MA362 — 复分析

Assignment 10

Instructor: 姚卫红

Author: 刘逸灏 (515370910207)

— SJTU (Fall 2019)

习题　六 (一)/4

求下列各积分之值:

(1) $\displaystyle\int_0^{2\pi} \frac{d\theta}{a+\cos\theta} \quad (a>1);$

(2) $\displaystyle\int_0^{2\pi} \frac{dx}{(2+\sqrt{3}\cos x)^2}.$

(1)

令 $z = e^{i\theta}$, 则 $d\theta = \dfrac{dz}{iz}$

$$\int_0^{2\pi} \frac{d\theta}{2a+\cos\theta} = \int_{|z|=1} \frac{2dz}{iz(a+z+z^{-1})} = \frac{2}{i}\int_{|z|=1} \frac{1}{z^2+2az+1}dz,$$

$$f(z) = \frac{1}{z^2+2az+1} = \frac{1}{(z+a-\sqrt{a^2-1})(z+a+\sqrt{a^2-1})}.$$

$-a+\sqrt{a^2-1}$ 为一阶极点, $-a-\sqrt{a^2-1}$ 为一阶极点, 只有 $-a+\sqrt{a^2-1}$ 在圆 $|z|<1$ 内.

$$\operatorname*{Res}_{z=-a+\sqrt{a^2-1}} f(z) = \left.\frac{1}{z+a+\sqrt{a^2-1}}\right|_{z=-a+\sqrt{a^2-1}} = \frac{1}{2\sqrt{a^2-1}}.$$

由留数定理得

$$\int_0^{2\pi} \frac{d\theta}{2a+\cos\theta} = \frac{2}{i}\cdot 2\pi i\cdot\frac{1}{2\sqrt{a^2-1}} = \frac{2\pi}{\sqrt{a^2-1}}.$$

(2)

令 $z = e^{ix}$, 则 $dx = \dfrac{dz}{iz}$

$$\int_0^{2\pi} \frac{dx}{(2+\sqrt{3}\cos x)^2} = \int_{|z|=1} \frac{4dz}{iz(4+\sqrt{3}z+\sqrt{3}z^{-1})^2} = \frac{4}{i}\int_{|z|=1} \frac{z}{(\sqrt{3}z^2+4z+\sqrt{3})^2}dz,$$

$$f(z) = \frac{z}{(\sqrt{3}z^2+4z+\sqrt{3})^2} = \frac{z}{(z+\sqrt{3})^2(\sqrt{3}z+1)^2}.$$

$-\sqrt{3}$ 为二阶极点, $-\dfrac{1}{\sqrt{3}}$ 为二阶极点, 只有 $-\dfrac{1}{\sqrt{3}}$ 在圆 $|z|<1$ 内.

$$\operatorname*{Res}_{z=-\frac{1}{\sqrt{3}}} f(z) = \left[\frac{z}{3(z+\sqrt{3})^2}\right]'\bigg|_{z=-\frac{1}{\sqrt{3}}} = \frac{\sqrt{3}-z}{3(\sqrt{3}+z)^3}\bigg|_{z=-\frac{1}{\sqrt{3}}} = \frac{1}{2}.$$

由留数定理得

$$\int_0^{2\pi} \frac{dx}{(2+\sqrt{3}\cos x)^2} = \frac{4}{i}\cdot 2\pi i\cdot\frac{1}{2} = 4\pi.$$

习题　六 (一)/5

求下列各积分:

(2) $\displaystyle\int_{-\infty}^{+\infty} \frac{x^2}{(x^2+a^2)^2}dx \quad (a>0)$;

(4) $\displaystyle\int_0^{+\infty} \frac{x\sin mx}{x^4+a^4}dx \quad (m>0, a>0)$.

(2)

$$f(z) = \frac{z^2}{(z^2+a^2)^2} = \frac{z^2}{(z-ia)^2(z+ia)^2}.$$

ia 为二阶极点, $-ia$ 为二阶极点, 只有 ia 在上半平面内.

$$\operatorname*{Res}_{z=ia} = \left[\frac{z^2}{(z+ia)^2}\right]'\bigg|_{z=ia} = -\frac{2az}{(a-iz)^3}\bigg|_{z=ia} = -\frac{i}{4a}.$$

由定理 6.7 得

$$\frac{x^2}{(x^2+a^2)^2}dx = 2\pi i\cdot -\frac{i}{4a} = \frac{\pi}{2a}.$$

(4)

