Introduction to Signals and Systems: V216

Lecture #16

Chapter 10: Z Transform

Z Transform

- In this chapter, we use the same approach for discrete time as we develop z-transform, which is the discrete-time counterpart of the Laplace transform.
- The motivations for and properties of the z-transform closely parallel those of the Laplace transform.
- However, we will encounter some important distinctions between the z-transform and the Laplace transform that arise from the fundamental differences between continuous-time and discrete-time signals and systems.

Discrete Time EigenFunctions

Consider a discrete-time input sequence (z is a complex number):

$x[n] = z^n$

Then using discrete-time convolution for an LTI system:

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

$$= \sum_{k=-\infty}^{\infty} h[k]z^{n-k}$$

$$= z^n \sum_{k=-\infty}^{\infty} h[k]z^{-k}$$

$$= H(z)z^n = H(z)x[n]$$

Z-transform of the impulse response

$$H(z) = \sum_{k=-\infty}^{\infty} h[k]z^{-k}$$

But this is just the input signal multiplied by $H(z)$, the **z-transform of the impulse response,** which is a complex function of z.

z^n is an **eigenfunction of a DT LTI** system

z-Transform of a Discrete-Time Signal

The **z-transform** of a discrete time signal is defined as:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

This is analogous to the CT Laplace Transform, and is denoted:

$$x[n] \overset{Z}{\leftrightarrow} X(z)$$

To understand this relationship, put *z* in **polar coords**, i.e. $z=re^{j\omega}$

$$X(re^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n](re^{j\omega})^{-n}$$

$$= \sum_{n=-\infty}^{\infty} (x[n]r^{-n})e^{-j\omega n}$$

Therefore, this is just equivalent to the scaled **DT Fourier Series**:

$$X(re^{j\omega}) = F\{x[n]r^{-n}\}$$

Geometric Interpretation & Convergence

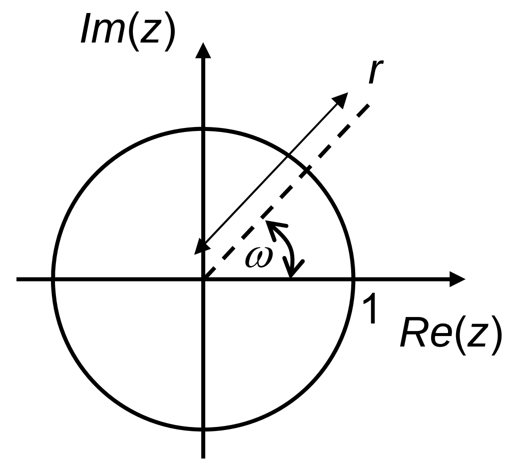
The relationship between the z-transform and Fourier transform for DT signals, closely parallels the discussion for CT signals

The z-transform reduces to the DT Fourier transform when the magnitude is unity r=1 (rather than $Re\{s\}$=0 or purely imaginary for the CT Fourier transform)

For the z-transform convergence, we require that the Fourier transform of $x[n]r^{-n}$ converges. This will generally converge for some values of r and not for others.

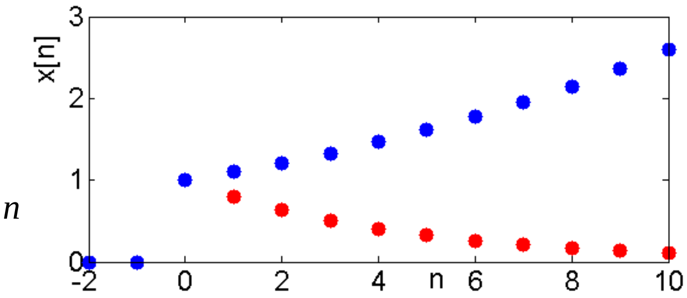
In general, the z-transform of a sequence has an associated range of values of z for which $X(z)$ converges.

This is referred to as the Region of Convergence (ROC). If it includes the unit circle, the DT Fourier transform also converges.



z-plane

Example 1: *z*-Transform of Power Signal



Consider the signal $x[n] = a^n u[n]$

Then the *z*-transform is:

$$X(z) = \sum_{n=-\infty}^{\infty} a^n u[n] z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n$$

For convergence of $X(z)$, we require

$$\sum_{n=0}^{\infty} (az^{-1})^n < \infty$$

The region of convergence (ROC) is

$$\left|az^{-1}\right| < 1 \quad \text{or} \quad |z| > |a|$$

and the Laplace transform is:

$$X(z) = \sum_{n=0}^{\infty} (az^{-1})^n = \frac{1}{1 - az^{-1}} = \frac{z}{z-a}, \qquad |z| > |a|$$

When $x[n]$ is the unit step sequence $a=1$

$$X(z) = \frac{1}{1 - z^{-1}}, \qquad |z| > 1$$

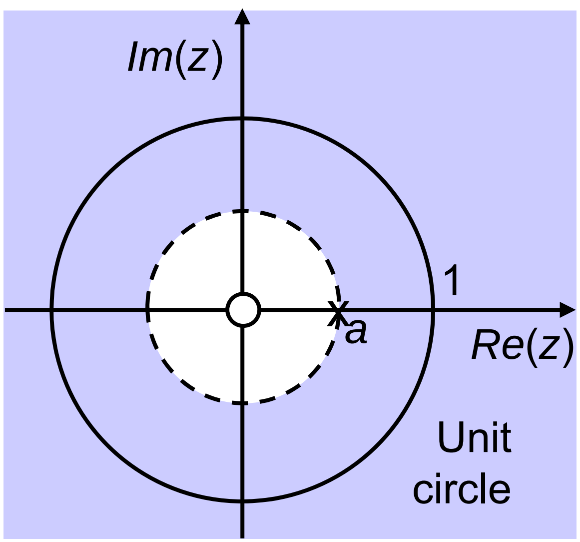
Example 1: Region of Convergence

The *z*-transform $X(z) = z/(z-a)$ is a rational function so it can be characterized by its **zeros** (numerator polynomial roots) and its **poles** (denominator polynomial roots)

For this example there is one zero at z=0, and one pole at z=a.

