MA426 — 微分几何

Assignment 7

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习题　4.2/1

设悬链面的方程是

$$\mathbf{r} = \left(\sqrt{u^2+a^2}\cos v, \sqrt{u^2+a^2}\sin v, a\log\left(u+\sqrt{u^2+a^2}\right)\right),$$

求它的第一基本形式和第二基本形式, 并求它在点 $(0,0)$ 处, 沿切向量 $d\mathbf{r} = 2\mathbf{r}_u + \mathbf{r}_v$ 的法曲率.

$$\mathbf{r}_u(u,v) = \left(\frac{u\cos v}{\sqrt{u^2+a^2}}, \frac{u\sin v}{\sqrt{u^2+a^2}}, \frac{a}{\sqrt{u^2+a^2}}\right),$$

$$\mathbf{r}_v(u,v) = \left(-\sqrt{u^2+a^2}\sin v, \sqrt{u^2+a^2}\cos v, 0\right),$$

$$E(u,v) = \mathbf{r}_u \cdot \mathbf{r}_u = \frac{u^2\cos^2 v + u^2\sin^2 v + a^2}{u^2+a^2} = 1,$$

$$F(u,v) = \mathbf{r}_u \cdot \mathbf{r}_v = -u\sin v\cos v + u\sin v\cos v = 0,$$

$$G(u,v) = \mathbf{r}_v \cdot \mathbf{r}_v = (u^2+a^2)\sin^2 v + (u^2+a^2)\cos^2 v = u^2+a^2.$$

第一基本形式为

$$I = E(u,v)(du)^2 + 2F(u,v)dudv + G(u,v)dv^2 = (du)^2 + (u^2+a^2)(dv)^2.$$

$$\mathbf{r}_u \times \mathbf{r}_v = (-a\cos v, -a\sin v, u),$$

$$|\mathbf{r}_u \times \mathbf{r}_v| = \sqrt{a^2\cos^2 v + a^2\sin^2 v + u^2} = \sqrt{u^2+a^2},$$

$$\mathbf{n}(u,v) = \frac{\mathbf{r}_\varphi \times \mathbf{r}_\theta}{|\mathbf{r}_\varphi \times \mathbf{r}_\theta|} = \frac{1}{\sqrt{u^2+a^2}}(-a\cos v, -a\sin v, u),$$

$$\mathbf{r}_{uu}(u,v) = \left(\frac{a^2\cos v}{(u^2+a^2)^{3/2}}, \frac{a^2\sin v}{(u^2+a^2)^{3/2}}, -\frac{au}{(u^2+a^2)^{3/2}}\right),$$

$$\mathbf{r}_{uv}(u,v) = \left(-\frac{u\sin v}{\sqrt{u^2+a^2}}, \frac{u\cos v}{\sqrt{u^2+a^2}}, 0\right),$$

$$\mathbf{r}_{vv}(u,v) = \left(-\sqrt{u^2+a^2}\cos v, -\sqrt{u^2+a^2}\sin v, 0\right),$$

$$L(u,v) = \mathbf{r}_{uu} \cdot \mathbf{n} = \frac{-a^3\cos^2 v - a^3\sin^2 v - au^2}{(u^2+a^2)^2} = -\frac{a}{u^2+a^2},$$

$$M(u,v) = \mathbf{r}_{uv} \cdot \mathbf{n} = au\sin v\cos v - au\sin v\cos v = 0,$$

$$N(u,v)=\mathbf{r}_{vv}\cdot\mathbf{n}=a\cos^2 v+a\sin^2 v=a.$$

第二基本形式为

$$II=L(u,v)(du)^2+2M(u,v)dudv+N(u,v)(dv)^2=-\frac{a}{u^2+a^2}(du)^2+a(dv)^2.$$

在 $d\mathbf{r}=2\mathbf{r}_u+\mathbf{r}_v$ 上, $du=2dv$

$$\kappa_n(u,v)=\frac{II}{I}=\frac{-\frac{a}{u^2+a^2}(du)^2+a(dv)^2}{(du)^2+(u^2+a^2)(dv)^2}=\frac{a(u^2+a^2-4)}{(u^2+a^2)(u^2+a^2+4)},$$

$$\kappa_n(0,0)=\frac{a(a^2-4)}{a^2(a^2+4)}=\frac{a^2-4}{a(a^2+4)}.$$

习题　4.2/5

求下列曲面上的已知曲线的法曲率:

（5）曲面 $\mathbf{r}=(u,v,kuv)$ 上的曲线 $u=v^2$.

(5)

$$\mathbf{r}_u(u,v)=(1,0,kv),$$

$$\mathbf{r}_v(u,v)=(0,1,ku),$$

$$E(u,v)=\mathbf{r}_u\cdot\mathbf{r}_u=1+k^2v^2,$$

$$F(u,v)=\mathbf{r}_u\cdot\mathbf{r}_v=k^2uv,$$

$$G(u,v)=\mathbf{r}_v\cdot\mathbf{r}_v=1+k^2u^2.$$

第一基本形式为

$$I=E(u,v)(du)^2+2F(u,v)dudv+G(u,v)dv^2=(1+k^2v^2)(du)^2+2k^2uvdudv+(1+k^2u^2)(dv)^2.$$

$$\mathbf{r}_u\times\mathbf{r}_v=(-kv,-ku,1),$$

$$|\mathbf{r}_u\times\mathbf{r}_v|=\sqrt{1+k^2u^2+k^2v^2},$$

$$\mathbf{n}(u,v)=\frac{\mathbf{r}_\varphi\times\mathbf{r}_\theta}{|\mathbf{r}_\varphi\times\mathbf{r}_\theta|}=\frac{1}{\sqrt{1+k^2u^2+k^2v^2}}(-kv,-ku,1),$$

$$\mathbf{r}_{uu}(u,v)=(0,0,0),$$

$$\mathbf{r}_{uv}(u,v)=(0,0,k),$$

$$\mathbf{r}_{vv}(u,v)=(0,0,0),$$

$$L(u,v)=\mathbf{r}_{uu}\cdot\mathbf{n}=0,$$

$$M(u,v)=\mathbf{r}_{uv}\cdot\mathbf{n}=\frac{k}{\sqrt{1+k^2u^2+k^2v^2}},$$
$$N(u,v)=\mathbf{r}_{vv}\cdot\mathbf{n}=0.$$

第二基本形式为

$$II=L(u,v)(du)^2+2M(u,v)dudv+N(u,v)(dv)^2=\frac{2k}{\sqrt{1+k^2u^2+k^2v^2}}dudv.$$

在 $u=v^2$ 上, $du=2vdv$

$$\begin{aligned}\kappa_n(u,v)=\frac{II}{I}&=\frac{\frac{2k}{\sqrt{1+k^2u^2+k^2v^2}}dudv}{(1+k^2v^2)(du)^2+2k^2uvdudv+(1+k^2u^2)(dv)^2}\\&=\frac{\frac{2k}{\sqrt{1+k^2v^4+k^2v^2}}2vdv\cdot dv}{(1+k^2v^2)(2vdv)^2+2k^2v^2\cdot v\cdot 2vdv\cdot dv+(1+k^2v^4)(dv)^2}\\&=\frac{4kv}{(1+4v^2+9k^2v^4)\sqrt{1+k^2v^2+k^2v^4}}\end{aligned}$$