MA362 — 复分析

Assignment 1

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习题 2.1/1

(i)

对于任意 $z \in \mathbf{C}$ 有

$$\frac{f(z+h)-f(z)}{h} = \frac{|z+h|-|z|}{h} = \frac{|z+h|^2-|z|^2}{h(|z+h|+|z|)} = \frac{z\bar{h}+\bar{z}h+|h|^2}{h(|z+h|+|z|)}.$$

当 $z \neq 0$ 时

$$\lim_{h\to 0}\frac{f(z+h)-f(z)}{h} = \lim_{h\to 0}\frac{z\bar{h}+\bar{z}h}{2|z|h} = \lim_{h\to 0}\frac{z\bar{h}/h+\bar{z}}{2|z|}.$$

如果让 h 取实数, 则上述极限为 $\dfrac{z+\bar{z}}{2|z|}$; 如果让 h 取纯虚数, 则上述极限为 $\dfrac{-z+\bar{z}}{2|z|}$.

当 $z = 0$ 时

$$\lim_{h\to 0}\frac{f(z+h)-f(z)}{h} = \lim_{h\to 0}\frac{|h|^2}{h|h|} = \lim_{h\to 0}\frac{|h|}{h}.$$

如果让 h 取实数, 则上述极限为 1; 如果让 h 取纯虚数, 则上述极限为 -1.

因此, 当 $h \to 0$ 时上述极限不存在, 因而在 $\mathbf{C}$ 中处处不可微.

(ii)

对于任意 $z \in \mathbf{C}$ 有

$$\frac{f(z+h)-f(z)}{h} = \frac{|z+h|^2-|z|^2}{h} = \frac{z\bar{h}+\bar{z}h+|h|^2}{h}.$$

当 $z \neq 0$ 时

$$\lim_{h\to 0}\frac{f(z+h)-f(z)}{h} = \lim_{h\to 0}\frac{z\bar{h}+\bar{z}h}{h} = \lim_{h\to 0}(z\bar{h}/h+\bar{z}).$$

如果让 h 取实数, 则上述极限为 $z+\bar{z}$; 如果让 h 取纯虚数, 则上述极限为 $-z+\bar{z}$.

当 $z = 0$ 时

$$\lim_{h\to 0}\frac{f(z+h)-f(z)}{h} = \lim_{h\to 0}\frac{|h|^2}{h} = 0.$$

因此, 当 $z \neq 0, h \to 0$ 时上述极限不存在, 因而仅在 $z = 0$ 处可微.

(iii)

对于任意 $z \in \mathbf{C}$ 有

$$\lim_{h\to 0}\frac{f(z+h)-f(z)}{h} = \lim_{h\to 0}\frac{\mathrm{Re}(z+h)-\mathrm{Re}(z)}{h} = \lim_{h\to 0}\frac{\mathrm{Re}(h)}{h}.$$

如果让 h 取实数, 则上述极限为 1; 如果让 h 取纯虚数, 则上述极限为 0.
因此, 当 $h \to 0$ 时上述极限不存在, 因而在 $\mathbf{C}$ 中处处不可微.

(v)

设 $f(z) = z_0$, 对于任意 $z \in \mathbf{C}$ 有

$$\lim_{h\to 0} \frac{f(z+h) - f(z)}{h} = \lim_{h\to 0} \frac{z_0 - z_0}{h} = 0.$$

因此, 当 $h \to 0$ 时上述极限存在, 因而在 $\mathbf{C}$ 中处处可微.

习题 2.1/2

由 f 和 g 都在 z_0 处可微有

$$f'(z_0) = \lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0},$$

$$g'(z_0) = \lim_{z\to z_0} \frac{g(z) - g(z_0)}{z - z_0}.$$

由 $f(z_0) = g(z_0) = 0$, $g'(z_0) \neq 0$ 可得

$$\frac{f'(z_0)}{g'(z_0)} = \lim_{z\to z_0} \frac{f(z) - f(z_0)}{g(z) - g(z_0)} = \lim_{z\to z_0} \frac{f(z)}{g(z)}.$$

习题 2.1/4

对于任意 $z \in \mathbf{G}$ 有 $\bar{z} \in \mathbf{D}$, 设 $z + h \in \mathbf{G}$, 则 $\overline{z+h} \in \mathbf{D}$. 由 f 是 $\mathbf{D}$ 上的全纯函数可得

$$f'(\bar{z}) = \lim_{\bar{h}\to 0} \frac{f(\overline{z+h}) - f(\bar{z})}{\bar{h}},$$

$$\overline{f'(\bar{z})} = \overline{\lim_{\bar{h}\to 0} \frac{f(\overline{z+h}) - f(\bar{z})}{\bar{h}}} = \lim_{h\to 0} \frac{\overline{f(\overline{z+h}) - f(\bar{z})}}{h} = \lim_{h\to 0} \frac{\overline{f(\overline{z+h})} - \overline{f(\bar{z})}}{h}.$$

因此 $\overline{f(\bar{z})}$ 是 $\mathbf{G}$ 上的全纯函数.