The pole-zero and ROC plot is shown here

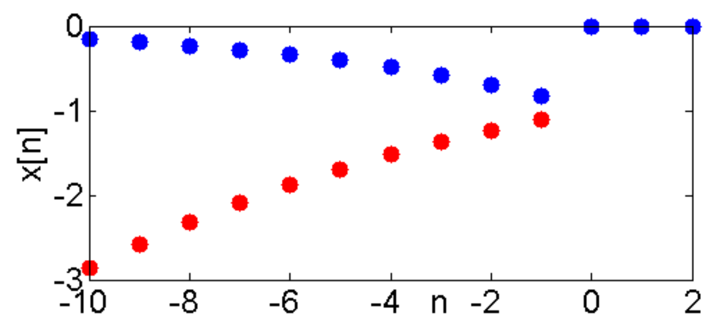
For |a|>1, the ROC does not include the unit circle, for those values of *a*, the discrete time Fourier transform of $a^n u[n]$ does not converge.



Example 2: *z*-Transform of Power Signal

Now consider the signal $x[n] = -a^n u[-n-1]$

Then the Laplace transform is:



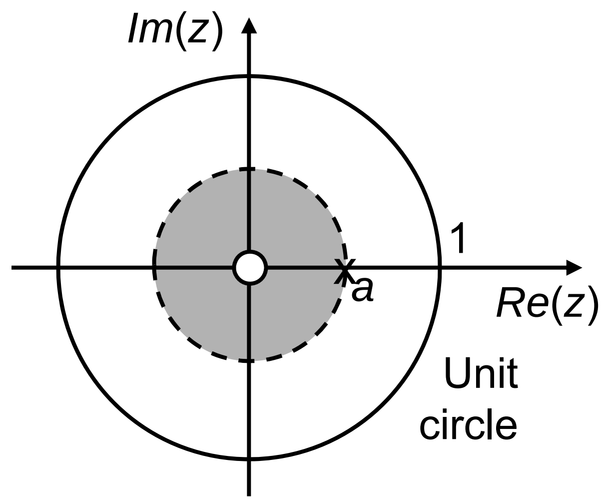
$$X(z) = -\sum_{n=-\infty}^{\infty} a^n u[-n-1] z^{-n} = -\sum_{n=-\infty}^{-1} a^n z^{-n}$$

$$= -\sum_{n=1}^{\infty} a^{-n} z^n = 1 - \sum_{n=0}^{\infty} (a^{-1}z)^n$$

If $|a^{-1}z|<1$, or equivalently, $|z|<|a|$, this sum converges to:

$$X(z) = 1 - \frac{1}{1-a^{-1}z} = \frac{1}{1-az^{-1}} = \frac{z}{z-a}, \qquad |z| < |a|$$

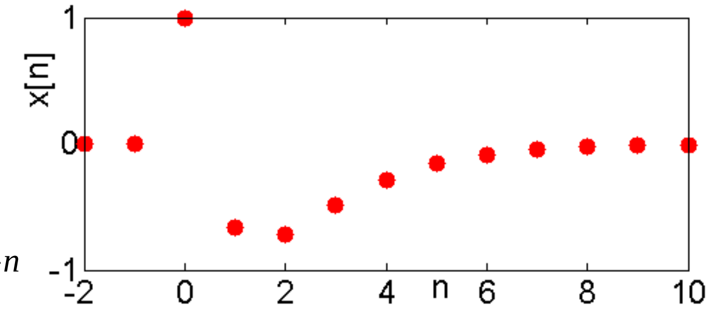
The pole-zero plot and ROC is shown right for $0<a<1$



Example 3: Sum of Two Exponentials

Consider the input signal

$$x[n] = 7(1/3)^n u[n] - 6(1/2)^n u[n]$$



The *z*-transform is then:

$$X(z) = \sum_{n=-\infty}^{\infty} \{7(1/3)^n u[n] - 6(1/2)^n u[n]\} z^{-n}$$

$$= 7\sum_{n=0}^{\infty} (1/3)^n z^{-n} - 6\sum_{n=0}^{\infty} (1/2)^n z^{-n}$$

$$= \frac{7z}{z - \frac{1}{3}} - \frac{6z}{z - \frac{1}{2}}$$

$$= \frac{z(z - \frac{3}{2})}{(z - \frac{1}{3})(z - \frac{1}{2})}$$

For the region of convergence we require both summations to converge $|z|>1/3$ and $|z|>1/2$, so

$|z|>1/2$

Example 4

- Example 10.4: Determine the z-transform of

$$x[n] = \left(\frac{1}{3}\right)^n \sin\left(\frac{\pi}{4}n\right)u[n]$$

- With the use of Euler expression, the algebraic expression for the Laplace transform is then,

$$X(z) = \sum_{n=-\infty}^{\infty}\left\{\frac{1}{2j}\left(\frac{1}{3}e^{j\pi/4}\right)^n u[n]\right\}z^{-n} - \sum_{n=-\infty}^{\infty}\left\{\frac{1}{2j}\left(\frac{1}{3}e^{-j\pi/4}\right)^n u[n]\right\}z^{-n}$$

- From Example 10.1, we know that

$$\left(\frac{1}{3}e^{j\pi/4}\right)^n u[n] \xleftrightarrow{Z} = \frac{1}{1-\frac{1}{3}e^{j\pi/4}z^{-1}}, \quad |z| > \frac{1}{3}$$

