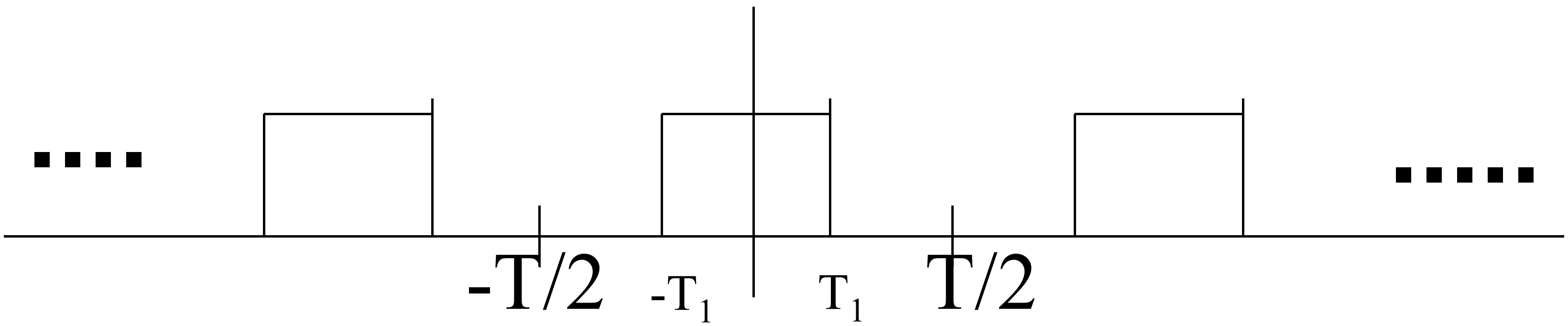
# Introduction to Signals and Systems: V216

## Lecture #8

## Chapter 4: The Continuous-Time Fourier Transform

# Fourier Series




This periodic signal x(t) repeats every T seconds.
$x(t)=1$, for $|t|<T_1$, and $x(t)=0$, for $T_1 <|t|< T/2$

Fundamental period= T,
Fundamental frequency $\omega_0 =2\pi/T$.
Choosing the period of integration to be between -T/2 and +T/2. Use eqn 3.39 to get at Fourier Series Coefficients.

# Fourier Series

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt = \frac{1}{T}\int_T x(t)e^{-jk\frac{2\pi}{T}t}dt$$

Let us get the dc, constant term or average value over a period, first, i.e. $k = 0$

$$a_0 = \frac{1}{T}\int_{-T_1}^{T_1} dt = \frac{2T_1}{T}$$

# Fourier Series

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt = \frac{1}{T}\int_T x(t)e^{-jk\frac{2\pi}{T}t}dt$$

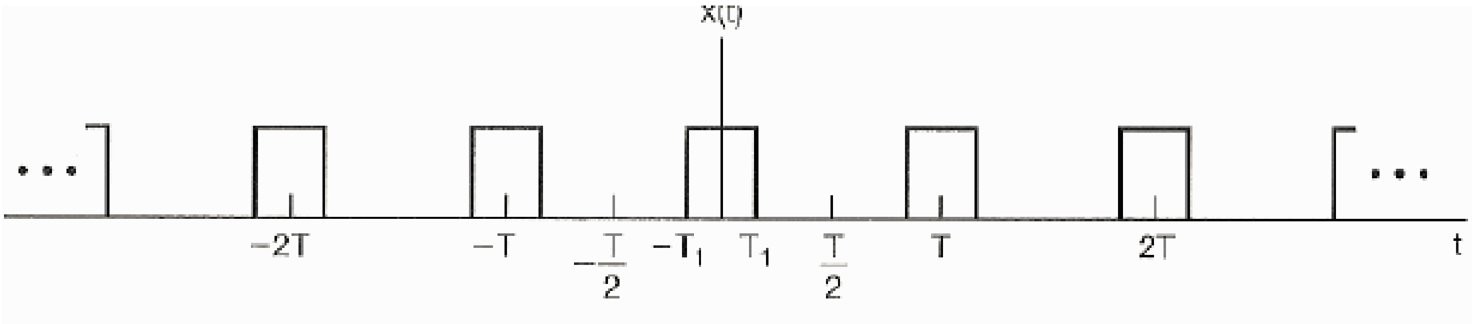
For fundamental first order and higher order harmonics :-

we have $k \neq 0$.

$$a_k = \frac{1}{T}\int_{-T_1}^{T_1} e^{-jk\omega_0 t}dt = -\frac{1}{jk\omega_0 T}e^{-jk\omega_0 t}\Big|_{-T_1}^{T_1},$$

$$a_k = \frac{2}{k\omega_0 T}\left[\frac{e^{jk\omega_0 T_1} - e^{-jk\omega_0 T_1}}{2j}\right] = \frac{2\sin(k\omega_0 T_1)}{k\omega_0 T} = \frac{\sin(k\omega_0 T_1)}{k\pi}$$

# The Continuous-Time Fourier Transform




$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & T_1 < |t| < T/2 \end{cases}$$

