Introduction to Signals and Systems: V216

Lecture #7

Chapter 3: Fourier Series Representation of Periodic Signals

Convergence of Fourier Series

- Can we get Fourier Series representation for all periodic signals.
- I.e. are the coefficients from eqn 3.39 finite or in other words the integrals do not diverge.
- If the coefficients are finite and substituted in the synthesis eqn, will the resulting infinite series converge or not to the original signal x(t).

Convergence of Fourier Series

Approximating the given periodic signal x(t) by a linear combination of a finite number of harmonically related complex exponentials : - that is, by a finite series of the form : -

$$x_N(t) = \sum_{k=-N}^{N} a_k e^{jk\omega_0 t}$$

Convergence of Fourier Series

Let $e_N(t)$ denote the approximation error.

that is $e_N(t) = x(t) - x_N(t) = x(t) - \sum_{k=-N}^{N} a_k e^{jk\omega_0 t}$

We are going to use the energy in the error over one period.

$$E = \int_T | e_N(t) |^2 \, dt.$$

The objective is to minimise the energy in the error.

It can be shown that for this to be so, the particular choice for the coefficient is :-

$a_k = \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt.$ which is our Fourier series coefficient.

Convergence of Fourier Series

If x(t) has a Fourier Series representation, the best approximation is obtained by truncating the Fourier series to the desired term

For a periodic signal to be representable by the Fourier series the signal must have finite energy over a single period : -

$$\int_T |x(t)|^2 < \infty.$$

Dirichlet Conditions

1) Over any period, x(t) must be absolutely integrable,

i.e. $\int_T |x(t)|\,dt < \infty$.

2) In any finite interval of time, x(t) is bounded variation;

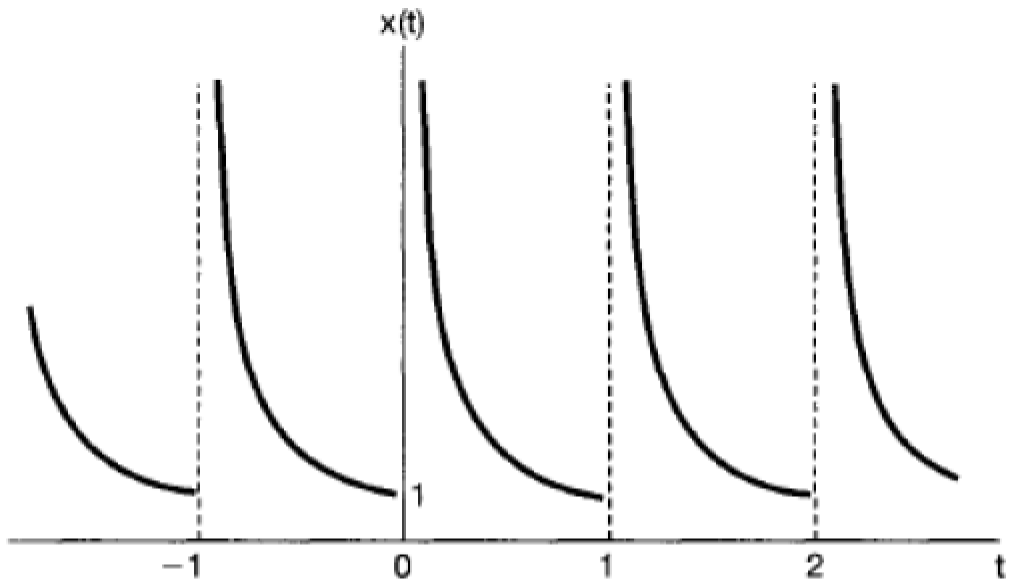
that is , there are no more than a finite number of maxima and minima during any single period of the signal. The function should not have an infinite number of maxima and minima in the interval.

3) In any finite interval of time, there are only a finite number of discontinuities. Furthermore, each of these discontinuities is finite. Figure 3.8 page 199 shows the examples of these 3 conditions being violated, so the signal cannot be representable by the Fourier Series.

Dirichlet Conditions

An Example That Violates Condition 1

$$x(t) = \sum_{n=-\infty}^{\infty} w(t-n),\ w(t) = \frac{1}{t}, \quad 0 < t \le 1$$

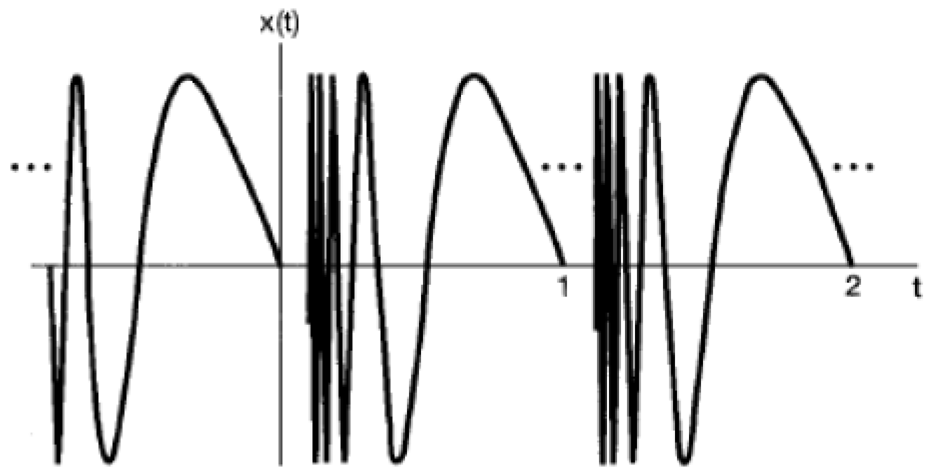


(a)

Dirichlet Conditions

An Example That Violates Condition 2

$$x(t)=\sum_{n=-\infty}^{\infty} w(t-n), \quad w(t)=\sin\left(\frac{2\pi}{t}\right), \quad 0<t\le 1$$



(b)

Dirichlet Conditions

An Example That Violates Condition 3 The signal, of period 8, is composed of an infinite number of sections, each of which is half the height and half the width of the previous section.

