

MA362 — 复分析

Assignment 1

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— SJTU (Fall 2019)

习题 2.1/1

(i)

对于任意 $z \in \mathbf{C}$ 有

$$\frac{f(z+h) - f(z)}{h} = \frac{|z+h| - |z|}{h} = \frac{|z+h|^2 - |z|^2}{h(|z+h| + |z|)} = \frac{z\bar{h} + \bar{z}h + |h|^2}{h(|z+h| + |z|)}.$$

当 $z \neq 0$ 时

$$\lim_{h \rightarrow 0} \frac{f(z+h) - f(z)}{h} = \lim_{h \rightarrow 0} \frac{z\bar{h} + \bar{z}h}{2|z|h} = \lim_{h \rightarrow 0} \frac{z\bar{h}/h + \bar{z}}{2|z|}.$$

如果让 h 取实数, 则上述极限为 $\frac{z + \bar{z}}{2|z|}$; 如果让 h 取纯虚数, 则上述极限为 $\frac{-z + \bar{z}}{2|z|}$.

当 $z = 0$ 时

$$\lim_{h \rightarrow 0} \frac{f(z+h) - f(z)}{h} = \lim_{h \rightarrow 0} \frac{|h|^2}{h|h|} = \lim_{h \rightarrow 0} \frac{|h|}{h}.$$

如果让 h 取实数, 则上述极限为 1; 如果让 h 取纯虚数, 则上述极限为 -1 .

因此, 当 $h \rightarrow 0$ 时上述极限不存在, 因而在 \mathbf{C} 中处处不可微.

(ii)

对于任意 $z \in \mathbf{C}$ 有

$$\frac{f(z+h) - f(z)}{h} = \frac{|z+h|^2 - |z|^2}{h} = \frac{z\bar{h} + \bar{z}h + |h|^2}{h}.$$

当 $z \neq 0$ 时

$$\lim_{h \rightarrow 0} \frac{f(z+h) - f(z)}{h} = \lim_{h \rightarrow 0} \frac{z\bar{h} + \bar{z}h}{h} = \lim_{h \rightarrow 0} (z\bar{h}/h + \bar{z}).$$

如果让 h 取实数, 则上述极限为 $z + \bar{z}$; 如果让 h 取纯虚数, 则上述极限为 $-z + \bar{z}$.

当 $z = 0$ 时

$$\lim_{h \rightarrow 0} \frac{f(z+h) - f(z)}{h} = \lim_{h \rightarrow 0} \frac{|h|^2}{h} = 0.$$

因此, 当 $z \neq 0, h \rightarrow 0$ 时上述极限不存在, 因而仅在 $z = 0$ 处可微.

(iii)

对于任意 $z \in \mathbf{C}$ 有

$$\lim_{h \rightarrow 0} \frac{f(z+h) - f(z)}{h} = \lim_{h \rightarrow 0} \frac{\operatorname{Re}(z+h) - \operatorname{Re}(z)}{h} = \lim_{h \rightarrow 0} \frac{\operatorname{Re}(h)}{h}.$$

如果让 h 取实数, 则上述极限为 1; 如果让 h 取纯虚数, 则上述极限为 0. 因此, 当 $h \rightarrow 0$ 时上述极限不存在, 因而在 \mathbf{C} 中处处不可微.

(v)

设 $f(z) = z_0$, 对于任意 $z \in \mathbf{C}$ 有

$$\lim_{h \rightarrow 0} \frac{f(z+h) - f(z)}{h} = \lim_{h \rightarrow 0} \frac{z_0 - z_0}{h} = 0.$$

因此, 当 $h \rightarrow 0$ 时上述极限存在, 因而在 \mathbf{C} 中处处可微.

习题 2.1/2

由 f 和 g 都在 z_0 处可微有

$$f'(z_0) = \lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0},$$

$$g'(z_0) = \lim_{z \rightarrow z_0} \frac{g(z) - g(z_0)}{z - z_0}.$$

由 $f(z_0) = g(z_0) = 0$, $g'(z_0) \neq 0$ 可得

$$\frac{f'(z_0)}{g'(z_0)} = \lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{g(z) - g(z_0)} = \lim_{z \rightarrow z_0} \frac{f(z)}{g(z)}.$$

习题 2.1/4

对于任意 $z \in \mathbf{G}$ 有 $\bar{z} \in \mathbf{D}$, 设 $z+h \in \mathbf{G}$, 则 $\overline{z+h} \in \mathbf{D}$. 由 f 是 \mathbf{D} 上的全纯函数可得

$$f'(\bar{z}) = \lim_{\bar{h} \rightarrow 0} \frac{f(\overline{z+h}) - f(\bar{z})}{\bar{h}},$$

$$\overline{f'(\bar{z})} = \overline{\lim_{\bar{h} \rightarrow 0} \frac{f(\overline{z+h}) - f(\bar{z})}{\bar{h}}} = \lim_{h \rightarrow 0} \frac{\overline{f(\overline{z+h}) - f(\bar{z})}}{h} = \lim_{h \rightarrow 0} \frac{\overline{f(\overline{z+h})} - \overline{f(\bar{z})}}{h}.$$

因此 $\overline{f'(\bar{z})}$ 是 \mathbf{G} 上的全纯函数.

