MA426 — 微分几何

Assignment 4

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习题 3.1/4

求由圆螺旋线的主法线所构成的曲面的参数方程. 这是一张什么曲线?

设圆螺旋线的方程为

$$\mathbf{r}(t) = (a\cos t, a\sin t, bt).$$

则主法线所构成的曲面的参数方程为

$$\mathbf{r}(t, u) = ((a+u)\cos t, (a+u)\sin t, bt).$$

显然这是一张正螺旋面.

习题 3.2/5

设 S 是圆锥面 $\mathbf{r} = (v\cos u, v\sin u, v)$, C 是 S 上的一条曲线, 其方程是 $u = \sqrt{2}t, v = e^t$.

(1) 将曲线 C 的切向量用 $\mathbf{r}_u$, $\mathbf{r}_v$ 的线性组合表示出来;

(2) 证明: C 的切向量平分了 $\mathbf{r}_u$ 和 $\mathbf{r}_v$ 的夹角.

(1)

C 的参数方程和切向量为

$$\mathbf{r}_c(t) = (e^t\cos\sqrt{2}t, e^t\sin\sqrt{2}t, e^t),$$

$$\mathbf{r}_c'(t) = (e^t(\cos\sqrt{2}t - \sqrt{2}\sin\sqrt{2}t), e^t(\sin\sqrt{2}t + \sqrt{2}\cos\sqrt{2}t, e^t).$$

代入 $\mathbf{r}_u$, $\mathbf{r}_v$ 可得

$$\mathbf{r}_u(t) = \mathbf{r}_u(u, v) = (-v\sin u, v\cos u, 0), \quad \mathbf{r}_v(t) = \mathbf{r}_v(u, v) = (\cos u, \sin u, 1)$$

$$\mathbf{r}_u(\sqrt{2}t, e^t) = (-e^t\sin\sqrt{2}t), e^t\cos\sqrt{2}t, 0), \quad \mathbf{r}_v(\sqrt{2}t, e^t) = (\cos\sqrt{2}t, \sin\sqrt{2}t, 1),$$

$$\mathbf{r}_c'(t) = \sqrt{2}\mathbf{r}_u(t) + e^t\mathbf{r}_v(t).$$

(2)

$$|\mathbf{r}_u(t)| = e^t, \quad |\mathbf{r}_v(t)| = \sqrt{2},$$

$$|\mathbf{r}_c'(t)| = \sqrt{2\mathbf{r}_u^2(t) + e^{2t}\mathbf{r}_v^2(t)} = 2e^t.$$

代入夹角公式得

$$\angle(\mathbf{r}_c'(t), \mathbf{r}_u(t)) = \arccos\frac{\mathbf{r}_c'(t)\cdot\mathbf{r}_u(t)}{|\mathbf{r}_c'(t)||\mathbf{r}_u(t)|} = \arccos\frac{\sqrt{2}|\mathbf{r}_u(t)|^2}{|\mathbf{r}_c'(t)||\mathbf{r}_u(t)|} = \arccos\frac{\sqrt{2}e^{2t}}{2e^{2t}} = \frac{\pi}{4},$$

$$\angle(\mathbf{r}_c'(t), \mathbf{r}_v(t)) = \arccos\frac{\mathbf{r}_c'(t)\cdot\mathbf{r}_v(t)}{|\mathbf{r}_c'(t)||\mathbf{r}_v(t)|} = \arccos\frac{e^t|\mathbf{r}_v(t)|^2}{|\mathbf{r}_c'(t)||\mathbf{r}_v(t)|} = \arccos\frac{2e^t}{2\sqrt{2}e^t} = \frac{\pi}{4}.$$

故 C 的切向量平分了 $\mathbf{r}_u$ 和 $\mathbf{r}_v$ 的夹角.