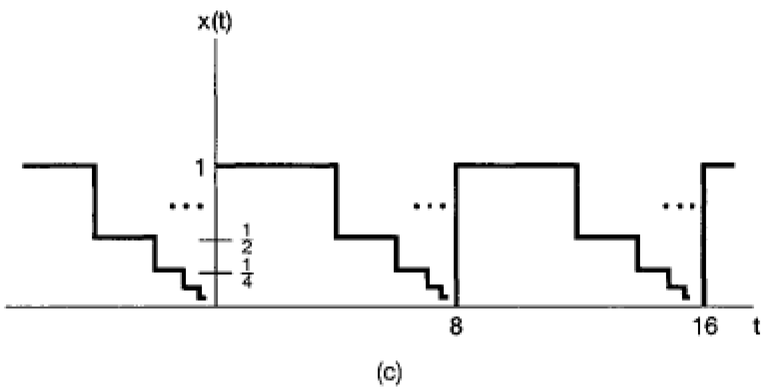
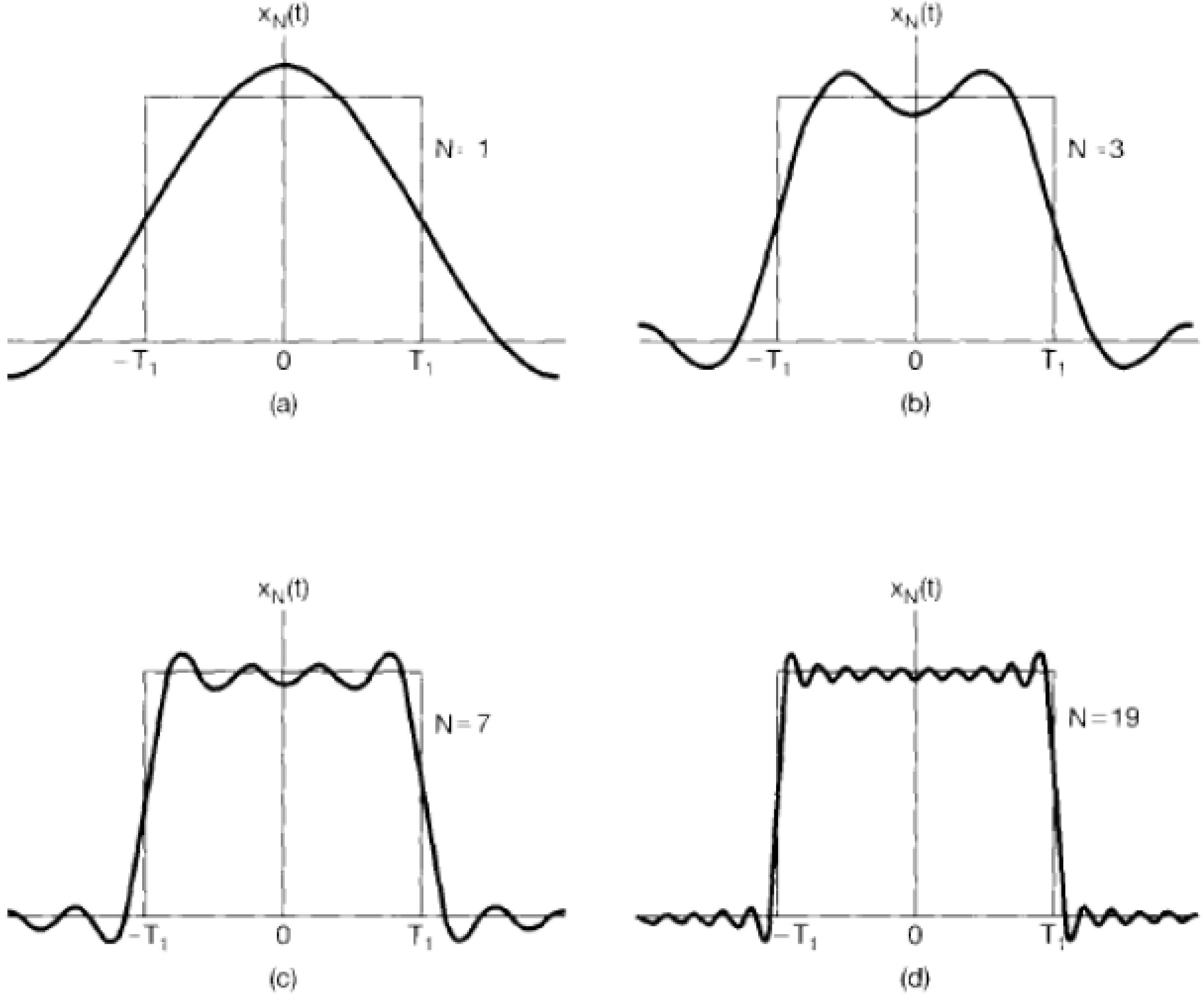


Figure 3.8 Signals that violate the Dirichlet conditions: (a) the signal $x(t) = 1/t$ for $0 < t \leq 1$, a periodic signal with period 1 (this signal violates the first Dirichlet condition); (b) the periodic signal of eq. (3.57), which violates the second Dirichlet condition; (c) a signal periodic with period 8 that violates the third Dirichlet condition [for $0 \leq t < 8$, the value of $x(t)$ decreases by a factor of 2 whenever the distance from t to 8 decreases by a factor of 2; that is, $x(t) = 1$, $0 \leq t < 4$, $x(t) = 1/2$, $4 \leq t < 6$, $x(t) = 1/4$, $6 \leq t < 7$, $x(t) = 1/8$, $7 \leq t < 7.5$, etc.].

Gibbs Phenomenon

- The partial sum in the vicinity of the discontinuity exhibits ripples.
- The peak amplitudes of these ripples does not seem to decrease with increasing N.
- As N increases, the ripples in the partial sums become compressed towards the discontinuity, but for any finite value of N, the peak amplitude of the ripples remains constant. Fig 3.9 pg 201 illustrates this phenomenon.

Gibbs Phenomenon



Properties of Continuous-time Fourier Series.

(1) Linearity : -

Let $x_1(t)$ and $x_2(t)$ denotes two periodic signals with the same period T.

$$x_1(t) \Leftrightarrow a_k$$

$$x_2(t) \Leftrightarrow b_k,$$

where $\Leftrightarrow$ denotes the Fourier Series represenation.

Any linear combination of the two signals will also be periodic with period T.

$$z(t) = Ax_1(t) + Bx_2(t).$$

$$z(t) \Leftrightarrow c_k,$$

Then $c_k = Aa_k + Bb_k$

Properties of Continuous-time Fourier Series.

(2) Time Shifting : -

When a time shift is applied to a periodic signal x(t), the period T of the signal is preserved.

$$\mathrm{y(t) = x(t - t_0).}$$

$$\mathrm{x(t)} \Leftrightarrow \mathrm{a_k},$$

$$\mathrm{y(t)} \Leftrightarrow \mathrm{b_{k,}}$$

$$\mathrm{b_k} = \frac{1}{\mathrm{T}} \int_T \mathrm{x(t - t_0)} e^{-jk\omega_0 t} dt.$$

Letting $\tau = \mathrm{t - t_0}$, and τ will also range over an interval of duration T, we have : -

$$\frac{1}{\mathrm{T}} \int_T x(\tau) e^{-jk\omega_0(\tau + t_0)} d\tau = \mathrm{e}^{-\mathrm{jk}\omega_0 t_0} \frac{1}{\mathrm{T}} \int_T x(\tau) e^{-jk\omega_0 \tau} d\tau$$

$$= \mathrm{e}^{-\mathrm{jk}\omega_0 t_0} a_k = e^{-jk\frac{2\pi}{T} t_0} a_k,$$

Then $\mathrm{y(t) = x(t - t_0)} \Leftrightarrow \mathrm{e}^{-\mathrm{jk}\omega_0 t_0} \mathrm{a_k} = \mathrm{e}^{-\mathrm{jk}\frac{2\pi}{\mathrm{T}} t_0} \mathrm{a_k}$

Properties of Continuous-time Fourier Series.

(3) Time Reversal : -

When a time reversal is applied to a periodic signal x(t), the period T of the signal is preserved.

$$y(t) = x(-t).$$

$$x(t) \Leftrightarrow a_k,$$

$$y(t) \Leftrightarrow b_{k,}$$

$$x(-t) = \sum_{k=-\infty}^{\infty} a_k e^{-jk2\frac{\pi}{T}t}$$

Substituting k = -m, we have : -

$$y(t) = x(-t) = \sum_{m=-\infty}^{\infty} a_{-m} e^{jm2\frac{\pi}{T}t}$$

$$\therefore b_k = a_{-k}.$$

That is, if $x(t) \Leftrightarrow a_k,$

$$x(-t) \Leftrightarrow a_{-k}.$$

Properties of Continuous-time Fourier Series.

