Introduction to Signals and Systems: V216

Lecture #6

Chapter 3: Fourier Series Representation of Periodic Signals

Fourier Series Representation of Periodic Signals

- Signals can be represented as linear combinations of basic signals with the following 2 properties.
- The set of basic signals can be used to construct a broad and useful class of signals.
- The response of an LTI system is a combination of the responses to these basic signals at the input.

Fourier Analysis

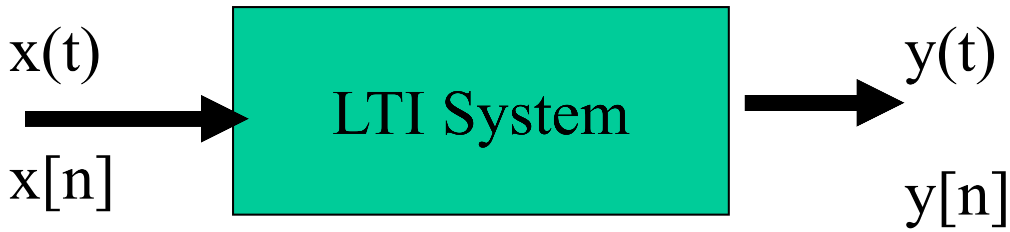
- Both properties are provided for by the complex exponential signals in the continuous and discrete time
- I.e. Signal of the form:-

e^{st} in continuous time.

z^n in the discrete time.

where s & z are complex numbers.

The principle of superposition for linear systems.



$$\text{if } \mathrm{x(t)} = \mathrm{a}_1\phi_1(t) + \mathrm{a}_2\phi_2(t) + \ldots\ldots$$

$$\phi_{\mathrm{k}}(t) \Rightarrow \varphi_k(t)$$

and system is linear

$$\text{Then : } \mathrm{y(t)} = \mathrm{a}_1\varphi_1(t) + \mathrm{a}_2\varphi_2(t) + \ldots$$

Similarly for the discrete-time case.

Criteria for choosing a set of basic signals in terms of which to decompose the input

$$\text{if } x(t) = a_1\phi_1(t) + a_2\phi_2(t) + \ldots\ldots$$

$$\phi_k(t) \Rightarrow \varphi_k(t)$$

and system is linear

$$\text{Then}: y(t) = a_1\varphi_1(t) + a_2\varphi_2(t) + \ldots$$

Choose $\phi_k(t)$ or $\phi_k[n]$ so that:

- a broad class of signals can be constructed as a linear combination of ϕ_k's.
- responses to ϕ_k's are easy to compute

Choice for basic signals that led to the convolution integral and convolution sum for LTI Systems

$$\mathrm{C\text{-}T}: \phi_{\mathrm{k}}(t) = \delta(t - k\Delta)$$

$$\varphi_k(t) = h(t - k\Delta)$$

$$\Rightarrow \text{Convolution Integral}$$

$$\mathrm{D\text{-}T}: \phi_{\mathrm{k}}[n] = \delta[n - k]$$

$$\varphi_k[n] = h[n - k]$$

$$\Rightarrow \text{ Convolution Sum}$$

Complex exponentials as a set of basic signals

$$\phi_k(t) = e^{s_k t} \qquad \mathrm{s_k} \text{ complex}$$

$$\phi_\mathrm{k}[n] = z_k{}^n \qquad \mathrm{z_k} \text{ complex}$$

Fourier Analysis:

$$\text{C-T}: \mathrm{s_k} = j\omega_k \qquad \phi_\mathrm{k}(t) = e^{j\omega_k t}$$

$$\text{D-T}: |\mathrm{z_k}| = 1 \qquad \phi_\mathrm{k}[n] = e^{j\Omega_k n}$$

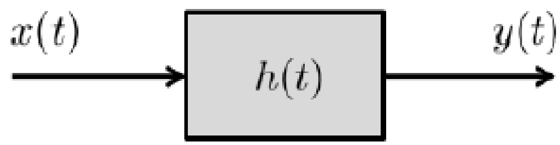
$$\mathrm{s_k} \text{ complex} \Rightarrow \text{Laplace transforms}$$

$$\mathrm{z_k} \text{ complex} \Rightarrow \text{z-transforms.}$$

Eigenfunction (e^{st} , z^{n}) & Eigenvalues (H(s) , H(z))

For an LTI system, if the output is a scaled version of its input, then the input function is called an eigenfunction of the system. The scaling factor is called the eigenvalue of the system.

