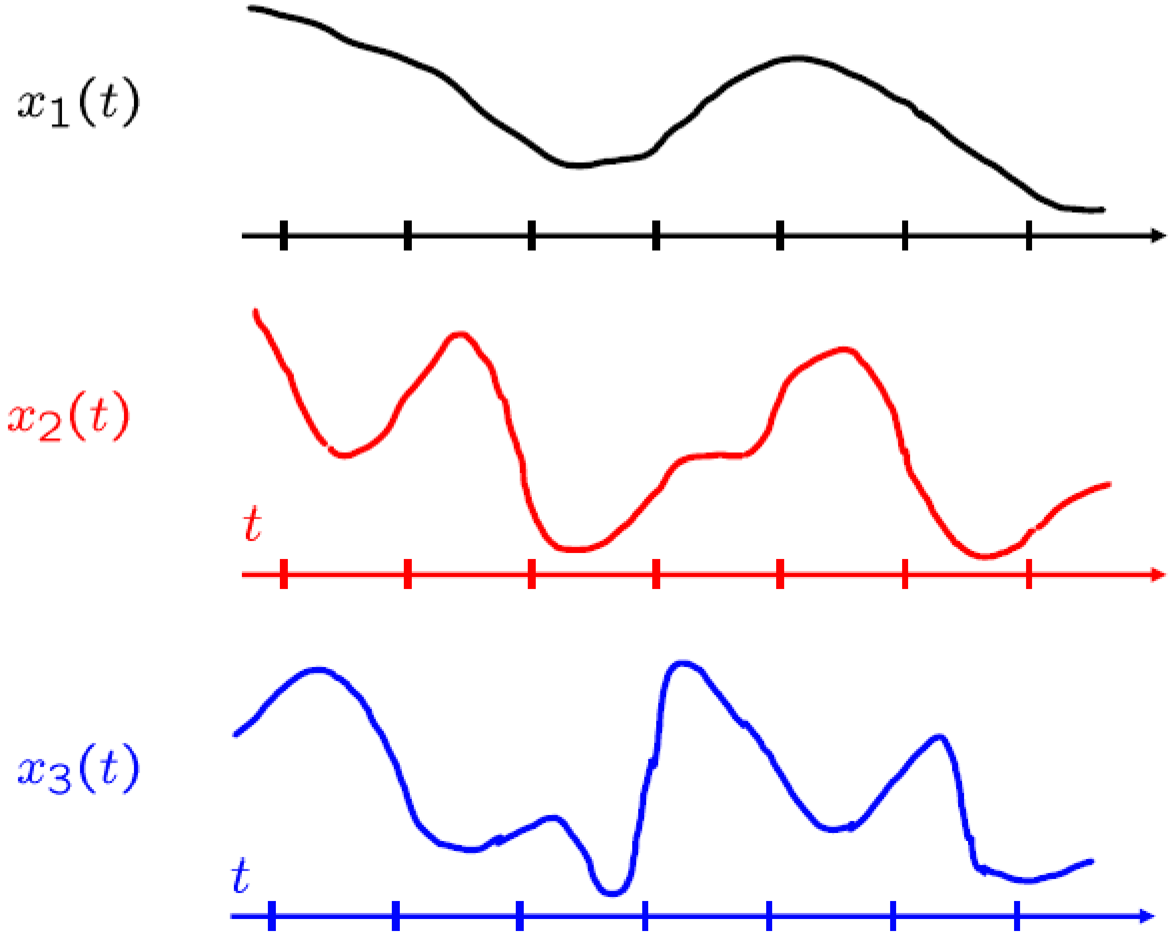
Introduction to Signals and Systems: V216

Lecture #12
Chapter 7: Sampling

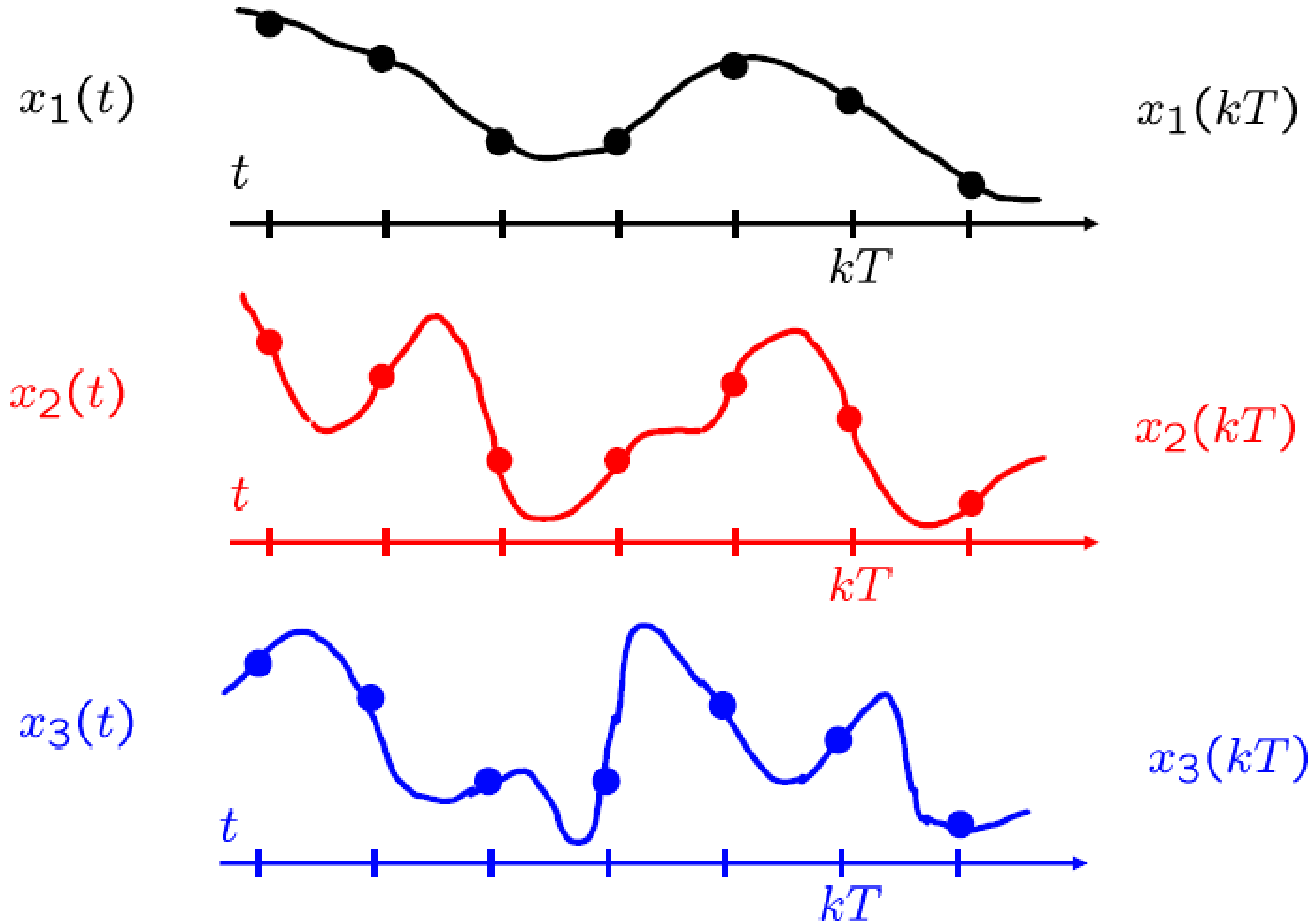
The Sampling Theorem

- Representation of CT Signals by its Samples



The Sampling Theorem

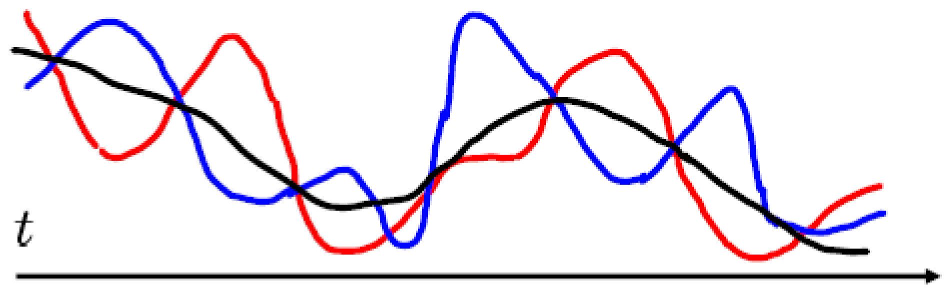
- Representation of CT Signals by its Samples