(4) Time Scaling : -

The operation of time scaling changes the period of the signal.

If x(t) is periodic with T being the period and fundamental frequency

$$\omega_0 = \frac{2\pi}{T},$$

then x(α t) is periodic with period $\frac{\mathrm{T}}{\alpha}$,

and fundamental frequency $\alpha\, \omega_0$ for positive α .

The Fourier coefficients remain the same.

$$\mathrm{x}(\alpha\ \mathrm{t}) = \sum_{\mathrm{k}=-\infty}^{\infty} \mathrm{a}_{\mathrm{k}} e^{jk(\alpha\omega_0)t}$$

Properties of Continuous-time Fourier Series.

(5) Multiplication : -

$x(t) \Leftrightarrow a_k$,

$y(t) \Leftrightarrow b_{k,}$

$$x(t)y(t) \Leftrightarrow h_k = \sum_{l=-\infty}^{\infty} a_l b_{k-l}.$$

Multiplication in the time domain is equivalent to convolution in the frequency domain.

Properties of Continuous-time Fourier Series.

(6) Conjugation and conjugate symmetry : -

$\mathrm{x(t)} \Leftrightarrow \mathrm{a}_{\mathrm{k}}$,

$then\ x^{*}(t) \Leftrightarrow a^{*}_{-k}$

$If\ \ \mathrm{x(t)\ is\ real,\ x(t) = x}^{*}(t).$

The Fourier series coefficients will be conjugate symmetry : -

$\mathrm{a}_{\text{-k}} = a^{*}_{k}$

Properties of Continuous-time Fourier Series.

(7) Parseval's Relation : -

$$\frac{1}{T}\int_T |\, x(t)\,|^2 \, dt = \sum_{k=-\infty}^{\infty} |\, a_k \,|^2 \, .$$

The total average power in a periodic signal equals the sum of the average powers in all of its harmonic components.

Properties of Continuous-time Fourier Series

Property	Section	Periodic Signal	Fourier Series Coefficients
		$\left.\begin{array}{l} x(t) \\ y(t) \end{array}\right\}$ Periodic with period T and fundamental frequency $\omega_0 = 2\pi/T$	a_k b_k
Linearity	3.5.1	$Ax(t) + By(t)$	$Aa_k + Bb_k$
Time Shifting	3.5.2	$x(t - t_0)$	$a_k e^{-jk\omega_0 t_0} = a_k e^{-jk(2\pi/T)t_0}$
Frequency Shifting		$e^{jM\omega_0 t} = e^{jM(2\pi/T)t} x(t)$	a_{k-M}
Conjugation	3.5.6	$x^*(t)$	a^*_{-k}
Time Reversal	3.5.3	$x(-t)$	a_{-k}
Time Scaling	3.5.4	$x(\alpha t), \alpha > 0$ (periodic with period T/α)	a_k
Periodic Convolution		$\int_T x(\tau)y(t-\tau)d\tau$	$Ta_k b_k$
Multiplication	3.5.5	$x(t)y(t)$	$\sum_{l=-\infty}^{+\infty} a_l b_{k-l}$
Differentiation		$\frac{dx(t)}{dt}$	$jk\omega_0 a_k = jk\frac{2\pi}{T}a_k$
Integration		$\int_{-\infty}^{t} x(t)\,dt$ (finite valued and periodic only if $a_0 = 0$)	$\left(\frac{1}{jk\omega_0}\right)a_k = \left(\frac{1}{jk(2\pi/T)}\right)a_k$
Conjugate Symmetry for Real Signals	3.5.6	$x(t)$ real	$\begin{cases} a_k = a^*_{-k} \\ \mathcal{Re}\{a_k\} = \mathcal{Re}\{a_{-k}\} \\ \mathcal{Im}\{a_k\} = -\mathcal{Im}\{a_{-k}\} \\ \lvert a_k \rvert = \lvert a_{-k} \rvert \\ \measuredangle a_k = -\measuredangle a_{-k} \end{cases}$
Real and Even Signals	3.5.6	$x(t)$ real and even	a_k real and even
Real and Odd Signals	3.5.6	$x(t)$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals		$\begin{cases} x_e(t) = \mathcal{E}v\{x(t)\} \quad [x(t) \text{ real}] \\ x_o(t) = \mathcal{O}d\{x(t)\} \quad [x(t) \text{ real}] \end{cases}$	$\mathcal{Re}\{a_k\}$ $j\mathcal{Im}\{a_k\}$

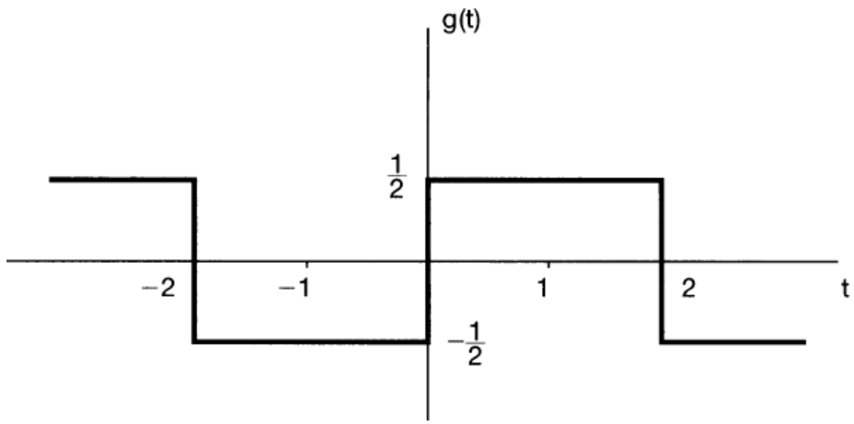
Parseval's Relation for Periodic Signals

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{+\infty} |a_k|^2$$

Example 3.6

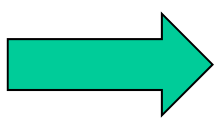
Consider the signal $g(t)$ with a fundamental period of 4, shown in Figure 3.10. We could determine the Fourier series representation of $g(t)$ directly from the analysis equation (3.39). Instead, we will use the relationship of $g(t)$ to the symmetric periodic square wave $x(t)$ in Example 3.5. Referring to that example, we see that, with $T = 4$ and $T_1 = 1$,

$$g(t) = x(t-1) - 1/2. \tag{3.69}$$



$$a_k = \frac{\sin(\pi k/2)}{k\pi}, \quad k \neq 0,$$

x(t)



$$a_0 = \frac{1}{2}.$$

The time-shift property in Table 3.1 indicates that, if the Fourier Series coefficients of $x(t)$ are denoted by a_k, the Fourier coefficients of $x(t-1)$ may be expressed as

$$b_k = a_k e^{-jk\pi/2}. \tag{3.70}$$

The Fourier coefficients of the *dc offset* in $g(t)$—i.e., the term $-1/2$ on the right-hand side of eq. (3.69)—are given by

$$c_k = \begin{cases} 0, & \text{for } k \neq 0 \\ -\frac{1}{2}, & \text{for } k = 0 \end{cases}. \tag{3.71}$$

