Introduction to Signals and Systems: V216

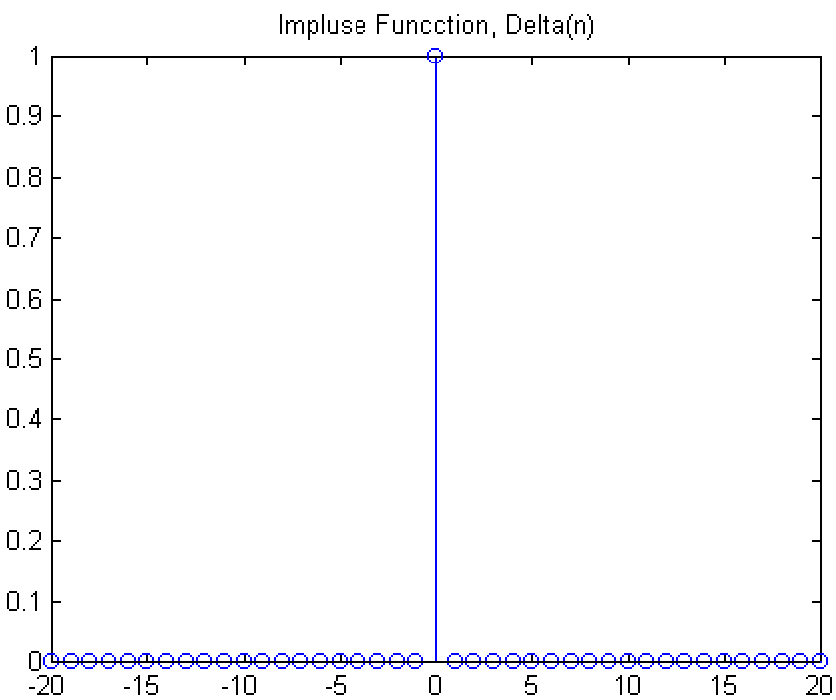
Lecture #2

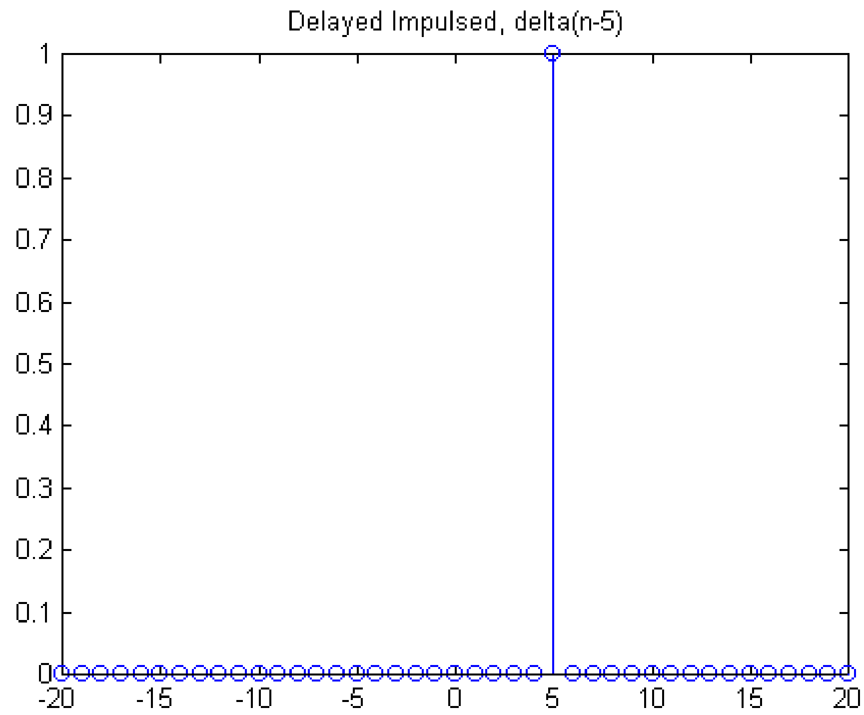
Chapter 1: Signals and Systems

Transformation of Independent Variable (time axes)

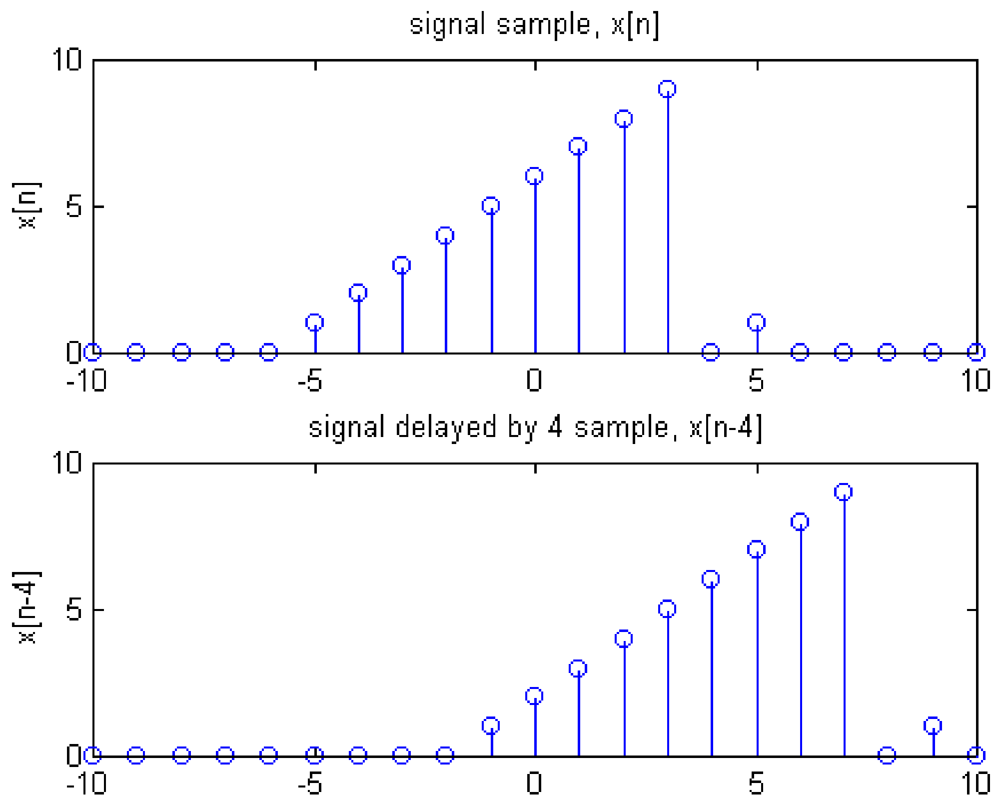
- Introducing several basic properties of signals & systems through elementary transformations.
- Examples of elementary transformation:-
  - time shift, x(t-t0), x[n-n0]
  - time reversal, x(-t), x[-n].
  - time scaling, x(0.5t), x[2n].
  - and combinations of these. x(at+b), x[an-b], where a & b are signed constants.