习题 2.2/1

设 $f(z) = u(x, y) + iv(x, y)$, 由 $f \in H(D)$ 可知

$$f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} + i \frac{\partial u}{\partial y} = 0.$$

故

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = \frac{\partial u}{\partial y} = 0.$$

因此 $u(x, y)$ 和 $v(x, y)$ 都为常数, 因而 f 是一常数.

习题 2.2/2

设 $f(z) = u(x, y) + iv(x, y)$, 由 $f \in H(D)$ 可知

$$f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} + i \frac{\partial u}{\partial y}, \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

(i)

由 $\operatorname{Re} f(z)$ 是常数可得

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0.$$

故

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = \frac{\partial u}{\partial y} = 0.$$

因此 $u(x, y)$ 和 $v(x, y)$ 都为常数, 因而 f 是一常数.

(ii)

由 $\operatorname{Im} f(z)$ 是常数可得

$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0.$$

故

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = \frac{\partial u}{\partial y} = 0.$$

因此 $u(x, y)$ 和 $v(x, y)$ 都为常数, 因而 f 是一常数.

(iii)

由 $|f(z)|$ 是常数 $k \in \mathbf{R}$ 可得

$$u^2(x, y) + v^2(x, y) = k^2.$$

若 $k = 0$ 易知 $u(x, y) = v(x, y) = 0, f(z) = 0$.

若 $k \neq 0$ 且存在, 分别对 x, y 求偏导可得

$$2u(x, y) \frac{\partial u}{\partial x} + 2v(x, y) \frac{\partial v}{\partial x} = u(x, y) \frac{\partial u}{\partial x} - v(x, y) \frac{\partial u}{\partial y} = 0,$$

$$2u(x, y) \frac{\partial u}{\partial y} + 2v(x, y) \frac{\partial v}{\partial y} = u(x, y) \frac{\partial u}{\partial y} + v(x, y) \frac{\partial u}{\partial x} = 0.$$

解方程组可得

$$[u^2(x, y) + v^2(x, y)] \frac{\partial u}{\partial x} = [u^2(x, y) + v^2(x, y)] \frac{\partial u}{\partial y} = 0,$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0.$$

易知 $u(x, y)$ 是常数, 由 (i) 可得 f 为常数. 因而 f 是一常数.

(iv)

由 $\arg f(z)$ 是常数 $k \in \mathbf{R}$ 可得

$$\frac{v(x, y)}{u(x, y)} = \tan k.$$

若 $\tan k = 0$ 或无穷易知 $u(x, y)$ 或 $v(x, y)$ 是常数, 由 (i)(ii) 可得 f 为常数.

若 $\tan k \neq 0$ 且存在, 分别对 x, y 求偏导可得

$$\frac{\partial v}{\partial x} = \tan k \frac{\partial u}{\partial x} = -\frac{\partial u}{\partial y}, \quad \frac{\partial v}{\partial y} = \tan k \frac{\partial u}{\partial y} = \frac{\partial u}{\partial x}.$$

解方程组可得

$$(1 + \tan^2 k) \frac{\partial u}{\partial x} = (1 + \tan^2 k) \frac{\partial u}{\partial y} = 0,$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0.$$

易知 $u(x, y)$ 是常数, 由 (i) 可得 f 为常数. 因而 f 是一常数.

习题 2.2/3

设 $f(z) = u(x, y) + iv(x, y) = \sqrt{xy}$, $u(x, y) = \sqrt{xy}$, $v(x, y) = 0$.

$$\frac{\partial u}{\partial x} = \lim_{h \rightarrow 0} \frac{u(h, 0) - u(0, 0)}{h} = 0,$$

$$\frac{\partial u}{\partial y} = \lim_{h \rightarrow 0} \frac{u(0, h) - u(0, 0)}{h} = 0,$$

$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0,$$

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2} \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) + \frac{i}{2} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) = 0.$$

故 $f(z)$ 在 $z = 0$ 处满足 Cauchy-Riemann 方程.

设 $h = x + xi$

$$\lim_{h \rightarrow 0} \frac{f(h) - f(0)}{h} = \lim_{x \rightarrow 0} \frac{|x|}{x + xi}.$$

如果让 x 取正数, 则上述极限为 $\frac{1}{1+i}$; 如果让 h 取负数, 则上述极限为 $-\frac{1}{1+i}$.

因此, 当 $h \rightarrow 0$ 时上述极限不存在, 因而 f 在 $z = 0$ 中处不可微.

