VE203 Assignment 9

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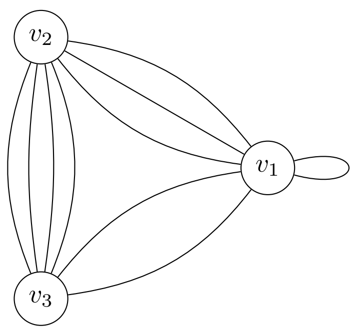
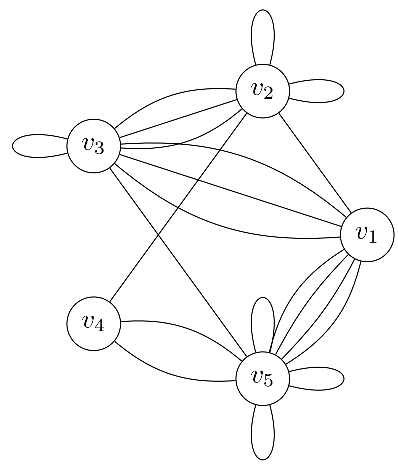
Exercise 9.1

i)



ii)



Exercise 9.2

i)

$$\phi : u_1 \to v_1, u_2 \to v_4, u_3 \to v_2, u_4 \to v_5, u_5 \to v_3$$

$$\begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{array} \begin{array}{c} \begin{array}{ccccc} u_1 & u_2 & u_3 & u_4 & u_5 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array} = \begin{array}{c} \\ v_1 \\ v_4 \\ v_2 \\ v_5 \\ v_3 \end{array} \begin{array}{c} \begin{array}{ccccc} v_1 & v_4 & v_2 & v_5 & v_3 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

So it is an isomorphism.

ii)

$$\phi : u_1 \to v_1, u_2 \to v_3, u_3 \to v_2, u_4 \to v_5, u_5 \to v_4$$

$$
\begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{array}
\begin{array}{c} \begin{array}{ccccc} u_1 & u_2 & u_3 & u_4 & u_5 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}
=
\begin{array}{c} \\ v_1 \\ v_3 \\ v_2 \\ v_5 \\ v_4 \end{array}
\begin{array}{c} \begin{array}{ccccc} v_1 & v_3 & v_2 & v_5 & v_4 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}
$$

So it is an isomorphism.

iii) There are two triangle subgraph $\{u_1, u_2, u_3\}$ and $\{u_5, u_6, u_7\}$ in the first graph, but there isn't any in the second graph, so it isn't an isomorphism.

iv)

$$\phi : u_1 \to v_3, u_2 \to v_1, u_3 \to v_2, u_4 \to v_5, u_5 \to v_6, u_6 \to v_8, u_7 \to v_7, u_8 \to v_4$$

$$
\begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \end{array}
\begin{array}{c} \begin{array}{cccccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 & u_7 & u_8 \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{pmatrix} \end{array}
=
\begin{array}{c} \\ v_3 \\ v_1 \\ v_2 \\ v_5 \\ v_6 \\ v_8 \\ v_7 \\ v_4 \end{array}
\begin{array}{c} \begin{array}{cccccccc} v_3 & v_1 & v_2 & v_5 & v_6 & v_8 & v_7 & v_4 \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{pmatrix} \end{array}
$$

So it is an isomorphism.

Exercise 9.3

i) For a complete graph, every two vertices are connected.
When $n = 2k + 1, k \in N$, each of the vertices has $2k$ edges, so the graph has an Euler circuit.
When $n = 2k, k \in N^+$, each of the vertices has $2k + 1$ edges, so the graph hasn't an Euler circuit.
So $n \in \{2k + 1, k \in N\}$

ii) For a wheel graph, the vertex in the center has $n-1$ edges and the other vertices have $n-2$ edges. Since $n - 1$ is odd or $n - 2$ is odd, the graph never has an Euler circuit.

iii) For a cycle graph, each vertex have two edges, so the graph always has an Euler circuit.

iv) For a cycle graph, each vertex have n edges, when n is even, the graph has an Euler circuit.
So $n \in \{2k + 2, k \in N\}$