Delayed Impulse

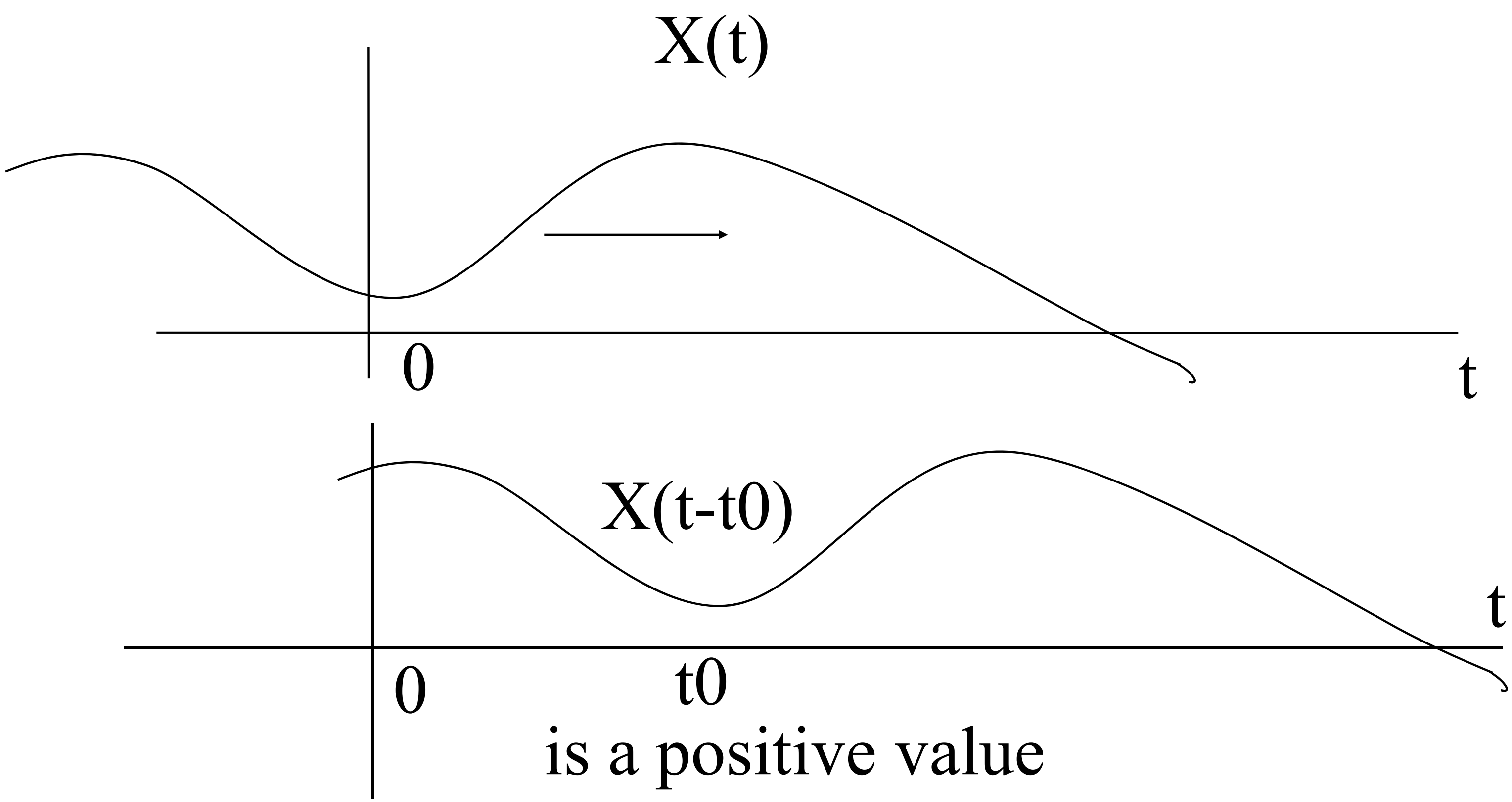




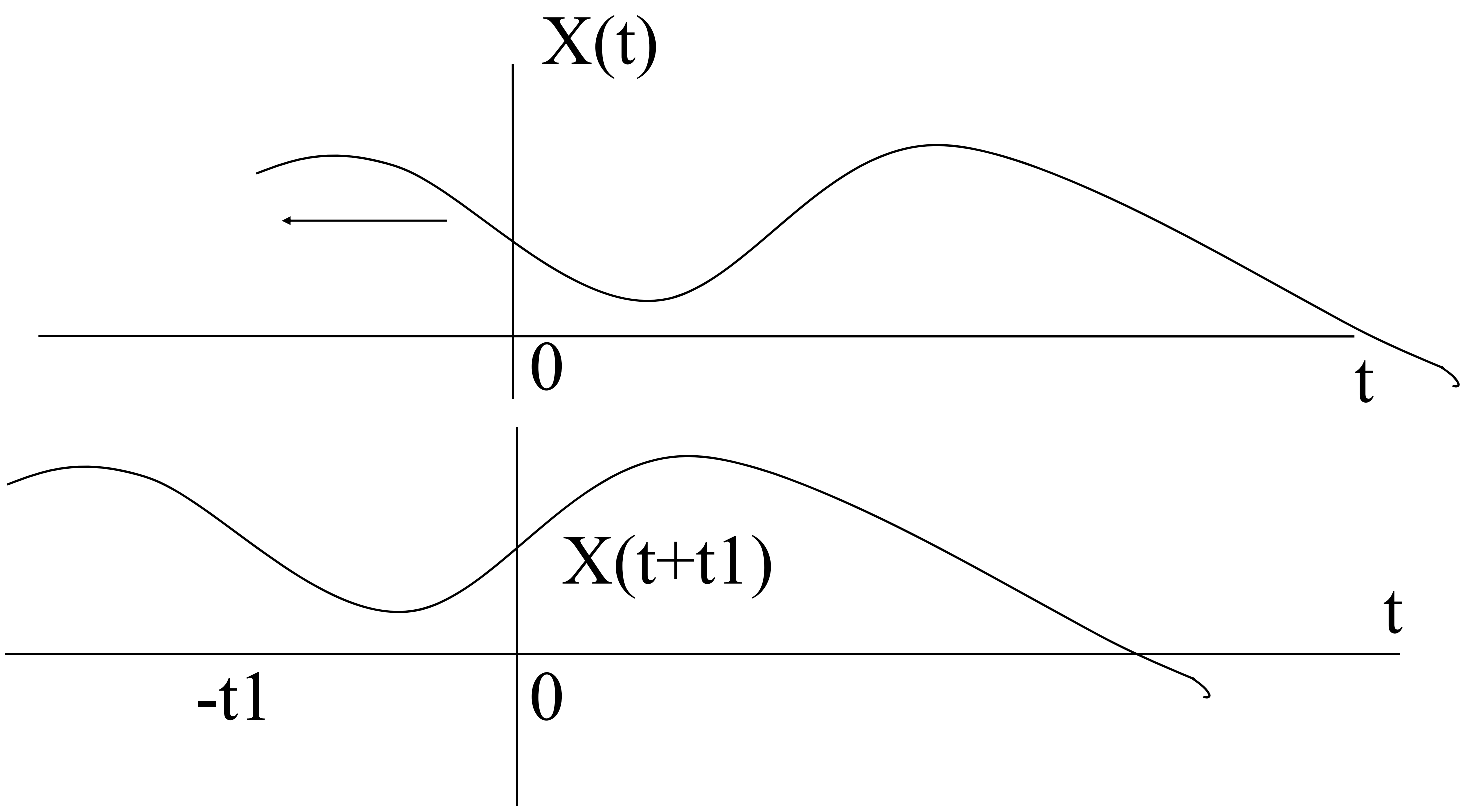
Example of Delayed Signal



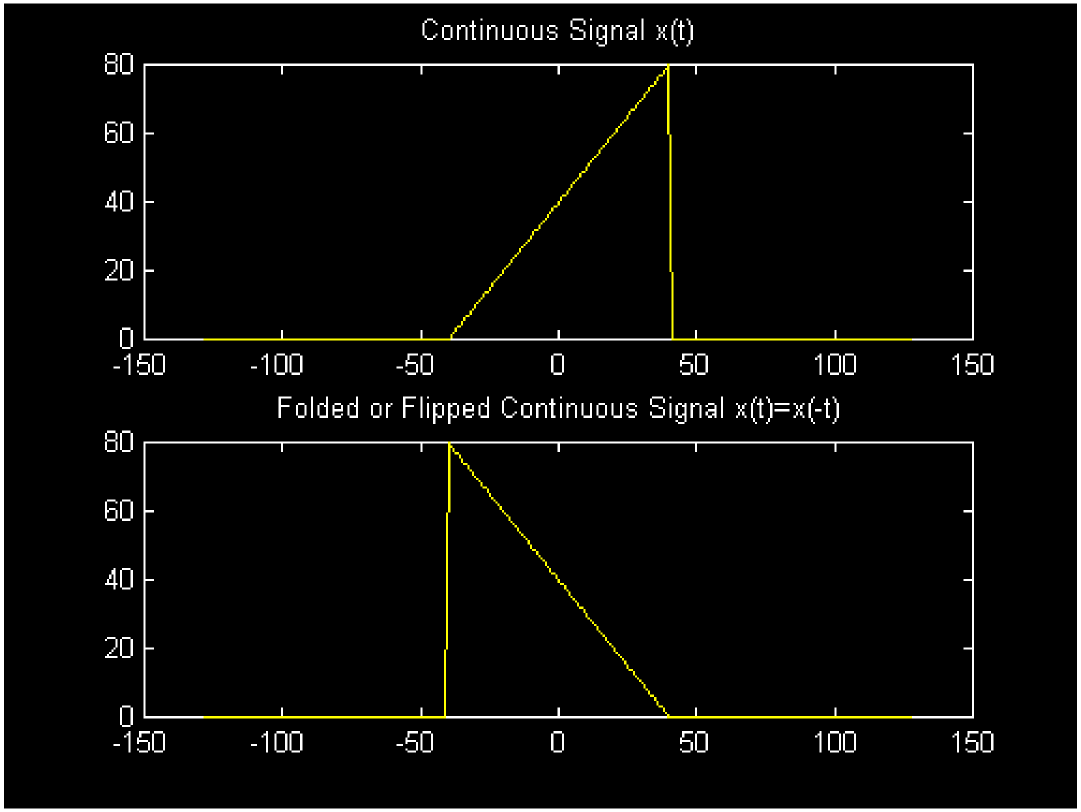
Shifting right or lagging signal x(t)



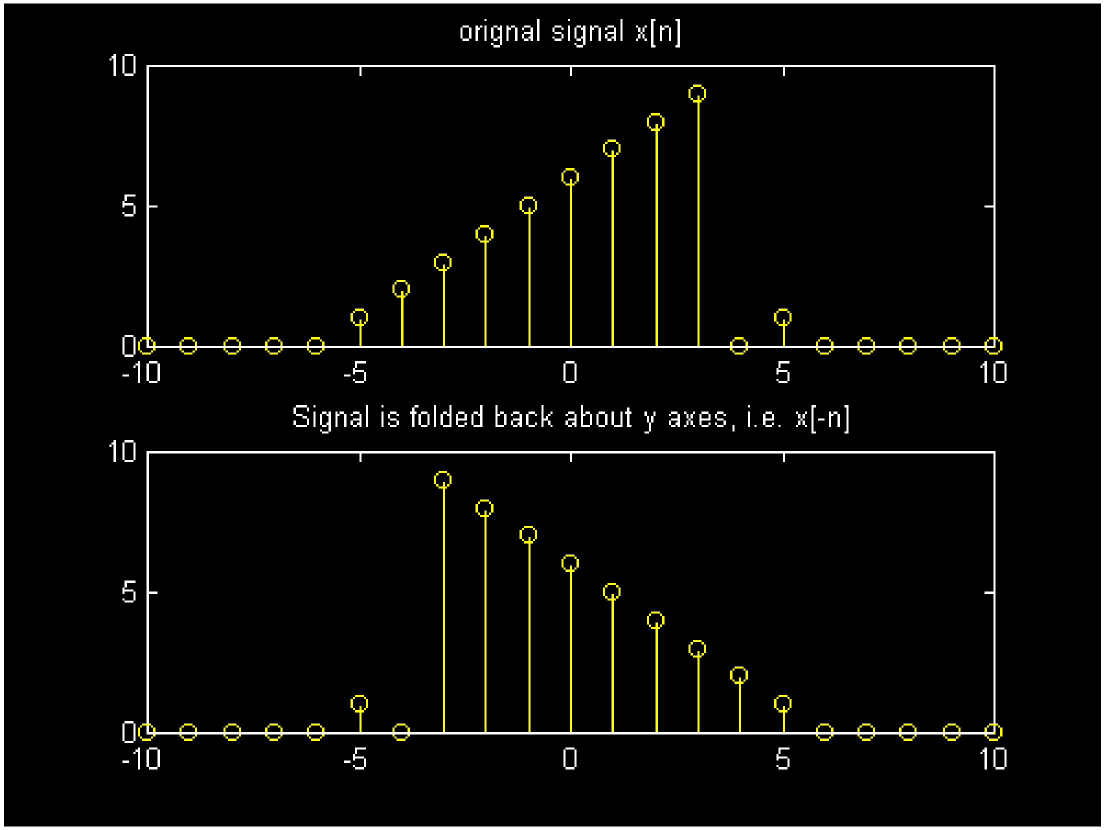
Shifting left or leading signal x(t)



