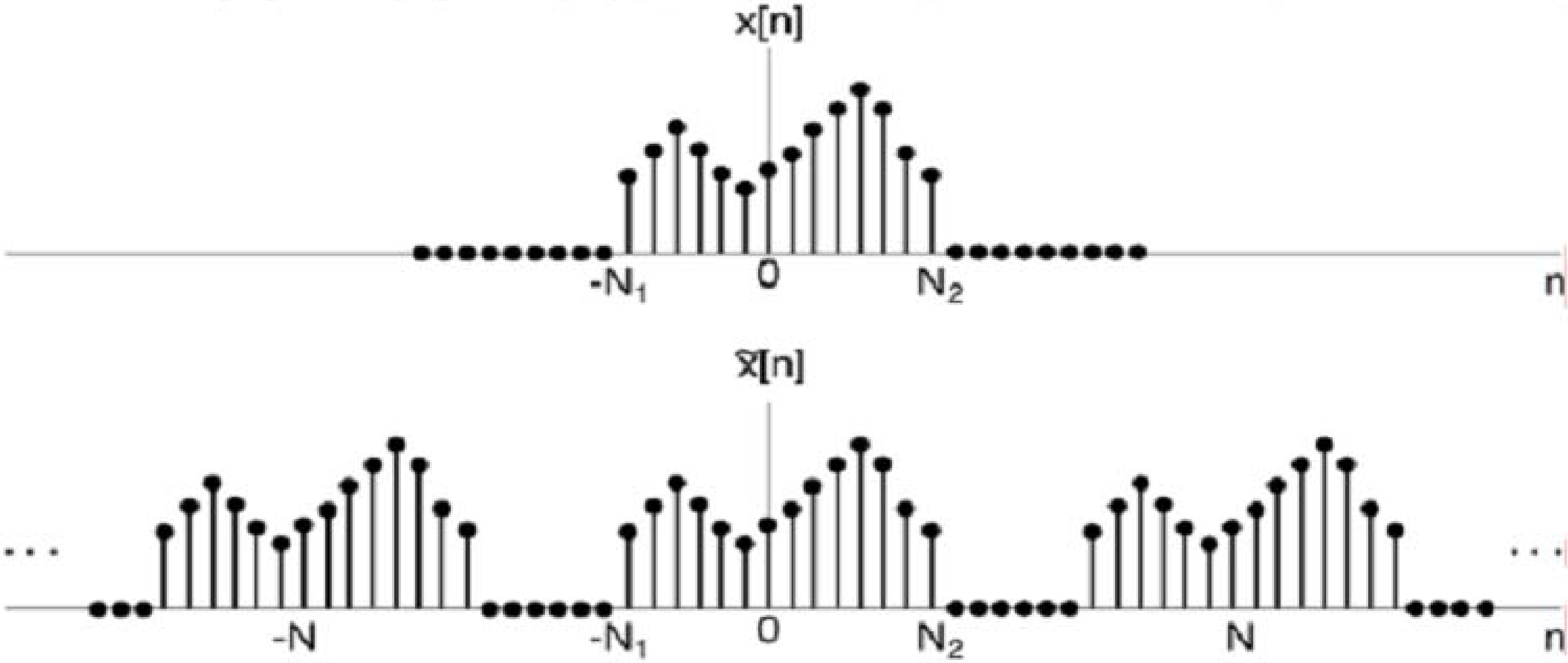
Introduction to Signals and Systems: V216

Lecture #10
Chapter 5: DT Fourier Transform

The DTFT

Derivation:

- $x[n]$ - aperiodic and (for simplicity) of finite duration
- N is large enough so that $x[n] = 0$ if $|n| \geq N/2$
- $\tilde{x}[n] = x[n]$ for $|n| \leq N/2$ and periodic with period N



$$\tilde{x}[n] = x[n] \text{ for } any\ n \text{ as } N \to \infty$$

The DTFT

$$
\begin{aligned}
\tilde{x}[n] &= \sum_{k=<N>} a_k e^{jk\omega_0 n}, \ \omega_0 = \frac{2\pi}{N} && \text{DTFS synthesis eq.} \\
a_k &= \frac{1}{N} \sum_{n=<N>} \tilde{x}[n] e^{-jk\omega_0 n} && \text{DTFS analysis eq.} \\
&= \frac{1}{N} \sum_{n=-N_1}^{N_2} \tilde{x}[n] e^{-jk\omega_0 n} = \frac{1}{N} \sum_{n=-\infty}^{\infty} x[n] e^{-jk\omega_0 n}
\end{aligned}
$$

Define

$$
\begin{aligned}
X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \quad \boxed{-\text{ periodic in } \omega \text{ with period } 2\pi} \\
&\Downarrow \\
a_k &= \frac{1}{N} X(e^{jk\omega_0})
\end{aligned}
$$

The DTFT

$$\tilde{x}[n] = \sum_{k=<N>} \underbrace{\frac{1}{N} X(e^{jk\omega_0})}_{a_k} e^{jk\omega_0 n} = \frac{1}{2\pi} \sum_{k=<N>} X(e^{jk\omega_0}) e^{jk\omega_0 n} \omega_0 \quad (*)$$

As $N \to \infty$: $\quad \tilde{x}[n] \to x[n]$ for every n

$$\omega_0 \to 0, \sum \omega_0 \to \int d\omega$$

The sum in $(*) \to$ an integral

$\Downarrow$ The DTFT Pair

$$\boxed{\begin{array}{ll} x[n] = \dfrac{1}{2\pi} \displaystyle\int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega & \text{Synthesis equation} \\ X(e^{j\omega}) = \displaystyle\sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} & \text{Analysis equation} \end{array}}$$

