Introduction to Signals and Systems: V216

Lecture #4

Chapter 2: Linear Time-Invariant Systems

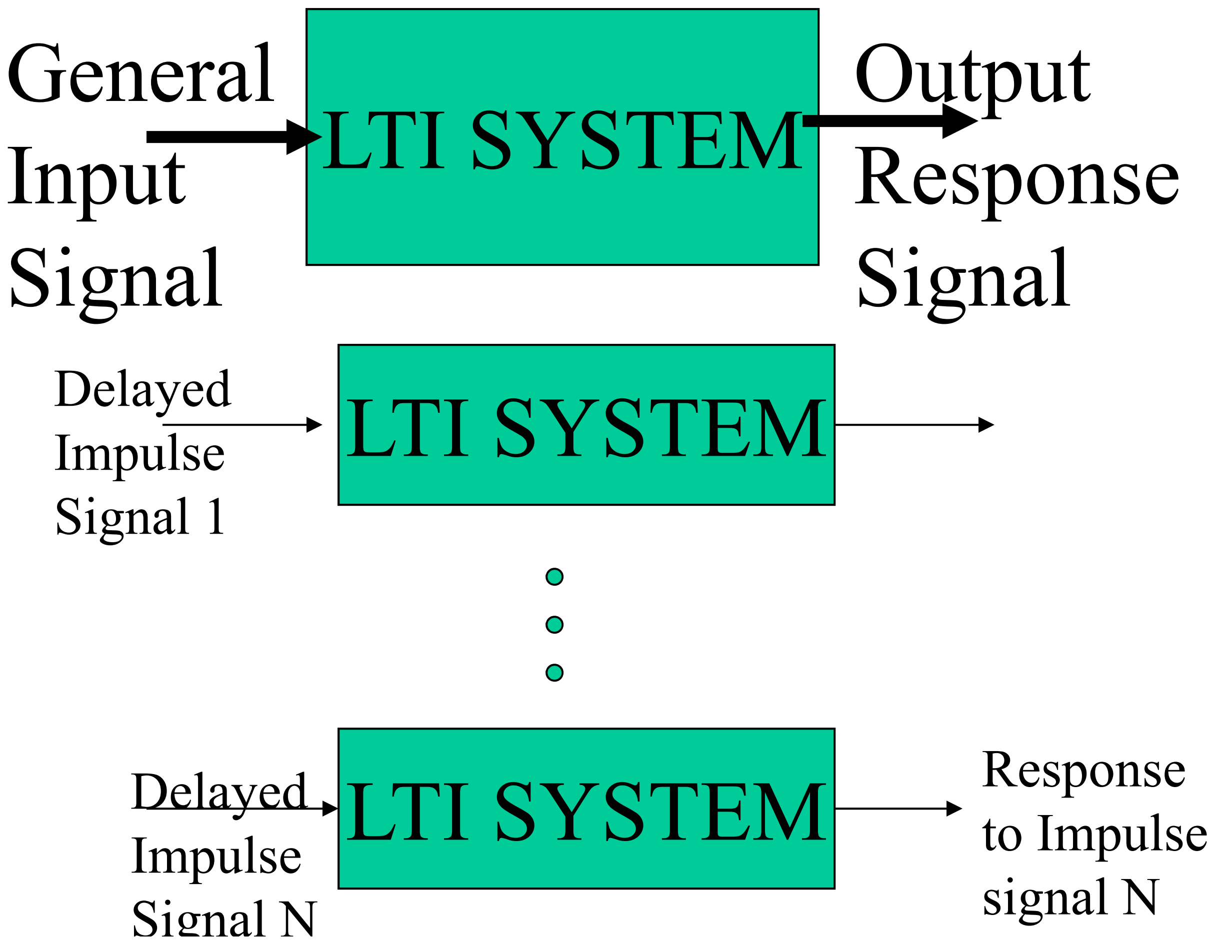
Computing the output response of LTI Systems.

- By breaking or decomposing and representing the input signal to the LTI system into terms of a linear combination of a set of basic signals.
- Using the superposition property of LTI system to compute the output of the system in terms of its response to these basic signals.

General Signal Representations By Basic Signal

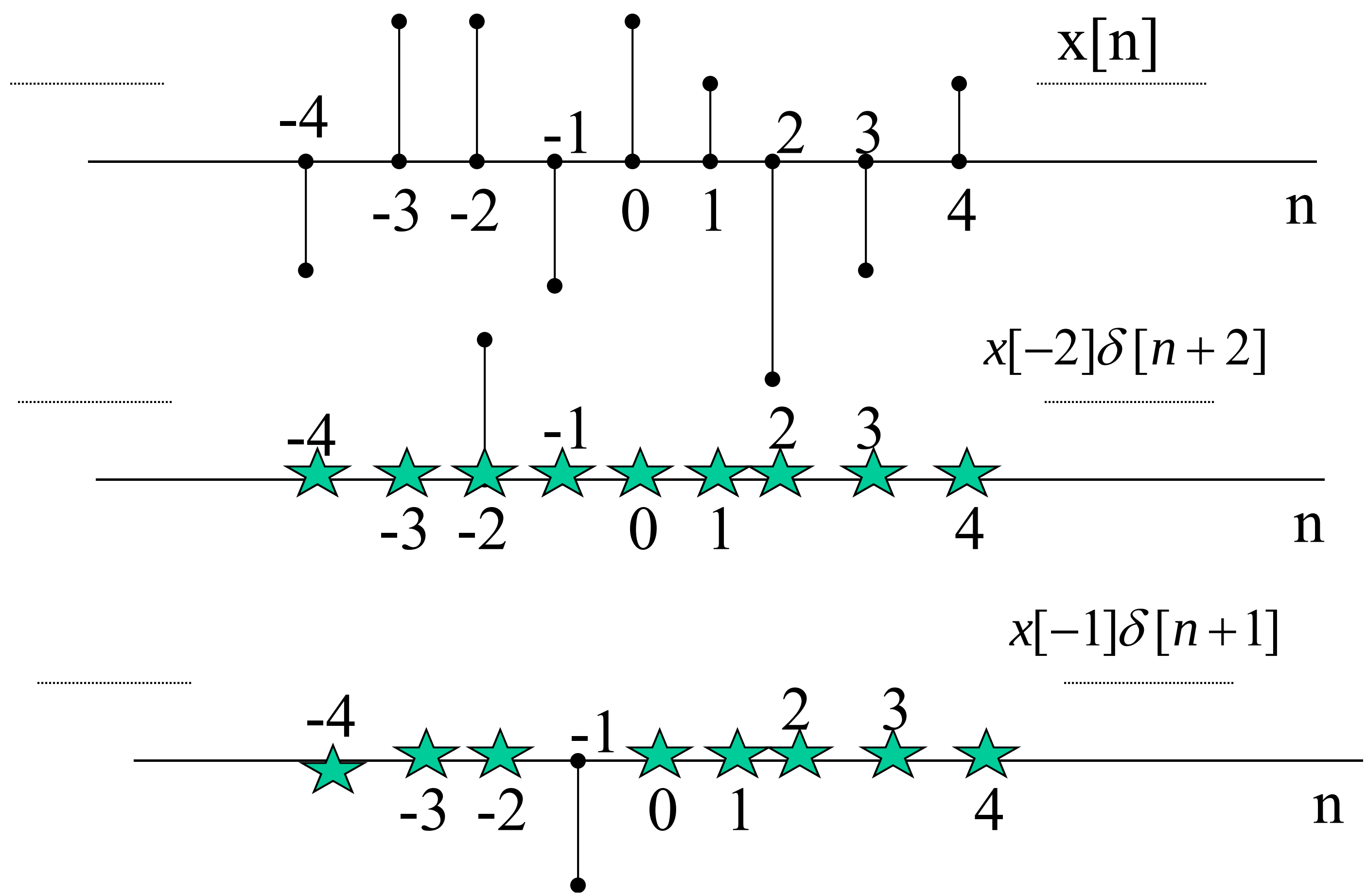
- The basic signal - in particular the unit **impulse** can be used to decompose and represent the general form of any signal.
- Linear combination of delayed impulses can represent these general signals.

