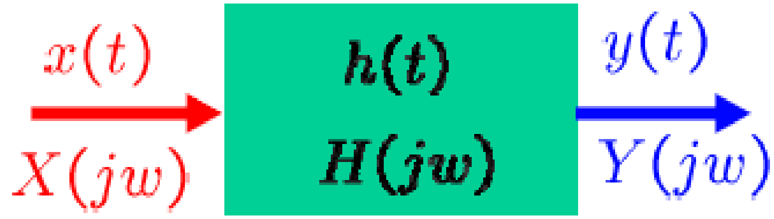
Introduction to Signals and Systems: V216

Lecture #9

Chapter 4: The Continuous-Time Fourier Transform

Convolution Property

- <u>Convolution Property:</u>



$$y(t) = x(t) * h(t) \overset{\mathcal{F}}{\longleftrightarrow} Y(jw) = X(jw)H(jw)$$

$$= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

Convolution Property

- From Convolution Integral:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(jw)e^{jwt}dw$$

$$X(jw) = \int_{-\infty}^{\infty} x(t)e^{-jwt}dt$$

$$x(t-t_0) \stackrel{\mathcal{F}}{\longleftrightarrow} e^{-jwt_0}X(jw)$$

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$$

$$\Rightarrow\ Y(jw) = \mathcal{F}\{y(t)\} = \int_{-\infty}^{+\infty}\left[\int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau\right]e^{-jwt}dt$$

$$= \int_{-\infty}^{+\infty} x(\tau)\left[\int_{-\infty}^{+\infty} h(t-\tau)e^{-jwt}dt\right]d\tau$$

$$= \int_{-\infty}^{+\infty} x(\tau)\left[e^{-jw\tau}H(jw)\right]d\tau$$

$$= H(jw)\int_{-\infty}^{+\infty} x(\tau)e^{-jw\tau}d\tau$$

$$\rightarrow\ Y(jw) = H(jw)X(jw)$$

Convolution Property

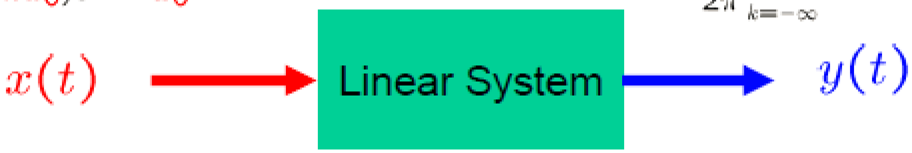
- From Superposition (or Linearity):

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(jw)e^{jwt}dw$$

$$= \lim_{w_0 \to 0} \frac{1}{2\pi} \sum_{k=-\infty}^{+\infty} X(jkw_0)e^{jkw_0 t}w_0$$

$$\frac{1}{2\pi}\sum_{k=-\infty}^{+\infty} X(jkw_0)e^{jkw_0 t}w_0$$

$$\frac{1}{2\pi}\sum_{k=-\infty}^{+\infty} X(jkw_0)H(jkw_0)e^{jkw_0 t}w_0$$



$$H(jkw_0) = \int_{-\infty}^{\infty} h(t)e^{-jkw_0 t}dt$$

$$y(t) = \lim_{w_0 \to 0} \frac{1}{2\pi} \sum_{k=-\infty}^{+\infty} X(jkw_0)H(jkw_0)e^{jkw_0 t}w_0$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(jw)H(jw)e^{jwt}dw$$

Convolution Property

- From Superposition (or Linearity):

$$\frac{1}{2\pi}\sum_{k=-\infty}^{+\infty} X(jkw_0)e^{jkw_0t}w_0 \longrightarrow \frac{1}{2\pi}\sum_{k=-\infty}^{+\infty} X(jkw_0)H(jkw_0)e^{jkw_0t}w_0$$

$$y(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(jw)H(jw)e^{jwt}dw$$

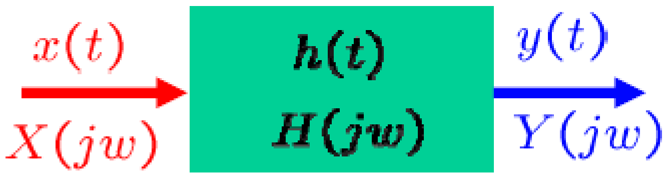
Since $\quad y(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} Y(jw)e^{jwt}dw$

$$\Rightarrow Y(jw) = X(jw)H(jw)$$

$$y(t) = x(t) * h(t) \overset{\mathcal{F}}{\longleftrightarrow} Y(jw) = X(jw)H(jw)$$

