

## MA362 — 复分析

### Assignment 4

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## 习题 二/11

试证

- (1)  $\overline{e^z} = e^{\bar{z}}$ ;
- (3)  $\overline{\cos z} = \cos \bar{z}$ .

(1)

设  $z = x + yi$

$$\overline{e^z} = \overline{e^{x+yi}} = \overline{e^x(\cos y + i \sin y)} = e^x \overline{\cos y + i \sin y} = e^x(\cos y - i \sin y) = e^{x-yi} = e^{\bar{z}}.$$

(3)

$$\overline{\cos z} = \overline{\frac{1}{2}(e^{iz} + e^{-iz})} = \frac{1}{2}\overline{e^{iz}} + \frac{1}{2}\overline{e^{-iz}} = \frac{1}{2}e^{i\bar{z}} + \frac{1}{2}e^{-i\bar{z}} = \cos \bar{z}.$$

## 习题 二/13

试求下面各式之值

- (1)  $e^{3+i}$ ;
- (2)  $\cos(1-i)$ .

(1)

$$e^{3+i} = e^3(\cos 1 + i \sin 1) = e^3 \cos 1 + ie^3 \sin 1.$$

(2)

$$\cos(1-i) = \frac{1}{2}e^{i(1-i)} + \frac{1}{2}e^{-i(1-i)} = \frac{1}{2}[e(\cos 1 + i \sin 1) + e^{-1}(\cos 1 - i \sin 1)] = (e + e^{-1}) \cos 1 + i(e - e^{-1}) \sin 1.$$

## 习题 二/20

试解方程:

- (1)  $e^z = 1 + \sqrt{3}i$ ;
- (2)  $\ln z = \frac{\pi i}{2}$ ;
- (3)  $1 + e^z = 0$ .

(1)

$$z = \ln(1 + \sqrt{3}) = \ln|1 + \sqrt{3}| + i \arg(1 + \sqrt{3}) + 2k\pi i = \ln 2 + \left(2k + \frac{1}{3}\right)\pi i, \quad k \in \mathbb{Z}.$$

(2)

$$z = e^{\frac{\pi i}{2}} = \cos \frac{\pi}{2} + i \sin \frac{\pi}{2} = i.$$

(3)

$$e^z = -1,$$

$$z = \ln(-1) = \ln|1| + i \arg(-1) + 2k\pi i = (2k + 1)\pi i, \quad k \in \mathbb{Z}.$$

## 习题 二/21

设  $z = re^{i\theta}$ , 试证

$$\operatorname{Re}[\ln(z - 1)] = \frac{1}{2} \ln(1 + r^2 - 2r \cos \theta).$$

$$\begin{aligned} \operatorname{Re}[\ln(z - 1)] &= \ln|z - 1| = \ln|re^{i\theta} - 1| \\ &= \ln|r \cos \theta + ir \sin \theta - 1| \\ &= \ln \sqrt{r^2 \cos^2 \theta - 2r \cos \theta + 1 + r^2 \sin^2 \theta} \\ &= \frac{1}{2} \ln(1 + r^2 - 2r \cos \theta). \end{aligned}$$

## 习题 二/22

设  $w = \sqrt[3]{z}$  确定在从原点  $z = 0$  起沿正实轴割破了的  $z$  平面上, 并且  $w(i) = -i$ , 试求  $w(-i)$  之值.

设

$$w(z) = \sqrt[3]{z} = \sqrt[3]{|z|} e^{i \frac{\arg z + 2k\pi}{3}}, \quad k = 0, 1, 2.$$

代入得

$$w(i) = \sqrt[3]{|i|} e^{i \frac{\arg i + 2k\pi}{3}} = e^{i \frac{(2k+1/2)\pi}{3}} = e^{i \frac{3}{2}\pi} = -i,$$
$$\frac{2k+1/2}{3}\pi = \frac{3}{2}\pi \implies k = 2.$$

故

$$w(z) = \sqrt[3]{|z|} e^{i \frac{\arg z + 4\pi}{3}},$$
$$w(-i) = \sqrt[3]{|-i|} e^{i \frac{\arg(-i) + 4\pi}{3}} = e^{i \frac{(4+3/2)\pi}{3}} = e^{i \frac{11}{6}\pi} = \cos \frac{11}{6}\pi + i \sin \frac{11}{6}\pi = \frac{\sqrt{3}}{2} - \frac{1}{2}i.$$

## 习题 二/23

设  $w = \sqrt[3]{z}$  确定在从原点  $z = 0$  起沿负实轴割破了的  $z$  平面上, 并且  $w(-2) = -\sqrt[3]{2}$  (这是边界上岸点对应的函数值), 试求  $w(i)$  之值.

设

$$w(z) = \sqrt[3]{z} = \sqrt[3]{|z|} e^{i \frac{\arg z + 2k\pi}{3}}, \quad k = 0, 1, 2.$$

代入得

$$w(-2) = \sqrt[3]{|-2|} e^{i \frac{\arg(-2) + 2k\pi}{3}} = \sqrt[3]{2} e^{i \frac{(2k+1)\pi}{3}} = \sqrt[3]{2} e^{i\pi} = -\sqrt[3]{2},$$
$$\frac{2k+1}{3}\pi = \pi \implies k = 1.$$

故

$$w(z) = \sqrt[3]{|z|} e^{i \frac{\arg z + 2\pi}{3}},$$
$$w(i) = \sqrt[3]{|i|} e^{i \frac{\arg i + 2\pi}{3}} = e^{i \frac{(2+1/2)\pi}{3}} = e^{i \frac{5}{6}\pi} = \cos \frac{5}{6}\pi + i \sin \frac{5}{6}\pi = -\frac{\sqrt{3}}{2} + \frac{1}{2}i.$$

## 习题 二/24

试求  $(1+i)^i$  和  $3^i$  之值.

$$(1+i)^i = e^{i \ln(1+i)} = e^{i[\ln|1+i| + i \arg(1+i) + 2k\pi i]} = e^{-(2k+1/4)\pi} e^{i \ln \sqrt{2}}, \quad k \in \mathbb{Z}.$$

$$3^i = e^{i \ln 3} = e^{i(\ln|3| + i \arg 3 + 2k\pi i)} = e^{-2k\pi} e^{i \ln 3}, \quad k \in \mathbb{Z}.$$