Response of LTI System to General Input Signal

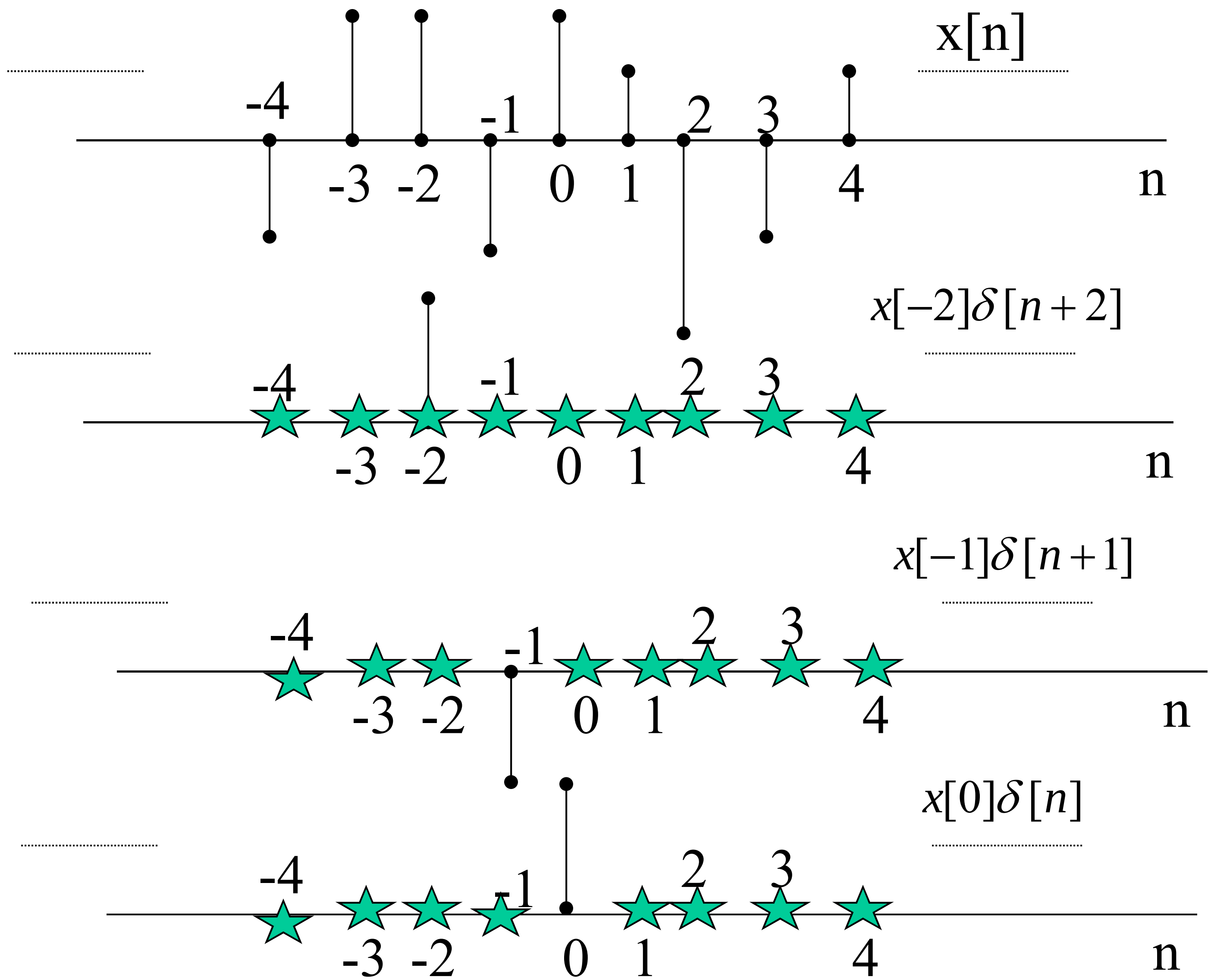


Representation of Discrete-time Signals in Terms of Impulses.

Discrete-time signals are sequences of individual impulses.



Discrete-time signals are sequences of individual scaled unit impulses.



Shifted Scaled Impulses:

$$x[n] = \ldots + x[-3]\delta[n+3] + x[-2]\delta[n+2] + x[-1]\delta[n+1] + x[0]\delta[n]$$
$$+ x[1]\delta\ [n-1] + x[2]\delta[n-2] + x[3]\delta[n-3] + \ldots..$$

Generally:-

$$x[n] = \sum_{k=-\infty}^{+\infty} x[k]\delta[n-k],$$

The arbitrary sequence is represented by a linear combination of shifted unit impulses δ[n-k] , where the weights in this linear combination are x[k].
The above equation is called the **shifting property** of discrete-time unit impulse.

As Example consider unit step signal x[n]=u[n]:-

$$x[n]=u[n]=...+0.\delta[n+3]+0.\delta[n+2]+0.\delta[n+1]+1.\delta[n]$$
$$+1.\delta\ [n-1]+1.\delta[n-2]+1.\delta[n-3]+.....$$

Generally:-

$$u[n]=\sum_{k=0}^{+\infty}1.\delta[n-k],$$

The unit step sequence is represented by a linear combination of shifted unit impulses δ[n-k] , where the weights in this linear combination are ones from k=0 right up to k=∞

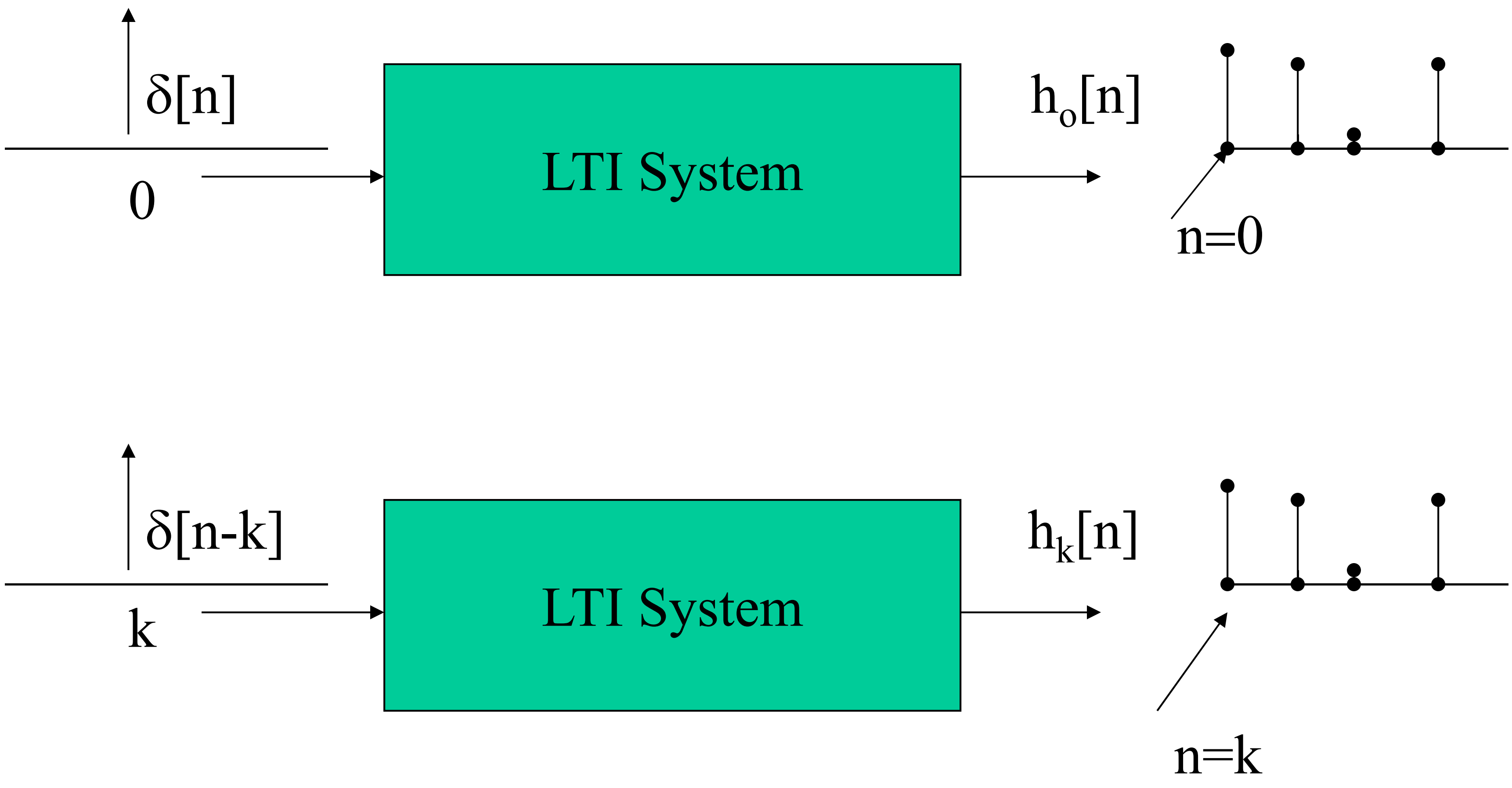
The Discrete-time Unit Impulse Responses and the Convolution Sum Representation

- To determine the output response of an LTI system to an arbitrary input signal x[n], we make use of the shifting property for input signal and the superposition and time-invariant properties of LTI system.

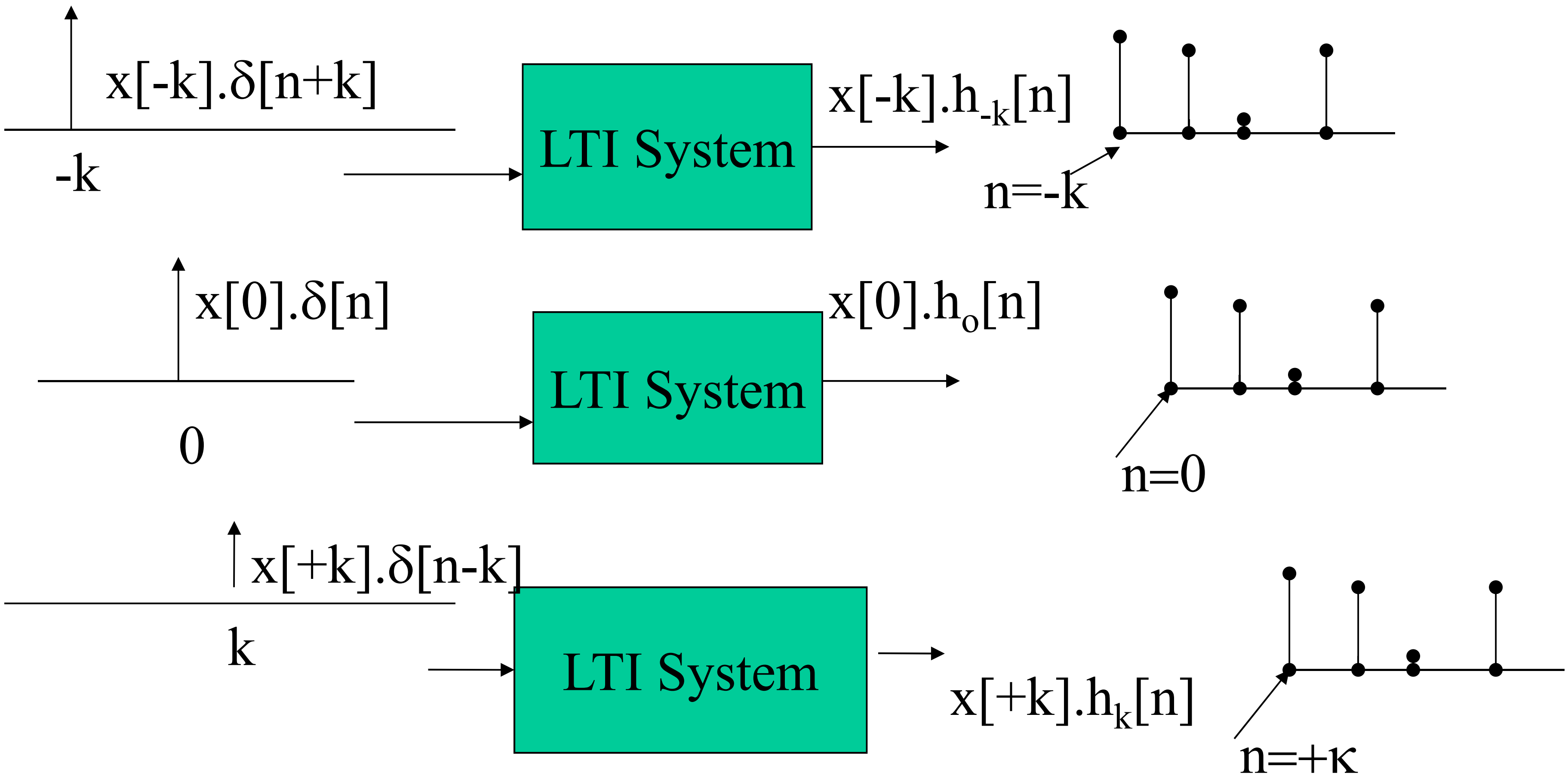
Convolution Sum Representation

- The response of a linear system to x[n] will be the superposition of the scaled responses of the system to each of these shifted impulses.
- From the time-invariant property, the response of LTI system to the time-shifted unit impulses are simply time-shifted responses of one another.

