MA426 — 微分几何

Assignment 3

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定理 2.7.4

设正则参数曲线 C 的参数方程是 $\mathbf{r}(s)$, s 是弧长参数, 则 C 的渐缩线的参数方程是

$$\mathbf{r} = \mathbf{r}(s) + \frac{1}{\kappa(s)}\boldsymbol{\beta}(s) - \frac{1}{\kappa(s)}\left(\tan\int\tau(s)ds\right)\boldsymbol{\gamma}(s).$$

设

$$\mathbf{r}_1(s) = \mathbf{r}(s) + \lambda(s)\boldsymbol{\beta}(s) + \mu(s)\boldsymbol{\gamma}(s)$$

是曲线的渐缩线, 则

$$\lambda(s)\boldsymbol{\beta}(s) + \mu(s)\boldsymbol{\gamma}(s)$$

是曲线 $\mathbf{r}_1(s)$ 的切向量, 对 $\mathbf{r}_1(s)$ 求导可得

$$\mathbf{r}_1'(s) = \mathbf{r}'(s) + \lambda'(s)\boldsymbol{\beta}(s) + \lambda(s)\boldsymbol{\beta}'(s) + \mu'(s)\boldsymbol{\gamma}(s) + \mu(s)\boldsymbol{\gamma}'(s).$$

代入 Frenet 公式

$$\mathbf{r}'(s) = \boldsymbol{\alpha}(s), \quad \boldsymbol{\beta}'(s) = -\kappa(s)\boldsymbol{\alpha}(s) + \tau(s)\boldsymbol{\gamma}(s), \quad \boldsymbol{\gamma}'(s) = -\tau(s)\boldsymbol{\beta}(s)$$

可得

$$\mathbf{r}_1'(s) = [1-\lambda(s)\kappa(s)]\boldsymbol{\alpha}(s) + [\lambda'(s) - \mu(s)\tau(s)]\boldsymbol{\beta}(s) + [\mu'(s) + \lambda(s)\tau(s)]\boldsymbol{\gamma}(s).$$

由 $\mathbf{r}_1'(s)$ 和 $\mathbf{r}_1(s)$ 的切向量平行可知

$$1-\lambda(s)\kappa(s) = 0, \quad \frac{\lambda(s)}{\lambda'(s)-\mu(s)\tau(s)} = \frac{\mu(s)}{\mu'(s)+\lambda(s)\tau(s)},$$

$$(\lambda^2(s)+\mu^2(s))\tau(s) = \lambda'(s)\mu(s) - \mu'(s)\lambda(s),$$

$$\tau(s) = -\frac{\mu'(s)\lambda(s)-\lambda'(s)\mu(s)}{\lambda^2(s)+\mu^2(s)} = -\frac{d}{ds}\arctan\frac{\mu(s)}{\lambda(s)},$$

$$\arctan\frac{\mu(s)}{\lambda(s)} = -\int\tau(s)ds$$

故

$$\lambda(s) = \frac{1}{\kappa(s)}, \quad \mu(s) = -\lambda(s)\tan\int\tau(s)ds = -\frac{1}{\kappa(s)}\left(\tan\int\tau(s)ds\right).$$