Convolution Property

- Equivalent LTI Systems:

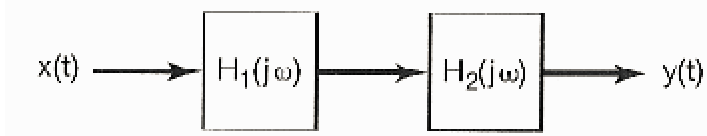


$$h(t) \overset{\mathcal{F}}{\longleftrightarrow} H(jw)$$

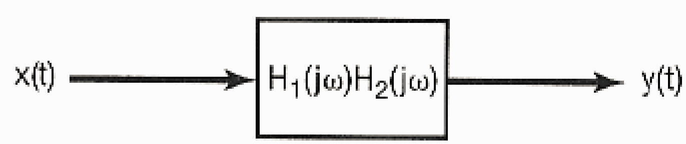
impulse response | frequency response

$$y(t) = x(t) * h(t)$$

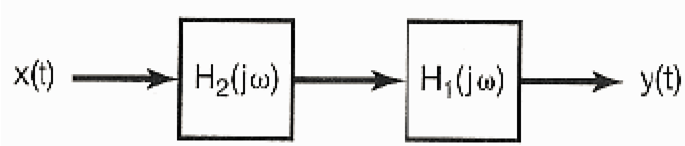
$$Y(jw) = X(jw)H(jw)$$



(a)



(b)



(c)