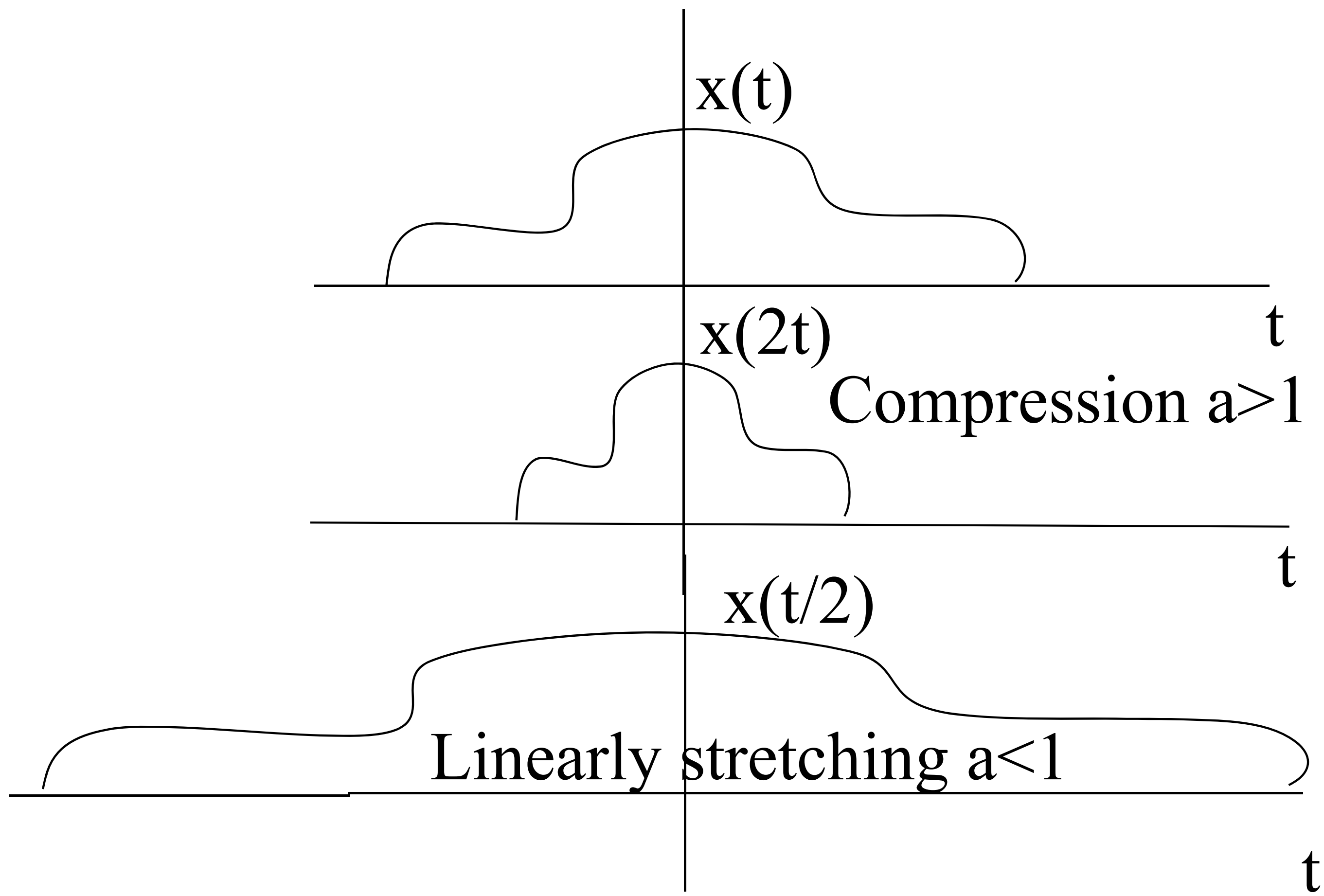
Folded or Flipped x(t) =x(-t), time reversal



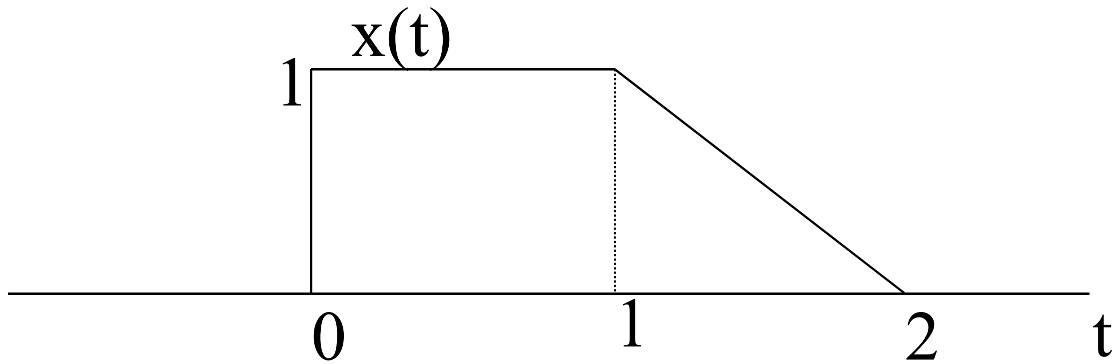
Signal Flip about y- axes X[-n], time reversal

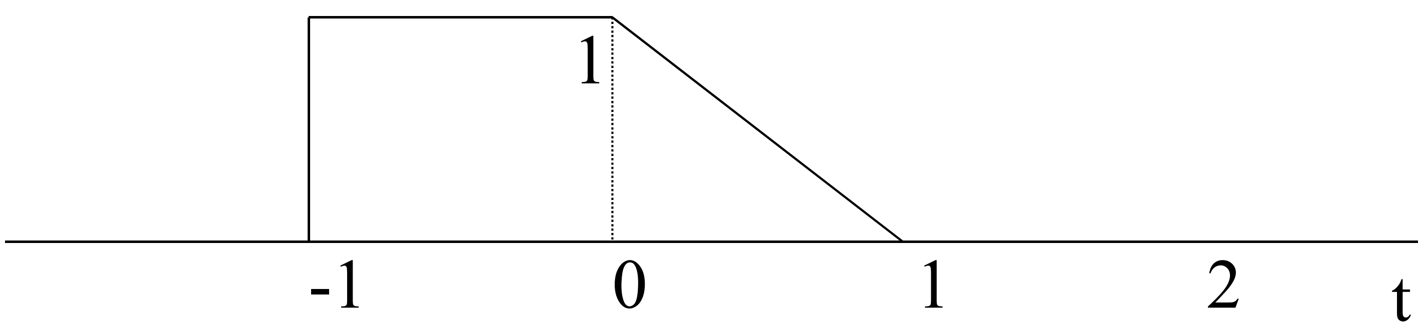


Time scaling of continuous signal



Example 1
sketch signal x(-t+1)

