Introduction to Signals and Systems: V216

Lecture #15
Chapter 9: Laplace Transform

Differentiation in the Time Domain

Consider the Laplace transform derivative in the time domain

$$x(t) \overset{L}{\leftrightarrow} X(s) \qquad \text{ROC}=R$$

$$x(t) = \frac{1}{2\pi j}\int_{\sigma - j\infty}^{\sigma + j\infty} X(s)e^{st}\,ds$$

$$\frac{dx(t)}{dt} = \frac{1}{2\pi j}\int_{\sigma - j\infty}^{\sigma + j\infty} sX(s)e^{st}\,ds$$

$$\frac{dx(t)}{dt} \overset{L}{\leftrightarrow} sX(s) \qquad \text{ROC} \supseteq R$$

$sX(s)$ has an extra zero at 0, and may cancel out a corresponding pole of $X(s)$, so ROC may be larger

Widely used to solve when the system is described by LTI differential equations

Example: System Impulse Response

Consider trying to find the system response (potentially unstable) for a second order system with an impulse input $x(t)=\delta(t)$, $y(t)=h(t)$

$$a\frac{d^2y(t)}{dt^2}+b\frac{dy(t)}{dt}+cy(t)=x(t)$$

Taking Laplace transforms of both sides and using the linearity property

$$aL\left\{\frac{d^2y(t)}{dt^2}\right\}+bL\left\{\frac{dy(t)}{dt}\right\}+cL\{y(t)\}=L\{\delta(t)\}$$

$$L\{y(t)\}(as^2+bs+c)=1$$

$$L\{y(t)\}=H(s)=\frac{1}{as^2+bs+c}=\frac{1}{a(s-r_1)(s-r_2)}=\frac{k_1}{(s-r_1)}+\frac{k_2}{(s-r_2)}$$

where r_1 and r_2 are distinct roots, and calculating the inverse transform

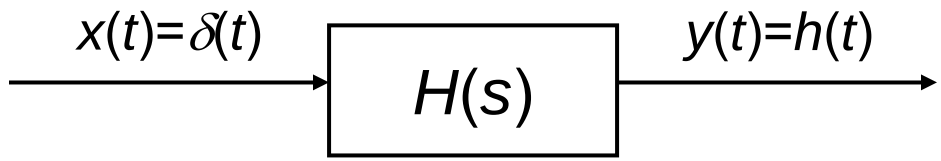
$$y(t)=k_1e^{r_1t}u(t)+k_2e^{r_2t}u(t)$$

The general solution to a second order system can be expressed as the sum of two complex (possibly real) exponentials

Continuous-Time Transfer Functions

System/Transfer Functions

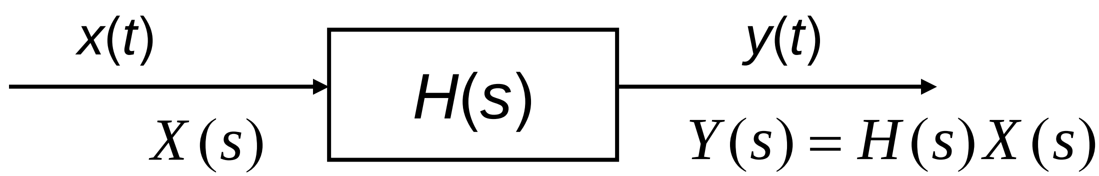
The system/transfer function, $H(s)$, is defined as the Laplace transform of the system's impulse response



$$H(s) = \int_{-\infty}^{\infty} h(t) e^{-st} \, dt$$

When $s=j\omega$, this is the **Fourier transform** and more generally, this is the **Laplace transform**

The transfer function is very important because



the unknown system output (Laplace transform) is given by the **multiplication** $X(s)$ and $H(s)$

Example 1: First Order System

Consider a general LTI, first order, differential equation with an impulse input

$$a\frac{dy(t)}{dt} + by(t) = x(t)$$

$$a\frac{dh(t)}{dt} + bh(t) = \delta(t)$$

Taking Laplace transforms

$$L\{h(t)\}(as + b) = 1$$

$$H(s) = L\{h(t)\} = \frac{1}{a(s + b/a)} \qquad \mathrm{Re}\{s\} > -b/a$$

which gives the system's transfer function, *H*(*s*). This can be solved to show that (see earlier examples)

$$h(t) = a^{-1}e^{-(b/a)t}u(t)$$

Example 2: Second Order System

Consider a general LTI, second order, differential equation with an impulse input

$$a\frac{d^2y(t)}{dt^2}+b\frac{dy(t)}{dt}+cy(t)=x(t)$$

$$a\frac{d^2h(t)}{dt^2}+b\frac{dh(t)}{dt}+ch(t)=\delta(t)$$

Taking Laplace transforms

$$L\{h(t)\}(as^2+bs+c)=1 \qquad \mathrm{Re}\{s\}>\max\{\mathrm{Re}\{-r_1\},\mathrm{Re}\{-r_2\}\}$$

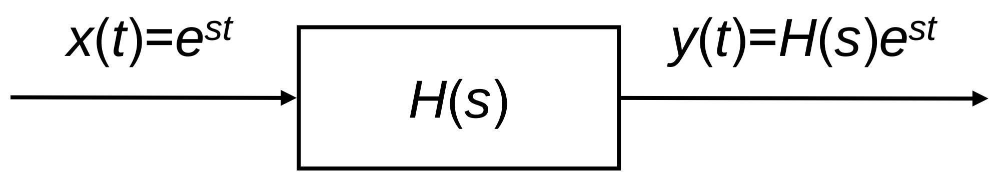
$$H(s)=L\{h(t)\}=\frac{1}{as^2+bs+c}=\frac{1}{a(s-r_1)(s-r_2)}$$

which gives the **system's transfer function**, *H*(*s*). This can be solved, using partial fractions, to show that

$$h(t)=a^{-1}k_1e^{r_1t}u(t)+a^{-1}k_2e^{r_2t}u(t)$$

Transfer Functions & System Eigenfuctions

Remember that $x(t)=e^{st}$ is an eigenfuction of an LTI system with corresponding eigenvalue $H(s)$



In addition, by Laplace theory, most input signals $x(t)$ can be expressed as a linear combination of basis signals e^{st} (inverse Laplace transform):

$$x(t) = \frac{1}{2\pi j}\int_{\sigma - j\infty}^{\sigma + j\infty} X(s)e^{st}\,ds$$

Therefore the system's output can be expressed as

$$y(t) = \frac{1}{2\pi j}\int_{\sigma - j\infty}^{\sigma + j\infty} H(s)X(s)e^{st}\,ds = \frac{1}{2\pi j}\int_{\sigma - j\infty}^{\sigma + j\infty} Y(s)e^{st}\,ds$$

which is why $Y(s)=H(s)X(s)$, again using the transfer function

Frequency Response Analysis

Of particular interest is **frequency response analysis**. This corresponds to input signals of the form

$$x(t) = e^{j\omega t}$$

and the corresponding transfer function is

$$H(j\omega) = \int_{-\infty}^{\infty} h(t) e^{-j\omega t} dt$$

In this case *s*=*j*ω, the **transfer function** corresponds to the **Fourier transform**. It is a complex function of frequency.

