VE216 RC5

Chapter 7 & 8

Some of contents are from professor's lecture slides.

Sampling

Under certain conditions, a continuous-time signal can be completely represented by and recoverable from knowledge of its values, or samples, at points equally spaced in time.

The concept of sampling suggests an extremely attractive and widely employed method for using discrete-time system technology to implement continuous-time systems and process continuous-time signals: we exploit sampling to convert a continuous-time signal to a discrete-time signal, process the discrete-time signal using a discrete-time system, and then convert back to continuous time.

The Sampling Theorem

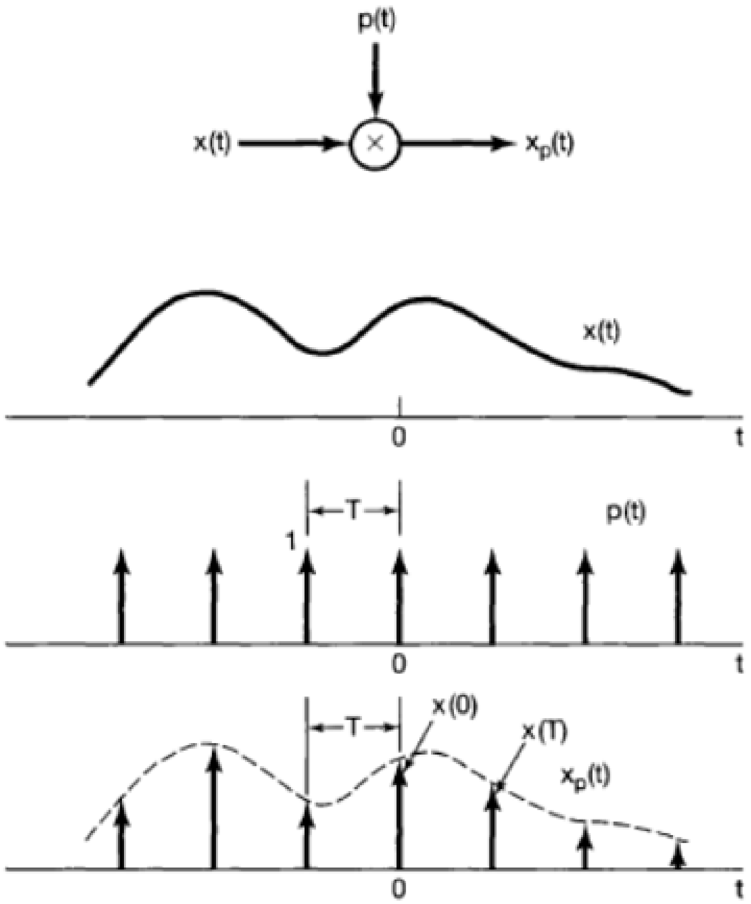


Figure 7.2 Impulse-train sampling.

The Sampling Theorem

We may mathematically represent the sampled signal as the product of the original continuous-time signal and an impulse train.

$$x_p(t) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t-nT) = \sum_{n=-\infty}^{\infty} x(t)\delta(t-nT)$$

$$= x(t)\sum_{n=-\infty}^{\infty} \delta(t-nT) = x(t)\,p(t)$$

This representation is commonly termed *impulse sampling*.

The Sampling Theorem

$$x_p(t) = x(t)p(t) \leftrightarrow X_p(j\omega) = \frac{1}{2\pi} X(j\omega) * P(j\omega)$$

$$X_p(j\omega) = \frac{1}{2\pi} X(j\omega) * \left(\frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s) \right)$$

$$= \frac{1}{T} \sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s))$$

$X_p(j\omega)$ is a periodic function of ω consisting of a superposition of shifted replicas of $X(j\omega)$, scaled by $1/T$.

The Sampling Theorem

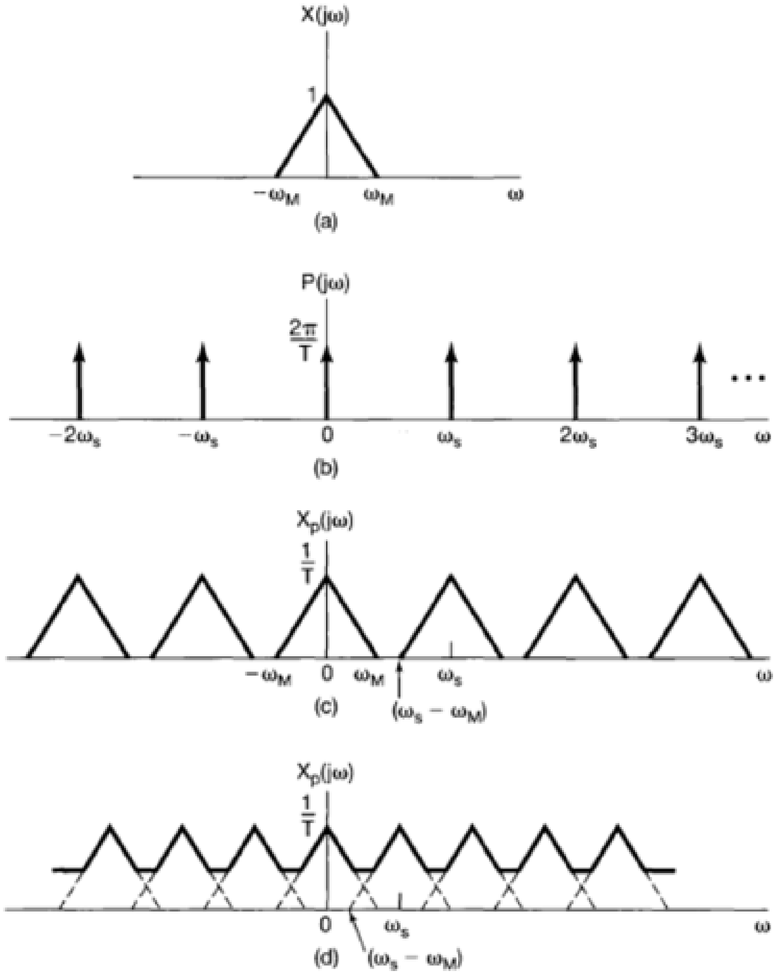


Figure 7.3 Effect in the frequency domain of sampling in the time domain: (a) spectrum of original signal; (b) spectrum of sampling function; (c) spectrum of sampled signal with $\omega_s > 2\omega_M$; (d) spectrum of sampled signal with $\omega_s < 2\omega_M$.

The Sampling Theorem

Key points for impulse sampling:

(1) The FT of the sampled signal is given by an infinite sum of shifted versions of the original signal's FT.

(2) The shifted versions are offset by integer multiples of ω_s, the sampling frequency.

(3) The shifted versions of $X(j\omega)$ may overlap with each other if ω_s is not large enough compared with the frequency extent, or bandwidth (W), of $X(j\omega)$. Overlap in the shifted replicas of $X(j\omega)$ is termed aliasing.

(4) Aliasing is prevented by choosing the sampling interval T so that $\omega_s > 2\omega_M$. It is known as the *sampling theorem.*

The Sampling Theorem

Sampling Theorem Let $x(t)$ be a bandlimited signal with $X(j\omega) = 0$ for $|\omega| > W$. Then $x(t)$ is uniquely determined by its samples $x(nT)$, $n = 0, \pm 1, \pm 2, \cdots$, if

$$\omega_s > 2\omega_M$$

where $\omega_s = \dfrac{2\pi}{T}$

Example Determine the sampling frequency corresponding to the signal $x(t) = 1 + \cos(2000\pi t) + \sin(3000\pi t)$.

Example Let $x(t)$ be a band-limited signal with $X(j\omega) = 0$ for $|\omega| > \omega_M$. Then the sampling frequency for $y(t) = \frac{d}{dt}x(t) + 2x(t-1)$ should be at least ____________ .

Answer $\omega_s = 2\omega_M = 6000\pi$

Solution $Y(j\omega) = j\omega X(j\omega) + 2e^{-j\omega} X(j\omega)$

$$= \left(j\omega + 2e^{-j\omega} \right) X(j\omega).$$

It is obvious that $Y(j\omega)$ is also bandlimited to ω_M. So $\omega_{s(\min)} = 2\omega_M$

Reconstruction of A Signal from Its Samples Using Interpolation

Ideal Reconstruction The goal of reconstruction is to apply some operation on $X_p(j\omega)$ that converts it back to $X(j\omega)$. Any such operation must eliminate the replicas of $X(j\omega)$ that centered at $k\omega_s$. This is acomplished by multiplying $X_p(j\omega)$ by

$$H(j\omega) = \begin{cases} T, & |\omega| \le \omega_c = \omega_s/2 \\ 0, & |\omega| > \omega_c = \omega_s/2 \end{cases}$$

Ideal Reconstruction

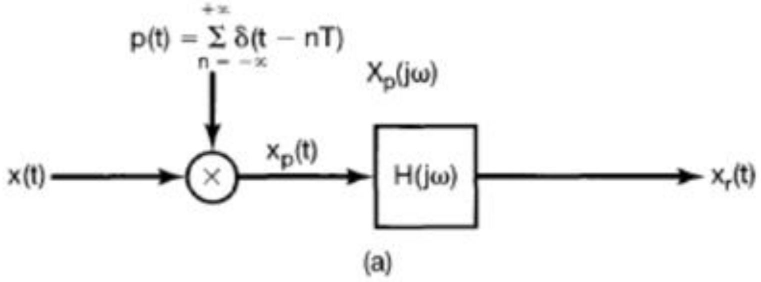
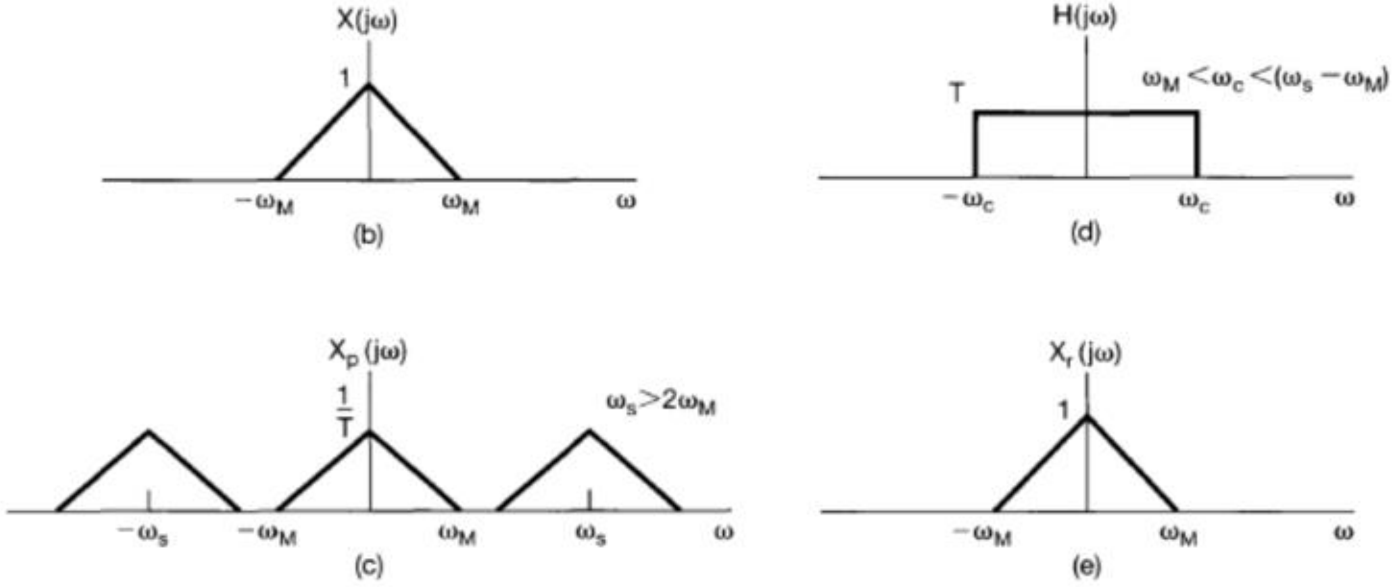


Figure 7.4 Exact recovery of a continuous-time signal from its samples using an ideal lowpass filter: (a) system for sampling and reconstruction; (b) representative spectrum for $x(t)$; (c) corresponding spectrum for $x_p(t)$; (d) ideal lowpass filter to recover $X(j\omega)$ from $X_p(j\omega)$; (e) spectrum of $x_r(t)$.