Exercise 9.4

If all simple circuits have even length, for two vertices u and v, choosing a vertex u and setting $S = \{v : 2|d(u, v)\}$ and $T = \{v : 2 \not| d(u, v)\}$. Suppose $v_1 \in S$ and $v_2 \in T$, and $d(v_1, v_2) = d(u, v_1) + d(u, v_2)$ or $d(v_1, v_2) = k - d(u, v_1) - d(u, v_2)$ where k is the length of the simplest circuit containing u, v_1, v_2, where k is even. So $d(u, v_1) + d(u, v_2)$ and $k - d(u, v_1) - d(u, v_2)$ are both odd, which means $d(v_1, v_2)$ is odd. So S and T defines a bipartition of the graph.

If a simple circuit have odd length, then the subgraph of the circuit isn't bipartite, so the graph is bipartite only if all simple circuits have even length.

Exercise 9.5

Define $sum(V) = the\ addition\ of\ digits\ of\ the\ vertex$, then the bipartition is

$$(\{sum(V) \equiv 0(\text{mod } 2)\}, \{sum(V) \equiv 1(\text{mod } 2)\})$$

For $n = 1$, the bipartition of Q_1 is $(\{0\}, \{1\})$
For $n = k, k \in N, k > 1$, suppose Q_n bipartite for all $n \in N$ and the bipartition is listed above.
For $n = k + 1$, we have two subgraphs formed from Q_n that adding either 0 or 1 in front of every vertex and then connect them together. We can find that each of the two newly connected vertices are in different partitions since the subgraph adding 1 in the front have totally opposite bipartition. So it is proved.

Exercise 9.6

i) The total possible number of edges in a graph containing n vertices is

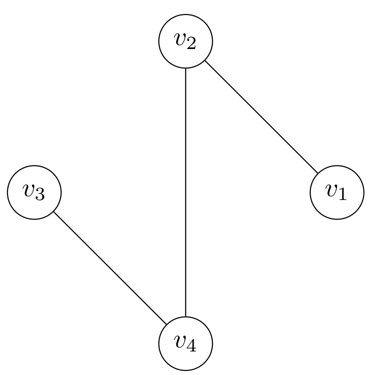
$$N = \frac{n(n-1)}{2}$$

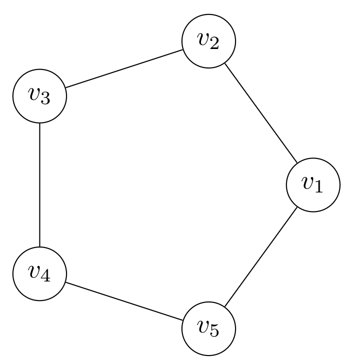
Since $N_E = N_{E^C}$,

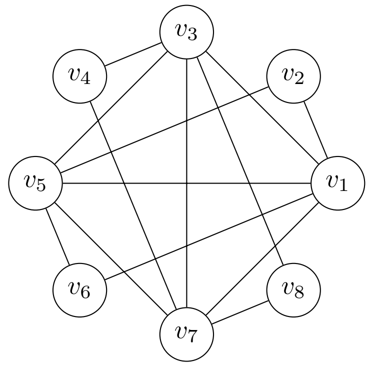
$$N_E = N_{E^C} = \frac{n(n-1)}{4}$$

So only when $n = 4m$ or $4m + 1, m \in N$, N_E and N_{E^C} are integers.

ii)







Exercise 9.7

i) Suppose the addition of a single edge will never produce a Hamilton circuit in H, but if we add edges to form a complete graph, there must exist a Hamilton circuit in H, which reaches a contradiction. So it is proved.

ii) Since there exists a Hamilton circuit in H, then we can remove one added edge in the circuit so that the vertices on the two sides of the edge are not adjacent, then we get a Hamilton path in H.

iii) Since in G, $deg(x) + deg(y) > n$, and $deg_H(v) \geqslant deg_G(v)$, so $deg(v_1) + deg(v_n) > n$ in H.
We know $n - deg(v_n) < deg(v_1)$, and $n - deg(v_n)$ is the number of vertices not adjacent to v_n, so there are at most $deg(v_1)$ vertices not adjacent to v_n.