Example when *x*(*t*) is periodic, with fundamental frequency ω_0, the Fourier transform is given by:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

The system's response is given by

$$y(t) = \sum_{k=-\infty}^{\infty} a_k H(jk\omega_0) e^{jk\omega_0 t}$$

Example: 1st Order System and cos() Input

The transfer function of a 1st order system is given by:

$$H(j\omega) = \frac{1}{(j\omega + b)}$$

Assume *a*=1

The input signal $x(t)=\cos(\omega_0 t)$, which has fundamental frequency ω_0 is:

$$x(t) = \tfrac{1}{2}(e^{j\omega_0 t} + e^{-j\omega_0 t})$$

The (stable) system's output is:

$$\begin{aligned}
y(t) &= \tfrac{1}{2}\left(e^{j\omega_0 t}\frac{1}{b + j\omega_0} + e^{-j\omega_0 t}\frac{1}{b - j\omega_0} \right) \\
&= \tfrac{1}{2(b^2+\omega_0^2)}\left(e^{j\omega_0 t}(b - j\omega_0) + e^{-j\omega_0 t}(b + j\omega_0) \right) \\
&= \tfrac{b}{(b^2+\omega_0^2)}\cos(\omega_0 t) + \tfrac{\omega_0}{(b^2+\omega_0^2)}\sin(\omega_0 t)
\end{aligned}$$

Gain and Phase Transfer Function Analysis

For a complex number/function, we can represent it in polar form by calculating the magnitude (gain) and angle (phase):

$$H(j\omega) = |H(j\omega)|e^{j\angle H(j\omega)}$$

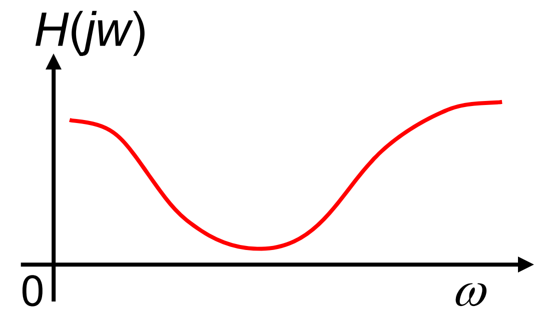
In filter/system analysis and design, we're interested in how certain frequencies are magnified or suppressed -

These are of particular interest by plotting the system properties (amplitude/phase) against frequency:

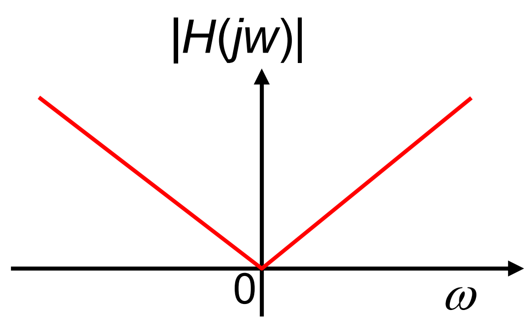
- Frequency shaping
- Frequency selection
- Low pass
- High pass

Frequency Shaping Filters

It is often necessary to change the relative magnitude of a signal at different frequencies, which is referred to as **filtering**. LTI systems that change the shape of the spectrum are referred to as **frequency shaping filters**.



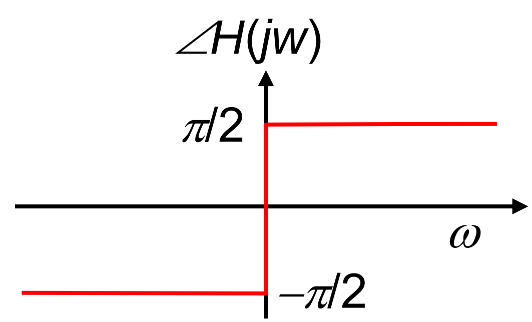
Audio systems often contain frequency shaping filters (LTI systems) to change the relative amount of bass (low frequency) and treble (high frequency). Real valued and often plotted on log scaling (dB = $20\log_{10}|H(j\omega)|$)



Complex differentiating filters are defined by

$$H(j\omega) = j\omega$$

which are useful for enhancing rapid variations in a signal. Both the magnitude and phase are plotted against frequency

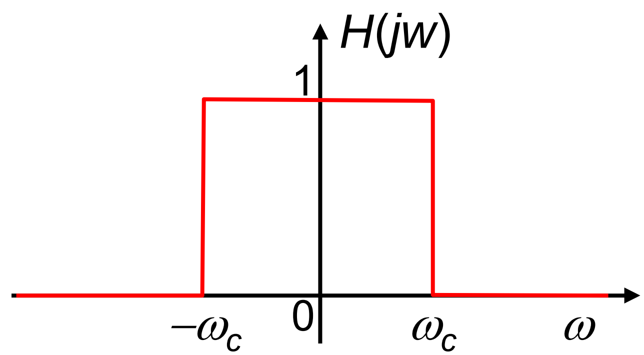


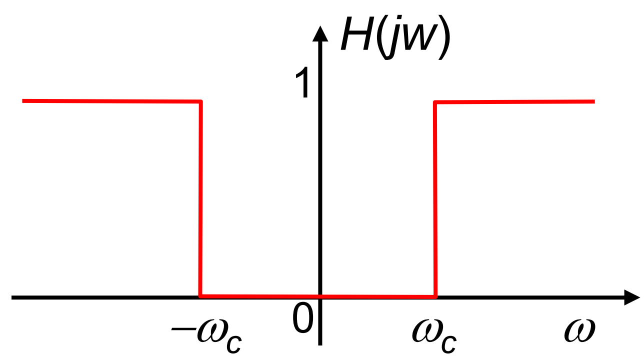
Frequency Selection Filters

Frequency selection filters are specially designed to accept some frequencies and reject others. Noise in an audio recording can be removed by low pass filtering, multiple communication signals can be encoded at different frequencies and then recovered by selecting particular frequencies

Low pass filters are designed to reject/attenuate high frequency "noise" while passing on the low frequencies

High pass filters are designed to reject/attenuate low frequency signal components while passing on high frequency





Electrical Low Pass Filter

Differential equation for the LTI system

$$RC\frac{dv_c(t)}{dt}+v_c(t)=v_s(t)$$

The frequency response transfer function $H(j\omega)$ can be determined using the eigensystem property **or** using its impulse response definition

$$RCj\omega H(j\omega)+H(j\omega)=1$$

$$H(j\omega)=\frac{1}{1+j\omega RC}$$

Magnitude-phase plot shown right

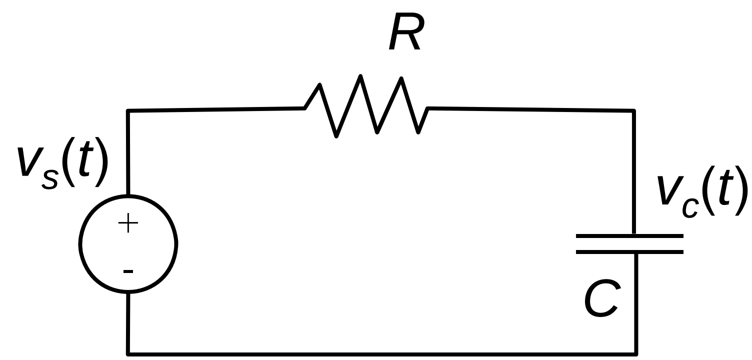
Step response:

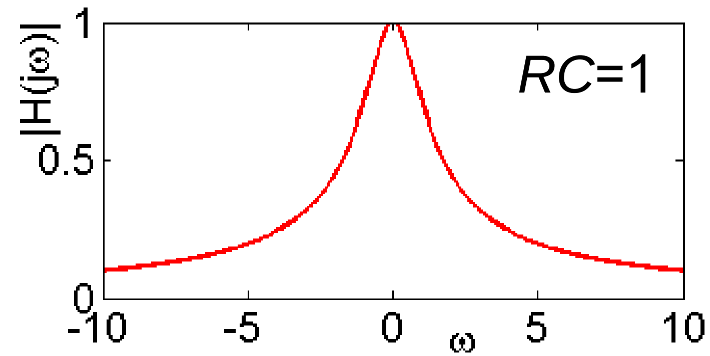
$$v_c(t)=(1-e^{-t/RC})u(t)$$

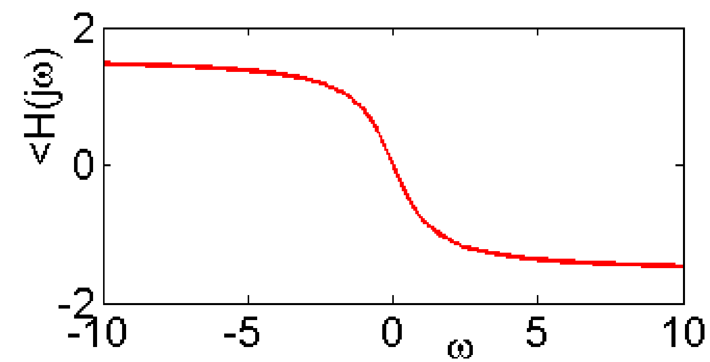
High RC – good frequency selection

Low RC – fast time response

Inevitable time/frequency design compromise







Electrical High Pass Filter

Differential equation for the LTI system

$$RC\frac{dv_r(t)}{dt}+v_r(t)=RC\frac{dv_s(t)}{dt}$$

The frequency response transfer function $H(j\omega)$ can be determined using eigenfunction property or impulse response