Ideal Reconstruction

$$X_r(j\omega) = H(j\omega)X_p(j\omega)$$

$$x_r(t) = h(t) * x_p(t)$$

where $h(t) = T\dfrac{\sin(\omega_c t)}{\pi t} = \dfrac{2\pi}{\omega_s}T\dfrac{\sin(\omega_c t)}{\pi t}$

$$\overset{\text{Let } \omega_c = \omega_s/2}{=} \frac{2\pi}{\omega_s}T\frac{\sin(\omega_s t/2)}{\pi t} = \mathrm{Sa}(\omega_s t/2)$$

Note: $\mathrm{Sa}(x) = \dfrac{\sin x}{x}$

$$x_r(t) = h(t) * x_p(t) = h(t) * \sum_{n=-\infty}^{\infty} x(nT)\delta(t-nT)$$

$$= \sum_{n=-\infty}^{\infty} x(nT)h(t-nT) = \sum_{n=-\infty}^{\infty} x(nT)\mathrm{Sa}\left(\omega_s(t-nT)/2\right)$$

(Ideal bandlimited interpolation or Shannon's sampling theorem)

Ideal Reconstruction

Zero - Order Hold

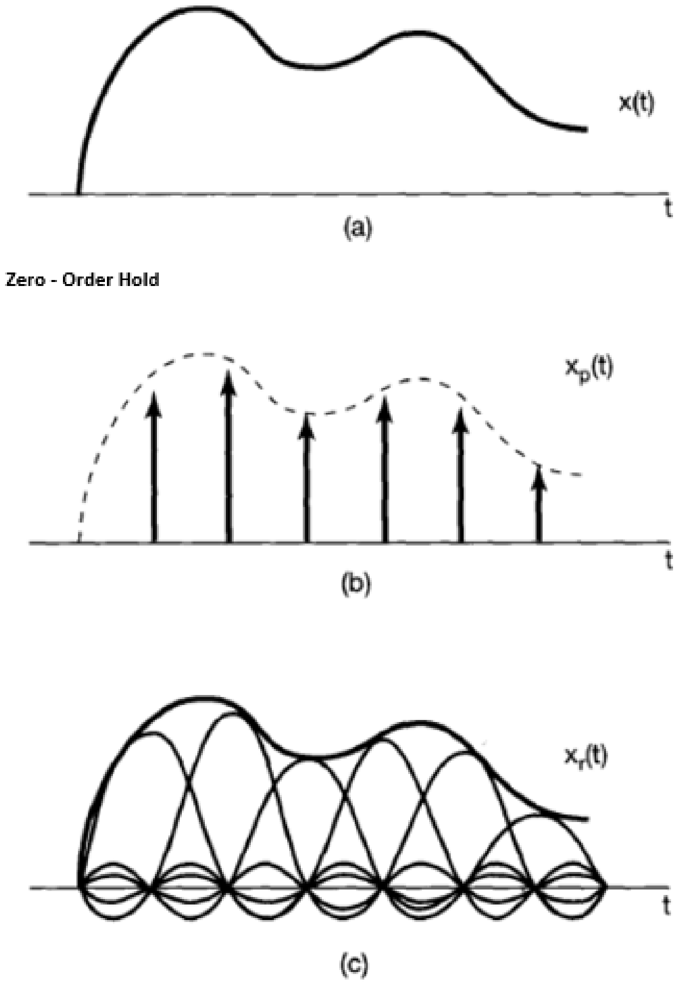


Figure 7.10 Ideal band-limited interpolation using the sinc function: (a) band-limited signal $x(t)$; (b) impulse train of samples of $x(t)$; (c) ideal band-limited interpolation in which the impulse train is replaced by a superposition of sinc functions [eq. (7.11)].

Zero - Order Hold

Reconstruction via Zero - Order Hold In practice, we obviously don't sample with impulses or implement ideal lowpass filters.
One practical example is the zero-order hold. It is a very rough approximation, although in some cases it is sufficient. For example, if additional lowpass filter is (naturally) applied in a given application.

Zero - Order Hold

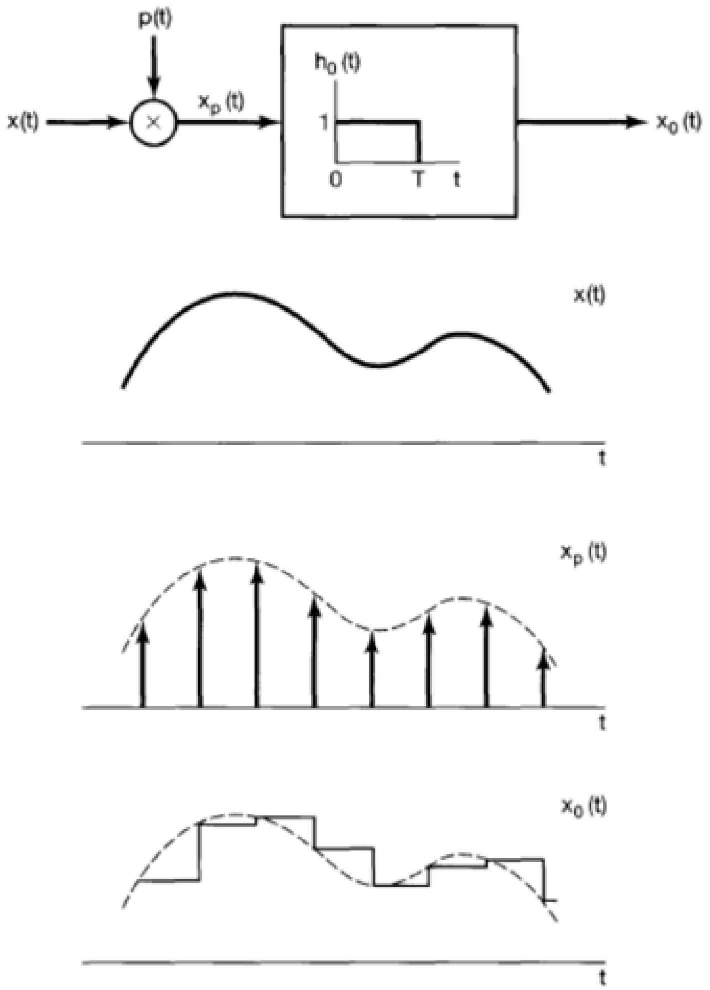


Figure 7.6 Zero-order hold as impulse-train sampling followed by an LTI system with a rectangular impulse response.

Zero - Order Hold

$$x_o(t) = \sum_{n=-\infty}^{\infty} x(nT) h_o(t - nT)$$

where

$$h_o(t) = \begin{cases} 1, & 0 \le t \le T \\ 0, & \text{otherwise} \end{cases} \leftrightarrow H_o(j\omega) = T\,\mathrm{Sa}(\omega T/2) e^{-j\omega T/2}$$

$$H_r(j\omega) = \begin{cases} e^{j\omega T/2} / [T\,\mathrm{Sa}(\omega T/2)], & |\omega| \le \omega_s / 2 \\ 0, & \text{otherwise} \end{cases}$$

Zero - Order Hold

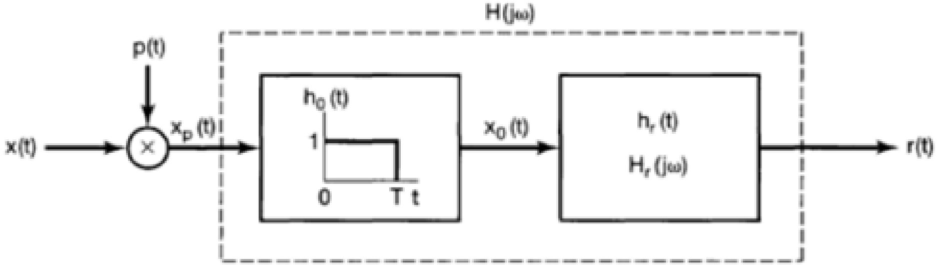


Figure 7.7 Cascade of the representation of a zero-order hold (Figure 7.6) with a reconstruction filter.

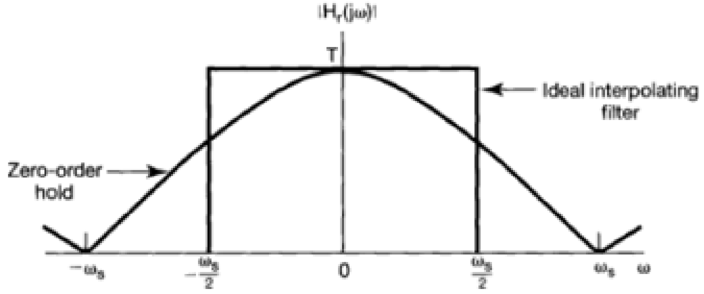


Figure 7.11 Transfer function for the zero-order hold and for the ideal interpolating filter.