Any 2π interval in ω

DTFT Examples

1) $x[n] = \delta[n]$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \delta[n] e^{-j\omega n} = 1$$

2) $x[n] = \delta[n - n_0]$ - shifted unit sample

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \delta[n - n_0] e^{-j\omega n} = e^{-j\omega n_0}$$

Same amplitude (=1) as above,
but with a *linear* phase $-\omega n_0$

$x[n] = a^n u[n], |a| < 1$ - Exponentially decaying function

$$X(e^{j\omega}) = \sum_{n=0}^{\infty} a^n e^{-j\omega n} = \sum_{n=0}^{\infty} \underbrace{\left(ae^{-j\omega}\right)^n}_{|ae^{-j\omega}|<1}$$

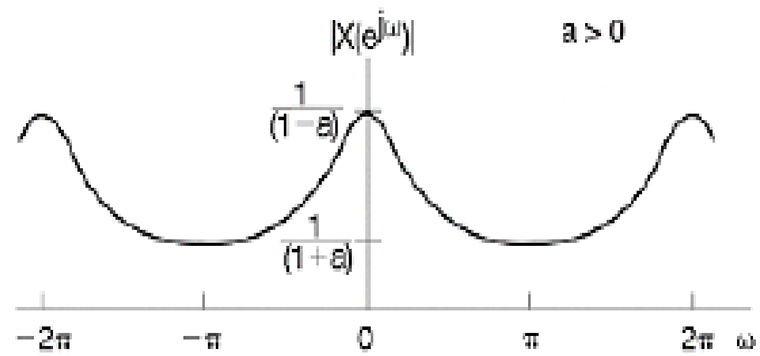
Infinite sum formula

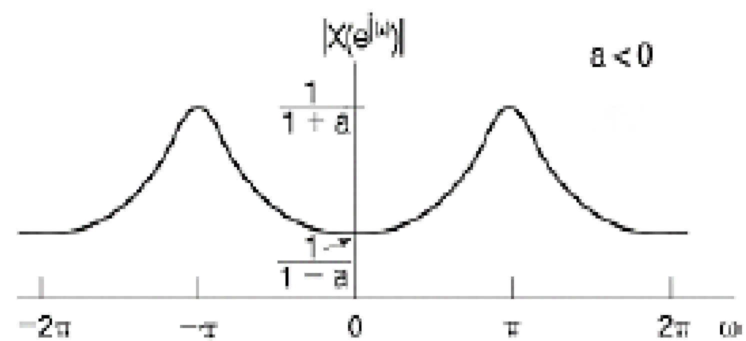
$$= \frac{1}{1 - ae^{-j\omega}} = \frac{1}{(1 - a\cos\omega) + ja\sin\omega}$$

$$|X(e^{j\omega})| = \frac{1}{\sqrt{1 - 2a\cos\omega + a^2}}$$

$$\omega = 0 : X(e^{j\omega}) = \frac{1}{\sqrt{1 - 2a + a^2}} = \frac{1}{1-a}$$

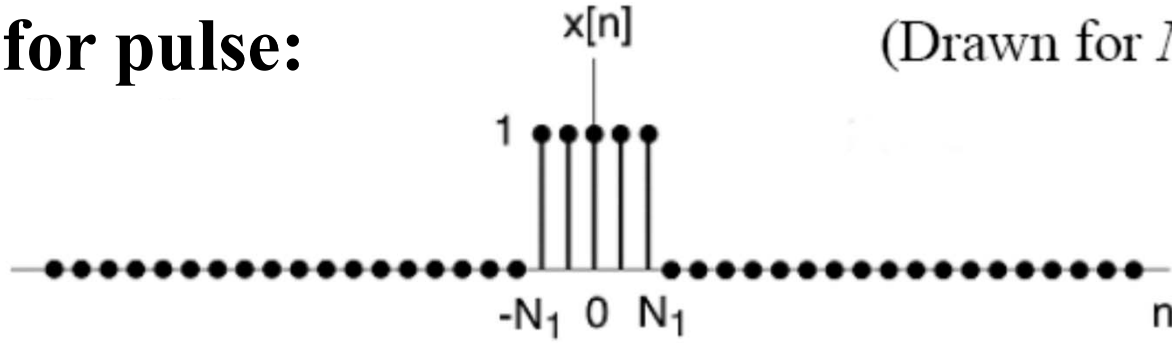
$$\omega = \pi : X(e^{j\omega}) = \frac{1}{\sqrt{1 + 2a + a^2}} = \frac{1}{1+a}$$