Showing complex exponential as eigenfunction of CT system



Suppose that $x(t) = e^{st}$ for some $s \in \mathbb{C}$, then the output is given by

$$
\begin{aligned}
y(t) = h(t) * x(t) &= \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau \\
&= \int_{-\infty}^{\infty} h(\tau)e^{s(t-\tau)}d\tau \\
&= \left[\int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau\right] e^{st} = H(s)e^{st} = H(s)x(t),
\end{aligned}
$$

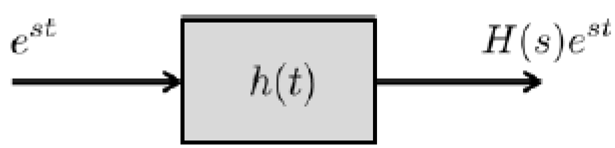
where $H(s)$ is defined as

$$H(s) = \int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau.$$

Showing complex exponential as eigenfunction of CT system

The function $H(s)$ is known as the *transfer function* of the continuous-time LTI system. Note that $H(s)$ is defined by the impulse response $h(t)$, and is a function in s (independent of t). Therefore, $H(s)x(t)$ can be regarded as a scalar $H(s)$ multiplied to the function $x(t)$.

From the derivation above, we see that if the input is $x(t) = e^{st}$, then the output is a scaled version $y(t) = H(s)e^{st}$:

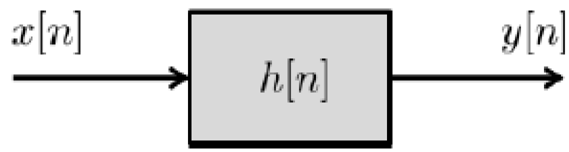


Therefore, using the definition of eigenfunction, we show that

1. e^{st} is an eigenfunction of any continuous-time LTI system, and

2. $H(s)$ is the corresponding eigenvalue.

Showing complex exponential as eigenfunction of DT system



Suppose that the impulse response is given by $h[n]$ and the input is $x[n] = z^n$, then the output $y[n]$ is

$$
\begin{aligned}
y[n] = h[n] * x[n] &= \sum_{k=-\infty}^{\infty} h[k]x[n-k] \\
&= \sum_{k=-\infty}^{\infty} h[k]z^{n-k} \\
&= z^n \sum_{k=-\infty}^{\infty} h[k]z^{-k} = H(z)z^n,
\end{aligned}
$$

where we defined

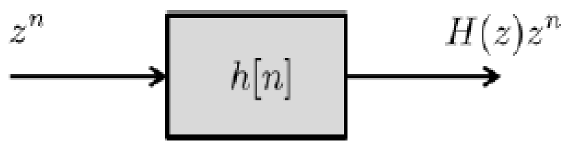
$$H(z) = \sum_{k=-\infty}^{\infty} h[k]z^{-k},$$

Showing complex exponential as eigenfunction of DT system

and $H(z)$ is known as the transfer function of the discrete-time LTI system.

Similar to the continuous-time case, this result indicates that

1. z^n is an eigenfunction of a discrete-time LTI system, and
2. $H(z)$ is the corresponding eigenvalue.

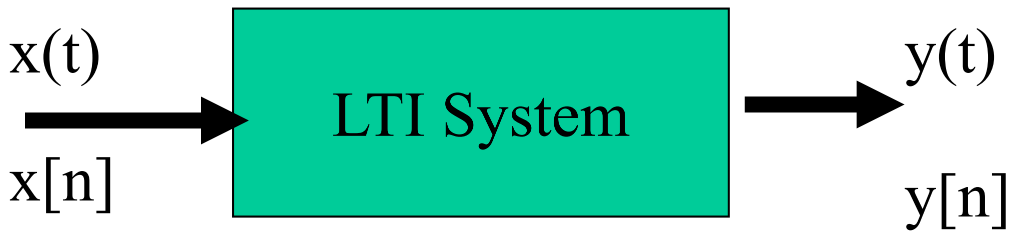


Considering the subclass of periodic complex exponentials $e^{-j(2\pi/N)n}$ by setting $z = e^{j2\pi/N}$, we have

$$H(z)|_{z=e^{j\Omega}} = H(e^{j\Omega}) = \sum_{k=-\infty}^{\infty} h[k]e^{-j\Omega k},$$

where $\Omega = \frac{2\pi}{N}$, and $H(e^{j\Omega})$ is called the *frequency response* of the system.