iv) Since S is the set of vertices preceding each vertex adjacent to v_1, so S contains $deg(v_1)$ vertices.
Since $v_1, ..., v_n$ is a Hamilton path, v_n is the last vertex, v_{n+1} doesn't exists. So $v_n \notin S$.

v) Since there are $deg(v_1)$ vertices in S and $deg(v_n)$ vertices adjacent to v_n, and $deg(v_1)+deg(v_n) > n$, according to the Pigeonhole Principle, there is at least a vertex v_k both in S and adjacent to v_n. So $v_k \in S$ and v_k is adjacent to v_n, then according to the definition of S, v_{k+1} is adjacent to v_1. Since $v_1, ..., v_n$ is a Hamilton path, v_k is adjacent to v_{k+1}. So v_1 and v_{k+1} and v_k and v_n are connected.

vi) There are edges between $v_1, v_2, ..., v_{k-1}, v_k, v_n, v_{n-1}, ..., v_{k+1}, v_1$, so it is a Hamilton circuit, which reaches a contradiction with the suppose in part iii). So the Ore's Theorem holds.

Exercise 9.8

0^{nd} Iteration:
$S_0 = \emptyset$

1^{st} Iteration:
$S_1 = \{U\}$
$U : 0$, $D : 7$ (U), $E : 10$ (U), $Q : 4$ (U), $Z : 6$ (U)

2^{nd} Iteration:
$S_2 = \{U, D\}$
$U : 0$, $D : 7$ (U), $E : 10$ (U), $Q : 4$ (U), $Z : 6$ (U), $C : 17$ (U, D)

3^{rd} Iteration:
$S_3 = \{U, D, E\}$
$U : 0$, $D : 7\ (U)$, $E : 10\ (U)$, $Q : 4\ (U)$, $Z : 6\ (U)$, $C : 17\ (U, D)$, $F : 20\ (U, E)$

4^{th} Iteration:
$S_4 = \{U, D, E, Q\}$
$U : 0$, $D : 7\ (U)$, $E : 10\ (U)$, $Q : 4\ (U)$, $Z : 6\ (U)$, $C : 17\ (U, D)$, $F : 14\ (U, Q)$, $P : 9\ (U, Q)$, $T : 11\ (U, Q)$

5^{th} Iteration:
$S_5 = \{U, D, E, Q, Z\}$
$U : 0$, $D : 7\ (U)$, $E : 10\ (U)$, $Q : 4\ (U)$, $Z : 6\ (U)$, $C : 17\ (U, D)$, $F : 14\ (U, Q)$, $P : 9\ (U, Q)$, $T : 11\ (U, Q)$

6^{th} Iteration:
$S_6 = \{U, D, E, Q, Z, C\}$
$U : 0$, $D : 7\ (U)$, $E : 10\ (U)$, $Q : 4\ (U)$, $Z : 6\ (U)$, $C : 17\ (U, D)$, $F : 14\ (U, Q)$, $P : 9\ (U, Q)$, $T : 11\ (U, Q)$, $B : 25\ (U, D, C)$

7^{th} Iteration:
$S_7 = \{U, D, E, Q, Z, C, F\}$
$U : 0$, $D : 7\ (U)$, $E : 10\ (U)$, $Q : 4\ (U)$, $Z : 6\ (U)$, $C : 17\ (U, D)$, $F : 14\ (U, Q)$, $P : 9\ (U, Q)$, $T : 11\ (U, Q)$, $B : 25\ (U, D, C)$, $G : 20\ (U, Q, F)$, $X : 23\ (U, Q, F)$

8^{th} Iteration:
$S_8 = \{U, D, E, Q, Z, C, F, P\}$
$U : 0$, $D : 7\ (U)$, $E : 10\ (U)$, $Q : 4\ (U)$, $Z : 6\ (U)$, $C : 16\ (U, Q, P)$, $F : 14\ (U, Q)$, $P : 9\ (U, Q)$, $T : 11\ (U, Q)$, $B : 25\ (U, D, C)$, $G : 20\ (U, Q, F)$, $X : 23\ (U, Q, F)$, $O : 15\ (U, Q, P)$, $S : 15\ (U, Q, P)$