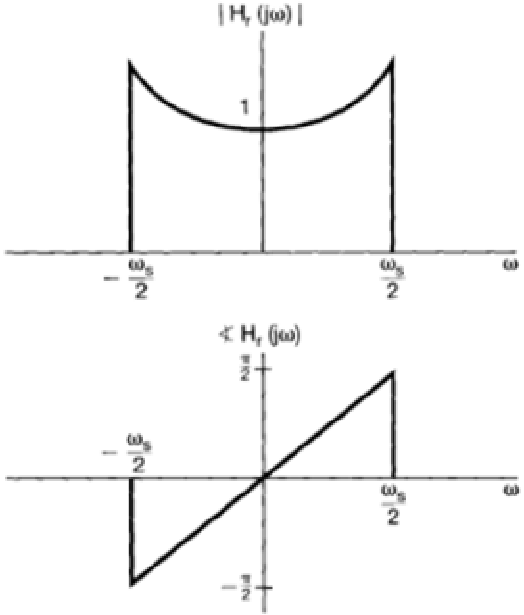
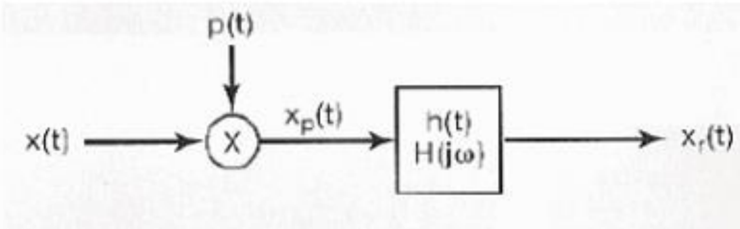
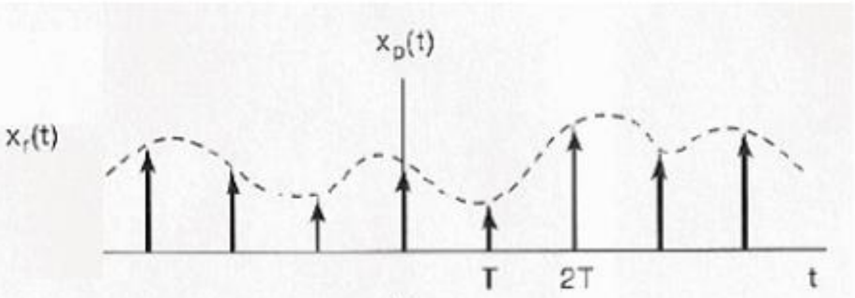


Figure 7.8 Magnitude and phase for the reconstruction filter for a zero-order hold.

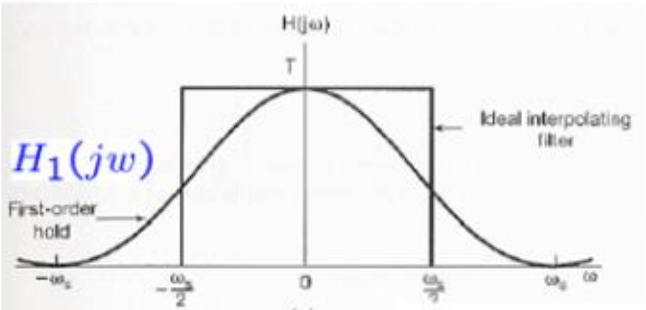
Higher - Order Hold



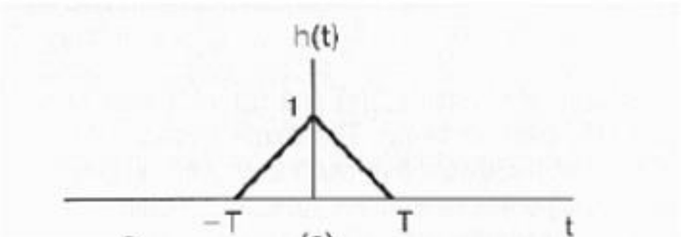
(a)



(b)

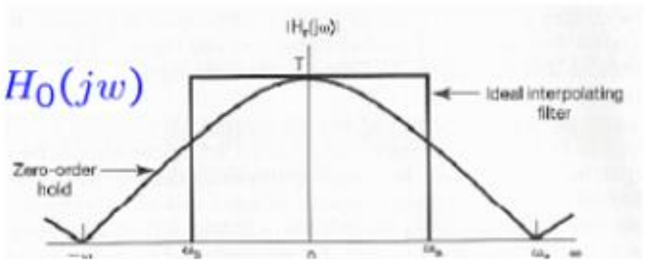


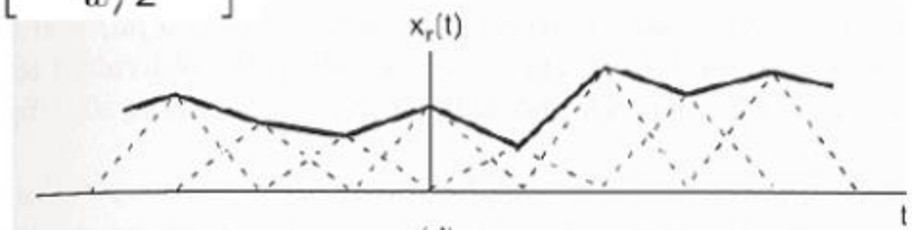
(e)



(c)

$$H_1(jw) = \frac{1}{T}\left[\frac{\sin(wT/2)}{w/2}\right]^2$$





(d)

The Effect of Undersampling: Aliasing

Some insight into the relationship between $x(t)$ and $x_r(t)$ when $\omega_s < 2\omega_M$ is provided by considering in more detail the comparatively simple case of a sinusoidal signal. Thus, let

$$x(t) = \cos \omega_0 t$$

with Fourier transform

$$X(j\omega) = \pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$$

The Effect of Undersampling: Aliasing

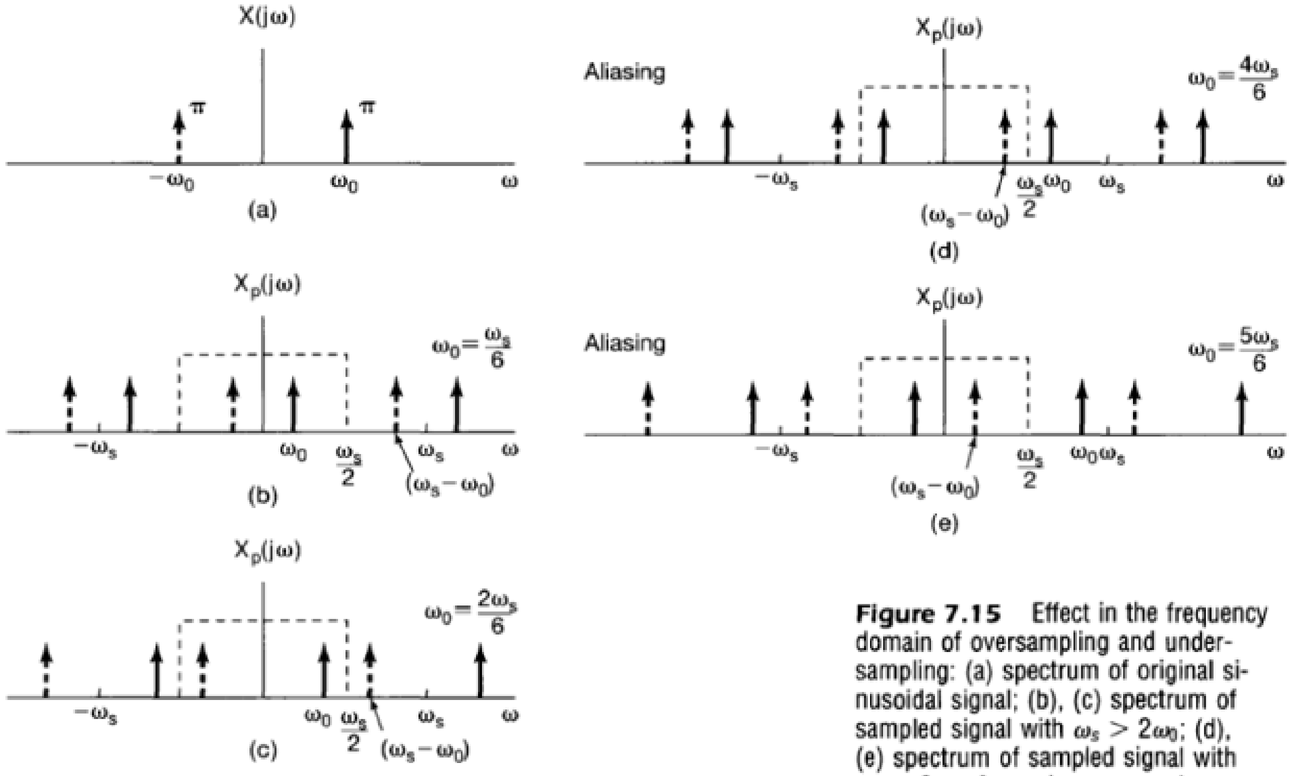


Figure 7.15 Effect in the frequency domain of oversampling and undersampling: (a) spectrum of original sinusoidal signal; (b), (c) spectrum of sampled signal with $\omega_s > 2\omega_0$; (d), (e) spectrum of sampled signal with $\omega_s < 2\omega_0$. As we increase ω_0 in moving from (b) through (d), the impulses drawn with solid lines move to the right, while the impulses drawn with dashed lines move to the left. In (d) and (e), these impulses have moved sufficiently that there is a change in the ones falling within the passband of the ideal lowpass filter.

The Effect of Undersampling: Aliasing

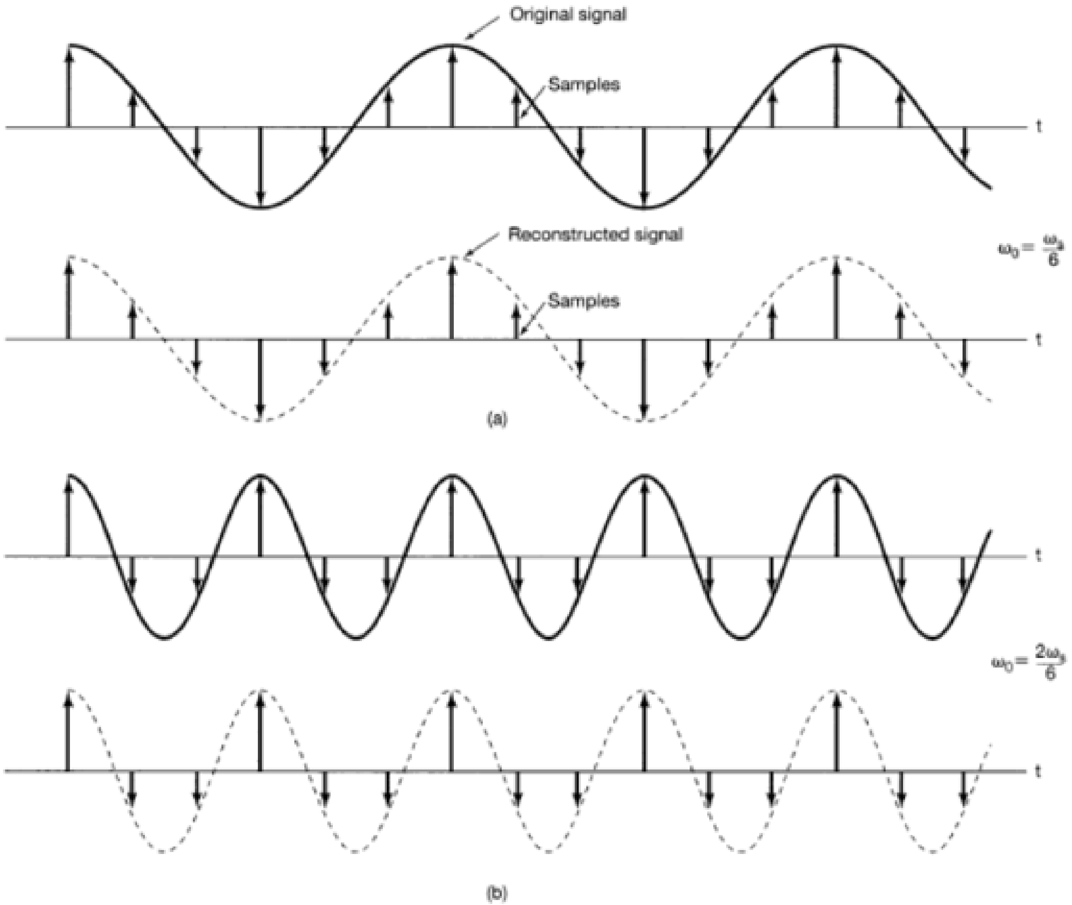


Figure 7.16 Effect of aliasing on a sinusoidal signal. For each of four values of ω_0, the original sinusoidal signal (solid curve), its samples, and the reconstructed signal (dashed curve) are illustrated: (a) $\omega_0 = \omega_s/6$; (b) $\omega_0 = 2\omega_s/6$; (c) $\omega_0 = 4\omega_s/6$; (d) $\omega_0 = 5\omega_s/6$. In (a) and (b) no aliasing occurs, whereas in (c) and (d) there is aliasing.