$$RCj\omega H(j\omega)+H(j\omega)=RCj\omega.1$$

$$H(j\omega)=\frac{j\omega RC}{1+j\omega RC}$$

Magnitude-phase plot shown right

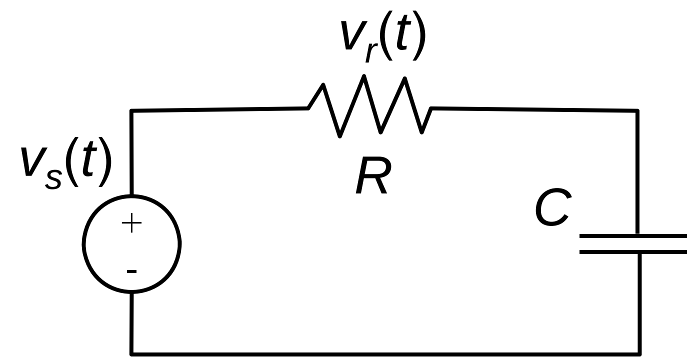
Step response:

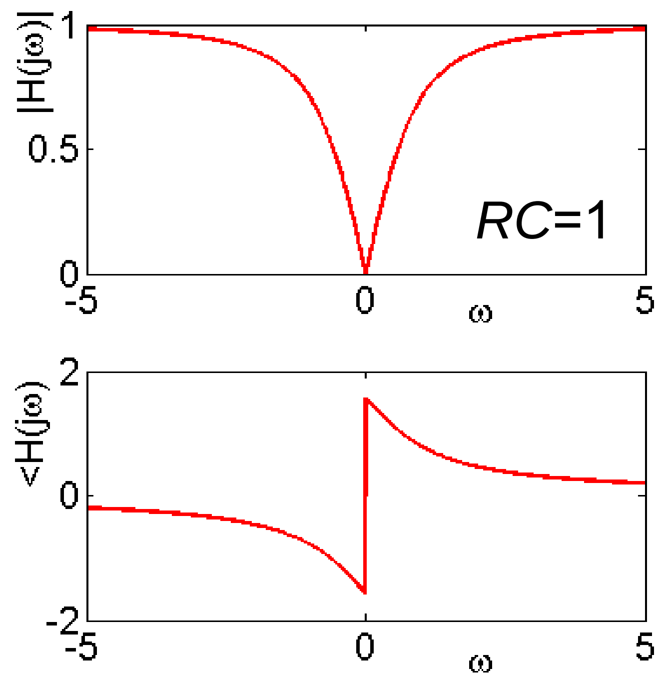
$$v_c(t)=e^{-t/RC}u(t)$$

High RC – good frequency selection

Low RC – fast time response

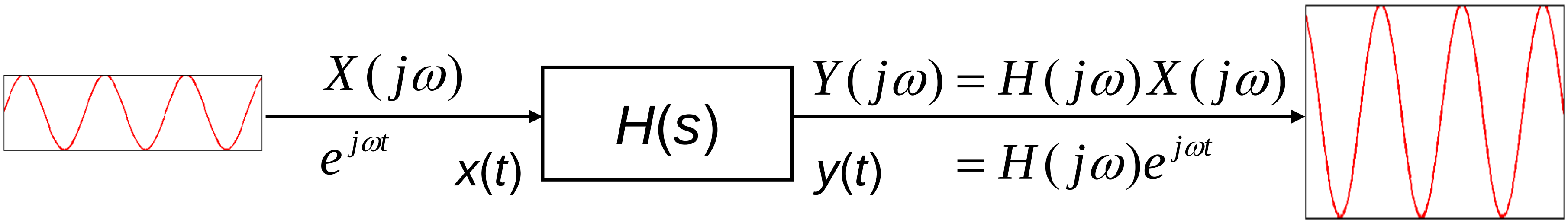
Inevitable time/frequency design compromise





Continuous-Time Transfer Functions

Introduction: Transfer Functions & Frequency Response



We can use the Fourier (Laplace) transfer function $H(j\omega)$ ($H(s)$) in a variety of ways:

- Design a system/filter with appropriate frequency domain characteristics
- Calculate the system's time domain response using $Y(j\omega)=H(j\omega)X(j\omega)$ and taking the inverse Fourier transform

However, we can also get a lot of information from studying $H(j\omega)$ directly and representing it in polar fashion as

$$H(j\omega) = |H(j\omega)|e^{j\angle H(j\omega)}$$

Example: 1st Order System and Cos Input

The 1st order system transfer function is: (a>0, $h(t)=e^{-at}u(t)$)

$$H(j\omega) = \frac{1}{(j\omega + a)}$$

The input signal $x(t)=\cos(\omega_0 t)$, which has fundamental frequency ω_0, has Fourier transform:

$$X(j\omega) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$$

The (stable) system's output is:

$$Y(j\omega) = \frac{\pi}{a + j\omega}\delta(\omega - \omega_0) + \frac{\pi}{a + j\omega}\delta(\omega + \omega_0)$$

$$= \frac{\pi(a - j\omega)}{a^2 + \omega^2}\delta(\omega - \omega_0) + \frac{\pi(a - j\omega)}{a^2 + \omega^2}\delta(\omega + \omega_0)$$

$$y(t) = \tfrac{1}{2\pi}\int_{-\infty}^{\infty} Y(j\omega)e^{j\omega t}d\omega = \frac{(a - j\omega_0)e^{j\omega_0 t} + (a + j\omega_0)e^{-j\omega_0 t}}{2(a^2 + \omega_0^2)}$$

$$= \frac{a\cos(\omega_0 t) + \omega_0 \sin(\omega_0 t)}{(a^2 + \omega_0^2)}$$

System Gain and Phase Shift

In the frequency domain, the effect of the system on the input signal for the frequency component ω is:

$$Y(j\omega) = |H(j\omega)|e^{j\angle H(j\omega)}\,|X(j\omega)|e^{j\angle X(j\omega)}$$

$$|Y(j\omega)| = |H(j\omega)||X(j\omega)|$$

$$\angle Y(j\omega) = \angle H(j\omega) + \angle X(j\omega)$$

The effect of a system, $H(j\omega)$, has on the Fourier transform of an input signal is to:

- **Scale the magnitude** by $|H(j\omega)|$. This is commonly referred to as the **system gain**.
- **Shift the phase** of the input signal by adding $\angle H(j\omega)$ to it. This is commonly referred to as the **phase shift**.

These modifications (**magnitude and phase distortions**) may be desirable/undesirable and must be understood in system analysis and design.