Applying the linearity property in Table 3.1, we conclude that the coefficients for $g(t)$ may be expressed as

$$d_k = \begin{cases} a_k e^{-jk\pi/2}, & \text{for } k \neq 0 \\ a_0 - \frac{1}{2}, & \text{for } k = 0 \end{cases},$$

where each a_k may now be replaced by the corresponding expression from eqs. (3.45) and (3.46), yielding

$$d_k = \begin{cases} \frac{\sin(\pi k/2)}{k\pi} e^{-jk\pi/2}, & \text{for } k \neq 0 \\ 0, & \text{for } k = 0 \end{cases}. \tag{3.72}$$

Example 3.9

Suppose we are given the following facts about a signal $x(t)$:

1. $x(t)$ is a real signal.
2. $x(t)$ is periodic with period $T = 4$, and it has Fourier series coefficients a_k.
3. $a_k = 0$ for $|k| > 1$.
4. The signal with Fourier coefficients $b_k = e^{-j\pi k/2} a_{-k}$ is odd.
5. $\frac{1}{4}\int_4 |x(t)|^2 dt = 1/2$.

Let us show that this information is sufficient to determine the signal $x(t)$ to within a sign factor. According to Fact 3, $x(t)$ has at most three nonzero Fourier series coefficients a_k: a_0, a_1, and a_{-1}. Then, since $x(t)$ has fundamental frequency $\omega_0 = 2\pi/4 = \pi/2$, it follows that

$$x(t) = a_0 + a_1 e^{j\pi t/2} + a_{-1} e^{-j\pi t/2}.$$

Since $x(t)$ is real (Fact 1), we can use the symmetry properties in Table 3.1 to conclude that a_0 is real and $a_1 = a*_{-1}$. Consequently,

$$x(t) = a_0 + a_1 e^{j\pi t/2} + (a_1 e^{j\pi t/2})^* = a_0 + 2\Re e\{a_1 e^{j\pi t/2}\}. \tag{3.81}$$

Let us now determine the signal corresponding to the Fourier coefficients b_k given in Fact 4. Using the time-reversal property from Table 3.1, we note that a_{-k} corresponds to the signal $x(-t)$. Also, the time-shift property in the table indicates that multiplication of the kth Fourier coefficient by $e^{-jk\pi/2} = e^{-jk\omega_0}$ corresponds to the underlying signal being shifted by 1 to the right (i.e., having t replaced by $t-1$). We conclude that the coefficients b_k correspond to the signal $x(-(t-1)) = x(-t+1)$, which, according to Fact 4, must be odd. Since $x(t)$ is real, $x(-t+1)$ must also be real. From Table 3.1, it then follows that the Fourier coefficients of $x(-t+1)$ must be purely imaginary and odd. Thus, $b_0 = 0$ and $b_{-1} = -b_1$. Since time-reversal and time-shift operations cannot change the average power per period, Fact 5 holds even if $x(t)$ is replaced by $x(-t+1)$. That is,

$$\frac{1}{4}\int_4 |x(-t+1)|^2 dt = 1/2. \tag{3.82}$$

We can now use Parseval's relation to conclude that

$$|b_1|^2 + |b_{-1}|^2 = 1/2. \tag{3.83}$$

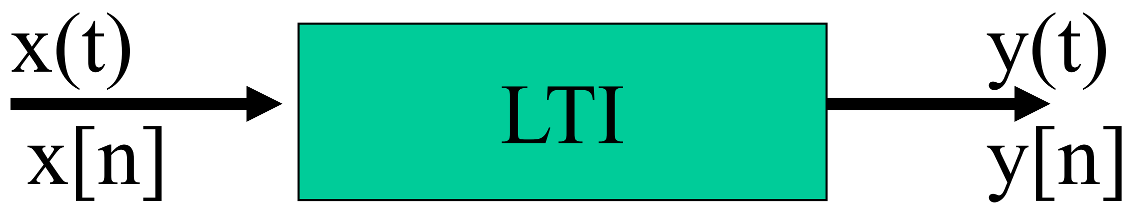
Substituting $b_1 = -b_{-1}$ in this equation, we obtain $|b_1| = 1/2$. Since b_1 is also known to be purely imaginary, it must be either $j/2$ or $-j/2$.

Now we can translate these conditions on b_0 and b_1 into equivalent statements on a_0 and a_1. First, since $b_0 = 0$, Fact 4 implies that $a_0 = 0$. With $k = 1$, this condition implies that $a_1 = e^{-j\pi/2}b_{-1} = -jb_{-1} = jb_1$. Thus, if we take $b_1 = j/2$, then $a_1 = -1/2$, and therefore, from eq. (3.81), $x(t) = -\cos(\pi t/2)$. Alternatively, if we take $b_1 = -j/2$, then $a_1 = 1/2$, and therefore, $x(t) = \cos(\pi t/2)$.

The Discrete-time Fourier Series

- In the previous lectures you have learned about:-
  - Fourier Series Pair Equations for the continuous-time periodic signals.
  - And also their properties.
- The derivation is through using:-
  - Signals as represented by linear combinations of basic signals with the following 2 properties.
  - The set of basic signals can be used to construct a broad and useful class of signals.
  - The response of an LTI system is a combination of the responses to these basic signals at the input.

Parallel Between The Continous-time & The Discrete-time.



Decompose Input as : -

$$x = a_1\phi_1 + a_2\phi_2 + a_3\phi_3 + \ldots\ldots\ldots Fi$$

$$\Downarrow \qquad \Downarrow \qquad \Downarrow \qquad \Downarrow$$

$$\text{Then } y = a_1\varphi + a_2\varphi_2 + a_3\varphi_3 + \ldots\ldots \text{ Sai}$$

Parallel Between The Continous-time & The Discrete-time.

Choose $\phi_k(t)$ or $\phi_{\mathrm{k}}[n]$ so that :-

– Broad Class of Signals can be constructed.

- Response to ϕ's easy to compute.

Continuous - Time :-

$$\phi_{\mathrm{k}}(t) = e^{j\omega_k t}$$

$$e^{j\omega_k t} \Rightarrow e^{j\omega_k t}\int_{-\infty}^{\infty} h(t)e^{-j\omega_k \tau}d\tau$$

$$\Rightarrow e^{j\omega_k t}H(\omega_k)$$

Discrete-time

$$\phi_k[n] = e^{j\Omega_k n}$$

$$e^{j\Omega_k n} \Rightarrow e^{j\Omega_k n} \sum_{r=-\infty}^{\infty} h[r] e^{-j\Omega_k r}$$

$$\Rightarrow e^{j\Omega_k n} H(\Omega_k)$$

Eigen Function x Eigen Value

Linear Combinations of Harmonically Related Complex Exponentials,

A discrete - time signal x[n] is periodic with period N

if x[n] = x[n + N].

The fundemental period is the smallest positive integer N,

$\Omega_0 = \dfrac{2\pi}{N}$ is the fundamental frequency.

E.g. the complex exponential $e^{j(2\pi/N)n}$ is periodic with N.

Set of all discrete - time complex exponential signals with period N is given by : -

$\phi_k[n] = e^{jk\Omega_0 n} = e^{jk(2\pi/N)n}$, k = 0, ±1, ±2, ... ± (N - 1).