$$a_k = \frac{2\sin(k w_0 T_1)}{k w_0 T}$$

Fourier series coefficients

$$T a_k = \left.\frac{2\sin(w T_1)}{w}\right|_{w = k w_0}$$

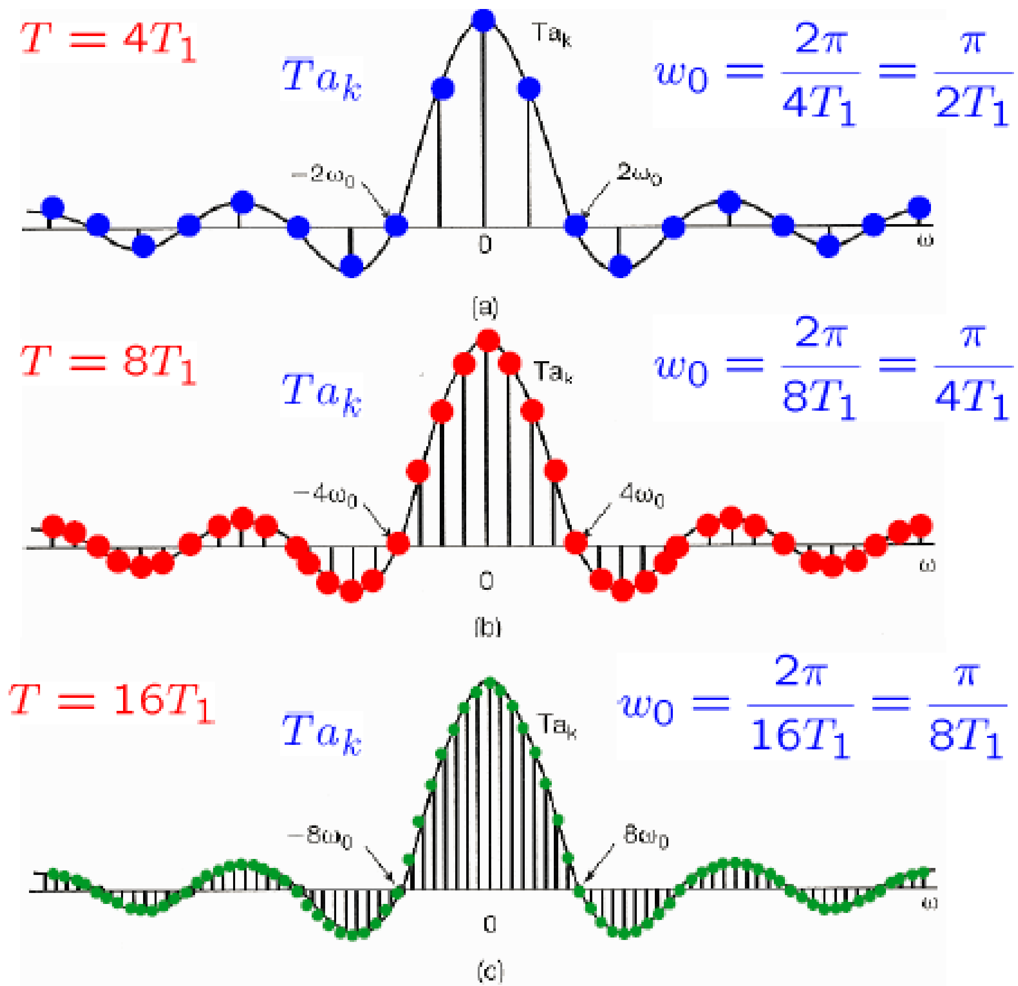
$w$ as a continuous variable

# The Continuous-Time Fourier Transform

$$Ta_k = \frac{2\sin(wT_1)}{w}$$

$$w = kw_0 = k\frac{2\pi}{T}$$

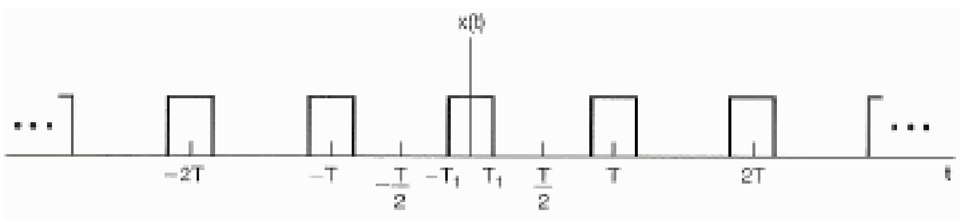
$$w_0 = \frac{2\pi}{T}$$

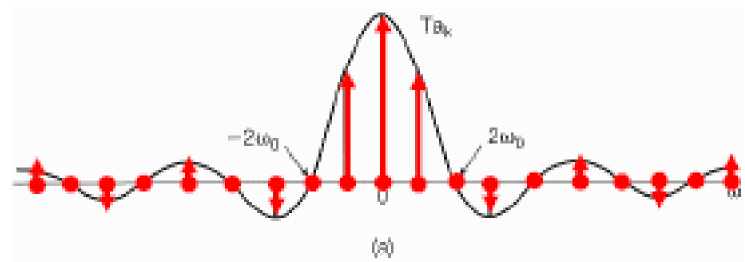


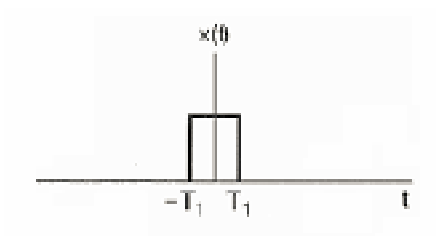
# The Continuous-Time Fourier Transform

$$w = kw_0 = k\frac{2\pi}{T}$$

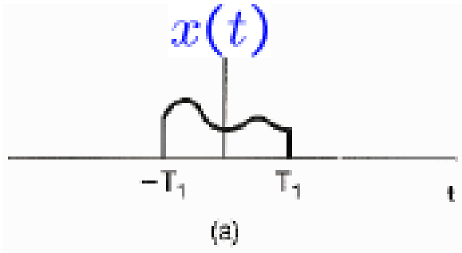
$$T \rightarrow \infty \;\Rightarrow\; \{Ta_k\} \rightarrow \left.\frac{2\sin(wT_1)}{w}\right|_{w=kw_0}$$



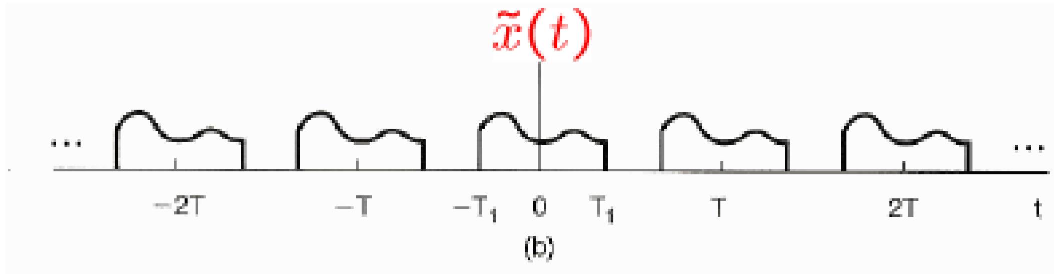




# The Continuous-Time Fourier Transform




an aperiodic signal




a periodic signal

$$\tilde{x}(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t}$$

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} \tilde{x}(t) e^{-jkw_0 t} dt$$

$$\Rightarrow a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t) e^{-jkw_0 t} dt = \frac{1}{T}\int_{-\infty}^{+\infty} x(t) e^{-jkw_0 t} dt$$

$T \rightarrow \infty \Rightarrow$

# The Continuous-Time Fourier Transform

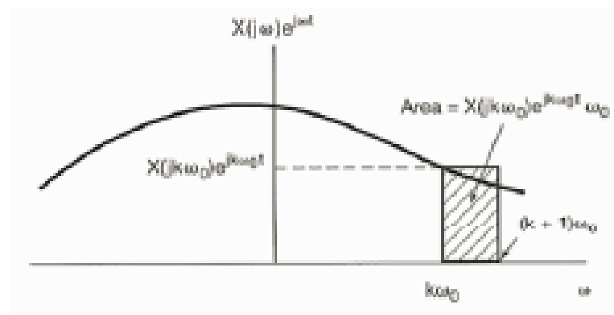
- Define the envelope $X(jw)$ of $Ta_k$ as

$$Ta_k = \frac{2\sin(wT_1)}{w}$$

$$X(jw) = \int_{-\infty}^{+\infty} x(t) e^{-jwt} dt$$

- Then,

$$a_k = \frac{1}{T} X(jkw_0)$$




- Hence,

$$w_0 = \frac{2\pi}{T}$$

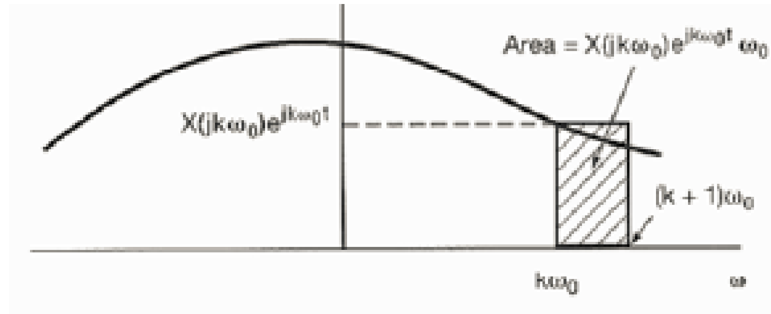
$$\tilde{x}(t) = \sum_{k=-\infty}^{+\infty} \frac{1}{T} X(jkw_0) e^{jkw_0 t}$$

$$\frac{1}{T} = \frac{1}{2\pi} w_0$$

$$= \frac{1}{2\pi} \sum_{k=-\infty}^{+\infty} X(jkw_0) e^{jkw_0 t} w_0$$

# The Continuous-Time Fourier Transform

also $w_0 \rightarrow 0$




$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(jw)e^{jwt}dw$$

- inverse Fourier transform eqn

- synthesis eqn

$$X(jw) = \int_{-\infty}^{+\infty} x(t)e^{-jwt}dt$$

- $X(jw)$: Fourier Transform of $x(t)$
  spectrum

- analysis eqn

$$a_k = \frac{1}{T}X(jw)\Big|_{w=kw_0}$$

# The Continuous-Time Fourier Transform

- Sufficient conditions for the convergence of FT

$$x(t) \xrightarrow{\mathcal{CTFT}} X(jw) \qquad X(jw) = \int_{-\infty}^{+\infty} x(t)e^{-jwt}dt$$

$$\hat{x}(t) \xleftarrow{\mathcal{CTIFT}} X(jw) \qquad \hat{x}(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(jw)e^{jwt}dw$$

$$e(t) = \hat{x}(t) - x(t)$$

- If $x(t)$ has finite energy

i.e., square integrable, $\int_{-\infty}^{+\infty} |x(t)|^2 dt < \infty$

$\Rightarrow \quad X(jw)$ is finite

$$\Rightarrow \quad \int_{-\infty}^{+\infty} |e(t)|^2 dt = 0$$

# The Continuous-Time Fourier Transform

- Sufficient conditions for the convergence of FT
  - Dirichlet conditions:

1. x(t) be absolutely integrable; that is, $$\int_{-\infty}^{+\infty} |x(t)|\,dt < \infty$$

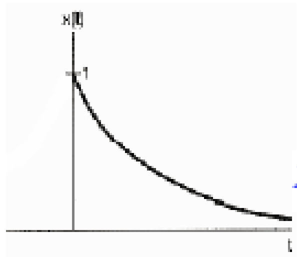
2. x(t) have a finite number of maxima and minima within any finite interval

3. x(t) have a finite number of discontinuities within any finite interval
Furthermore, each of these discontinuities must be finite