Example: Cos Input to a 1st Order System

Consider a sinusoidal input signal to a first order, LTI, stable system

$$h(t) = e^{-at}u(t), \qquad a > 0$$

$$x(t) = \cos(\omega_0 t)$$

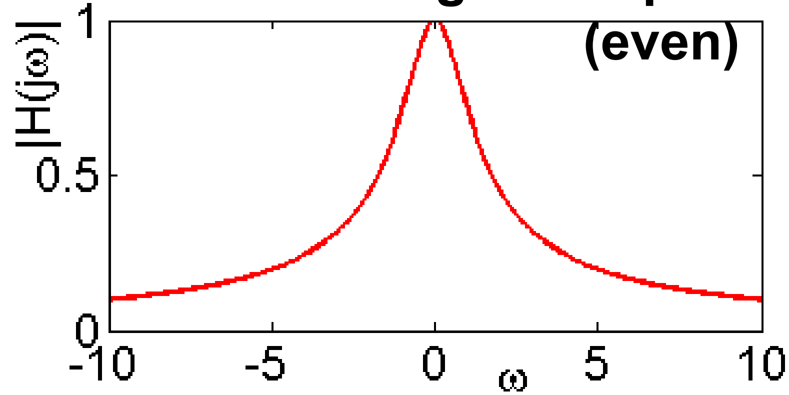
$$H(j\omega) = \frac{1}{j\omega + a}$$

When ω_0 is close to zero, its magnitude is passed on scaled by $1/a$

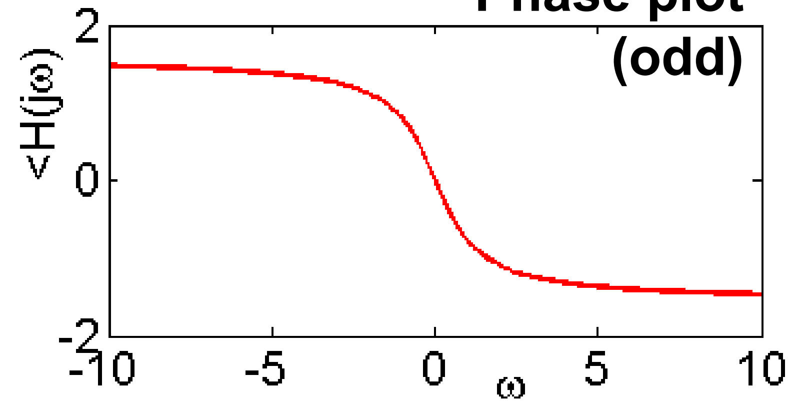
When the $|\omega_0|$ is high, the signal is substantially suppressed

i.e. it is a low pass filter …

Magnitude plot (even)



Phase plot (odd)



We deduce the properties solely by looking at the **transfer function in the frequency domain**

The Effect of Phase …

The effect of the transfer function's magnitude is fairly easy to see – it magnifies/suppresses the input signal

The effect of the change in phase is a bit less obvious to imagine.

Consider when the phase shift is a linear function of ω:

$$H(j\omega) = e^{-j\omega t_0} \qquad |H(j\omega)| = 1$$

$$\angle H(j\omega) = -\omega t_0$$

This system corresponds to a **pure time shift** of the input

$$y(t) = x(t - t_0)$$

Slope of the phase corresponds to the time delay

When the phase is not a linear function, it is slightly more complex

Log-Magnitude and Phase Plots

When analysing system responses, it is typical to use a **log scaling** for the magnitude

$$\log(|Y(j\omega)|) = \log(|H(j\omega)|) + \log(|X(j\omega)|)$$

So the gain effect is **additive**: **0 means "no change"**

If the log magnitude is plotted, the effect can be interpreted as adding each individual component (like the time-delayed phase)

Often units are **decibels (dB)** $\mathbf{20\log_{10}}$

Similarly, taking logs of frequency allows us to view detail over a much greater range (which is important for frequency selective filters)

Note that taking a log of the frequency, we typically only consider positive frequency values (as the **magnitude is even**, and the **phase is odd**)

Bode Plots

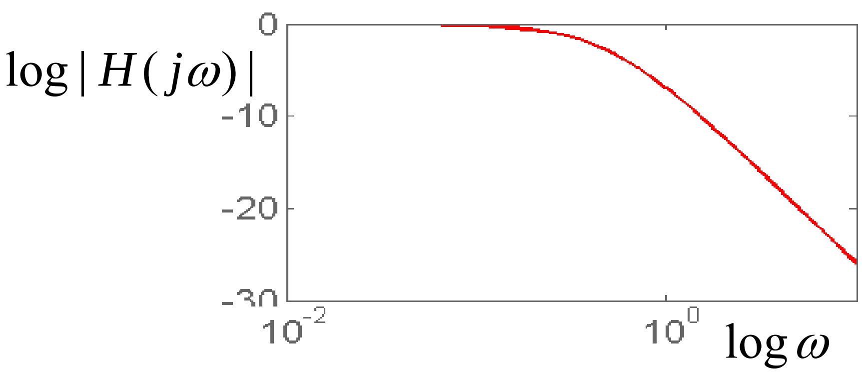
A **Bode Plot** for a system is simply plots of log magnitude and phase against log frequency

Both the log magnitude and phase effects are now **additive**

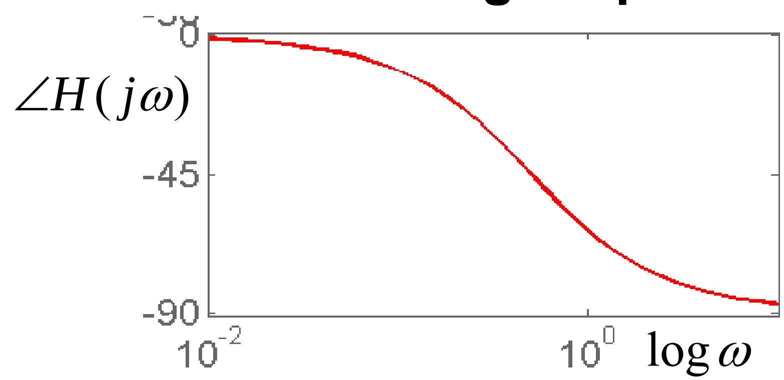
Widely used for **analysis and design** of **filters** and **controllers**

Example

Log mag v log freq



Phase v log freq



Low pass, unity filter

Example 1: Bode Plot 1st Order System

Consider a LTI first order system
described by:

$$\tau \frac{dy(t)}{dt} + y(t) = x(t), \qquad \tau > 0$$

Fourier transfer function is:

$$H(j\omega) = \frac{1}{\tau j\omega + 1}$$

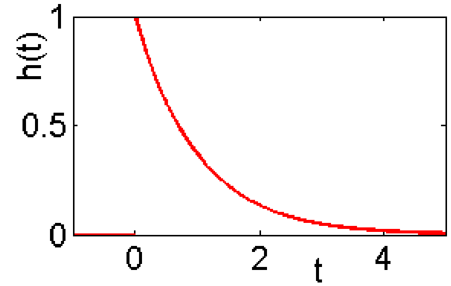
the impulse response is:

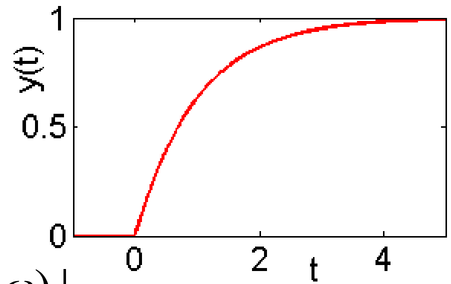
$$h(t) = \tfrac{1}{\tau} e^{-t/\tau} u(t)$$

