VE203 Assignment 6

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Exercise 6.1

i)

$$
\begin{aligned}
f(n) &= af\left(\frac{n}{b}\right) + cn^d \\
&= a\left[af\left(\frac{n}{b^2}\right) + c\left(\frac{n}{b}\right)^d\right] + cn^d \\
&= a^2 f\left(\frac{n}{b^2}\right) + ac\left(\frac{n}{b}\right)^d + cn^d \\
&= ... \\
&= a^{\log_b n} f(1) + \sum_{i=1}^{log_b n} cn^d \left(\frac{a}{b^d}\right)^{i-1} \\
&= n^{\log_b a} f(1) + cn^d \sum_{i=1}^{\log_b n} \\
&= f(1)n^d + cn^d \log_b n
\end{aligned}
$$

ii)

$$
\lim_{n\to\infty} \frac{f(1)n^d + cn^d \log_b n}{n^d logn} = \lim_{n\to\infty} c\frac{\log_b n}{\log_b n / \log_b 10} = c\log_b 10 = C
$$

iii)

$$
\begin{aligned}
f(n) &= a^{\log_b n} f(1) + \sum_{i=1}^{log_b n} cn^d \left(\frac{a}{b^d}\right)^{i-1} \\
&= n^{\log_b a} f(1) + cn^d \frac{1 - \left(\frac{a}{b^d}\right)^{\log_b n}}{1 - \frac{a}{b^d}} \\
&= n^{\log_b a} f(1) + cn^d \frac{b^d - b^d\left(\frac{a}{b^d}\right)^{\log_b n}}{b^d - a} \\
&= n^{\log_b a} f(1) + \frac{b^d c}{b^d - a} n^d + cn^d \frac{b^d \frac{a^{\log_b n}}{n^d}}{a - b^d} \\
&= \frac{b^d c}{b^d - a} n^d + \left[f(1) + \frac{b^d c}{a - b^d}\right] n^{\log_b a}
\end{aligned}
$$

iv)

$$
a < b^d \Longrightarrow \log_b a < d
$$

$$\lim_{n\to\infty}\frac{C_1 n^d + C_2 n^{\log_b a}}{n^d} = C_1 = C$$

v)

$$a > b^d \implies \log_b a > d$$

$$\lim_{n\to\infty}\frac{C_1 n^d + C_2 n^{\log_b a}}{n^{\log_b a}} = C_2 = C$$

Exercise 6.2

i)

$$f(n) = f(n/2) + 1$$

ii)

$$O(\log n)$$

Exercise 6.3

We can simply find that in a bit string, whenever a series of $''0''$ is switched to $''1''$, we get a $''01''$ string, and whenever a series of $''1''$ is switched to $''0''$, we get a $''10''$ string. Then we only need to consider the first and last bit of the string.

When the first and last bit is the same (both $''0''$ or both $''1''$), the number of switching from $''0''$ to $''1''$ and $''1''$ to $''0''$ is the same, so $''10''$ and $''01''$ occurs the same time.

When the first bit is $''0''$ and the last bit is $''1''$, the number of switching from $''0''$ to $''1''$ is one more than that of $''1''$ to $''0''$, so $''01''$ occurs one more time than $''10''$.

Similarly, when the first bit is $''1''$ and the last bit is $''0''$, $''10''$ occurs one more time than $''01''$.

In conclusion, in a bit string, the string 01 occurs at most one more time than the string 10.

Exercise 6.4

i) When $q_{n-1} - 1 \neq 0$,

$$(p_n, q_n) = (p_{n-1} + 1, q_{n-1} - 1)$$

When $q_{n-1} - 1 = 0$,

$$(p_n, q_n) = (1, p_{n-1} + 1)$$

Forming them together, we get

$$(p_n, q_n) = ((p_{n-1} + 1) \text{ mod } \mathrm{p_{n-1}, q_{n-1} - 1 + [1 - u(q_{n-1} - 1)](p_{n-1} + 1)})$$

ii) For $p = 1$ and $q = 1$, $(p, q) = (p_1, q_1)$

For $p = 1$ and $q = k, k \in N^+$, suppose we can find $(p, q) = (p_m, q_m)$
Then for $p = 1$ and $q = k + 1$, we can also find $(p, q) = (p_{m+k}, q_{m+k})$ according to the recurrence relation $(p_n, q_n) = (1, p_{n-1} + 1)$