9^{th} Iteration:
$S_9 = \{U, D, E, Q, Z, C, F, P, T\}$
$U : 0$, $D : 7\ (U)$, $E : 10\ (U)$, $Q : 4\ (U)$, $Z : 6\ (U)$, $C : 16\ (U, Q, P)$, $F : 14\ (U, Q)$, $P : 9\ (U, Q)$, $T : 11\ (U, Q)$, $B : 25\ (U, D, C)$, $G : 20\ (U, Q, F)$, $X : 14\ (U, Q, T)$, $O : 15\ (U, Q, P)$, $S : 15\ (U, Q, P)$

10^{th} Iteration:
$S_{10} = \{U, D, E, Q, Z, C, F, P, T, B\}$
$U : 0$, $D : 7\ (U)$, $E : 10\ (U)$, $Q : 4\ (U)$, $Z : 6\ (U)$, $C : 16\ (U, Q, P)$, $F : 14\ (U, Q)$, $P : 9\ (U, Q)$, $T : 11\ (U, Q)$, $B : 25\ (U, D, C)$, $G : 20\ (U, Q, F)$, $X : 14\ (U, Q, T)$, $O : 15\ (U, Q, P)$, $S : 15\ (U, Q, P)$, $A : 31\ (U, D, C, B)$

11^{th} Iteration:
$S_{11} = \{U, D, E, Q, Z, C, F, P, T, B, G\}$
$U : 0$, $D : 7\ (U)$, $E : 10\ (U)$, $Q : 4\ (U)$, $Z : 6\ (U)$, $C : 16\ (U, Q, P)$, $F : 14\ (U, Q)$, $P : 9\ (U, Q)$, $T : 11\ (U, Q)$, $B : 25\ (U, D, C)$, $G : 20\ (U, Q, F)$, $X : 14\ (U, Q, T)$, $O : 15\ (U, Q, P)$, $S : 15\ (U, Q, P)$, $A : 31\ (U, D, C, B)$, $H : 31\ (U, Q, F, G)$

12^{th} Iteration:
$S_{12} = \{U, D, E, Q, Z, C, F, P, T, B, G, X\}$
$U : 0$, $D : 7\ (U)$, $E : 10\ (U)$, $Q : 4\ (U)$, $Z : 6\ (U)$, $C : 16\ (U, Q, P)$, $F : 14\ (U, Q)$, $P : 9\ (U, Q)$, $T : 11\ (U, Q)$, $B : 25\ (U, D, C)$, $G : 20\ (U, Q, F)$, $X : 14\ (U, Q, T)$, $O : 15\ (U, Q, P)$, $S : 15\ (U, Q, P)$, $A : 31\ (U, D, C, B)$, $H : 17\ (U, Q, T, X)$, $W : 23\ (U, Q, T, X)$

13^{th} Iteration:
$S_{13} = \{U, D, E, Q, Z, C, F, P, T, B, G, X, O\}$
$U : 0$, $D : 7\ (U)$, $E : 10\ (U)$, $Q : 4\ (U)$, $Z : 6\ (U)$, $C : 16\ (U, Q, P)$, $F : 14\ (U, Q)$, $P : 9\ (U, Q)$, $T : 11\ (U, Q)$, $B : 24\ (U, Q, P, O)$, $G : 20\ (U, Q, F)$, $X : 14\ (U, Q, T)$, $O : 15\ (U, Q, P)$, $S : 15\ (U, Q, P)$, $A : 31\ (U, D, C, B)$, $H : 17\ (U, Q, T, X)$, $W : 23\ (U, Q, T, X)$, $N : 20\ (U, Q, P, O)$, $Y : 19\ (U, Q, P, O)$