Linear Combinations of Harmonically Related Complex Exponentials.

A more general periodic sequence x[n] can be represented by a linear combinations of harmonically related complex exponentials

$\phi_k[n] = e^{jk\Omega_0 n} = e^{jk(2\pi/N)n}$, as follows:-

$$x[n] = \sum_{k} a_k \phi_k[n] = \sum_{k} a_k e^{jk\Omega_0 n} = \sum_{k} a_k e^{jk(2\pi/N)n}$$

where $k = 0,1,2,.....N-1$.

Notation for Discrete-time Fourier Series is given by:-

$$x[n] = \sum_{k=(N)} a_k \phi_k[n] = \sum_{k=(N)} a_k e^{jk\Omega_0 n} = \sum_{k=(N)} a_k e^{jk(2\pi/N)n}$$

where the summation can take on any combinations of any N integer values.

a_k's are the Fourier series coefficients.

Discrete-time Fourier Series

x[n] periodic in time with period N

Fundamental frequency $\Omega_0 = \dfrac{2\pi}{N}$,

$e^{jk\Omega_0 n} = e^{j(k+N)\Omega_0 n}$ is periodic in frequency.

where $e^{\mathrm{jN}\frac{2\pi}{\mathrm{N}}n} = 1$, since n is an integer

$$x[n] = \sum_{k} a_k e^{jk\Omega_0 n}, \ \mathrm{k} = 0,1,2,\ldots\mathrm{N}-1$$

$$\mathrm{x[n]} = \sum_{k=<N>} a_k e^{jk\Omega_0 n}$$

N equations with N unknowns

$$\mathrm{a_k} = \frac{1}{N} \sum_{<N>} x[n] e^{-jk\Omega_0 n}$$

Discrete-time Fourier series pair

Using back the same symbol for frequency

$\Omega_0 = \omega_0$.

$$x[n] = \sum_{k=<N>} a_k e^{jk\omega_0 n} = \sum_{k=<N>} a_k e^{jk(2\pi/N)n},$$

$$a_k = \frac{1}{N} \sum_{n=<N>} x[n] e^{-jk\omega_0 n} = \frac{1}{N} \sum_{n=<N>} x[n] e^{-jk(2\pi/N)n}.$$

$$a_k = a_{k+N}.$$

Continuous-time & Discrete-time Fourier Series

Continuous - time FS : -

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t},$$

$$a_k = \frac{1}{T_0} \int_{T_0} x(t) e^{-jk\omega_0 t} dt.$$

Discrete - time FS : -

Synthesis : -

$$\mathrm{x[n]} = \sum_{k=<N>} a_k e^{jk\omega_0 n}$$

$$\mathrm{a_k} = \frac{1}{N} \sum_{n=<N>} x[n] e^{-jk\omega_0 n}$$

Convergence of Fourier Series

- Continuous time:
  - x(t) square integrable
  - or Dirichlet conditions.

$$\text{Discrete - time}$$

$$\mathrm{x[n]} = \sum_{k=<N>} a_k e^{jk\omega_0 n}$$

$$\hat{x}[n] = \sum_{\text{p terms}} a_k e^{jk\omega_0 n}, \mathrm{p} = \mathrm{N}, \hat{x}[n] = x[n]$$

Fourier Series Representation of Discrete-time Periodic Signals

- Fourier series of continuous-time periodic signals are infinite series.
- Fourier series of discrete-time periodic signals are of finite series in nature.
- So for the fourier series of discrete-time periodic signals, mathematical issues of convergence do not arise.

Example 3.10

Consider the signal $x[n] = \sin\omega_0 n$, $x[n]$ is periodic only if $2\pi/\omega_0$ is an integer or a ratio of integers. For case when $2\pi/\omega_0$ is an integer N i.e. when $\omega_0 = 2\pi/N$, x[n] is periodic with fundamental period N and result is exactly analogous to continuous - time $x(t) = \sin\omega_0 t$. Using the eigen relationship and expandin the signal as a sum of two complex exponentials, $x[n] = \frac{1}{2j}e^{j(2\pi/N)n} - \frac{1}{2j}e^{-j(2\pi/N)n}$

$\therefore$ *From synthesis equation* $a_1 = \frac{1}{2j}$ *and* $a_{-1} = -\frac{1}{2j}$,

Coefficients repeat with N thus $a_{N+1} = 1/2j$ and $a_{N-1} = -1/2j$.

Example 3.11

Consider the signal $\mathrm{x}[\mathrm{n}] = 1 + \sin \omega_0 n + 3\cos \omega_0 n + \cos(2\omega_0 n + \pi / 2)$

when $\omega_0 = 2\pi / N$, x[n] is periodic with fundamental period N.

Using the eigen relationship and expanding

the signal as complex exponentials,

$$\mathrm{x}[\mathrm{n}] = 1 + \frac{1}{2j}[e^{j(2\pi/N)n} - e^{-j(2\pi/N)n}] + \frac{3}{2}[e^{j(2\pi/N)n} + e^{-j(2\pi/N)n}]$$

$$+ \frac{1}{2}[e^{j(4\pi n/N + \pi/2)} + e^{-j(4\pi n/N + \pi/2)}].$$

Collecting terms :-

$$\mathrm{x}[\mathrm{n}] = 1 + (\frac{3}{2} + \frac{1}{2j})e^{j(2\pi/N)n} + (\frac{3}{2} - \frac{1}{2j})e^{-j(2\pi/N)n}$$

$$+ (\frac{1}{2}e^{j\pi/2})e^{j2(2\pi/N)n} + (\frac{1}{2}e^{-j\pi/2})e^{-j2(2\pi/N)n}.$$

Example 3.11

$\therefore$ *From synthesis equation* : –

$a_0 = 1,$

$a_1 = (\frac{3}{2} + \frac{1}{2j})$

$a_{-1} = (\frac{3}{2} - \frac{1}{2j})$

$a_2 = \frac{1}{2} j,$

$a_{-2} = -\frac{1}{2} j,$

with $a_k = 0$ for other valuse of k.

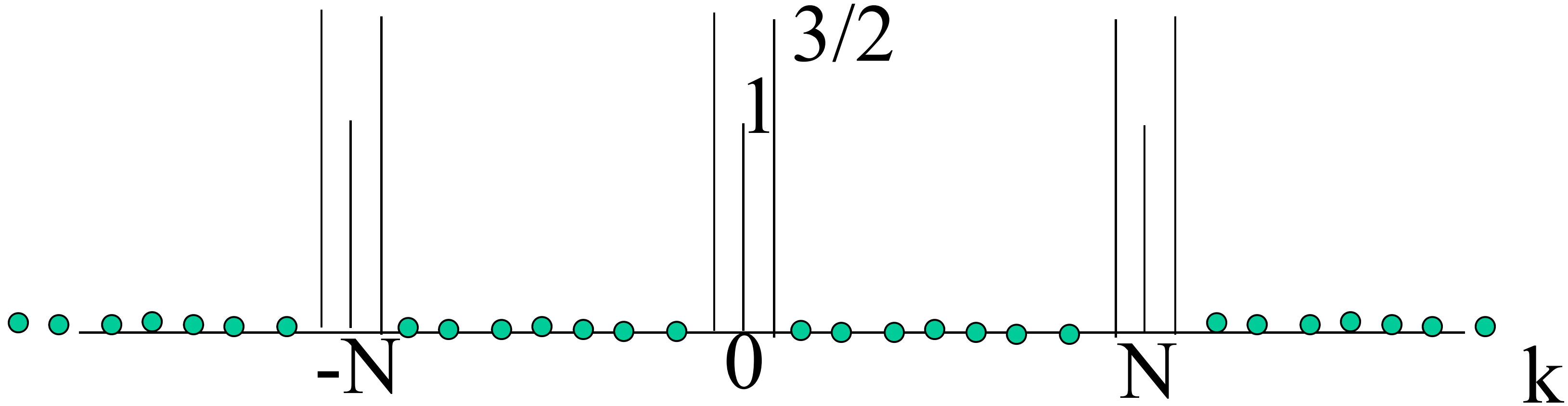
Coefficients repeat with N thus $\mathrm{a}_{\mathrm{N}} = a_0 = 1$, and $\mathrm{a}_{\mathrm{N-1}} = a_{-1} = (\frac{3}{2} - \frac{1}{2j}),$