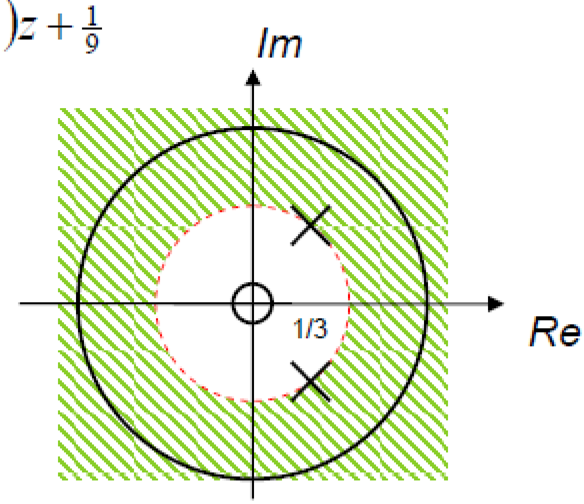
$$\left(\frac{1}{3}e^{-j\pi/4}\right)^n u[n] \xleftrightarrow{Z} = \frac{1}{1-\frac{1}{3}e^{-j\pi/4}z^{-1}}, \quad |z| > \frac{1}{3}$$

Example 4

- $X(z)$ converges for which the z-transform of both terms converges. In summary, we have

$$X(z)=\frac{1}{2j}\frac{1}{1-\frac{1}{3}e^{j\pi/4}z^{-1}}-\frac{1}{2j}\frac{1}{1-\frac{1}{3}e^{-j\pi/4}z^{-1}},\quad |z|>\frac{1}{3}$$

$$=\frac{\left(\frac{2}{3}\sin\frac{\pi}{4}\right)z}{(z-\frac{1}{3}e^{j\pi/4})(z-\frac{1}{3}e^{-j\pi/4})}=\frac{\frac{1}{3\sqrt{2}}z}{z^2-\left(\frac{2}{3}\cos\frac{\pi}{4}\right)z+\frac{1}{9}}$$



Properties of z-Transform

Linearity
Time Shifting
Scaling in the z-domain
Time Reversal
Time Expansion
Conjugation
Convolution
Differentiation in the z-domain
The initial-value Theorem.

z-T Property associated with Linearity

$x_1[n] \xleftrightarrow{z-T} X_1(z)$ with ROC denoted by R_1

$x_2[n] \xleftrightarrow{z-T} X_2(z)$ with ROC denoted by R_2

$ax_1[n]+bx_2[n] \xleftrightarrow{LT} aX_1(z)+bX_2(z)$ with ROC containing $\mathrm{R}_1 \bigcap R_2$.

$\bigcap$ is the symbol for intersect with.

Linearity of the z-Transform

If $\quad x_1[n] \overset{Z}{\leftrightarrow} X_1(z) \quad$ ROC=R_1

and $\quad x_2[n] \overset{Z}{\leftrightarrow} X_2(z) \quad$ ROC=R_2

Then $\quad ax_1[n] + bx_2[n] \overset{Z}{\leftrightarrow} aX_1(z) + bX_2(z) \quad$ ROC= $R_1 \cap R_2$

This follows directly from the definition of the *z*-transform (as the summation operator is linear, see Example 3). It is easily extended to a linear combination of an arbitrary number of signals

Time Shifting & *z*-Transforms

If $$x[n] \overset{Z}{\leftrightarrow} X(z) \qquad \text{ROC}=R$$

Then $$x[n-n_0] \overset{Z}{\leftrightarrow} z^{-n_0} X(z) \qquad \text{ROC}=R$$

Proof
$$Z\{x[n-1]\} = \sum_{n=-\infty}^{\infty} x[n-1] z^{-n}$$
$$= z^{-1} \sum_{n=-\infty}^{\infty} x[n-1] z^{-(n-1)}$$
$$= z^{-1} \sum_{m=-\infty}^{\infty} x[m] z^{-m} = z^{-1} Z\{x[n]\}$$

This is very important for producing the **z-transform transfer function of a difference equation** which uses the property:

$$x[n-1] \overset{Z}{\leftrightarrow} z^{-1} X(z)$$

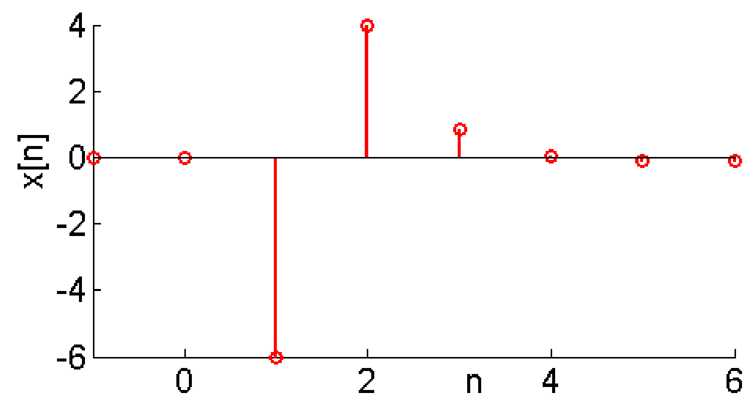
Example: Linear & Time Shift

Consider the input signal

$$x[n] = 7(1/3)^{n-2}u[n-2] - 6(1/2)^{n-1}u[n-1]$$

We know that

$$a^n u[n] \overset{Z}{\leftrightarrow} \frac{z}{z-a}$$



So

$$X(z) = 7z^{-2}\frac{z}{z-1/3} - 6z^{-1}\frac{z}{z-1/2}$$

$$= 7\frac{1}{z^2-1/3z} - 6\frac{1}{z-1/2}$$

ZT Properties continued

$$x[n] \xleftrightarrow{zT} X(z), \quad \text{with ROC} = \text{R},$$

Time Shifting : -

$$\text{then } x[n-n_0] \xleftrightarrow{zT} z^{-n_0} X(z), \quad \text{with ROC} = \text{R, except for possible addition or deletion of the origin or infinity.}$$

Scaling in the z - domain : -

$$\text{then } z_0^n x[n] \xleftrightarrow{zT} X\left(\frac{z}{z_0}\right), \quad \text{with ROC} = |z_0|\,\text{R.}$$

Time Reversal : -

$$\text{then } x[-n] \xleftrightarrow{zT} X\left(\frac{1}{z}\right), \quad \text{with ROC} = \frac{1}{R}.$$

