VE216 RC2

Chapter 2

Some of contents are from professor's lecture slides.

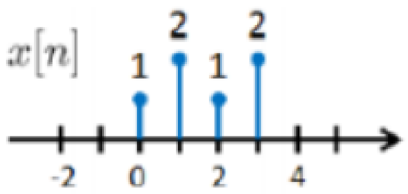
Discrete-Time LTI Systems: The Convolution Sum

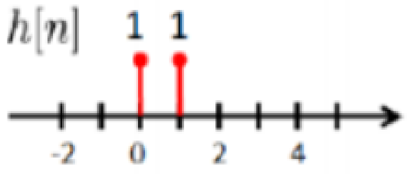
The output of an LTI system is the convolution sum of the input to the system and the impulse response of the system.

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Example

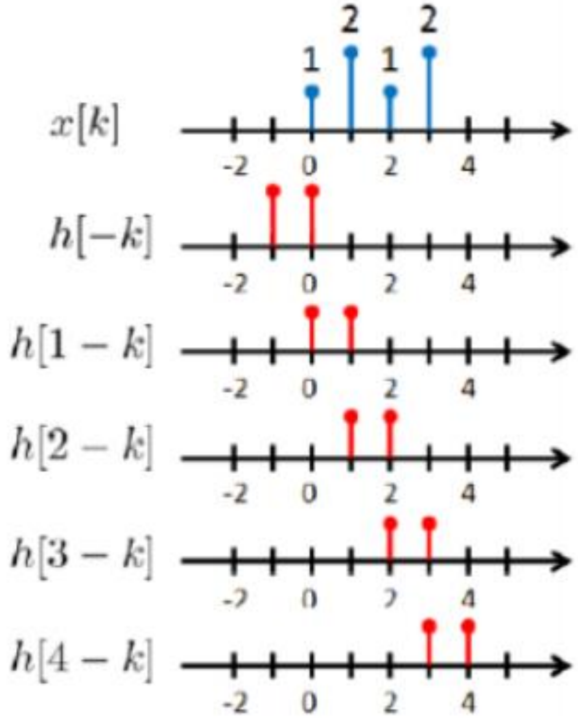
Example 1 : Consider the signal x[n] and the impulse response h[n] shown below.





Method 1

1. Flip
2. Shift
3. Multiply and Add



$y[0] = 1$

$y[1] = 1 + 2 = 3$

$y[2] = 2 + 1 = 3$

$y[3] = 1 + 2 = 3$

$y[4] = 2$

Method 2

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{\substack{i,j \\ i+j=n}} x[i]h[j]$$

x[i]h[j]		x[i]			
		1	2	1	2
h[j]	1	1	2	1	2
	1	1	2	1	2

y[n]={1, 3, 3, 3, 2}

Continuous-Time LTI Systems: The Convolution Integral

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau.$$

Convolution Integral Evaluation Procedure

(a) Graph $x(\tau)$ and $h(t-\tau)$.

(b) Shift $h(t-\tau)$ to the far left on the time axis.

(c) Write the mathematical representation of $w_t(\tau) = x(\tau)h(t-\tau)$

(d) Increase t until $w_t(\tau)$ changes.

(e) Integrate $w_t(\tau)$ for all τ to obtain $y(t)$.

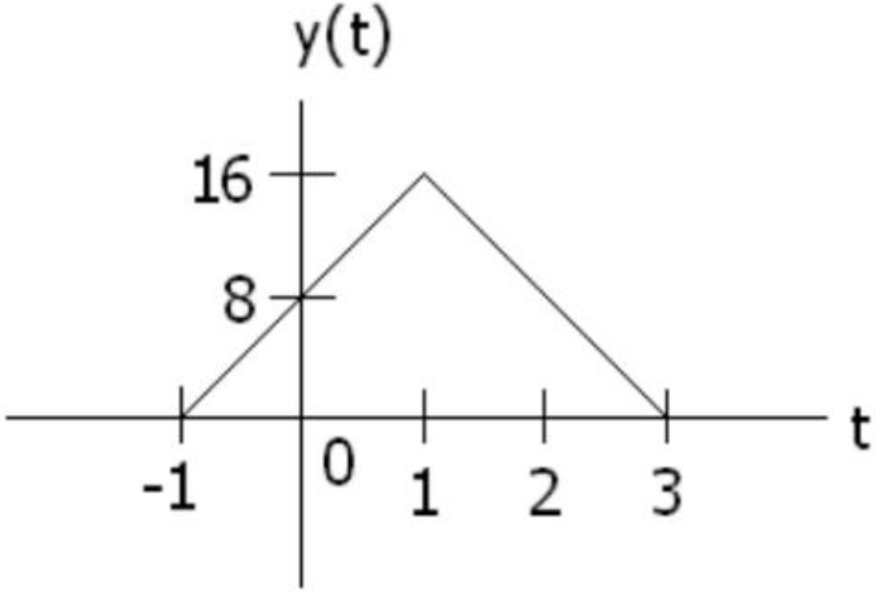
Example Given that

$$x(t) = 4\left[u(t+1) - u(t-1)\right]$$

$$h(t) = 2\left[u(t) - u(t-2)\right]$$

Find $y(t) = x(t) * h(t)$.