习题 2.2/1

设 $f(z) = u(x, y) + iv(x, y)$, 由 $f \in H(D)$ 可知

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} + i\frac{\partial u}{\partial y} = 0.$$

故

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = \frac{\partial u}{\partial y} = 0.$$

因此 $u(x, y)$ 和 $v(x, y)$ 都为常数, 因而 f 是一常数.

习题 2.2/2

设 $f(z)=u(x,y)+iv(x,y)$, 由 $f\in H(D)$ 可知

$$f'(z)=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}=\frac{\partial v}{\partial y}+i\frac{\partial u}{\partial y},\quad \frac{\partial u}{\partial x}=\frac{\partial v}{\partial y},\quad \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}.$$

(i)

由 $\operatorname{Re} f(z)$ 是常数可得

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial y}=0.$$

故

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial x}=\frac{\partial v}{\partial y}=\frac{\partial u}{\partial y}=0.$$

因此 $u(x,y)$ 和 $v(x,y)$ 都为常数, 因而 f 是一常数.

(ii)

由 $\operatorname{Im} f(z)$ 是常数可得

$$\frac{\partial v}{\partial x}=\frac{\partial v}{\partial y}=0.$$

故

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial x}=\frac{\partial v}{\partial y}=\frac{\partial u}{\partial y}=0.$$

因此 $u(x,y)$ 和 $v(x,y)$ 都为常数, 因而 f 是一常数.

(iii)

由 $|f(z)|$ 是常数 $k\in\mathbf{R}$ 可得

$$u^2(x,y)+v^2(x,y)=k^2.$$

若 $k=0$ 易知 $u(x,y)=v(x,y)=0,f(z)=0$.
若 $k=0$ 且存在, 分别对 x, y 求偏导可得

$$2u(x,y)\frac{\partial u}{\partial x}+2v(x,y)\frac{\partial v}{\partial x}=u(x,y)\frac{\partial u}{\partial x}-v(x,y)\frac{\partial u}{\partial y}=0,$$

$$2u(x,y)\frac{\partial u}{\partial y}+2v(x,y)\frac{\partial v}{\partial y}=u(x,y)\frac{\partial u}{\partial y}+v(x,y)\frac{\partial u}{\partial x}=0.$$

解方程组可得

$$\left[u^2(x,y)+v^2(x,y)\right]\frac{\partial u}{\partial x}=\left[u^2(x,y)+v^2(x,y)\right]\frac{\partial u}{\partial y}=0,$$

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial y}=0.$$

易知 $u(x,y)$ 是常数, 由 (i) 可得 f 为常数. 因而 f 是一常数.

(iv)

由 $\arg f(z)$ 是常数 $k \in \mathbf{R}$ 可得

$$\frac{v(x,y)}{u(x,y)} = \tan k.$$

若 $\tan k = 0$ 或无穷易知 $u(x,y)$ 或 $v(x,y)$ 是常数, 由 (i)(ii) 可得 f 为常数.
若 $\tan k \neq 0$ 且存在, 分别对 x, y 求偏导可得

$$\frac{\partial v}{\partial x} = \tan k \frac{\partial u}{\partial x} = -\frac{\partial u}{\partial y}, \quad \frac{\partial v}{\partial y} = \tan k \frac{\partial u}{\partial y} = \frac{\partial u}{\partial x}.$$

解方程组可得

$$(1+\tan^2 k)\frac{\partial u}{\partial x} = (1+\tan^2 k)\frac{\partial u}{\partial y} = 0,$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0.$$

易知 $u(x,y)$ 是常数, 由 (i) 可得 f 为常数. 因而 f 是一常数.

习题 2.2/3

设 $f(z) = u(x,y) + iv(x,y) = \sqrt{xy}$, $u(x,y) = \sqrt{xy}$, $v(x,y) = 0$.

$$\frac{\partial u}{\partial x} = \lim_{h\to 0} \frac{u(h,0)-u(0,0)}{h} = 0,$$

$$\frac{\partial u}{\partial y} = \lim_{h\to 0} \frac{u(0,h)-u(0,0)}{h} = 0,$$

$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0,$$

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = 0.$$

故 $f(z)$ 在 $z=0$ 处满足 Cauchy-Riemann 方程.
设 $h = x + xi$

$$\lim_{h\to 0} \frac{f(h)-f(0)}{h} = \lim_{x\to 0} \frac{|x|}{x+xi}.$$

如果让 x 取正数, 则上述极限为 $\dfrac{1}{1+i}$; 如果让 h 取负数, 则上述极限为 $-\dfrac{1}{1+i}$.
因此, 当 $h \to 0$ 时上述极限不存在, 因而 f 在 $z=0$ 中处不可微.