Time Expansion : -

$$x_{(k)}[n] = x[n/k] \text{ if n is a multiple of k,}$$

$$= 0. \text{ if n is not a multiple of k.}$$

$$\text{then } x_{(k)}[n] \xleftrightarrow{zT} X(z^k), \quad \text{with ROC} = R^{1/k}$$

Conjugation : -

$$\text{then } x^*[n] \xleftrightarrow{zT} X^*(z^*), \quad \text{with ROC} = \text{R.}$$

ZT Properties continued

$$x[n] \xleftrightarrow{zT} X(z), \quad \text{with ROC} = \text{R},$$

$$\text{Differentiation in the z - Domain :-}$$

$$\text{then } \mathrm{n}x[n] \xleftrightarrow{zT} -z\frac{dX(z)}{dz}, \quad \text{with ROC} = \text{R}.$$

$$\text{Initial - value theorem :-}$$

$$\text{If } \mathrm{x[n]} = 0 \text{ for } \mathrm{n} < 0\,,$$

$$then\ x[0] = \lim_{z \to \infty} X(z).$$

ZT Properties continued

Property	Signal	z-Transform	ROC
	$x[n]$	$X(z)$	R
	$x_1[n]$	$X_1(z)$	R_1
	$x_2[n]$	$X_2(z)$	R_2
Linearity	$ax_1[n] + bx_2[n]$	$aX_1(z) + bX_2(z)$	At least the intersection of R_1 and R_2
Time shifting	$x[n - n_0]$	$z^{-n_0}X(z)$	R, except for the possible addition or deletion of the origin
Scaling in the z-domain	$e^{j\omega_0 n}x[n]$	$X(e^{-j\omega_0}z)$	R
	$z_0^n x[n]$	$X\left(\frac{z}{z_0}\right)$	$z_0 R$
	$a^n x[n]$	$X(a^{-1}z)$	Scaled version of R (i.e., $\|a\|R$ = the set of points $\{\|a\|z\}$ for z in R)
Time reversal	$x[-n]$	$X(z^{-1})$	Inverted R (i.e., R^{-1} = the set of points z^{-1}, where z is in R)
Time expansion	$x_{(k)}[n] = \begin{cases} x[r], & n = rk \\ 0, & n \neq rk \end{cases}$ for some integer r	$X(z^k)$	$R^{1/k}$ (i.e., the set of points $z^{1/k}$, where z is in R)
Conjugation	$x^*[n]$	$X^*(z^*)$	R
Convolution	$x_1[n] * x_2[n]$	$X_1(z)X_2(z)$	At least the intersection of R_1 and R_2
First difference	$x[n] - x[n-1]$	$(1 - z^{-1})X(z)$	At least the intersection of R and $\|z\| > 0$

ZT Properties continued

Accumulation	$\sum_{k=-\infty}^{n} x[k]$	$\frac{1}{1-z^{-1}} X(z)$	At least the intersection of R and $\lvert z\rvert > 1$
Differentiation in the z-domain	$nx[n]$	$-z\frac{dX(z)}{dz}$	R

Initial Value Theorem

If $x[n] = 0$ for $n < 0$, then

$$x[0] = \lim_{z\to\infty} X(z)$$

Some common z-transform pairs

Signal	Transform	ROC
1. $\delta[n]$	1	All z
2. $u[n]$	$\dfrac{1}{1 - z^{-1}}$	$\lvert z\rvert > 1$
3. $-u[-n-1]$	$\dfrac{1}{1 - z^{-1}}$	$\lvert z\rvert < 1$
4. $\delta[n-m]$	z^{-m}	All z, except 0 (if $m > 0$) or ∞ (if $m < 0$)
5. $\alpha^n u[n]$	$\dfrac{1}{1 - \alpha z^{-1}}$	$\lvert z\rvert > \lvert\alpha\rvert$
6. $-\alpha^n u[-n-1]$	$\dfrac{1}{1 - \alpha z^{-1}}$	$\lvert z\rvert < \lvert\alpha\rvert$
7. $n\alpha^n u[n]$	$\dfrac{\alpha z^{-1}}{(1 - \alpha z^{-1})^2}$	$\lvert z\rvert > \lvert\alpha\rvert$
8. $-n\alpha^n u[-n-1]$	$\dfrac{\alpha z^{-1}}{(1 - \alpha z^{-1})^2}$	$\lvert z\rvert < \lvert\alpha\rvert$
9. $[\cos \omega_0 n]u[n]$	$\dfrac{1 - [\cos \omega_0]z^{-1}}{1 - [2\cos \omega_0]z^{-1} + z^{-2}}$	$\lvert z\rvert > 1$
10. $[\sin \omega_0 n]u[n]$	$\dfrac{[\sin \omega_0]z^{-1}}{1 - [2\cos \omega_0]z^{-1} + z^{-2}}$	$\lvert z\rvert > 1$
11. $[r^n \cos \omega_0 n]u[n]$	$\dfrac{1 - [r\cos \omega_0]z^{-1}}{1 - [2r\cos \omega_0]z^{-1} + r^2 z^{-2}}$	$\lvert z\rvert > r$
12. $[r^n \sin \omega_0 n]u[n]$	$\dfrac{[r\sin \omega_0]z^{-1}}{1 - [2r\cos \omega_0]z^{-1} + r^2 z^{-2}}$	$\lvert z\rvert > r$

The inverse z-transform

Since $X(z) = X(re^{j\omega}) = F\{x[n]r^{-n}\}$, we have:

$$x[n]r^{-n} = F^{-1}\{X(re^{j\omega})\} = \frac{1}{2\pi}\int_{2\pi} X(re^{j\omega})e^{j\omega n}d\omega$$

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(re^{j\omega})\left(re^{j\omega}\right)^n d\omega$$