Unit Impulse Response h[n]



Response to scaled unit impulse input x[n]δ[n-k]

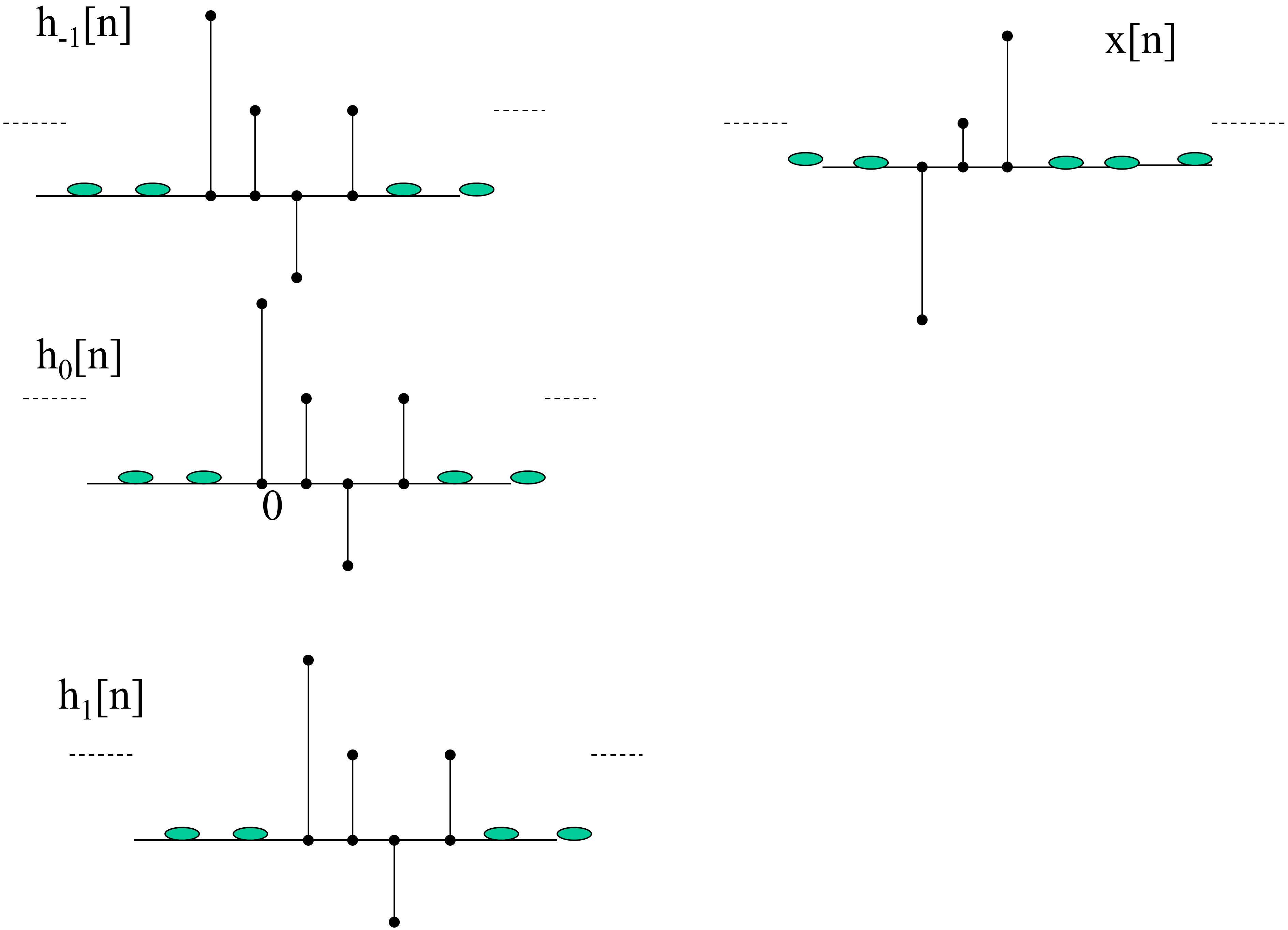


Output y[n] of LTI System

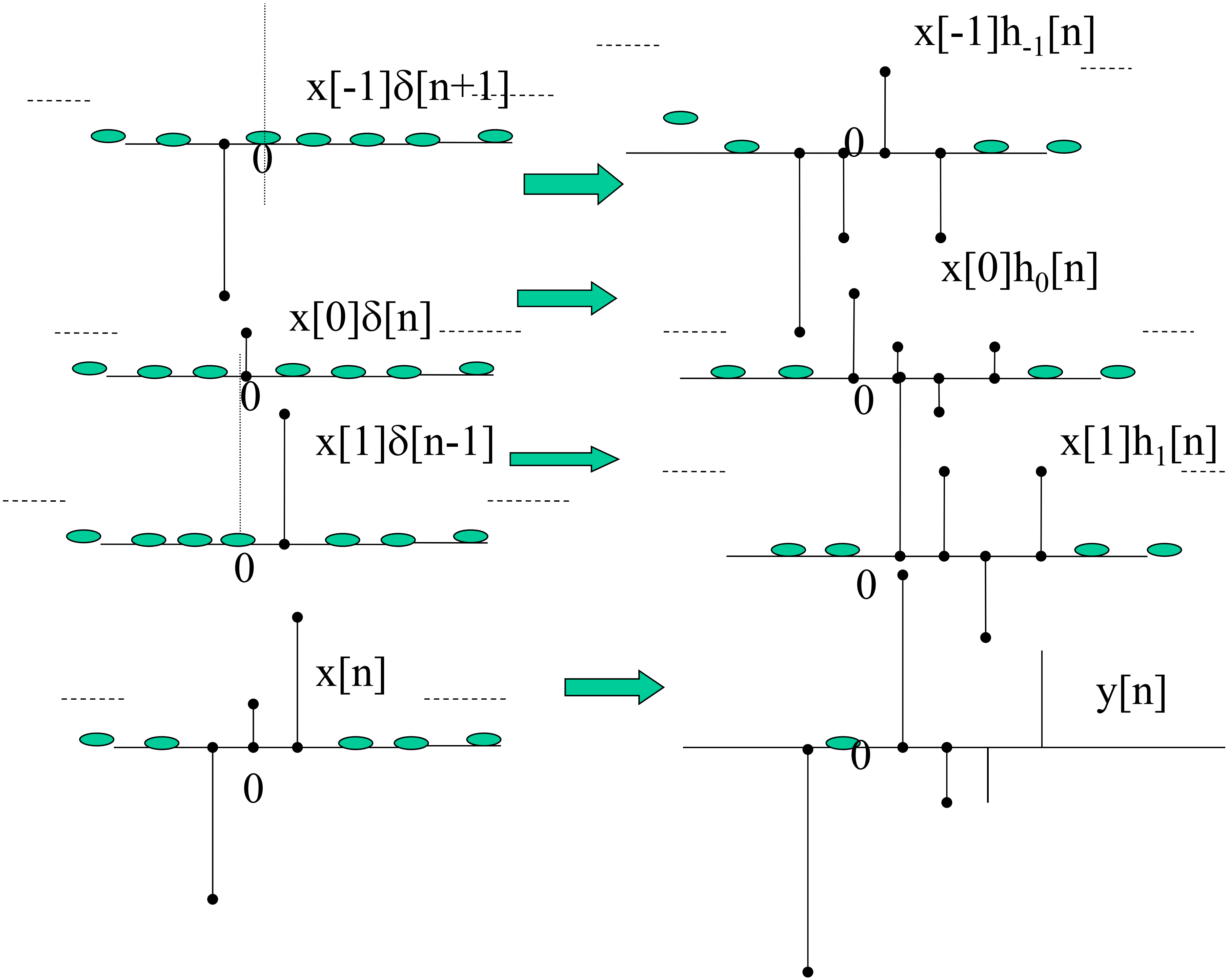
With the input x[n] being expressed as the delayed train of scaled impulses we have : -

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k].h_k[n].$$

Thus, if we know the response of a linear system **h(n)** to the set of shifted unit impulses, we can construct the response y[n] to an arbitrary input signal x[n].



$h_{-1}[n]$
$h_0[n]$
0
$h_1[n]$
$x[n]$



x[-1]δ[n+1]
0
x[0]δ[n]
0
x[1]δ[n-1]
0
x[n]
0
x[-1]h-1[n]
0
x[0]h0[n]
0
x[1]h1[n]
0
y[n]
0

- In general, the response $h_k[n]$ need not be related to each other for different values of k.
- If the linear system is also time-invariant system, then these responses $h_k[n]$ to time shifted unit impulse are all time-shifted versions of each other.I.e. $h_k[n]=h_0[n-k]$.
- For notational convenience we drop the subscript on $h_0[n] = h[n]$.
- **h[n] is defined as the unit impulse response**

Convolution sum or Superposition sum.