$\mathrm{a}_{\mathrm{N+1}} = a_1 = (\frac{3}{2} + \frac{1}{2j}), a_{N-2} = a_{-2} = -\frac{1}{2} j,\ a_{N+2} = a_2 = \frac{1}{2} j$

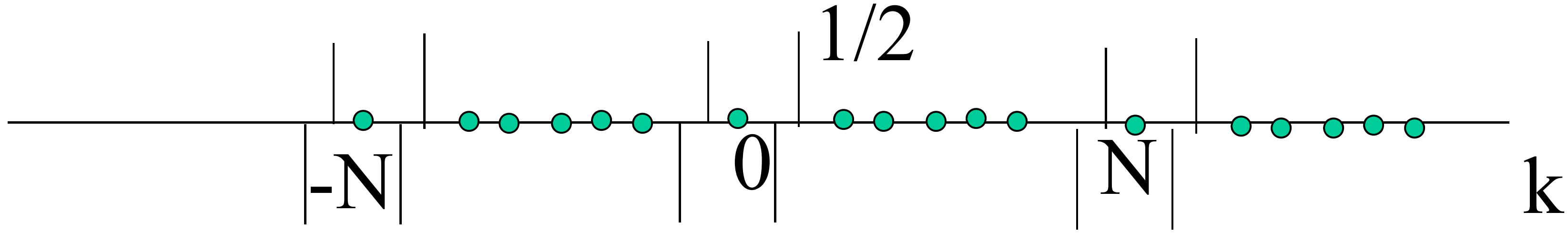
Figure 3.15 in the book shows real/ imaginary or magnitude/ phase of these Fourier coefficients.