So for $p = 1$ and $q \in N^+$, we can find $(p, q) = (p_n, q_n)$

For $p = k, k < q, k \in N^+$ and $q \in N^+$, suppose we can find $(p, q) = (p_m, q_m)$ for all of q
Then for $p = k+1$, we can also find $(p, q) = (p_{m+k+q}, q_{m+k+q})$ according to the recurrence relation $(p_n, q_n) = (p_{n-1} + 1, q_{n-1} - 1)$ and $(p_n, q_n) = (1, p_{n-1} + 1)$
So for $p < q, p \in N^+$ and $q \in N^+$, we can find $(p, q) = (p_n, q_n)$

iii) For $n = 1$ we can find $\phi(p_1/q_1) = 1$, it is true.
For $n = k$, suppose that $\phi(p_n/q_n) = n$
For $n = k + 1$, we should discuss it in two situations
Firstly, for $q_n = 1$, $p_{n+1} = 1$, $q_{n+1} = p_n + 1$

$$\begin{aligned}
\phi\left(\frac{p_{n+1}}{q_{n+1}}\right) &= \frac{(p_{n+1} + q_{n+1} - 1)(p_{n+1} + q_{n+1} - 2)}{2} + p_{n+1} \\
&= \frac{(q_n + p_n)(q_n + p_n - 1)}{2} + 1 \\
&= \frac{(q_n + p_n - 2)(q_n + p_n - 1)}{2} + 1 + q_n + p_n - 1 \\
&= \frac{(p_n + q_n - 1)(p_n + q_n - 2)}{2} + p_n + 1 \\
&= n + 1
\end{aligned}$$

Secondly, for $q_n > 1$, $p_{n+1} = p_n + 1$, $q_{n+1} = q_n - 1$

$$\begin{aligned}
\phi\left(\frac{p_{n+1}}{q_{n+1}}\right) &= \frac{(p_{n+1} + q_{n+1} - 1)(p_{n+1} + q_{n+1} - 2)}{2} + p_{n+1} \\
&= \frac{(p_n + q_n - 1)(p_n + q_n - 2)}{2} + p_n + 1 \\
&= n + 1
\end{aligned}$$

So it is proved.

iv) Since ϕ is a function from a set p/q to a set n, it is surjective.
Since it can be reversed as is proved in part v), it is injective.
So it is bijective.

v) When $k \in N^+$, suppose

$$\frac{k(k+1)}{2} < n \leqslant \frac{(k+1)(k+2)}{2}$$

where

$$\frac{(k+1)(k+2)}{2} - \frac{k(k+1)}{2} = k + 1$$

and

$$k^2 + k - 2n < 0$$
$$k \in \left(\frac{-1 - \sqrt{1+8n}}{2}, \frac{-1 + \sqrt{1+8n}}{2}\right)$$

The maximum integer k is what we need here, so

$$k = \left[\frac{-1 + \sqrt{1+8n}}{2}\right]$$

According to the recurrence we find

$$p = n - \frac{k(k+1)}{2} \quad q = k + 2 - p$$

So we find the inverse ϕ^{-1}

$$\phi^{-1}(n) = \frac{n - \frac{k(k+1)}{2}}{k + 2 - n + \frac{k(k+1)}{2}} \text{ where k} = \left[\frac{-1 + \sqrt{1 + 8\text{n}}}{2}\right]$$

vi)

$$\phi(p, q) = \frac{(p+q)(p+q-1)}{2} + p$$

vii)

$$\phi(p, q) = (|p| + |q|)(|p| + |q| - 1) + 2p - u(pq)$$

Exercise 6.5

Exercise 6.6

Since card $M =$ card N, the number of elements in set M and N is the same.

Since $M \in N$, if there is an element which is in N but is not in M, then the number of elements in M is more than that in N, which leads to a contradiction.

So there isn't any element which is in N but is not in M, which means $M = N$

Exercise 6.7

If $f : M \to N$ is surjective, according to theorem 2.4.21, if card $M \neq$ card N, f is injective.

If $f : M \to N$ is not surjective, then we can find a $f' : M \to N'$ which is surjective so that card $N' <$ card N, so we can use the theorem again. If card $M \neq$ card N', f' is injective. So f is also injective.

So it is proved.