# The Continuous-Time Fourier Transform

- Example 4.1:

$$x(t) = e^{-at}\, u(t), \quad a > 0$$




$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(jw)e^{jwt}dw$$

$$X(jw) = \int_{-\infty}^{\infty} x(t)e^{-jwt}dt$$

$$\Rightarrow X(jw) = \int_{-\infty}^{\infty} x(t)e^{-jwt}dt$$

$$= \int_{-\infty}^{\infty} e^{-at}\, u(t)\, e^{-jwt}dt$$

$$= \int_{0}^{\infty} e^{-at}\, e^{-jwt}dt$$

$$= \int_{0}^{\infty} e^{-(a+w)t}\, dt$$

$$= -\frac{1}{a+jw}e^{-(a+jw)t}\Big|_{0}^{\infty}$$

$$= 0 - (-\frac{1}{a+jw}e^{-(a+jw)0})$$

$$= \frac{1}{a+jw}, \quad a > 0$$

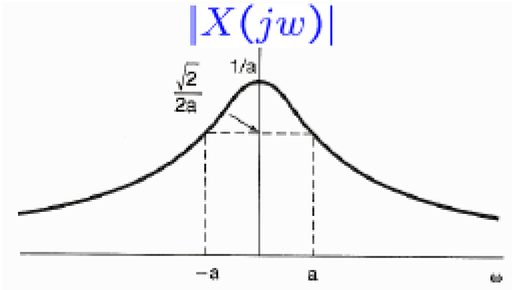
# The Continuous-Time Fourier Transform

- Example 4.1:

$$\Rightarrow \; X(jw) = \frac{1}{a + jw}, \quad a > 0$$

$$\Rightarrow \; |X(jw)| = \frac{1}{\sqrt{a^2 + w^2}}$$

$$\Rightarrow \; \sphericalangle X(jw) = -\tan^{-1}\left(\frac{w}{a}\right)$$



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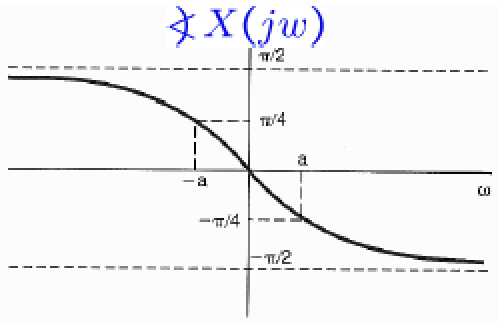


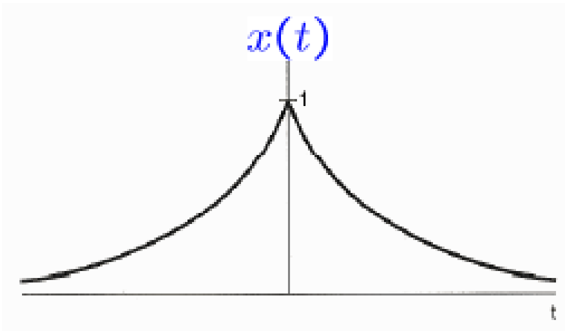
(b)

# The Continuous-Time Fourier Transform

- Example 4.2:

$$X(jw) = \int_{-\infty}^{\infty} x(t) e^{-jwt} dt$$

$$x(t) = e^{-a|t|}, \quad a > 0$$

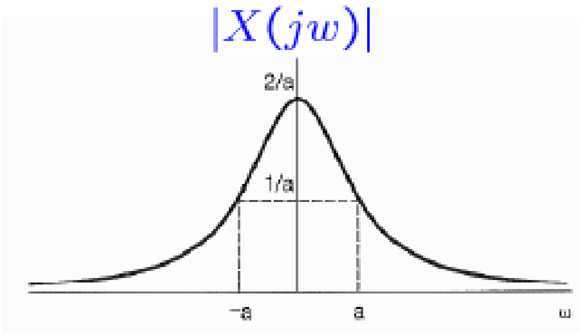



$$\Rightarrow \ X(jw) = \int_{-\infty}^{\infty} e^{-a|t|} \, e^{-jwt} dt$$

$$= \int_{-\infty}^{0} e^{at} \, e^{-jwt} dt \quad + \int_{0}^{\infty} e^{-at} \, e^{-jwt} dt$$

$$= \frac{1}{a - jw} + \frac{1}{a + jw}$$

$$= \frac{2a}{a^2 + w^2}$$

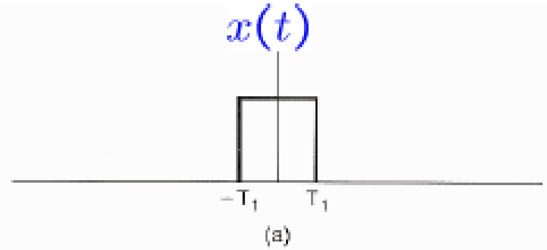


# The Continuous-Time Fourier Transform

- Example 4.4:

$$X(jw) = \int_{-\infty}^{\infty} x(t) e^{-jwt} dt$$

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & |t| > T_1 \end{cases}$$



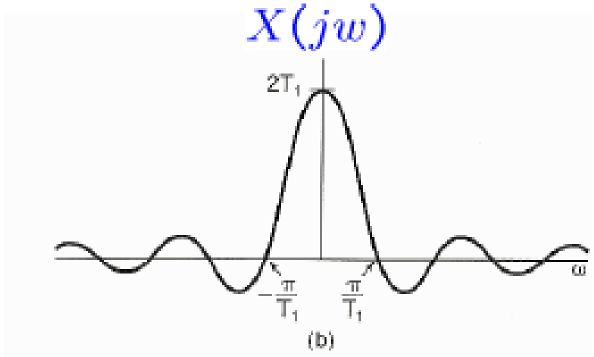

$$\Rightarrow X(jw) = \int_{-\infty}^{\infty} x(t) e^{-jwt} dt = \int_{-T_1}^{T_1} e^{-jwt} dt$$

$$= \frac{1}{-jw} e^{-jwt} \Big|_{-T_1}^{T_1}$$

$$= \frac{1}{-jw} \left( e^{-jwT_1} - e^{jwT_1} \right)$$

$$= \frac{1}{jw} \left( e^{jwT_1} - e^{-jwT_1} \right)$$

$$= 2 \frac{\sin(wT_1)}{w}$$



# The Continuous-Time Fourier Transform

- Example 4.5:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(jw)e^{jwt}dw$$

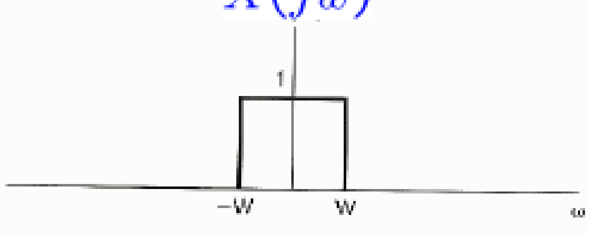
$$X(jw) = \int_{-\infty}^{\infty} x(t)e^{-jwt}dt$$

$$X(jw) = \begin{cases} 1, & |w| < W \\ 0, & |w| > W \end{cases}$$

$$\Rightarrow x(t) = \frac{1}{2\pi}\int_{-W}^{W} e^{jwt}dw$$

$$= \frac{\sin(Wt)}{\pi t}$$

$X(jw)$



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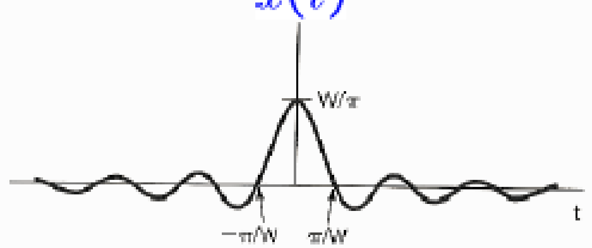
$x(t)$



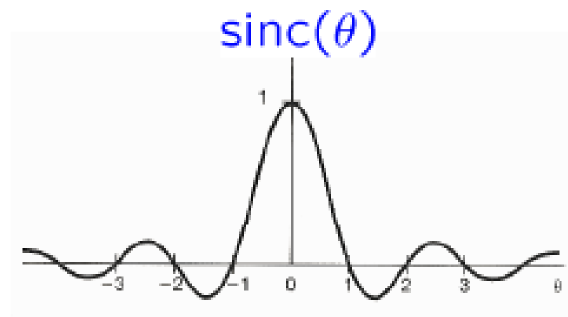
(b)