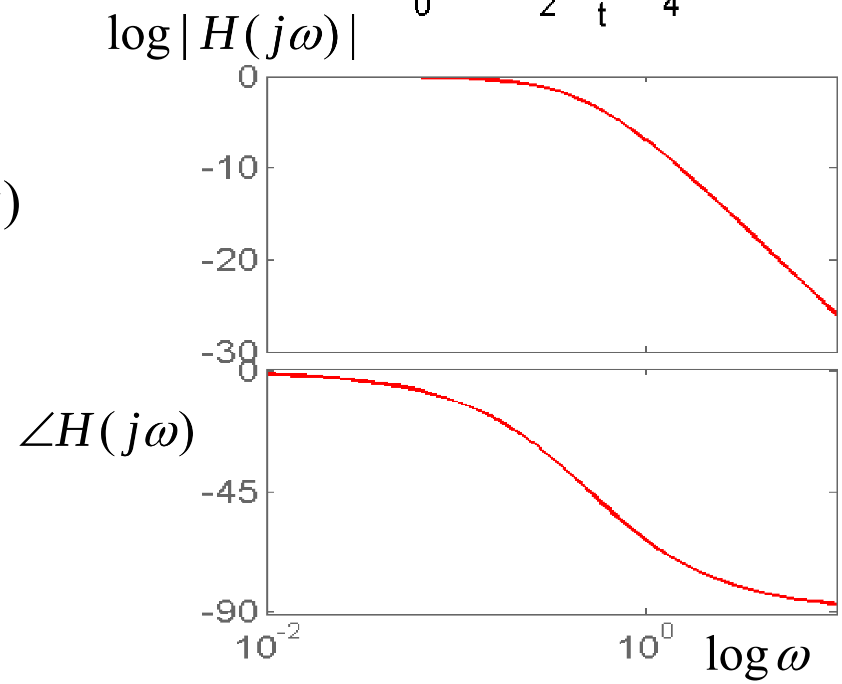
and the step response is:

$$y(t) = h(t) * u(t) = (1 - \tfrac{1}{\tau} e^{-t/\tau}) u(t)$$

Bode diagrams are shown as
log/log plots on the x and y
axis with τ=2.





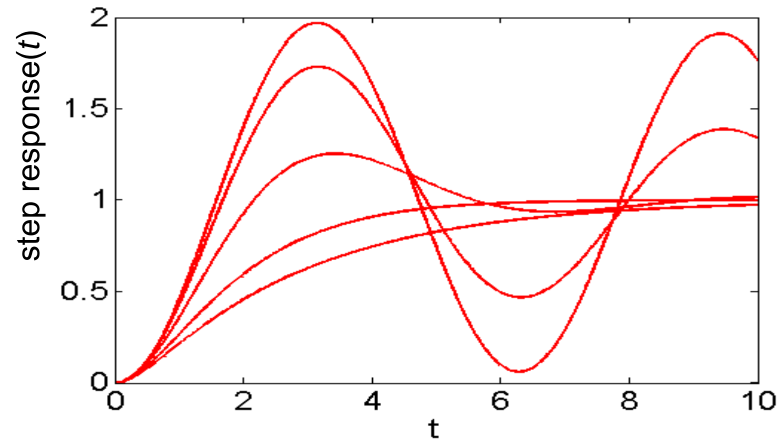


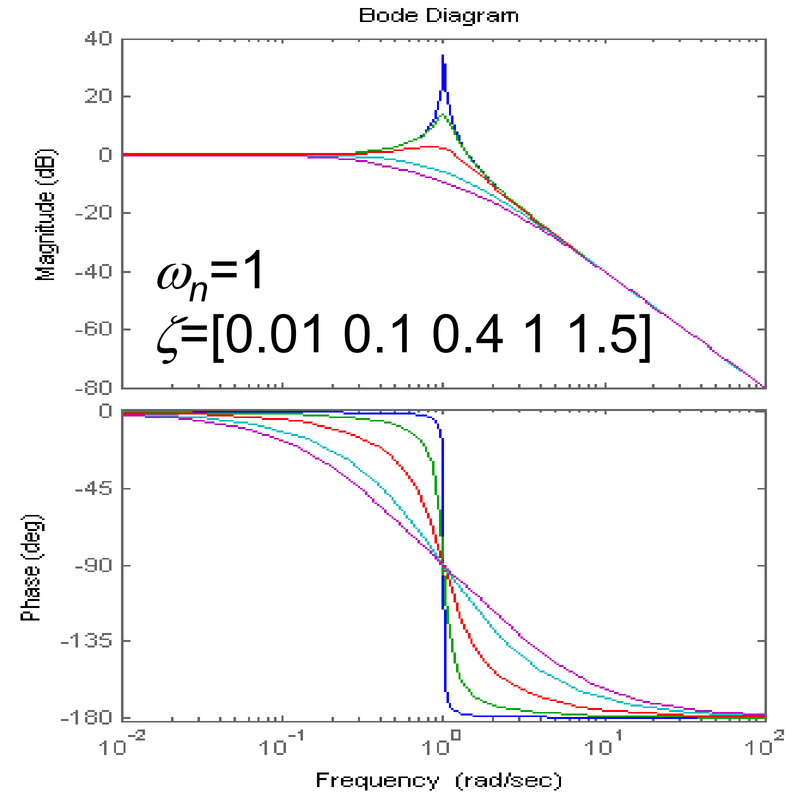
Example 2: Bode Plot 2nd Order System

The LTI 2nd order differential equation

$$\frac{d^2 y(t)}{dt^2} + 2\zeta\omega_n \frac{dy(t)}{dt} + \omega_n^2 y(t) = \omega_n^2 x(t)$$

which can represent the response of mass-spring systems and RLC circuits, amongst other things





ω_n is the undamped natural frequency

ζ is the damping ratio

$$H(j\omega) = \frac{\omega_n^2}{(j\omega)^2 + 2\zeta\omega_n(j\omega) + \omega_n^2}$$

$$= \frac{\omega_n^2}{(j\omega - p_1)(j\omega - p_2)}$$

$$p_1 = -\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1}$$

$$p_2 = -\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}$$

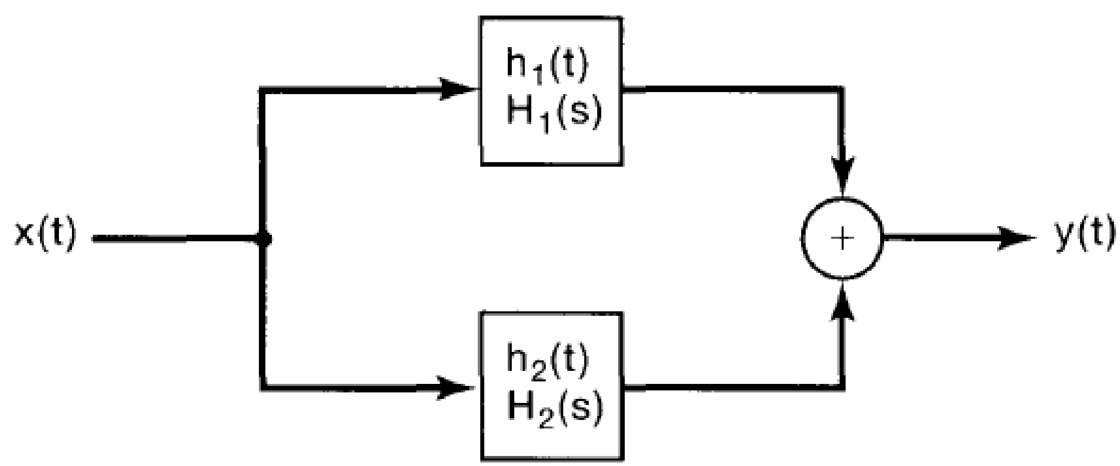
SYSTEM FUNCTION ALGEBRA AND BLOCK DIAGRAM REPRESENTATIONS

Consider the parallel interconnection of two systems, as shown in Figure 9.30(a). The impulse response of the overall system is

$$h(t) = h_1(t) + h_2(t), \tag{9.155}$$

and from the linearity of the Laplace transform,

$$H(s) = H_1(s) + H_2(s). \tag{9.156}$$



(a)