Example 3.11

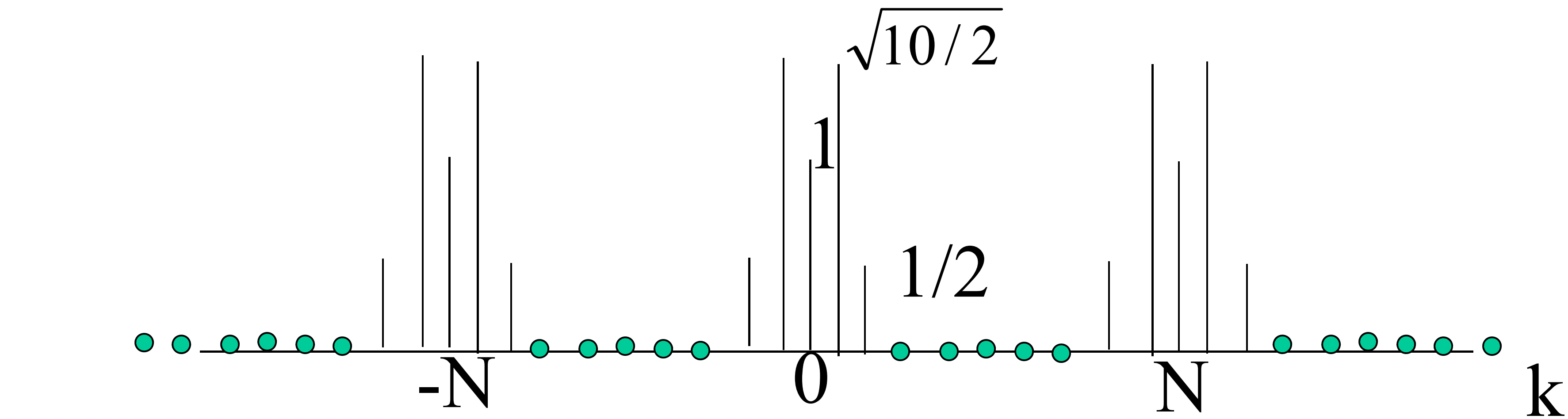
$Re(a_k)$

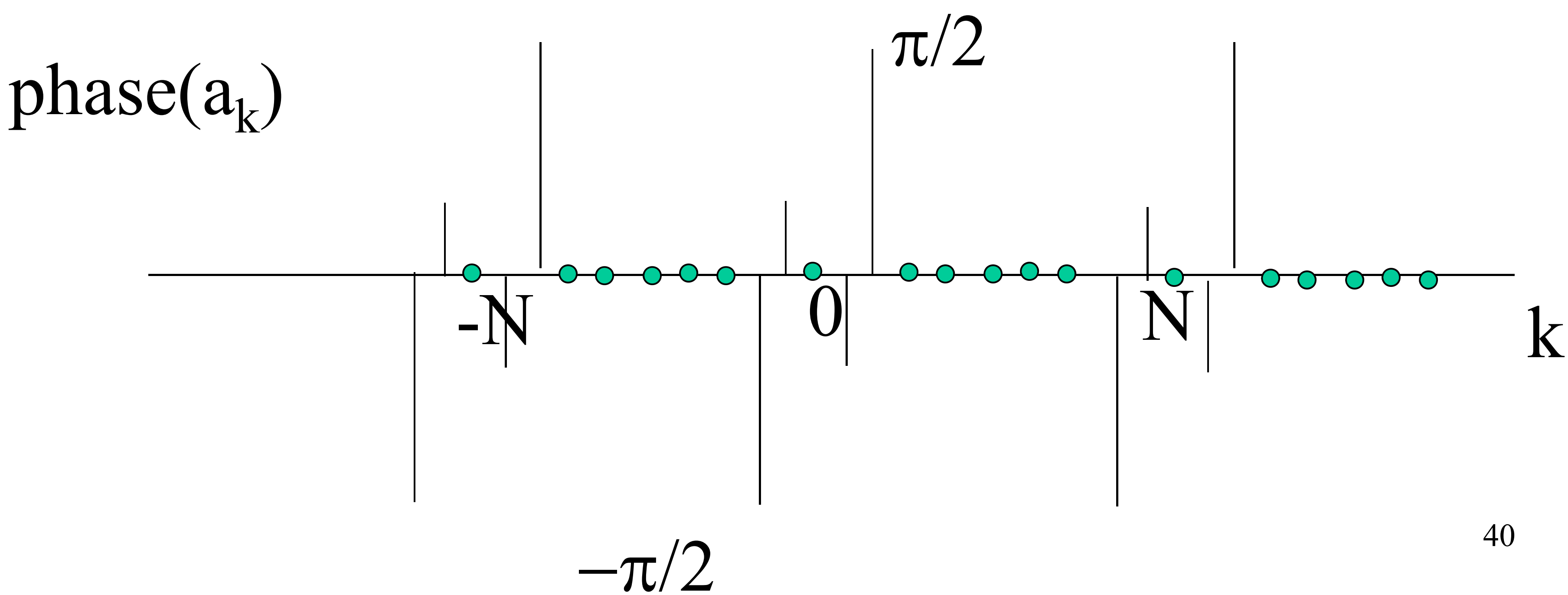


$Im\ (a_k)$

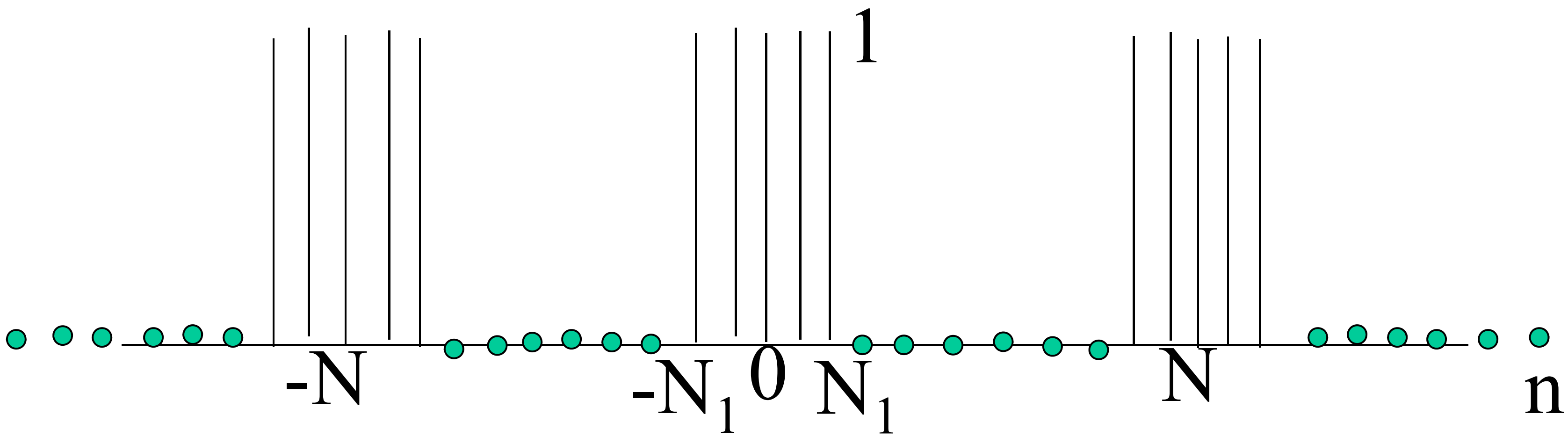


Example 3.11





Example 3.12



From analysis equation :-

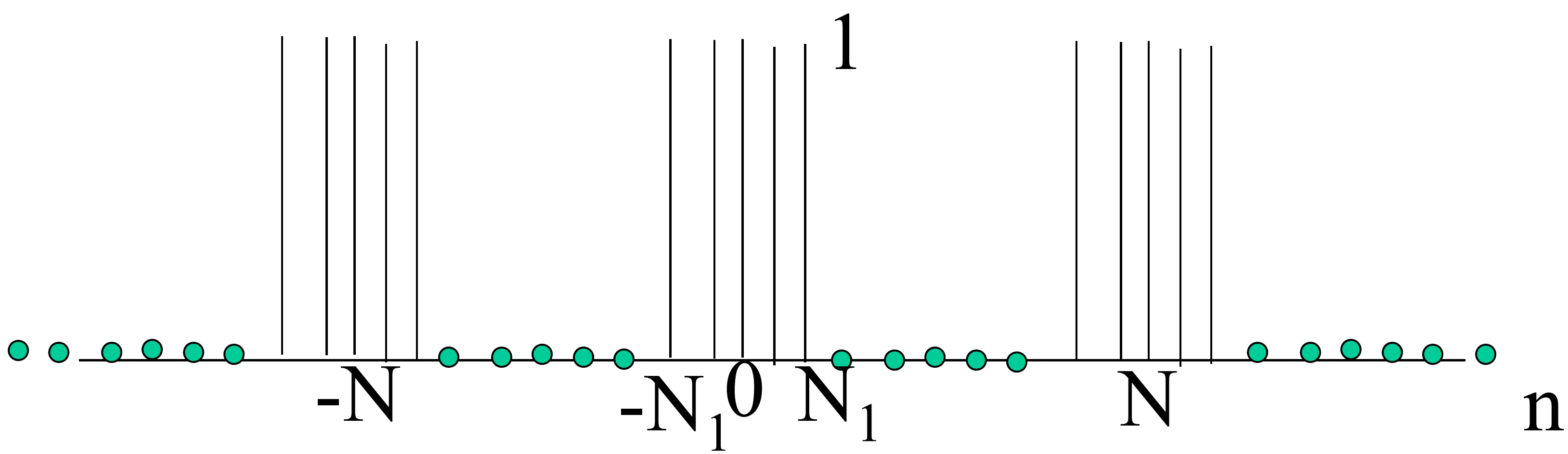
$$a_k = \frac{1}{N}\sum_{n=-N_1}^{n=N_1} e^{-jk(2\pi/N)n}$$

$$Let \quad m = n + N_1.$$

$$a_k = \frac{1}{N}\sum_{m=0}^{2N_1} e^{-jk(2\pi/N)(m-N_1)}.$$

$$a_k = \frac{1}{N} e^{jk(2\pi/N)N_1} \sum_{m=0}^{2N_1} e^{jk(2\pi/N)m}.$$

Example 3.12



$$a_k = \frac{1}{N}\frac{\sin[2\pi k(N_1+1/2)/N]}{\sin(\pi k/N)}, \quad \mathrm{k} \neq 0, \pm \mathrm{N}, \pm 2N, \ldots$$

and

$$a_k = \frac{2N_1+1}{N}, \quad \mathrm{k} = 0, \pm\mathrm{N}, \pm 2\mathrm{N}, \ldots.$$

