MA426 — 微分几何

Assignment 2

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习题　2.5/3

证明: 下列每一条曲线的曲率和和挠率相等.

(1) $\mathbf{r}(t) = (a(3t - t^3), 3at^2, a(3t + t^3))$;

(2) $\mathbf{r}(t) = \left(t, \dfrac{t^2}{2a}, \dfrac{t^3}{6a^2}\right), \quad a \neq 0$;

(3) $\mathbf{r}(t) = \left(t + \dfrac{a^2}{t}, t - \dfrac{a^2}{t}, 2a\log\dfrac{t}{a}\right), \quad t, a > 0.$

(1)

$$
\begin{aligned}
\mathbf{r}(t) &= (a(3t - t^3), 3at^2, a(3t + t^3)),\\
\mathbf{r}'(t) &= (a(3 - 3t^2), 6at, a(3 + 3t^2)),\\
\mathbf{r}''(t) &= (-6at, 6a, 6at),\\
\mathbf{r}'''(t) &= (-6a, 0, 6a),\\
|\mathbf{r}'(t)| &= \sqrt{a^2(3-3t)^2 + (6at)^2 + a^2(3+3t)^2} = 3\sqrt{2}|a|(t^2+1),\\
\mathbf{r}'(t) \times \mathbf{r}''(t) &= (18a^2(t^2-1), -36a^2t, 18a^2(t^2+1)),\\
|\mathbf{r}'(t) \times \mathbf{r}''(t)| &= 18a^2\sqrt{(t^2-1)^2 + (-2t)^2 + (t^2+1)^2} = 18\sqrt{2}a^2(t^2+1),\\
\kappa(t) &= \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \frac{18\sqrt{2}a^2(t^2+1)}{[3\sqrt{2}|a|(t^2+1)]^3} = \frac{1}{3|a|(t^2+1)^2},\\
\tau(t) &= \frac{(\mathbf{r}'(t), \mathbf{r}''(t), \mathbf{r}'''(t))}{|\mathbf{r}'(t) \times \mathbf{r}''(t)|^2} = 18a^2 \cdot \frac{(t^2-1, -2t, t^2+1)\cdot(-6a, 0, 6a)}{[18\sqrt{2}a^2(t^2+1)]^2} = \frac{1}{3a(t^2+1)^2}.
\end{aligned}
$$

(2)

$$
\begin{aligned}
\mathbf{r}(t) &= (t, t^2/2a, t^3/6a^2),\\
\mathbf{r}'(t) &= (1, t/a, t^2/2a),\\
\mathbf{r}''(t) &= (0, 1/a, t/a^2),\\
\mathbf{r}'''(t) &= (0, 0, 1/a^2),
\end{aligned}
$$

$$
\begin{aligned}
|\mathbf{r}'(t)| &= \sqrt{1+(t/a)^2+(t^2/2a)^2} = \frac{2a^2+t^2}{2a^2},\\
\mathbf{r}'(t)\times\mathbf{r}''(t) &= (t^2/2a^3, -t/a^2, 1/a),\\
|\mathbf{r}'(t)\times\mathbf{r}''(t)| &= \sqrt{(t^2/2a^3)^2+(-t/a^2)^2+(1/a)^2} = \frac{2a^2+t^2}{2|a|^3},\\
\kappa(t) &= \frac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \frac{\frac{2a^2+t^2}{2|a|^3}}{[\frac{2a^2+t^2}{2a^2}]^3} = \frac{4|a|^3}{(2a^2+t^2)^2},\\
\tau(t) &= \frac{(\mathbf{r}'(t),\mathbf{r}''(t),\mathbf{r}'''(t))}{|\mathbf{r}'(t)\times\mathbf{r}''(t)|^2} = \frac{(t^2/2a^3, -t/a^2, 1/a)\cdot(0,0,1/a^2)}{[\frac{2a^2+t^2}{2|a|^3}]^2} = \frac{4a^3}{(2a^2+t^2)^2}.
\end{aligned}
$$

(3)

$$
\begin{aligned}
\mathbf{r}(t) &= (t+a^2/t, t-a^2/t, 2a\log(t/a)),\\
\mathbf{r}'(t) &= (1-a^2/t^2, 1+a^2/t^2, 2a/t),\\
\mathbf{r}''(t) &= (2a^2/t^3, -2a^2/t^3, -2a/t^2),\\
\mathbf{r}'''(t) &= (-6a^2/t^4, 6a^2/t^4, 4a/t^3),\\
|\mathbf{r}'(t)| &= \sqrt{(1-a^2/t^2)^2+(1+a^2/t^2)^2+(2a/t)^2} = \frac{\sqrt{2}(a^2+t^2)}{t^2},\\
\mathbf{r}'(t)\times\mathbf{r}''(t) &= (2a^3/t^4-2a/t^2, 2a^3/t^4+2a/t^2, -4a^2/t^3),\\
|\mathbf{r}'(t)\times\mathbf{r}''(t)| &= \sqrt{(2a^3/t^4-2a/t^2)^2+(2a^3/t^4+2a/t^2)^2+(-4a^2/t^3)^2} = \frac{2\sqrt{2}a(a^2+t^2)}{t^4},\\
\kappa(t) &= \frac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \frac{\frac{2\sqrt{2}a(a^2+t^2)}{t^4}}{[\frac{\sqrt{2}(a^2+t^2)}{t^2}]^3} = \frac{at^2}{(a^2+t^2)^2},\\
\tau(t) &= \frac{(\mathbf{r}'(t),\mathbf{r}''(t),\mathbf{r}'''(t))}{|\mathbf{r}'(t)\times\mathbf{r}''(t)|^2} = \frac{(\frac{2a^3}{t^4}-\frac{2a}{t^2}, \frac{2a^3}{t^4}+\frac{2a}{t^2}, -\frac{4a^2}{t^3})\cdot(-\frac{6a^2}{t^4}, \frac{6a^2}{t^4}, \frac{4a}{t^3})}{[\frac{2\sqrt{2}a(a^2+t^2)}{t^4}]^2} = \frac{at^2}{(a^2+t^2)^2}.
\end{aligned}
$$