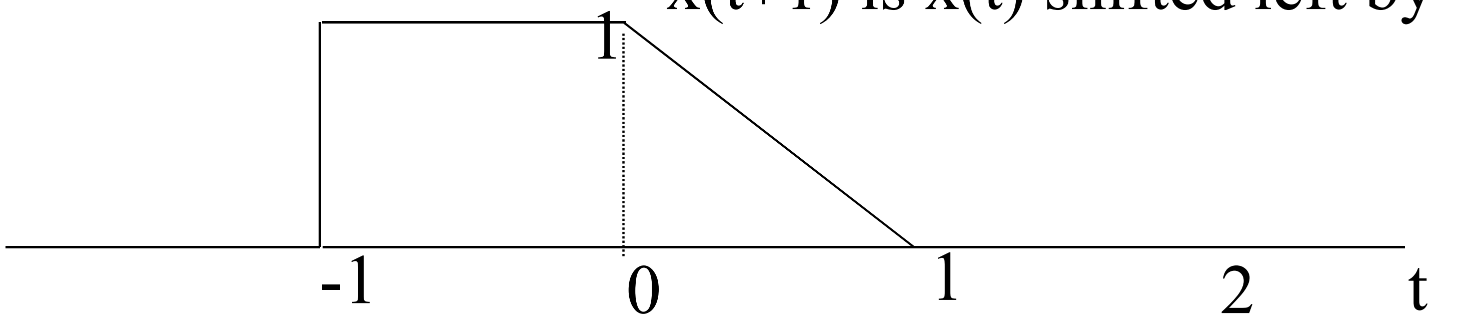


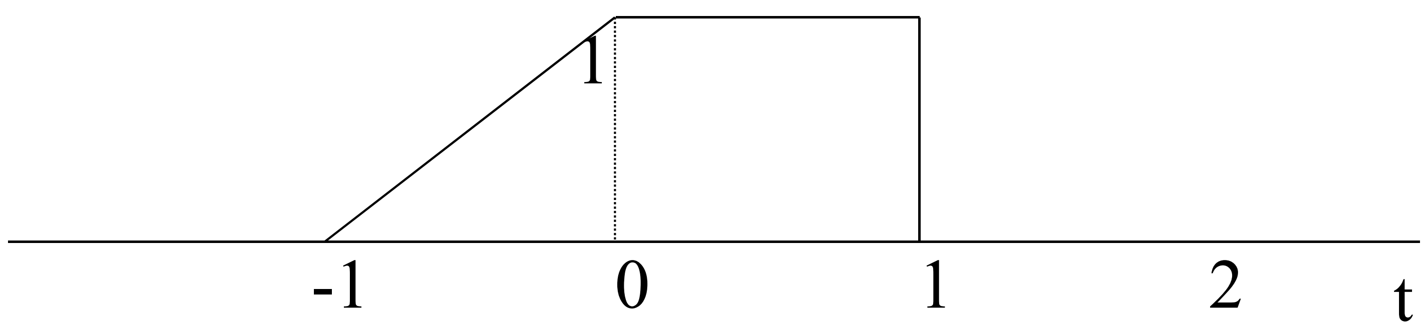


x(t+1), x(t) shifted left by 1sec

Example 1

x(t+1) is x(t) shifted left by 1

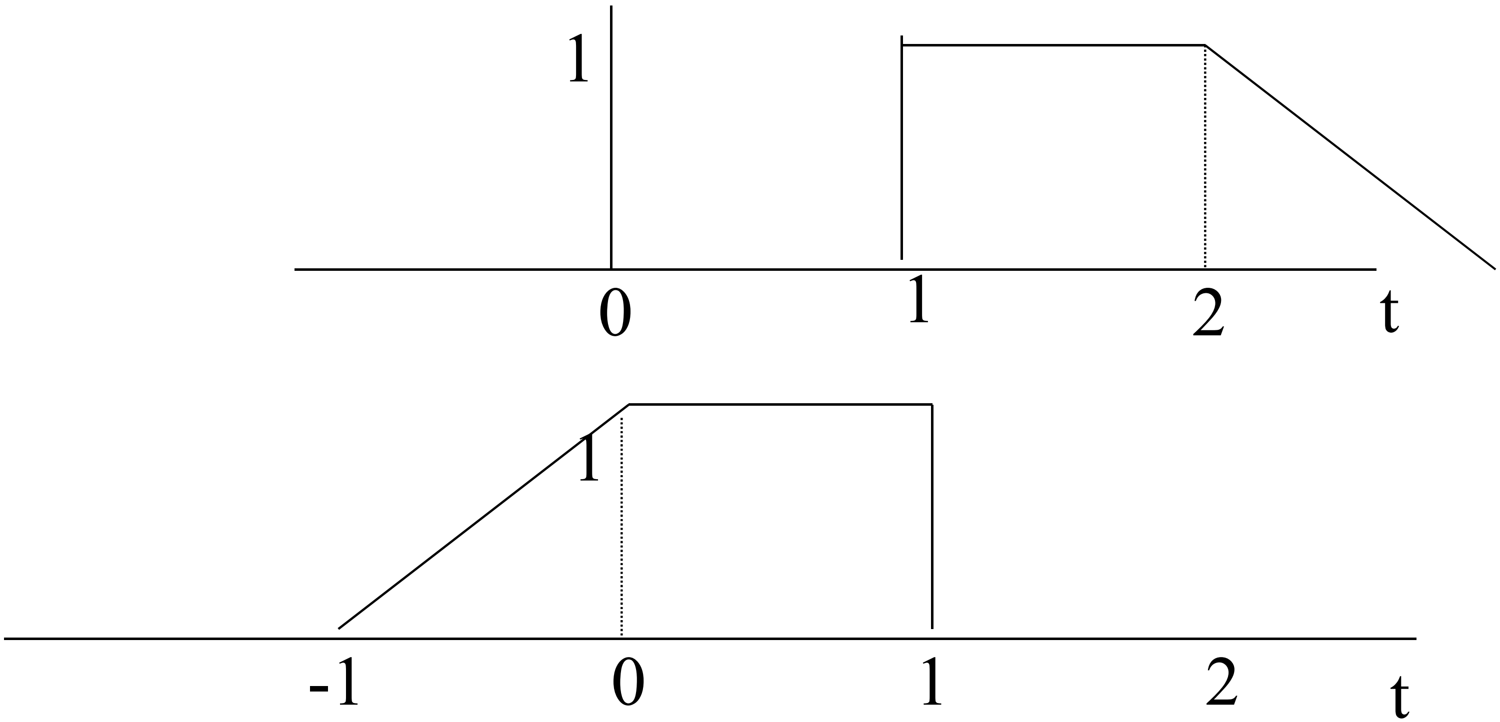




x(-t+1) is x(t+1) flipped about t=0

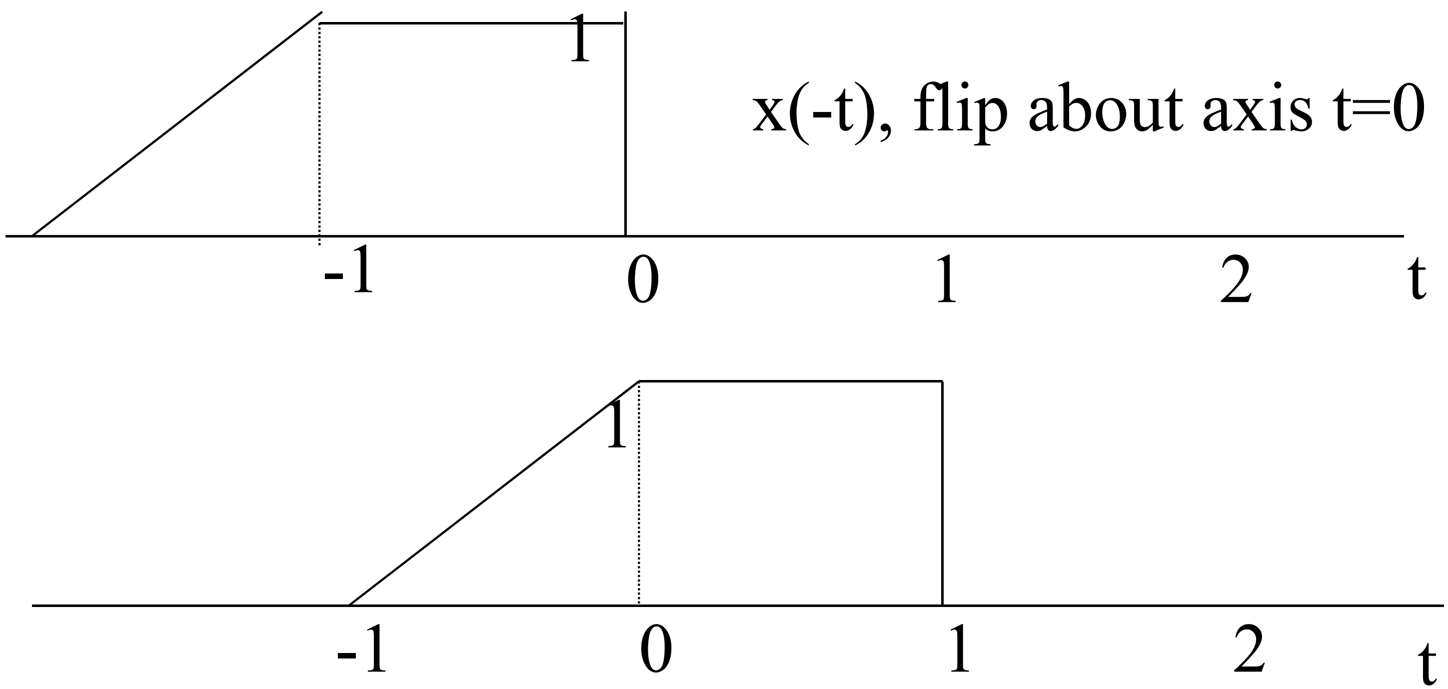
Example 1 (Method 2)

x(t-1) is x(t) shifted right by 1sec



x(-t+1)=x(-1(t-1)) , Flip about axis t=1

Example 1, (Method 3)

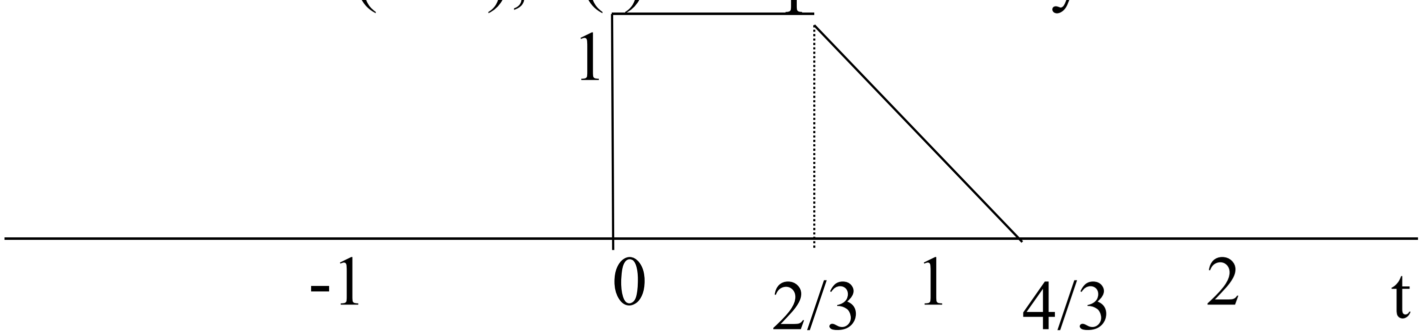


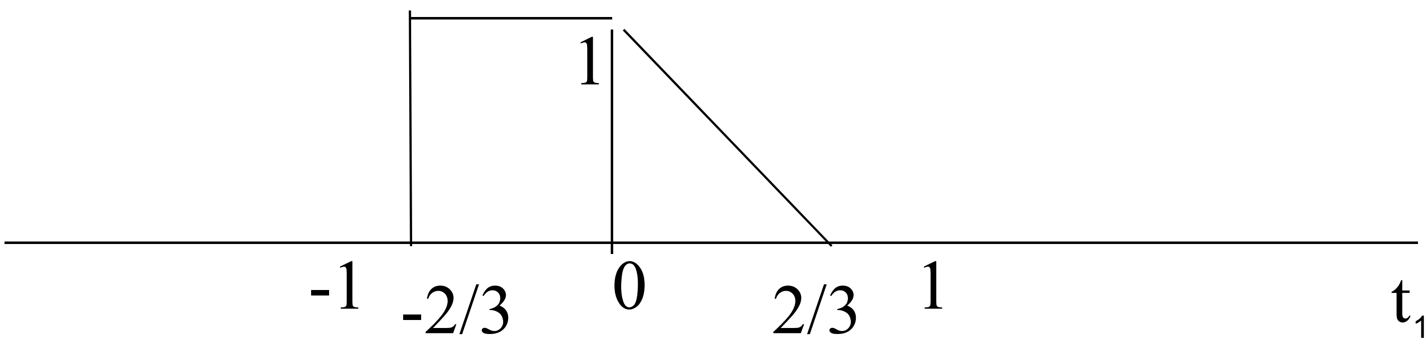
x(-t), flip about axis t=0

x(-t+1), shift right (because -t) by 1

Example 2 Sketch x(3t/2 +1)

x(3t/2), x(t) compressed by 2/3





x((3/2)*(t+2/3)), x(t) compressed by 2/3 & shifted left by 2/3

Periodic Complex Exponential & Sinusoidal Signal

Using Euler's relation, the complex exponential signal can be written in terms of sinusoidal signals with the fundamental period :-

$$e^{j\omega_o t} = \cos \omega_o t + j \sin \omega_o t$$

Similarly the sinusoidal signal can be written in terms of periodic complex exponential, with fundamental period :-

$$A\cos(\omega_o t+\theta)=\frac{A}{2}e^{j\theta}e^{j\omega_o t}+\frac{A}{2}e^{-j\theta}e^{-j\omega_o t}$$