Eigenfunction (e^{st} , z^{n}) & Eigenvalues (H(s) , H(z))



$$\textit{continuous}\ \text{time}: \text{x(t)} = \text{e}^{\text{st}} \Rightarrow y(t) = H(s)e^{st}$$

$$\text{discrete time}: \text{x[n]} = \text{z}^{\text{n}} \Rightarrow \text{y[n]} = \text{H(z)z}^{\text{n}}$$

The importance of complex exponentials in the study of LTI systems stems from the fact that the response of an LTI system to a complex exponential input is the same complex exponential with only a change in amplitude.

Example x(t) to be linear combination of 3 complex exponentials

$$\mathrm{x(t)} = \mathrm{a}_1 e^{s_1 t} + \mathrm{a}_2 e^{s_2 t} + \mathrm{a}_3 e^{s_3 t}$$

From the eigenfunction property,

the response to each separately is :-

$$\mathrm{a}_1 e^{s_1 t} \rightarrow \mathrm{a}_1 H(s_1) e^{s_1 t}$$

$$\mathrm{a}_2 e^{s_2 t} \rightarrow \mathrm{a}_2 H(s_2) e^{s_2 t}$$

$$\mathrm{a}_3 e^{s_3 t} \rightarrow \mathrm{a}_3 H(s_3) e^{s_3 t}$$

Example x(t) is linear combination of 3 complex exponentials

$$\mathrm{x(t)} = \mathrm{a}_1 e^{s_1 t} + \mathrm{a}_2 e^{s_2 t} + \mathrm{a}_3 e^{s_3 t}$$

From the superposition property,

the response to the sum is the sum of the responses :-

$$\mathrm{y(t)} = \mathrm{a}_1 H(s_1) e^{s_1 t} + \mathrm{a}_2 H(s_2) e^{s_2 t} + \mathrm{a}_3 H(s_3) e^{s_3 t}$$

Generally, if input $\mathrm{x(t)} = \sum_{\mathrm{k}} \mathrm{a}_{\mathrm{k}} e^{s_k t}$,

then the output will be $\mathrm{y(t)} = \sum_{\mathrm{k}} \mathrm{a}_{\mathrm{k}} H(s_k) e^{s_k t}$

similarly for D-time if input $\mathrm{x[n]} = \sum_{\mathrm{k}} \mathrm{a}_{\mathrm{k}} z_k{}^n$,

then the output will be $\mathrm{y[n]} = \sum_{\mathrm{k}} \mathrm{a}_{\mathrm{k}} H(z_k) z_k{}^n$.

Why is eigenfunction important?

then the output is again a sum of complex exponentials:

$$y(t) = \sum_{k=-\infty}^{\infty} a_k H(s_k) e^{s_k t}.$$

Similarly for discrete-time signals, if

$$x[n] = \sum_{k=-\infty}^{\infty} a_k z_k^n,$$

then

$$x[n] = \sum_{k=-\infty}^{\infty} a_k H(z_k) z_k^n.$$

This is an important observation, because as long as we can express a signal $x(t)$ as a linear combination of eigenfunctions, then the output $y(t)$ can be easily determined by looking at the transfer function (which is fixed for an LTI system!). Now, the question is : How do we express a signal $x(t)$ as a linear combination of complex exponentials?

Fourier Series Representation of Continuous-time Periodic Signals.

In general, not every signal x(t) can be decomposed as a linear combination of complex exponentials.
However, such decomposition is still possible for an extremely large class of signals.
We want to study one class of signals that allows the decomposition. They are the periodic signals

A signal is periodic if $x(t)=x(t+T)$, for all t.
T is the fundamental period.

Fourier Series Representation of Continuous-time Periodic Signals.

A signal is periodic if x(t)=x(t+T), for all t.
T is the fundamental period.

$\omega_0 = \dfrac{2\pi}{T}$, is the fundamental radian frequency.

We have studied before 2 basic periodic signals, sinusoidal $\mathrm{x(t)} = \cos(\omega_0 t)$

and complex exponential $\mathrm{x(t)} = \mathrm{e}^{\mathrm{j}\omega_0 t}$, Associated with this basic complex exponential signal is

the set of harmonically related complex exponentials,

$$\phi_k(t) = e^{jk\omega_0 t} = e^{jk\frac{2\pi}{T}t}, \mathrm{k} = 0, \pm 1, \pm 2, \ldots$$

Each of these signals has a fundamental frequency that is a multiple of ω_0, and each is periodic with T (for k $\geq$ 2, the fundamental period is a fraction of T).