14^{th} Iteration:
$S_{14} = \{U, D, E, Q, Z, C, F, P, T, B, G, X, O, S\}$
$U : 0$, $D : 7\ (U)$, $E : 10\ (U)$, $Q : 4\ (U)$, $Z : 6\ (U)$, $C : 16\ (U, Q, P)$, $F : 14\ (U, Q)$, $P : 9\ (U, Q)$, $T : 11\ (U, Q)$, $B : 24\ (U, Q, P, O)$, $G : 20\ (U, Q, F)$, $X : 14\ (U, Q, T)$, $O : 15\ (U, Q, P)$, $S : 15\ (U, Q, P)$, $A : 31\ (U, D, C, B)$, $H : 17\ (U, Q, T, X)$, $W : 23\ (U, Q, T, X)$, $N : 20\ (U, Q, P, O)$, $Y : 18\ (U, Q, P, S)$, $R : 22\ (U, Q, P, S)$

15^{th} Iteration:
$S_{15} = \{U, D, E, Q, Z, C, F, P, T, B, G, X, O, S, A\}$
$U : 0$, $D : 7\ (U)$, $E : 10\ (U)$, $Q : 4\ (U)$, $Z : 6\ (U)$, $C : 16\ (U, Q, P)$, $F : 14\ (U, Q)$, $P : 9\ (U, Q)$, $T : 11\ (U, Q)$, $B : 24\ (U, Q, P, O)$, $G : 20\ (U, Q, F)$, $X : 14\ (U, Q, T)$, $O : 15\ (U, Q, P)$, $S : 15\ (U, Q, P)$, $A : 31\ (U, D, C, B)$, $H : 17\ (U, Q, T, X)$, $W : 23\ (U, Q, T, X)$, $N : 20\ (U, Q, P, O)$, $Y : 18\ (U, Q, P, S)$, $R : 22\ (U, Q, P, S)$, $M : 38\ (U, D, C, B, A)$

16^{th} Iteration:
$S_{16} = \{U, D, E, Q, Z, C, F, P, T, B, G, X, O, S, A, H\}$
$U : 0$, $D : 7\ (U)$, $E : 10\ (U)$, $Q : 4\ (U)$, $Z : 6\ (U)$, $C : 16\ (U, Q, P)$, $F : 14\ (U, Q)$, $P : 9\ (U, Q)$, $T : 11\ (U, Q)$, $B : 24\ (U, Q, P, O)$, $G : 20\ (U, Q, F)$, $X : 14\ (U, Q, T)$, $O : 15\ (U, Q, P)$, $S : 15\ (U, Q, P)$, $A : 31\ (U, D, C, B)$, $H : 17\ (U, Q, T, X)$, $W : 23\ (U, Q, T, X)$, $N : 20\ (U, Q, P, O)$, $Y : 18\ (U, Q, P, S)$, $R : 22\ (U, Q, P, S)$, $M : 38\ (U, D, C, B, A)$, $I : 27\ (U, Q, T, X, H)$

17^{th} Iteration:
$S_{17} = \{U, D, E, Q, Z, C, F, P, T, B, G, X, O, S, A, H, W\}$
$U : 0$, $D : 7\ (U)$, $E : 10\ (U)$, $Q : 4\ (U)$, $Z : 6\ (U)$, $C : 16\ (U, Q, P)$, $F : 14\ (U, Q)$, $P : 9\ (U, Q)$, $T : 11\ (U, Q)$, $B : 24\ (U, Q, P, O)$, $G : 20\ (U, Q, F)$, $X : 14\ (U, Q, T)$, $O : 15\ (U, Q, P)$, $S : 15\ (U, Q, P)$, $A : 31\ (U, D, C, B)$, $H : 17\ (U, Q, T, X)$, $W : 23\ (U, Q, T, X)$, $N : 20\ (U, Q, P, O)$, $Y : 18\ (U, Q, P, S)$, $R : 22\ (U, Q, P, S)$, $M : 38\ (U, D, C, B, A)$, $I : 27\ (U, Q, T, X, H)$, $V : 27\ (U, Q, T, X, W)$

So the length is 23 and the path is (U, Q, T, X, W)