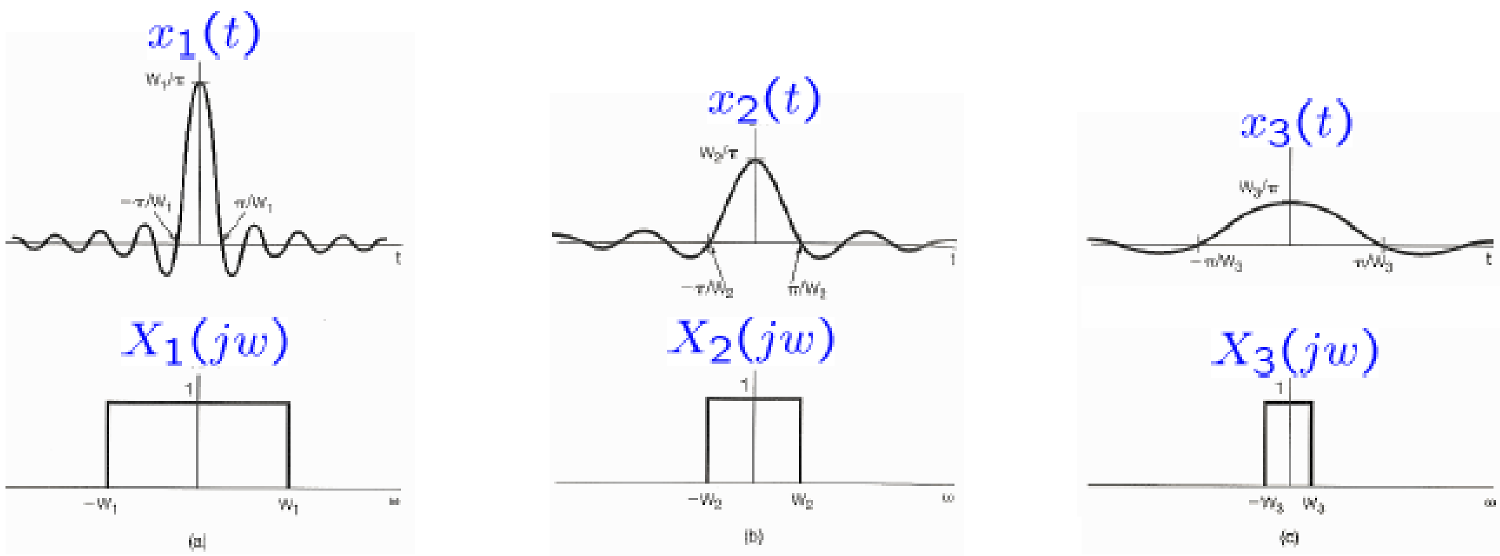
# The Continuous-Time Fourier Transform

- sinc functions:

$$\text{sinc}(\theta) = \frac{\sin(\pi\theta)}{\pi\theta}$$

$$\frac{\sin(Wt)}{\pi t} = \frac{W}{\pi}\text{sinc}\left(\frac{Wt}{\pi}\right)$$



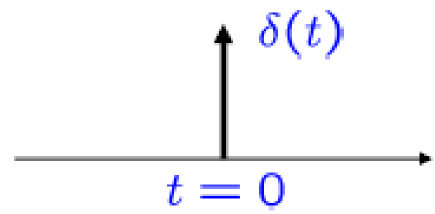



# The Continuous-Time Fourier Transform

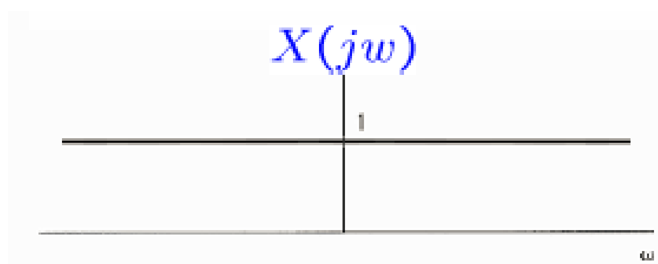
- Example 4.3:

$$X(jw) = \int_{-\infty}^{\infty} x(t) e^{-jwt} dt$$

$x(t) = \delta(t)$, i.e., unit impules



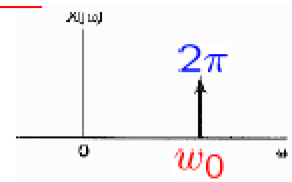

$$\Rightarrow X(jw) = \int_{-\infty}^{\infty} \delta(t)\, e^{-jwt} dt = 1$$



# The Fourier Transform for Periodic Signals

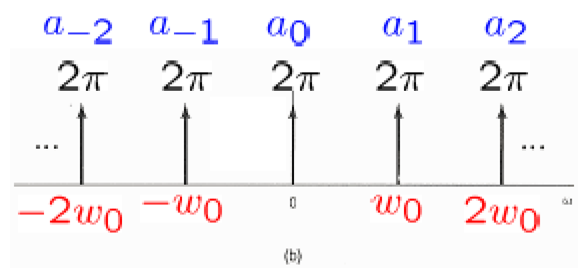
- Fourier Transform from Fourier Series:

$$X(jw) = 2\pi \quad \delta( w - w_0)$$




$$\Rightarrow x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} 2\pi \quad \delta(w - w_0)e^{jwt}dw$$

$$= e^{jw_0 t}$$




- more generally,

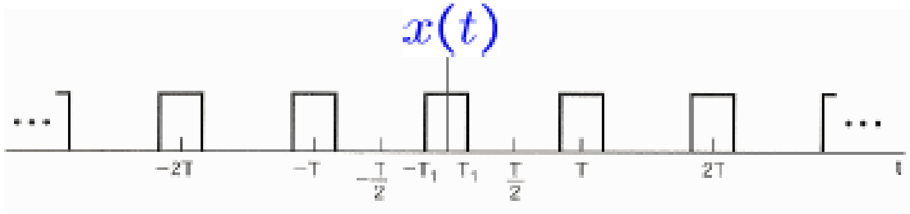
$$X(jw) = \sum_{k=-\infty}^{+\infty} 2\pi a_k \delta(w - kw_0)$$

$$\Rightarrow x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t}$$

Fourier series represntation of a periodic signal

# The Fourier Transform for Periodic Signals

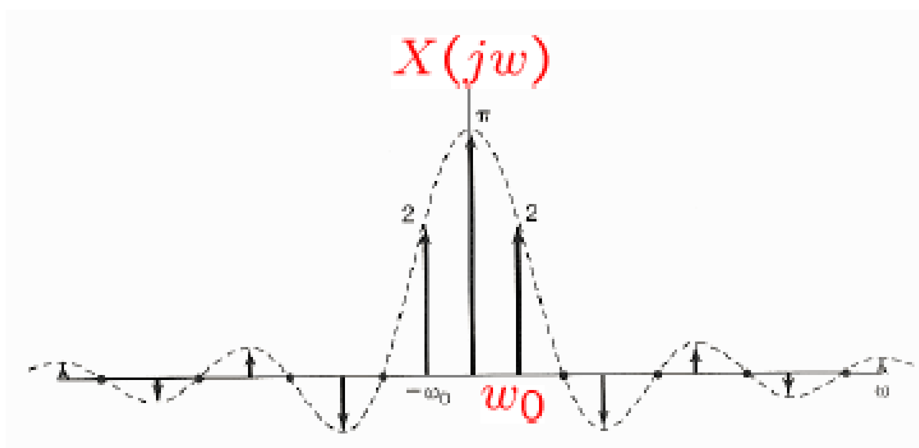
- Example 4.6:




$$\Rightarrow \quad a_k = \frac{\sin(kw_0T_1)}{\pi k}$$

$$\Rightarrow \quad X(jw) = \sum_{k=-\infty}^{+\infty} 2\pi a_k \delta(w - kw_0)$$

$$= \sum_{k=-\infty}^{+\infty} \frac{2\sin(kw_0T_1)}{k} \delta(w - kw_0)$$



# The Fourier Transform for Periodic Signals

- Example 4.7:

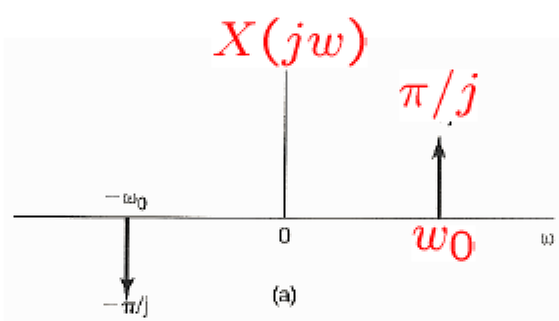
$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t}$$

$$x(t) = \sin(w_0 t) = \frac{e^{jw_0 t} - e^{-jw_0 t}}{2j}$$

$$\Rightarrow a_1 = \frac{1}{2j} \qquad a_{-1} = -\frac{1}{2j} \qquad a_k = 0, \quad k \neq 1, -1$$

$$x(t) = \cos(w_0 t) = \frac{e^{jw_0 t} + e^{-jw_0 t}}{2}$$

$$\Rightarrow a_1 = \frac{1}{2} \qquad a_{-1} = \frac{1}{2} \qquad a_k = 0, \quad k \neq 1, -1$$



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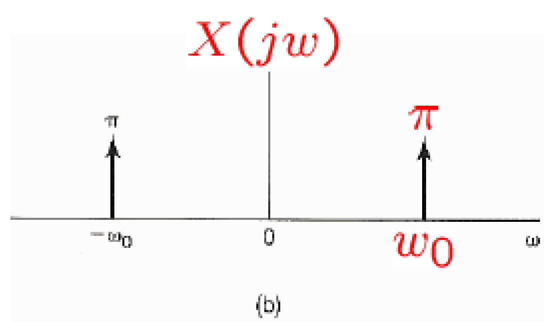

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# The Fourier Transform for Periodic Signals

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t}$$

$$a_k = \frac{1}{T}\int_{-T/2}^{+T/2} \tilde{x}(t) e^{-jkw_0 t} dt$$

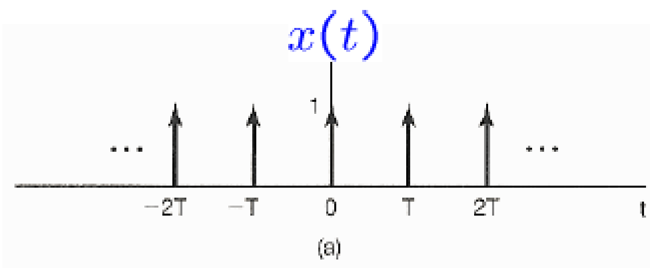
$$X(jw) = \sum_{k=-\infty}^{+\infty} 2\pi a_k \delta(w - kw_0)$$

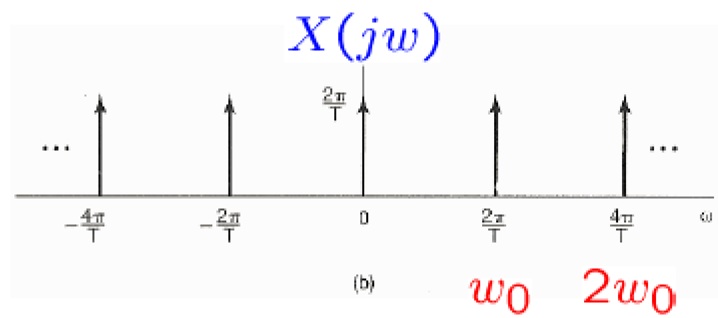
- Example 4.8:

$$x(t) = \sum_{k=-\infty}^{+\infty} \delta(t - kT)$$

$$\Rightarrow \quad a_k = \frac{1}{T}\int_{-T/2}^{+T/2} \delta(t) e^{-jkw_0 t} dt = \frac{1}{T}$$

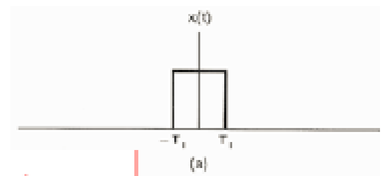
$$\Rightarrow \quad X(jw) = \frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \delta(w - \frac{2\pi}{T}k)$$



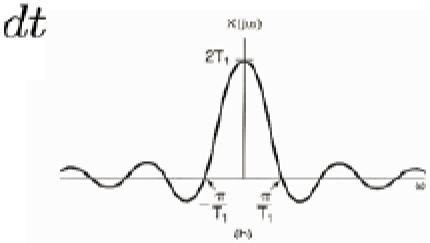
# Properties of the Continuous-Time Fourier Transform

- Fourier Transform Pair:




- Synthesis equation: $x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(jw)e^{jwt}dw$

- Analysis equation: $X(jw) = \int_{-\infty}^{+\infty} x(t)e^{-jwt}dt$




- Notations:

$$X(jw) = \mathcal{F}\{x(t)\}$$

$$\frac{1}{a+jw} = \mathcal{F}\{e^{-at}u(t)\}$$

$$x(t) = \mathcal{F}^{-1}\{X(jw)\}$$

$$e^{-at}u(t) = \mathcal{F}^{-1}\{\frac{1}{a+jw}\}$$

$$x(t) \xleftrightarrow{\mathcal{CTFT}} X(jw)$$

$$e^{-at}u(t) \xleftrightarrow{\mathcal{CTFT}} \frac{1}{a+jw}$$

# Properties of the Continuous-Time Fourier Transform

- Linearity:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(jw)e^{jwt}dw$$

$$X(jw) = \int_{-\infty}^{+\infty} x(t)e^{-jwt}dt$$

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(jw)$$

$$y(t) \overset{\mathcal{F}}{\longleftrightarrow} Y(jw)$$

$$\Rightarrow \quad a\,x(t)+b\,y(t) \overset{\mathcal{F}}{\longleftrightarrow} a\,X(jw)+b\,Y(jw)$$

# Properties of the Continuous-Time Fourier Transform

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(jw)e^{jwt}dw$$

$$X(jw) = \int_{-\infty}^{+\infty} x(t)e^{-jwt}dt$$

- Time Shifting:

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(jw)$$

$$\Rightarrow\ x(t-t_0) \overset{\mathcal{F}}{\longleftrightarrow} e^{-jwt_0}X(jw)$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(jw)e^{jwt}dw$$

$$x(t-t_0) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(jw)e^{jw(t-t_0)}dw$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \left(e^{-jwt_0}X(jw)\right)e^{jwt}dw$$

$$Y(jw) = \int_{-\infty}^{+\infty} x(t-t_0)e^{-jwt}dt$$

$$= \int_{-\infty}^{+\infty} x(\tau)e^{-jw(\tau+t_0)}d\tau$$

$$= e^{-jwt_0}\int_{-\infty}^{+\infty} x(\tau)e^{-jw\tau}d\tau$$

# Properties of the Continuous-Time Fourier Transform

- Time Shift $\rightarrow$ Phase Shift:

$$\mathcal{F}\{x(t)\} = X(jw) = |X(jw)|e^{j\sphericalangle X(jw)}$$

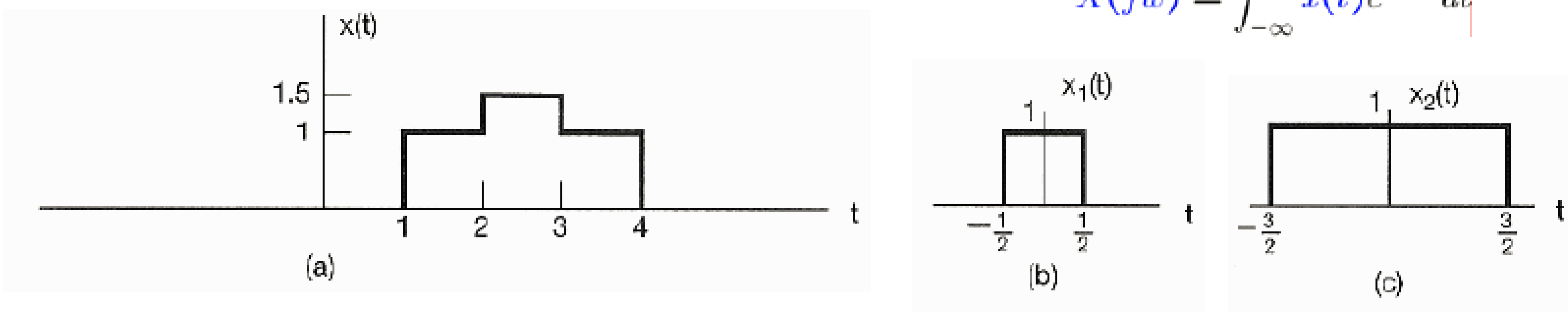
$$\mathcal{F}\{x(t-t_0)\} = e^{-jwt_0}X(jw) = |X(jw)|e^{j[\sphericalangle X(jw)-wt_0]}$$