习题 2.2/4

由 $z = x + yi = r(\cos \theta + i \sin \theta)$ 得

$$x(r, \theta) = r \cos \theta, \quad y(r, \theta) = r \sin \theta,$$

$$\frac{\partial x}{\partial r} = \cos \theta, \quad \frac{\partial x}{\partial \theta} = -r \sin \theta,$$

$$\frac{\partial y}{\partial r} = \sin \theta, \quad \frac{\partial y}{\partial \theta} = r \cos \theta.$$

$f(z) = u(x, y) + iv(x, y) = u(r, \theta) + iv(r, \theta)$, 由链式求导法则可得

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial r} = \frac{\partial u}{\partial x} \cos \theta + \frac{\partial u}{\partial y} \sin \theta,$$

$$\frac{1}{r} \frac{\partial v}{\partial \theta} = \frac{1}{r} \left(\frac{\partial v}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial v}{\partial y} \frac{\partial y}{\partial \theta} \right) = -\frac{\partial v}{\partial x} \sin \theta + \frac{\partial v}{\partial y} \cos \theta,$$

$$\frac{\partial v}{\partial r} = \frac{\partial v}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial v}{\partial y} \frac{\partial y}{\partial r} = \frac{\partial v}{\partial x} \cos \theta + \frac{\partial v}{\partial y} \sin \theta,$$

$$-\frac{1}{r} \frac{\partial u}{\partial \theta} = \frac{1}{r} \left(\frac{\partial u}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial \theta} \right) = \frac{\partial u}{\partial x} \sin \theta - \frac{\partial u}{\partial y} \cos \theta.$$

代入 Cauchy-Riemann 方程

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

可知

$$\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta}, \quad \frac{\partial v}{\partial r} = -\frac{1}{r} \frac{\partial u}{\partial \theta}.$$

习题 2.2/5

同上题, 由链式求导法则可得

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial r} = \frac{\partial f}{\partial x} \cos \theta + \frac{\partial f}{\partial y} \sin \theta,$$

$$\frac{i}{r} \frac{\partial f}{\partial \theta} = \frac{i}{r} \left(\frac{\partial f}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial \theta} \right) = -\frac{\partial f}{\partial x} i \sin \theta + \frac{\partial f}{\partial y} i \cos \theta.$$

代入可知

$$\frac{1}{2} e^{i\theta} \left(\frac{\partial f}{\partial r} + \frac{i}{r} \frac{\partial f}{\partial \theta} \right) = \frac{1}{2} e^{i\theta} \left[\frac{\partial f}{\partial x} (\cos \theta - i \sin \theta) + \frac{\partial f}{\partial y} (\sin \theta + i \cos \theta) \right] = \frac{1}{2} \left(\frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} \right) = \frac{\partial f}{\partial \bar{z}},$$

$$\frac{1}{2} e^{-i\theta} \left(\frac{\partial f}{\partial r} - \frac{i}{r} \frac{\partial f}{\partial \theta} \right) = \frac{1}{2} e^{-i\theta} \left[\frac{\partial f}{\partial x} (\cos \theta + i \sin \theta) + \frac{\partial f}{\partial y} (\sin \theta - i \cos \theta) \right] = \frac{1}{2} \left(\frac{\partial f}{\partial x} - i \frac{\partial f}{\partial y} \right) = \frac{\partial f}{\partial z}.$$

习题 2.2/8

由 $f \in H(D)$ 可知

$$\frac{\partial f}{\partial \bar{z}} = f'(z), \quad \frac{\partial f}{\partial z} = 0.$$

由 $\overline{f(\bar{z})} \in H(D)$ 由此可知

$$\overline{\overline{f(\bar{z}_0 + \Delta z)}} - \overline{f(\bar{z}_0)} = \frac{\partial \bar{f}}{\partial z} \Delta z + \frac{\partial \bar{f}}{\partial \bar{z}} \overline{\Delta z} + o(|\Delta z|).$$

$$\frac{\partial \bar{f}}{\partial \bar{z}} = \overline{f'(z)}, \quad \frac{\partial \bar{f}}{\partial z} = 0.$$

故

$$\begin{aligned}& \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) |f(z)|^p \\&= \Delta |f(z)\overline{f(z)}|^{p/2} \\&= 4 \frac{\partial^2}{\partial z \partial \bar{z}} |f(z)\overline{f(z)}|^{p/2} \\&= 4 \frac{\partial}{\partial \bar{z}} \frac{\partial}{\partial z} |f(z)\overline{f(z)}|^{p/2} \\&= 4 \frac{\partial}{\partial \bar{z}} \frac{p}{2} |f(z)|^{p/2-1} |f'(z)| |\overline{f(z)}|^{p/2} \\&= 4 \cdot \frac{p}{2} |f(z)|^{p/2-1} |f'(z)| \cdot \frac{p}{2} |\overline{f(z)}|^{p/2-1} |\overline{f'(z)}| \\&= p^2 |f(z)\overline{f(z)}|^{p/2-1} |f'(z)\overline{f'(z)}| \\&= p^2 |f(z)|^{p-2} |f'(z)|^2.\end{aligned}$$