DIFFERENTIAL GEOMETRY HOMEWORK 1

Each of the first six problems is worth 10 points, the last problem is worth 5 points for extra credit.

1. (Cycloid) A disk of radius a rolls along a straight line without sliding. Deduce the parametric equation of the curve traced by a point M on the boundary circle.

2. Compute the curvature and the torsion of the following curves:
   - $x = e^t, y = e^{-t}, z = t\sqrt{2}$;
   - $x = \cos^3 t, y = \sin^3 t, z = \cos(2t)$.

3. (Catenary) The plane curve $\alpha(t) = (t, \cosh t), t \in \mathbb{R}$ is called the catenary. Find its signed curvature.

4. Let $\alpha(t)$ be a regular parametric space curve. t is not necessarily arc length parameter. Denote $\frac{d\alpha}{dt} = \alpha'$ and $\frac{d^2\alpha}{dt^2} = \alpha''$, show that the curvature and the torsion are

$$k(t) = \frac{|\alpha' \wedge \alpha''|}{|\alpha'|^3}, \quad \tau(t) = -\frac{(\alpha' \wedge \alpha'') \cdot \alpha'''}{|\alpha' \wedge \alpha''|^2}.$$

5. Show the ellipse $x = a\cos t, y = b\sin t$ has exactly four vertices. Where are these vertices at?

6. Let $\alpha(s), s \in [0, l]$ be a closed convex plane curve positively oriented. s is the arc length parameter.

$$\beta(s) = \alpha(s) - rn(s),$$

is called the parallel curve to α, where $r > 0$ is a fixed small number. Show
   - The length of β is the length of $\alpha + 2\pi r$.
   - The area enclosed by β is the area enclosed by $\alpha + rl + \pi r^2$.
   - $k_\beta(s) = k_\alpha(s)/(1 + rk_\alpha(s))$.

7. (extra) Prove the integral of the curvature of an infinite convex plane curve is not greater than π.

Date: Due 10/8.