 $z = 2k\pi i,\ k = 0, \pm 1, \cdots$.

习题 2.4/4

(i)

由 $f'(z) = f(z)$ 可得

$$[f(z)e^{-z}]' = f'(z)e^{-z} - f(z)e^{-z} = 0.$$

根据习题 2.2/1, $f(z)e^{-z}$ 为常数, 且已知 $f(0) = 1$, 可得

$$f(z)e^{-z} = f(0)e^0 = 1,$$

$$f(z) \equiv e^z.$$

(ii)

由 $f(z+\omega) = f(z)f(\omega)$ 可得

$$[f(z+\omega)]' = f'(z)f(\omega) + f(z)\cdot 0 = f'(z)f(\omega).$$

当 $z = 0$ 时

$$f'(\omega) = f'(0)f(\omega) = f(\omega).$$

根据 (i) 可得 $f(z) \equiv e^z$.

习题　2.4/7

(i)

由 $f'(z) = e^{-f(z)}$ 可得

$$[e^{f(z)} - z]' = f'(z)e^{f(z)} - 1 = 0.$$

根据习题 2.2/1, $e^{f(z)} - z$ 为常数, 且已知 $f(1) = 0$, 可得

$$e^{f(z)} - z = e^{f(1)} - 1 = 0,$$

$$f(z) \equiv \ln z.$$

(ii)

由 $f(z\omega) = f(z) + f(\omega)$ 可得

$$[f(z\omega)]' = \omega f'(z\omega) = f'(z)$$

当 $z = 1$ 时

$$\omega f'(\omega) = 1,$$
$$f'(\omega) = \frac{1}{\omega},$$
$$f(\omega) = \ln \omega + C.$$

代入 $f(1) = 0$ 可得 $f(z) \equiv \ln z$.

习题　2.4/8

$\forall z_1 \neq z_2$, $z_1, z_2 \in B(0, 1)$

$$f(z) = z^2 + 2z + 3 = (z + 1)^2 + 2,$$

$$f(z_1) - f(z_2) = (z_1 + 1)^2 - (z_2 + 1)^2 = (z_1 - z_2)(z_1 + z_2 + 2).$$

$$|z_1 + z_2 + 2| \geqslant 2 - |z_1 + z_2|,$$

$$|z_1 + z_2| \leqslant |z_1| + |z_2| < 2,$$

$$|z_1 + z_2 + 2| > 0.$$

故 $f(z_1) - f(z_2) \neq 0$, $f(z)$ 在 $B(0, 1)$ 单叶.

习题　2.4/12

(i)

由 $f'(z)=\mu\dfrac{f(z)}{z}$ 可得

$$[f(z)z^{-\mu}]'=f'(z)z^{-\mu}-\mu f(z)z^{-\mu-1}=z^{-\mu}\left(f'(z)-\mu\frac{f(z)}{z}\right)=0.$$

根据习题 2.2/1, $f(z)z^{-\mu}$ 为常数, 且已知 $f(1)=1$, 可得

$$f(z)z^{-\mu}=f(1)\cdot 1^{-\mu}=1,$$

$$f(z)\equiv z^{\mu}=e^{\mu\ln z}=|z|^{\mu}e^{i\mu\arg z}e^{i2k\pi\mu}.$$

取 $k=0$ 时的单叶性域

$$f(z)\equiv|z|^{\mu}e^{i\mu\arg z}.$$

(ii)

由 $f(z\omega)=f(z)f(\omega)$ 可得

$$[f(z\omega)]'=\omega f'(z\omega)=f'(z)f(\omega)+f(z)\cdot 0=f'(z)f(\omega).$$

当 $z=1$ 时

$$\omega f'(\omega)=\mu f(\omega),$$

$$f'(\omega)=\mu\frac{f(\omega)}{\omega}.$$

根据 (i) 可得 $f(z)\equiv|z|^{\mu}e^{i\mu\arg z}$.

习题　2.4/17

$\forall z_1\neq z_2$, $z_1,z_2\in D$, 即证

$$\cos z_1-\cos z_2=-2\sin\frac{z_1+z_2}{2}\sin\frac{z_1-z_2}{2}\neq 0,$$

$$\sin z_1-\sin z_2=2\cos\frac{z_1+z_2}{2}\sin\frac{z_1-z_2}{2}\neq 0.$$

等价于当 $k\in Z$

$$\cos z\ \text{在}D\text{中单叶}:\quad z_1+z_2\neq 2k\pi,\quad z_1-z_2\neq 2k\pi,$$

$$\sin z\ \text{在}D\text{中单叶}:\quad z_1+z_2\neq(2k+1)\pi,\quad z_1-z_2\neq 2k\pi.$$

(i)

$$D = \{x \in \mathbf{C} : \theta_0 < \operatorname{Re} z < \theta_0 + \pi\}.$$

$$\operatorname{Re}(z_1 - z_2) \in (-\pi, 0) \cup (0, \pi) \Longrightarrow z_1 - z_2 \neq 2k\pi.$$

$$\operatorname{Re}(z_1 + z_2) \in (2\theta_0, 2\theta_0 + 2\pi) \Longrightarrow \begin{cases} z_1 + z_2 \neq 2k\pi \text{ 当且仅当 } \theta_0 = k\pi \\ z_1 + z_2 \neq (2k+1)\pi \text{ 当且仅当 } \theta_0 = \left(k + \dfrac{1}{2}\right)\pi \end{cases}.$$

故 $\theta_0 = k\pi$ 时, $\cos z$ 在 D 中单叶; $\theta_0 = \left(k + \dfrac{1}{2}\right)\pi$ 时, $\sin z$ 在 D 中单叶.

(ii)

$$D = \{x \in \mathbf{C} : \theta_0 < \operatorname{Re} z < \theta_0 + 2\pi, \operatorname{Im} z > 0\}.$$

$$\operatorname{Re}(z_1 - z_2) \in (-2\pi, 0) \cup (0, 2\pi) \Longrightarrow z_1 - z_2 \neq 2k\pi.$$

$$\operatorname{Im}(z_1 + z_2) = \operatorname{Im} z_1 + \operatorname{Im} z_2 > 0 \Longrightarrow z_1 + z_2 \neq k\pi.$$

故 $\cos z$ 和 $\sin z$ 在 D 中单叶.

(iii)

$$D = \{x \in \mathbf{C} : \theta_0 < \operatorname{Re} z < \theta_0 + 2\pi, \operatorname{Im} z < 0\}.$$

$$\operatorname{Re}(z_1 - z_2) \in (-2\pi, 0) \cup (0, 2\pi) \Longrightarrow z_1 - z_2 \neq 2k\pi.$$

$$\operatorname{Im}(z_1 + z_2) = \operatorname{Im} z_1 + \operatorname{Im} z_2 < 0 \Longrightarrow z_1 + z_2 \neq k\pi.$$

故 $\cos z$ 和 $\sin z$ 在 D 中单叶.