

MA426 — 微分几何

Assignment 6

Instructor: 陈优民

Author: 刘逸灏 (515370910207)

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习题 4.1/1

求下列曲线的第二基本形式:

(1) $\mathbf{r} = (a \cos \varphi \cos \theta, a \cos \varphi \sin \theta, b \sin \varphi)$ (椭球面);

(1)

在椭球面中 $a, b > 0, 0 < \varphi < \pi, 0 < \theta < 2\pi$.

$$\mathbf{r}_\varphi = (-a \sin \varphi \cos \theta, -a \sin \varphi \sin \theta, b \cos \varphi),$$

$$\mathbf{r}_\theta = (-a \cos \varphi \sin \theta, a \cos \varphi \cos \theta, 0),$$

$$\mathbf{r}_\varphi \times \mathbf{r}_\theta = (-ab \cos^2 \varphi \cos \theta, -ab \cos^2 \varphi \sin \theta, -a^2 \cos \varphi \sin \varphi),$$

$$|\mathbf{r}_\varphi \times \mathbf{r}_\theta| = a \cos \varphi \sqrt{b^2 \cos^2 \varphi \cos^2 \theta + b^2 \cos^2 \varphi \sin^2 \theta + a^2 \sin^2 \varphi} = a \cos \varphi \sqrt{a^2 \sin^2 \varphi + b^2 \cos^2 \varphi},$$

$$\mathbf{n} = \frac{\mathbf{r}_\varphi \times \mathbf{r}_\theta}{|\mathbf{r}_\varphi \times \mathbf{r}_\theta|} = \frac{1}{\sqrt{a^2 \sin^2 \varphi + b^2 \cos^2 \varphi}} (-b \cos \varphi \cos \theta, -b \cos \varphi \sin \theta, -a \sin \varphi),$$

$$\mathbf{r}_{\varphi\varphi} = (-a \cos \varphi \cos \theta, -a \cos \varphi \sin \theta, -b \sin \varphi),$$

$$\mathbf{r}_{\varphi\theta} = (a \sin \varphi \sin \theta, -a \sin \varphi \cos \theta, 0),$$

$$\mathbf{r}_{\theta\theta} = (-a \cos \varphi \cos \theta, -a \cos \varphi \sin \theta, 0),$$

$$L = \mathbf{r}_{\varphi\varphi} \cdot \mathbf{n} = \frac{ab \cos^2 \varphi \cos^2 \theta + ab \cos^2 \varphi \sin^2 \theta + ab \sin^2 \varphi}{\sqrt{a^2 \sin^2 \varphi + b^2 \cos^2 \varphi}} = \frac{ab}{\sqrt{a^2 \sin^2 \varphi + b^2 \cos^2 \varphi}},$$

$$M = \mathbf{r}_{\varphi\theta} \cdot \mathbf{n} = \frac{-ab \sin \varphi \cos \varphi \sin \theta \cos \theta + ab \sin \varphi \cos \varphi \sin \theta \cos \theta}{\sqrt{a^2 \sin^2 \varphi + b^2 \cos^2 \varphi}} = 0,$$

$$N = \mathbf{r}_{\theta\theta} \cdot \mathbf{n} = \frac{ab \cos^2 \varphi \cos^2 \theta + ab \cos^2 \varphi \sin^2 \theta}{\sqrt{a^2 \sin^2 \varphi + b^2 \cos^2 \varphi}} = \frac{ab \cos^2 \varphi}{\sqrt{a^2 \sin^2 \varphi + b^2 \cos^2 \varphi}}.$$

第二基本形式为

$$II = L(d\varphi^2) + 2Md\varphi d\theta + N(d\theta)^2 = \frac{ab(d\varphi^2) + ab \cos^2 \varphi (d\theta)^2}{\sqrt{a^2 \sin^2 \varphi + b^2 \cos^2 \varphi}}.$$

习题 4.1/3

求曲线 $\mathbf{r} = \mathbf{r}(s)$ 的切线面的第二基本形式, 其中 s 是该曲线的弧长参数.

$\mathbf{r} = \mathbf{r}(s)$ 的切线面方程为

$$\mathbf{r}(s, t) = \mathbf{r}(s) + t\mathbf{r}'(s) = \mathbf{r}(s) + t\boldsymbol{\alpha}(s).$$

故

$$\mathbf{r}_s = \mathbf{r}_s(s, t) = \mathbf{r}'(s) + t\boldsymbol{\alpha}'(s) = \boldsymbol{\alpha}(s) + t\kappa(s)\boldsymbol{\beta}(s),$$

$$\mathbf{r}_t = \mathbf{r}_t(s, t) = \boldsymbol{\alpha}(s),$$

$$\mathbf{r}_s \times \mathbf{r}_t = (\boldsymbol{\alpha}(s) + t\kappa(s)\boldsymbol{\beta}(s)) \times \boldsymbol{\alpha}(s) = t\kappa(s)\boldsymbol{\gamma}(s),$$

$$|\mathbf{r}_s \times \mathbf{r}_t| = |t\kappa(s)\boldsymbol{\gamma}(s)| = t\kappa(s),$$

$$\mathbf{n} = \frac{\mathbf{r}_s \times \mathbf{r}_t}{|\mathbf{r}_s \times \mathbf{r}_t|} = \frac{t\kappa(s)\boldsymbol{\gamma}(s)}{t\kappa(s)} = \boldsymbol{\gamma}(s),$$

$$\mathbf{r}_{ss} = \boldsymbol{\alpha}'(s) + t\kappa'(s)\boldsymbol{\beta}(s) + t\kappa(s)\boldsymbol{\beta}'(s) = -t\kappa(s)^2\boldsymbol{\alpha}(s) + (\kappa(s) + t\kappa'(s))\boldsymbol{\beta}(s) + t\kappa(s)\tau(s)\boldsymbol{\gamma}(s),$$

$$\mathbf{r}_{st} = \kappa(s)\boldsymbol{\beta}(s),$$

$$\mathbf{r}_{tt} = 0,$$

$$L = \mathbf{r}_{ss} \cdot \mathbf{n} = -t\kappa(s)^2\boldsymbol{\alpha}(s)\boldsymbol{\gamma}(s)(\kappa(s) + t\kappa'(s))\boldsymbol{\beta}(s)\boldsymbol{\gamma}(s) + t\kappa(s)\tau(s)\boldsymbol{\gamma}(s)^2 = t\kappa(s)\tau(s),$$

$$M = \mathbf{r}_{st} \cdot \mathbf{n} = \kappa(s)\boldsymbol{\beta}(s)\boldsymbol{\gamma}(s) = 0,$$

$$N = \mathbf{r}_{tt} \cdot \mathbf{n} = 0.$$

第二基本形式为

$$II = L(ds)^2 + 2Mdsdt + N(dt)^2 = t\kappa(s)\tau(s)(ds)^2.$$