# Properties of the Continuous-Time Fourier Transform

- Example 4.9:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(jw)e^{jwt}dw$$

$$X(jw) = \int_{-\infty}^{\infty} x(t)e^{-jwt}dt$$




$$x(t) = \frac{1}{2}x_1(t-2.5) + x_2(t-2.5)$$

$$X_1(jw) = \frac{2\sin(w/2)}{w}$$

$$X_2(jw) = \frac{2\sin(3w/2)}{w}$$

$$\Rightarrow\ X(jw) = e^{-j5w/2}\left\{\frac{\sin(w/2) + 2\sin(3w/2)}{w}\right\}$$

# Properties of the Continuous-Time Fourier Transform

**3. Conjugation property**

FT $\quad x^*(t) \leftrightarrow X^*(-j\omega)$

FS $\quad x^*(t) \leftrightarrow X^*[-k]$

Proof:

$$X^*(j\omega) = \left[ \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \right]^* = \int_{-\infty}^{\infty} \left( x(t) e^{-j\omega t} \right)^* dt$$

$$= \int_{-\infty}^{\infty} x^*(t) e^{j\omega t} dt$$

$$X^*(-j\omega) = \int_{-\infty}^{\infty} x^*(t) e^{-j\omega t} dt$$

# Properties of the Continuous-Time Fourier Transform

**4. Conjugate symmetry property**

If the signal is $x(t)$ real, then the Fourier representation is complex-conjugate symmetric.

FT $\quad X^*(j\omega) = X(-j\omega)$

FS $\quad X^*[k] = X[-k]$

Proof:

$$x(t) \leftrightarrow X(j\omega),\ x^*(t) \leftrightarrow X^*(-j\omega)$$

$$x(t) = x^*(t) \Rightarrow X(j\omega) = X^*(-j\omega)$$

$$X^*(j\omega) = X(-j\omega)$$

# Properties of the Continuous-Time Fourier Transform

- Conjugation & Conjugate Symmetry:

If $x(t)$ is a real function

$$x(t) = \mathcal{E}v\{x(t)\} + \mathcal{O}d\{x(t)\} = x_e(t) + x_o(t)$$

- $$\Rightarrow \mathcal{F}\{x(t)\} = \mathcal{F}\{x_e(t)\} + \mathcal{F}\{x_o(t)\}$$

$\Rightarrow \mathcal{F}\{x_e(t)\}$ : a real function

- $\Rightarrow \mathcal{F}\{x_o(t)\}$ : a purely imaginary function

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(jw)$$

$$\mathcal{E}v\{x(t)\} \overset{\mathcal{F}}{\longleftrightarrow} \mathcal{R}e\{X(jw)\}$$

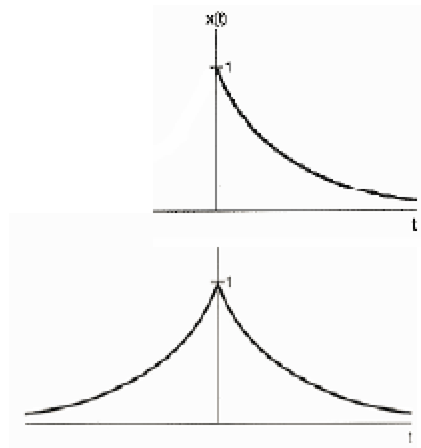
- $$\mathcal{O}d\{x(t)\} \overset{\mathcal{F}}{\longleftrightarrow} j\,\mathcal{I}m\{X(jw)\}$$

# Properties of the Continuous-Time Fourier Transform

- Example 4.10:

$$e^{-at}u(t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{a+jw}$$

$$e^{-a|t|} \overset{\mathcal{F}}{\longleftrightarrow} ?$$




$$x(t) = e^{-a|t|} = e^{-at}u(t) + e^{at}u(-t)$$

$$= 2\left[\frac{e^{-at}u(t) + e^{at}u(-t)}{2}\right] \qquad = 2\mathcal{E}v\left\{e^{-at}u(t)\right\}$$

$$\mathcal{E}v\left\{e^{-at}u(t)\right\} \overset{\mathcal{F}}{\longleftrightarrow} \mathcal{R}e\left\{\frac{1}{a+jw}\right\}$$

$$\mathcal{O}d\left\{e^{-at}u(t)\right\} \overset{\mathcal{F}}{\longleftrightarrow} j\,\mathcal{I}m\left\{\frac{1}{a+jw}\right\}$$

$$X(jw) = 2\mathcal{R}e\left\{\frac{1}{a+jw}\right\} = \frac{2a}{a^2+w^2}$$

# Properties of the Continuous-Time Fourier Transform

- <u>Differentiation & Integration:</u>

$$X(jw) = \int_{-\infty}^{+\infty} x(t) e^{-jwt} dt$$

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(jw)$$

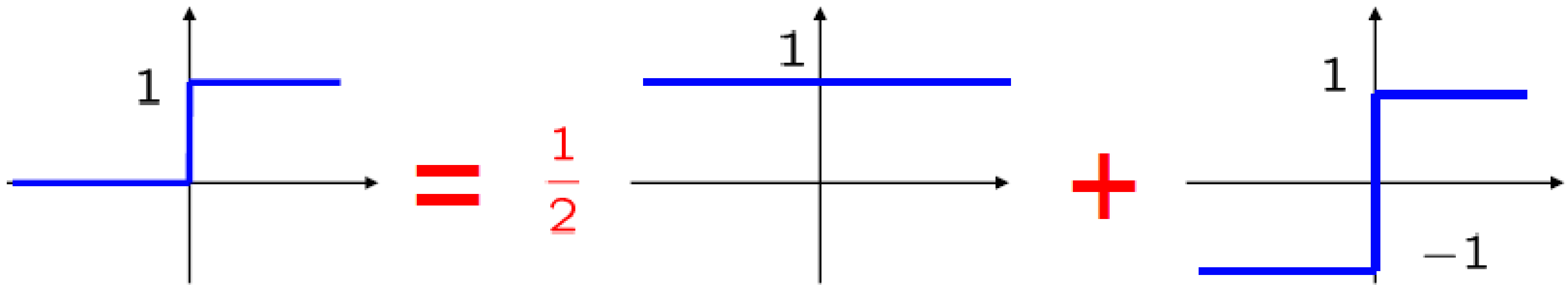
$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(jw) e^{jwt} dw$$

$$\frac{d}{dt} x(t) \overset{\mathcal{F}}{\longleftrightarrow} jwX(jw)$$

$$\int_{-\infty}^{t} x(\tau) d\tau \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{jw} X(jw) + \pi X(0)\delta(w)$$

dc or average value

# Properties of the Continuous-Time Fourier Transform




$u(t)$ $\quad\quad$ $\frac{1}{2}$ $\quad$ 1 $\quad\quad$ sgn(t)

$$1 \overset{\mathcal{FT}}{\longleftrightarrow} 2\pi\delta(jw) \qquad \text{sgn(t)} \overset{\mathcal{FT}}{\longleftrightarrow} S(jw)$$

$$\frac{d}{dt}\,\text{sgn(t)} \overset{\mathcal{FT}}{\longleftrightarrow} jw\, S(jw)$$

$$2\,\delta(t) \overset{\mathcal{FT}}{\longleftrightarrow} jw\, S(jw)$$

$$\Rightarrow U(jw) =$$

$$\Rightarrow S(jw) =$$

$$u(t) \overset{FT}{\leftrightarrow} \frac{1}{j\omega} + \pi\delta(\omega)$$

# Properties of the Continuous-Time Fourier Transform

$$\text{sign}(t) \overset{FT}{\leftrightarrow} \frac{2}{j\omega}$$

Proof:

$$\text{sign}(t) = 2u(t) - 1$$

$$u(t) \leftrightarrow U(j\omega) = \frac{1}{j\omega} + \pi\delta(\omega); 1 \leftrightarrow 2\pi\delta(\omega)$$

$$2u(t) - 1 \leftrightarrow 2\left(\frac{1}{j\omega} + \pi\delta(\omega)\right) - 2\pi\delta(\omega)$$