Thus, a linear combination of harmonically related complex exponential $x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk\frac{2\pi}{T}t}$, is the Fourier series and is also periodic with period T. k $= 1$ is the first harmonic component, k $= 2$ is the second harmonics components.

Example 3.2

$$x(t) = \sum_{k=-3}^{+3} a_k e^{jk2\pi t},$$

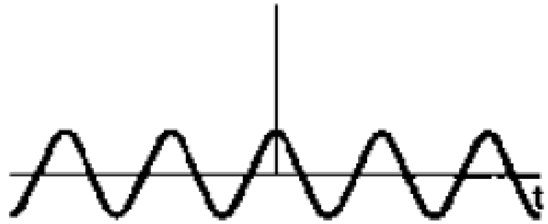
where $a_0 = 1,\ a_1 = a_{-1} = \frac{1}{4}, \quad a_2 = a_{-2} = \frac{1}{2}, \quad a_3 = a_{-3} = \frac{1}{3}$

use Euler relationship, $\cos\theta = \frac{1}{2}(\mathrm{e}^{\mathrm{j}\theta} + \mathrm{e}^{-\mathrm{j}\theta})$,

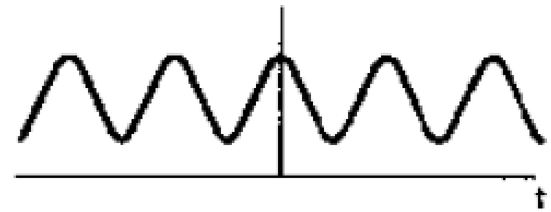
to get the cosine form below :-

$$x(t) = 1 + \frac{1}{2}\cos 2\pi t + \cos 4\pi t + \frac{2}{3}\cos 6\pi t.$$

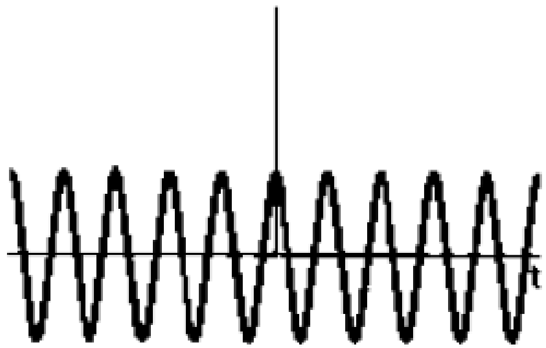
$x_1(t) = \frac{1}{2} \cos 2\pi t$



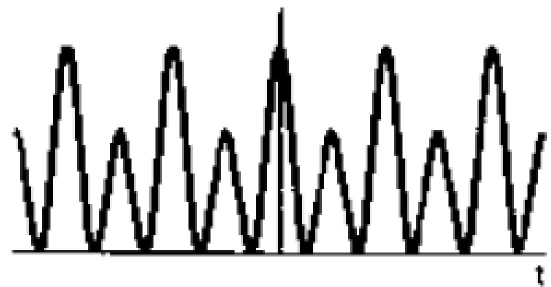
$x_0(t) + x_1(t)$



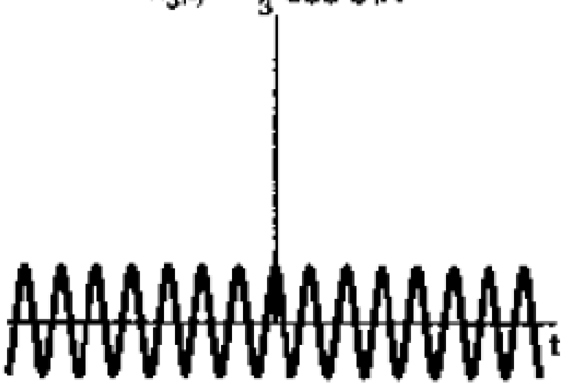
$x_2(t) = \cos 4\pi t$



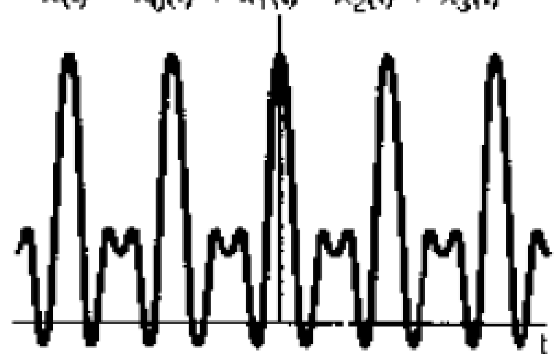
$x_0(t) + x_1(t) + x_2(t)$



$x_3(t) = \frac{2}{3} \cos 6\pi t$



$x(t) = x_0(t) + x_1(t) - x_2(t) + x_3(t)$



Trigonometric forms of Fourier Series.

