MA426 — 微分几何

Assignment 9

Instructor: **陈优民**

Author: **刘逸灏** (515370910207)

— SJTU (Fall 2019)

习题　4.3/3

求下列曲面的 Guass 曲率 K 和平均曲率 H:

(1) $\mathbf{r} = (u\cos v, u\sin v, ku^2)$.

$$\mathbf{r}_u(u,v) = (\cos v, \sin v, 2ku),$$

$$\mathbf{r}_v(u,v) = (-u\sin v, u\cos v, 0),$$

$$E(u,v) = \mathbf{r}_u \cdot \mathbf{r}_u = \cos^2 u + \sin^2 u + 4k^2u^2 = 4k^2u^2 + 1,$$

$$F(u,v) = \mathbf{r}_u \cdot \mathbf{r}_v = -u\sin v\cos v + u\sin v\cos v = 0,$$

$$G(u,v) = \mathbf{r}_v \cdot \mathbf{r}_v = u^2\sin^2 v + u^2\cos^2 v = u^2.$$

$$\mathbf{r}_u \times \mathbf{r}_v = (-2ku^2\cos v, -2ku^2\sin v, u),$$

$$|\mathbf{r}_u \times \mathbf{r}_v| = u\sqrt{4k^2u^2\cos^2 v + 4k^2u^2\sin^2 v + 1} = u\sqrt{4k^2u^2+1},$$

$$\mathbf{n}(u,v) = \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|} = \frac{1}{\sqrt{4k^2u^2+1}}(-2ku\cos v, -2ku\sin v, 1),$$

$$\mathbf{r}_{uu}(u,v) = (0,0,2k),$$

$$\mathbf{r}_{uv}(u,v) = (-\sin v, \cos v, 0),$$

$$\mathbf{r}_{vv}(u,v) = (-u\cos v, -u\sin v, 0),$$

$$L(u,v) = \mathbf{r}_{uu} \cdot \mathbf{n} = \frac{2k}{\sqrt{4k^2u^2+1}},$$

$$M(u,v) = \mathbf{r}_{uv} \cdot \mathbf{n} = \frac{2ku\sin v\cos v - 2ku\sin v\cos v}{\sqrt{4k^2u^2+1}} = 0,$$

$$N(u,v) = \mathbf{r}_{vv} \cdot \mathbf{n} = \frac{2ku^2\cos^2 v + 2ku^2\sin^2 v}{\sqrt{4k^2u^2+1}} = \frac{2ku^2}{\sqrt{4k^2u^2+1}}.$$

故 Guass 曲率 K 和平均曲率 H 为

$$K = \frac{LN - M^2}{EG - F^2} = \frac{\dfrac{4k^2u^2}{4k^2u^2+1}}{u^2(4k^2u^2+1)} = \frac{4k^2}{(4k^2u^2+1)^2},$$

$$H=\frac{LG-2MF+NE}{2(EG-F^2)}=\frac{\frac{2ku^2}{\sqrt{4k^2u^2+1}}+2ku^2\sqrt{4k^2u^2+1}}{2u^2(4k^2u^2+1)}=\frac{2k(2k^2u^2+1)}{(4k^2u^2+1)^{\frac{3}{2}}}.$$

习题 4.3/4

求双曲抛物面 $\mathbf{r}=(a(u+v),b(u-v),2uv)$ 的 Guass 曲率 K, 平均曲率 H, 主曲率 κ_1, κ_2 和它们对应的主方向.

$$\mathbf{r}_u(u,v)=(a,b,2v),$$

$$\mathbf{r}_v(u,v)=(a,-b,2u),$$

$$E(u,v)=\mathbf{r}_u\cdot\mathbf{r}_u=a^2+b^2+4v^2,$$

$$F(u,v)=\mathbf{r}_u\cdot\mathbf{r}_v=a^2-b^2+4uv,$$

$$G(u,v)=\mathbf{r}_v\cdot\mathbf{r}_v=a^2+b^2+4u^2.$$

$$\mathbf{r}_u\times\mathbf{r}_v=(2b(u+v),-2a(u-v),-2ab),$$

$$|\mathbf{r}_u\times\mathbf{r}_v|=2\sqrt{b^2(u+v)^2+a^2(u-v)^2+a^2b^2},$$

$$\mathbf{n}(u,v)=\frac{\mathbf{r}_u\times\mathbf{r}_v}{|\mathbf{r}_u\times\mathbf{r}_v|}=\frac{1}{\sqrt{b^2(u+v)^2+a^2(u-v)^2+a^2b^2}}(b(u+v),-a(u-v),-ab),$$