Alternativ ely we can express :-

$$A\cos(\omega_o t+\theta)=A.\mathrm{Re}\{e^{j(\omega_o t+\theta)}\}$$

$$or\ A\sin(\omega_o t+\theta)=A.IM\{e^{j(\omega_o t+\theta)}\}$$

$T_o =$ *fundamental* period

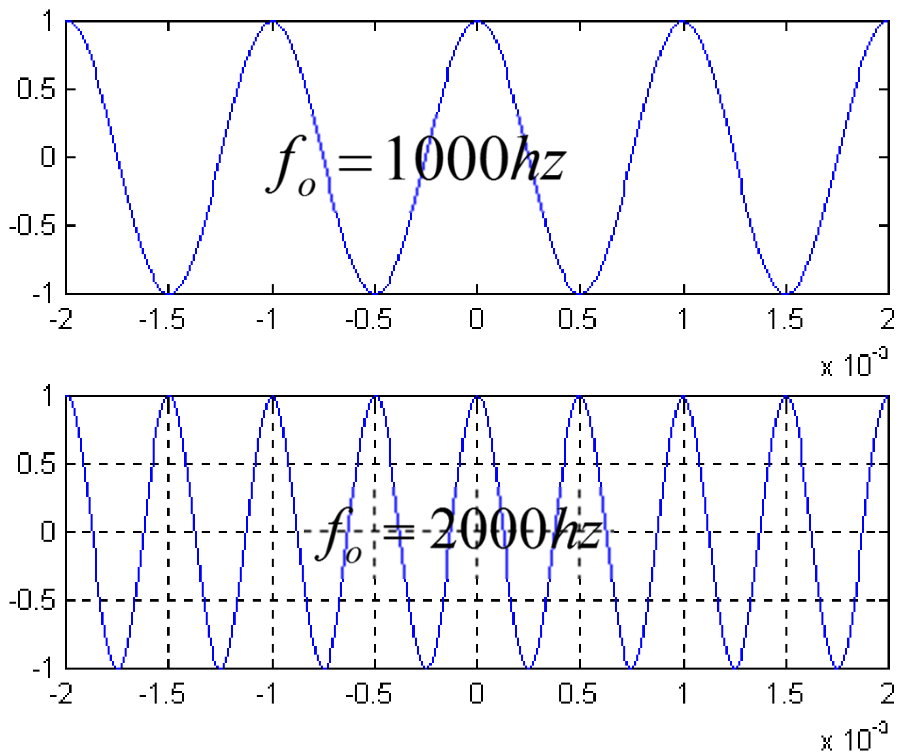
$\omega_\mathrm{o} = 2\pi \mathrm{f}_\mathrm{o}$ is fundamenta l angular frequency

in radians per second.

f_o is the fundamenta l frequency in

hertz or cycles per second.

Increasing w_o , increase the rate of oscillation.



Case $w_o=0$
x(t) is a constant

Therefore is periodic with period T
for any positive value of T.
Thus fundamental period is undefined.
On other hand can define fundamental
frequency to be zero.
I.e. constant signal(d.c) has zero rate of
oscillation.

Infinite Total Energy & finite average Power of Periodic Signal

Energy & Average Power

for Periodic Exponentia l Signal

over one period :-

$$\mathrm{E}_{\text{period}} = \int_0^{T_o} |e^{j\omega_o t}|^2 \, dt$$

$$= \int_0^{\mathrm{T_o}} 1.\mathrm{dt} = \mathrm{T_o}$$

$$P_{period} = \frac{1}{T_o} E_{period} = 1.$$

$$E_\infty = \text{infinite}$$

$$P_\infty = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |e^{j\omega_o t}|^2 \, dt = 1$$

Periodic Complex Exponential

Play a central role in signals & systems.
Serve as building block for many other signals.
Sets of harmonically related complex exponential are periodic with a common period T_o.

Periodic Complex Exponential

A necessary condition for a complex exponential $e^{j\omega_o t}$ to be periodic with period T_o is:- $e^{j\omega T_o} = 1$

which implies that ωT_o is a multiple of 2π. i.e. $\omega \mathrm{T_o} = 2\pi\, k, k = 0, \pm 1, \pm 2, \ldots$

Harmonically Complex Exponential Signals.

If we define $\omega_o = \dfrac{2\pi}{T_o}$,

then ω must be an integer multiple of ω_o i.e. $\omega = k\omega_o$.

A harmonically related set of complex exponential is a set of periodic exponentials with fundamental frequencies that are multiples of a single positive frequency ω_o.

Harmonically Complex Exponential Signals.

$$\Phi_k(t) = e^{jk\omega_o t}, \ \mathrm{k} = 0, \pm 1, \pm 2, ...$$

For $\mathrm{k} = 0$, $\Phi_k(t)$ is a constant.

$\mathrm{k} \neq 0$, $\Phi_k(t)$ is periodic with fundamenta l

frequencie s $|\mathrm{k}|\omega_{\mathrm{o}}$ and fundamenta l period

$$\frac{2\pi}{|\mathrm{k}|\omega_{\mathrm{o}}} = \frac{T_o}{|k|}$$

General Complex Exponential Signals

$x(t) = Ce^{at}$, where both 'C' and 'a' are complex numbers.

If C is expressed in polar form and a in cartesian form :-

$\mathrm{C} = |\mathrm{C}| e^{j\theta}$ and $\mathrm{a} = \mathrm{r} + \mathrm{j}\omega_{\mathrm{o}}$.

Then $x(t) = Ce^{at}$

$= |\mathrm{C}| e^{j\theta} e^{(r + j\omega_o)t} = |C| e^{rt} e^{j(\omega_o t + \theta)}$.

General Complex Exponential Signals.

Using Euler' s relation,

$x(t) = Ce^{at} = |C| e^{rt} \cos(\omega_o t + \theta) + j |C| e^{rt} \sin(\omega_o t + \theta)$

If $r = 0$,

the real & imaginary parts are sinusoidal .

If $r > 0$,

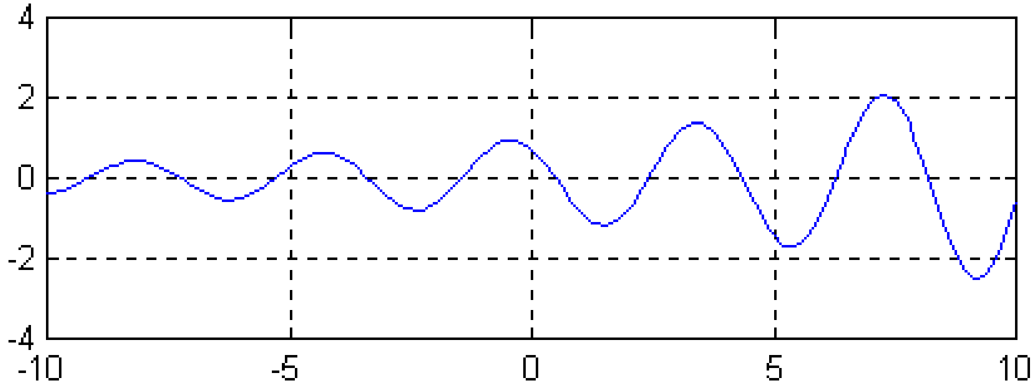
x(t) is sinusoidal signals multipled by growing exponentia l.

If $r < 0$,

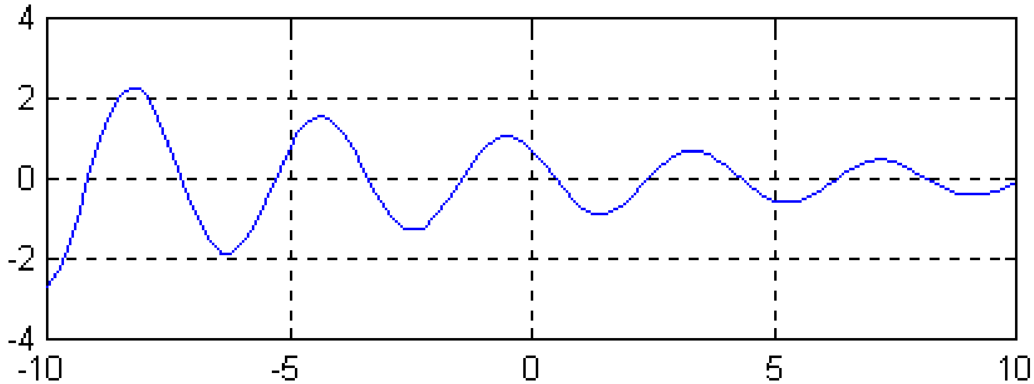
x(t) is sinusoidal signals multipled by decaying exponentia l.

Growing & Decaying Sinusoidal Signals.

r>0



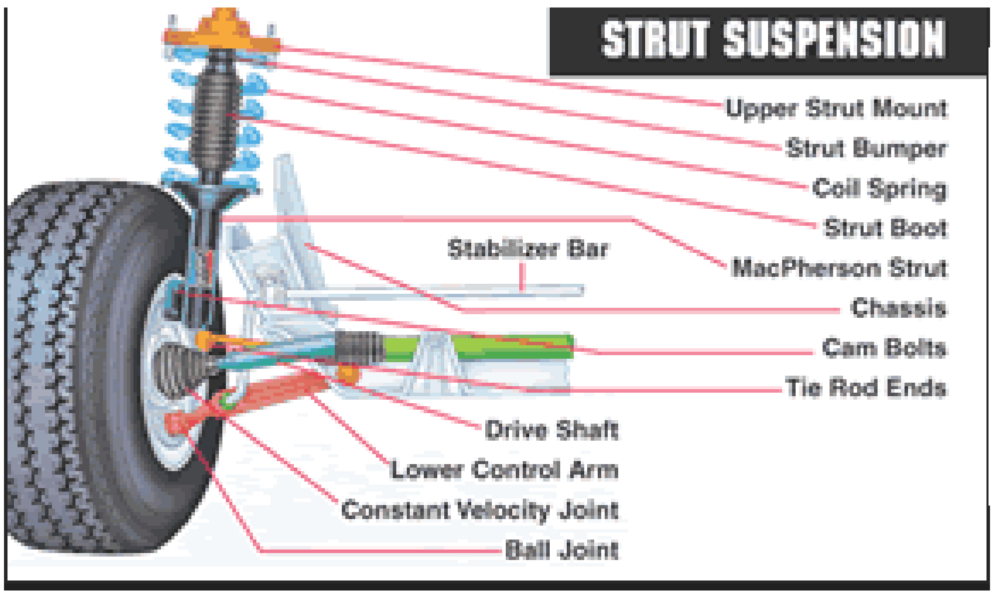
r<0

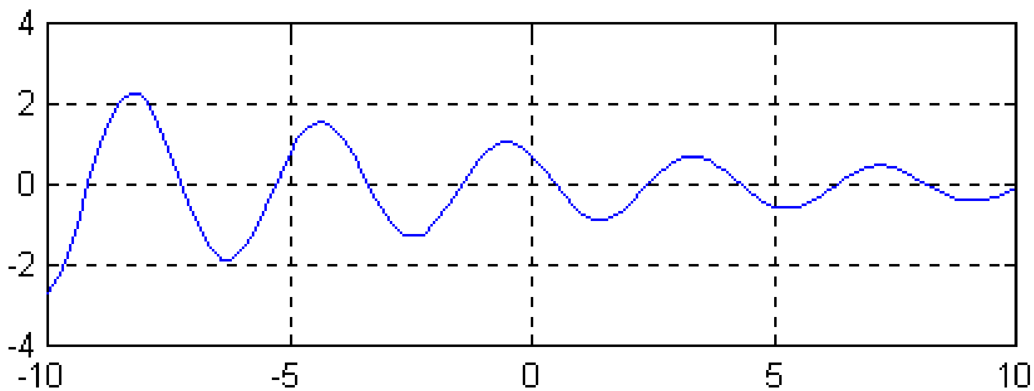


Decaying or Damped Sinusoids

Response of RLC circuits.