Answer.



Example Given that

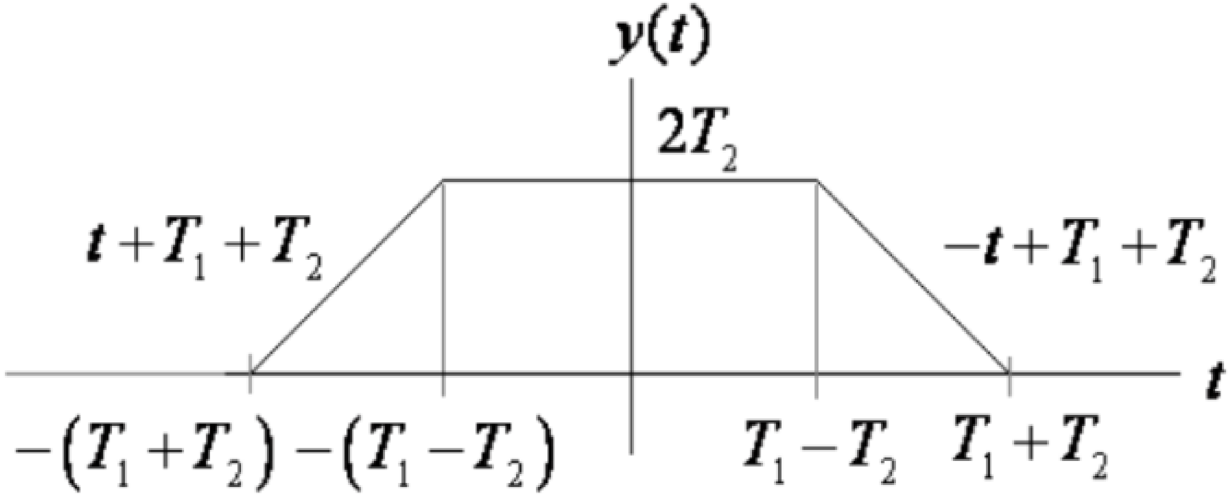
$$x(t) = u(t + T_1) - u(t - T_1)$$

$$h(t) = u(t + T_2) - u(t - T_2)$$

$$T_1 > T_2$$

Find $y(t) = x(t) * h(t)$.

Answer.



Properties of Convolution

Distributive Property

$$x(t) * h_1(t) + x(t) * h_2(t) = x(t) * \{h_1(t) + h_2(t)\}$$

Associative Property

$$\{x(t) * h_1(t)\} * h_2(t) = x(t) * \{h_1(t) * h_2(t)\}$$

Commutative Property

$$h_1(t) * h_2(t) = h_2(t) * h_1(t)$$

Relations between LTI System Properties and the Impulse Response

An LTI system is memoryless if and only if

$h[n] = c\delta[n]$

or

$h(t) = c\delta(t)$

An LTI system is causal if and only if

$h[n] = 0 \text{ for } n < 0$

or

$h(t) = 0 \text{ for } t < 0$

An LTI system is stable if and only if

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

or

$$\int_{-\infty}^{\infty} |h(t)|\, dt < \infty$$

Invertible LTI systems : The impulse response $h^{inv}[n]$ of the inverse system for an LTI system with impulse response $h[n]$ must satisfy

$$h[n] * h^{inv}[n] = \delta[n]$$

or

$$h(t) * h^{inv}(t) = \delta(t)$$

Differential Equations

$$\sum_{k=0}^{N} a_k \frac{d^k}{dt^k} y(t) = \sum_{k=0}^{M} b_k \frac{d^k}{dt^k} x(t)$$

The procedure is as follows:

- Find the form of homogeneous solution from the roots of the characteristic equation
- Find a particular solution by assuming that it is of the same form as the input, yet is independent of all terms in the homogeneous solution
- Determine the coefficients in the homogeneous solution so that the complete solution satisfies the initial conditions

The solution $y(t)$ consists of two parts -- a particular solution $y_p(t)$ (or the forced response of the system) plus a homogeneous solution $y_h(t)$ (or the natural response of the system).

$$y(t) = y_p(t) + y_h(t)$$

The *homogeneous solution* (or *general solution*) is the solution of the homogeneous equation

$$\sum_{k=0}^{N} a_k \frac{d^k y_h(t)}{dt^k} = 0$$

The same coefficients yield the characteristic equation

$$\sum_{k=0}^{N} a_k r^k = 0$$

whose N roots play a crucial role in finding $y_h(t)$.