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k].h[n-k]$$

Convolution operation

notation given by : -

$$y[n] = x[n] * h[n]$$

How to Evaluate Convolution?

Evaluate the convolution graphically

1-Flip
2- Slide
3- Multiply
4- Add

$$h[k] \xrightarrow{\text{Flip}} h[-k] \xrightarrow{\text{Slide}} h[n-k] \xrightarrow{\text{Multiply}} x[k]h[n-k]$$

$$\xrightarrow{\text{Sum}} \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

How to Evaluate Convolution?

Evaluate the convolution some graphically

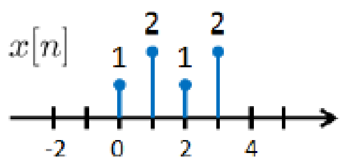
To evaluate convolution, there are three basic steps:

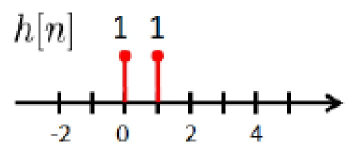
1. Flip

2. Shift

3. Multiply and Add

Example 1 : Consider the signal x[n] and the impulse response h[n] shown below.





Let's compute the output $y[n]$ one by one. First, consider $y[0]$:

$$y[0] = \sum_{k=-\infty}^{\infty} x[k]h[0-k] = \sum_{k=-\infty}^{\infty} x[k]h[-k] = 1.$$

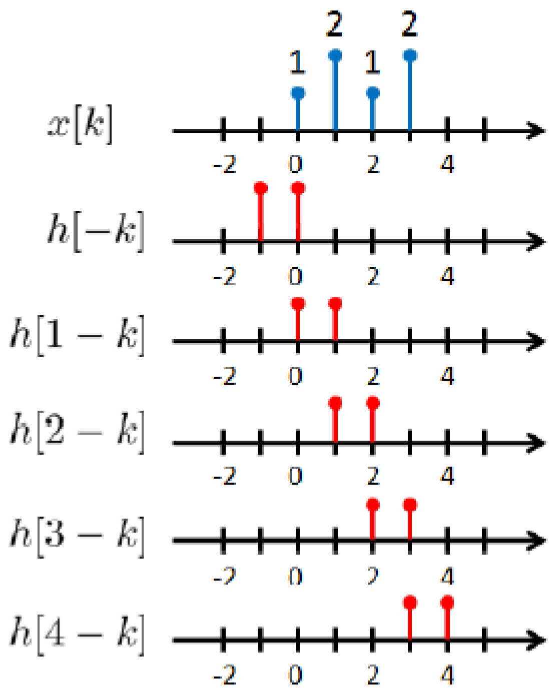
Note that $h[-k]$ is the flipped version of $h[k]$, and $\sum_{k=-\infty}^{\infty} x[k]h[-k]$ is the multiply-add between $x[k]$ and $h[-k]$.

How to Evaluate Convolution?

To calculate $y[1]$, we flip $h[k]$ to get $h[-k]$, shift $h[-k]$ go get $h[1-k]$, and multiply-add to get $\sum_{k=-\infty}^{\infty} x[k]h[1-k]$. Therefore,

$$y[1] = \sum_{k=-\infty}^{\infty} x[k]h[1-k] = \sum_{k=-\infty}^{\infty} x[k]h[1-k] = 1 \times 1 + 2 \times 1 = 3.$$

Pictorially, the calculation is shown in the figure below.



$y[0] = 1$

$y[1] = 1 + 2 = 3$

$y[2] = 2 + 1 = 3$

$y[3] = 1 + 2 = 3$

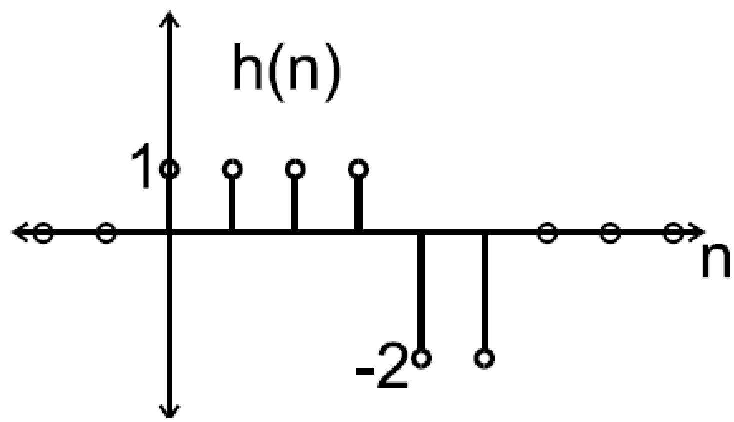
$y[4] = 2$

How to Evaluate Convolution?

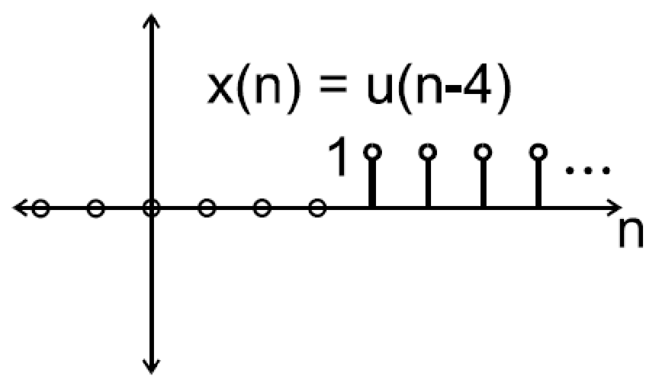
We are given the impulse response shown below.

Example 2:

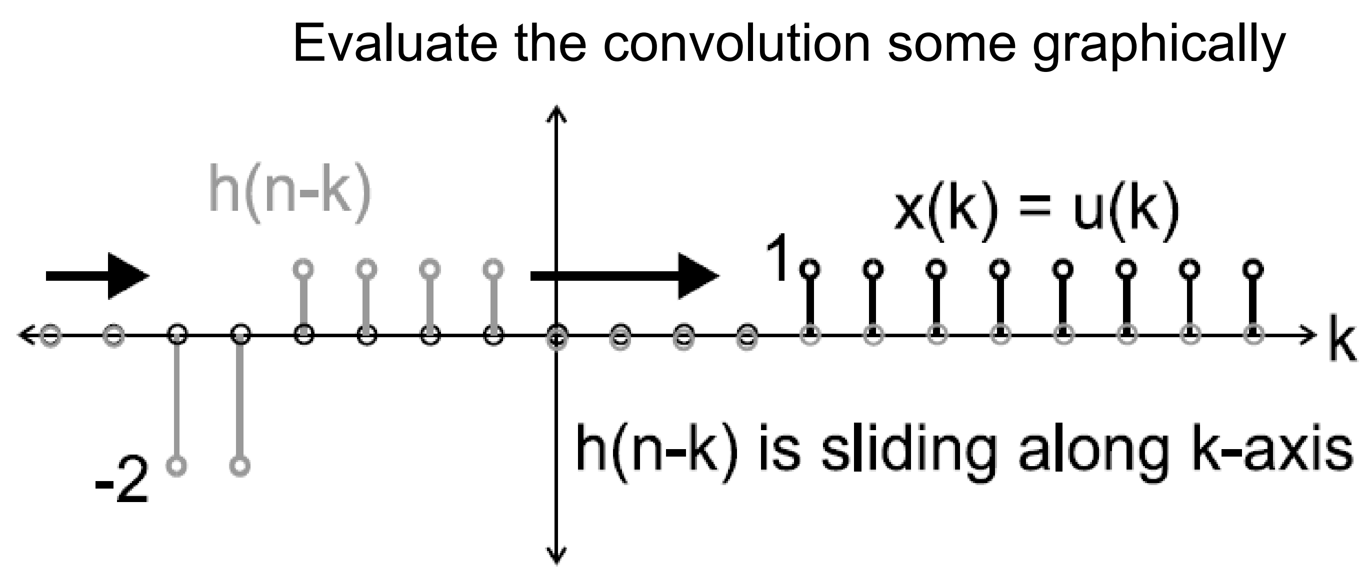
$$h(n) = \begin{cases} 0 & \text{for} \quad n < 0 \\ 1 & \text{for} \quad 0 \le n \le 3 \\ -2 & \text{for} \quad 4 \le n \le 5 \\ 0 & \text{for} \quad n > 5 \end{cases}$$



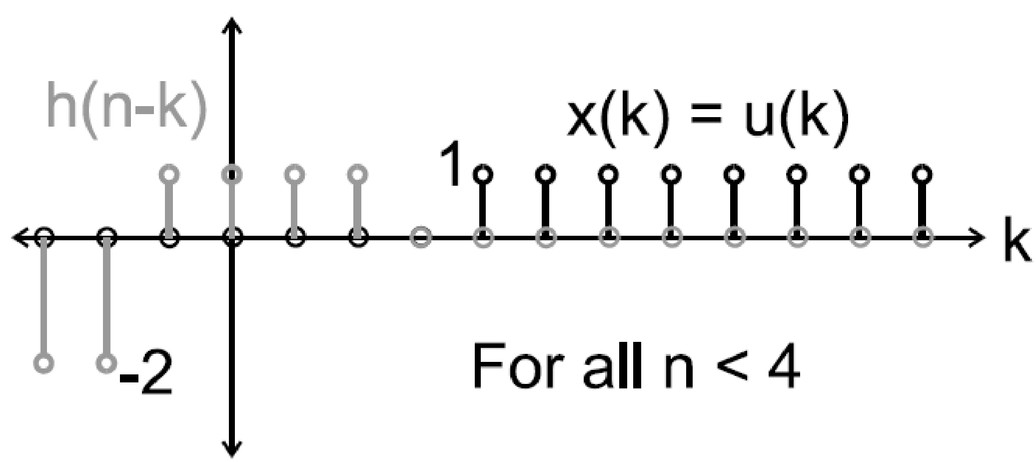
Let $x(n) = u(n-4)$.



How to Evaluate Convolution?