Let $z = re^{j\omega}$ and *r* fixed, then we have $dz = jre^{j\omega}d\omega = jzd\omega$. Therefore,

$$x[n] = \frac{1}{2\pi j}\oint X(z)z^{n-1}dz$$

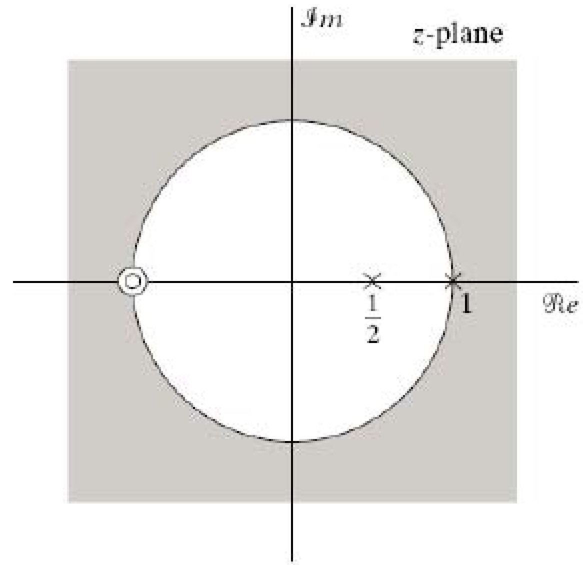
➔integration over a counter-clockwise closed circular contour centred at the origin and with radius *r*, i.e., $|z|=r$.

For rational z-transform, we don't need to use contour integration. Instead, there are two alternative methods: partial fraction expansion and power series expansion.

The inverse z-transform

- Example: Find the inverse z-transform of

$$X(z)=\frac{1+2z^{-1}+z^{-2}}{1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2}}=\frac{\left(1+z^{-1}\right)^2}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-z^{-1}\right)},\qquad |z|>1$$



- Solution:
  - Since $M = N = 2$, $X(z)$ can be expressed as

$$X(z)=B_0+\frac{A_1}{1-\frac{1}{2}z^{-1}}+\frac{A_2}{1-z^{-1}}$$

  - The constant B_0 can be found by long division:

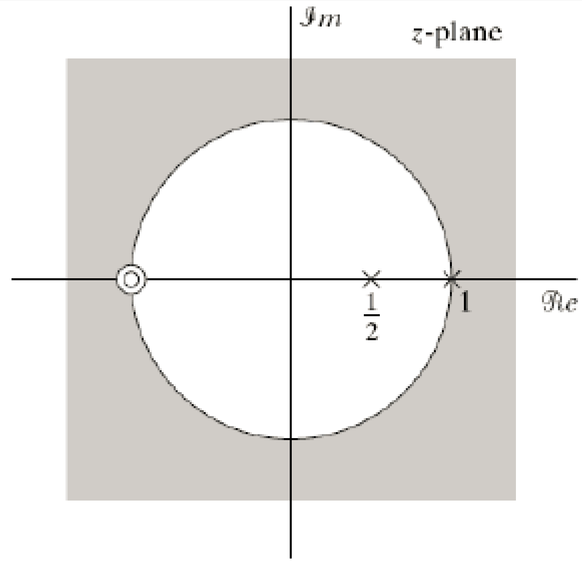
$$\begin{array}{r} 2\\ \tfrac{1}{2}z^{-2}-\tfrac{3}{2}z^{-1}+1\,\big)\,\overline{z^{-2}+2z^{-1}+1}\\ \underline{z^{-2}-3z^{-1}+2}\\ +5z^{-1}-1 \end{array}$$

Since the remainder after one step of long division is of degree 1 in the variable z^{-1}, it is not necessary to continue to divide.

The inverse z-transform

- Example: (Cont'd)
  - Thus, $X(z)$ can be written as

$$X(z) = 2 + \frac{-1 + 5z^{-1}}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 - z^{-1}\right)} = 2 + \frac{A_1}{1 - \frac{1}{2}z^{-1}} + \frac{A_2}{1 - z^{-1}}$$



  - The coefficient of A_1 and A_2 can be found by

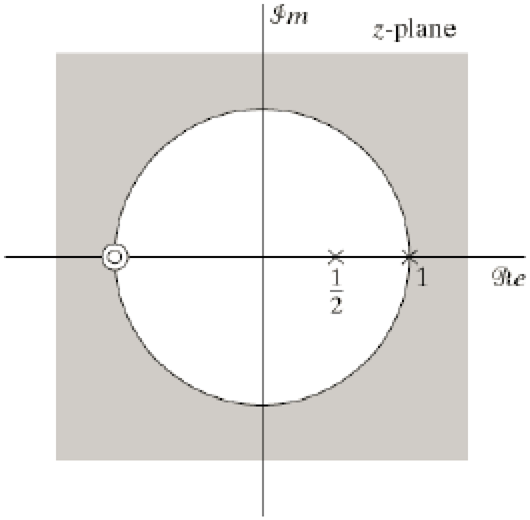
$$A_1 = \left(1 - \tfrac{1}{2}z^{-1}\right)X(z)\Big|_{z=\frac{1}{2}} =$$

$$A_2 = \left(1 - z^{-1}\right)X(z)\Big|_{z=1} =$$

The inverse z-transform

- Therefore,

$$X(z) = 2 - \frac{9}{1-\frac{1}{2}z^{-1}} + \frac{8}{1-z^{-1}}$$



- We have,

$$\delta[n] \xleftrightarrow{z} 1 \qquad |z| > 1 \geq 0$$

$$\left(\tfrac{1}{2}\right)^n u[n] \xleftrightarrow{z} \frac{1}{1-\frac{1}{2}z^{-1}} \qquad |z| > 1 > \tfrac{1}{2}$$

$$u[n] \xleftrightarrow{z} \frac{1}{1-z^{-1}} \qquad |z| > 1$$

- Therefore,

$$X(z) = 2 - \frac{9}{1-\frac{1}{2}z^{-1}} + \frac{8}{1-z^{-1}} \xleftrightarrow{z} x[n] = 2\delta[n] - 9\left(\tfrac{1}{2}\right)^n u[n] + 8u[n]$$