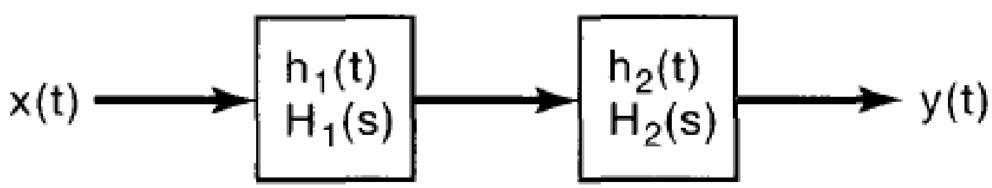
SYSTEM FUNCTION ALGEBRA AND BLOCK DIAGRAM REPRESENTATIONS

Similarly, the impulse response of the series interconnection in Figure 9.30(b) is

$$h(t) = h_1(t) * h_2(t), \tag{9.157}$$

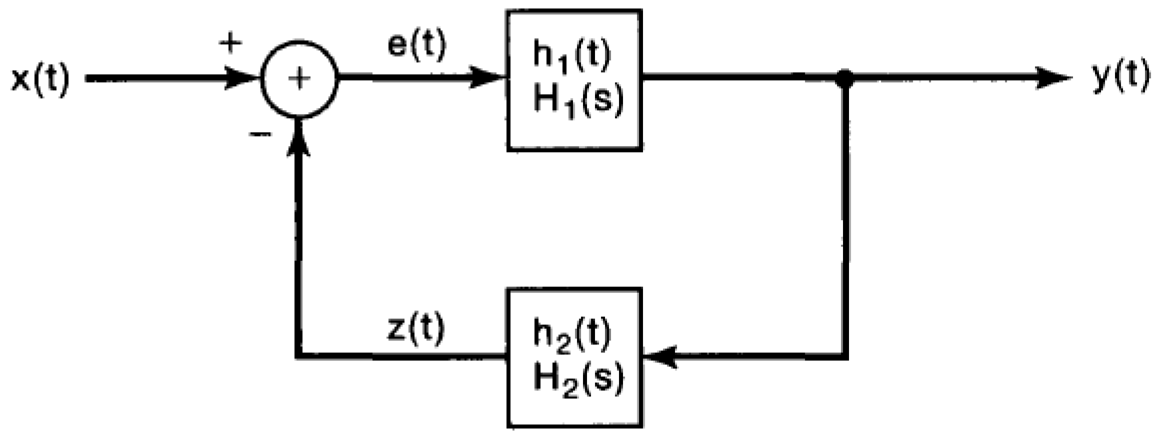
and the associated system function is

$$H(s) = H_1(s)H_2(s). \tag{9.158}$$



(b)

SYSTEM FUNCTION ALGEBRA AND BLOCK DIAGRAM REPRESENTATIONS



$$Y(s) = H_1(s)E(s),$$
$$E(s) = X(s) - Z(s),$$

Feedback interconnection of two LTI systems.

$$Z(s) = H_2(s)Y(s),$$

from which we obtain the relation

$$Y(s) = H_1(s)[X(s) - H_2(s)Y(s)],$$

or

$$\frac{Y(s)}{X(s)} = H(s) = \frac{H_1(s)}{1 + H_1(s)H_2(s)}.$$

SYSTEM FUNCTION ALGEBRA AND BLOCK DIAGRAM REPRESENTATIONS

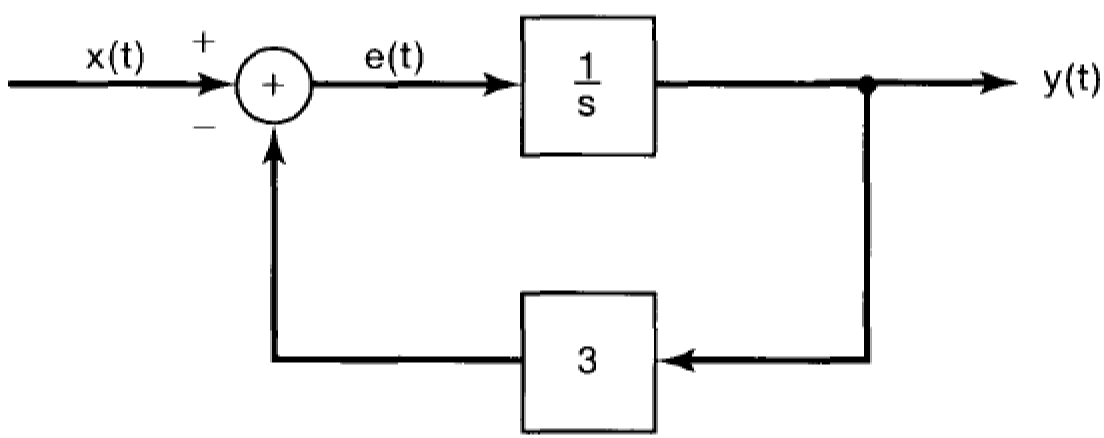
Example 9.28

Consider the causal LTI system with system function

$$H(s) = \frac{1}{s+3}.$$

From Section 9.7.3, we know that this system can also be described by the differential equation

$$\frac{dy(t)}{dt} + 3y(t) = x(t),$$



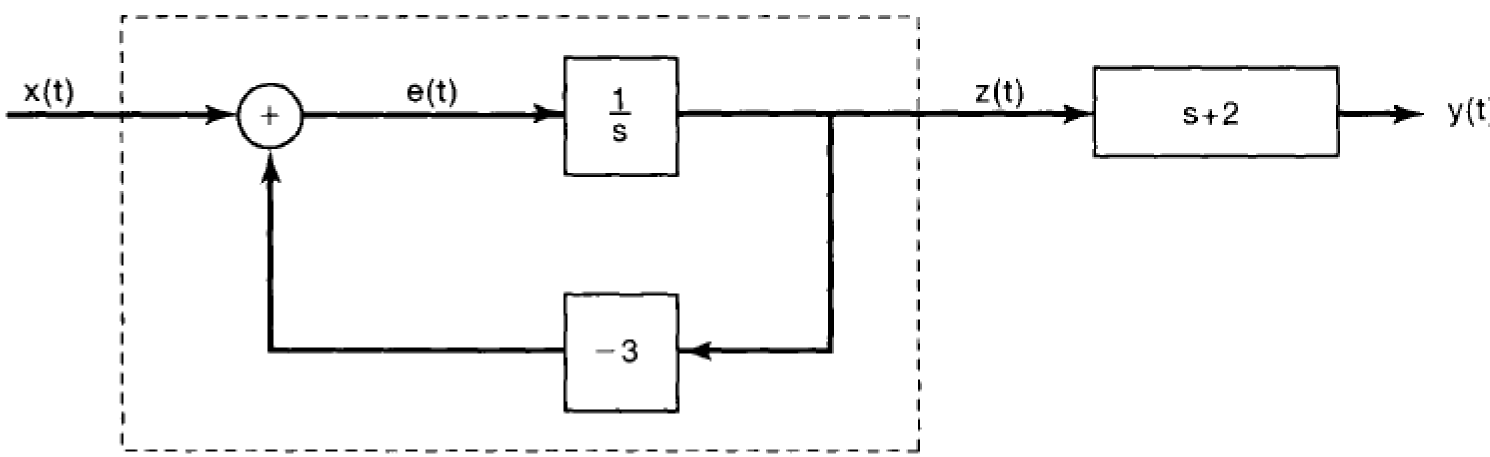
SYSTEM FUNCTION ALGEBRA AND BLOCK DIAGRAM REPRESENTATIONS

Example 9.29

Consider now the causal LTI system with system function

$$H(s) = \frac{s+2}{s+3} = \left(\frac{1}{s+3}\right)(s+2).$$

As suggested by eq. (9.164), this system can be thought of as a cascade of a system with system function $1/(s+3)$ followed by a system with system function $s+2$, and we have illustrated this in Figure 9.33(a), in which we have used the block diagram in Figure 9.32(a) to represent $1/(s+3)$.