The Effect of Undersampling: Aliasing

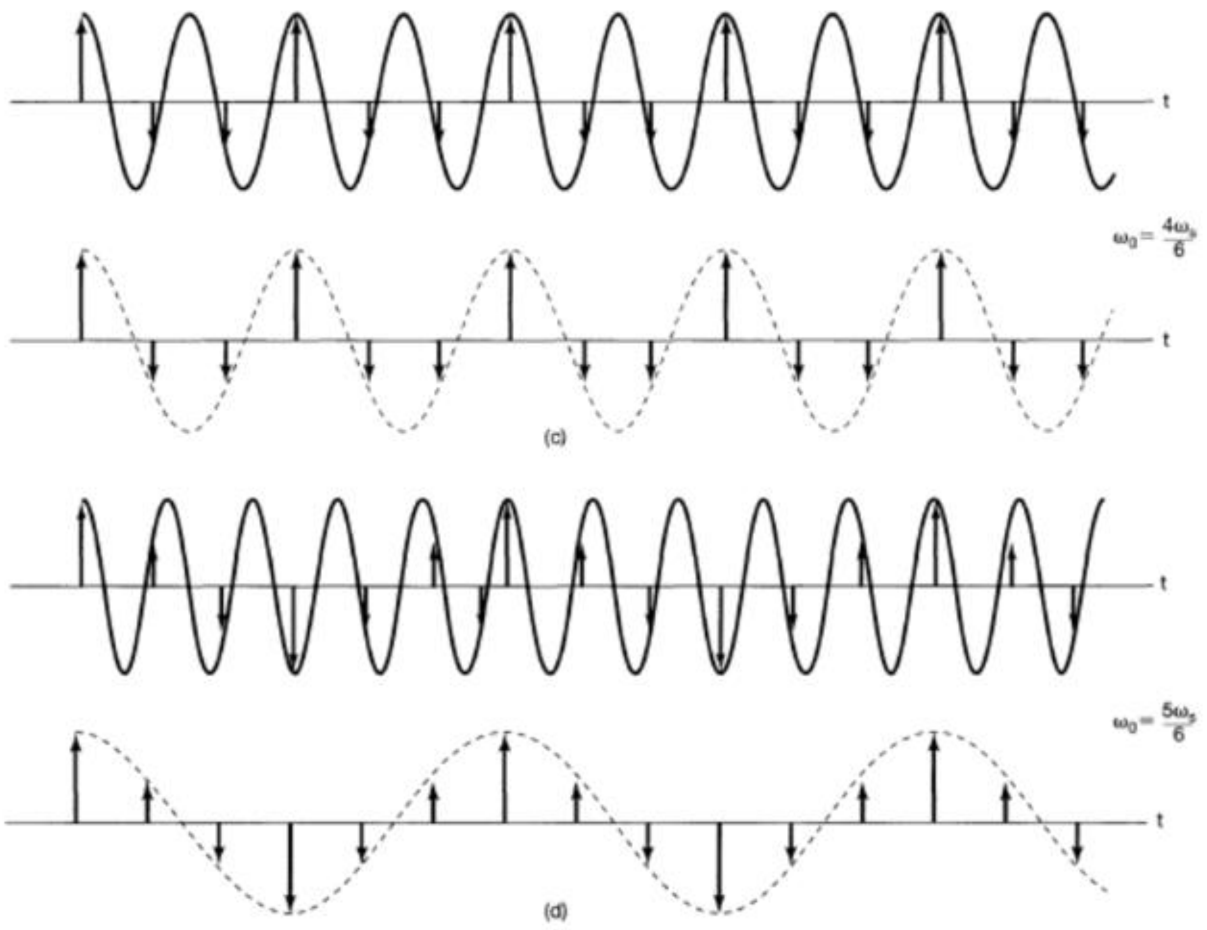
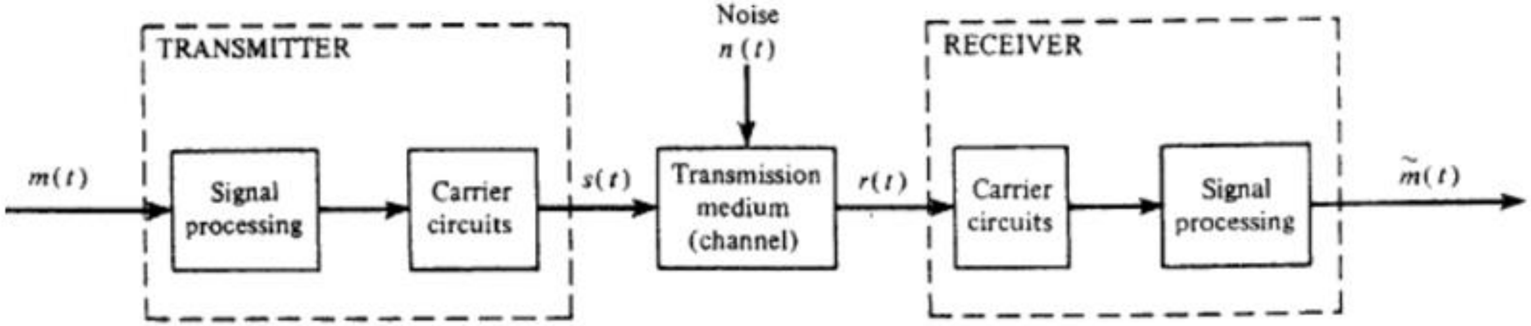


Figure 7.16 *Continued*

Communication Systems

Communication systems play a key role in our modem world in transmitting information between people, systems, and computers.



Communication system

Communication Systems

In general terms, in all communication systems the information-bearing signal at the source is first processed by a transmitter to change it into a form suitable for transmission over the communication channel. At the receiver, the signal is then recovered through appropriate processing.

The general process of embedding an information-bearing signal into a second signal is typically referred to as modulation. Extracting the information-bearing signal is known as demodulation.

There are a wide variety of modulation methods used in practice.

We may identify two main classes of modulation: continuous-wave and pulse modulation.

Modulation

Continuous-wave modulation

- Amplitude modulation
  - Amplitude modulation (AM)
  - Double sideband-suppressed carrier modulation (DSB-SC)
  - Single sideband modulation (SSB)
- Angle modulation
  - Frequency modulation (FM)
  - Phase modulation (PM)

Pulse modulation

- Pulse-amplitude modulation (PAM)
- Pulse-duration modulation or pulse-width modulation (PWM)
- Pulse-position modulation (PPM)

Find examples in your text book & professor's lecture slides.

DSB - SC

DSB - SC In DSB-SC, a sinusoidal signal $c(t)$ has its amplitude multiplied (modulated) by the information-bearing signal $x(t)$. The signal $x(t)$ is typically referred to as the *modulating* signal and the signal $c(t)$ as the *carrier* signal. The *modulated* signal $y(t)$ is then the product of these two signals:

$$y(t) = x(t)c(t)$$

DSB - SC

Suppose

$$c(t) = \cos(\omega_c t + \theta_c) \qquad (8.2)$$

The frequency ω_c is referred to as the *carrier frequency*. For convenience, let us choose $\theta_c = 0$, so that the modulated signal is

$$y(t) = x(t)\cos\omega_c t$$

DSB - SC

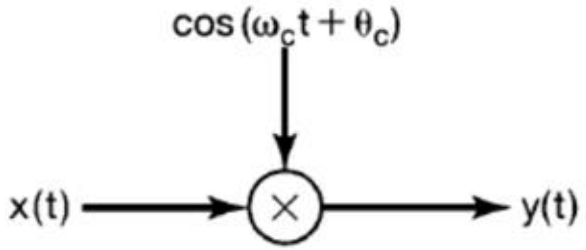


Figure 8.3 Amplitude modulation with a sinusoidal carrier.

The spectrum of the carrier signal is

$$C(j\omega) = \pi[\delta(\omega - \omega_c) + \delta(\omega + \omega_c)] \qquad (8.9)$$

and thus,

$$Y(j\omega) = \frac{1}{2\pi} X(j\omega) * C(j\omega)$$

$$= \frac{1}{2}[X(j(\omega - \omega_c)) + X(j(\omega + \omega_c))] \qquad (8.10)$$

DSB - SC

With $X(j\omega)$ as depicted in Fig. 8.4(a), the spectrum of $y(t)$ is that shown in Fig. 8.4(c). Note that there is now a replication of the spectrum of the original signal, centered around both $+\omega_c$ and $-\omega_c$. As a consequence, $x(t)$ is recoverable from $y(t)$ if $\omega_c > \omega_M$.

DSB - SC

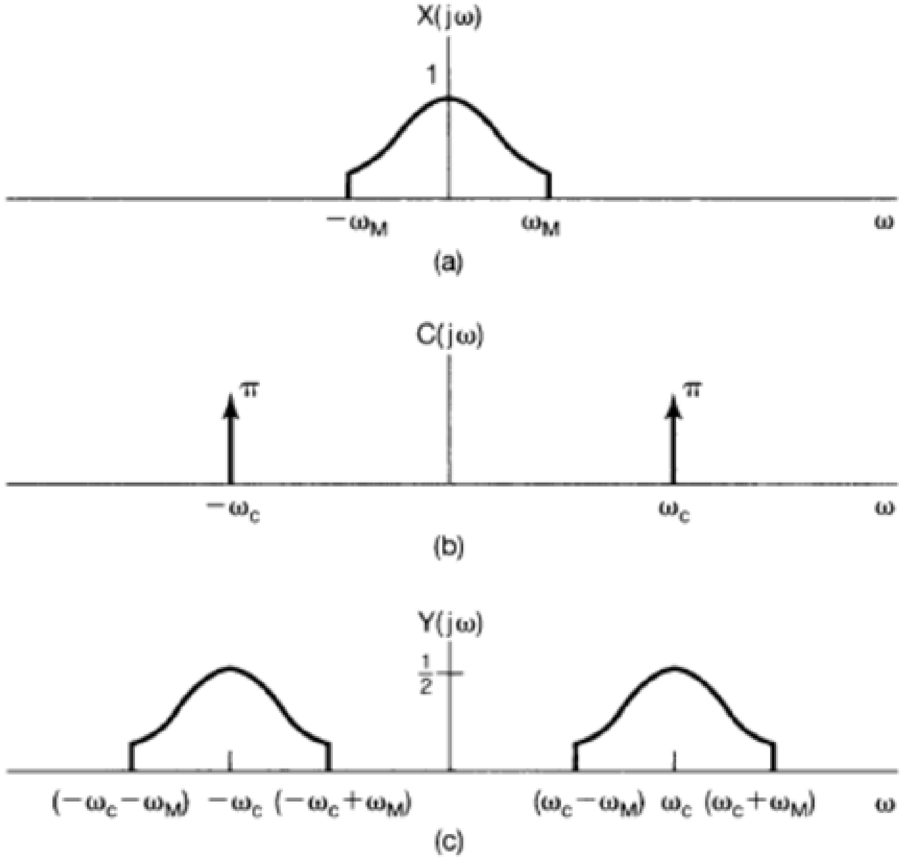


Figure 8.4 Effect in the frequency domain of amplitude modulation with a sinusoidal carrier: (a) spectrum of modulating signal $x(t)$; (b) spectrum of carrier $c(t) = \cos \omega_c t$; (c) spectrum of amplitude-modulated signal.