$$\mathbf{r}_{uu}(u,v)=(0,0,0),$$

$$\mathbf{r}_{uv}(u,v)=(0,0,2),$$

$$\mathbf{r}_{vv}(u,v)=(0,0,0),$$

$$L(u,v)=\mathbf{r}_{uu}\cdot\mathbf{n}=0,$$

$$M(u,v)=\mathbf{r}_{uv}\cdot\mathbf{n}=-\frac{2ab}{\sqrt{b^2(u+v)^2+a^2(u-v)^2+a^2b^2}},$$

$$N(u,v)=\mathbf{r}_{vv}\cdot\mathbf{n}=0.$$

故 Guass 曲率 K 和平均曲率 H 为

$$EG-F^2=(a^2+b^2+4v^2)(a^2+b^2+4u^2)-(a^2-b^2+4uv)^2=4[b^2(u+v)^2+a^2(u-v)^2+a^2b^2],$$

$$K=\frac{LN-M^2}{EG-F^2}=-\frac{M^2}{EG-F^2}=-\frac{a^2b^2}{[b^2(u+v)^2+a^2(u-v)^2+a^2b^2]^2},$$

$$H=\frac{LG-2MF+NE}{2(EG-F^2)}=-\frac{MF}{EG-F^2}=\frac{ab(a^2-b^2+4uv)}{2[b^2(u+v)^2+a^2(u-v)^2+a^2b^2]^{\frac{3}{2}}}.$$

主曲率 κ_1, κ_2 为

$$\sqrt{H^2-K}=\sqrt{\frac{M^2F^2}{(EG-F^2)^2}+\frac{M^2}{EG-F^2}}=\frac{M}{EG-F^2}\sqrt{EG},$$

$$\kappa_1 = H + \sqrt{H^2 - K} = -\frac{M(F - \sqrt{EG})}{EG - F^2} = -\frac{ab\left[a^2 - b^2 + 4uv - \sqrt{(a^2 + b^2 + 4v^2)(a^2 + b^2 + 4u^2)}\right]}{2[b^2(u+v)^2 + a^2(u-v)^2 + a^2b^2]^{\frac{3}{2}}},$$

$$\kappa_2 = H - \sqrt{H^2 - K} = -\frac{M(F + \sqrt{EG})}{EG - F^2} = -\frac{ab\left[a^2 - b^2 + 4uv + \sqrt{(a^2 + b^2 + 4v^2)(a^2 + b^2 + 4u^2)}\right]}{2[b^2(u+v)^2 + a^2(u-v)^2 + a^2b^2]^{\frac{3}{2}}}.$$

相对于 κ_1 的主方向为

$$\frac{du}{dv} = -\frac{M - \kappa_1 F}{L - \kappa_1 E} = -\frac{F}{E} + \frac{1}{\kappa_1}\frac{M}{E} = -\frac{F}{E} + \frac{F + \sqrt{EG}}{M}\frac{M}{E} = \sqrt{\frac{G}{E}} = \sqrt{\frac{a^2 + b^2 + 4u^2}{a^2 + b^2 + 4v^2}}.$$

相对于 κ_2 的主方向为

$$\frac{du}{dv} = -\frac{M - \kappa_2 F}{L - \kappa_2 E} = -\frac{F}{E} + \frac{1}{\kappa_2}\frac{M}{E} = -\frac{F}{E} + \frac{F - \sqrt{EG}}{M}\frac{M}{E} = -\sqrt{\frac{G}{E}} = -\sqrt{\frac{a^2 + b^2 + 4u^2}{a^2 + b^2 + 4v^2}}.$$