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} \text{........} eqn.3.25 \text{ (Synthesis Equation)}$$

is the Complex Exponential Form of

the Fourier Series.

Taking conjugation of both side of eqn. 3.25, we have :-

$$x^*(\mathrm{t}) = \sum_{k=-\infty}^{+\infty} a_k^* e^{-jk\omega_0 t}, \text{ For real x(t), } \mathrm{x}^*(\mathrm{t}) = \mathrm{x(t)}.$$

$$\therefore x(\mathrm{t}) = \sum_{k=-\infty}^{+\infty} a_k^* e^{-jk\omega_0 t}, \text{ replacing k by - k,}$$

$$x(\mathrm{t}) = \sum_{k=-\infty}^{+\infty} a_{-k}^* e^{jk\omega_0 t} \text{ and comparing with eqn. 3.25,}$$

$$a_k = a_{\text{-k}}^*, \text{ or } a_k^* = a_{-k}.$$

$$\text{From Eqn 3.25 : - } x(t) = a_0 + \sum_{\mathrm{k}=1}^{\infty} [a_k e^{jk\omega_0 t} + a_{-k} e^{-jk\omega_0 t}]$$

Trigonometric forms of Fourier Series.

$$x(t) = a_0 + \sum_{\mathrm{k}=1}^{\infty} [a_k e^{jk\omega_0 t} + a^* e^{-jk\omega_0 t}] = a_0 + \sum_{k=1}^{\infty} 2Re\{a_k e^{jk\omega_0 t}\}.$$

Expressing $\mathrm{a_k} = A_k e^{j\theta_k}$, $x(t) = a_0 + \sum_{k=1}^{\infty} 2Re\{A_k e^{j(k\omega_0 t + \theta_k)}\}.$

$$\therefore \mathrm{x(t) = a_0} + 2\sum_{\mathrm{k}=1}^{\infty} \mathrm{A_k} \cos(k\omega_0 t + \theta_k),$$

the other form by letting $\mathrm{a_k} = B_k + jC_k$,

$$x(t) = a_0 + 2\sum_{k=1}^{\infty} [B_k \cos k\omega_0 t - C_k \sin k\omega_0 t].$$

Solving for Fourier Series Coefficients(Analysis)

Multiplying both side of the Fourier Series Synthesis eqn by $e^{-jn\omega_0 t}$: -

$$x(t)e^{-jn\omega_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} e^{-jn\omega_0 t}.$$

Integrating both sides of equation from 0 to $\mathrm{T} = \dfrac{2\pi}{\omega_0}$, we have

$$\int_0^{\mathrm{T}} x(t)e^{-jn\omega_0 t}dt = \int_0^{\mathrm{T}} \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} e^{-jn\omega_0 t}dt$$

$$\int_0^{\mathrm{T}} x(t)e^{-jn\omega_0 t}dt = \sum_{k=-\infty}^{+\infty} a_k [\int_0^{T} e^{jk\omega_0 t} e^{-jn\omega_0 t}dt]$$

$$\int_0^{\mathrm{T}} x(t)e^{-jn\omega_0 t}dt = \sum_{k=-\infty}^{+\infty} a_k [\int_0^{T} e^{j(k-n)\omega_0 t}dt] \quad \text{.........equation 3.34}$$

However $\int_0^{T} e^{j(k-n)\omega_0 t}dt = \int_0^{T} \cos(k-n)\omega_0 t dt + j\int_0^{T} \sin(k-n)\omega_0 t dt$

Solving for Fourier Series Coefficients(Analysis)

For $k \neq n$, $\cos(k-n)\omega_0 t$ and $\sin(k-n)\omega_0 t$ are periodic sinusoids

with fundamental period $\left(\dfrac{T}{|k-n|}\right)$.

The integrations over whole integer numbers of fundamental periods

for sine and cosine term are zeros.