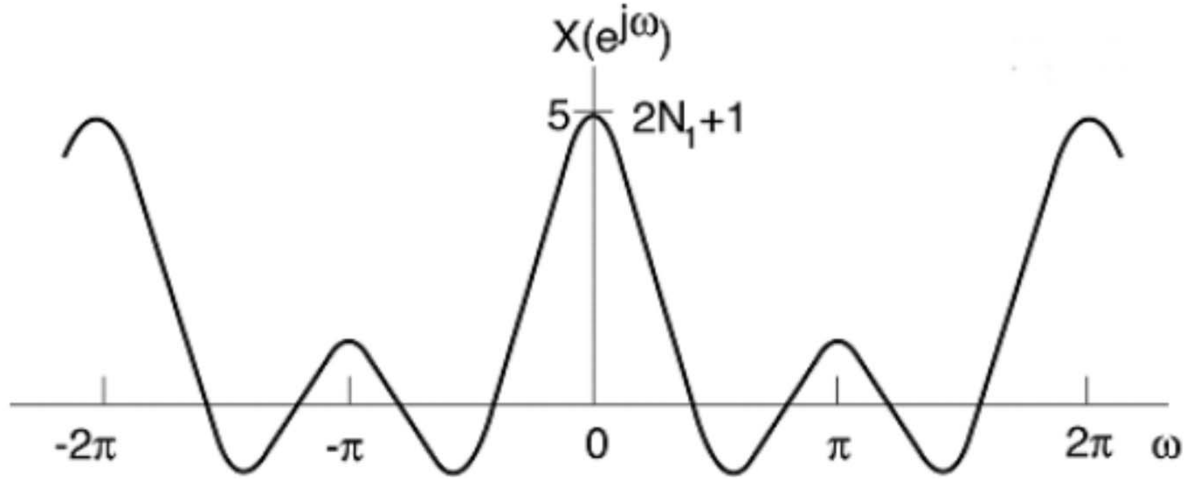


The DTFT for pulse:

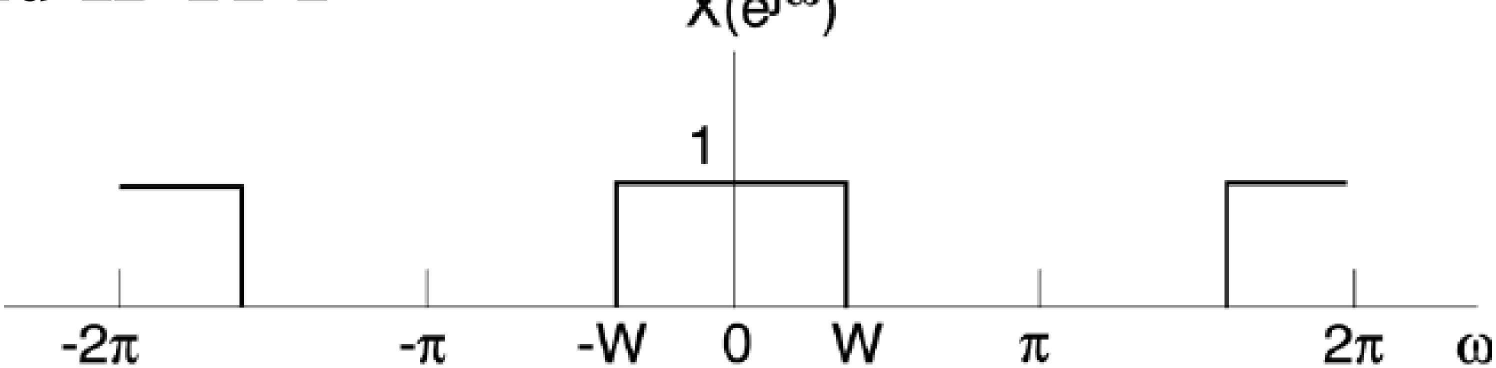
(Drawn for $N_1 = 2$)



$$X(e^{j\omega}) = \sum_{n=-N_1}^{N_1} e^{-j\omega n} = \sum_{n=-N_1}^{N_1} (e^{-j\omega})^n = \frac{\sin \omega \left(N_1 + \frac{1}{2}\right)}{\sin(\omega/2)} = X(e^{j(\omega - 2\pi)})$$

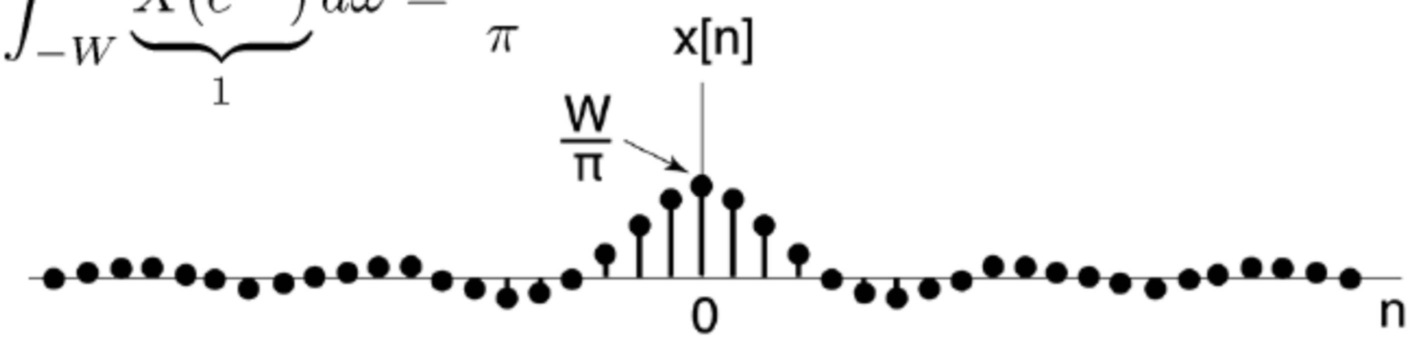


Find IDTFT



$$x[n] = \frac{1}{2\pi}\int_{-W}^{W} e^{j\omega n}\,d\omega = \frac{\sin Wn}{\pi n}$$

$$x[0] = \frac{1}{2\pi}\int_{-W}^{W} \underbrace{X(e^{j\omega})}_{1}\,d\omega = \frac{W}{\pi}$$