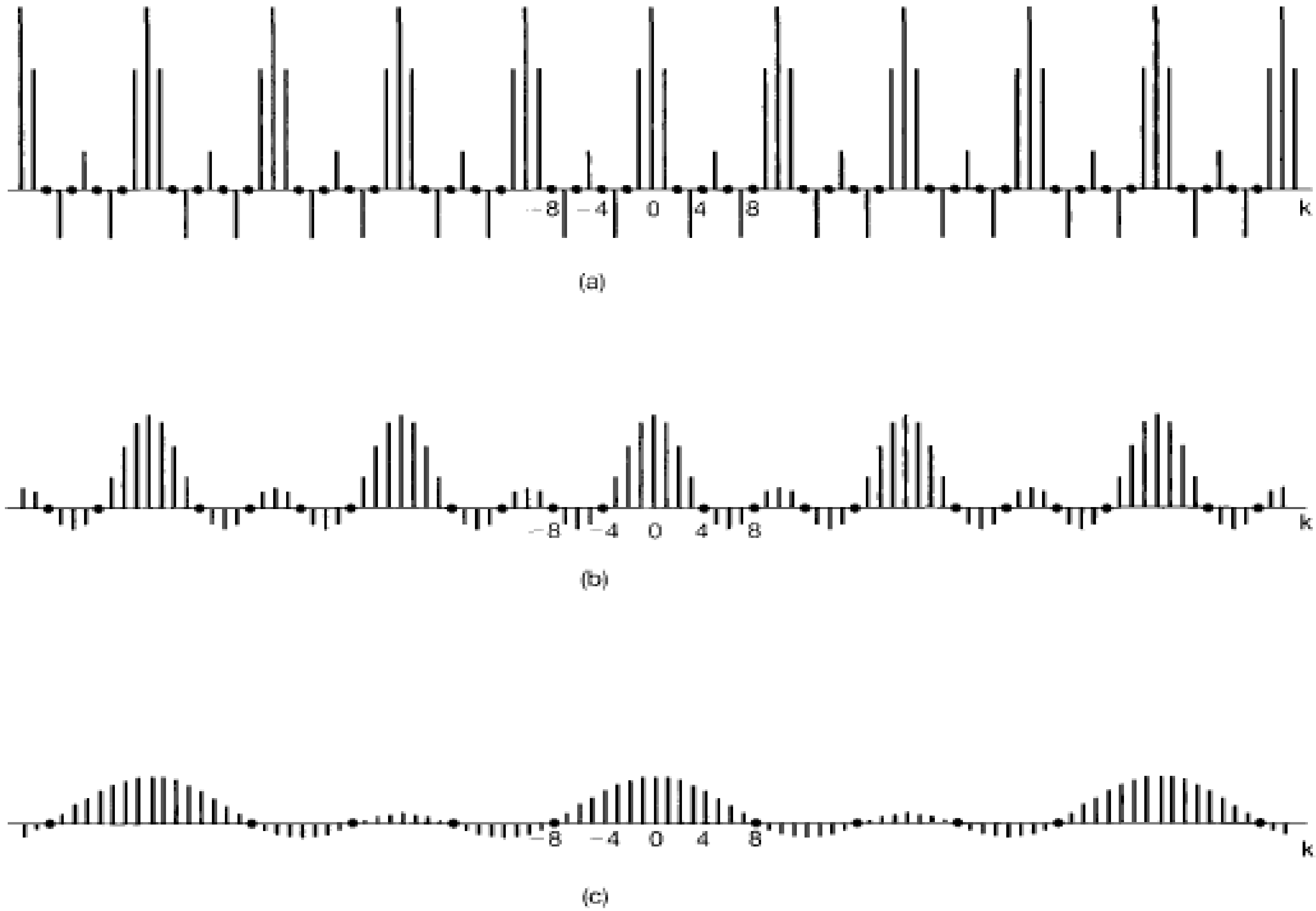
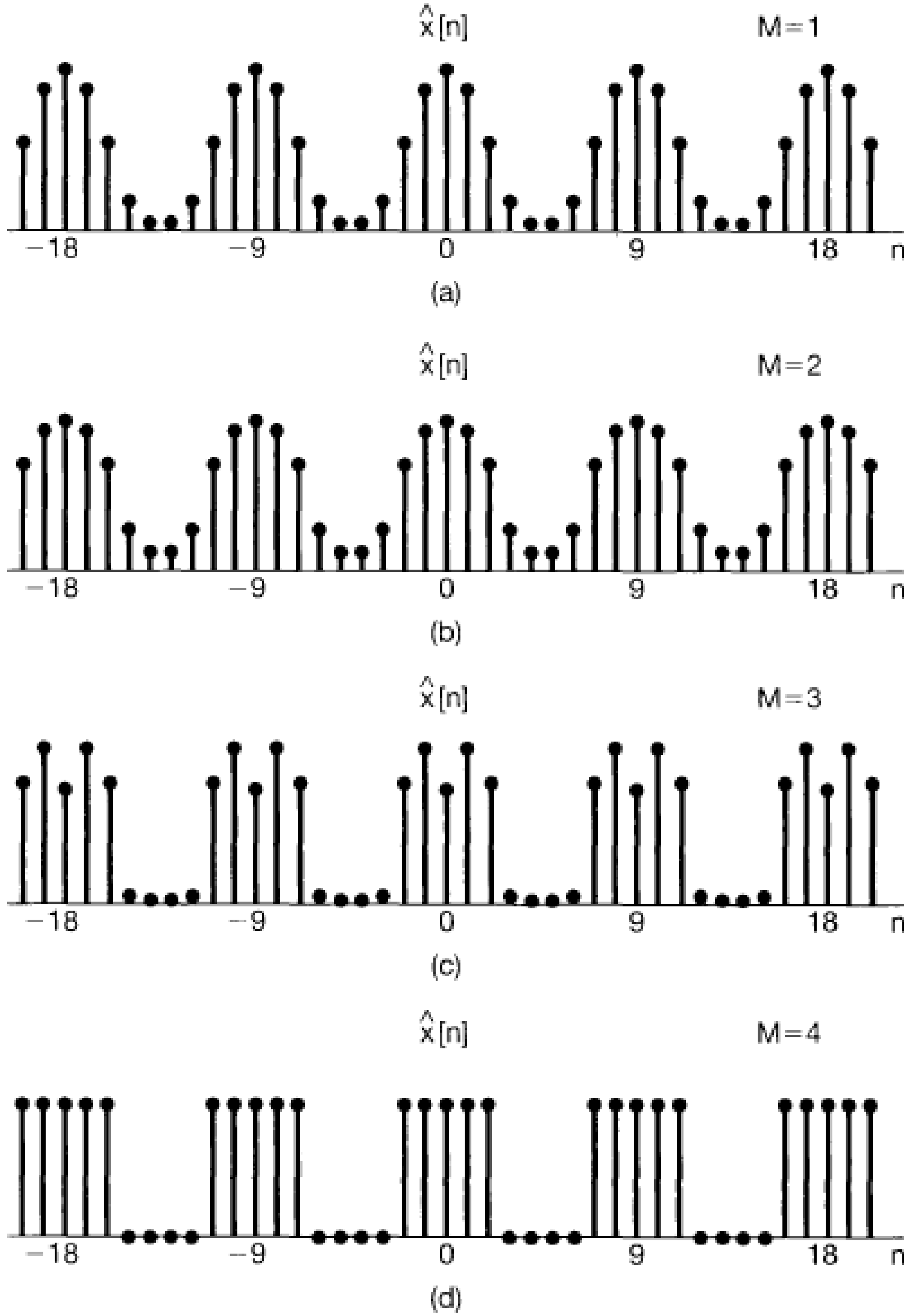


Figure 3.17 Fourier series coefficients for the periodic square wave of Example 3.12; plots of Na_k for $2N_1 + 1 = 5$ and (a) $N = 10$; (b) $N = 20$; and (c) $N = 40$.



$$\hat{x}[n] = \sum_{k=-M}^{M} a_k e^{jk(2\pi/N)n}$$

Figure 3.18 Partial sums of eqs. (3.106) and (3.107) for the periodic square wave of Figure 3.16 with $N = 9$ and $2N_1 + 1 = 5$: (a) $M = 1$; (b) $M = 2$; (c) $M = 3$; (d) $M = 4$.