DSB - SC

At the receiver, $x(t)$ can be recovered by modulating $y(t)$ with the same sinusoidal carrier, i.e.,

$$w(t) = y(t)\cos\omega_c t \tag{8.12}$$

and applying a lowpass filter to the result, as illustrated in Fig. 8.8. This process is referred to as *synchronous demodulation,* in which the demodulator is synchronized with the modulator.

DSB - SC

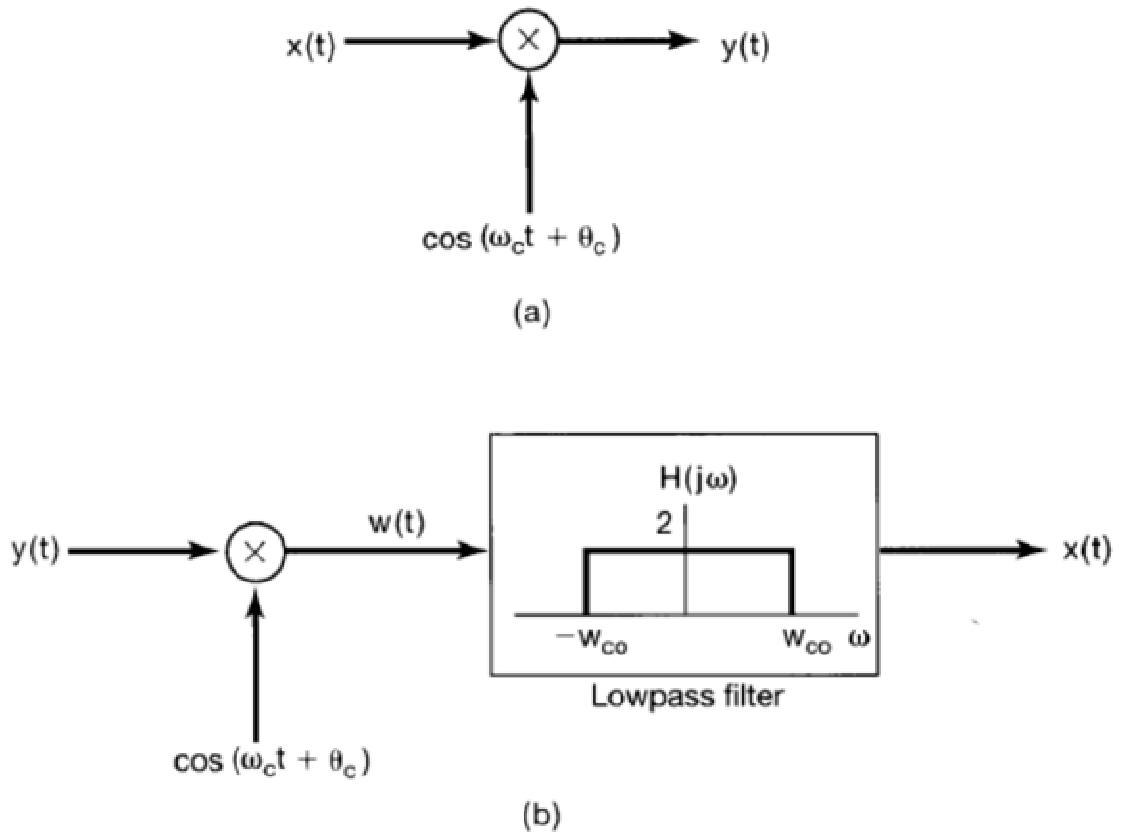


Figure 8.8 Amplitude modulation and demodulation with a sinusoidal carrier: (a) modulation system; (b) demodulation system. The lowpass filter cutoff frequency w_{co} is greater than ω_M and less than $2\omega_c - \omega_M$.

DSB - SC

$$w(t) = y(t)\cos\omega_c t = x(t)\cos^2\omega_c t$$

$$= x(t)\frac{1+\cos 2\omega_c t}{2} = \frac{1}{2}x(t) + \frac{1}{2}x(t)\cos 2\omega_c t$$

$$W(j\omega) = \frac{1}{2}X(j\omega) + \frac{1}{4}X(j(\omega - 2\omega_c)) + \frac{1}{4}X(j(\omega + 2\omega_c))$$

$$H(j\omega) = \begin{cases} 2, & |\omega| \le \omega_{co} \\ 0, & |\omega| > \omega_{co} \end{cases}, \quad \omega_M < \omega_{co} < 2\omega_c - \omega_M$$

$$H(j\omega)W(j\omega) = X(j\omega)$$

Figure 8.6 shows the spectra of $y(t)$ and $w(t)$.

DSB - SC

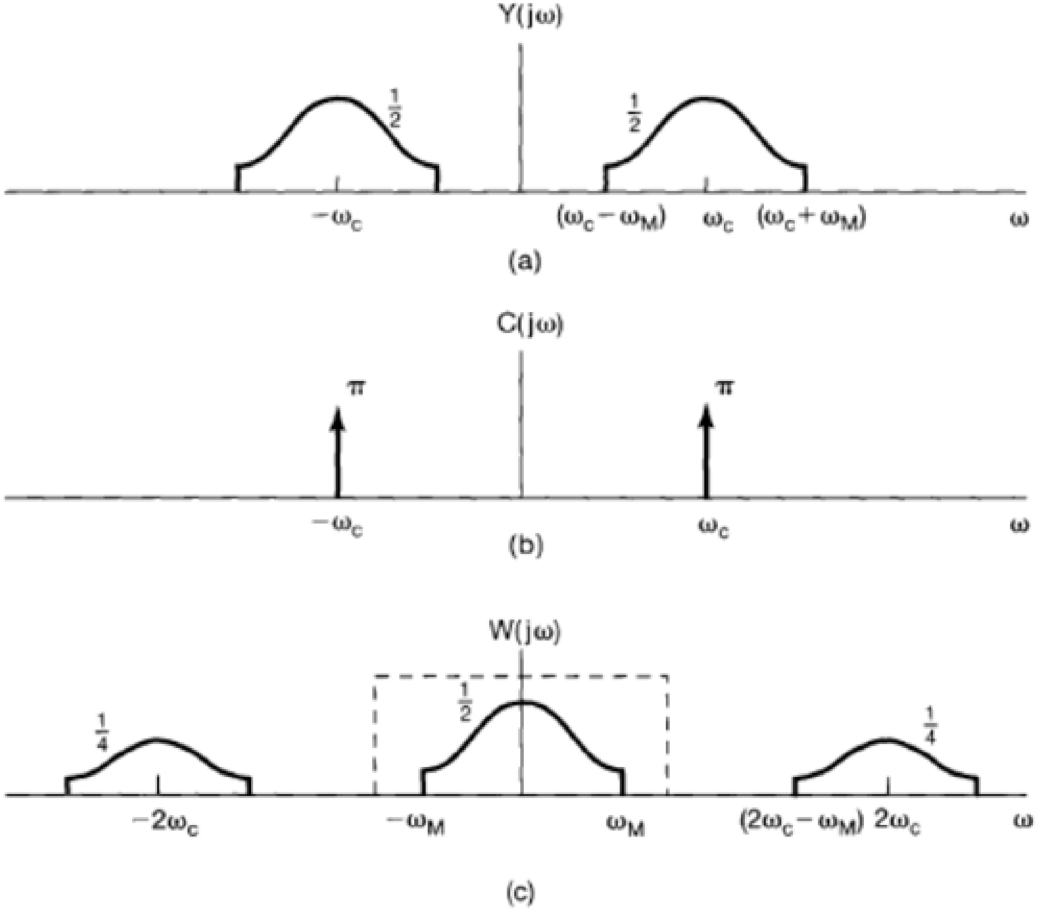


Figure 8.6 Demodulation of an amplitude-modulated signal with a sinusoidal carrier: (a) spectrum of modulated signal; (b) spectrum of carrier signal; (c) spectrum of modulated signal multiplied by the carrier. The dashed line indicates the frequency response of a lowpass filter used to extract the demodulated signal.

AM

AM In standard AM, the carrier signal $c(t)$ has its amplitude multiplied (modulated) by the quantity

$$x(t) + A$$

where

$$|x(t)| < A$$

Figure 8.12 shows the modulator.

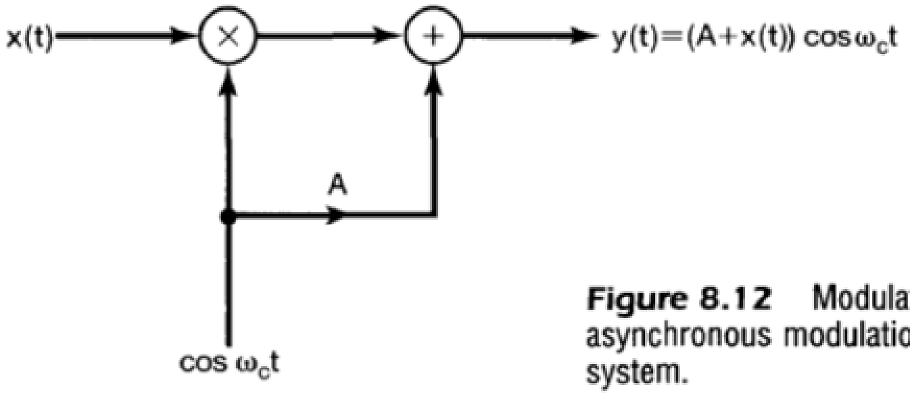


Figure 8.12 Modulator for an asynchronous modulation-demodulation system.