The inverse z-transform

- Example: Find the inverse z-transform of

$$X(z) = z^2\left(1-0.5z^{-1}\right)\left(1+z^{-1}\right)\left(1-z^{-1}\right)$$

- Solution:
  - By expanding $X(z)$,

$$X(z) = z^2\left(1-0.5z^{-1}\right)\left(1+z^{-1}\right)\left(1-z^{-1}\right) = z^2\left(1-0.5z^{-1}\right)\left(1-z^{-2}\right)$$
$$= z^2\left(1-0.5z^{-1}-z^{-2}+0.5z^{-3}\right) = z^2-0.5z-1+0.5z^{-1}$$

  - Comparing to the coefficients in $X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$

$$x[n] = \begin{cases} 1, & n=-2 \\ -0.5, & n=-1 \\ -1, & n=0 \\ 0.5, & n=1 \\ 0, & otherwise \end{cases} \Leftrightarrow x[n] = \delta[n+2]-0.5\delta[n+1]-\delta[n]+0.5\delta[n-1]$$

The inverse z-transform

Power Series Expansion

Given $X(z)$, suppose that we can write $X(z)$ as a power series in z^{-1}

$$X(z) = \sum_{n=-\infty}^{\infty} a_n (z^{-1})^n$$

$$= \cdots + x[-2]z^2 + x[-1]z + x[0] + x[1]z^{-1} + x[2]z^{-2} + \cdots$$

- Comparing the above expression with the analysis equation, we concluded that $x[n] = a_n$
- Therefore, given a power series expansion for $X(z)$, we can obtain $x[n]$ from the coefficients of the power series.

The inverse z-transform

- Example 10.14: Find the inverse z-transform of

$$X(z) = \log\left(1 + az^{-1}\right), \quad |z| > |a|$$

- Solution:
  - With the property 4, $x[n]$ is a right-sided sequence.
  - Using the power series expansion for $\log(1+\lambda)$, with $|\lambda|<1$, we obtain

$$\log(1+\lambda) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\lambda^n}{n} \Rightarrow X(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} a^n z^{-n}}{n}$$

  - This gives

$$x[n] = \begin{cases} (-1)^{n+1}\dfrac{a^n}{n}, & n \geq 1 \\ 0, & otherwise \end{cases}$$

OR

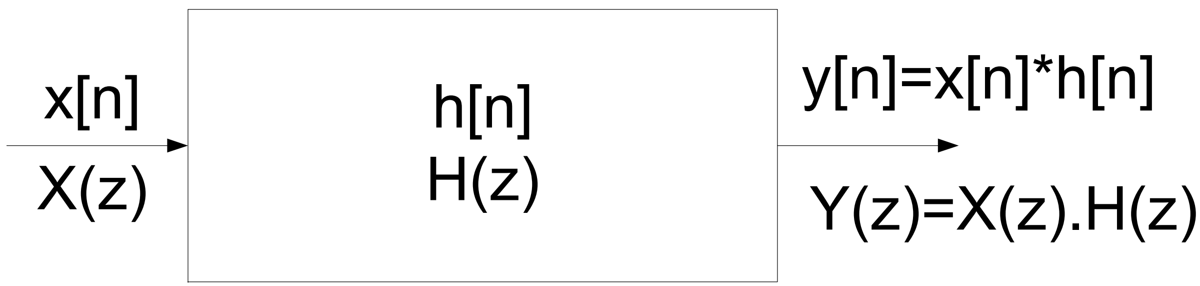
$$x[n] = (-1)^{n+1}\frac{a^n}{n}u[n-1]$$

Analysis & Characterization of LTI systems using z-Transforms.

$$\text{If } x[n] = z^n,\ y[n] = H(z).z^n,$$

$$\text{where } z^n \text{ is eigenfunction of LTI system,}$$

$$H(z) \text{ is eigenvalue of LTI system}$$



H(z) is known as System Function or Transfer Function

$$\text{Frequency response} = H(z) \text{ with } z = e^{j\omega}\ (\textit{unit circle in ROC})$$

Causality.

(1) A LTI system is causal if : -

Its impulse response $h[n] = 0$ for $n < 0$ ie. right - sided.

or in other words : -

ROC of its system function $H(z)$ is the exterior of the circle including infinity.

(2) A discrete - time LTI system with rational system function $H(z)$

is causal if and only if : -

(a) the ROC is the exterior of a circle outside the outermost pole;

and (b) with $H(z)$ expressed as a ratio of polynomials in z,

the order of the numerator cannot be greater than the order

of the denominator.

Stability.

An LTI system is stable if and only if:-

- Its Impulse response h[n] is absolutely summable.
- Or Fourier Transform of h[n] converges.
- Or the ROC of its system function H(z) includes the unit circle, |z|=1

For causal LTI system, all poles of H(z) must be in the unit circle, |z|=1

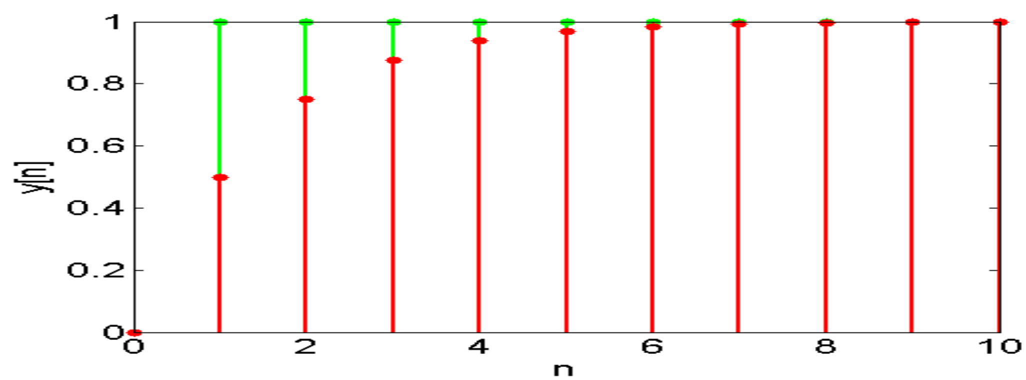
Introduction to Discrete Time Transfer Fns

