MA426 — 微分几何

Assignment 6

Instructor: **陈优民**

Author: **刘逸灏** (515370910207)

— SJTU (Fall 2019)

习题 4.1/1

求下列曲线的第二基本形式:

(1) $\mathbf{r} = (a\cos\varphi\cos\theta, a\cos\varphi\sin\theta, b\sin\varphi)$(椭球面);

(1)

在椭球面中 $a, b > 0$, $0 < \varphi < \pi$, $0 < \theta < 2\pi$.

$$\mathbf{r}_\varphi = (-a\sin\varphi\cos\theta, -a\sin\varphi\sin\theta, b\cos\varphi),$$

$$\mathbf{r}_\theta = (-a\cos\varphi\sin\theta, a\cos\varphi\cos\theta, 0),$$

$$\mathbf{r}_\varphi \times \mathbf{r}_\theta = (-ab\cos^2\varphi\cos\theta, -ab\cos^2\varphi\sin\theta, -a^2\cos\varphi\sin\varphi),$$

$$|\mathbf{r}_\varphi \times \mathbf{r}_\theta| = a\cos\varphi\sqrt{b^2\cos^2\varphi\cos^2\theta + b^2\cos^2\varphi\sin^2\theta + a^2\sin^2\varphi} = a\cos\varphi\sqrt{a^2\sin^2\varphi + b^2\cos^2\varphi},$$

$$\mathbf{n} = \frac{\mathbf{r}_\varphi \times \mathbf{r}_\theta}{|\mathbf{r}_\varphi \times \mathbf{r}_\theta|} = \frac{1}{\sqrt{a^2\sin^2\varphi + b^2\cos^2\varphi}}(-b\cos\varphi\cos\theta, -b\cos\varphi\sin\theta, -a\sin\varphi),$$

$$\mathbf{r}_{\varphi\varphi} = (-a\cos\varphi\cos\theta, -a\cos\varphi\sin\theta, -b\sin\varphi),$$

$$\mathbf{r}_{\varphi\theta} = (a\sin\varphi\sin\theta, -a\sin\varphi\cos\theta, 0),$$

$$\mathbf{r}_{\theta\theta} = (-a\cos\varphi\cos\theta, -a\cos\varphi\sin\theta, 0),$$

$$L = \mathbf{r}_{\varphi\varphi}\cdot\mathbf{n} = \frac{ab\cos^2\varphi\cos^2\theta + ab\cos^2\varphi\sin^2\theta + ab\sin^2\theta}{\sqrt{a^2\sin^2\varphi + b^2\cos^2\varphi}} = \frac{ab}{\sqrt{a^2\sin^2\varphi + b^2\cos^2\varphi}},$$

$$M = \mathbf{r}_{\varphi\theta}\cdot\mathbf{n} = \frac{-ab\sin\varphi\cos\varphi\sin\theta\cos\theta + ab\sin\varphi\cos\varphi\sin\theta\cos\theta}{\sqrt{a^2\sin^2\varphi + b^2\cos^2\varphi}} = 0,$$

$$N = \mathbf{r}_{\theta\theta}\cdot\mathbf{n} = \frac{ab\cos^2\varphi\cos^2\theta + ab\cos^2\varphi\sin^2\theta}{\sqrt{a^2\sin^2\varphi + b^2\cos^2\varphi}} = \frac{ab\cos^2\varphi}{\sqrt{a^2\sin^2\varphi + b^2\cos^2\varphi}}.$$

第二基本形式为

$$II = L(d\varphi^2) + 2Md\varphi d\theta + N(d\theta)^2 = \frac{ab(d\varphi^2) + ab\cos^2\varphi(d\theta)^2}{\sqrt{a^2\sin^2\varphi + b^2\cos^2\varphi}}.$$

习题　4.1/3

求曲线 $\mathbf{r}=\mathbf{r}(s)$ 的切线面的第二基本形式, 其中 s 是该曲线的弧长参数.

$\mathbf{r}=\mathbf{r}(s)$ 的切线面方程为

$$\mathbf{r}(s,t)=\mathbf{r}(s)+t\mathbf{r}'(s)=\mathbf{r}(s)+t\boldsymbol{\alpha}(s).$$

故

$$\mathbf{r}_s=\mathbf{r}_s(s,t)=\mathbf{r}'(s)+t\boldsymbol{\alpha}'(s)=\boldsymbol{\alpha}(s)+t\kappa(s)\boldsymbol{\beta}(s),$$

$$\mathbf{r}_t=\mathbf{r}_t(s,t)=\boldsymbol{\alpha}(s),$$

$$\mathbf{r}_s\times\mathbf{r}_t=(\boldsymbol{\alpha}(s)+t\kappa(s)\boldsymbol{\beta}(s))\times\boldsymbol{\alpha}(s)=t\kappa(s)\boldsymbol{\gamma}(s),$$

$$|\mathbf{r}_s\times\mathbf{r}_t|=|t\kappa(s)\boldsymbol{\gamma}(s)|=t\kappa(s),$$

$$\mathbf{n}=\frac{\mathbf{r}_s\times\mathbf{r}_t}{|\mathbf{r}_s\times\mathbf{r}_t|}=\frac{t\kappa(s)\boldsymbol{\gamma}(s)}{t\kappa(s)}=\boldsymbol{\gamma}(s),$$

$$\mathbf{r}_{ss}=\boldsymbol{\alpha}'(s)+t\kappa'(s)\boldsymbol{\beta}(s)+t\kappa(s)\boldsymbol{\beta}'(s)=-t\kappa(s)^2\boldsymbol{\alpha}(s)+(\kappa(s)+t\kappa'(s))\boldsymbol{\beta}(s)+t\kappa(s)\tau(s)\boldsymbol{\gamma}(s),$$

$$\mathbf{r}_{st}=\kappa(s)\boldsymbol{\beta}(s),$$

$$\mathbf{r}_{tt}=0,$$

$$L=\mathbf{r}_{ss}\cdot\mathbf{n}=-t\kappa(s)^2\boldsymbol{\alpha}(s)\boldsymbol{\gamma}(s)(\kappa(s)+t\kappa'(s))\boldsymbol{\beta}(s)\boldsymbol{\gamma}(s)+t\kappa(s)\tau(s)\boldsymbol{\gamma}(s)^2=t\kappa(s)\tau(s),$$

$$M=\mathbf{r}_{st}\cdot\mathbf{n}=\kappa(s)\boldsymbol{\beta}(s)\boldsymbol{\gamma}(s)=0,$$

$$M=\mathbf{r}_{tt}\cdot\mathbf{n}=0.$$

第二基本形式为

$$II=L(ds)^2+2Mdsdt+N(dt)^2=t\kappa(s)\tau(s)(ds)^2.$$