MA426 DIFFERENTIAL GEOMETRY HOMEWORK 2

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Ex 1.

$$r = (x, y, z) = (x, y, f(x, y))$$
$$r_x = (1, 0, f_x(x, y))$$
$$r_y = (0, 1, f_y(x, y))$$
$$r_x \wedge r_y = (-f_x, -f_y, 1)$$
$$|r_x \wedge r_y| = \sqrt{1 + f_x^2 + f_y^2}$$
$$A = \iint_\Omega |r_x \wedge r_y| dxdy = \iint_\Omega \sqrt{1 + f_x^2 + f_y^2} dxdy$$

Ex 2.

For any constant v, the length curve $(u_1, u_2) \to R^3$ given by $X(u, v)$ is equal, let the length be k_v, then

$$\int_{u_1}^{u_2} \sqrt{E(u, v)} du = k_v$$

Take partial difference of v on both sides, we can get

$$\int_{u_1}^{u_2} \frac{\partial \sqrt{E(u, v)}}{\partial v} du = 0$$

Since

$$\frac{\partial \sqrt{E(u, v)}}{\partial v} \geqslant 0$$
$$\frac{\partial E}{\partial v} = 0$$

For any constant u, the length curve $(v_1, v_2) \to R^3$ given by $X(u, v)$ is also equal, similarly, we can get

$$\frac{\partial G}{\partial u} = 0$$

Ex 3.

$$r = (x, y, z) = (x, y, x^3)$$
$$r_x = (1, 0, 3x^2), r_y = (0, 1, 0)$$
$$E = r_x \cdot r_x = 1 + 9x^4, F = r_x \cdot r_y = 0, G = r_y \cdot r_y = 1$$
$$n = \frac{r_x \wedge r_y}{|r_x \wedge r_y|} = \frac{(-3x^2, 0, 1)}{\sqrt{1 + 9x^4}}$$
$$r_{xx} = 0, 0, 6x, r_{xy} = (0, 0, 0), r_{yy} = (0, 0, 0)$$
$$L = r_{xx} \cdot n = \frac{6x}{\sqrt{1 + 9x^4}}, M = r_{xy} \cdot n = 0, N = r_{yy} \cdot n = 0$$

The mean curvature is

$$H = \frac{LG - 2MF + NE}{2(EG - F^2)} = \frac{L}{2E} = \frac{3x}{(1 + 9x^4)^{1.5}}$$

The Gauss curvature is

$$K = \frac{LN - M^2}{EG - F^2} = 0$$

Ex 4.

$$r = (t, v, X) = (t, v, \alpha(t) + v\omega(t))$$
$$r_t = (1, 0, \alpha'(t) + v\omega'(t)), r_v = (0, 1, \omega(t))$$
$$E = r_t \cdot r_t = 1 + [\alpha'(t) + v\omega'(t)]^2, F = r_t \cdot r_v = \omega(t)[\alpha'(t) + v\omega'(t)], G = r_v \cdot r_v = 1 + \omega^2(t)$$
$$n = \frac{r_t \wedge r_v}{|r_t \wedge r_v|} = \frac{(-[\alpha'(t) + v\omega'(t)], -\omega(t), 1)}{\sqrt{1 + \omega^2(t) + [\alpha'(t) + v\omega'(t)]^2}}$$
$$r_{tt} = (0, 0, \alpha''(t) + v\omega''(t)), r_{tv} = (0, 0, \omega'(t)), r_{vv} = (0, 0, 0)$$
$$L = r_{tt} \cdot n = \frac{\alpha''(t) + v\omega''(t)}{\sqrt{1 + \omega^2(t) + [\alpha'(t) + v\omega'(t)]^2}}, M = r_{tv} \cdot n = \frac{\omega'(t)}{\sqrt{1 + \omega^2(t) + [\alpha'(t) + v\omega'(t)]^2}}, N = r_{vv} \cdot n = (0, 0, 0)$$

Let

$$a = \alpha'(t) + v\omega'(t), b = \omega(t)$$

The Gauss curvature is

$$K = \frac{LN - M^2}{EG - F^2} = \frac{-M^2}{EG - F^2} = -\frac{[\omega'(t)]^2}{(1 + a^2 + b^2)[(1 + a^2)(1 + b^2) - a^2b^2]} = -\frac{[\omega'(t)]^2}{\{1 + [\alpha'(t) + v\omega'(t)]^2 + \omega^2(t)\}^2}$$

Ex 5.

Let $z = \phi(s)$,

$$r(x, y, z) = (\phi(s)\cos\theta, \phi(s)\sin\theta, \phi(s))$$