If the roots $r_1, r_2, \ldots, r_N$ are all distinct, then the solution takes the form

$$y_h(t) = \sum_{i=1}^{N} c_i e^{r_i t}$$

where $c_i, i = 1, 2, \ldots, N$ are undetermined coefficients.

The form of the homogeneous solution changes slightly when the characteristic equation has repeated roots. If root r_j is repeated p times, then $y_h(t)$ contains

$$e^{r_j t}(c_0 + c_1 t + \cdots + c_{p-1} t^{p-1})$$

If $r_j = \alpha \pm j\beta$, then $y_h(t)$ contains

$$e^{\alpha t}\left(c_1 \cos \beta t + c_2 \sin \beta t\right)$$

If $r = \alpha \pm j\beta$ with multiplicity p, then $y_h(t)$ contains

$$e^{\alpha t}[(c_0 + c_1 t + \cdots + c_{p-1} t^{p-1}) \cos \beta t + (d_0 + d_1 t + \cdots + d_{p-1} t^{p-1}) \sin \beta t]$$

$x(t)$	$y_p(t)$
1	c
t^m	$c_m t^m + c_{m-1} t^{m-1} + \cdots + c_1 t + c_0$
e^{at}	ce^{at}
$\cos(\omega t + \varphi)$	$c\cos(\omega t + \phi) = c_1 \cos(\omega t) + c_2 \sin(\omega t)$

When $x(t)$ is of the same form as one of the components of $y_h(t)$, we must assume a $y_p(t)$ that is independent of all terms in $y_h(t)$ – we multiply the form of the particular solution by the lowest power of t that will give an component not included in $y_h(t)$:

$x(t)$	$y_p(t)$
$e^{r_j t}$	$cte^{r_j t}$, r_j is single
$e^{r_j t}$	$ct^p e^{r_j t}$, r_j is a root with multiplicity p

Example Solve the differential equation

$$\frac{d^2y(t)}{dt^2}+5\frac{dy(t)}{dt}+6y(t)=x(t)$$

$$x(t)=e^{-t}u(t)$$

$$y(t)\Big|_{t=0^-}=1,\ \frac{dy(t)}{dt}\Bigg|_{t=0^-}=2$$

Solution.

(1) Determine the initial conditions at time $t=0^+$

$$y(t)\big|_{t=0^+}=y(t)\big|_{t=0^-}=1,\ \frac{d}{dt}y(t)\bigg|_{t=0^+}=\frac{d}{dt}y(t)\bigg|_{t=0^-}=2$$

(2) The form of the homogeneous solution

$$r^2+5r+6=0$$

$$(r+2)(r+3)=0$$

$$r_1=-2,\ r_2=-3$$

$$y_h(t)=c_1e^{-2t}+c_2e^{-3t}$$

(3) The particular solution

$$x(t)=e^{-t},t>0\Rightarrow y_p(t)=ce^{-t},t>0$$

$$\frac{d}{dt}y_p(t)=-ce^{-t},\ \frac{d^2}{dt^2}y_p(t)=ce^{-t}$$

$$ce^{-t}+5\left(-ce^{-t}\right)+6\left(ce^{-t}\right)=e^{-t}\Rightarrow c=\frac{1}{2}$$

$$y_p(t)=\frac{1}{2}e^{-t},t>0$$

(4) The form of the complete solution

$$y(t)=y_h(t)+y_p(t)=c_1e^{-2t}+c_2e^{-3t}+\frac{1}{2}e^{-t},t>0$$

(5) The coefficients

$$\frac{d}{dt}y(t)=-2c_1e^{-2t}-3c_2e^{-3t}-\frac{1}{2}e^{-t},t>0$$

$$c_1+c_2+\frac{1}{2}=y(t)\Big|_{t=0^+}=1$$

$$-2c_1-3c_2-\frac{1}{2}=\frac{d}{dt}y(t)\bigg|_{t=0^+}=2$$

$$c_1=4,\ c_2=-\frac{7}{2}$$

(6) The complete solution

$$y(t)=4e^{-2t}-\frac{7}{2}e^{-3t}+\frac{1}{2}e^{-t},t>0$$

Difference Equations

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

The procedure is as follows:

- Find the form of homogeneous solution from the roots of the characteristic equation
- Find a particular solution by assuming that it is of the same form as the input, yet is independent of all terms in the homogeneous solution
- Determine the coefficients in the homogeneous solution so that the complete solution satisfies the initial conditions

The solution $y[n]$ consists of two parts -- a particular solution plus a homogeneous solution.