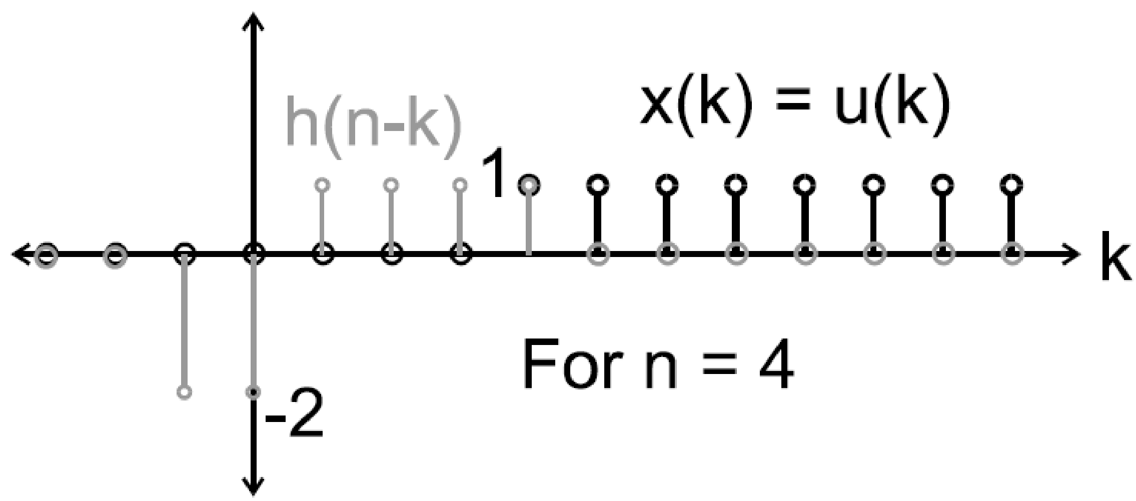


For $n < 4$, $y(n) = 0$ since there is only zero overlap between the two signals. This is illustrated below.

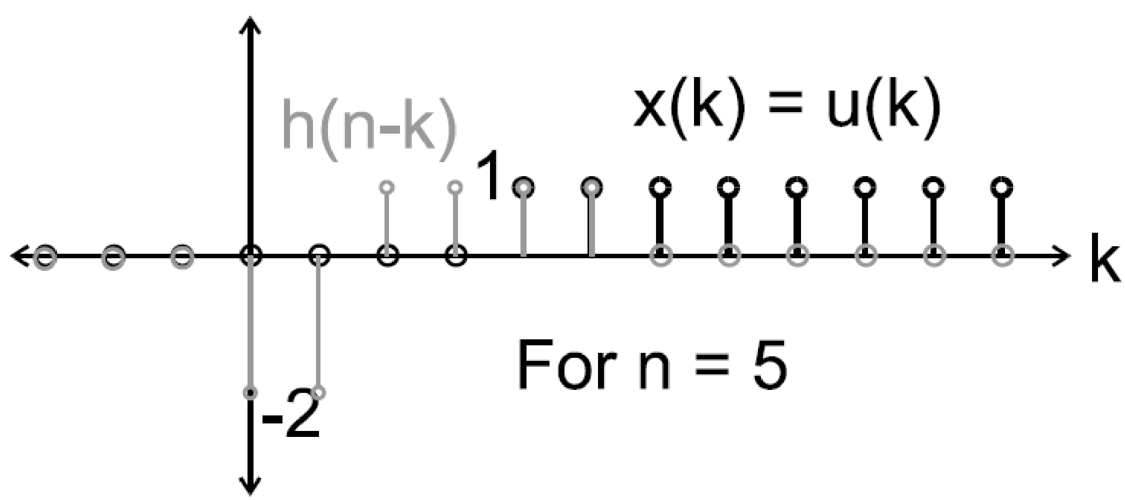


How to Evaluate Convolution?

When $n = 4$, $y(4) = 1$ since only one point of the two signals overlaps, and $1 \cdot 1 = 1$. This is shown in the figure below.



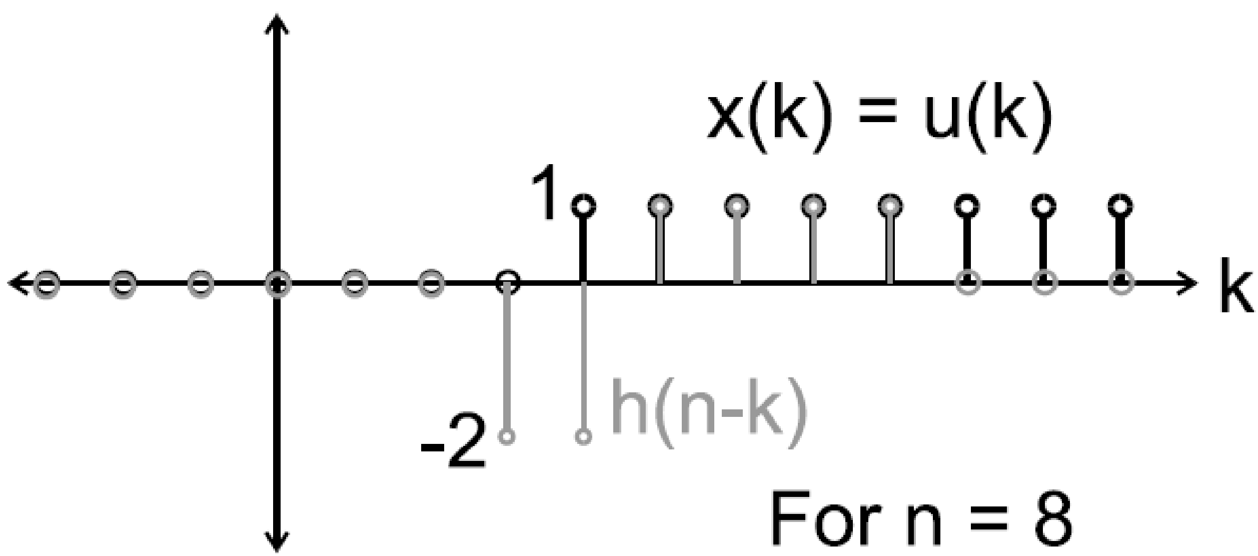
When $n = 5$, $y(5) = 2$ since it is the sum of the two overlapping points. This is shown in the figure below.



How to Evaluate Convolution?

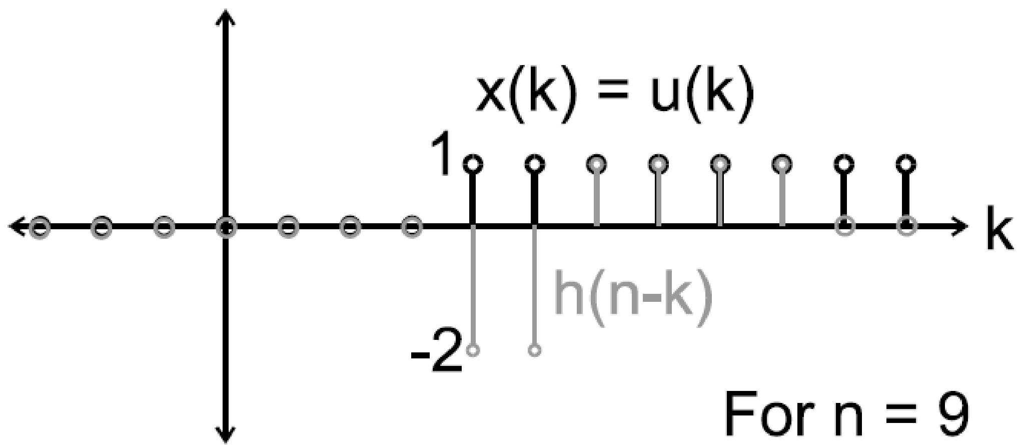
Similarly, $y(6) = 3$ and $y(7) = 4$.

Note that when $n = 8$, we have a negative overlap, and so $y(8) = 2$. This is shown below.

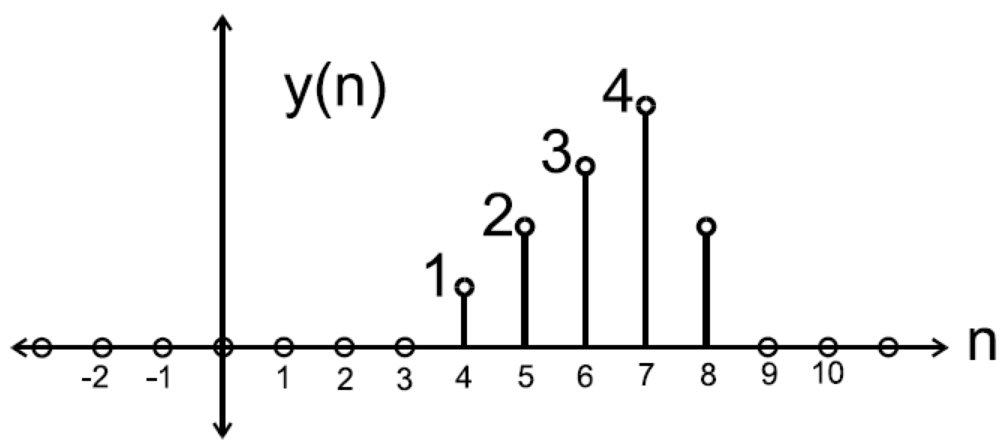


How to Evaluate Convolution?

For the case where $n \geq 9$, $y(n) = 0$ since it is summing over the entire length of the impulse response. This is shown in the figure below.



Thus, we can plot our overall $y(n)$ as shown here.



Thus, we have evaulated the convolution some graphically by taking advantage of this shifting and flipping behaviour.