# Properties of the Continuous-Time Fourier Transform

- Example 4.11:

$$x(t) = u(t) \stackrel{\mathcal{F}}{\longleftrightarrow} X(jw) = ?$$

$$g(t) = \delta(t) \stackrel{\mathcal{F}}{\longleftrightarrow} G(jw) = 1$$

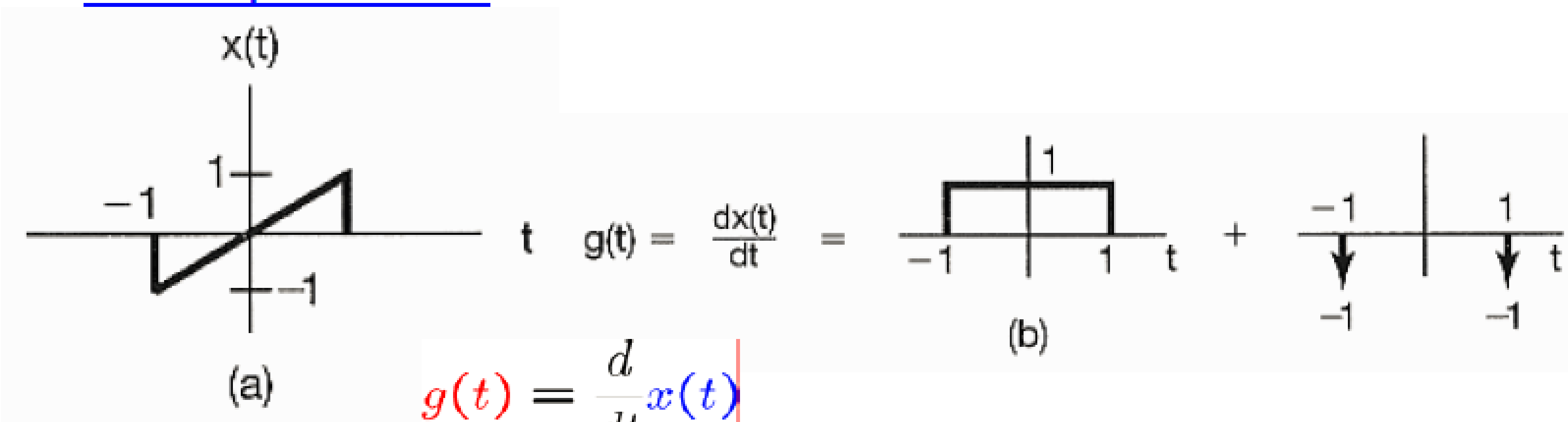
$$x(t) = \int_{-\infty}^{t} g(\tau)d\tau \qquad X(jw) = \frac{1}{jw}G(jw) + \pi G(0)\delta(w)$$

$$= \frac{1}{jw} + \pi\delta(w)$$

$$\delta(t) = \frac{d}{dt}u(t) \stackrel{\mathcal{F}}{\longleftrightarrow} jw\left[\frac{1}{jw} + \pi\delta(w)\right] = 1$$

# Properties of the Continuous-Time Fourier Transform

- Example 4.12:




$$g(t) = \frac{d}{dt}x(t)$$

$$G(jw) = \frac{2\sin(w)}{w} - e^{jw} - e^{-jw}$$

$$\Rightarrow X(jw) = \frac{G(jw)}{jw} + \pi G(0)\delta(w)$$

$$= \frac{2\sin(w)}{jw^2} - \frac{2\cos(w)}{jw}$$

# Properties of the Continuous-Time Fourier Transform

- Time & Frequency Scaling:

$$X(jw) = \int_{-\infty}^{+\infty} x(t) e^{-jwt} dt$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(jw) e^{jwt} dw$$

$$x(t) \stackrel{\mathcal{F}}{\longleftrightarrow} X(jw)$$

$$x(at) \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{1}{|a|} X\left(\frac{jw}{a}\right)$$

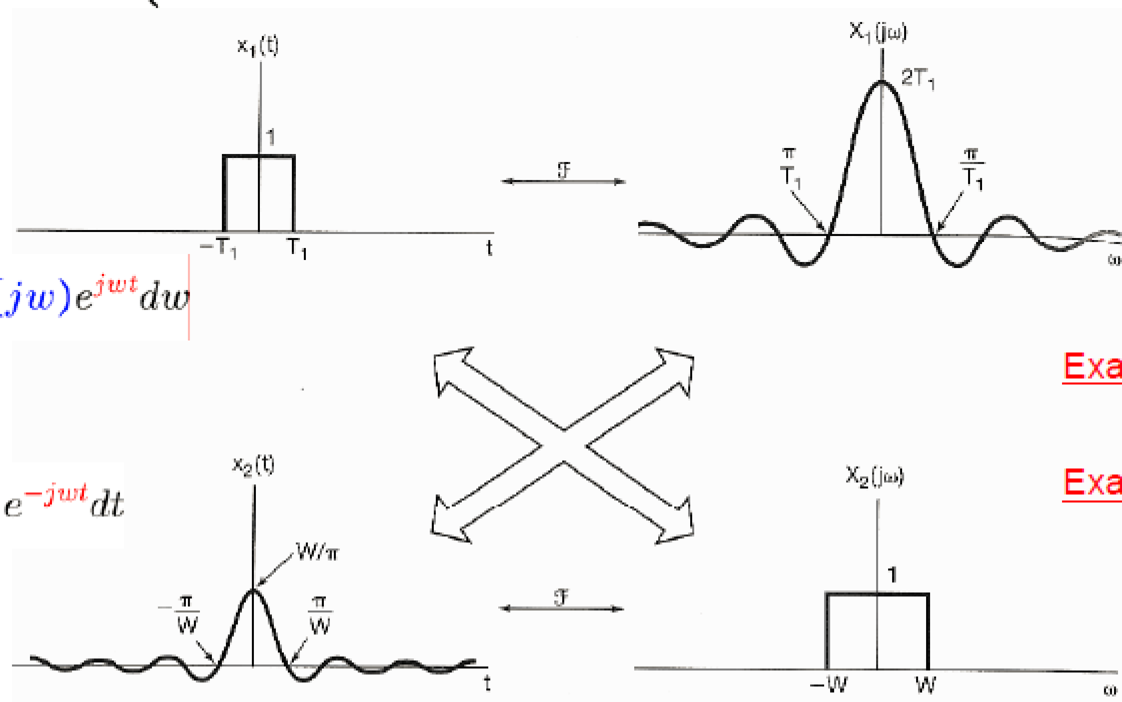
$$\frac{1}{|b|} x\left(\frac{t}{b}\right) \stackrel{\mathcal{F}}{\longleftrightarrow} X(jbw)$$

$$x(-t) \stackrel{\mathcal{F}}{\longleftrightarrow} X(-jw)$$

# Properties of the Continuous-Time Fourier Transform

- Duality:

$$x_1(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & |t| > T_1 \end{cases} \xleftrightarrow{\mathcal{F}} X_1(jw) = \frac{2\sin(wT_1)}{w}$$




$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(jw)e^{jwt}dw$$

Example 4.4

$$X(jw) = \int_{-\infty}^{+\infty} x(t)e^{-jwt}dt$$

Example 4.5

$$x_2(t) = \frac{\sin(Wt)}{\pi t} \xleftrightarrow{\mathcal{F}} X_2(jw) = \begin{cases} 1, & |w| < W \\ 0, & |w| > W \end{cases}$$

# Properties of the Continuous-Time Fourier Transform

- Duality:

$$x(t) \stackrel{\mathcal{F}}{\longleftrightarrow} X(jw)$$

$$x(t-t_0) \stackrel{\mathcal{F}}{\longleftrightarrow} e^{-jwt_0}X(jw)$$

$$\frac{d}{dt}x(t) \stackrel{\mathcal{F}}{\longleftrightarrow} jwX(jw)$$

$$\int_{-\infty}^{t} x(\tau)d\tau \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{1}{jw}X(jw) + \pi X(0)\delta(w)$$

$$-jtx(t) \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{d}{dw}X(jw)$$

$$e^{jw_0t}x(t) \stackrel{\mathcal{F}}{\longleftrightarrow} X(j(w-w_0))$$

$$-\frac{1}{jt}x(t) + \pi x(0)\delta(t) \stackrel{\mathcal{F}}{\longleftrightarrow} \int_{-\infty}^{w} X(\eta)d\eta$$

# Properties of the Continuous-Time Fourier Transform

## Parseval relationships (Parseval's Theorem)

The Parseval relationships state that the energy (or power) in the time-domain representation of a signal is equal to the energy (or power) in the frequency-domain representation.

$$W_x = \int_{-\infty}^{\infty} |x(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 \, d\omega$$