DTFT for Periodic Signals

Recall the following DTFT pair:

$$e^{jk\omega_0 n} \longleftrightarrow 2\pi \sum_{m=-\infty}^{\infty} \delta(\omega - k\omega_0 - 2\pi m)$$

Represent periodic signal $x[n]$ in terms of DTFS:

$$x[n] = x[n+N] = \sum_{k=<N>} a_k e^{jk\omega_0 n}, \; \omega_0 = \frac{2\pi}{N}$$

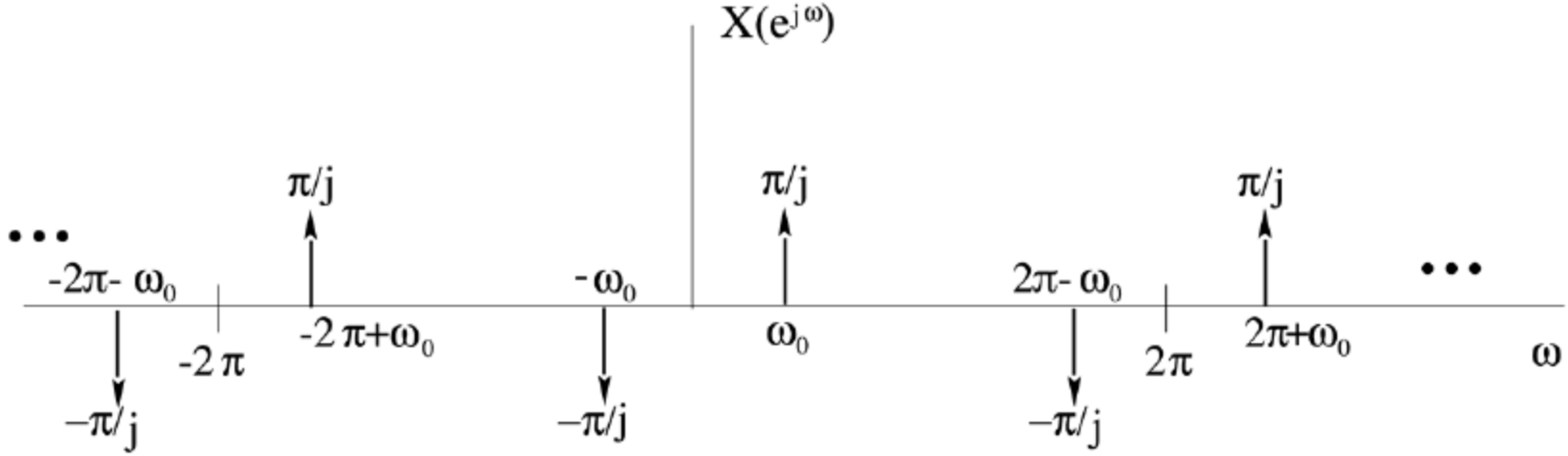
$$\begin{aligned} X(e^{j\omega}) &= \sum_{k=<N>} a_k \left[2\pi \sum_{m=-\infty}^{\infty} \delta(\omega - k\omega_0 - 2\pi m) \right] \\ &= 2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\omega - k\omega_0) = 2\pi \sum_{k=-\infty}^{\infty} a_k \delta\left(\omega - \frac{2\pi k}{N}\right) \end{aligned}$$

Linearity of DTFT

DTFT for Periodic Signals

$$x[n] = \sin \omega_0 n = \frac{1}{2j} e^{j\omega_0 n} - \frac{1}{2j} e^{-j\omega_0 n}$$

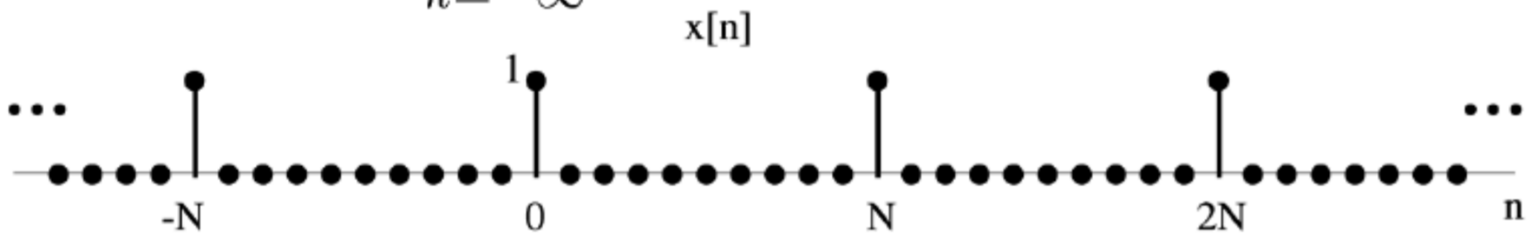
$$X(e^{j\omega}) = \frac{\pi}{j} \sum_{m=-\infty}^{\infty} \delta(\omega - \omega_0 - 2\pi m) - \frac{\pi}{j} \sum_{m=-\infty}^{\infty} \delta(\omega + \omega_0 - 2\pi m)$$