AM

The modulated signal $y(t)$ has the general form illustrated in Fig. 8.10.

$$y(t) = [x(t) + A]\cos(\omega_c t) = x(t)\cos\omega_c t + A\cos\omega_c t$$

$$Y(j\omega) = \frac{1}{2}[X(j(\omega - \omega_c)) + X(j(\omega + \omega_c))] +$$

$$\pi A[\delta(\omega - \omega_c) + \delta(\omega + \omega_c)]$$

In Fig. 8.14, we show a comparison of the spectra associated with the DSB-SC signal and AM signal.

AM

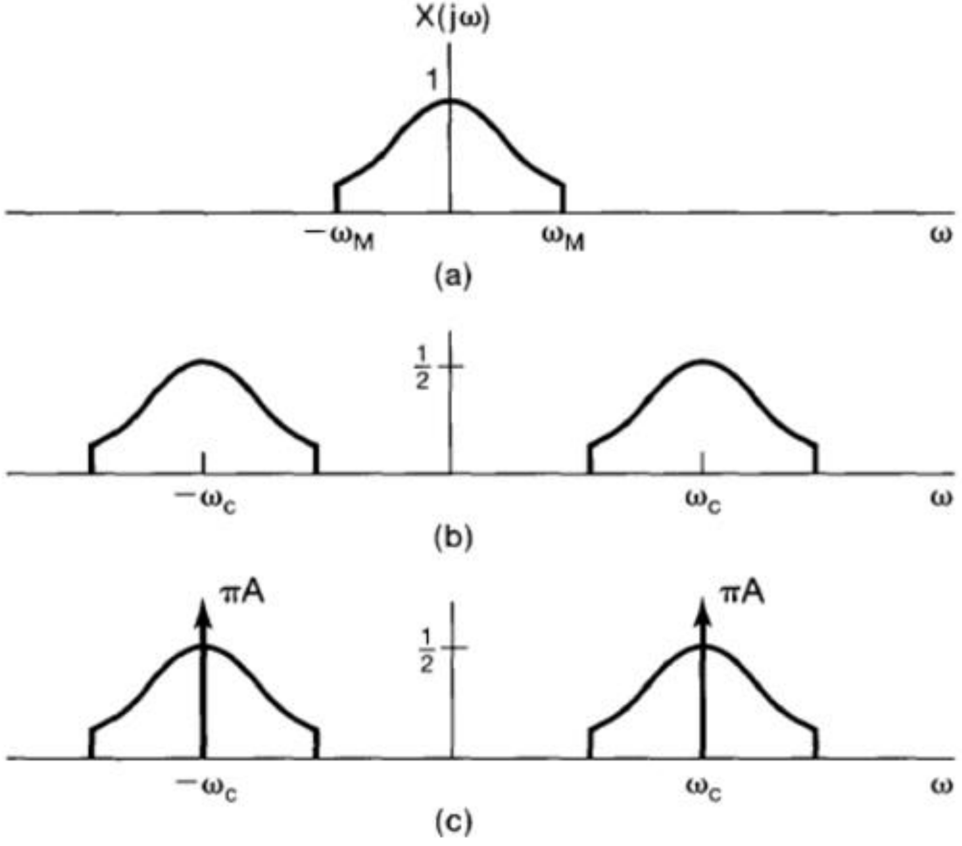
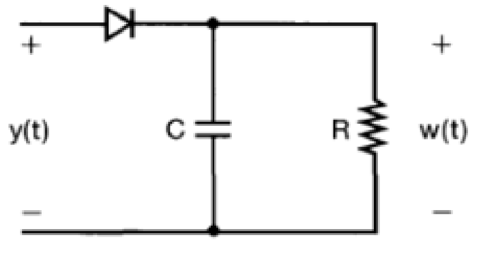


Figure 8.14 Comparison of spectra for synchronous and asynchronous sinusoidal amplitude modulation systems: (a) spectrum of modulating signal; (b) spectrum of $x(t) \cos \omega_c t$ representing modulated signal in a synchronous system; (c) spectrum of $[x(t) + A] \cos \omega_c t$ representing modulated signal in an asynchronous system.

AM

Standard AM signal can be demodulated by using asynchronous demodulation (or envelope detection). One example envelope detection is shown in Fig. 8.11.



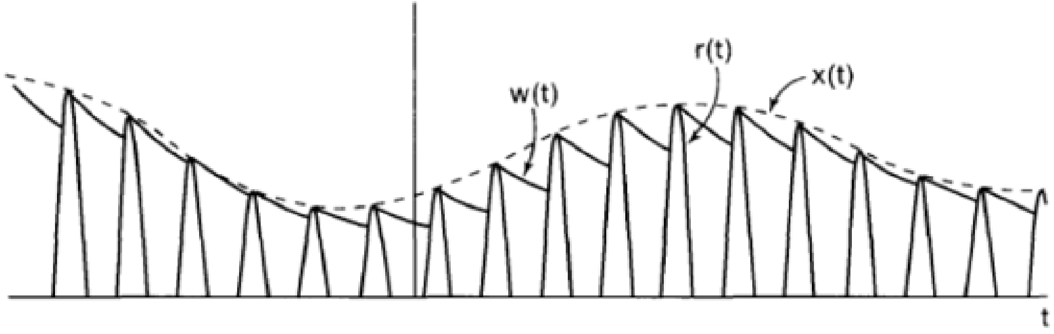


Figure 8.11 Demodulation by envelope detection: (a) circuit for envelope detection using half-wave rectification; (b) waveforms associated with the envelope detector in (a): $r(t)$ is the half-wave rectified signal, $x(t)$ is the true envelope, and $w(t)$ is the envelope obtained from the circuit in (a). The relationship between $x(t)$ and $w(t)$ has been exaggerated in (b) for purposes of illustration. In a practical asynchronous demodulation system, $w(t)$ would typically be a much closer approximation to $x(t)$ than depicted here.

AM

Suppose we receive the radio signal

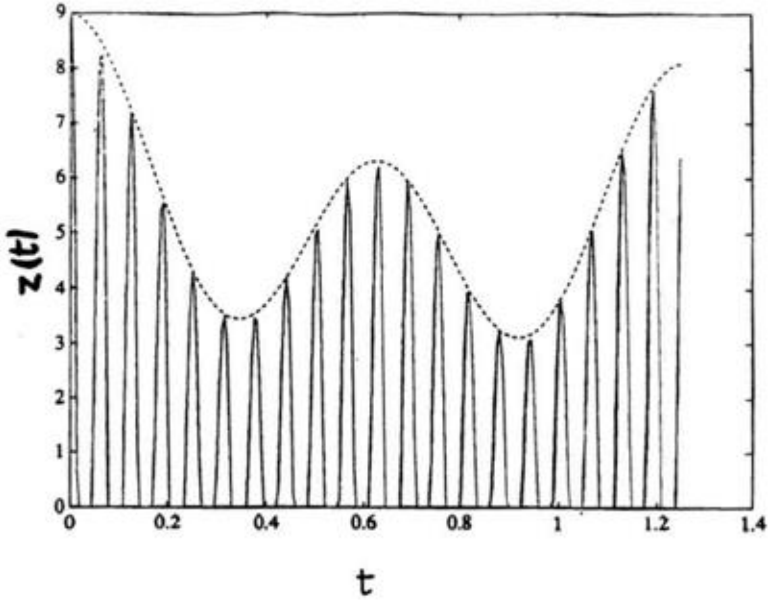
$y(t) = [x(t) + A]\cos(\omega_c t)$

How can we recover $x(t)$? Lets "rectify" $y(t)$

by using a "half-wave rectifier" with the

following input-output relation

$$z(t) = \begin{cases} y(t), & y(t) > 0 \\ 0, & y(t) < 0 \end{cases}$$



AM

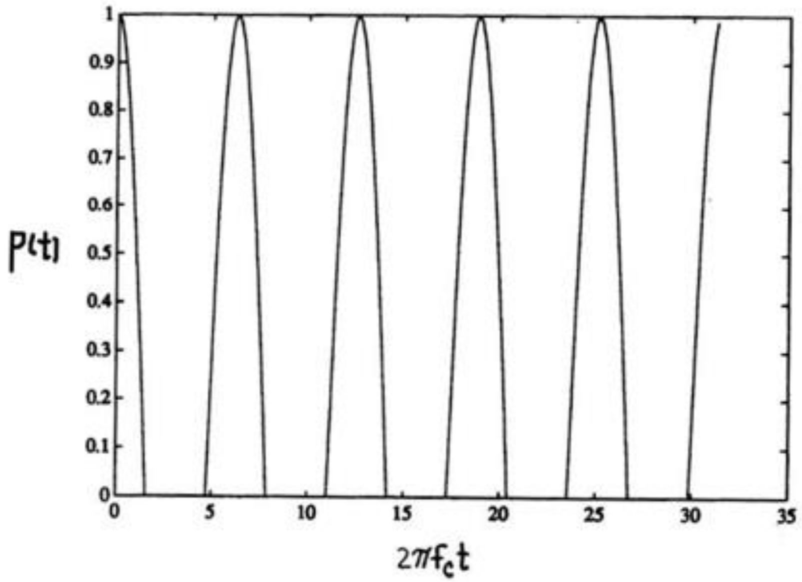
Since $[x(t)+A]>0$, we can write

$$z(t)=\begin{cases}[x(t)+A]\cos(\omega_c t), & \cos(\omega_c t)>0\\ 0, & \cos(\omega_c t)<0\end{cases}$$

$$=[x(t)+A]p(t)$$

where

$$p(t)=\begin{cases}\cos(\omega_c t), & \cos(\omega_c t)>0\\ 0, & \cos(\omega_c t)<0\end{cases}$$



AM

Note that $p(t)$ is a periodic function with $T = 2\pi / \omega_c$. Thus $p(t)$ has an FS representation