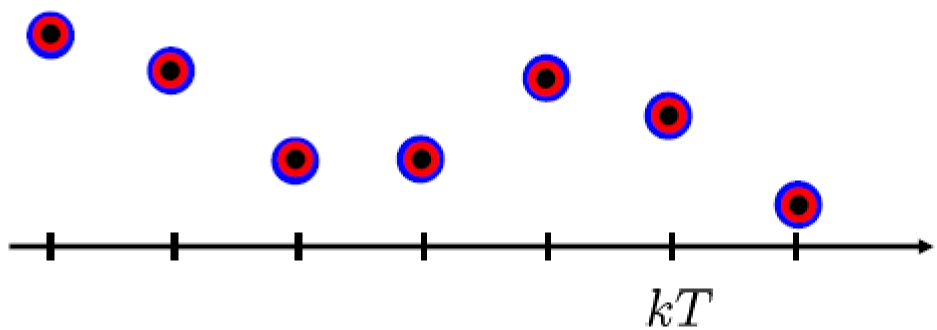
The Sampling Theorem

- Representation of CT Signals by its Samples

$x_1(t) \neq x_2(t) \neq x_3(t)$

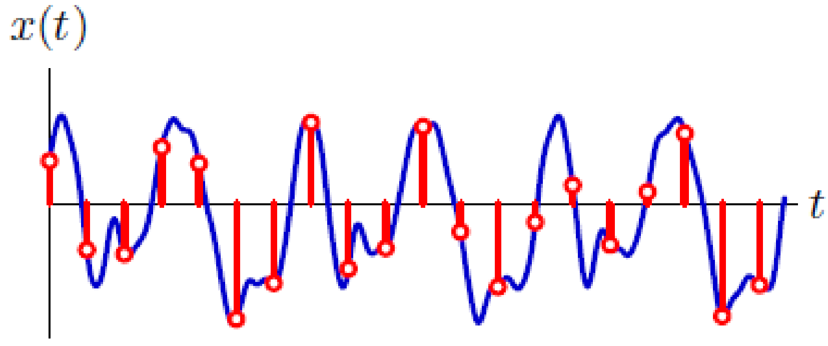


$x_1(kT) = x_2(kT) = x_3(kT)$

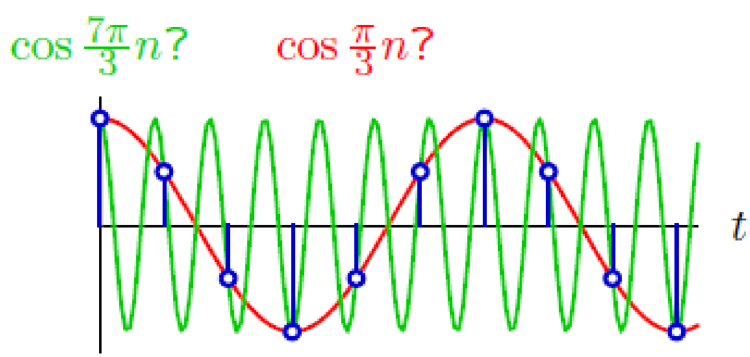


The Sampling Theorem

We would like to sample in a way that preserves information, which may not seem possible.



Information between samples is lost. Therefore, the same samples can represent multiple signals.



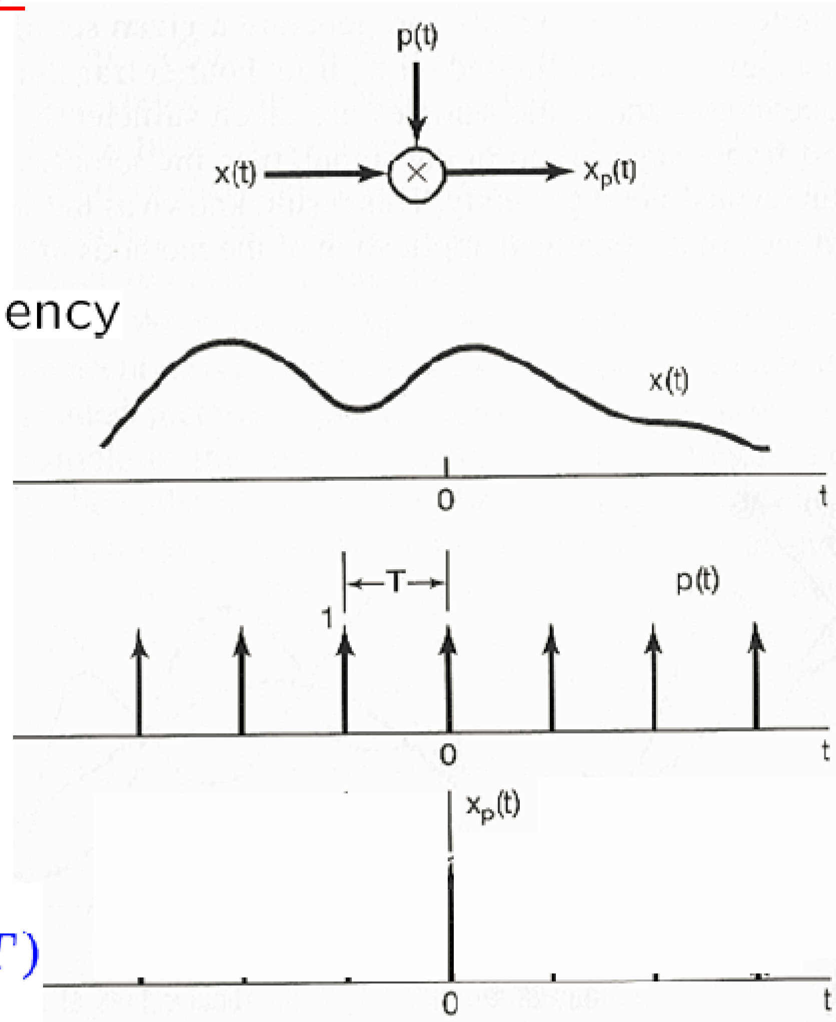
The Sampling Theorem

- Impulse-Train Sampling:

$p(t)$: sampling function

T : sampling period

$w_s = \dfrac{2\pi}{T}$: sampling frequency



$$\Rightarrow \; x_p(t) = x(t)\, p(t)$$

$$p(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT)$$

$$x_p(t) = \sum_{n=-\infty}^{+\infty} x(nT)\delta(t - nT)$$

The Sampling Theorem

- Impulse-Train Sampling: From multiplication property,

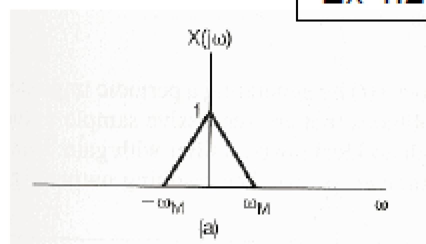
$$x_p(t) = x(t)\,p(t) \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad X_p(jw) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\theta)P(j(w-\theta))d\theta$$

Eq 4.70, p. 322

$$= \frac{1}{T}\sum_{k=-\infty}^{+\infty} X(j(w-kw_s))$$

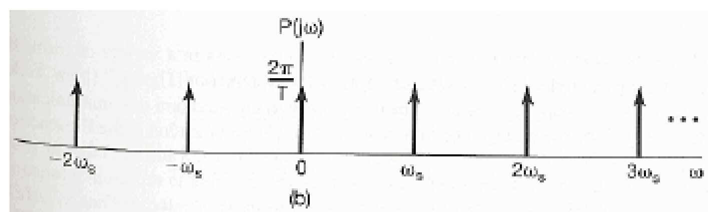
Ex 4.21, p. 323

$$x(t) \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad X(jw)$$



$$p(t) \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad P(jw) = \frac{2\pi}{T}\sum_{k=-\infty}^{+\infty} \delta(w-kw_s)$$

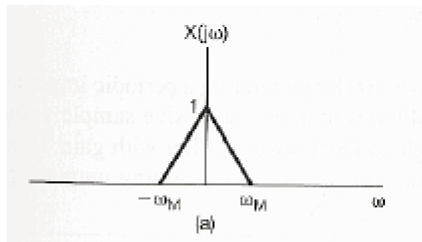
Ex 4.8, pp. 299-300

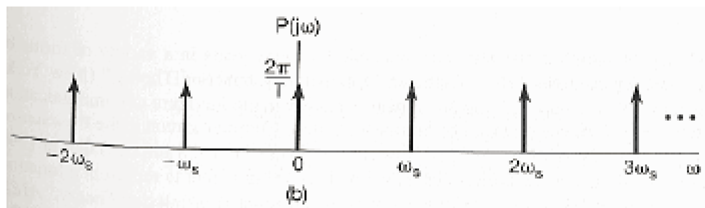


The Sampling Theorem

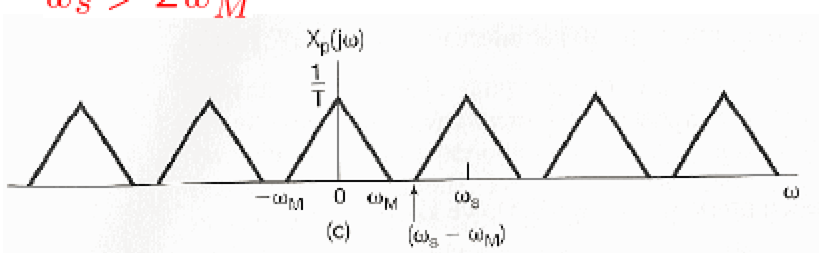
- Impulse-Train Sampling:

Ex 4.21, 4.22, pp. 323-4

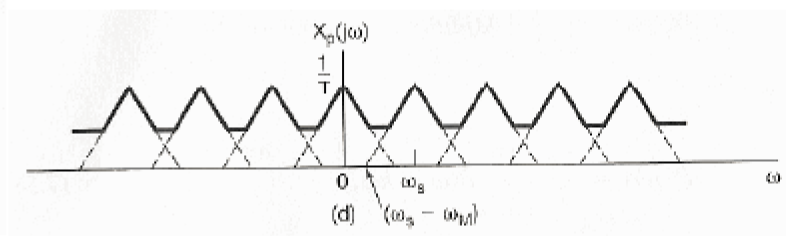




$w_s > 2w_M$



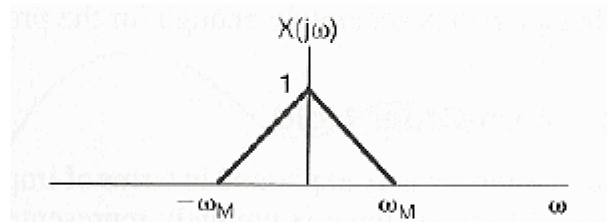
$w_s < 2w_M$



The Sampling Theorem

- <u>The Sampling Theorem:</u>

  $x(t)$: a band-limited signal

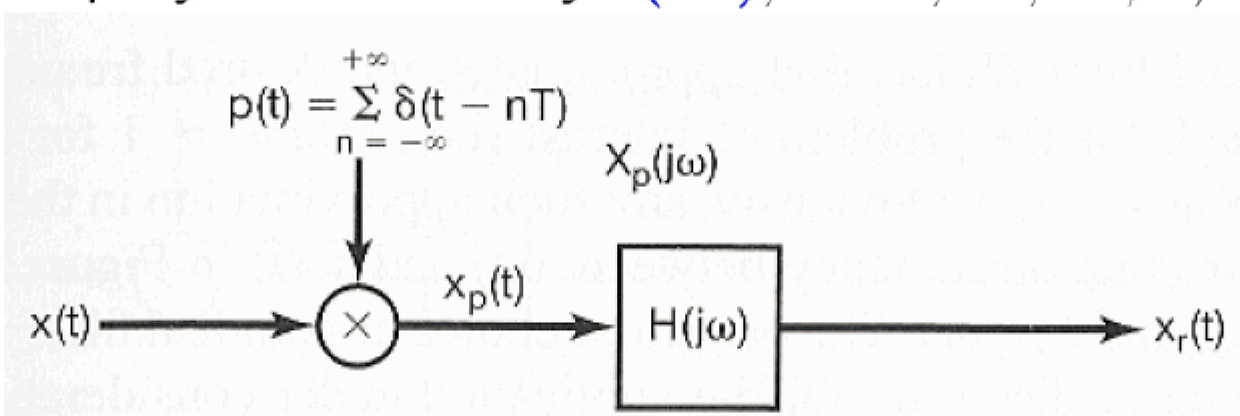
  

  (b)

  with $X(jw) = 0$ for $|w| > w_M$

  if $w_s > 2w_M$ where $w_s = \dfrac{2\pi}{T}$

  $\Rightarrow$ $x(t)$ is uniquely determined by $x(nT), n = 0, \pm 1, \pm 2, ...,$

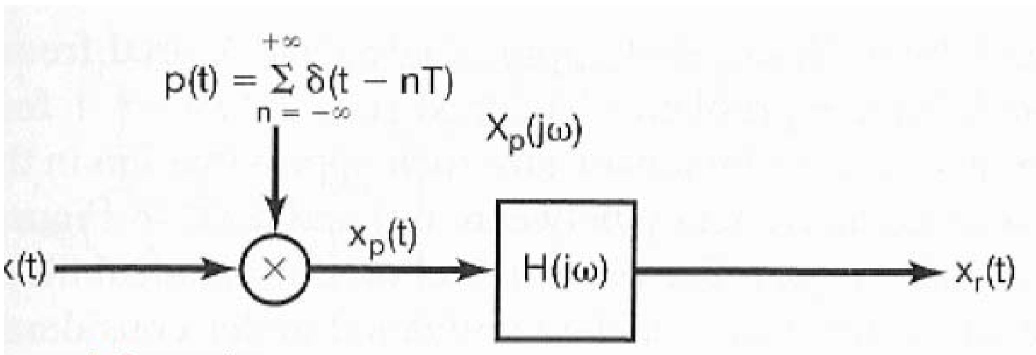
  (a)

