MA362 — 复分析

Assignment 4

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习题 二/11

试证

(1) $\overline{e^z} = e^{\bar{z}}$;

(3) $\overline{\cos z} = \cos \bar{z}$.

(1)

设 $z = x + yi$

$$\overline{e^z} = \overline{e^{x+yi}} = \overline{e^x(\cos y + i\sin y)} = e^x\overline{\cos y + i\sin y} = e^x(\cos y - i\sin y) = e^{x-yi} = e^{\bar{z}}.$$

(3)

$$\overline{\cos z} = \overline{\frac{1}{2}(e^{iz} + e^{-iz})} = \frac{1}{2}\overline{e^{iz}} + \frac{1}{2}\overline{e^{-iz}} = \frac{1}{2}e^{i\bar{z}} + \frac{1}{2}e^{-i\bar{z}} = \cos \bar{z}.$$

习题 二/13

试求下面各式之值

(1) e^{3+i};

(2) $\cos(1-i)$.

(1)

$$e^{3+i} = e^3(cos 1 + i\sin 1) = e^3\cos 1 + ie^3\sin 1.$$

(2)

$$\cos(1-i) = \frac{1}{2}e^{i(1-i)} + \frac{1}{2}e^{-i(1-i)} = \frac{1}{2}[e(\cos 1 + i\sin 1) + e^{-1}(\cos 1 - i\sin 1)] = (e+e^{-1})\cos 1 + i(e-e^{-1})\sin 1.$$

习题 二/20

试解方程:

(1) $e^z = 1+\sqrt{3}i$;
(2) $\ln z = \dfrac{\pi i}{2}$;
(3) $1+e^z=0$.

(1)

$$z = \ln\left(1+\sqrt{3}\right) = \ln|1+\sqrt{3}| + i\arg(1+\sqrt{3}) + 2k\pi i = \ln 2 + \left(2k+\frac{1}{3}\right)\pi i, \quad k\in Z.$$

(2)

$$z = e^{\frac{\pi i}{2}} = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = i.$$

(3)

$$e^z = -1,$$

$$z = \ln(-1) = \ln|1| + i\arg(-1) + 2k\pi i = (2k+1)\pi i, \quad k\in Z.$$

习题 二/21

设 $z = re^{i\theta}$, 试证

$$\mathrm{Re}[\ln(z-1)] = \frac{1}{2}\ln\left(1+r^2-2r\cos\theta\right).$$

$$\begin{aligned}
\mathrm{Re}[\ln(z-1)] &= \ln|z-1| = \ln|re^{i\theta}-1| \\
&= \ln|r\cos\theta + ir\sin\theta - 1| \\
&= \ln\sqrt{r^2\cos^2\theta - 2r\cos\theta + 1 + r^2\sin^2\theta} \\
&= \frac{1}{2}\ln\left(1+r^2-2r\cos\theta\right).
\end{aligned}$$

习题 二/22

设 $w=\sqrt[3]{z}$ 确定在从原点 $z=0$ 起沿正实轴割破了的 z 平面上, 并且 $w(i)=-i$, 试求 $w(-i)$ 之值.

设

$$w(z)=\sqrt[3]{z}=\sqrt[3]{|z|}e^{i\frac{\arg z+2k\pi}{3}},\quad k=0,1,2.$$

代入得

$$w(i)=\sqrt[3]{|i|}e^{i\frac{\arg i+2k\pi}{3}}=e^{i\frac{(2k+1/2)\pi}{3}}=e^{i\frac{3}{2}\pi}=-i,$$

$$\frac{2k+1/2}{3}\pi=\frac{3}{2}\pi\Longrightarrow k=2.$$

故

$$w(z)=\sqrt[3]{|z|}e^{i\frac{\arg z+4\pi}{3}},$$

$$w(-i)=\sqrt[3]{|-i|}e^{i\frac{\arg(-i)+4\pi}{3}}=e^{i\frac{(4+3/2)\pi}{3}}=e^{i\frac{11}{6}\pi}=\cos\frac{11}{6}\pi+i\sin\frac{11}{6}\pi=\frac{\sqrt{3}}{2}-\frac{1}{2}i.$$

习题　二/23

设 $w=\sqrt[3]{z}$ 确定在从原点 $z=0$ 起沿负实轴割破了的 z 平面上，并且 $w(-2)=-\sqrt[3]{2}$(这是边界上岸点对应的函数值)，试求 $w(i)$ 之值.

设

$$w(z)=\sqrt[3]{z}=\sqrt[3]{|z|}e^{i\frac{\arg z+2k\pi}{3}},\quad k=0,1,2.$$

代入得

$$w(-2)=\sqrt[3]{|-2|}e^{i\frac{\arg(-2)+2k\pi}{3}}=\sqrt[3]{2}e^{i\frac{(2k+1)\pi}{3}}=\sqrt[3]{2}e^{i\pi}=-\sqrt[3]{2},$$

$$\frac{2k+1}{3}\pi=\pi\Longrightarrow k=1.$$

故

$$w(z)=\sqrt[3]{|z|}e^{i\frac{\arg z+2\pi}{3}},$$

$$w(i)=\sqrt[3]{|i|}e^{i\frac{\arg i+2\pi}{3}}=e^{i\frac{(2+1/2)\pi}{3}}=e^{i\frac{5}{6}\pi}=\cos\frac{5}{6}\pi+i\sin\frac{5}{6}\pi=-\frac{\sqrt{3}}{2}+\frac{1}{2}i.$$