$$\Rightarrow Y(jw) = H_1(jw)H_2(jw)X(jw)$$

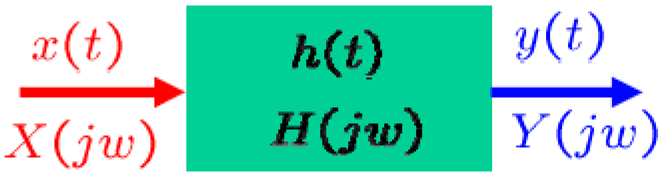
Convolution Property

- Example 4.15: Time Shift

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(jw)e^{jwt}dw$$

$$X(jw) = \int_{-\infty}^{\infty} x(t)e^{-jwt}dt$$

$$x(t-t_0) \overset{\mathcal{F}}{\longleftrightarrow} e^{-jwt_0}X(jw)$$



$$h(t) = \delta(t-t_0)$$

$$\Rightarrow H(jw) = e^{-jwt_0}$$

$$\begin{aligned} Y(jw) &= H(jw)X(jw) \\ &= e^{-jwt_0}X(jw) \end{aligned}$$

$$\Rightarrow y(t) = x(t-t_0)$$

Convolution Property

- Examples 4.16 & 17: Differentiator & Integrator

$$y(t) = \frac{d}{dt}x(t) \quad \Rightarrow \quad Y(jw) = jwX(jw)$$

$$\Rightarrow \quad H(jw) = jw$$

$$y(t) = \int_{-\infty}^{t} x(\tau)d\tau \quad \Rightarrow \quad h(t) = u(t)$$

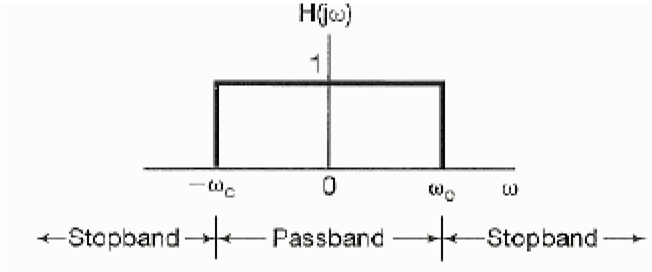
impulse response

$$\Rightarrow \quad H(jw) = \frac{1}{jw} + \pi\delta(w)$$

Convolution Property

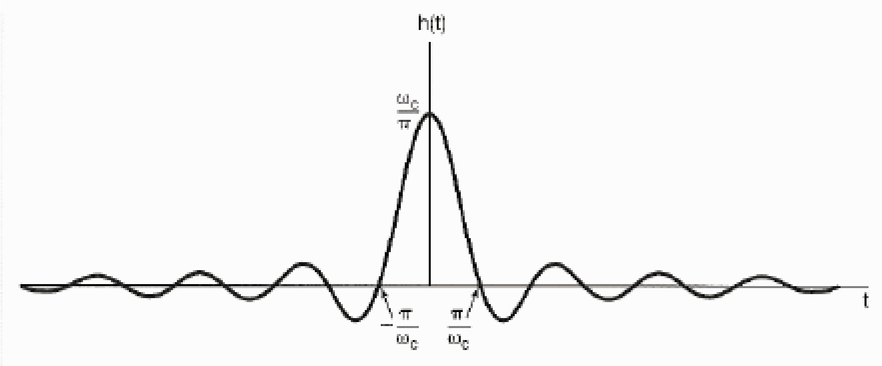
- Example 4.18: Ideal Lowpass Filter

$$H(jw) = \begin{cases} 1, & |w| < w_c \\ 0, & |w| > w_c \end{cases}$$



$$\Rightarrow\ h(t) = \frac{1}{2\pi}\int_{-w_c}^{+w_c} e^{jwt}dw$$

$$= \frac{\sin(w_c t)}{\pi t}$$



$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(jw)e^{jwt}dw$$

$$X(jw) = \int_{-\infty}^{\infty} x(t)e^{-jwt}dt$$

Convolution Property

- Filter Design:

$$H(jw) = \int_{-\infty}^{\infty} h(t)e^{-jwt}dt$$

$\delta(t)$

$x(t)$

$h(t)$

$$y(t) = h(t) * x(t)$$
$$= \int_{-\infty}^{\infty} h(t-\tau)x(\tau)d\tau$$

$$H(jw) = \int_{-\infty}^{\infty} h(t)e^{-jwt}dt$$

$$X(jw) = \int_{-\infty}^{\infty} x(t)e^{-jwt}dt$$

$$\Rightarrow Y(jw) = H(jw)X(jw)$$

$$\Rightarrow y(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} Y(jw)e^{jwt}dw$$

Convolution Property

- Filter Design:

$$H(jw) = \int_{-\infty}^{\infty} h(t)e^{-jwt}dt$$

$\delta(t)$

$x(t)$

$h(t)$

$y(t) = h(t) * x(t)$

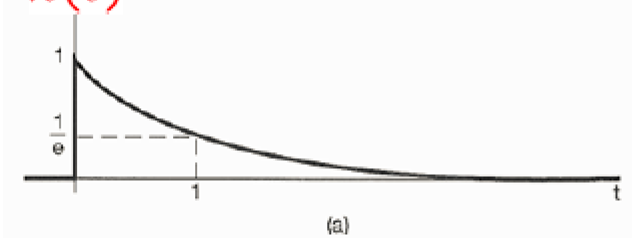
$Y(jw) = H(jw)X(jw)$

$$h(t) = e^{-t}u(t) \overset{\mathcal{F}}{\longleftrightarrow} H(jw) = \frac{1}{jw+1}$$

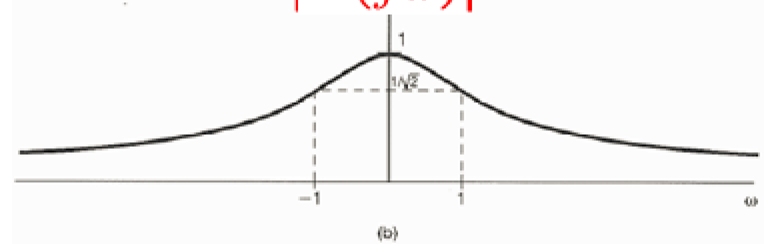
impulse response

frequency response

$h(t)$



$|H(jw)|$



Convolution Property

- Example 4.19:

$$h(t) = e^{-at}u(t), \quad a > 0 \qquad \Rightarrow \quad H(jw) = \frac{1}{a + jw}$$

$$x(t) = e^{-bt}u(t), \quad b > 0 \qquad \Rightarrow \quad X(jw) = \frac{1}{b + jw}$$

$$\Rightarrow \; Y(jw) = H(jw)X(jw) \quad = \frac{1}{a + jw}\,\frac{1}{b + jw}$$

if $a \neq b$

$$= \frac{1}{b - a}\left[\frac{1}{a + jw} - \frac{1}{b + jw}\right]$$

Convolution Property

- Example 4.19:

if $a \neq b$ $\quad Y(jw) = \dfrac{1}{b-a}\left[\dfrac{1}{a+jw} - \dfrac{1}{b+jw}\right]$