SYSTEM FUNCTION ALGEBRA AND BLOCK DIAGRAM REPRESENTATIONS

Example 9.29

It is also possible to obtain an alternative block diagram representation for the system in eq. (9.164). Using the linearity and differentiation properties of the Laplace transform, we know that $y(t)$ and $z(t)$ in Figure 9.33 (a) are related by

$$y(t) = \frac{dz(t)}{dt} + 2z(t).$$

However, the input $e(t)$ to the integrator is exactly the derivative of the output $z(t)$, so that

$$y(t) = e(t) + 2z(t),$$

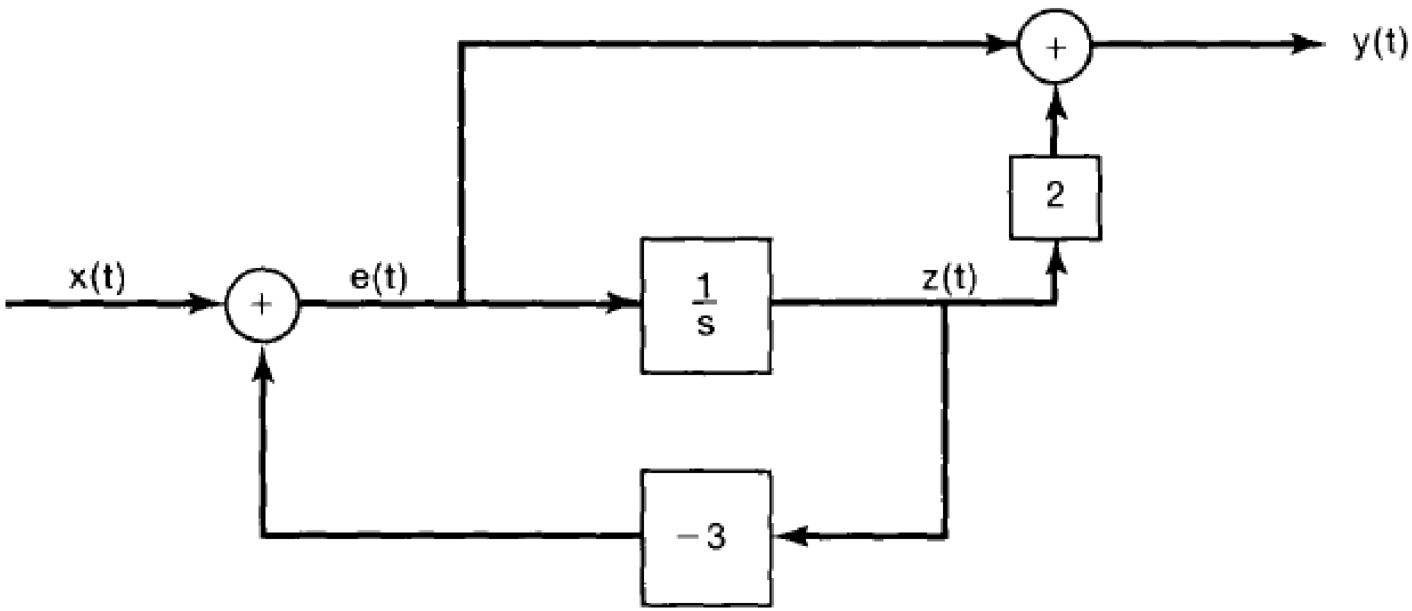
which leads directly to the alternative block diagram representation shown in Figure 9.33(b). Note that the block diagram in Figure 9.33(a) requires the differentiation of $z(t)$, since

$$y(t) = \frac{dz(t)}{dt} + 2z(t)$$

In contrast, the block diagram in Figure 9.33(b) does not involve the explicit differentiation of any signal.

SYSTEM FUNCTION ALGEBRA AND BLOCK DIAGRAM REPRESENTATIONS

Example 9.29



The Unilateral Laplace Transform

- There are many applications of Laplace transforms in which it is reasonable to assume that the signals involved are causal. In such problems, it is advantageous to define the unilateral or one-sided Laplace transform, which is based only on the nonnegative-time portions of a signal.

- Any signal that is identically zero for $t<0$ has identical bilateral and unilateral Laplace transforms.

- The ROC for a unilateral Laplace transform must always be a right-half plane.

-The unilateral Laplace transform is of considerable value in analyzing causal systems and particularly, systems specified by linear constant-coefficient differential equations with nonzero initial conditions.

The Unilateral Laplace Transform

- Time Scaling

$$x(t) \leftrightarrow X(s)$$

$$x(at) \leftrightarrow \frac{1}{a} X\left(\frac{s}{a}\right), a > 0$$

- Time Shift

$$x(t) \leftrightarrow X(s)$$

$$x(t-\tau) \leftrightarrow e^{-s\tau} X(s)$$

for all τ such that $x(t-\tau)u(t) = x(t-\tau)u(t-\tau)$

The Unilateral Laplace Transform

- Differentiation in the Time Domain

$$x(t) \leftrightarrow X(s)$$

$$\boxed{\frac{d}{dt}x(t) \leftrightarrow sX(s) - x(0^-)}$$

Proof. $\int_{0^-}^{\infty} \frac{d}{dt}x(t)e^{-st}dt = \int_{0^-}^{\infty} e^{-st}dx(t) = x(t)e^{-st}\Big|_{0^-}^{\infty}$

$$-\int_{0^-}^{\infty} x(t)de^{-st} = x(t)e^{-st}\Big|_{t=\infty} - x(t)e^{-st}\Big|_{t=0^-} + s\int_{0^-}^{\infty} x(t)e^{-st}dt$$

$$= sX(s) - x(0^-) \quad (X(s) \text{ exists} \Rightarrow x(t)e^{-st}\Big|_{t=\infty} = 0)$$

The Unilateral Laplace Transform

$$\frac{d^2}{dt^2}x(t) \leftrightarrow s\left(sX(s) - x(0^-)\right) - x'(0^-)$$

$$= s^2 X(s) - sx(0^-) - x'(0^-)$$

$$\frac{d^3}{dt^3}x(t) \leftrightarrow s\left(s^2 X(s) - sx(0^-) - x'(0^-)\right) - x''(0^-)$$

$$= s^3 X(s) - s^2 x(0^-) - sx'(0^-) - x''(0^-)$$

The Unilateral Laplace Transform

- Integration

$$x(t) \leftrightarrow X(s)$$

$$\int_{-\infty}^{t} x(\tau)d\tau \leftrightarrow \frac{1}{s}\int_{-\infty}^{0^-} x(\tau)d\tau + \frac{X(s)}{s}$$

$$\text{Proof. } y(t) = \int_{-\infty}^{t} x(\tau)d\tau \leftrightarrow Y(s)$$

$$\frac{d}{dt}y(t) = x(t) \leftrightarrow sY(s) - y(0^-) = X(s)$$

$$Y(s) = \frac{y(0^-) + X(s)}{s}$$

The Unilateral Laplace Transform

Example. Find the inverse Laplace transform of

$$X(s) = \frac{3s+4}{(s+1)(s+2)^2}$$

Solution. $X(s) = \dfrac{1}{s+1} - \dfrac{1}{s+2} + \dfrac{2}{(s+2)^2}$

$$x(t) = e^{-t}u(t) - e^{-2t}u(t) + 2te^{-2t}u(t)$$

The Unilateral Laplace Transform

Example Use the ULT to determine the output of a system represented by

$$\frac{d^2}{dt^2}y(t)+5\frac{d}{dt}y(t)+6y(t)=\frac{d}{dt}x(t)+6x(t)$$

in response to the input $x(t)=u(t)$. Assume that the initial conditions on the system are

$y(0^-)=1$ and $y'(0^-)=2$.