DTFT for Periodic Signals

Find DTFT for Periodic Impulse Train

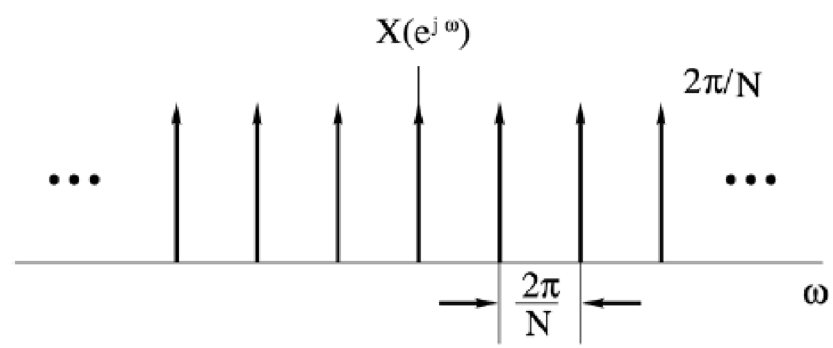
$$x[n] = \sum_{k=-\infty}^{\infty} \delta[n - kN] \qquad \omega_0 = 2\pi/N$$



The DTFS coefficients for this signal are:

$$a_k = \frac{1}{N} \sum_{n=<N>} x[n] e^{-jk\omega_0 n} = \frac{1}{N} \sum_{n=0}^{N-1} \underbrace{x[n]}_{=\delta[n]} e^{-jk\omega_0 n} = \frac{1}{N}$$

$$X(e^{j\omega}) = \frac{2\pi}{N} \sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi k}{N}\right)$$



Properties of the DT Fourier Transform

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \qquad \text{– Analysis equation}$$

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}\,d\omega \qquad \text{– Synthesis equation}$$

1) Periodicity: $X(e^{j(\omega+2\pi)}) = X(e^{j\omega})$

2) Linearity: $ax_1[n] + bx_2[n] \longleftrightarrow aX_1(e^{j\omega}) + bX_2(e^{j\omega})$

More Properties

3) **Time Shifting:** $x[n - n_0] \longleftrightarrow e^{-j\omega n_0} X(e^{j\omega})$

4) **Frequency Shifting:** $e^{j\omega_0 n} x[n] \longleftrightarrow X(e^{j(\omega-\omega_0)})$

5) Time Reversal:

$$x[-n] \longleftrightarrow X(e^{-j\omega})$$

6) Conjugate Symmetry:

$$x[n] \text{ real} \Rightarrow X(e^{j\omega}) = X^*(e^{-j\omega})$$

$$\Downarrow$$

$$|X(e^{j\omega})| \text{ and } \Re e\left\{X(e^{j\omega})\right\} \text{ are even functions}$$
$$\angle X(e^{j\omega}) \text{ and } \Im m\left\{X(e^{j\omega})\right\} \text{ are odd functions}$$

and

$$x[n] \text{ real and even} \Leftrightarrow X(e^{j\omega}) \text{ real and even}$$
$$x[n] \text{ real and odd} \Leftrightarrow X(e^{j\omega}) \text{ purely imaginary and odd}$$

7) Time Expansion

Recall CT property: $x(at) \longleftrightarrow \frac{1}{|a|} X\left(j\left(\frac{\omega}{a}\right)\right)$

Time scale in CT is infinitely fine

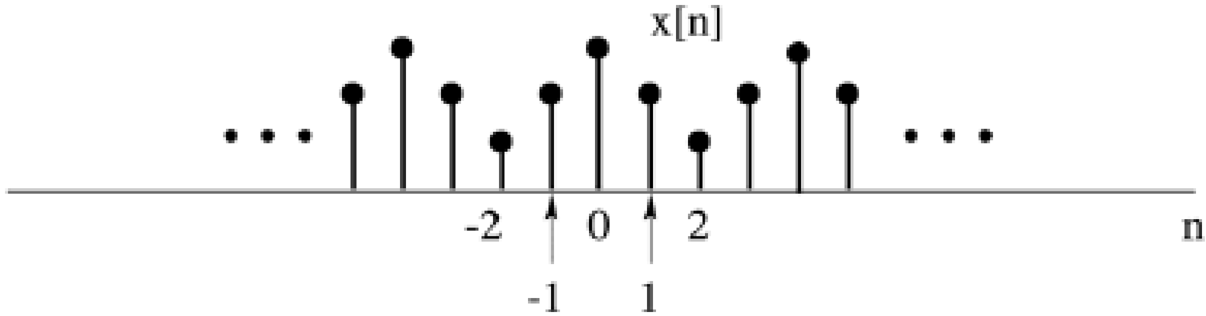
But in DT: $x[n/2]$ makes no sense

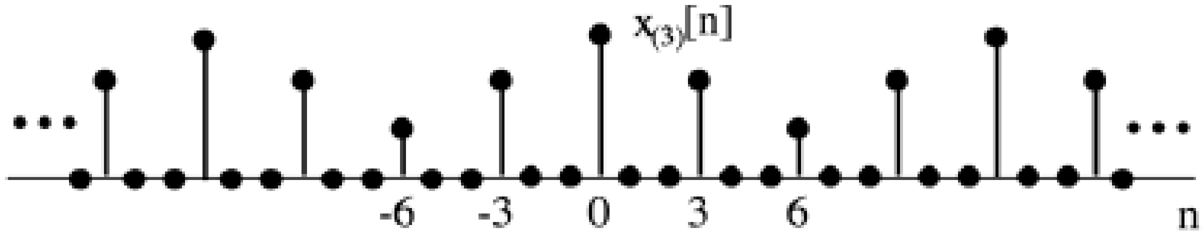
$x[2n]$ misses odd values of $x[n]$

But we can "slow" a DT signal down by inserting zeros:

k — an integer ≥ 1

$x_{(k)}[n]$ — insert $(k - 1)$ zeros between successive values





Insert two zeros in this example (k=3)