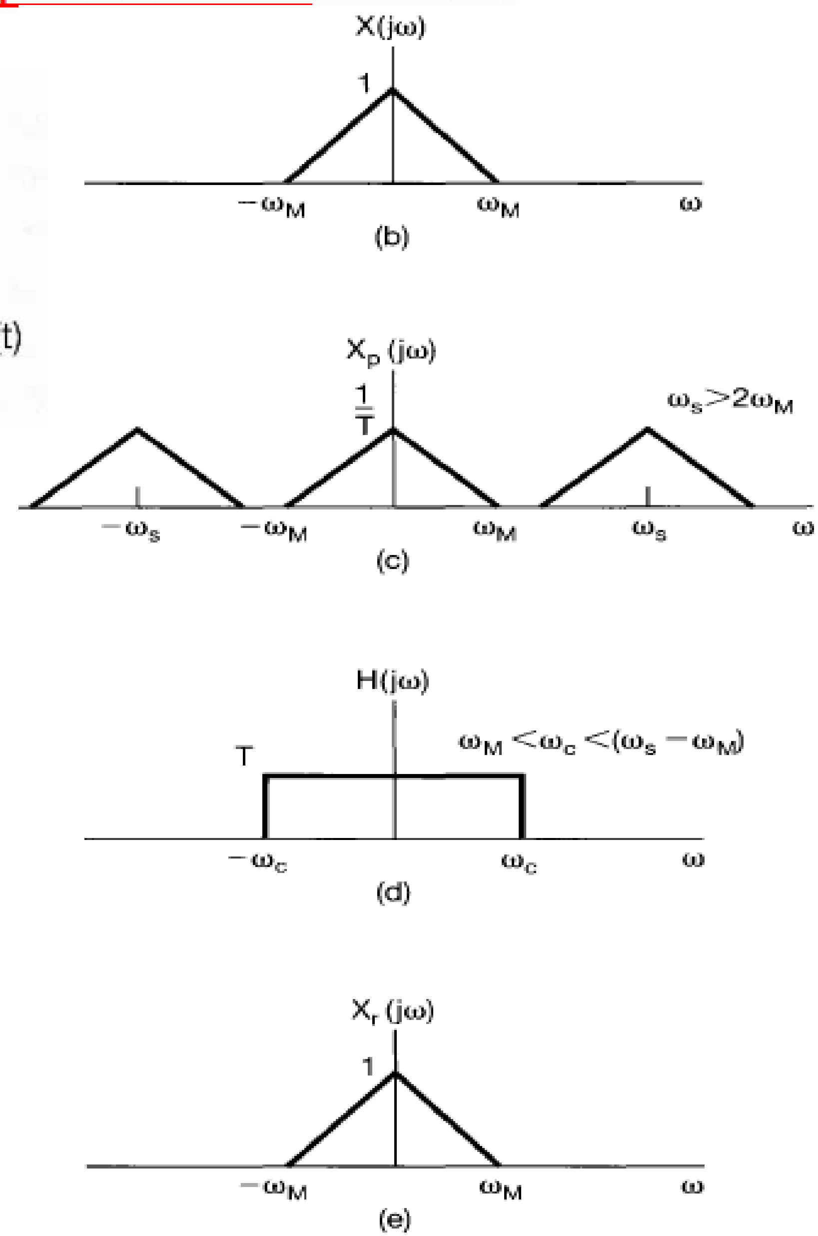
  $\Rightarrow$ $2w_M$: Nyquist rate

  w_M : Nyquist frequency

The Sampling Theorem

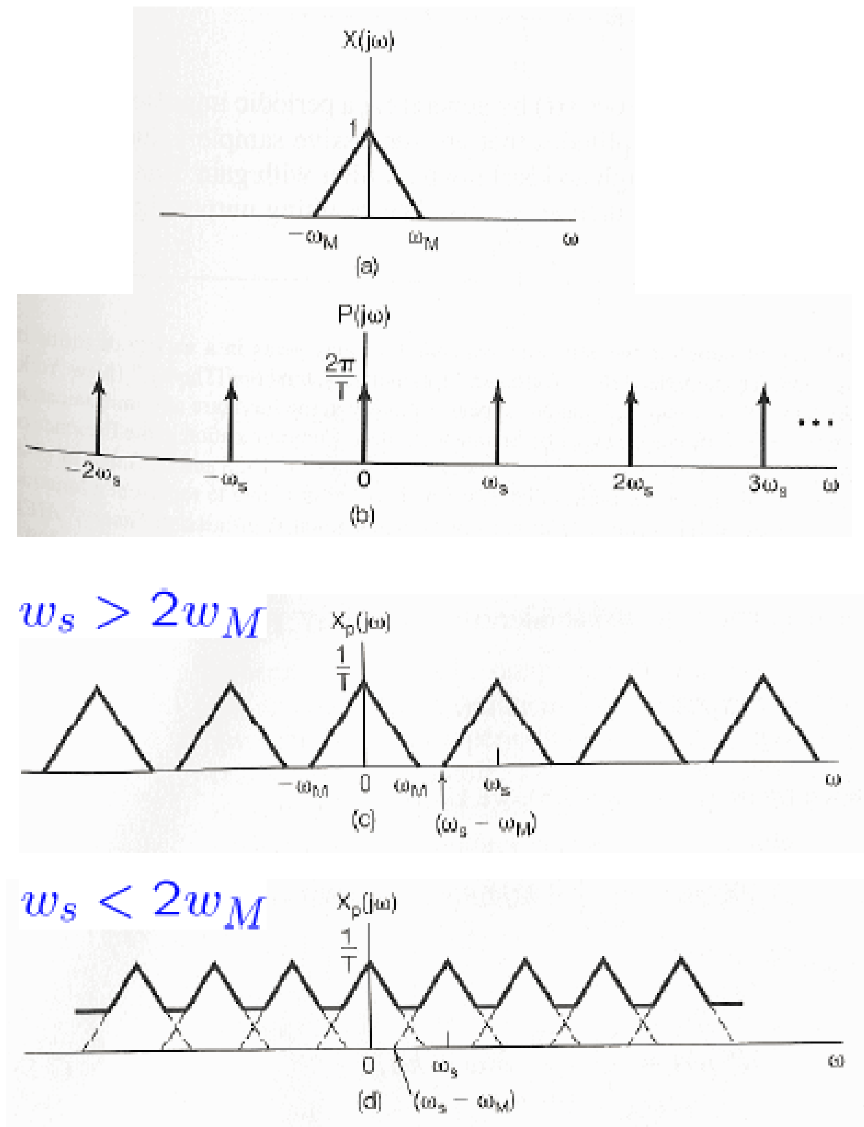
- Exact Recovery by an Ideal Lowpass Filter:





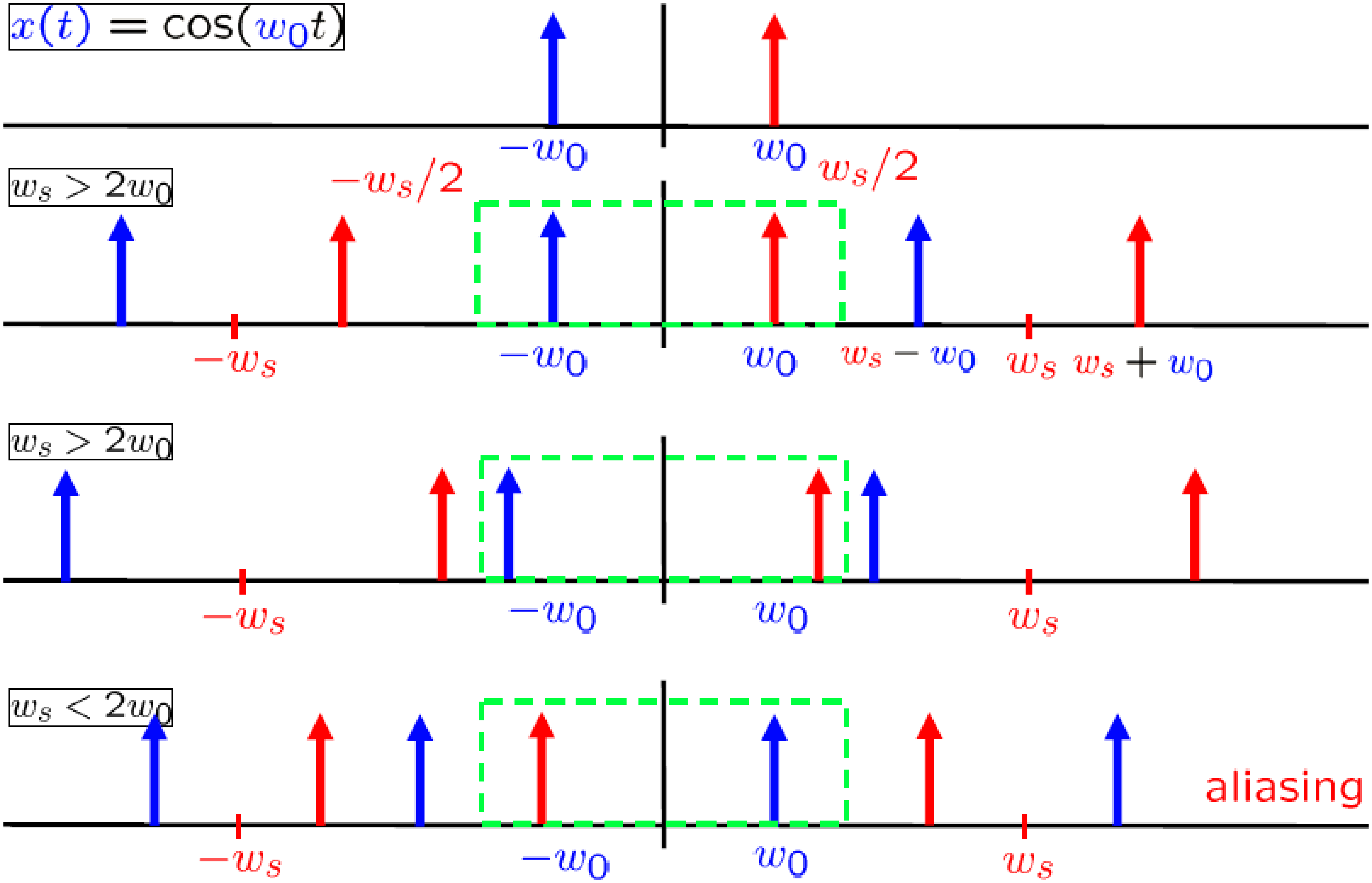
Effect of Under-sampling: Aliasing

- <u>Overlapping in Frequency-Domain: Aliasing</u>



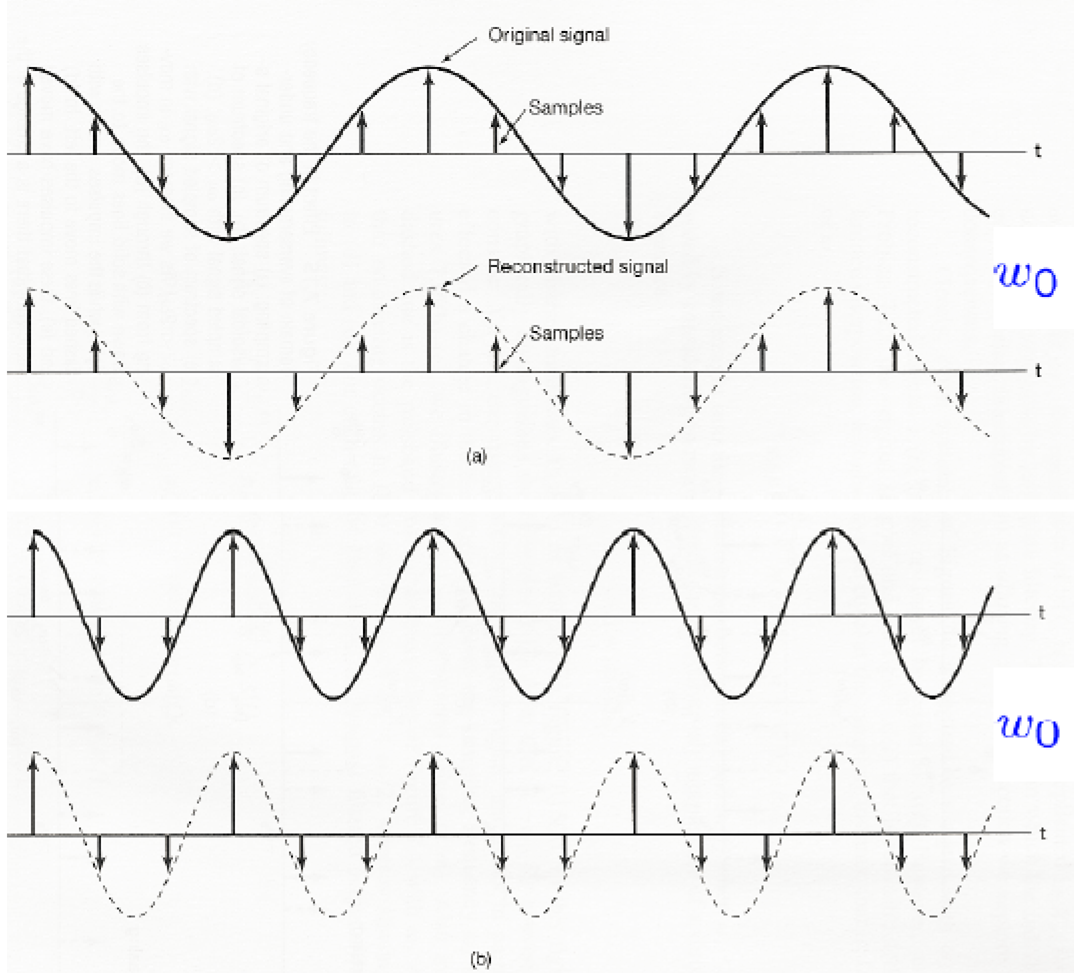
Effect of Under-sampling: Aliasing

- Overlapping in Frequency-Domain: Aliasing



Effect of Under-sampling: Aliasing

- Overlapping in Frequency-Domain: Aliasing



(a)