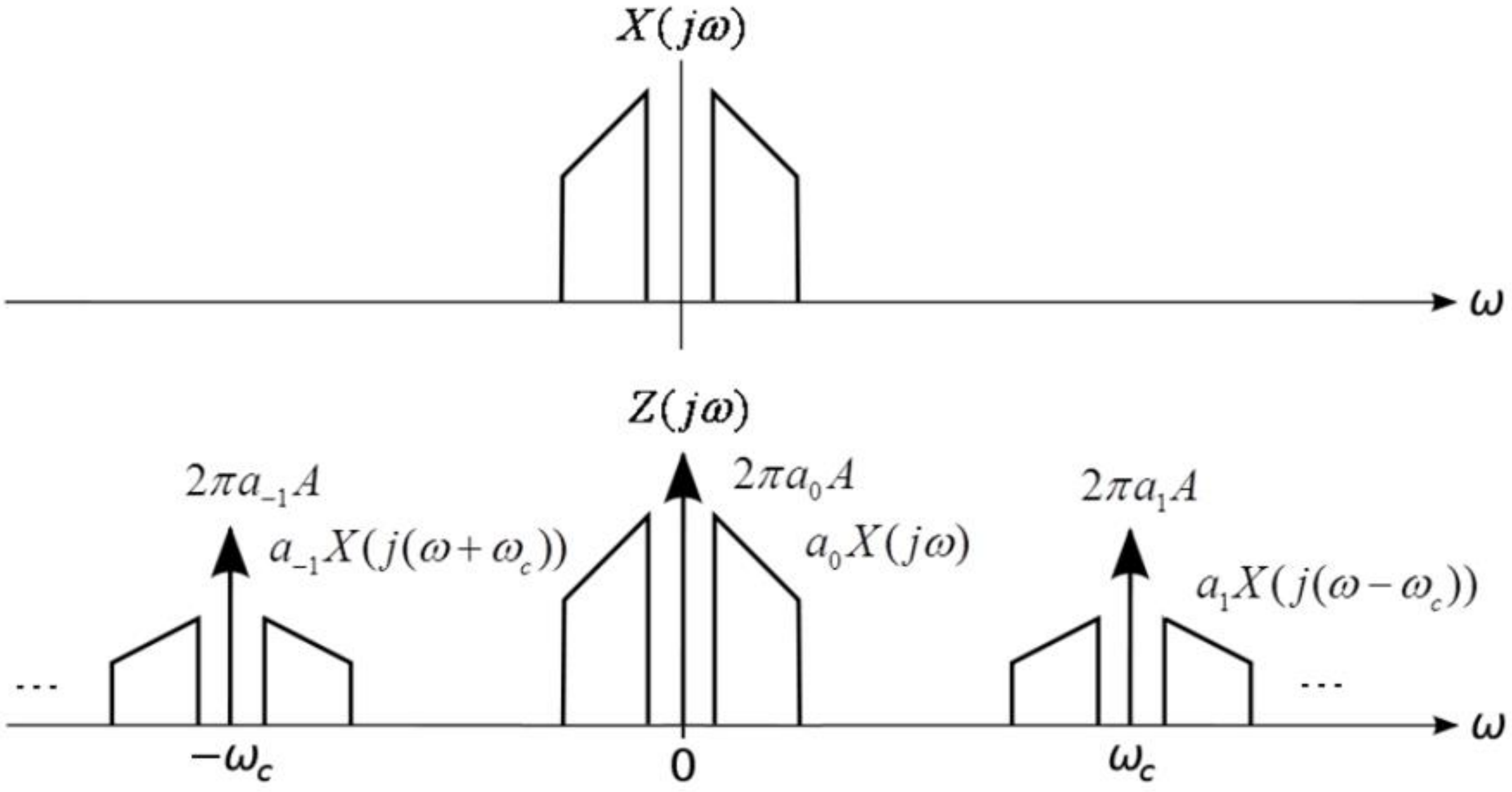
$$p(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_c t}$$

$$z(t) = [x(t) + A]p(t) = \sum_{k=-\infty}^{\infty} a_k [x(t) + A] e^{jk\omega_c t}$$

$$= \sum_{k=-\infty}^{\infty} a_k A e^{jk\omega_c t} + \sum_{k=-\infty}^{\infty} a_k x(t) e^{jk\omega_c t}$$

$$Z(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k A \delta(\omega - k\omega_c) + \sum_{k=-\infty}^{\infty} a_k X(j(\omega - k\omega_c))$$

AM



$X(j\omega)$
ω
$Z(j\omega)$
$2\pi a_{-1}A$
$a_{-1}X(j(\omega+\omega_c))$
$2\pi a_0 A$
$a_0 X(j\omega)$
$2\pi a_1 A$
$a_1 X(j(\omega-\omega_c))$
$\cdots$
$\cdots$
$-\omega_c$
0
ω_c
ω

AM

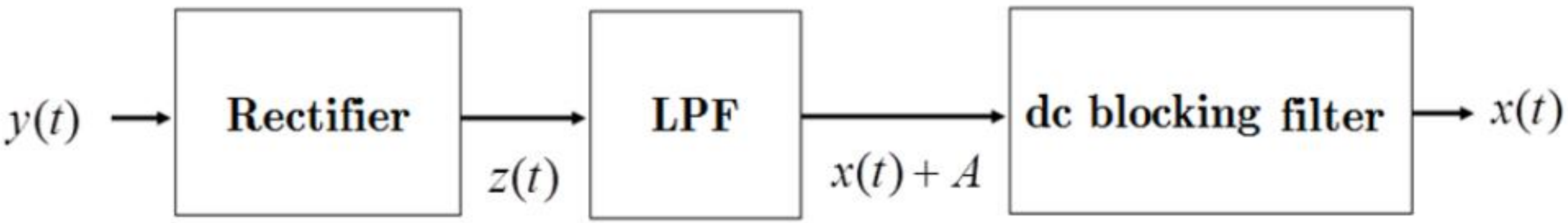
It follows from the above figure that we can recover $\left[x(t)+A\right]$ by lowpass filtering $z(t)$:

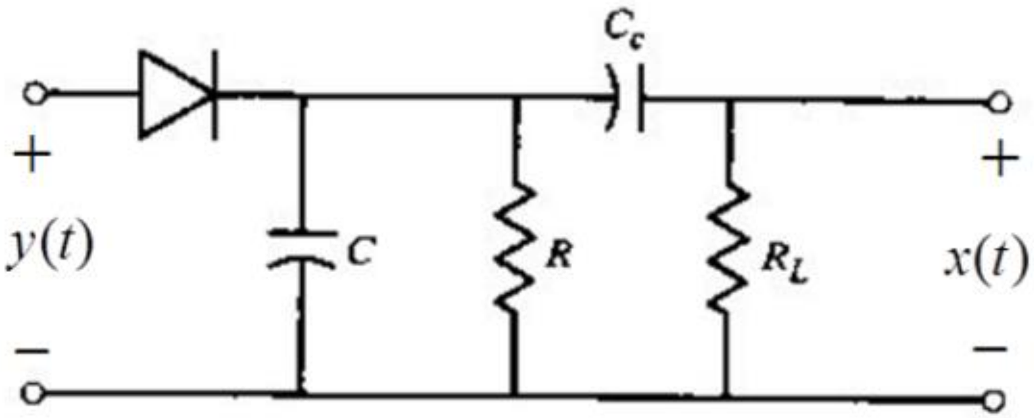
$$H(j\omega)=\begin{cases} a_0^{-1}, & |\omega|<\omega_{co} \\ 0, & |\omega|>\omega_{co} \end{cases}$$

where $\omega_M \ll \omega_{co} \ll \omega_c$.

And to recover $x(t)$ from $\left[x(t)+A\right]$, we need another filter which blocks the dc component.