Time Expansion (continued)

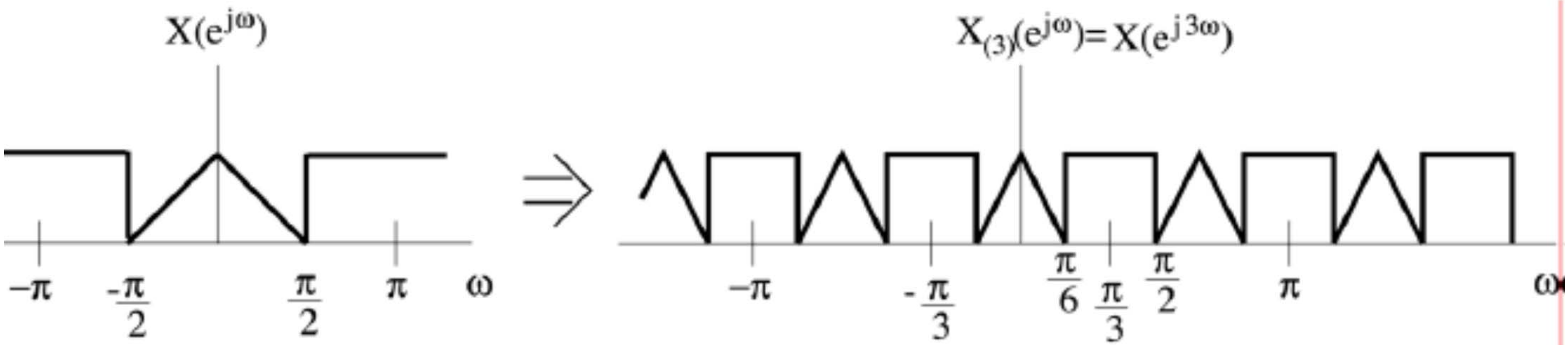
$$x_{(k)}[n] = \begin{cases} x[n/k] & \text{if } n \text{ is an integer multiple of k} \\ 0 & \text{otherwise} \end{cases}$$

— Stretched by a factor of *k* in time domain

$$X_{(k)}(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x_{(k)}[n] e^{-j\omega n} \overset{n=mk}{=} \sum_{m=-\infty}^{\infty} x_{(k)}[mk] e^{-j\omega mk}$$

$$= \sum_{m=-\infty}^{\infty} x[m] e^{-j(k\omega)m} = X(e^{jk\omega})$$

-compressed by a factor of *k* in frequency domain



8) Differentiation in Frequency

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

$$\frac{d}{d\omega}X(e^{j\omega}) = -j\sum_{n=-\infty}^{\infty} nx[n]e^{-j\omega n}$$

$\Downarrow$ multiply by j on both sides

Multiplication by n $\quad nx[n] \leftrightarrow j\frac{d}{d\omega}X(e^{j\omega}) \quad$ Differentiation in frequency

9) Parseval's Relation

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{2\pi} |X(e^{j\omega})|^2 d\omega$$

9) The Convolution Property

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n] = h[n] * x[n]$$

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$$

10) Multiplication Property

$$x_1[n]\,x_2[n] \longleftrightarrow 1/2\pi X_1(e^{j\omega}) \otimes X_2(e^{j\omega})$$

DTFT Properties

TABLE 5.1 PROPERTIES OF THE DISCRETE-TIME FOURIER TRANSFORM

Section	Property	Aperiodic Signal	Fourier Transform
		$x[n]$	$X(e^{j\omega})$ periodic with
		$y[n]$	$Y(e^{j\omega})$ period 2π
5.3.2	Linearity	$ax[n] + by[n]$	$aX(e^{j\omega}) + bY(e^{j\omega})$
5.3.3	Time Shifting	$x[n - n_0]$	$e^{-j\omega n_0}X(e^{j\omega})$
5.3.3	Frequency Shifting	$e^{j\omega_0 n}x[n]$	$X(e^{j(\omega-\omega_0)})$
5.3.4	Conjugation	$x^*[n]$	$X^*(e^{-j\omega})$
5.3.6	Time Reversal	$x[-n]$	$X(e^{-j\omega})$
5.3.7	Time Expansion	$x_{(k)}[n] = \begin{cases} x[n/k], & \text{if } n = \text{multiple of } k \\ 0, & \text{if } n \neq \text{multiple of } k \end{cases}$	$X(e^{jk\omega})$
5.4	Convolution	$x[n] * y[n]$	$X(e^{j\omega})Y(e^{j\omega})$
5.5	Multiplication	$x[n]y[n]$	$\frac{1}{2\pi}\int_{2\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$
5.3.5	Differencing in Time	$x[n] - x[n-1]$	$(1 - e^{-j\omega})X(e^{j\omega})$
5.3.5	Accumulation	$\sum_{k=-\infty}^{n} x[k]$	$\frac{1}{1-e^{-j\omega}}X(e^{j\omega}) + \pi X(e^{j0})\sum_{k=-\infty}^{+\infty}\delta(\omega - 2\pi k)$
5.3.8	Differentiation in Frequency	$nx[n]$	$j\frac{dX(e^{j\omega})}{d\omega}$
5.3.4	Conjugate Symmetry for Real Signals	$x[n]$ real	$\begin{cases} X(e^{j\omega}) = X^*(e^{-j\omega}) \\ \mathcal{R}e\{X(e^{j\omega})\} = \mathcal{R}e\{X(e^{-j\omega})\} \\ \mathcal{I}m\{X(e^{j\omega})\} = -\mathcal{I}m\{X(e^{-j\omega})\} \\ \lvert X(e^{j\omega})\rvert = \lvert X(e^{-j\omega})\rvert \\ \sphericalangle X(e^{j\omega}) = -\sphericalangle X(e^{-j\omega}) \end{cases}$
5.3.4	Symmetry for Real, Even Signals	$x[n]$ real an even	$X(e^{j\omega})$ real and even
5.3.4	Symmetry for Real, Odd Signals	$x[n]$ real and odd	$X(e^{j\omega})$ purely imaginary and odd
5.3.4	Even-odd Decomposition of Real Signals	$x_e[n] = \mathcal{E}v\{x[n]\}$ [$x[n]$ real] $x_o[n] = \mathcal{O}d\{x[n]\}$ [$x[n]$ real]	$\mathcal{R}e\{X(e^{j\omega})\}$ $j\mathcal{I}m\{X(e^{j\omega})\}$
5.3.9	Parseval's Relation for Aperiodic Signals	$\sum_{n=-\infty}^{+\infty} \lvert x[n]\rvert^2 = \frac{1}{2\pi}\int_{2\pi} \lvert X(e^{j\omega})\rvert^2 d\omega$	