习题 2.2/4

由 $z = x + yi = r(\cos\theta + i\sin\theta)$ 得

$$x(r,\theta) = r\cos\theta, \quad y(r,\theta) = r\sin\theta,$$

$$\frac{\partial x}{\partial r} = \cos\theta, \quad \frac{\partial x}{\partial \theta} = -r\sin\theta,$$

$$\frac{\partial y}{\partial r} = \sin\theta, \quad \frac{\partial y}{\partial \theta} = r\cos\theta.$$

$f(z) = u(x,y) + iv(x,y) = u(r,\theta) + iv(r,\theta)$, 由链式求导法则可得

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta,$$

$$\frac{1}{r}\frac{\partial v}{\partial \theta} = \frac{1}{r}\left(\frac{\partial v}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial \theta}\right) = -\frac{\partial v}{\partial x}\sin\theta + \frac{\partial v}{\partial y}\cos\theta,$$

$$\frac{\partial v}{\partial r} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial v}{\partial x}\cos\theta + \frac{\partial v}{\partial y}\sin\theta,$$

$$-\frac{1}{r}\frac{\partial u}{\partial \theta} = \frac{1}{r}\left(\frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta}\right) = \frac{\partial u}{\partial x}\sin\theta - \frac{\partial u}{\partial y}\cos\theta.$$

代入 Cauchy-Riemann 方程

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

可知

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}.$$

习题 2.2/5

同上题, 由链式求导法则可得

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta,$$

$$\frac{i}{r}\frac{\partial f}{\partial \theta} = \frac{i}{r} = \left(\frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta}\right) = -\frac{\partial f}{\partial x}i\sin\theta + \frac{\partial f}{\partial y}i\cos\theta.$$

代入可知

$$\frac{1}{2}e^{i\theta}\left(\frac{\partial f}{\partial r} + \frac{i}{r}\frac{\partial f}{\partial \theta}\right) = \frac{1}{2}e^{i\theta}\left[\frac{\partial f}{\partial x}(\cos\theta - i\sin\theta) + \frac{\partial f}{\partial y}(\sin\theta + i\cos\theta)\right] = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) = \frac{\partial f}{\partial \bar{z}},$$

$$\frac{1}{2}e^{-i\theta}\left(\frac{\partial f}{\partial r} - \frac{i}{r}\frac{\partial f}{\partial \theta}\right) = \frac{1}{2}e^{-i\theta}\left[\frac{\partial f}{\partial x}(\cos\theta + i\sin\theta) + \frac{\partial f}{\partial y}(\sin\theta - i\cos\theta)\right] = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) = \frac{\partial f}{\partial z}.$$

习题 2.2/8

由 $f \in H(D)$ 可知

$$\frac{\partial f}{\partial z} = f'(z), \quad \frac{\partial f}{\partial \bar{z}} = 0.$$

由 $\overline{f(\bar{z})} \in H(D)$ 由此可知

$$\overline{f(\overline{z_0 + \Delta z})} - \overline{f(\overline{z_0})} = \frac{\partial \bar{f}}{\partial z}\Delta z + \frac{\partial \bar{f}}{\partial \bar{z}}\overline{\Delta z} + o(|\Delta z|).$$

$$\frac{\partial \bar{f}}{\partial \bar{z}} = \overline{f'(z)}, \quad \frac{\partial \bar{f}}{\partial z} = 0.$$

故

$$
\begin{aligned}
&\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^p\\
=&\Delta|f(z)\overline{f(z)}|^{p/2}\\
=&4\frac{\partial^2}{\partial z\partial\bar{z}}|f(z)\overline{f(z)}|^{p/2}\\
=&4\frac{\partial^2}{\partial z\partial\bar{z}}|f(z)\overline{f(z)}|^{p/2}\\
=&4\frac{\partial}{\partial\bar{z}}\frac{p}{2}|f(z)|^{p/2-1}|f'(z)||\overline{f(z)}|^{p/2}\\
=&4\cdot\frac{p}{2}|f(z)|^{p/2-1}|f'(z)|\cdot\frac{p}{2}|\overline{f(z)}|^{p/2-1}|\overline{f'(z)}|\\
=&p^2|f(z)\overline{f(z)}|^{p/2-1}|f'(x)\overline{f'(x)}|\\
=&p^2|f(z)|^{p-2}|f'(x)|^2.
\end{aligned}
$$