MA426 DIFFERENTIAL GEOMETRY HOMEWORK 1

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Ex 1.

Suppose the angular velocity of the disk is ω, let the straight line be the x axis and suppose that the center of disk is $(0,a)$ and M is at (a,a) in the initial state. Then

$$\alpha(t) = (a\omega t + a\cos\omega t, a + a\sin\omega t)$$

Ex 2.

$$x = e^t, y = e^{-t}, z = t\sqrt{2}$$

$$\begin{aligned}
\kappa(t) &= \frac{\sqrt{(z''y' - y''z')^2 + (x''z' - z''x')^2 + (y''x' - x''y')^2}}{\sqrt{x'^2 + y'^2 + z'^2}^3} \\
&= \frac{\sqrt{2e^{-2t} + 2e^{2t} + 4}}{\sqrt{e^{2t} + e^{-2t} + 2}^3} = \frac{\sqrt{2}}{(e^t + e^{-t})^2}
\end{aligned}$$

$$\begin{aligned}
\tau(t) &= -\frac{x'''(z''y' - y''z') + y'''(x''z' - z''x') + z'''(y''x' - x''y')}{(z''y' - y''z')^2 + (x''z' - z''x')^2 + (y''x' - x''y')^2} \\
&= -\frac{-\sqrt{2} - \sqrt{2}}{2e^{-2t} + 2e^{2t} + 4} = \frac{\sqrt{2}}{(e^t + e^{-t})^2}
\end{aligned}$$

$$x = \cos^3 t, y = \sin^3 t, z = \cos 2t$$

$$\begin{aligned}
\kappa(t) &= \frac{\sqrt{(z''y' - y''z')^2 + (x''z' - z''x')^2 + (y''x' - x''y')^2}}{\sqrt{x'^2 + y'^2 + z'^2}^3} \\
&= \frac{15\sin^2 t(1 - \sin^2 t)}{(\frac{5}{2}\sin 2t)^3} = \frac{3}{25\sin t\cos t}
\end{aligned}$$

$$\begin{aligned}\tau(t) &= -\frac{x'''(z''y' - y''z') + y'''(x''z' - z''x') + z'''(y''x' - x''y')}{(z''y' - y''z')^2 + (x''z' - z''x')^2 + (y''x' - x''y')^2} \\ &= -\frac{\frac{9}{2}\sin^3 2t}{[15\sin^2 t(1-\sin^2 t)]^2} = -\frac{4}{25\sin t\cos t}\end{aligned}$$

Ex 3.

$$k(t) = \frac{x'y'' - y'x''}{\sqrt{x'^2 + y'^2}^3} = \frac{\cosh t}{\sqrt{1+\sinh^2 t}^3} = \frac{1}{\cosh^2 t}$$

Ex 4.

Let $t(s)$ be the unit tangent vector and $n(s)$ be the unit normal vector, from definition we know $t'(s) = k(s)n(s) = \dfrac{d\alpha/dt}{ds/dt}$, then

$$\alpha' = \frac{ds}{dt}t(s)$$

Differentiate both sides by t and substitute $t'(s)$ by $k(s)n(s)$

$$\alpha'' = \frac{d^2s}{dt^2}t(s) + \frac{ds}{dt}t'(s) = \frac{d^2s}{dt^2}t(s) + \frac{ds}{dt}k(s)n(s)$$

Multiple both sides by $t(s)$ as vector product

$$t(s) \wedge \alpha'' = \frac{d^2s}{dt^2}[t(s) \wedge t(s)] + \frac{ds}{dt}k(s)[t(s) \wedge n(s)]$$

Taking norms of both sides

$$\left|\frac{d\alpha}{dt}\frac{dt}{ds} \wedge \alpha''\right| = 0 + \left|\left(\frac{ds}{dt}\right)^2 k(t)\right|$$

Therefore

$$k(t) = \frac{|\alpha' \wedge \alpha''|}{|\alpha'|^3}$$

Ex 5.

$$k(t) = \frac{x'y'' - y'x''}{\sqrt{x'^2 + y'^2}^3} = \frac{ab\sin^2 t + ab\cos^2 t}{\sqrt{a^2\sin^2 t + b^2\cos^2 t}^3} = ab(a^2\sin^2 t + b^2\cos^2 t)^{-3/2}$$

$$k'(t) = ab(2a^2\sin t\cos t - 2b^2\sin t\cos t) \cdot -\frac{3}{2}(a^2\sin^2 t + b^2\cos^2 t)^{-5/2} = 0$$

Since it is an ellipse, we know that $ab \neq 0, a^2 - b^2 \neq 0$, and

$$a^2\sin^2 t + b^2\cos^2 t > \min\{a^2, b^2\} > 0$$

So

$$\sin t \cos t = 0$$

$$t = 2k\pi, \frac{(4k+1)\pi}{2}, (2k+1)\pi, \frac{(4k+3)\pi}{2} \quad k \in Z$$

There are exactly four vertices, they are at $(a, 0)$, $(0, b)$, $(-a, 0)$ and $(0, -b)$.

Ex 6.