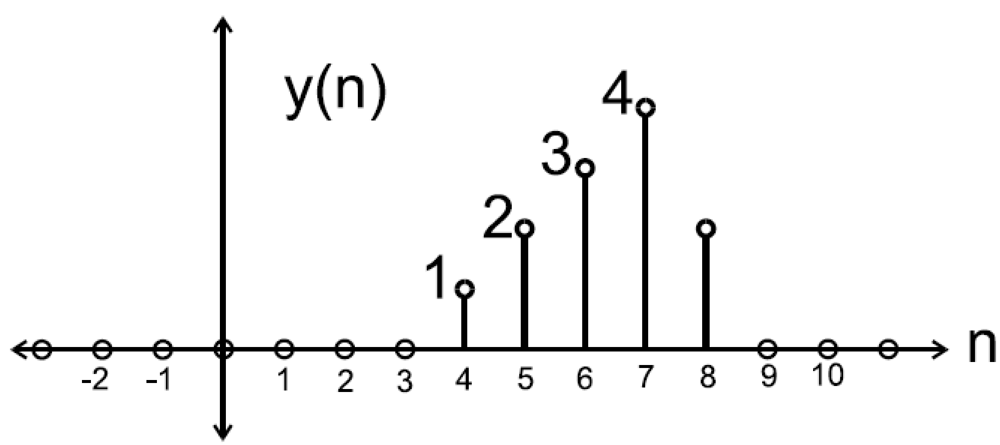
How to Evaluate Convolution?

$$
\begin{aligned}
y(n) = x(n) * h(n) &= \sum_{k=-\infty}^{\infty} x(k)h(n-k) \\
&= \sum_{k=-\infty}^{\infty} u(k-4)h(n-k)
\end{aligned}
$$

We can use $u(k-4)$ to change the summation limits, but it doesn't help much.

$$
= \sum_{k=4}^{\infty} 1 \cdot h(n-k)
$$

Thus, we can plot our overall $y(n)$ as shown here.

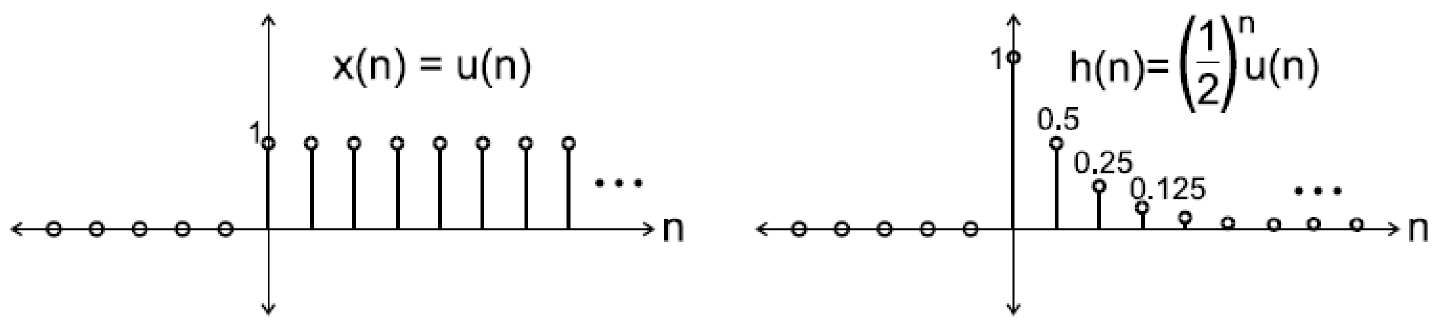


How to Evaluate Convolution?

Example 3:

Say we are given the following signal $x(n)$ and system impulse response $h(n)$.

$$x(n) = u(n) \quad \text{and} \quad h(n) = \left(\frac{1}{2}\right)^n u(n)$$



We wish to find the step response $s(n)$ of the system (i.e. the response of the system to the unit step input $x(n) = u(n)$. This is shown below.

$$s(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

How to Evaluate Convolution?

$$
\begin{aligned}
s(n) &= \sum_{k=-\infty}^{\infty} u(k) \left(\frac{1}{2}\right)^{n-k} u(n-k) \\
&= \sum_{k=0}^{n} 1 \cdot \left(\frac{1}{2}\right)^{n-k} \cdot 1 \\
&\quad \text{We can pull out any terms only in } n \\
&\quad \text{since that is not the summation variable.} \\
&= \sum_{k=0}^{n} \left(\frac{1}{2}\right)^{n} \left(\frac{1}{2}\right)^{-k} \\
&= \left(\frac{1}{2}\right)^{n} \sum_{k=0}^{n} \left(\frac{1}{2}\right)^{-k} \\
&= \left(\frac{1}{2}\right)^{n} \sum_{k=0}^{n} 2^{k}
\end{aligned}
$$

How to Evaluate Convolution?

Now we have a form consistent with a geometric series. We can use that to solve.

$$\text{Recall} \quad \sum_{k=0}^{n} 2^k = \frac{1-2^{n+1}}{1-2} = 2^{n+1} - 1$$

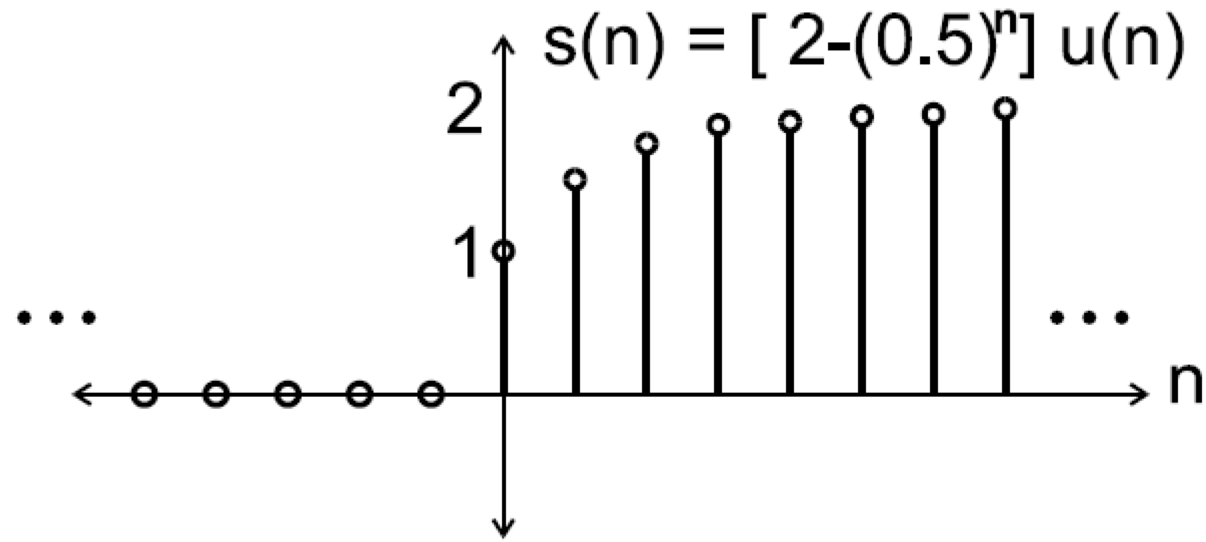
So we have $s(n)$ as follows.

$$\begin{aligned}
s(n) &= \left(\frac{1}{2}\right)^n \left(2^{n+1} - 1\right) \\
&= \left(\frac{1}{2}\right)^n \left(2 \cdot 2^n - 1\right) \\
&= \left(\frac{1}{2}\right)^n \left(2 \cdot \left(\frac{1}{2}\right)^{-n} - 1\right) \\
&= 2 \cdot \left(\frac{1}{2}\right)^{-n} \left(\frac{1}{2}\right)^n - 1 \cdot \left(\frac{1}{2}\right)^n \\
s(n) &= 2 - \left(\frac{1}{2}\right)^n
\end{aligned}$$

How to Evaluate Convolution?

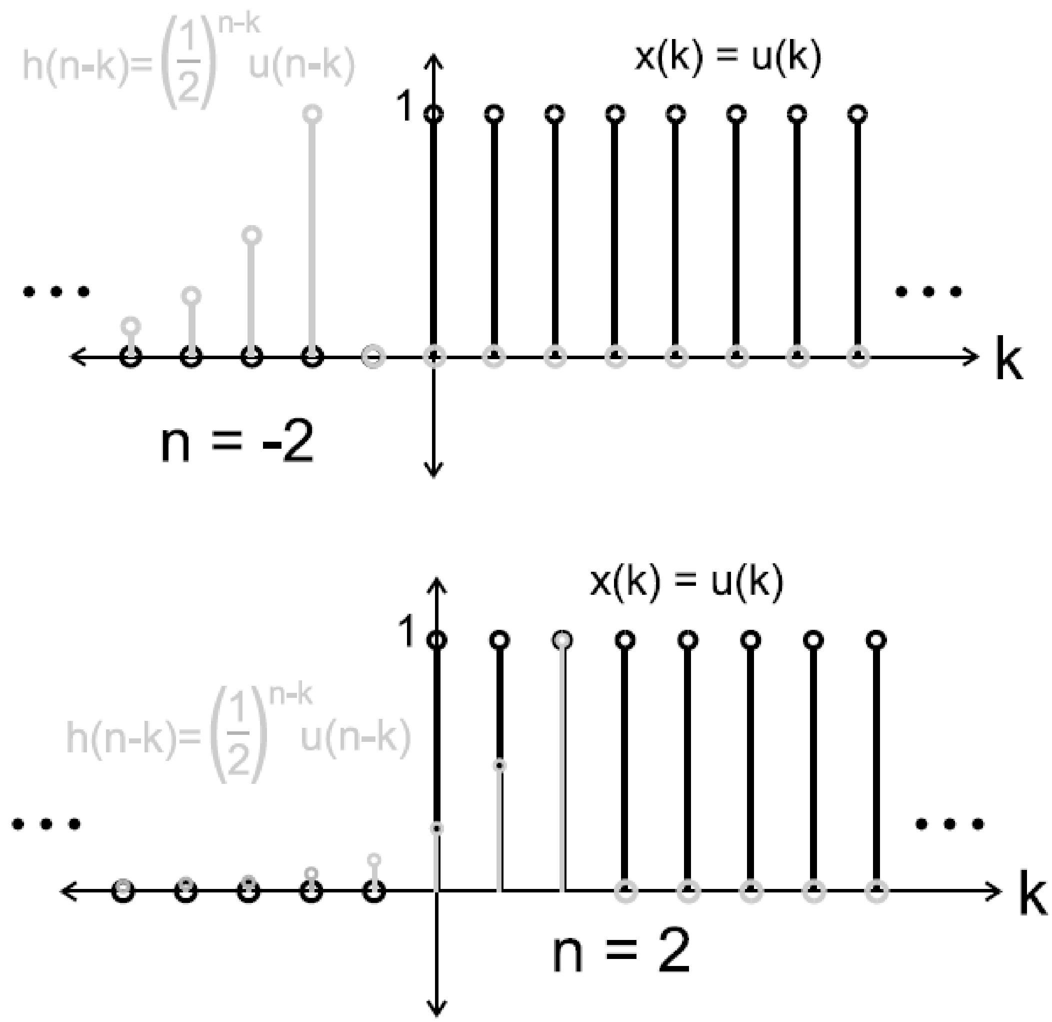
So, overall, we have the following step response.