Mechanical systems having both damping & restoring forces e.g. automotive suspension system.





Discrete-Time Complex Exponential Sequence.

$x[n] = C\alpha^n,$

where C and α are in general complex numbers.

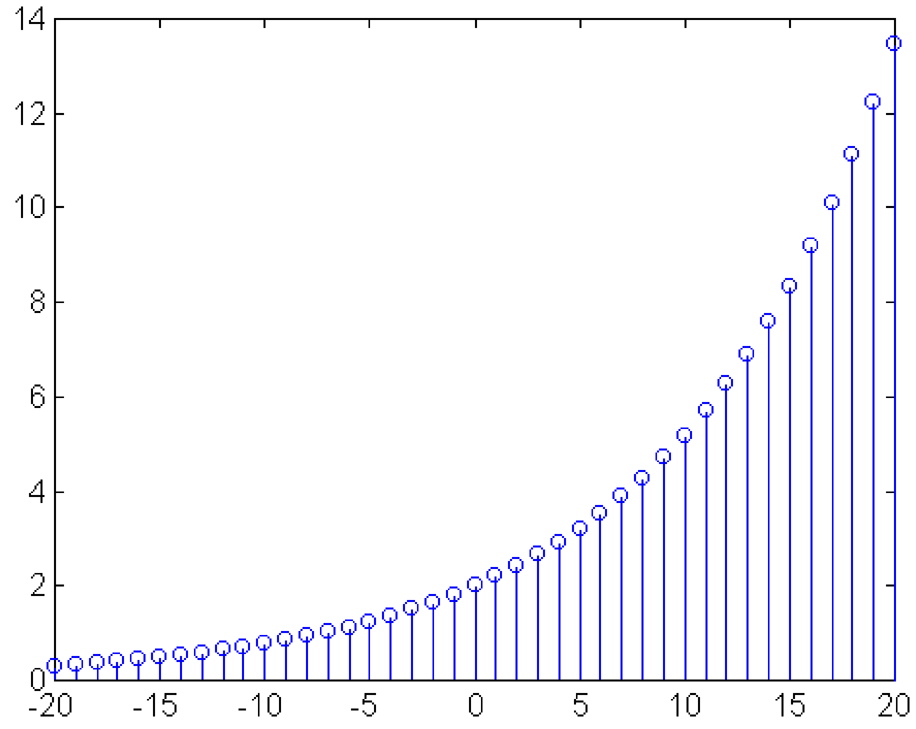
Alternativ ely we can express the sequence in the following form :-

$x[n] = Ce^{\beta n}$, where $\alpha = e^{\beta}$.

Although t his form is similar to the continuous - time exponentia l signal we have described previously , the former form is perferred when dealing with the discrete - time sequence.