A **discrete-time LTI system** can be represented as a (first order) **difference equation** of the form:

$$a_1 y[n] + a_2 y[n-1] = b_1 x[n] + b_2 x[n-1]$$

$$y[n] = (-a_2 y[n-1] + b_1 x[n] + b_2 x[n-1]) / a_1$$

This is analogous to a sampled CT differential equation

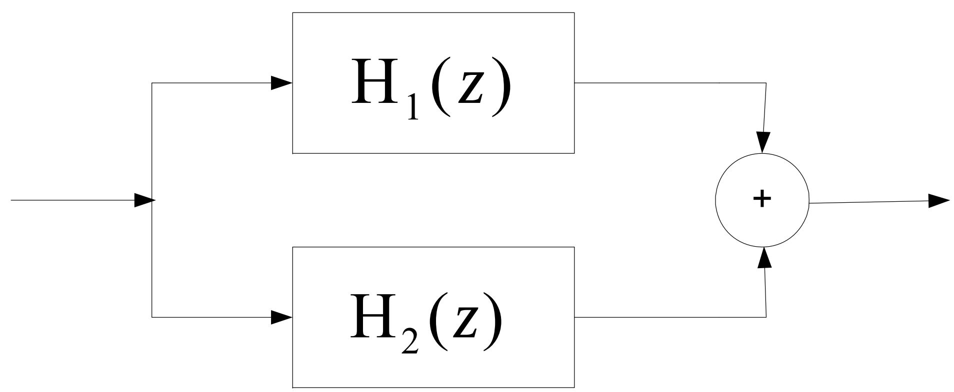


This is hard to solve analytically, and we'd like to be able to perform some form of analogous manipulation like continuous time **transfer functions**, i.e.

$Y(s) = H(s)X(s)$

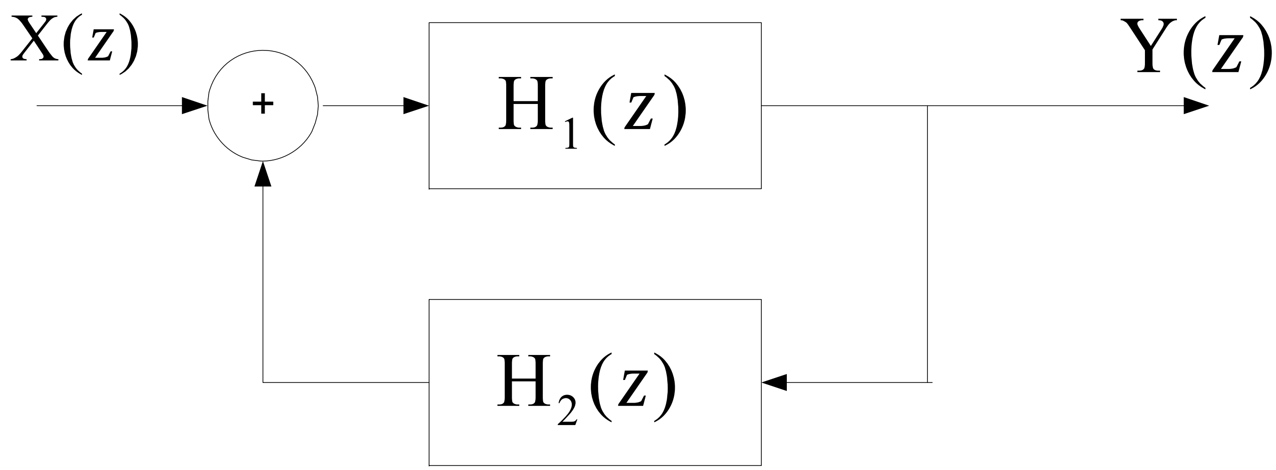
System Function for Interconnections of LTI Systems

$$H(z) = H_1(z).H_2(z)$$



$$H(z) = H_1(z) + H_2(z)$$

System Function for Interconnections of LTI Systems

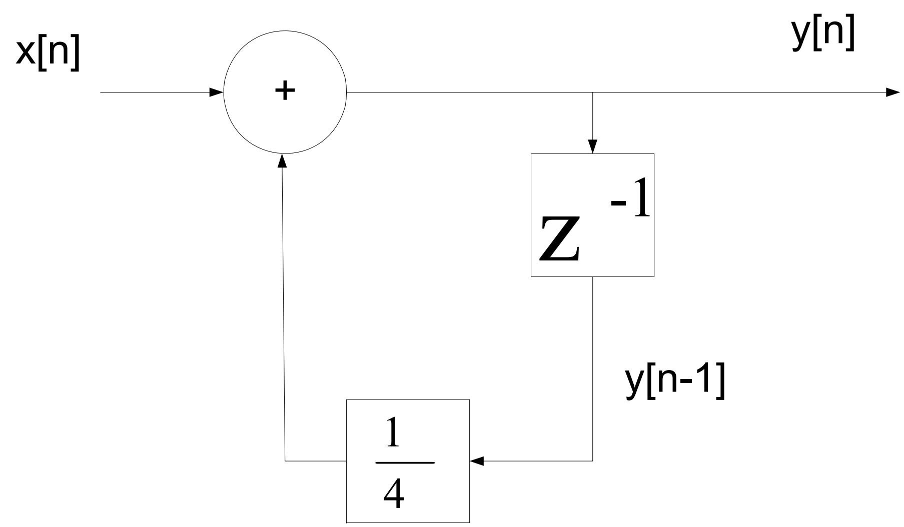


$$\frac{Y(z)}{X(z)} = H(z) = \frac{H_1(z)}{1+H_1(z)_2H(z)}$$

Block Diagram of Causal LTI systems described by Difference Equations and Rational System Functions.

