**MA426 — 微分几何**

*Assignment 8*

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## 习题 4.2/6

求下列曲面上的渐进曲线:

(2) 双曲抛物面: $\mathbf{r}=\left(\frac{u+v}{2},\frac{u-v}{2},uv\right)$.

### (2)

$$\mathbf{r}_u(u,v)=\left(\frac{1}{2},\frac{1}{2},v\right),$$

$$\mathbf{r}_v(u,v)=\left(\frac{1}{2},-\frac{1}{2},u\right),$$

$$\mathbf{r}_u\times\mathbf{r}_v=\left(\frac{u+v}{2},-\frac{u-v}{2},-\frac{1}{2}\right),$$

$$|\mathbf{r}_u\times\mathbf{r}_v|=\sqrt{\frac{1}{4}(u+v)^2+\frac{1}{4}(u-v)^2+\frac{1}{4}}=\frac{\sqrt{2u^2+2v^2+1}}{2},$$

$$\mathbf{n}(u,v)=\frac{\mathbf{r}_u\times\mathbf{r}_v}{|\mathbf{r}_u\times\mathbf{r}_v|}=\frac{1}{\sqrt{2u^2+2v^2+1}}(u+v,v-u,-1),$$

$$\mathbf{r}_{uu}(u,v)=(0,0,0),$$

$$\mathbf{r}_{uv}(u,v)=(0,0,1),$$

$$\mathbf{r}_{vv}(u,v)=(0,0,0),$$

$$L(u,v)=\mathbf{r}_{uu}\cdot\mathbf{n}=0,$$

$$M(u,v)=\mathbf{r}_{uv}\cdot\mathbf{n}=-\frac{1}{\sqrt{2u^2+2v^2+1}},$$

$$N(u,v)=\mathbf{r}_{vv}\cdot\mathbf{n}=0.$$

第二基本形式为

$$II=L(u,v)(du)^2+2M(u,v)dudv+N(u,v)(dv)^2=-\frac{1}{\sqrt{2u^2+2v^2+1}}dudv.$$

$II=0$ 的解为 $du=0$ 或 $dv=0$, 故渐进曲线为 $u=u_0$ 和 $v=v_0$.

## 定理 4.3.2

Weingarten 映射 $W$ 是从切空间 $T_pS$ 到它自身的自共轭映射, 即对于曲面 $S$ 在点 $u, v$ 的任意两个切方向 $d\mathbf{r}$ 和 $\delta\mathbf{r}$, 下面的公式

$$W(d\mathbf{r}) \cdot \delta\mathbf{r} = d\mathbf{r} \cdot W(\delta\mathbf{r})$$

成立.

设

$$d\mathbf{r} = \mathbf{r}_u du + \mathbf{r} v dv, \quad \delta\mathbf{r} = \mathbf{r}_u \delta u + \mathbf{r}_v \delta v.$$

则

$$W(d\mathbf{r}) = -(\mathbf{n}_u du + \mathbf{n}_v dv), \quad W(\delta\mathbf{r}) = -(\mathbf{n}_u \delta u + \mathbf{n}_v \delta v).$$

$$\begin{aligned} W(d\mathbf{r}) \cdot \delta\mathbf{r} &= -(\mathbf{n}_u du + \mathbf{n}_v dv) \cdot (\mathbf{r}_u \delta u + \mathbf{r}_v \delta v) \\ &= L du \delta u + M(du \delta v + dv \delta u) + N dv \delta v \\ &= (\mathbf{r}_u du + \mathbf{r} v dv) \cdot -(\mathbf{n}_u \delta u + \mathbf{n}_v \delta v) \\ &= d\mathbf{r} \cdot W(\delta\mathbf{r}). \end{aligned}$$

## 习题　4.3/4

证明: 曲面 $S$ 上任意一点 $p$ 的某个邻域内都有正交参数系 $(u, v)$, 使得参数曲线在点 $p$ 处的切方向是曲面 $S$ 在该点的两个彼此正交的主方向.

设曲面 $S$ 的正交参数系为 $(u, v)$, 则在点 $p(u_0, v_0)$ 处的两个正交的主方向单位向量为

$$\mathbf{e}_1 = \xi_1 \mathbf{r}_u(u_0, v_0) + \eta_1 \mathbf{r}_v(u_0, v_0),$$

$$\mathbf{e}_2 = \xi_2 \mathbf{r}_u(u_0, v_0) + \eta_2 \mathbf{r}_v(u_0, v_0).$$

现只需找到变换 $T : (u, v) \longrightarrow (u', v')$ 使得 $\mathbf{r}_{u'}(u_0, v_0) = \mathbf{e}_1$, $\mathbf{r}_{v'}(u_0, v_0) = \mathbf{e}_2$. 令

$$T: \quad u = \xi_1 u' + \xi_2 v', \quad v = \eta_1 u' + \eta_2 v',$$

$$\frac{\partial(u, v)}{\partial(u', v')} = \begin{vmatrix} \dfrac{\partial u}{\partial u'} & \dfrac{\partial v}{\partial u'} \\ \dfrac{\partial u}{\partial v'} & \dfrac{\partial v}{\partial v'} \end{vmatrix} = \begin{vmatrix} \xi_1 & \eta_1 \\ \xi_2 & \eta_2 \end{vmatrix} \neq 0,$$

$$\mathbf{r}_{u'} = \frac{\partial u}{\partial u'} \mathbf{r}_u(u_0, v_0) + \frac{\partial v}{\partial u'} \mathbf{r}_v(u_0, v_0) = \xi_1 \mathbf{r}_u(u_0, v_0) + \eta_1 \mathbf{r}_v(u_0, v_0) = \mathbf{e}_1,$$

$$\mathbf{r}_{v'} = \frac{\partial u}{\partial v'} \mathbf{r}_u(u_0, v_0) + \frac{\partial v}{\partial v'} \mathbf{r}_v(u_0, v_0) = \xi_2 \mathbf{r}_u(u_0, v_0) + \eta_2 \mathbf{r}_v(u_0, v_0) = \mathbf{e}_2.$$

故变换成立, $(u', v')$ 即为需证的正交参数系.