AM

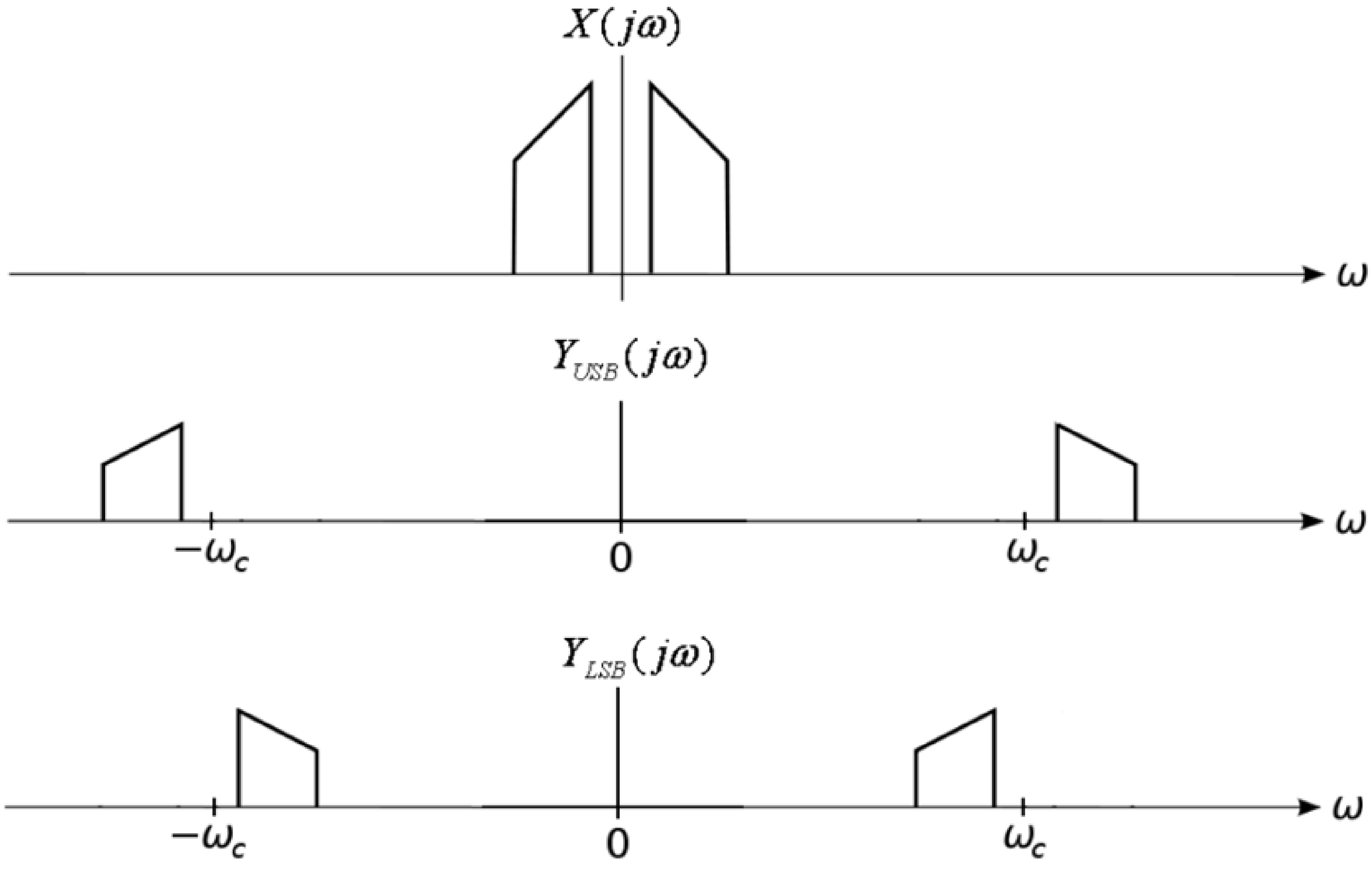




SSB

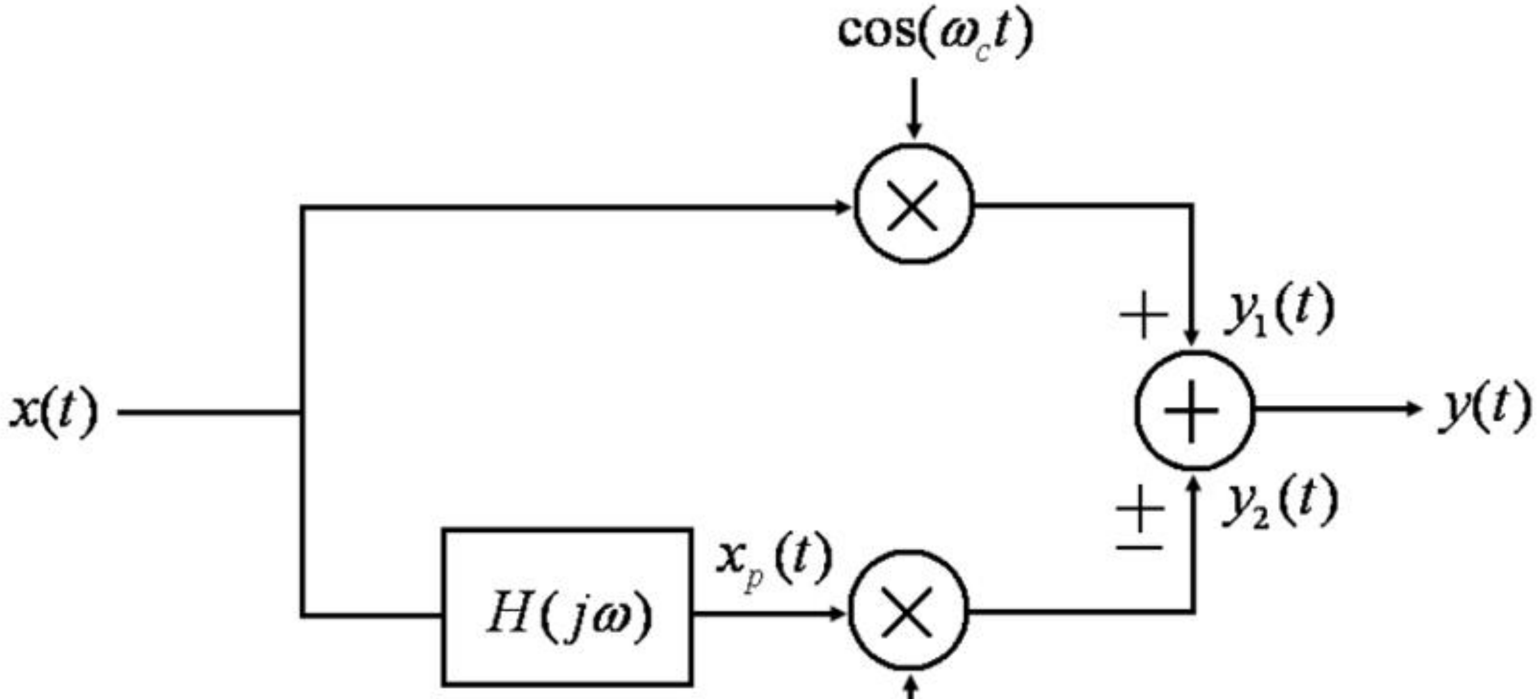
Single - Sideband AM In double-sideband modulation, there is a basic redundancy in the modulated signal. Using SSB modulation, we can remove the redundancy.

SSB


$X(j\omega)$
ω
$Y_{USB}(j\omega)$
ω
$-\omega_c$
0
ω_c
$Y_{LSB}(j\omega)$
ω
$-\omega_c$
0
ω_c

SSB

Generation of SSB signal.



$$H(j\omega)=\begin{cases}-j, & \omega>0\\ j, & \omega<0\end{cases}$$

SSB

$$\operatorname{sgn}(t) = \begin{cases} 1, & t > 0 \\ -1 & ,t < 0 \end{cases} \leftrightarrow \frac{2}{j\omega}$$

$$\frac{2}{jt} \leftrightarrow 2\pi \operatorname{sgn}(-\omega) = -2\pi \operatorname{sgn}(\omega)$$

$$H(j\omega) = -j \operatorname{sgn}(\omega) \Rightarrow \operatorname{sgn}(\omega) = \frac{H(j\omega)}{-j}$$

$$\frac{2}{jt} \leftrightarrow \frac{2\pi H(j\omega)}{j} \Rightarrow \frac{1}{\pi t} \leftrightarrow H(j\omega)$$

$$x_p(t) = \frac{1}{\pi t} * x(t) \leftrightarrow X_p(j\omega) = H(j\omega) X(j\omega)$$

$$h(t) = \frac{1}{\pi t} \leftrightarrow H(j\omega) = -j \operatorname{sgn}(\omega): \text{ Hilbert transformer}$$

SSB

$$X_p(j\omega) = -j\,\mathrm{sgn}(\omega)X(j\omega)$$

$$= -j\left[u(\omega) - u(-\omega)\right]X(j\omega)$$

$$= -jX(j\omega)u(\omega) + jX(j\omega)u(-\omega)$$

$$X_+(j\omega) \triangleq X(j\omega)u(\omega),\ X_-(j\omega) \triangleq X(j\omega)u(-\omega)$$

$$X(j\omega) = X_+(j\omega) + X_-(j\omega)$$

$$X_p(j\omega) = -jX_+(j\omega) + jX_-(j\omega)$$

SSB

$$y_1(t) = x(t)\cos(\omega_c t) \leftrightarrow$$

$$Y_1(j\omega) = \frac{1}{2}[X(j(\omega-\omega_c)) + X(j(\omega+\omega_c))]$$

$$= \frac{1}{2}[X_+(j(\omega-\omega_c)) + X_-(j(\omega-\omega_c))$$

$$+ X_+(j(\omega+\omega_c)) + X_-(j(\omega+\omega_c))]$$

SSB

$$y_2(t) = x_p(t)\sin(\omega_c t) \leftrightarrow$$

$$Y_2(j\omega) = \frac{1}{2j}[X_p(j(\omega-\omega_c)) - X_p(j(\omega+\omega_c))]$$

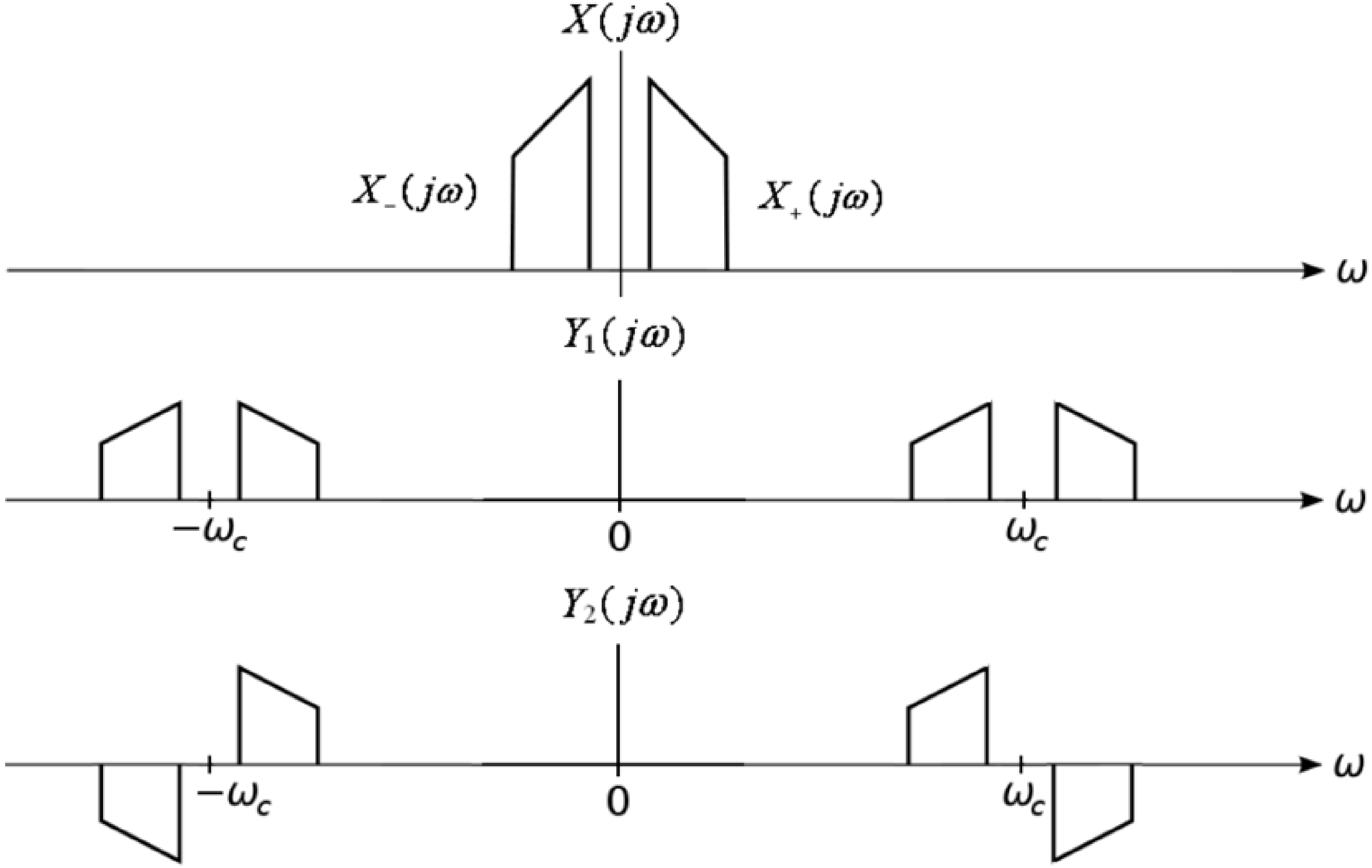
$$= \frac{1}{2j}[-jX_+(j(\omega-\omega_c)) + jX_-(j(\omega-\omega_c))$$

$$-(-jX_+(j(\omega+\omega_c)) + jX_-(j(\omega+\omega_c)))]$$

$$= \frac{1}{2}[-X_+(j(\omega-\omega_c)) + X_-(j(\omega-\omega_c))$$

$$+X_+(j(\omega+\omega_c)) - X_-(j(\omega+\omega_c))]$$

SSB



$X(j\omega)$
$X_-(j\omega)$
$X_+(j\omega)$
ω
$Y_1(j\omega)$
$-\omega_c$
0
ω_c
ω
$Y_2(j\omega)$
$-\omega_c$
0
ω_c
ω

SSB

$$y(t) = y_1(t) \pm y_2(t) \leftrightarrow$$

$$Y(j\omega) = Y_1(j\omega) \pm Y_2(j\omega)$$

$$Y_{USB}(j\omega) = Y_1(j\omega) + Y_2(j\omega) = \frac{1}{2}[X_+(j(\omega - \omega_c)) +$$

$$X_-(j(\omega - \omega_c)) + X_+(j(\omega + \omega_c)) + X_-(j(\omega + \omega_c))] +$$

$$\frac{1}{2}[-X_+(j(\omega - \omega_c)) + X_-(j(\omega - \omega_c)) + X_+(j(\omega + \omega_c))$$

$$-X_-(j(\omega + \omega_c))]$$

$$= X_-(j(\omega - \omega_c) + X_+(j(\omega + \omega_c))$$

SSB

$$Y_{LSB}(j\omega) = Y_1(j\omega) - Y_2(j\omega) = \frac{1}{2}[X_+(j(\omega-\omega_c)) +$$

$$X_-(j(\omega-\omega_c)) + X_+(j(\omega+\omega_c)) + X_-(j(\omega+\omega_c))] -$$

$$\frac{1}{2}[-X_+(j(\omega-\omega_c)) + X_-(j(\omega-\omega_c)) + X_+(j(\omega+\omega_c))$$

$$-X_-(j(\omega+\omega_c))]$$

$$= X_+(j(\omega-\omega_c) + X_-(j(\omega+\omega_c))$$