a duality relationship between the discrete-time Fourier transform and the continuous-time Fourier series. This relation is discussed in Section 5.7.2.

5.7.1 Duality in the Discrete-Time Fourier Series

Since the Fourier series coefficients a_k of a periodic signal $x[n]$ are themselves a periodic sequence, we can expand the sequence a_k in a Fourier series. The duality property for discrete-time Fourier series implies that the Fourier series coefficients for the periodic sequence a_k are the values of $(1/N)x[-n]$ (i.e., are proportional to the values of the original

DTFT Properties

TABLE 5.1 PROPERTIES OF THE DISCRETE-TIME FOURIER TRANSFORM

Section	Property	Aperiodic Signal	Fourier Transform
		$x[n]$ $y[n]$	$\left.\begin{array}{l}X(e^{j\omega})\\Y(e^{j\omega})\end{array}\right\}$ periodic with period 2π
5.3.2	Linearity	$ax[n]+by[n]$	$aX(e^{j\omega})+bY(e^{j\omega})$
5.3.3	Time Shifting	$x[n-n_0]$	$e^{-j\omega n_0}X(e^{j\omega})$
5.3.3	Frequency Shifting	$e^{j\omega_0 n}x[n]$	$X(e^{j(\omega-\omega_0)})$
5.3.4	Conjugation	$x^*[n]$	$X^*(e^{-j\omega})$
5.3.6	Time Reversal	$x[-n]$	$X(e^{-j\omega})$
5.3.7	Time Expansion	$x_{(k)}[n]=\begin{cases}x[n/k], & \text{if } n = \text{multiple of } k\\ 0, & \text{if } n \neq \text{multiple of } k\end{cases}$	$X(e^{jk\omega})$
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5.3.5	Differencing in Time	$x[n]-x[n-1]$	$(1-e^{-j\omega})X(e^{j\omega})$
5.3.5	Accumulation	$\sum_{k=-\infty}^{n}x[k]$	$\frac{1}{1-e^{-j\omega}}X(e^{j\omega})$ $+\pi X(e^{j0})\sum_{k=-\infty}^{+\infty}\delta(\omega-2\pi k)$
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5.3.4	Conjugate Symmetry for Real Signals	$x[n]$ real	$\begin{cases}X(e^{j\omega})=X^*(e^{-j\omega})\\ \mathcal{R}e\{X(e^{j\omega})\}=\mathcal{R}e\{X(e^{-j\omega})\}\\ \mathcal{I}m\{X(e^{j\omega})\}=-\mathcal{I}m\{X(e^{-j\omega})\}\\ \lvert X(e^{j\omega})\rvert=\lvert X(e^{-j\omega})\rvert\\ \sphericalangle X(e^{j\omega})=-\sphericalangle X(e^{-j\omega})\end{cases}$
5.3.4	Symmetry for Real, Even Signals	$x[n]$ real an even	$X(e^{j\omega})$ real and even
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5.3.9	Parseval's Relation for Aperiodic Signals $\sum_{n=-\infty}^{+\infty}\lvert x[n]\rvert^2=\frac{1}{2\pi}\int_{2\pi}\lvert X(e^{j\omega})\rvert^2 d\omega$		

a duality relationship between the discrete-time Fourier transform and the continuous-time Fourier series. This relation is discussed in Section 5.7.2.