被积函数是偶函数, 故

$$\int_0^{+\infty} \frac{x\sin mx}{x^4+a^4}dx = \frac{1}{2}\int_{-\infty}^{+\infty} \frac{x\sin mx}{x^4+a^4}dx,$$

$$f(z) = \frac{ze^{imz}}{z^4+a^4}.$$

有四个一阶极点

$$a_k = ae^{\frac{\pi+2k\pi}{4}i}, \quad (k=0,1,2,3),$$

$$\operatorname*{Res}_{z=a_k} f(z) = \frac{ze^{imz}}{(z^4+a^4)'}\bigg|_{z=a_k} = \frac{e^{imz}}{4z^2}\bigg|_{z=a_k} = \frac{e^{ima_k}}{4a_k^2}.$$

$f(z)$ 在上半平面内只有两个极点 a_0 和 a_1

$$\operatorname*{Res}_{z=a_0} f(z) = \frac{e^{-\frac{\sqrt{2}ma}{2}+i\frac{\sqrt{2}ma}{2}}}{4a^2 i} = \frac{e^{-\frac{\sqrt{2}ma}{2}}}{4a^2 i}\left(\cos\frac{\sqrt{2}ma}{2} + i\sin\frac{\sqrt{2}ma}{2}\right),$$

$$\operatorname*{Res}_{z=a_1} f(z) = \frac{e^{-\frac{\sqrt{2}ma}{2} - i\frac{\sqrt{2}ma}{2}}}{-4a^2 i} = -\frac{e^{-\frac{\sqrt{2}ma}{2}}}{4a^2 i}\left(\cos\frac{\sqrt{2}ma}{2} - i\sin\frac{\sqrt{2}ma}{2}\right).$$

由定理 6.8 得

$$\int_{-\infty}^{+\infty} \frac{xe^{imx}}{x^4+a^4}dx = 2\pi i \cdot \left[\operatorname*{Res}_{z=a_0} f(z) + \operatorname*{Res}_{z=a_1} f(z)\right] = 2\pi i \cdot \frac{e^{-\frac{\sqrt{2}ma}{2}}}{4a^2 i} \cdot 2i\sin\frac{\sqrt{2}ma}{2} = i\frac{\pi}{a^2}e^{-\frac{\sqrt{2}ma}{2}}\sin\frac{\sqrt{2}ma}{2}.$$

且

$$\int_{-\infty}^{+\infty} \frac{xe^{imx}}{x^4+a^4}dx = \int_{-\infty}^{+\infty} \frac{x\cos mx}{x^4+a^4}dx + i\int_{-\infty}^{+\infty} \frac{x\sin mx}{x^4+a^4}dx.$$

故

$$\int_{0}^{+\infty} \frac{x\sin mx}{x^4+a^4}dx = \frac{\pi}{2a^2}e^{-\frac{\sqrt{2}ma}{2}}\sin\frac{\sqrt{2}ma}{2}.$$

习题　六 (一)/6

仿照例 6.15 的方法计算下列积分:

(1) $\displaystyle\int_{0}^{+\infty} \frac{\sin x}{x(x^2+a^2)}dx \quad (a>0)$;

(2) $\displaystyle\int_{0}^{+\infty} \frac{\sin x}{x(x^2+1)^2}dx$.

(1)

被积函数是偶函数, 故

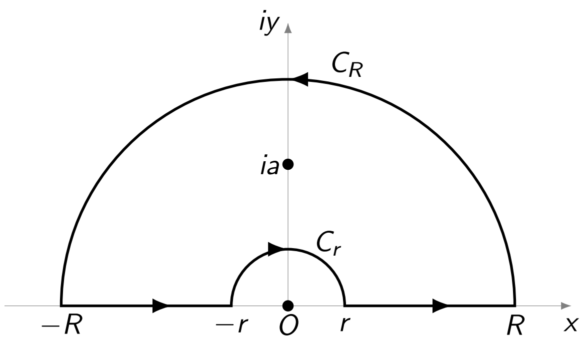
$$\int_{0}^{+\infty} \frac{\sin x}{x(x^2+a^2)}dx = \frac{1}{2}\text{P.V.}\int_{-\infty}^{+\infty} \frac{\sin x}{x(x^2+a^2)}dx,$$

$$f(z) = \frac{e^{iz}}{z(z^2+a^2)} = \frac{e^{iz}}{z(z-ia)(z+ia)}.$$

0 为一阶极点, ia 为一阶极点, $-ia$ 为一阶极点, 只有 ia 在上半平面内.

$$\operatorname*{Res}_{z=ia} = \left.\frac{e^{iz}}{z(z+ia)}\right|_{z=ia} = -\frac{e^{-a}}{2a^2}.$$

考虑 $f(z)$ 在上半平面内沿下图所示之闭曲线路径 C 的积分.