For $k = n$, $\cos(k-n)\omega_0 t = 1$ and $\sin(k-n)\omega_0 t = 0$

$$\therefore \int_0^T e^{j(k-n)\omega_0 t}\,dt = \int_0^T 1\,dt + j\int_0^T 0\,dt = T$$

Overall we have : -

$$\int_0^T e^{j(k-n)\omega_0 t}\,dt = T, \quad k = n$$

$$= 0. \quad k \neq n$$

From Equ 3.34 taking $k = n$: -

.

$$\therefore a_n = \frac{1}{T}\int_0^T x(t)e^{-jn\omega_0 t}\,dt$$

Summarizing for Fourier Series Pair Representation

Fourier Series Synthesis:- eqn 3.38

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk\frac{2\pi}{T}t}$$

Fourier Series Analysis or F.S. Coefficients 0r Spectral Coefficents:- eqn 3.39

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt = \frac{1}{T}\int_T x(t)e^{-jk\frac{2\pi}{T}t}dt$$

Example 3.4

$$x(t) = 1 + \sin\omega_0 t + 2\cos\omega_0 t + \cos(2\omega_0 t + \frac{\pi}{4})$$

use Euler relationship for $\cos\theta = \frac{1}{2}(\mathrm{e}^{\mathrm{j}\theta} + \mathrm{e}^{\text{-j}\theta})$

and $\sin\theta = \frac{1}{2j}(\mathrm{e}^{\mathrm{j}\theta} - \mathrm{e}^{\text{-j}\theta})$: -

$$\mathrm{x(t)} = 1 + \frac{1}{2\mathrm{j}}[e^{j\omega_0 t} - e^{-j\omega_0 t}] + [e^{j\omega_0 t} + e^{-j\omega_0 t}] + \frac{1}{2}[e^{j2\omega_0 t + \pi/4} + e^{-j2\omega_0 t + \pi/4}]$$

$$x(t) = 1 + (1 + \frac{1}{2j})e^{j\omega_0 t} + (1 - \frac{1}{2j})e^{-j\omega_0 t} + \frac{1}{2}e^{j(\pi/4)})e^{j2\omega_0 t} + \frac{1}{2}e^{-j(\pi/4)})e^{-j2\omega_0 t}$$

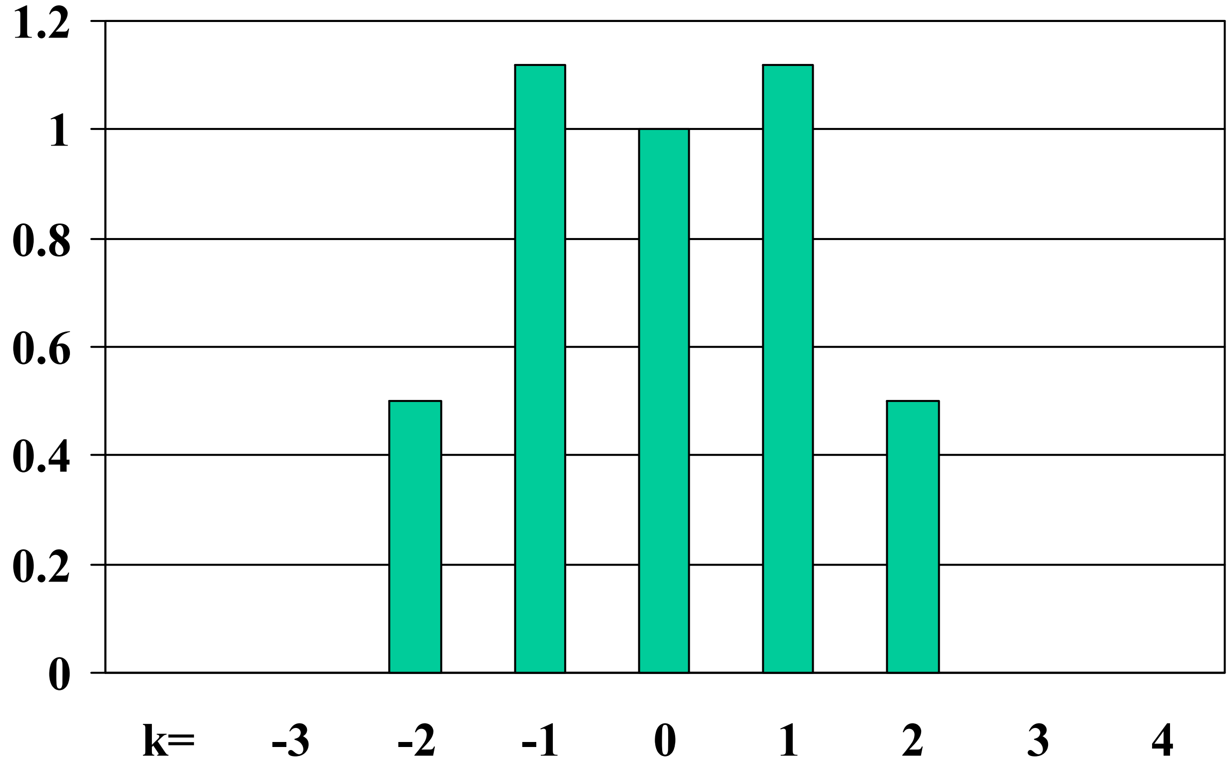
$\therefore$ Fourier series coefficients are : -