$$\Rightarrow y(t) = \frac{1}{b-a}\left[e^{-at}u(t) - e^{-bt}u(t)\right]$$

if $a = b$ $\quad Y(jw) = \dfrac{1}{(a+jw)^2}$

since $\quad e^{-at}u(t) \overset{\mathcal{F}}{\longleftrightarrow} \dfrac{1}{a+jw}$

and $\quad t\, e^{-at}u(t) \overset{\mathcal{F}}{\longleftrightarrow} j\dfrac{d}{dw}\left[\dfrac{1}{a+jw}\right] = \dfrac{1}{(a+jw)^2}$

$$\Rightarrow y(t) = te^{-at}u(t)$$

Convolution Property

- Example 4.20:

$$H(jw) = \int_{-\infty}^{\infty} h(t)e^{-jwt}dt$$

$$x(t) = \frac{\sin(w_i t)}{\pi t} \longrightarrow \boxed{\text{Filter LTI System}} \longrightarrow y(t) = ?$$

$$h(t) = \frac{\sin(w_c t)}{\pi t}$$

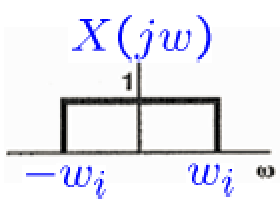
$$\Rightarrow Y(jw) = H(jw)X(jw)$$

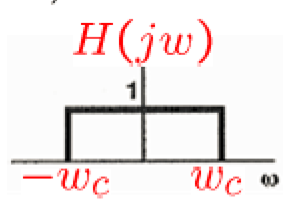
$$w_0 = \min(w_c, w_i)$$

$$\Rightarrow X(jw) = \begin{cases} 1, & |w| \le w_i \\ 0, & \text{otherwise} \end{cases}$$

$$\Rightarrow H(jw) = \begin{cases} 1, & |w| \le w_c \\ 0, & \text{otherwise} \end{cases}$$

$$\Rightarrow Y(jw) = \begin{cases} 1, & |w| \le w_0 \\ 0, & \text{otherwise} \end{cases}$$

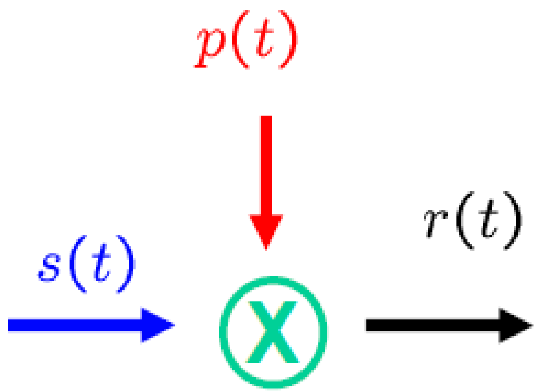




$$\Rightarrow y(t) = \begin{cases} \frac{\sin(w_c t)}{\pi t}, & w_c \le w_i \\ \frac{\sin(w_i t)}{\pi t}, & w_c \ge w_i \end{cases}$$

Multiplication Property

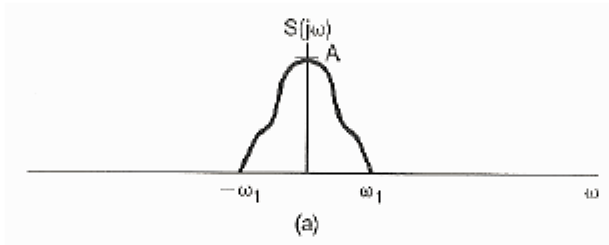
- Multiplication Property:

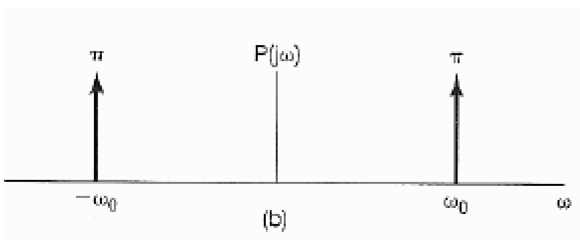


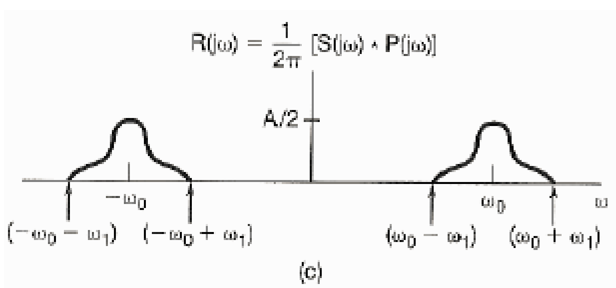
$$r(t) = s(t)p(t) \overset{\mathcal{F}}{\longleftrightarrow} R(jw) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(j\theta)P(j(w-\theta))d\theta$$