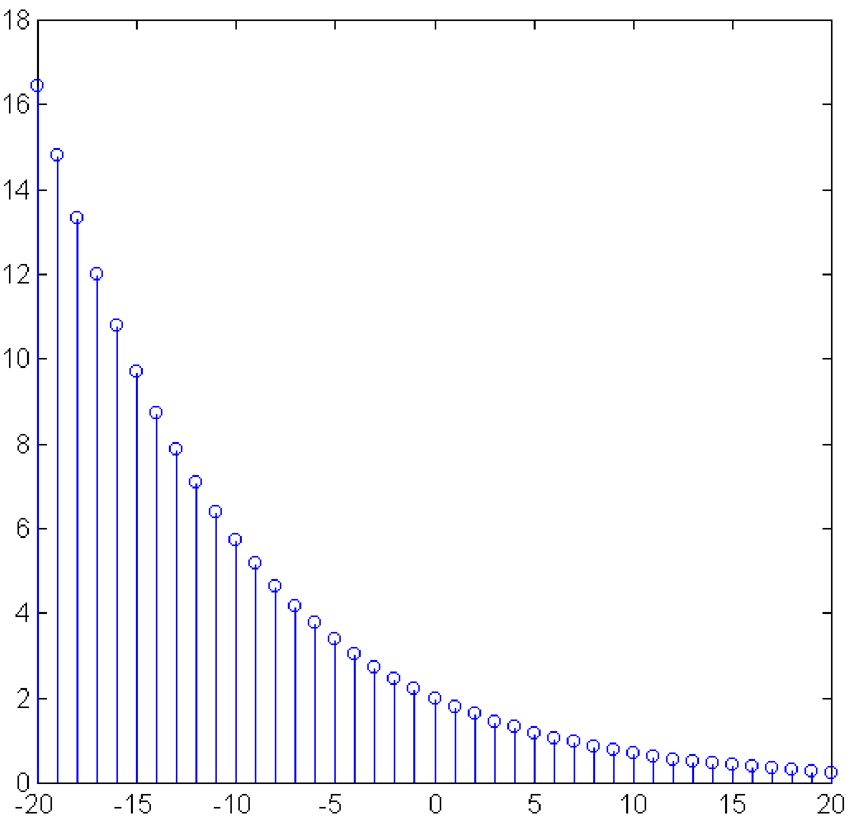
Real Exponential Signal

$x[n] = C * \alpha^n$ where $\alpha > 1$ $e.g.$ $x[n] = 2 * 1.1^n$



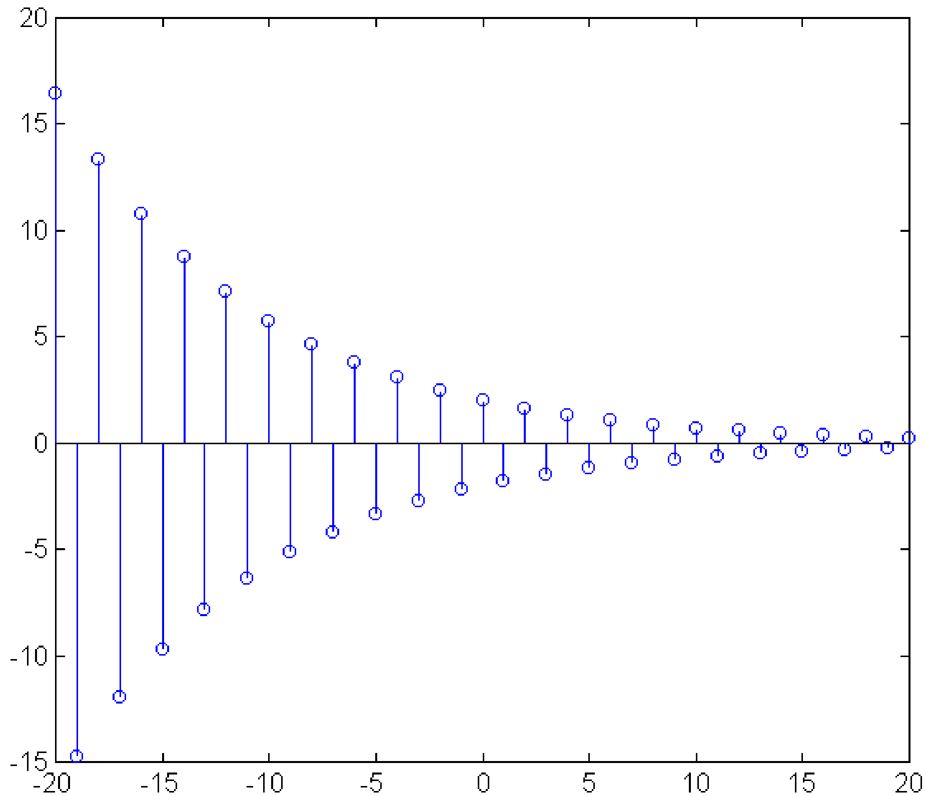
Real Exponential Signal

$x[n] = C * \alpha^n$ where $0 < \alpha < 1$ $\qquad x[n] = 2 * 0.9^n$



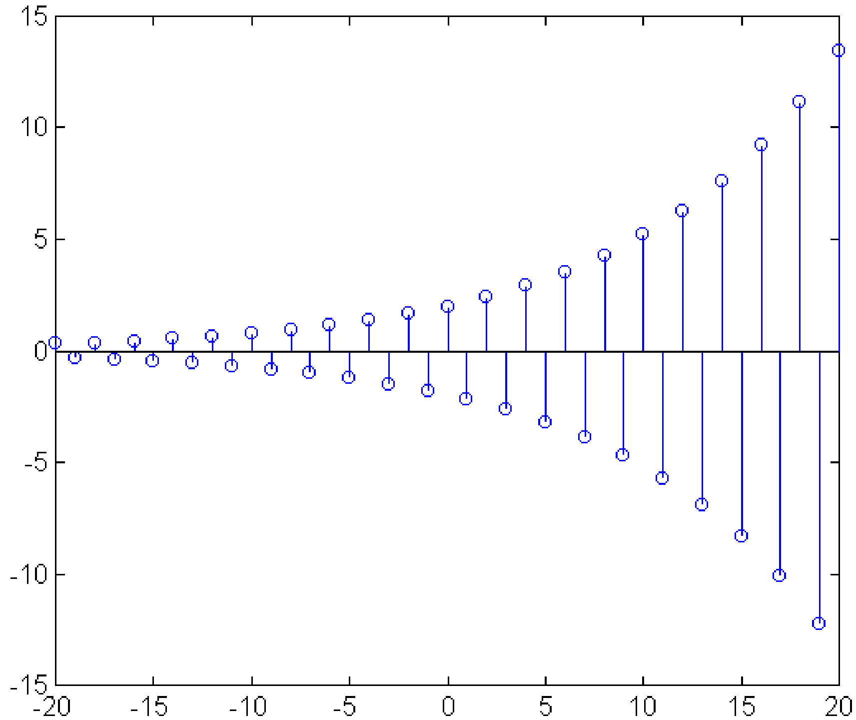
Real Exponential Signal

$x[n] = C * \alpha^n$ where $-1 < \alpha < 0$ $\quad x[n] = 2 * (-0.9)^n$



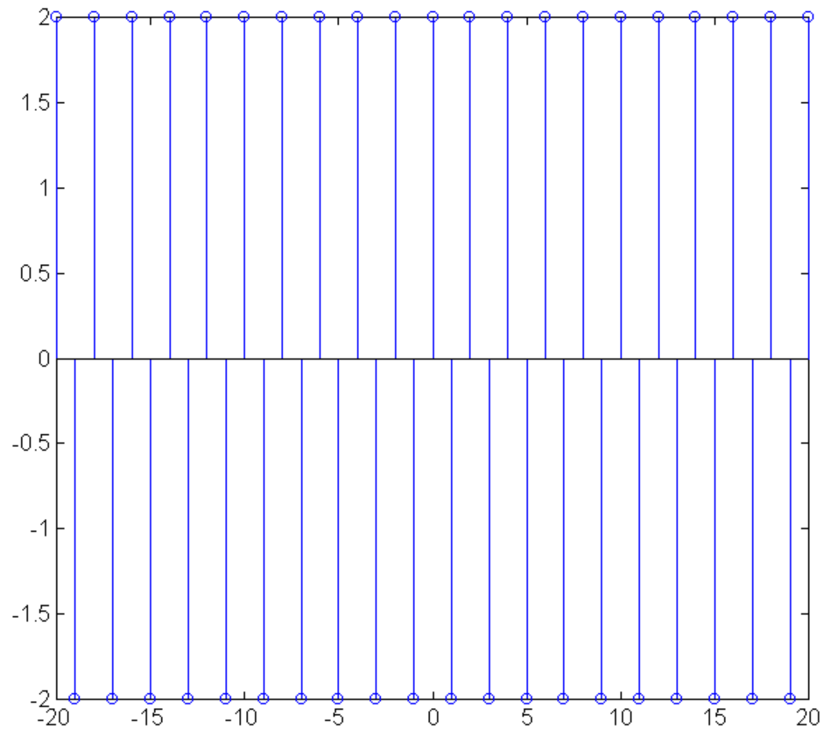
Real Exponential Signal

$x[n] = C * \alpha^n$ where $\alpha < -1$ $\quad x[n] = 2*(-1.1)^n$



Real Exponential Signal

$x[n] = C * \alpha^n$ where $\alpha = -1$ $\quad x[n] = 2 * (-1)^n$



Real Exponential Signal

Real-valued discrete exponentials are used to describe:-

1) Population growth as function of generation.
2) Total return on investment as a function of day, month or quarter.

Discrete-time Sinusoidal Signals

$x[n] = Ce^{\beta n}$, let $\mathrm{C} = 1$ & $\beta = \mathrm{j}\omega_o$ be purely an imaginary number.

$\therefore x[n] = e^{j\omega_o n}$.

This signal is closely related to sinusoidal signal :

$x[n] = A\cos(\varpi_o n + \phi)$.

Taking n as dimensionl ess, then both

ϖ_o and ϕ have units of radians.

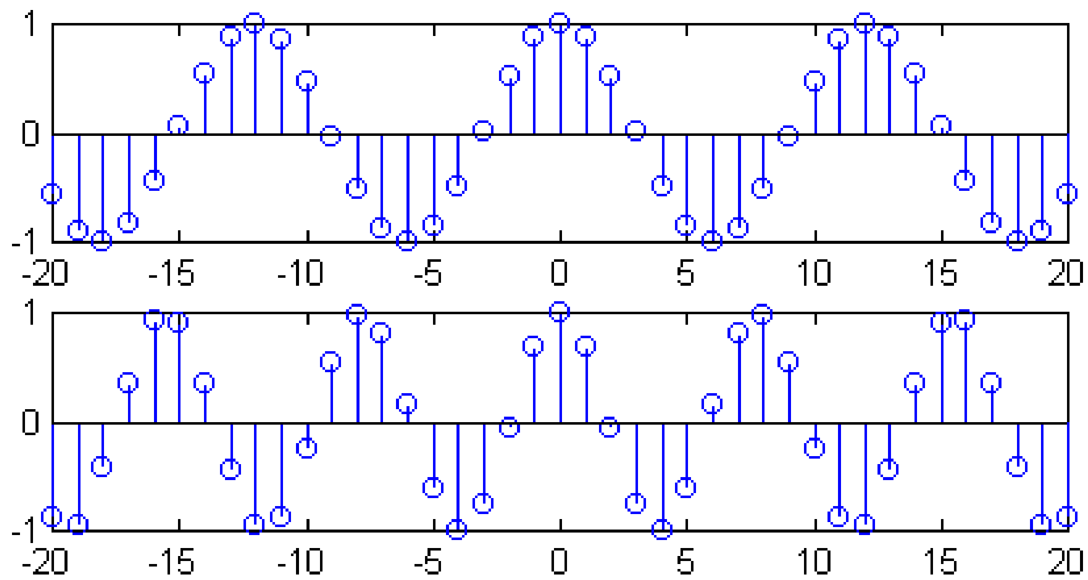
From Euler' s relation : -$e^{j\omega_o n} = \cos\omega_o n + j\sin\omega_o n$

$$A\cos(\omega_o n + \phi) = \frac{A}{2}e^{j\phi}e^{j\omega_o n} + \frac{A}{2}e^{-j\phi}e^{-j\omega_o n}$$

Discrete-time Sinusoidal Signals

$$x[n] = \cos(2\pi n / 12)$$

$$x[n] = \cos(8\pi n / 31)$$



Discrete-time Sinusoidal Signals

These discrete-time signals possessed:-

1) Infinite total energy
2) Finite average power.

General Complex Exponential Signals

The general discrete - time complex exponential can be interpreted in terms of real exponentials and sinusoidal signals.

Writing C and α in polar form : -

$$\mathrm{C} = |\mathrm{C}|\,\mathrm{e}^{\mathrm{j}\theta},\ \alpha = |\alpha|\,\mathrm{e}^{\mathrm{j}\omega_\mathrm{o}}.$$

$$x[n] = C\alpha^n = |C||\alpha|^n \cos(\omega_o n + \theta) + j|C||\alpha|^n \sin(\omega_o n + \theta)$$

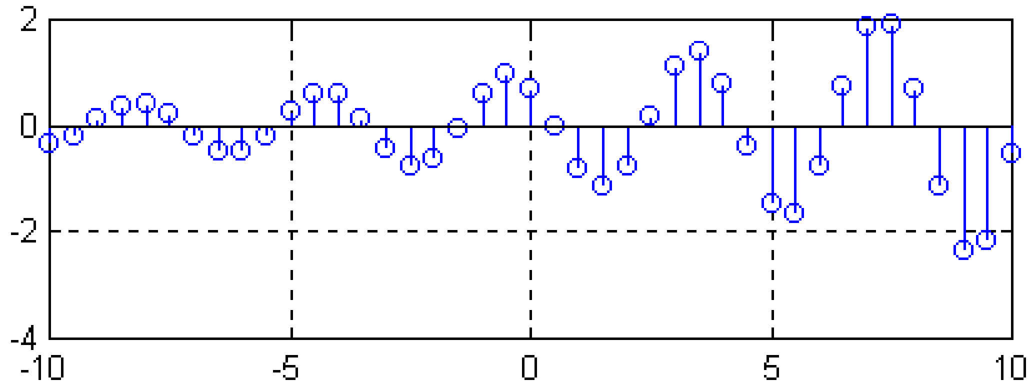
$|\alpha| = 1$, *real & imaginary* parts are sinusoidal.

$|\alpha| < 1$, sinusoidal decaying exponentially,

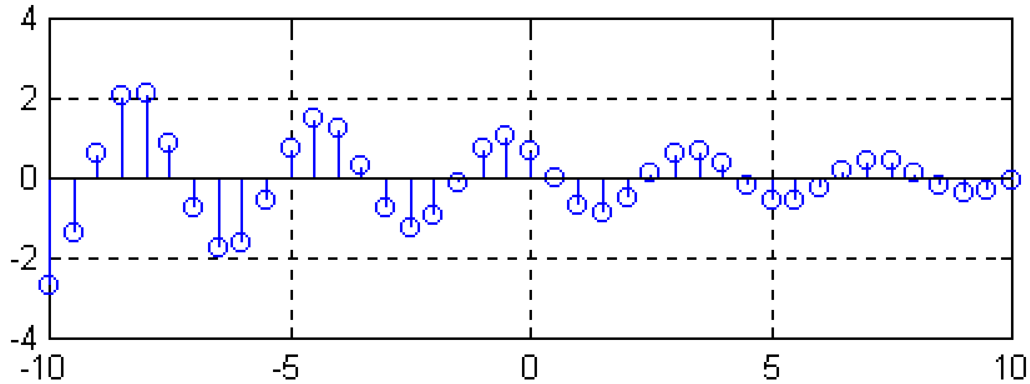
$|\alpha| > 1$, sinusoidal growing exponentially.

General Complex Exponential Signals

$\alpha > 1$



$\alpha < 1$



Periodicity Properties of Discrete-time Complex Exponentials

Two properties of continuous-time counterpart $e^{j\omega_o t}$

1) The Larger is ω_o, the higher is the rate of oscillation.

2) $e^{j\omega_o t}$ is periodic for any value of ω_o.

There are differences in each of the above properties for the discrete-time case of $e^{j\omega_o n}$.

Periodicity Properties of Discrete-time Complex Exponentials

Consider the discrete-time complex exponential with frequency $\omega_o + 2\pi$:

$$e^{j(\omega_o+2\pi)n} = e^{j2\pi n}e^{j\omega_o n} = e^{j\omega_o n}.$$

From this we conclude that the exponential at frequency $\omega_o + 2\pi$ is the same as that at frequency ω_o.

Similarly at frequencies $\omega_o \pm 2\pi, \omega_o \pm 4\pi$, and so on.

This is very different from the continuous-time case whereby the signals are all distinct for all distinct values of ω_o.

Because of this periodicity of 2π, we need only to consider frequency interval of 2π in the case for discrete-time signals.