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DTFT Pairs

Signal	Fourier Transform	Fourier Series Coefficients (if periodic)
$\sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$	a_k
$e^{j\omega_0 n}$	$2\pi \sum_{l=-\infty}^{+\infty} \delta(\omega - \omega_0 - 2\pi l)$	(a) $\omega_0 = \frac{2\pi m}{N}$ $a_k = \begin{cases} 1, & k = m, m \pm N, m \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$ (b) $\frac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic
$\cos \omega_0 n$	$\pi \sum_{l=-\infty}^{+\infty} \{\delta(\omega - \omega_0 - 2\pi l) + \delta(\omega + \omega_0 - 2\pi l)\}$	(a) $\omega_0 = \frac{2\pi m}{N}$ $a_k = \begin{cases} \frac{1}{2}, & k = \pm m, \pm m \pm N, \pm m \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$ (b) $\frac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic
$\sin \omega_0 n$	$\frac{\pi}{j} \sum_{l=-\infty}^{+\infty} \{\delta(\omega - \omega_0 - 2\pi l) - \delta(\omega + \omega_0 - 2\pi l)\}$	(a) $\omega_0 = \frac{2\pi r}{N}$ $a_k = \begin{cases} \frac{1}{2j}, & k = r, r \pm N, r \pm 2N, \ldots \\ -\frac{1}{2j}, & k = -r, -r \pm N, -r \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$ (b) $\frac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic
$x[n] = 1$	$2\pi \sum_{l=-\infty}^{+\infty} \delta(\omega - 2\pi l)$	$a_k = \begin{cases} 1, & k = 0, \pm N, \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$

DTFT Pairs

Periodic square wave $x[n] = \begin{cases} 1, & \lvert n\rvert \le N_1 \\ 0, & N_1 < \lvert n\rvert \le N/2 \end{cases}$ and $x[n+N] = x[n]$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$	$a_k = \frac{\sin[(2\pi k/N)(N_1 + \frac{1}{2})]}{N \sin[2\pi k/2N]}, \; k \ne 0, \pm N, \pm 2N, \ldots$ $a_k = \frac{2N_1 + 1}{N}, \; k = 0, \pm N, \pm 2N, \ldots$
$\sum_{k=-\infty}^{+\infty} \delta[n - kN]$	$\frac{2\pi}{N} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{N}\right)$	$a_k = \frac{1}{N}$ for all k
$a^n u[n], \quad \lvert a\rvert < 1$	$\frac{1}{1 - ae^{-j\omega}}$	—
$x[n] \begin{cases} 1, & \lvert n\rvert \le N_1 \\ 0, & \lvert n\rvert > N_1 \end{cases}$	$\frac{\sin[\omega(N_1 + \frac{1}{2})]}{\sin(\omega/2)}$	—
$\frac{\sin Wn}{\pi n} = \frac{W}{\pi} \operatorname{sinc}\left(\frac{Wn}{\pi}\right)$ $0 < W < \pi$	$X(\omega) = \begin{cases} 1, & 0 \le \lvert\omega\rvert \le W \\ 0, & W < \lvert\omega\rvert \le \pi \end{cases}$ $X(\omega)$ periodic with period 2π	—
$\delta[n]$	1	—
$u[n]$	$\frac{1}{1 - e^{-j\omega}} + \sum_{k=-\infty}^{+\infty} \pi \delta(\omega - 2\pi k)$	—
$\delta[n - n_0]$	$e^{-j\omega n_0}$	—
$(n+1)a^n u[n], \quad \lvert a\rvert < 1$	$\frac{1}{(1 - ae^{-j\omega})^2}$	—
$\frac{(n + r - 1)!}{n!(r-1)!} a^n u[n], \quad \lvert a\rvert < 1$	$\frac{1}{(1 - ae^{-j\omega})^r}$	—

SYSTEMS CHARACTERIZED BY LINEAR CONSTANT-COEFFICIENT DIFFERENCE EQUATIONS

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k].$$

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})}.$$

$$\sum_{k=0}^{N} a_k e^{-jk\omega} Y(e^{j\omega}) = \sum_{k=0}^{M} b_k e^{-jk\omega} X(e^{j\omega}),$$

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{\sum_{k=0}^{M} b_k e^{-jk\omega}}{\sum_{k=0}^{N} a_k e^{-jk\omega}}.$$

SYSTEMS CHARACTERIZED BY LINEAR CONSTANT-COEFFICIENT DIFFERENCE EQUATIONS

Example 5. 18

Consider the causal LTI system that is characterized by the difference equation

$$y[n] - ay[n-1] = x[n],$$

with $|a| < 1$. From eq. (5.80), the frequency response of this system is

$$H(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}}.$$

Thus, the impulse response of the system is

$$h[n] = a^n u[n].$$

SYSTEMS CHARACTERIZED BY LINEAR CONSTANT-COEFFICIENT DIFFERENCE EQUATIONS

Consider a causal LTI system that is characterized by the difference equation

$$y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = 2x[n].$$

From eq. (5.80), the frequency response is

$$H(e^{j\omega}) = \frac{2}{1 - \frac{3}{4}e^{-j\omega} + \frac{1}{8}e^{-j2\omega}}.$$

As a first step in obtaining the impulse response, we factor the denominator of

$$H(e^{j\omega}) = \frac{2}{(1 - \frac{1}{2}e^{-j\omega})(1 - \frac{1}{4}e^{-j\omega})}.$$

$H(e^{j\omega})$ can be expanded by the method of partial fractions, as in Example appendix. The result of this expansion is

$$H(e^{j\omega}) = \frac{4}{1 - \frac{1}{2}e^{-j\omega}} - \frac{2}{1 - \frac{1}{4}e^{-j\omega}}.$$

The inverse transform of each term can be recognized by inspection, with the

$$h[n] = 4\left(\frac{1}{2}\right)^n u[n] - 2\left(\frac{1}{4}\right)^n u[n].$$