Multiplication Property

- Example 4.21:







$$r(t) = s(t)p(t)$$

$$s(t) \xleftrightarrow{\mathcal{F}} S(jw)$$

$$p(t) \xleftrightarrow{\mathcal{F}} P(jw)$$

$$p(t) = \cos(w_0 t)$$

$$P(jw) = \pi\delta(w - w_0) + \pi\delta(w + w_0)$$

$$R(jw) = \frac{1}{2\pi}[S(jw) * P(jw)]$$

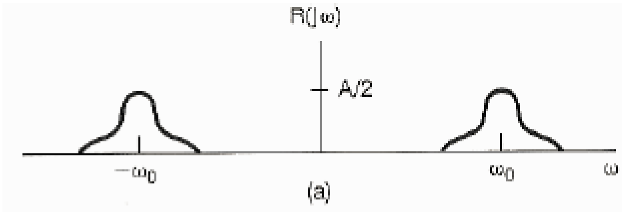
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} S(j\theta)P(j(w-\theta))d\theta$$

$$= \frac{1}{2}S\left(j(w - w_0)\right) + \frac{1}{2}S\left(j(w + w_0)\right)$$

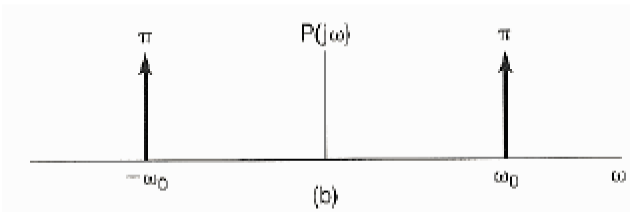
Multiplication Property

- Example 4.22:

$$g(t) = r(t)p(t)$$



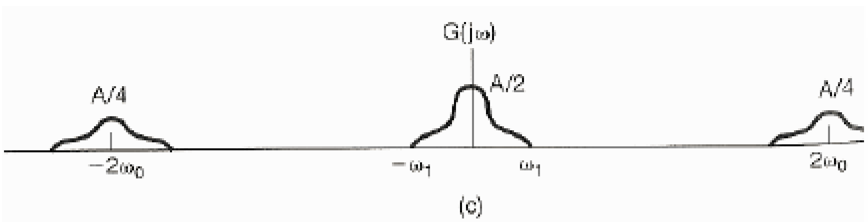
$$r(t) \overset{\mathcal{F}}{\longleftrightarrow} R(jw)$$



$$p(t) \overset{\mathcal{F}}{\longleftrightarrow} P(jw)$$

$$p(t) = \cos(w_0 t)$$

$$G(jw) = \frac{1}{2\pi}\left[R(jw) * P(jw)\right]$$



Multiplication Property

- Example 4.23:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(jw)e^{jwt}dw$$

$$X(jw) = \int_{-\infty}^{\infty} x(t)e^{-jwt}dt$$

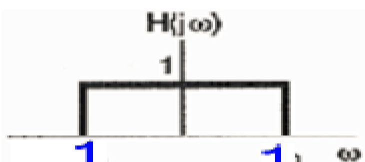
$$x(t) = \frac{\sin(t)\sin(t/2)}{\pi t^2}$$

$$X(jw) = \int_{-\infty}^{\infty} \frac{\sin(t)\sin(t/2)}{\pi t^2} e^{-jwt}dt$$

$$= \pi \left(\frac{\sin(t)}{\pi t}\right)\left(\frac{\sin(t/2)}{\pi t}\right)$$

$$\Rightarrow X(jw) = \frac{1}{2}\mathcal{F}\left\{\frac{\sin(t)}{\pi t}\right\} * \mathcal{F}\left\{\frac{\sin(t/2)}{\pi t}\right\}$$

$\frac{1}{2}$



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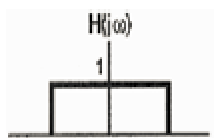