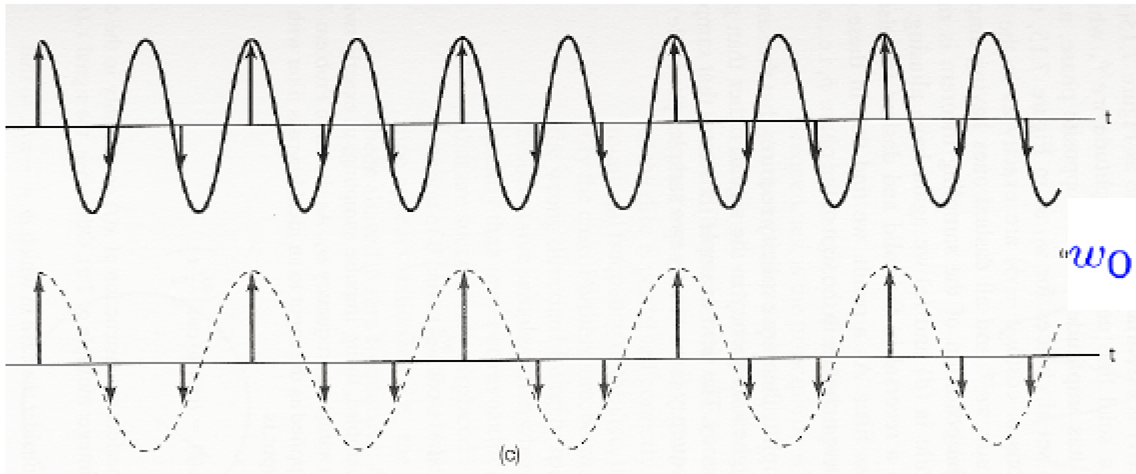
$$w_0 = \frac{w_s}{6}$$

(b)

$$w_0 = \frac{2w_s}{6}$$

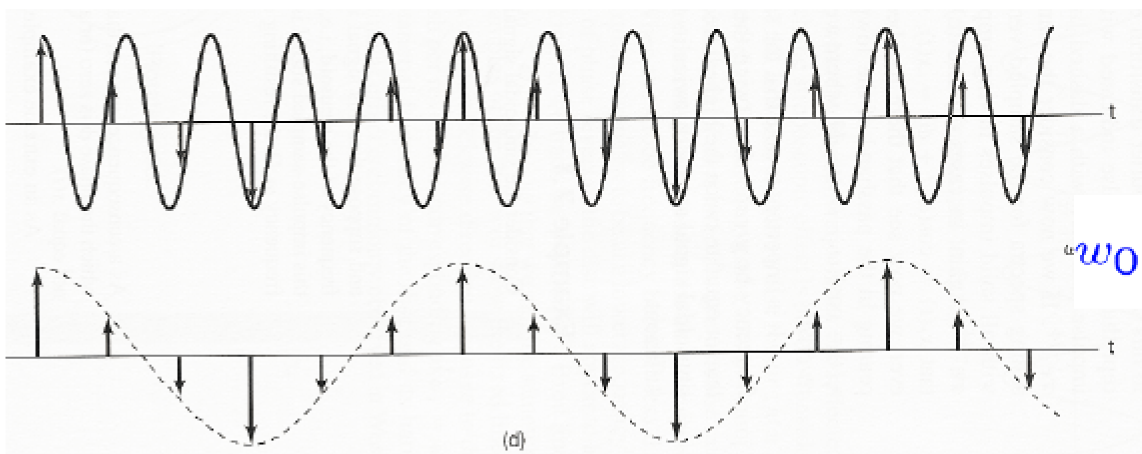
Effect of Under-sampling: Aliasing

- Overlapping in Frequency-Domain: Aliasing



$$w_0 = \frac{4w_s}{6}$$

(c)

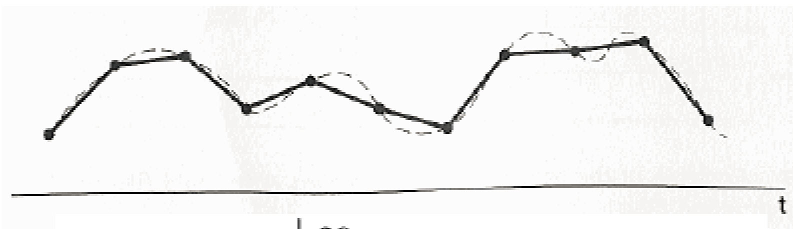


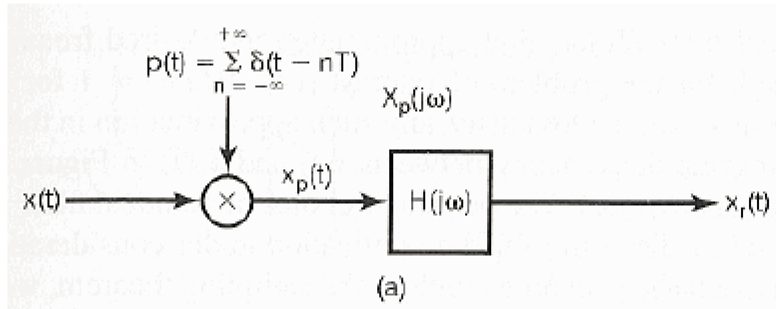
$$w_0 = \frac{5w_s}{6}$$

(d)

Reconstruction of a Signal from its Samples Using Interpolation

- Exact Interpolation:





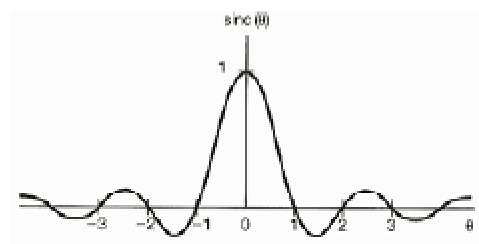
$$x_p(t) = \sum_{n=-\infty}^{+\infty} x(nT)\delta(t - nT)$$

ideal lowpass filter with a magnitude of T

$$x_r(t) = x_p(t) * h(t)$$

$$h(t) = T\,\frac{w_c}{\pi}\,\frac{\sin(w_c t)}{w_c t}$$

Ex 2.11, p. 110 $\quad x(t - t_0) = x(t) * \delta(t - t_0)$

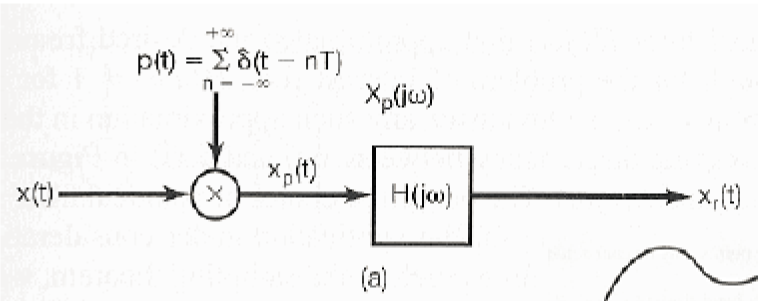


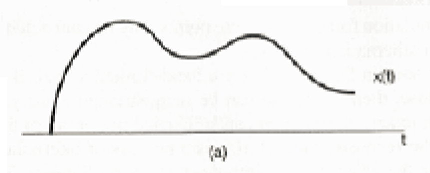
$$x_r(t) = \sum_{n=-\infty}^{+\infty} x(nT)h(t - nT)$$

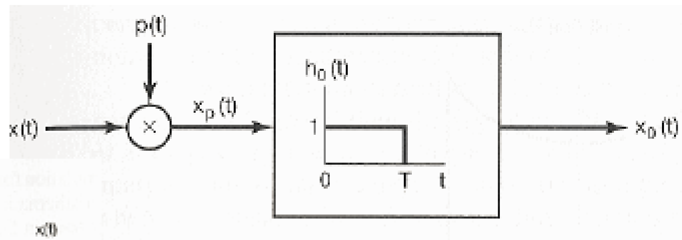
$$x_r(t) = \sum_{n=-\infty}^{+\infty} x(nT)\,\frac{w_c T}{\pi}\,\frac{\sin(w_c(t - nT))}{w_c(t - nT)}$$

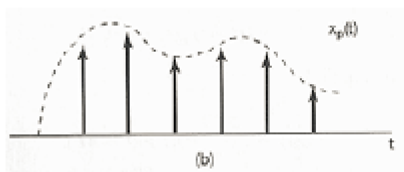
Reconstruction of a Signal from its Samples Using Interpolation

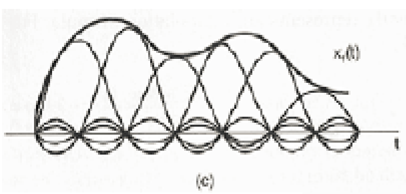
- Ideal Interpolating Filter & The Zero-Order Hold:

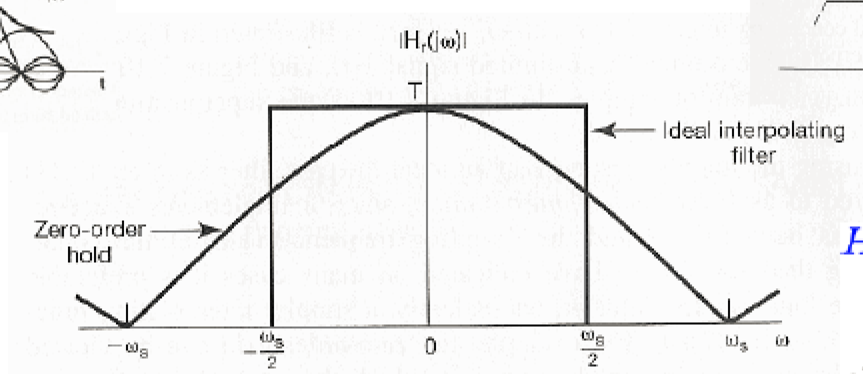


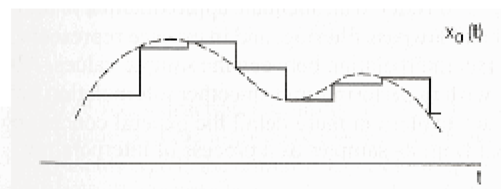












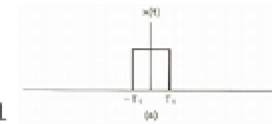
$$H_0(jw) = e^{-jwT/2}\left[\frac{2\sin(wT/2)}{w}\right]$$

Reconstruction of a Signal from its Samples Using Interpolation

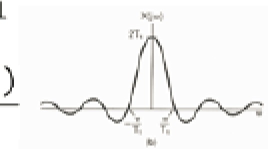
- Higher-Order Holds:

Ex 4.4, p. 293

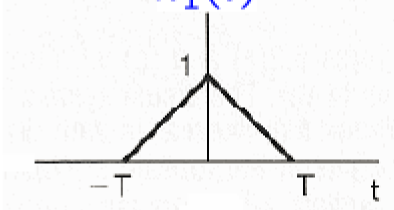
$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & |t| > T_1 \end{cases}$$



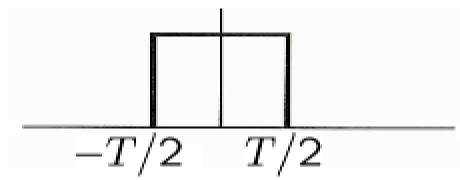
$$X(jw) = 2\frac{\sin(wT_1)}{w}$$



$h_1(t)$ $\qquad\qquad\qquad\qquad$ $h_{sq}(t)$ $\qquad\qquad$ $h_{sq}(t)$



$$= \frac{1}{T}$$



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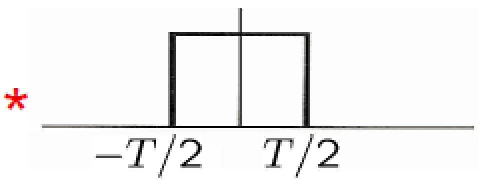
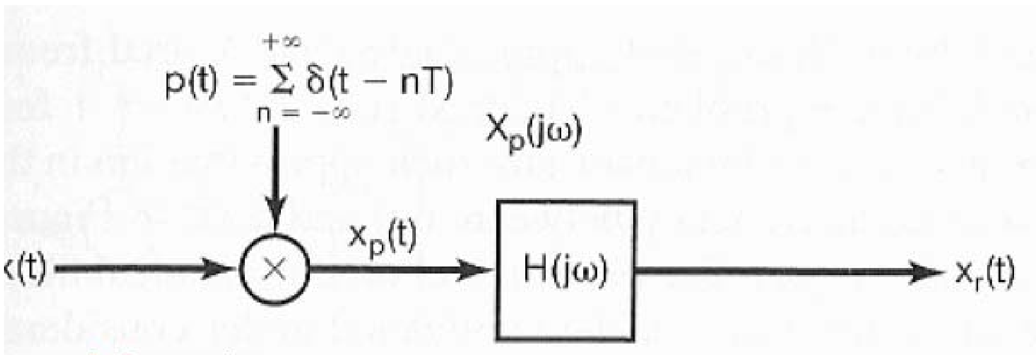


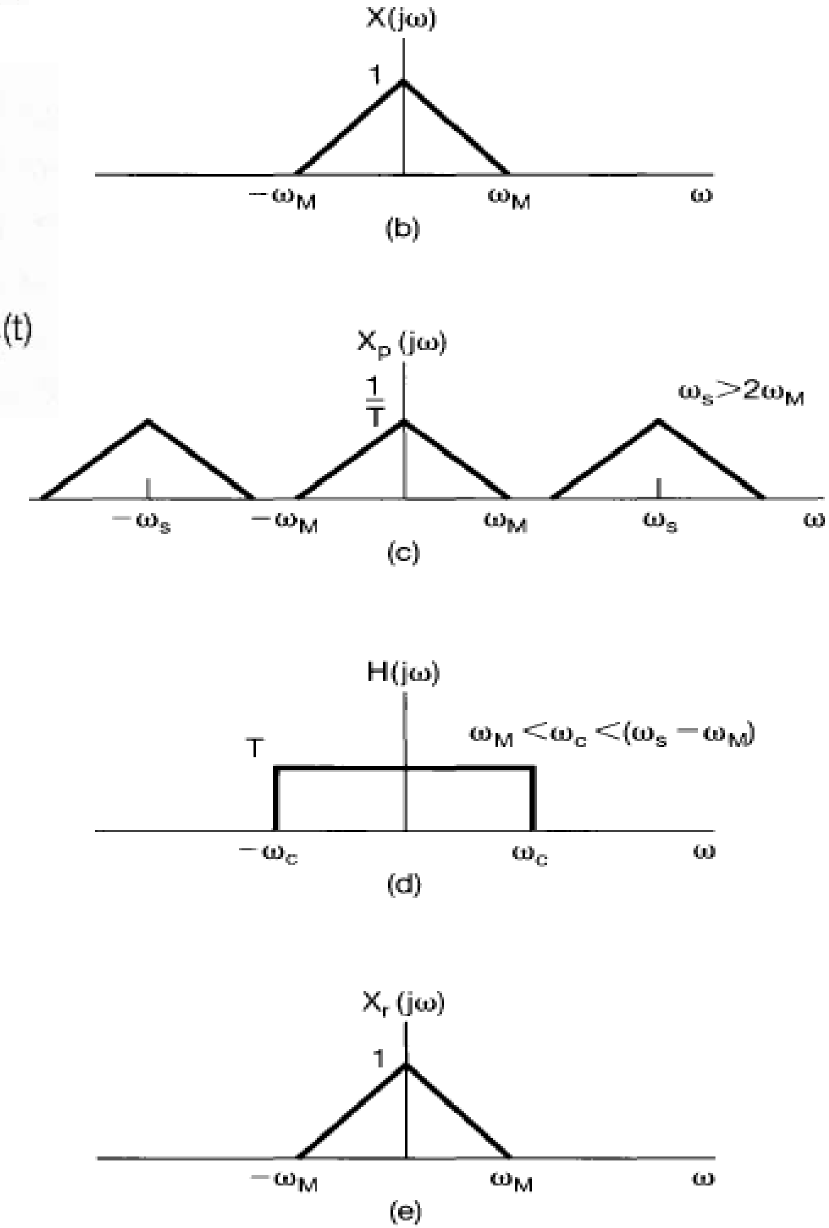
$$H_1(jw) = \frac{1}{T} \quad 2\frac{\sin(wT/2)}{w} \quad \text{X} \quad 2\frac{\sin(wT/2)}{w}$$

$$= \frac{1}{T}\left[\frac{\sin(wT/2)}{w/2}\right]^2$$

The Sampling Theorem

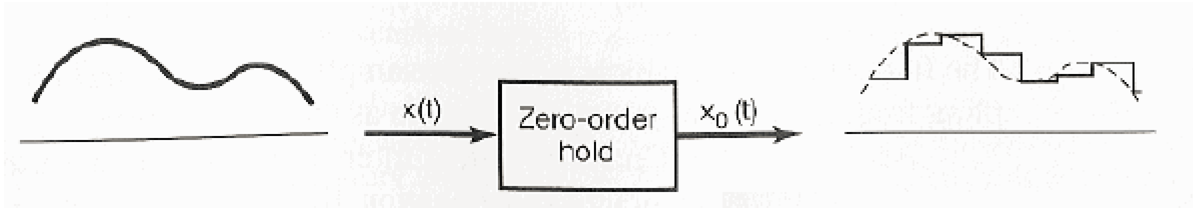
- Exact Recovery by an Ideal Lowpass Filter:

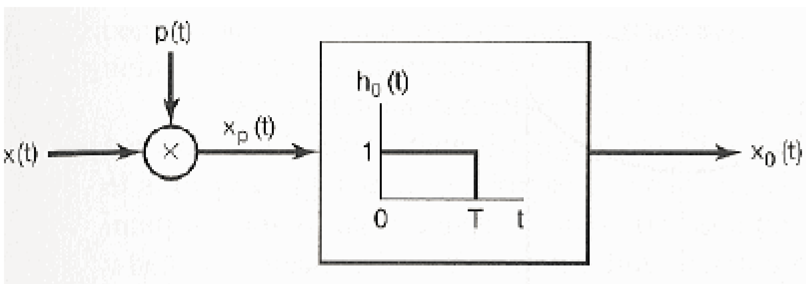


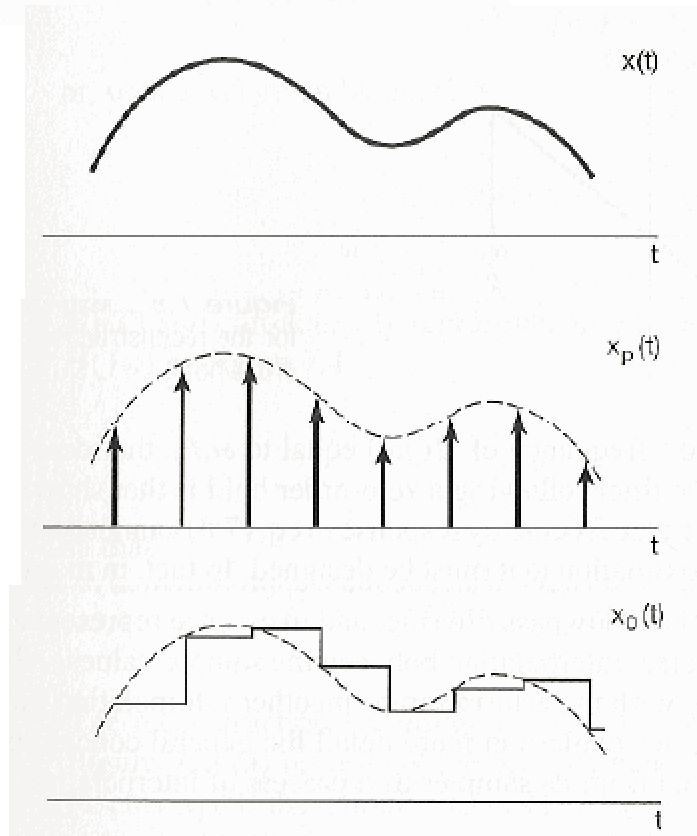


The Sampling Theorem

- Sampling with Zero-Order Hold:

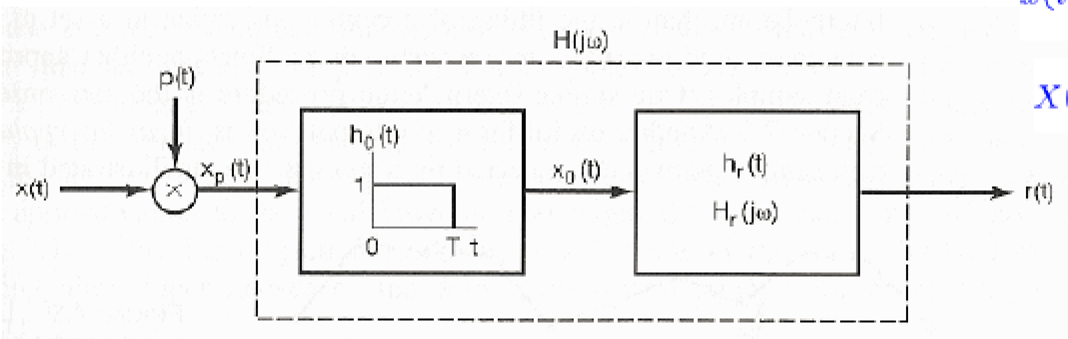






The Sampling Theorem

- Sampling with Zero-Order Hold:



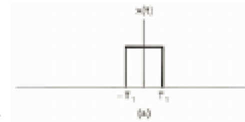
$$H_0(jw) = e^{-jwT/2}\left[\frac{2\sin(wT/2)}{w}\right]$$

$$H(jw) = H_0(jw)H_r(jw)$$

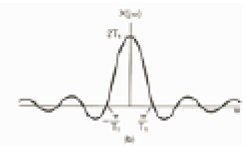
$$\Rightarrow H_r(jw) = \frac{e^{jwT/2}H(jw)}{\frac{2\sin(wT/2)}{w}}$$

Ex 4.4, p. 293

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & |t| > T_1 \end{cases}$$



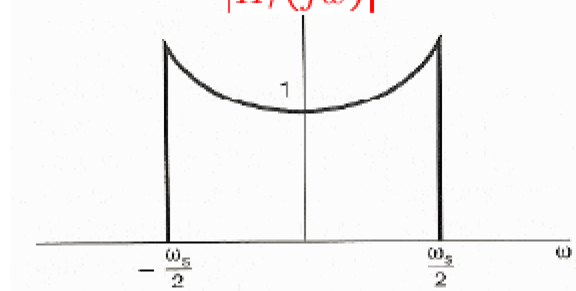
$$X(jw) = 2\frac{\sin(wT_1)}{w}$$



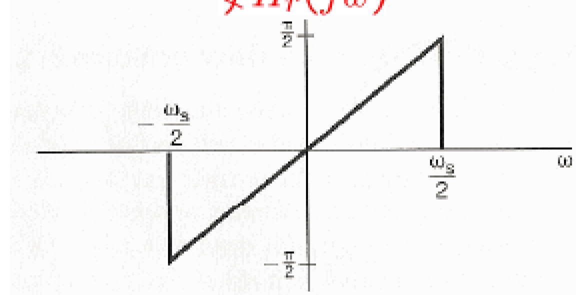
Eq 4.27, p. 301

$$x(t-t_0) \stackrel{\mathcal{F}}{\longleftrightarrow} e^{-jwt_0}X(jw)$$

$|H_r(jw)|$

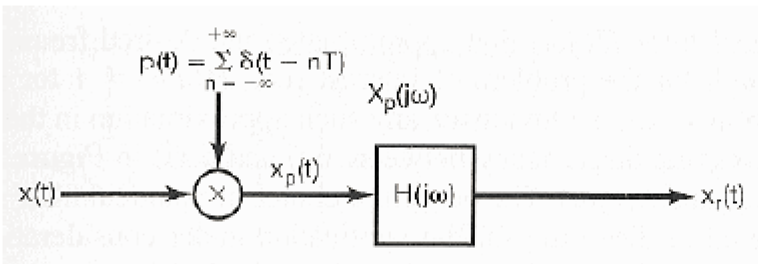


$\measuredangle H_r(jw)$

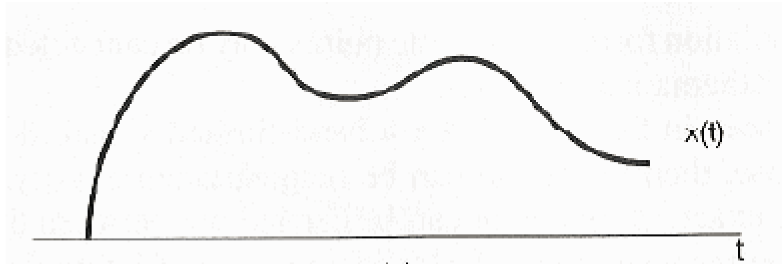


Reconstruction of a Signal from its Samples Using Interpolation

- Exact Interpolation:

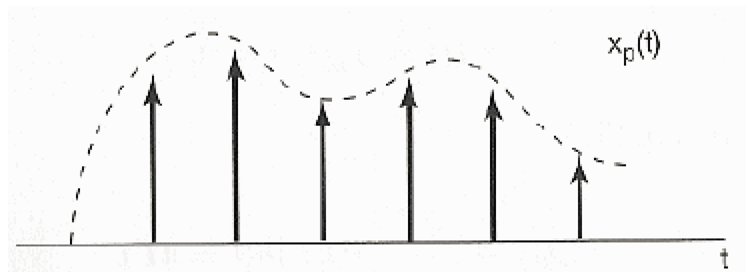


(a)

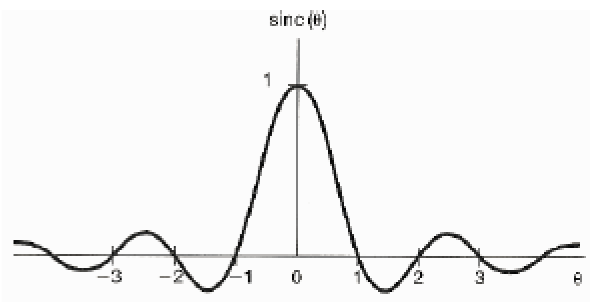


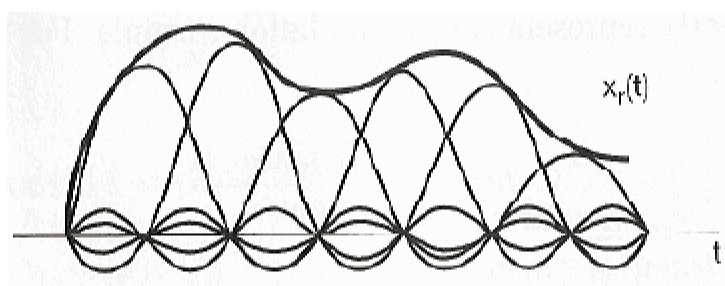
(a)

$$x_r(t) = \sum_{n=-\infty}^{+\infty} x(nT)\, \frac{w_c T}{\pi} \frac{\sin(w_c(t-nT))}{w_c(t-nT)}$$