$$s(n) = \left[2 - \left(\frac{1}{2}\right)^n\right] u(n)$$



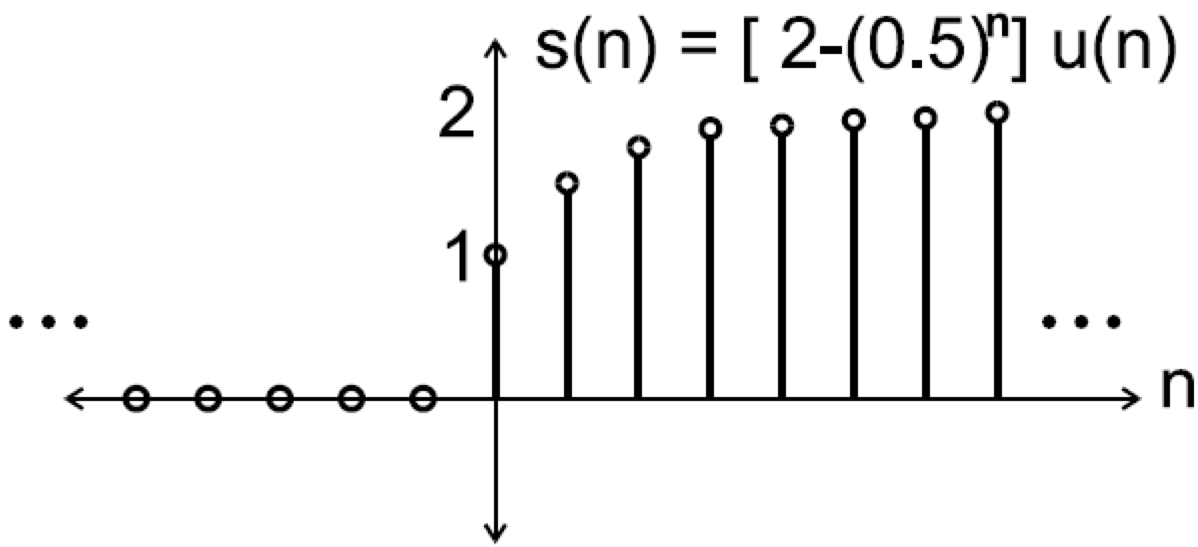
How to Evaluate Convolution?

Evaluate the convolution some graphically



How to Evaluate Convolution?

Evaluate the convolution some graphically



How to Evaluate Convolution?

Example Compute the convolution of $x[n]$ and $h[n]$,

where $x[n] = u[n]$ and $h[n] = \left(\frac{3}{4}\right)^n u[n]$.

Solution.

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

$$= \sum_{k=-\infty}^{\infty} u[k]\left(\tfrac{3}{4}\right)^{n-k} u[n-k] = \sum_{k=0}^{n} \left(\tfrac{3}{4}\right)^{n-k} = \sum_{l=0}^{n} \left(\tfrac{3}{4}\right)^{l}$$

$$= 4[1-\left(\tfrac{3}{4}\right)^{n+1}]$$

Properties of LTI System and the Impulse Response

- Any continuous/discrete-time **LTI system** is completely described by its impulse response through the convolution:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = x[n]*h[n]$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = x(t)*h(t)$$

- This only holds for LTI systems as follows:
- **Example**: The discrete-time impulse response

$$h[n] = \begin{cases} 1 & n = 0,1 \\ 0 & \text{otherwise} \end{cases}$$

Is completely described by the following LTI

$$y[n] = x[n] + x[n-1]$$

However, the following **non-linear systems** also have the same impulse response

$$y[n] = \left(x[n] + x[n-1]\right)^2$$

$$y[n] = \max\left(x[n], x[n-1]\right)$$

Therefore, if the system is **non-linear,** it is not completely characterised by the impulse response

Convolution Properties for LTI System

- Commutative Property
  - $x[n]*h[n]=h[n]*x[n]$
- Distributive Property
  - $x[n]*(h_1[n]+ h_2[n])= x[n]*h_1[n]+x[n]* h_2[n]$
- Associative Property
  - $x[n]*(h_1[n]* h_2[n])= (x[n]*h_1[n])* h_2[n]$

1- Commutative Property

- Convolution is a commutative operator (in both discrete and continuous time), i.e.:

$$x[n] * h[n] = h[n] * x[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

$$x(t) * h(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

- For example, in discrete-time:

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{r=-\infty}^{\infty} x[n-r]h[r] = h[n] * x[n]$$

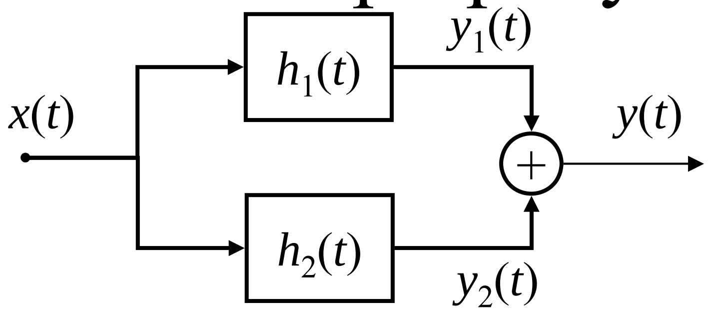
and similar for continuous time.

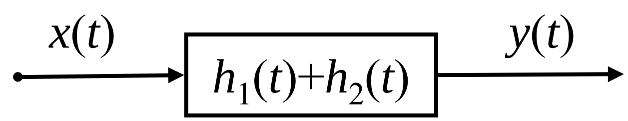
- Therefore, when calculating the response of a system to an input signal $x[n]$, we can imagine the signal being convolved with the unit impulse response $h[n]$, or vice versa, whichever appears the most straightforward.

2-Distributive Property (Parallel Systems)

$$x[n]*(h_1[n]+h_2[n]) = x[n]*h_1[n]+x[n]*h_2[n] = y_1[n]+y_2[n]$$
$$x(t)*(h_1(t)+h_2(t)) = x(t)*h_1(t)+x(t)*h_2(t) = y_1(t)+y_2(t)$$

- Another property of convolution is the distributive property





- The convolved sum of two impulse responses is equivalent to considering the two parallel systems equivalent for one system.

Example: Distributive Property

- Let $y[n]$ denote the convolution of the following two sequences:

$$x[n] = 0.5^n u[n] + 2^n u[-n]$$

$$h[n] = u[n]$$

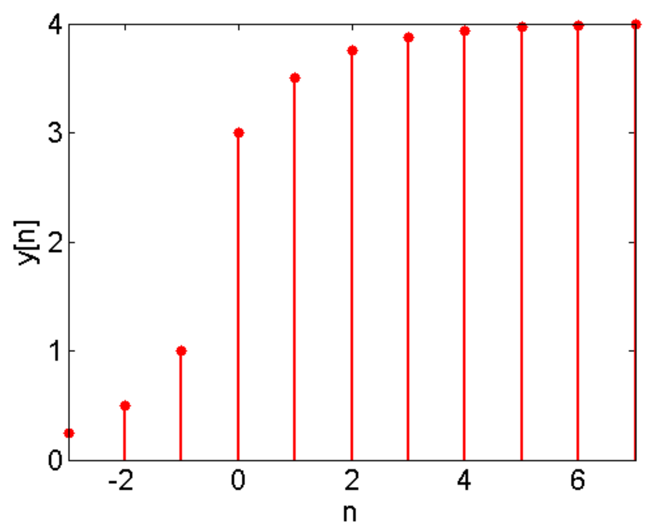
- $x[n]$ is non-zero for all n. We will use the **distributive** property to express $y[n]$ as the sum of two simpler convolution problems. Let $x_1[n] = 0.5^n u[n]$, $x_2[n] = 2^n u[-n]$, it follows that

$$y[n] = (x_1[n] + x_2[n]) * h[n]$$

- and $y[n] = y_1[n]+y_2[n]$, where $y_1[n] = x_1[n]*h[n]$, $y_1[n] = x_1[n]*h[n]$.

$$y_1[n] = \left(\frac{1-0.5^{n+1}}{1-0.5}\right) u[n]$$

$$y_2[n] = \begin{cases} 2^{n+1} & n \le 0 \\ 2 & n \ge 1 \end{cases}$$



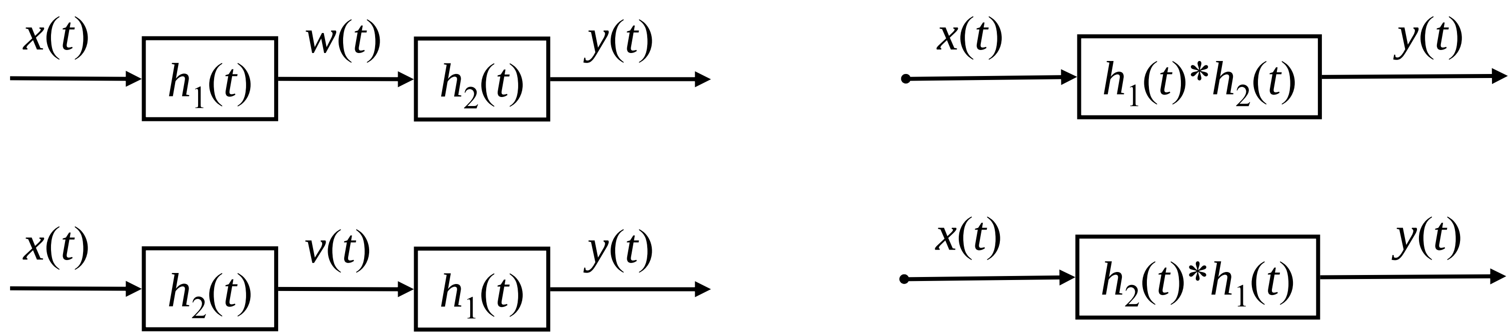
3- Associative Property (Serial Systems)