$-\frac{1}{2}$ $\frac{1}{2}$

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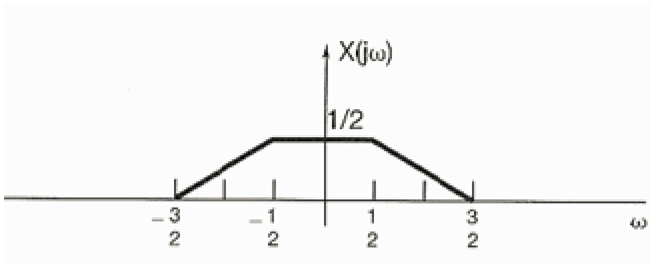
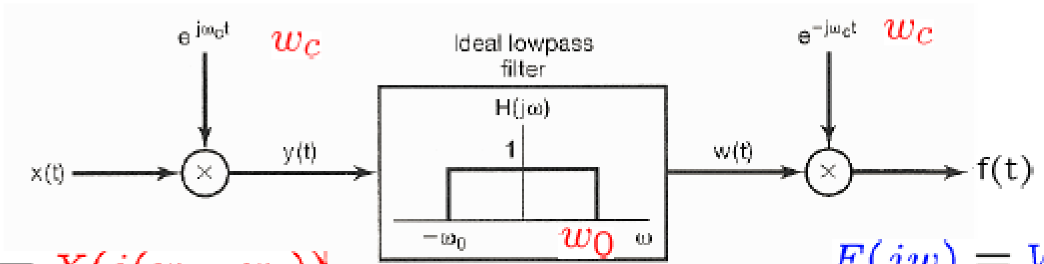


Multiplication Property

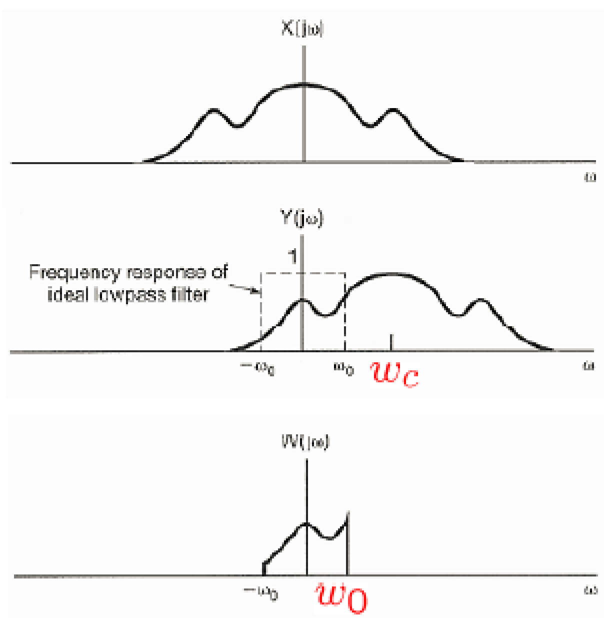
- Bandpass Filter Using Amplitude Modulation:

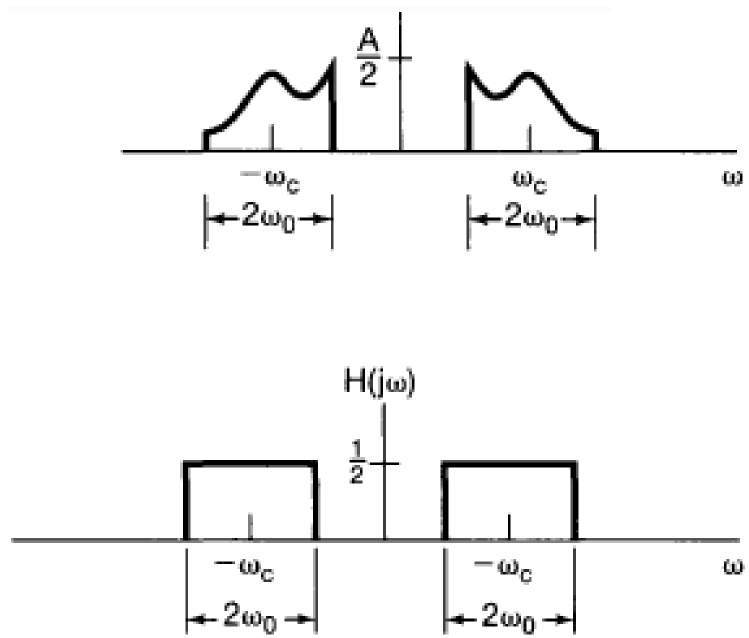
$$e^{jw_c t} \xleftrightarrow{\mathcal{F}} 2\pi\delta(w - w_c)$$