由留数定理得

$$\int_r^R f(x)dx + \int_{C_R} f(z)dz + \int_{-R}^{-r} f(x)dx - \int_{C_r} f(z)dz = 2\pi i \operatorname*{Res}_{z=ia} = 2\pi i \cdot -\frac{e^{-a}}{2a^2} = -i\pi\frac{e^{-a}}{a^2}.$$

由引理 6.2 知

$$\lim_{R\to+\infty}\int_{C_R} f(z)dz = \lim_{R\to+\infty}\int_{C_R}\frac{e^{iz}}{z(z^2+a^2)}dz = 0.$$

由引理 6.3 知

$$\lim_{r\to 0}\int_{C_r} f(z)dz = \lim_{r\to 0}\int_{C_r}\frac{e^{iz}}{z(z^2+a^2)}dz = i\pi\lim_{r\to 0}\frac{e^{iz}}{z^2+a^2} = i\pi\frac{1}{a^2}.$$

另 $r\to 0$, $R\to+\infty$ 可得

$$\text{P.V.}\int_{-\infty}^{+\infty}\frac{e^{ix}}{x(x^2+a^2)}dx = i\pi\frac{1-e^{-a}}{a^2}.$$

且

$$\int_{-\infty}^{+\infty}\frac{e^{ix}}{x(x^2+a^2)}dx = \int_{-\infty}^{+\infty}\frac{\cos x}{x(x^2+a^2)}dx + i\int_{-\infty}^{+\infty}\frac{\sin x}{x(x^2+a^2)}dx.$$

故

$$\int_0^{+\infty}\frac{\sin x}{x(x^2+a^2)}dx = \frac{\pi(1-e^{-a})}{2a^2}.$$

(2)

被积函数是偶函数, 故

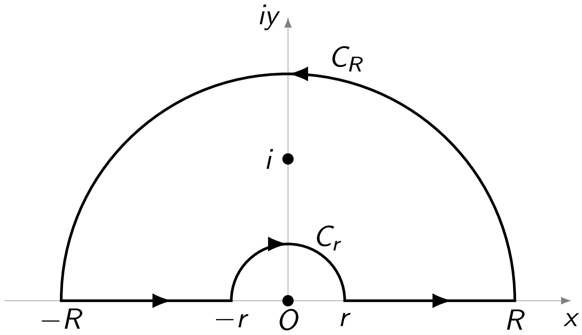
$$\int_0^{+\infty}\frac{\sin x}{x(x^2+1)^2}dx = \frac{1}{2}\text{P.V.}\int_{-\infty}^{+\infty}\frac{\sin x}{x(x^2+1)^2}dx,$$

$$f(z) = \frac{e^{iz}}{z(z^2+1)^2} = \frac{e^{iz}}{z(z-i)^2(z+i)^2}.$$

0 为一阶极点, i 为二阶极点, $-i$ 为二阶极点, 只有 i 在上半平面内.

$$\operatorname*{Res}_{z=i} = \left[\frac{e^{iz}}{z(z+i)^2}\right]'\bigg|_{z=i} = \frac{ie^{iz}(z^2+4iz-1)}{z^2(z+i)^3}\bigg|_{z=i} = -\frac{3}{4e}.$$

考虑 $f(z)$ 在上半平面内沿下图所示之闭曲线路径 C 的积分.



由留数定理得

$$\int_r^R f(x)dx + \int_{C_R} f(z)dz + \int_{-R}^{-r} f(x)dx - \int_{C_r} f(z)dz = 2\pi i \operatorname*{Res}_{z=ia} = 2\pi i \cdot -\frac{3}{4e} = -i\pi\frac{3}{2e}.$$

由引理 6.2 知
$$\lim_{R\to+\infty}\int_{C_R} f(z)dz = \lim_{R\to+\infty}\int_{C_R}\frac{e^{iz}}{z(z^2+1)^2}dz = 0.$$
由引理 6.3 知
$$\lim_{r\to 0}\int_{C_r} f(z)dz = \lim_{r\to 0}\int_{C_r}\frac{e^{iz}}{z(z^2+1)^2}dz = i\pi\lim_{r\to 0}\frac{e^{iz}}{(z^2+1)^2} = i\pi.$$
另 $r\to 0$, $R\to+\infty$ 可得
$$\text{P.V.}\int_{-\infty}^{+\infty}\frac{e^{ix}}{x(x^2+1)^2}dx = i\pi\left(1-\frac{3}{2e}\right).$$
且
$$\int_{-\infty}^{+\infty}\frac{e^{ix}}{x(x^2+1)^2}dx = \int_{-\infty}^{+\infty}\frac{\cos x}{x(x^2+1)^2}dx + i\int_{-\infty}^{+\infty}\frac{\sin x}{x(x^2+1)^2}dx.$$
故
$$\int_{0}^{+\infty}\frac{\sin x}{x(x^2+1)^2}dx = \pi\left(\frac{1}{2}-\frac{3}{4e}\right).$$