(b)

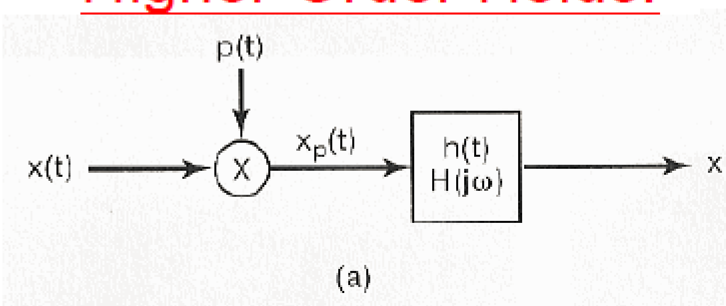


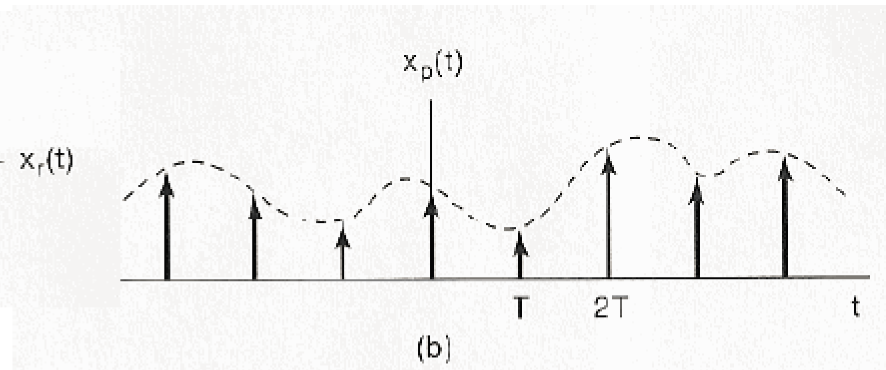


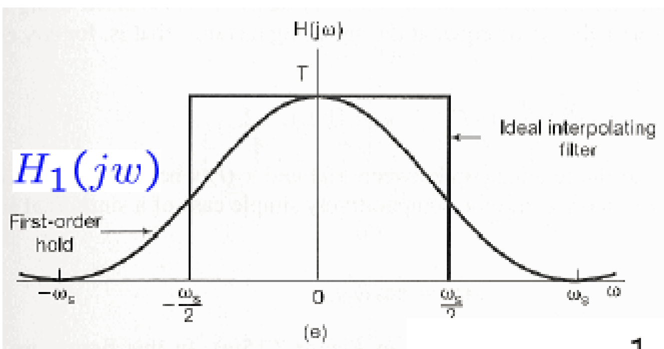
(c)

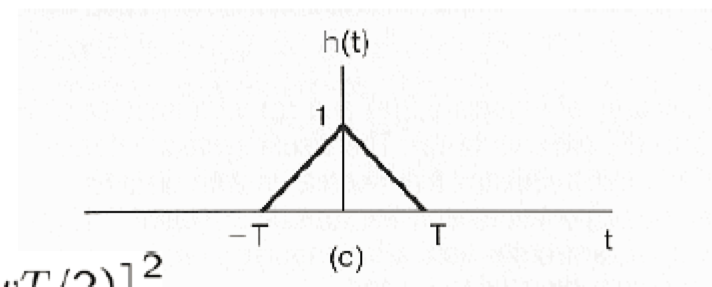
Reconstruction of a Signal from its Samples Using Interpolation

- Higher-Order Holds:

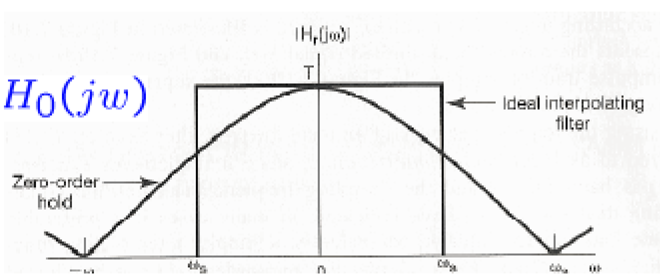


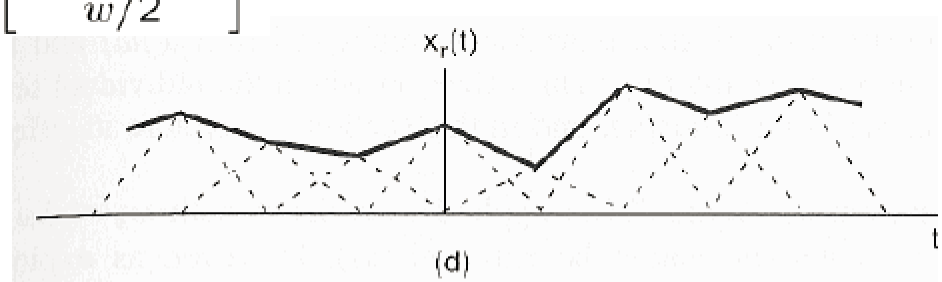






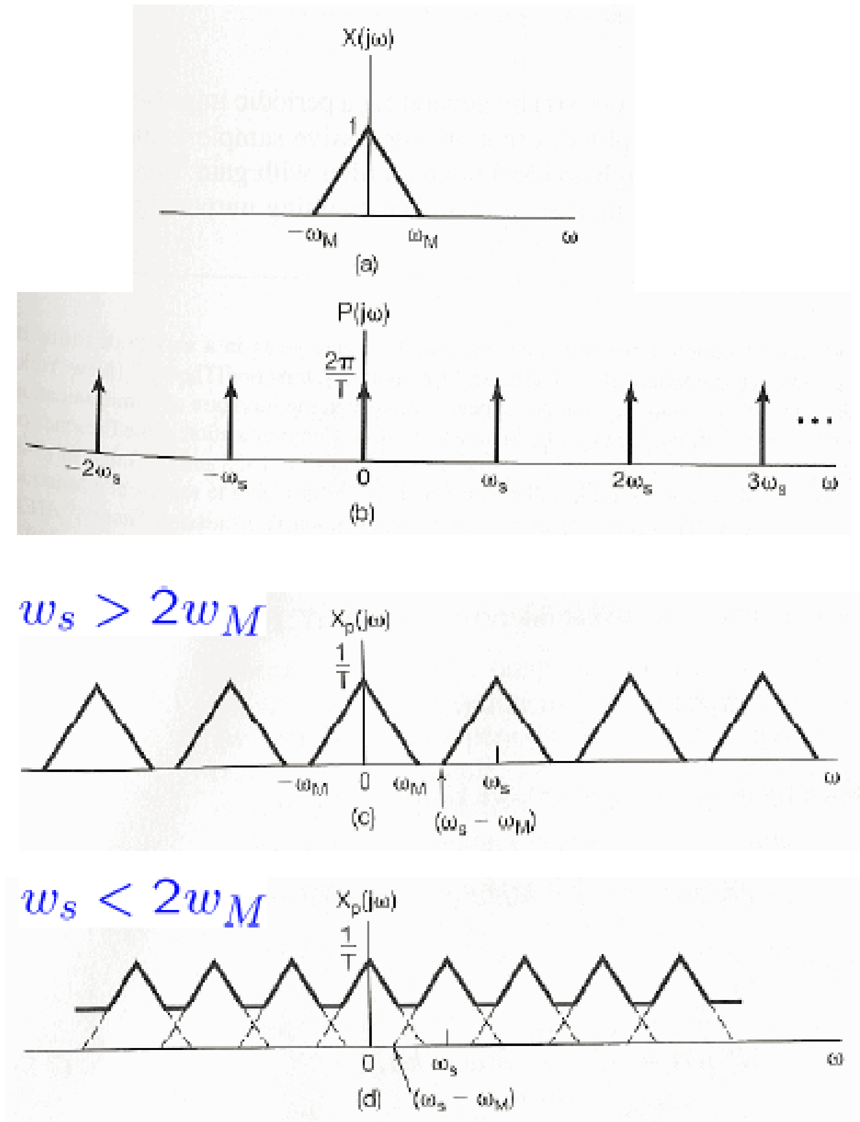
$$H_1(jw) = \frac{1}{T}\left[\frac{\sin(wT/2)}{w/2}\right]^2$$





Effect of Under-sampling: Aliasing

- Overlapping in Frequency-Domain: Aliasing



Summary

• The sampling theorem explicitly requires that the sampling rate is greater than twice the highest frequency in the signal

• In practice, an antialiasing filter is required before sampling in order to guarantee the elimination of high frequency components from the Signal

• In digital processing of signal samples, the computations required for generation of one output sample must be completed within the sampling period *T*

• The sampling frequency determines the computational requirements of the DSP implementation

• Thus, oversampling, i.e., increasing the sampling rate considerably above the required minimum, results in higher computational requirements