$$Y(jw) = X(j(w - w_c))$$

$$F(jw) = W(j(w + w_c))$$





Systems Characterized by Linear Constant Coefficient Differential Equations

- A useful class of CT LTI systems:

$$a_N \frac{d^N y(t)}{dt^N} + a_{N-1} \frac{d^{N-1} y(t)}{dt^{N-1}} + \cdots + a_1 \frac{dy(t)}{dt} + a_0 y(t)$$

$$= b_M \frac{d^M x(t)}{dt^M} + b_{M-1} \frac{d^{M-1} x(t)}{dt^{M-1}} + \cdots + b_1 \frac{dx(t)}{dt} + b_0 x(t)$$

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

$$x(t) \longrightarrow \boxed{\text{LTI System}} \longrightarrow y(t)$$

$$Y(jw) = X(jw)H(jw) \qquad H(jw) = \frac{Y(jw)}{X(jw)}$$

Systems Characterized by Linear Constant Coefficient Differential Equations

$$\mathcal{F}\left\{\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k}\right\} = \mathcal{F}\left\{\sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}\right\}$$

$$\sum_{k=0}^{N} a_k \mathcal{F}\left\{\frac{d^k y(t)}{dt^k}\right\} = \sum_{k=0}^{M} b_k \mathcal{F}\left\{\frac{d^k x(t)}{dt^k}\right\}$$

$$\sum_{k=0}^{N} a_k (jw)^k Y(jw) = \sum_{k=0}^{M} b_k (jw)^k X(jw)$$

$$Y(jw)\left[\sum_{k=0}^{N} a_k (jw)^k\right] = X(jw)\left[\sum_{k=0}^{M} b_k (jw)^k\right]$$

$$\Rightarrow \quad H(jw) = \frac{Y(jw)}{X(jw)} = \frac{\sum_{k=0}^{M} b_k (jw)^k}{\sum_{k=0}^{N} a_k (jw)^k} = \frac{b_M (jw)^M + \cdots + b_1 (jw) + b_0}{a_N (jw)^N + \cdots + a_1 (jw) + a_0}$$

Systems Characterized by Linear Constant Coefficient Differential Equations

- Examples 4.24 & 4.25:

$$H = \frac{Y}{X}$$

$$\frac{dy(t)}{dt} + ay(t) = x(t) \quad \Rightarrow \quad H(jw) = \frac{1}{jw + a}$$

$$(jw)Y(jw) + aY(jw) = X(jw) \quad \Rightarrow \quad h(t) = e^{-at}u(t)$$

$$\frac{d^2y(t)}{dt^2} + 4\,\frac{dy(t)}{dt} + 3\,y(t) = \frac{dx(t)}{dt} + 2\,x(t)$$

$$\Rightarrow \quad H(jw) = \frac{(jw)+2}{(jw)^2 + 4(jw) + 3} = \frac{(jw+2)}{(jw+1)(jw+3)}$$

$$= \frac{1/2}{jw+1} + \frac{1/2}{jw+3}$$

$$\Rightarrow \quad h(t) = \frac{1}{2}\,e^{-t}u(t) + \frac{1}{2}\,e^{-3t}u(t)$$

Systems Characterized by Linear Constant Coefficient Differential Equations

- Example 4.26:

$$x(t) = e^{-t}u(t) \longrightarrow \boxed{\text{LTI System}} \longrightarrow y(t) = ???$$

$$H(jw) = \frac{(jw+2)}{(jw+1)(jw+3)}$$

$$\Rightarrow \quad Y(jw) = X(jw)H(jw)$$

$$= \left[\frac{1}{jw+1}\right]\left[\frac{jw+2}{(jw+1)(jw+3)}\right]$$

$$= \frac{jw+2}{(jw+1)^2(jw+3)}$$

$$= \frac{\frac{1}{4}}{jw+1} + \frac{\frac{1}{2}}{(jw+1)^2} - \frac{\frac{1}{4}}{jw+3}$$

$$\Rightarrow \quad y(t) = \left[\frac{1}{4}e^{-t} + \frac{1}{2}t\,e^{-t} - \frac{1}{4}e^{-3t}\right]u(t)$$