$$y[n] = y_p[n] + y_h[n]$$

The homogeneous solution $y_h[n]$ is the solution of the homogeneous equation

$$\sum_{k=0}^{N} a_k y_h[n-k] = 0$$

The homogeneous solution is of the form

$$y_h[n] = \sum_{i=1}^{N} c_i r_i^n$$

The particular solution represents any solution of the difference equation for the given input. A particular solution is usually obtained by assuming an output of the same general form as the input:

$x[n]$	$y_p[n]$
1	c
n^m	$c_m n^m + c_{m-1} n^{m-1} + \cdots + c_1 n + c_0$
α^n	$c\alpha^n$
$\cos(\Omega n + \varphi)$	$c\cos(\Omega n + \phi) = c_1 \cos(\Omega n) + c_2 \sin(\Omega n)$

When the x[n] is of the same form as one of the components of $y_h[n]$:

$x[n] = r_j^n \Rightarrow y_p[n] = cnr_j^n$, r_j single

$x[n] = r_j^n \Rightarrow y_p[n] = cn^p r_j^n$, r_j repeated p times

Example Solve the difference equation

$$y[n]-\frac{1}{4}y[n-1]=x[n]$$

$$x[n]=\left(\frac{1}{2}\right)^n u[n]$$

$$y[-1]=8$$

Solution.

(1) Determine the initial condition at time $n=0$

$$y[0]=\frac{1}{4}y[-1]+x[0]=\frac{1}{4}\times 8+1=3$$

(2) The form of the homogeneous solution

$$r-\frac{1}{4}=0\Rightarrow r=\frac{1}{4}\Rightarrow y_h[n]=c_1\left(\frac{1}{4}\right)^n$$

(3) The particular solution

$$x[n]=\left(\frac{1}{2}\right)^n, n\geq 0\Rightarrow y_p[n]=c\left(\frac{1}{2}\right)^n, n\geq 0$$

$$c\left(\frac{1}{2}\right)^n-\frac{1}{4}c\left(\frac{1}{2}\right)^{n-1}=\left(\frac{1}{2}\right)^n\Rightarrow c=2$$

$$y_p[n]=2\left(\frac{1}{2}\right)^n, n\geq 0$$

(4) The form of the complete solution

$$y[n]=y_h[n]+y_p[n]=c_1\left(\frac{1}{4}\right)^n+2\left(\frac{1}{2}\right)^n, n\geq 0$$

(4) The coefficient

$$y[n]=c_1\left(\frac{1}{4}\right)^n+2\left(\frac{1}{2}\right)^n, n\geq 0$$

$$y[0]=c_1\left(\frac{1}{4}\right)^0+2\left(\frac{1}{2}\right)^0=c_1+2=3\Rightarrow c_1=1$$

(5) The complete solution

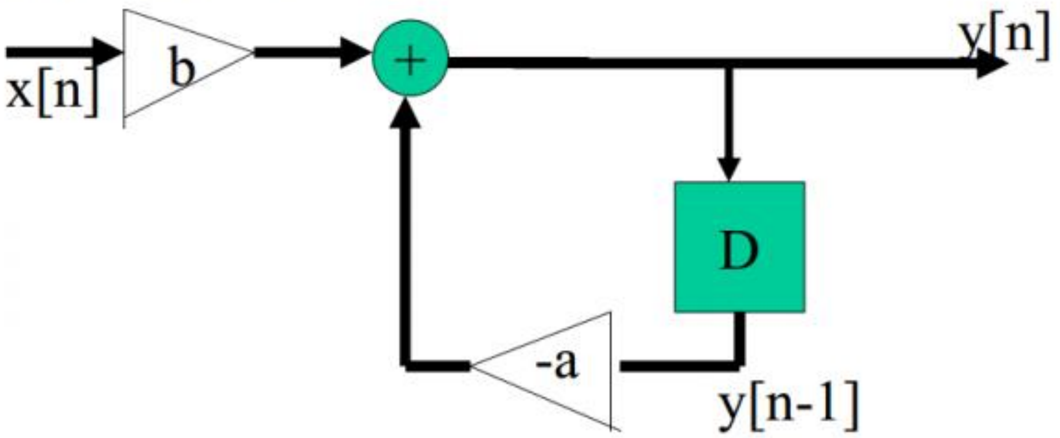
$$y[n]=\left(\frac{1}{4}\right)^n+2\left(\frac{1}{2}\right)^n, n\geq 0$$

Block Diagram Representations of First- Order Systems

First-Order Recursive Discrete-time System.

y[n]+ay[n-1]=bx[n]

y[n]=-ay[n-1]+bx[n]



First-Order Continuous-time System Described By Differential Equation

$$\frac{dy(t)}{dt} + ay(t) = bx(t)$$

$$\text{rewriting } y(t) = -\frac{1}{a}\frac{dy(t)}{dt} + \frac{b}{a}x(t)$$