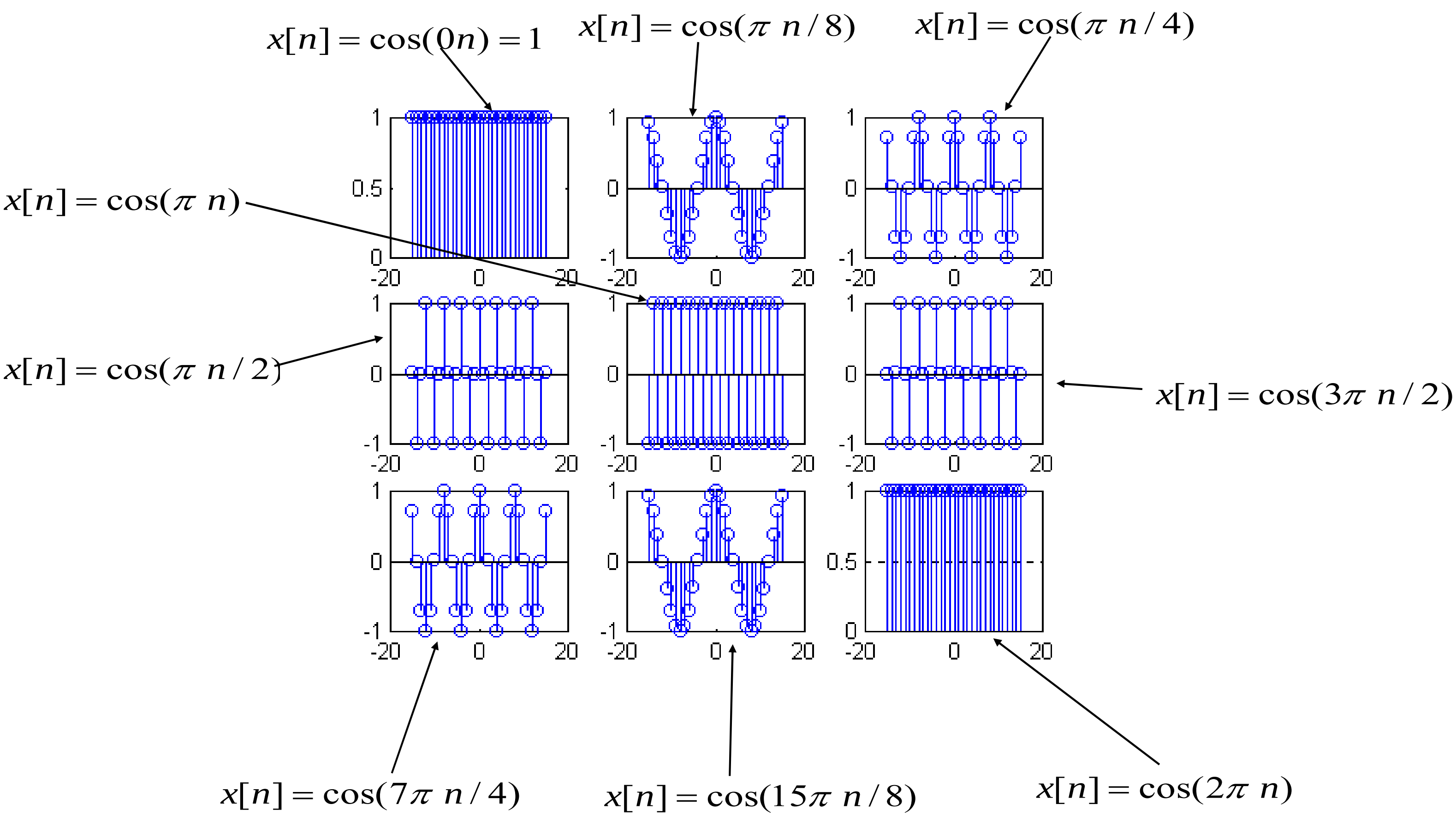
Periodicity Properties of Discrete-time Complex Exponentials

Because of this implied periodicity of discrete-time signal,
the signal $e^{j\omega_o n}$ does not have a continually increasing rate of oscillation
as ω_o is increased in magnitude.
Increasing ω_o from 0 (d.c., constant sequence, no oscillation)
the oscillation increases until $\omega_o = \pi$, thereafter the oscillation
will decrease to 0 i.e. a constant sequence or d.c. signal at $\omega_o = 2\pi$.

Periodicity Properties of Discrete-time Complex Exponentials

Therefore, low frequencie s occurs at $\omega_{\mathrm{o}} = 0,\ \pm 2\pi,$
$\pm$ even multiple of π.
High frequencie s are at $\omega_{\mathrm{o}} = \pm\pi, \pm 3\pi, \pm$odd multiple of π.
Note for $\omega_{\mathrm{o}} = \pi$, odd multiple of π, $\mathrm{e}^{\mathrm{j}\pi\,\mathrm{n}} = (e^{j\pi})^n = (-1)^n$,
the signal oscillates rapidly, changing sign at each point in time.

Periodicity Properties of Discrete-time Complex Exponentials



Periodicity Properties of Discrete-time Complex Exponentials

Second property concerns the periodicity of the discrete-time complex exponential.

In order for $e^{j\omega_o n}$ to be periodic with period $N > 0$,

$e^{j\omega_o (n+N)} = e^{j\omega_o n}$, or equivalently $e^{j\omega_o N} = 1$.

$\therefore \omega_o N$ must be a multiple of 2π.

i.e. $\omega_o N = 2\pi\, m$, or equvalently $\dfrac{\omega_o}{2\pi} = \dfrac{m}{N}$,

This means that the signal $e^{j\omega_o n}$ is periodic if $\omega_o / 2\pi$ is a rational number and is not periodic otherwise.

This is also true for the discrete-time sinusoids.

Fundamental Period & Frequency of discrete-time complex exponential

$x[n] = \cos(2\pi n/12)$

periodic because $\omega_o = 2\pi/12$,

$$\frac{\omega_o}{2\pi} = \frac{1}{12}$$

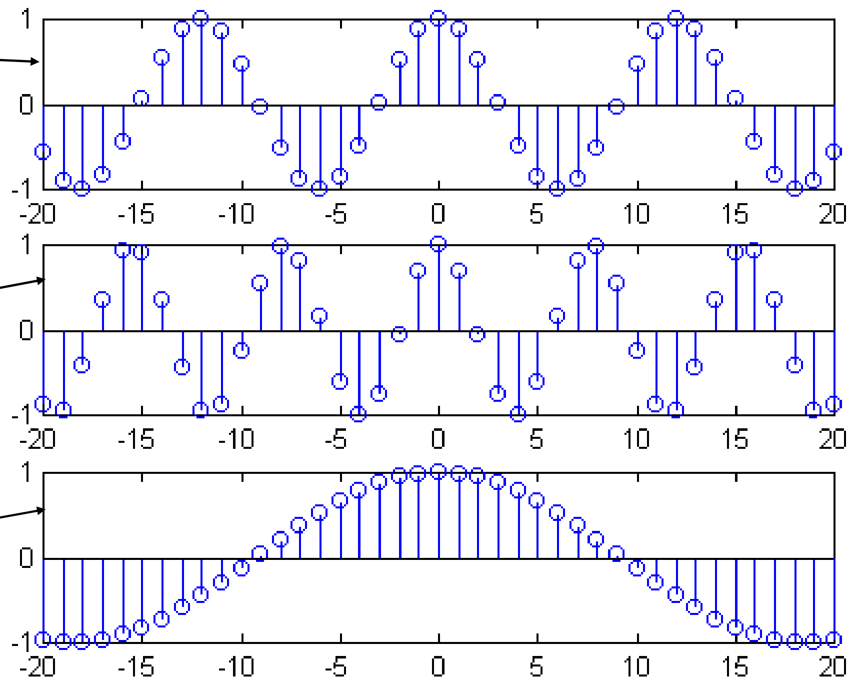
$x[n] = \cos(8\pi n/31)$

periodic because $\omega_o = 8\pi/31$, $\dfrac{\omega_o}{2\pi} = \dfrac{4}{31}$

$x[n] = \cos(n/6)$

not periodic because $\omega_o = 1/6$,

$$\frac{\omega_o}{2\pi} \neq \text{rational number}$$



Fundamental Period & Frequency of discrete-time complex exponential

If $x[n] = e^{j\omega_o n}$ is periodic with fundamental period N,

Its fundamental frequency is $\dfrac{2\pi}{N} = \dfrac{\omega_o}{m}$,

The fundamental period is written as :-

$$N = m\left(\frac{2\pi}{\omega_o}\right)$$

Comparison of the signal $e^{j\omega_o t}$ and $e^{j\omega_o n}$

$e^{j\omega_o t}$	$e^{j\omega_o n}$
Distinct signals for distinct values of ω_o.	Identical signals for values of ω_o separated by multiples of 2π
Periodic for any choice of ω_o.	Periodic only if $\omega_o = \frac{2\pi m}{N}$, for some integers $N > 0$ and m
Fundamental frequency ω_o	Fundamental frequency $\frac{\omega_0}{m}$
Fundamental period $\omega_o = 0: undefined$ $\omega_o \neq 0: \frac{2\pi}{\omega_o}$	Fundamental period $\omega_o = 0: undefined$ $\omega_o \neq 0: m(\frac{2\pi}{\omega_o})$

Harmonically related periodic exponential sequence

Considering periodic exponentials with common period N samples:

$$\phi_k[n] = e^{jk(2\pi/N)n}, \text{ for k} = 0, \pm 1, \ldots$$

This set of signals possess frequencies which are multiples of

$2\pi / N$

Harmonically related periodic exponential sequence

In continuous-time case $e^{jk(2\pi/T)t}$

are all distinct signals for $k = 0, \pm 1, \pm 2, \ldots\ldots$

$$\phi_{k+N}[n] = e^{j(k+N)(2\pi/N)n}$$

$$= \mathrm{e}^{\mathrm{jk}(2\pi/\mathrm{N})\mathrm{n}} e^{j2\pi n} = \phi_k[n]$$

Harmonically related periodic exponential sequence

Therefore , there are only N distinct periodic exponentials in the discrete harmonic sequences.

$$\phi_o[n] = 1, \phi_1[n] = e^{j2\pi n/N}, \phi_2[n] = e^{j4\pi n/N},$$

$$.......\phi_{N-1}[n] = e^{j2\pi(N-1)n/N}$$

Any other $\phi_{\mathrm{k}}[n]$ is identical to one of the above. (e.g. $\phi_{\mathrm{N}}[n] = \phi_0[n]$)