The Unilateral Laplace Transform

$$\left(s^2Y(s) - sy(0^-) - y'(0^-)\right) + 5\left(sY(s) - y(0^-)\right)$$

$$+6Y(s) = sX(s) + 6X(s)$$

$$y(0^-) = 1,\ y'(0^-) = 2,\ X(s) = \frac{1}{s}$$

$$(s^2 + 5s + 6)Y(s) = s + 7 + (s + 6)\frac{1}{s}$$

$$Y(s) = \underbrace{\frac{s+7}{s^2 + 5s + 6}}_{\text{zero-input component}} + \underbrace{\frac{s+6}{s(s^2 + 5s + 6)}}_{\text{zero-state component}}$$

The Unilateral Laplace Transform

$$Y_{zp}(s) = \frac{s+7}{s^2+5s+6} = \frac{s+7}{(s+2)(s+3)} = \frac{5}{s+2} - \frac{4}{s+3}$$

$$y_{zp}(t) = 5e^{-2t}u(t) - 4e^{-3t}u(t)$$

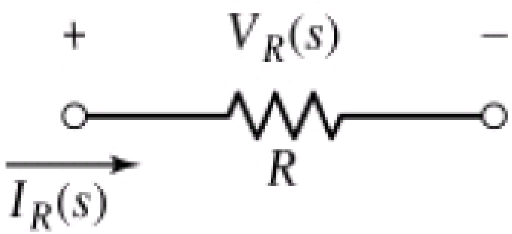
$$Y_{zs}(s) = \frac{s+6}{s(s^2+5s+6)} = \frac{1}{s} - \frac{2}{s+2} + \frac{1}{s+3}$$

$$y_{zs}(t) = u(t) - 2e^{-2t}u(t) + e^{-3t}u(t)$$

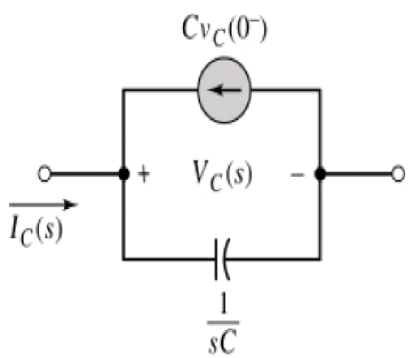
$$y(t) = y_{zp}(t) + y_{zs}(t) = u(t) + 3e^{-2t}u(t) - 3e^{-3t}u(t)$$

Laplace Transform in Circuit Analysis

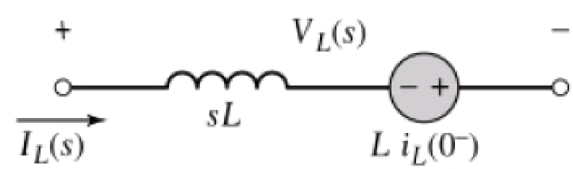
$$v_R(t) = Ri_R(t) \leftrightarrow V_R(s) = RI_R(s)$$



$$i_C(t) = C\frac{d}{dt}v_L(t) \leftrightarrow I_C(s) = sCV_C(s) - Cv_C(0^-)$$

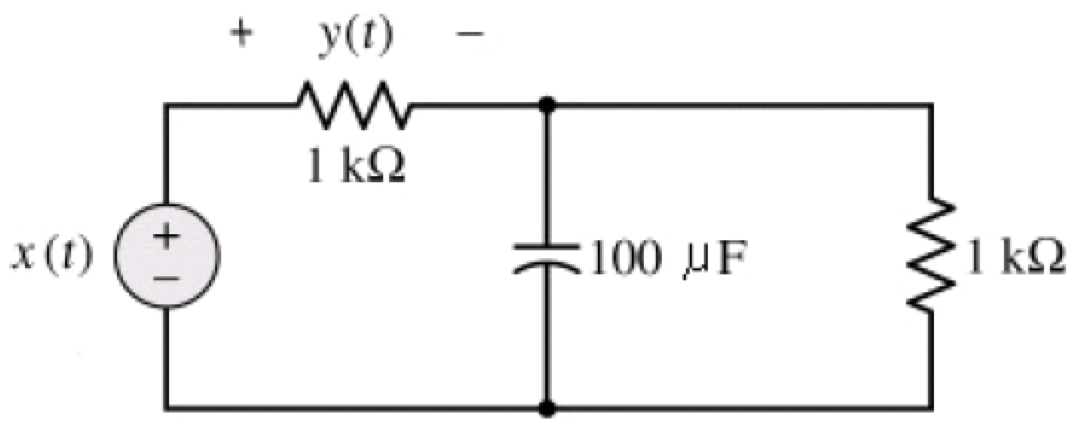


$$v_L(t) = L\frac{d}{dt}i_L(t) \leftrightarrow V_L(s) = sLI_L(s) - Li_L(0^-)$$

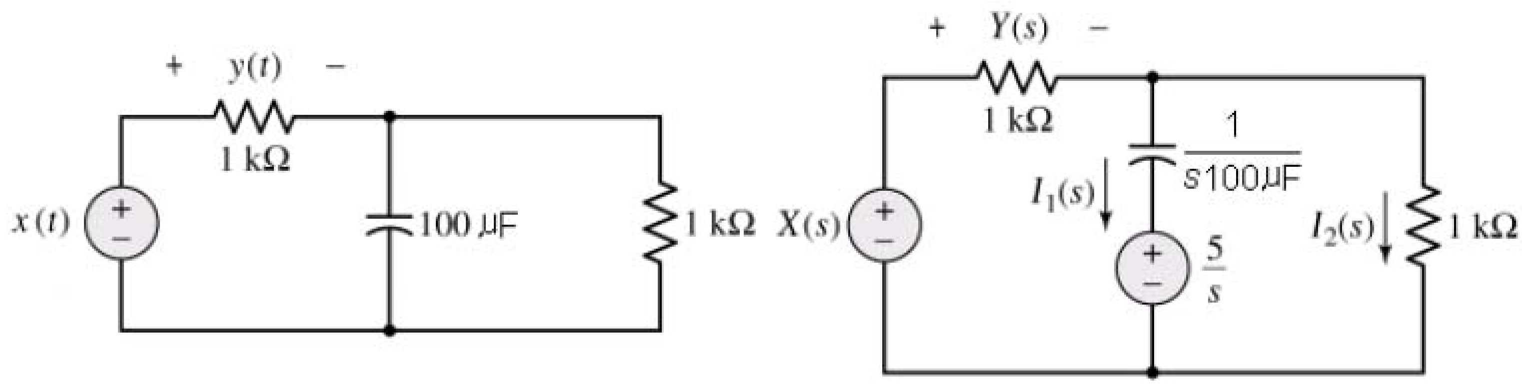


Laplace Transform in Circuit Analysis

Example Use Laplace transform circuit model to determine the voltage $y(t)$ in the circuit for an applied voltage $x(t) =$ 3 $e^{-10t}u(t)$ V. The voltage across the capacitor at time $t = 0^-$ is 5V.



Laplace Transform in Circuit Analysis



Solution

$$Y(s) = 1000[I_1(s) + I_2(s)]$$

$$X(s) = Y(s) + \frac{1}{s(10^{-4})} I_1(s) + \frac{5}{s}$$

$$X(s) = Y(s) + 1000 I_2(s)$$

Laplace Transform in Circuit Analysis

$$Y(s) = 1000I_1(s) + 1000I_2(s)$$

$$X(s) = Y(s) + \frac{10,000}{s} I_1(s) + \frac{5}{s}$$

$$0.1X(s) = 0.1Y(s) + 1000I_1(s) + 0.5$$

$$X(s) = Y(s) + 1000I_2(s)$$

$$Y(s) = \left[0.1sX(s) - 0.1sY(s) - 0.5\right] + \left[X(s) - Y(s)\right]$$

$$[0.1s + 2]Y(s) = [0.1s + 1]X(s) - 0.5$$

$$[s + 20]Y(s) = [s + 10]X(s) - 5$$

$$Y(s) = \frac{s+10}{s+20} X(s) - \frac{5}{s+20} = \frac{s+10}{s+20}\,\frac{3}{s+10} - \frac{5}{s+20} = -\frac{2}{s+20}$$

$$y(t) = -2e^{-20t}u(t) \text{ V.}$$