$$a_0 = 1, \quad a_1 = (1 + \frac{1}{2j}), \quad a_{-1} = (1 - \frac{1}{2j}),$$

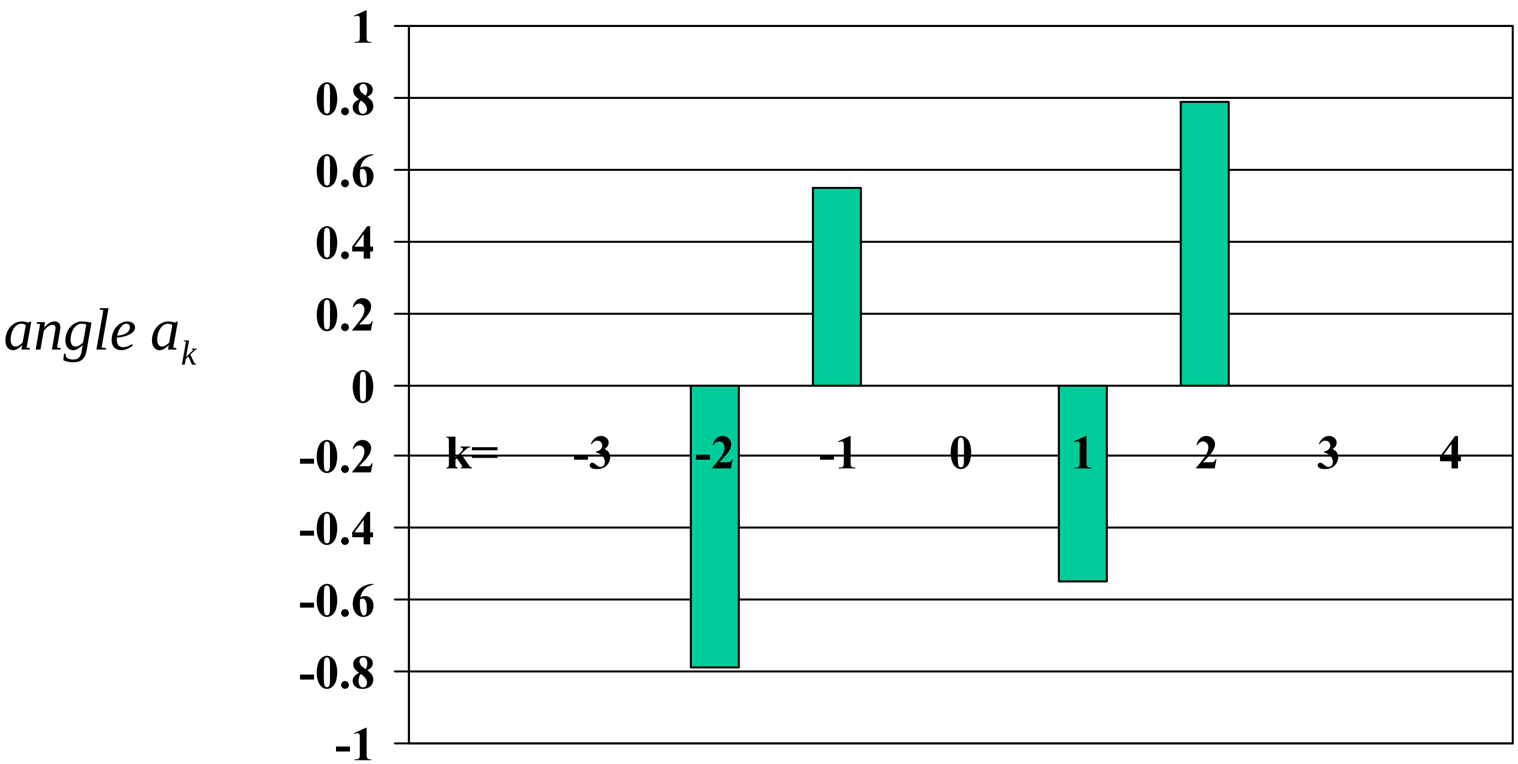
$$a_2 = \frac{1}{2}e^{j(\pi/4)}, \quad a_{-2} = \frac{1}{2}e^{-j(\pi/4)}, \quad a_k = 0 \ \mathit{for} \ |k| > 2.$$