Example 10.28 A Causal LTI system with system function :-

$$H(z) = \frac{1}{1 - \frac{1}{4} z^{-1}}$$



$$\frac{Y(z)}{X(z)} = \frac{1}{1 - \frac{1}{4} z^{-1}}$$

$$Y(z)\{1 - \frac{1}{4} z^{-1}\} = X(z)$$

Taking the inverse z - transform of the above equation,

$$y[n] - \frac{1}{4} y[n-1] = x[n] \quad or \quad y[n] = x[n] + \frac{1}{4} y[n-1]$$

Discrete Time Transfer Function

Consider a first order, LTI differential equation such as:

$$a_1 y[n] + a_2 y[n-1] = b_1 x[n] + b_2 x[n-1]$$

Then the **discrete time transfer function** is the **z-transform of the impulse response**, $H(z)$

As $Z\{\delta[n]\} = 1$, taking the z-transform of both sides of the equation (linearity we get), for the impulse response

$$Z\{a_1 h[n] + a_2 h[n-1]\} = Z\{b_1 \delta[n] + b_2 \delta[n-1]\}$$

$$(a_1 + a_2 z^{-1}) Z\{h[n]\} = (b_1 + b_2 z^{-1}) Z\{\delta[n]\}$$

$$H(z) = \frac{(b_1 + b_2 z^{-1})}{(a_1 + a_2 z^{-1})}$$

$$= \frac{(zb_1 + b_2 z)}{(za_1 + a_2)}$$

Discrete Time Transfer Function

The discrete-time transfer function of an LTI system is a **rational polynomial in *z***. (This is equivalent to the transfer function of a continuous time differential system which is a rational polynomial in *s*)

$$H(z) = \frac{(zb_1 + b_2)}{(za_1 + a_2)}$$

As usual the *z*-transform transfer function can computed by either:

1. If the difference equation is known, take the z-transform of each sides when the input signal is an impulse $\delta[n]$
2. If the discrete-time impulse response signal is known, calculate the z-transform of the signal $h[n]$.

In either case, the same result will be obtained.

Convolution using *z*-Transforms

The z-transform also has the multiplication property, i.e.

$$x[n] \overset{Z}{\leftrightarrow} X(z) \qquad \text{ROC}=R_1$$

$$h[n] \overset{Z}{\leftrightarrow} H(z) \qquad \text{ROC}=R_2$$

$$x[n] * h[n] \overset{Z}{\leftrightarrow} X(z)H(z) \qquad \text{ROC}\supseteq R_1 \cap R_2$$

Proof is "identical" to the Fourier/Laplace transform convolution and follows from eigensystem property

Note that pole-zero cancellation may occur between $H(z)$ and $X(z)$ which extends the ROC

While this is true for any two signals, it is particularly important as $H(z)$ represents the **transfer function** of discrete-time LTI system

z-T Property associated with Convolution

$x_1[n] \xleftrightarrow{zT} X_1(z)$ with ROC denoted by R_1

$x_2[n] \xleftrightarrow{zT} X_2(z)$ with ROC denoted by R_2

$x_1[n] * x_2[n] \xleftrightarrow{zT} X_1(z)X_2(z)$ with ROC containing $R_1 \bigcap R_2$.

$\bigcap$ is the symbol for intersect with.

Example 1: First Order Difference Equation

Calculate the output of a first order difference equation of a input signal $x[n] = 0.5^n u[n]$

$$0.5^n u[n] \overset{Z}{\leftrightarrow} X(z) = \frac{z}{z-0.5}$$

System transfer function (z-transform of the impulse response)

$$y[n] - 0.8y[n-1] = x[n]$$

$$H(z) = \frac{1}{1-0.8z^{-1}} = \frac{z}{z-0.8}$$

The (z-transform of the) output is therefore:

$$Y(z) = \frac{z^2}{(z-0.5)(z-0.8)}$$

ROC $|z|>0.8$

$$= \frac{1}{0.3}\left(\frac{0.8z}{(z-0.5)} - \frac{0.5z}{(z-0.8)}\right)$$

$$y[n] = (0.8 * 0.5^n u[n] - 0.5 * 0.8^n u[n]) / 0.3$$

Example 2: 2nd Order Difference Equation

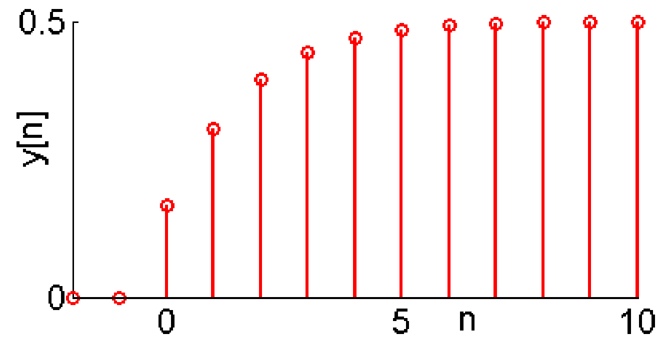
Consider the discrete time step input signal

$$u[n] \overset{z}{\leftrightarrow} X(z) = \frac{1}{1-z^{-1}}$$

to the 2nd order difference equation

$$6y[n] - 5y[n-1] + 1y[n-2] = x[n]$$

$$H(z) = \frac{1}{6-5z^{-1}+1z^{-2}} = \frac{1}{(2-z^{-1})(3-z^{-1})}$$



To calculate the solution, multiply and express as partial fractions

$$\begin{aligned} Y(z) &= \frac{1}{(3-z^{-1})(2-z^{-1})(1-z^{-1})} \\ &= 0.5\frac{1}{(3-z^{-1})} - \frac{1}{(2-z^{-1})} + 0.5\frac{1}{(1-z^{-1})} \\ &= 0.167\frac{1}{(1-1/3z^{-1})} - 0.5\frac{1}{(1-1/2z^{-1})} + 0.5\frac{1}{(1-z^{-1})} \end{aligned}$$

$$y[n] = (0.167(1/3)^n - 0.5(1/2)^n + 0.5)u[n]$$