Another property of (LTI) convolution is that it is associative

$$x[n]*(h_1[n]*h_2[n]) = (x[n]*h_1[n])*h_2[n]$$

$$x(t)*(h_1(t)*h_2(t)) = (x(t)*h_1(t))*h_2(t)$$

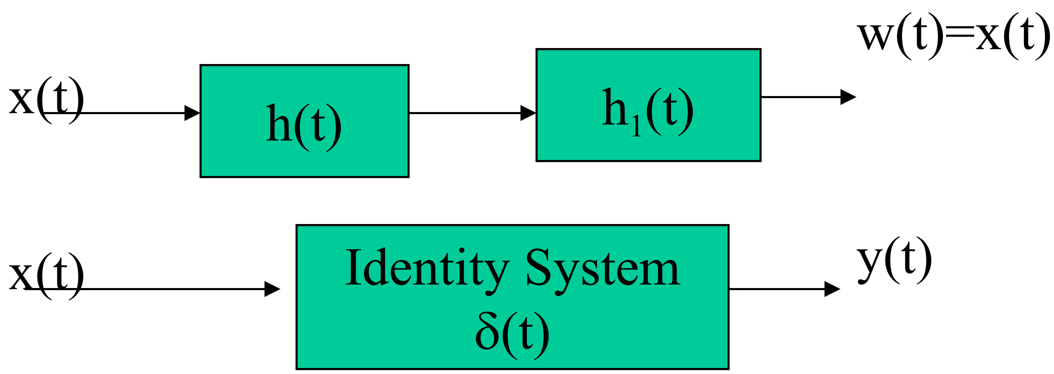
Therefore, the following four systems are all equivalent and $y[n] = x[n]*h_1[n]*h_2[n]$ is unambiguously defined.



This is not true for **non-linear** systems ($y_1[n] = 2x[n]$, $y_2[n] = x^2[n]$)

4- Invertibility

– An LTI system h[n] is invertible if and only if there exist $h_1[n]$ such that



Condition for $h_1[n]$ or $h_1(t)$ to be the inverse of h[n] or h(t) respectively

$$h[n] * h_1[n] = \delta[n]$$

$$\text{or } h(t) * h_1(t) = \delta(t)$$

5- Memory and Memoryless LTI system

Memoryless :

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k],$$

we conclude that

$$h[n-k] = 0, \quad \text{for all } k \neq n,$$

or equivalently,

$$h[n] = 0,, \quad \text{for all } n \neq 0.$$

This implies

$$y[n] = x[n]h[0] = ax[n],$$

where we have set $a = h[0]$.

An LTI system is memoryless if and only if

$h[n] = c\delta[n]$

or

$h(t) = c\delta(t)$

Example: Accumulator System

- Consider a DT LTI system with an impulse response

$$h[n] = u[n]$$

- Using convolution, the response to an arbitrary input $x[n]$:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

As $u[n\text{-}k] = 0$ for $n\text{-}k<0$ and 1 for $n\text{-}k\geq 0$, this becomes

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

- i.e. it acts as a running sum or accumulator. Therefore an inverse system can be expressed as:

$$y[n] = x[n] - x[n-1]$$

- A first difference (differential) operator, which has an impulse response

$$h_1[n] = \delta[n] - \delta[n-1]$$

6- Causality for LTI Systems

- Remember, a causal system only depends on **present and past** values of the input signal. We do not use knowledge about future information.

- For a discrete LTI system, convolution tells us that

  $h[n] = 0 \quad \text{for } n<0$

An LTI system is causal if and only if

$h[n] = 0 \text{ for } n < 0$

Causality for LTI Systems

For causal LTI systems, the convolution takes a new form.

$$y[n] = \sum_{k=-\infty}^{n} x[k]h[n-k] = \sum_{k=0}^{\infty} h[k]x[n-k]$$

or

$$y(t) = \int_{-\infty}^{t} x(\tau)h(t-\tau)d\tau = \int_{0}^{\infty} h(\tau)x(t-\tau)d\tau$$

- as $y[n]$ must not depend on $x[k]$ for $k>n$, as the impulse response must be zero before that.

$$x[n]*h[n] = \sum_{k=-\infty}^{n} x[k]h[n-k]$$

$$x(t)*h(t) = \int_{-\infty}^{t} x(\tau)h(t-\tau)d\tau$$

- Both the integrator and its inverse in the previous example are causal
- This is strongly related to inverse systems as we generally require our inverse system to be causal. If it is not causal, it is difficult to manufacture!

6- LTI System Stability

- Remember: A system is stable if every bounded input produces a bounded output

Therefore, consider a bounded input signal

$$|x[n]| < B \quad \text{for all } n$$

Applying convolution and taking the absolute value:

$$|y[n]| = \left| \sum_{k=-\infty}^{\infty} h[k]x[n-k] \right|$$

- Using the triangle inequality (magnitude of a sum of a set of numbers is no larger than the sum of the magnitude of the numbers):

$$|y[n]| \leq \sum_{k=-\infty}^{\infty} |h[k]||x[n-k]|$$

$$\leq B \sum_{k=-\infty}^{\infty} |h[k]|$$

- Therefore a DT **LTI system is stable** if and only if its impulse response is absolutely summable, ie

$$\sum_{k=-\infty}^{\infty} |h[k]| < \infty$$

Continuous-time system $\int_{-\infty}^{\infty} |h(\tau)| d\tau < \infty$

Example: System Stability

- Are the DT and CT pure time shift systems stable?

$$h[n] = \delta[n - n_0]$$
$$h(t) = \delta(t - t_0)$$

$$\sum_{k=-\infty}^{\infty} \left|h[k]\right| = \sum_{k=-\infty}^{\infty} \left|\delta[k - n_0]\right| = 1 < \infty$$

$$\int_{-\infty}^{\infty} \left|h(\tau)\right| d\tau = \int_{-\infty}^{\infty} \left|\delta(\tau - t_0)\right| d\tau = 1 < \infty$$

Therefore, both the CT and DT systems are **stable**: all finite input signals produce a finite output signal

- Are the discrete and continuous-time integrator systems stable?

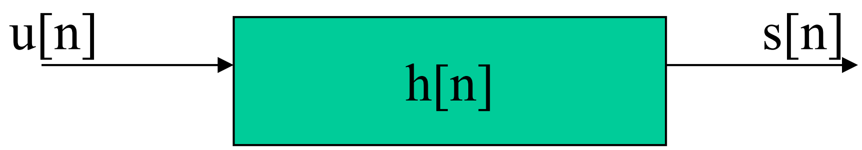
$$h[n] = u[n - n_0]$$
$$h(t) = u(t - t_0)$$

$$\sum_{k=-\infty}^{\infty} \left|h[k]\right| = \sum_{k=-\infty}^{\infty} \left|u[k - n_0]\right| = \sum_{k=n_0}^{\infty} \left|u[k]\right| = \infty$$

$$\int_{-\infty}^{\infty} \left|h(\tau)\right| d\tau = \int_{-\infty}^{\infty} \left|u(\tau - t_0)\right| d\tau = \int_{t_0}^{\infty} \left|u(\tau)\right| d\tau = \infty$$

Therefore, both the CT and DT systems are **unstable**: at least one finite input causes an infinite output signal

8- Unit Step Response of LTI System

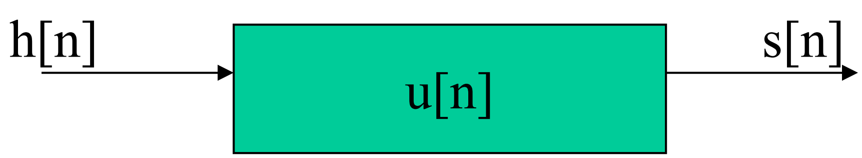


The step response of a discrete-time LTI system is the convolution of the unit step with the impulse response:-

s[n]=u[n]*h[n].

Via commutative property of convolution, s[n]=h[n]*u[n].

That means s[n] is the response to the input h[n] of a discrete-time LTI system with unit impulse response u[n].



Using the convolution sum:-

$$s[n] = \sum_{k=-\infty}^{\infty} h[k]\, u[n-k],$$

Since u[n - k] is 0 for n - k < 0, i.e. k > n and 1 for n - k ≥ 0, i.e. k ≤ n.

$$\therefore \mathrm{s[n]} = \sum_{k=-\infty}^{n} h[k],$$

That is, the step response of the discrete - time LTI system is the running sum of its impulse response.

$$\mathrm{s[n\text{ - }1]} = \sum_{k=-\infty}^{n-1} h[k],$$

$$\therefore s[n] - s[n-1] = \sum_{k=-\infty}^{n} h[k] - \sum_{k=-\infty}^{n-1} h[k],$$

$$s[n] - s[n-1] = \sum_{k=-\infty}^{n-1} h[k] + h[n] - \sum_{k=-\infty}^{n-1} h[k],$$

$$\therefore \mathrm{h[n] = s[n]\text{ - }s[n\text{ - }1]},$$

From here h[n] can be recovered from s[n], the impulse response of a discrete - time LTI system is the first difference of its step response.

Home Work

Chapter 2 problems: 2.7, 2.8, 2.11, 2.14, 2.18, 2.24, 2.29, 2.31, 2.38, 2.39, 2.53C

Book: Signals & Systems. by Alan V. Oppenheim, Alan S. Willsky with S. Hamid Nawab. Prentice-Hall, Second Edition, 1997.