$\frac{1}{2\pi}|X(j\omega)|^2$ : the energy spectrum of $x(t)$

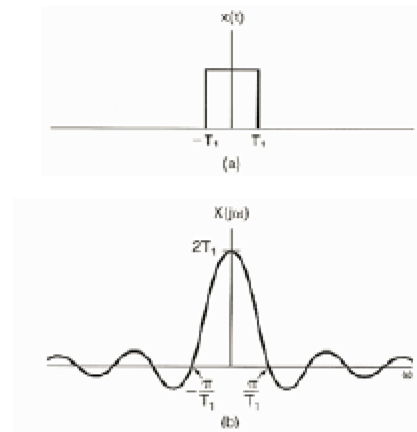
# Properties of the Continuous-Time Fourier Transform

## Parseval relationships (Parseval's Theorem)

- Parseval's relation:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(jw)e^{jwt}dw$$

$$X(jw) = \int_{-\infty}^{+\infty} x(t)e^{-jwt}dt$$




$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(jw)$$

$$\Rightarrow \int_{-\infty}^{+\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(jw)|^2 dw$$

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt = \int_{-\infty}^{+\infty} x(t)x^*(t)dt$$

$$= \int_{-\infty}^{+\infty} x(t)\left[\frac{1}{2\pi}\int_{-\infty}^{+\infty} X^*(jw)e^{-jwt}dw\right]dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} X^*(jw)\left[\int_{-\infty}^{+\infty} x(t)e^{-jwt}dt\right]dw$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(jw)|^2 dw$$

# Properties of the Continuous-Time Fourier Transform

**Parseval relationships (Parseval's Theorem)**

FT $$\int_{-\infty}^{\infty} |x(t)|^2\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|^2\, d\omega$$

FS $$\frac{1}{T}\int_T |x(t)|^2\, dt = \sum_{k=-\infty}^{\infty} |X[k]|^2$$

# Properties of the Continuous-Time Fourier Transform

| Section | Property | Aperiodic signal | Fourier transform |
|---|---|---|---|
| | | $x(t)$ | $X(j\omega)$ |
| | | $y(t)$ | $Y(j\omega)$ |
| 4.3.1 | Linearity | $ax(t) + by(t)$ | $aX(j\omega) + bY(j\omega)$ |
| 4.3.2 | Time Shifting | $x(t - t_0)$ | $e^{-j\omega t_0}X(j\omega)$ |
| 4.3.6 | Frequency Shifting | $e^{j\omega_0 t}x(t)$ | $X(j(\omega - \omega_0))$ |
| 4.3.3 | Conjugation | $x^*(t)$ | $X^*(-j\omega)$ |
| 4.3.5 | Time Reversal | $x(-t)$ | $X(-j\omega)$ |
| 4.3.5 | Time and Frequency Scaling | $x(at)$ | $\frac{1}{\|a\|}X\left(\frac{j\omega}{a}\right)$ |
| 4.4 | Convolution | $x(t) * y(t)$ | $X(j\omega)Y(j\omega)$ |
| 4.5 | Multiplication | $x(t)y(t)$ | $\frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\theta)Y(j(\omega-\theta))d\theta$ |
| 4.3.4 | Differentiation in Time | $\frac{d}{dt}x(t)$ | $j\omega X(j\omega)$ |
| 4.3.4 | Integration | $\int_{-\infty}^{t} x(t)dt$ | $\frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)$ |
| 4.3.6 | Differentiation in Frequency | $tx(t)$ | $j\frac{d}{d\omega}X(j\omega)$ |
| 4.3.3 | Conjugate Symmetry for Real Signals | $x(t)$ real | $\begin{cases} X(j\omega) = X^*(-j\omega) \\ \mathcal{R}e\{X(j\omega)\} = \mathcal{R}e\{X(-j\omega)\} \\ \mathcal{I}m\{X(j\omega)\} = -\mathcal{I}m\{X(-j\omega)\} \\ \|X(j\omega)\| = \|X(-j\omega)\| \\ \sphericalangle X(j\omega) = -\sphericalangle X(-j\omega) \end{cases}$ |
| 4.3.3 | Symmetry for Real and Even Signals | $x(t)$ real and even | $X(j\omega)$ real and even |
| 4.3.3 | Symmetry for Real and Odd Signals | $x(t)$ real and odd | $X(j\omega)$ purely imaginary and odd |

# Properties of the Continuous-Time Fourier Transform

| | | | |
|---|---|---|---|
| 4.3.3 | Conjugate Symmetry for Real Signals | $x(t)$ real | $\begin{cases} X(j\omega) = X^*(-j\omega) \\ \Re e\{X(j\omega)\} = \Re e\{X(-j\omega)\} \\ \Im m\{X(j\omega)\} = -\Im m\{X(-j\omega)\} \\ \lvert X(j\omega)\rvert = \lvert X(-j\omega)\rvert \\ \sphericalangle X(j\omega) = -\sphericalangle X(-j\omega) \end{cases}$ |
| 4.3.3 | Symmetry for Real and Even Signals | $x(t)$ real and even | $X(j\omega)$ real and even |
| 4.3.3 | Symmetry for Real and Odd Signals | $x(t)$ real and odd | $X(j\omega)$ purely imaginary and odd |
| 4.3.3 | Even-Odd Decomposition for Real Signals | $x_e(t) = \mathcal{E}v\{x(t)\}$ [$x(t)$ real]<br>$x_o(t) = \mathcal{O}d\{x(t)\}$ [$x(t)$ real] | $\Re e\{X(j\omega)\}$<br>$j\Im m\{X(j\omega)\}$ |

4.3.7 Parseval's Relation for Aperiodic Signals

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(j\omega)|^2 d\omega$$

# Basic Continuous-Time Fourier Transform Pairs

TABLE 4.2 BASIC FOURIER TRANSFORM PAIRS

| Signal | Fourier transform | Fourier series coefficients (if periodic) |
|---|---|---|
| $\sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$ | $2\pi \sum_{k=-\infty}^{+\infty} a_k \delta(\omega - k\omega_0)$ | $a_k$ |
| $e^{j\omega_0 t}$ | $2\pi\delta(\omega - \omega_0)$ | $a_1 = 1$<br>$a_k = 0$, otherwise |
| $\cos \omega_0 t$ | $\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$ | $a_1 = a_{-1} = \frac{1}{2}$<br>$a_k = 0$, otherwise |
| $\sin \omega_0 t$ | $\frac{\pi}{j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$ | $a_1 = -a_{-1} = \frac{1}{2j}$<br>$a_k = 0$, otherwise |
| $x(t) = 1$ | $2\pi\,\delta(\omega)$ | $a_0 = 1, \quad a_k = 0, \; k \neq 0$<br>(this is the Fourier series representation for any choice of $T > 0$) |
| Periodic square wave<br>$x(t) = \begin{cases} 1, & \lvert t\rvert < T_1 \\ 0, & T_1 < \lvert t\rvert \leq \frac{T}{2} \end{cases}$<br>and<br>$x(t + T) = x(t)$ | $\sum_{k=-\infty}^{+\infty} \frac{2 \sin k\omega_0 T_1}{k} \delta(\omega - k\omega_0)$ | $\frac{\omega_0 T_1}{\pi} \operatorname{sinc}\left(\frac{k\omega_0 T_1}{\pi}\right) = \frac{\sin k\omega_0 T_1}{k\pi}$ |
| $\sum_{n=-\infty}^{+\infty} \delta(t - nT)$ | $\frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)$ | $a_k = \frac{1}{T}$ for all $k$ |
| $x(t)\begin{cases} 1, & \lvert t\rvert < T_1 \\ 0, & \lvert t\rvert > T_1 \end{cases}$ | $\frac{2 \sin \omega T_1}{\omega}$ | — |

# Basic Continuous-Time Fourier Transform Pairs

| | | |
|---|---|---|
| $\dfrac{\sin Wt}{\pi t}$ | $X(j\omega) = \begin{cases} 1, & \lvert\omega\rvert < W \\ 0, & \lvert\omega\rvert > W \end{cases}$ | — |
| $\delta(t)$ | $1$ | — |
| $u(t)$ | $\dfrac{1}{j\omega} + \pi\delta(\omega)$ | — |
| $\delta(t - t_0)$ | $e^{-j\omega t_0}$ | — |
| $e^{-at}u(t), \mathcal{R}e\{a\} > 0$ | $\dfrac{1}{a + j\omega}$ | — |
| $te^{-at}u(t), \mathcal{R}e\{a\} > 0$ | $\dfrac{1}{(a + j\omega)^2}$ | — |
| $\frac{t^{n-1}}{(n-1)!}e^{-at}u(t)$, $\mathcal{R}e\{a\} > 0$ | $\dfrac{1}{(a + j\omega)^n}$ | — |