Magnitude of Fourier Coefficients.

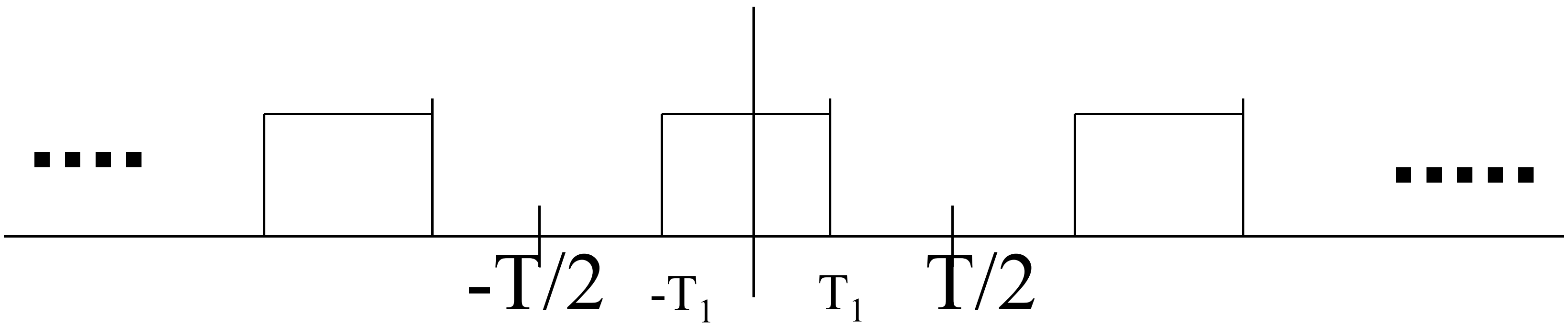
$| a_k |$



Phase of Fourier Coefficients.



Example 3.5



This periodic signal x(t) repeats every T seconds.
$x(t)=1$, for $|t|<T_1$, and $x(t)=0$, for $T_1 <|t|< T/2$

Fundamental period= T,
Fundamental frequency $\omega_0 = 2\pi/T$.
Choosing the period of integration to be between -T/2 and +T/2. Use eqn 3.39 to get at Fourier Series Coefficients.

Example 3.5 continued

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt = \frac{1}{T}\int_T x(t)e^{-jk\frac{2\pi}{T}t}dt$$

Let us get the dc, constant term or average value over a period, first, i.e. $k = 0$

$$a_0 = \frac{1}{T}\int_{-T_1}^{T_1} dt = \frac{2T_1}{T}$$

Example 3.5 continued

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt = \frac{1}{T}\int_T x(t)e^{-jk\frac{2\pi}{T}t}dt$$

For fundamental first order and higher order harmonics :-

we have $k \neq 0$.

$$a_k = \frac{1}{T}\int_{-T_1}^{T_1} e^{-jk\omega_0 t}dt = -\frac{1}{jk\omega_0 T}e^{-jk\omega_0 t}\Big|_{-T_1}^{T_1},$$

$$a_k = \frac{2}{k\omega_0 T}\left[\frac{e^{jk\omega_0 T_1} - e^{-jk\omega_0 T_1}}{2j}\right] = \frac{2\sin(k\omega_0 T_1)}{k\omega_0 T} = \frac{\sin(k\omega_0 T_1)}{k\pi}$$

Example 3.5 Continued

- In this example the coefficients are real values.
- Generally the coefficients are complex.
- In this case we can represent a single plot of magnitude of coefficient against k.
- Generally we will have the magnitude plot and the phase plot of the coefficients.

Example 3.5 Continued

$$\text{For T} = 4\text{T}_1, \omega_0 T_1 = \frac{2\pi}{T} T_1 = \frac{\pi}{2},$$

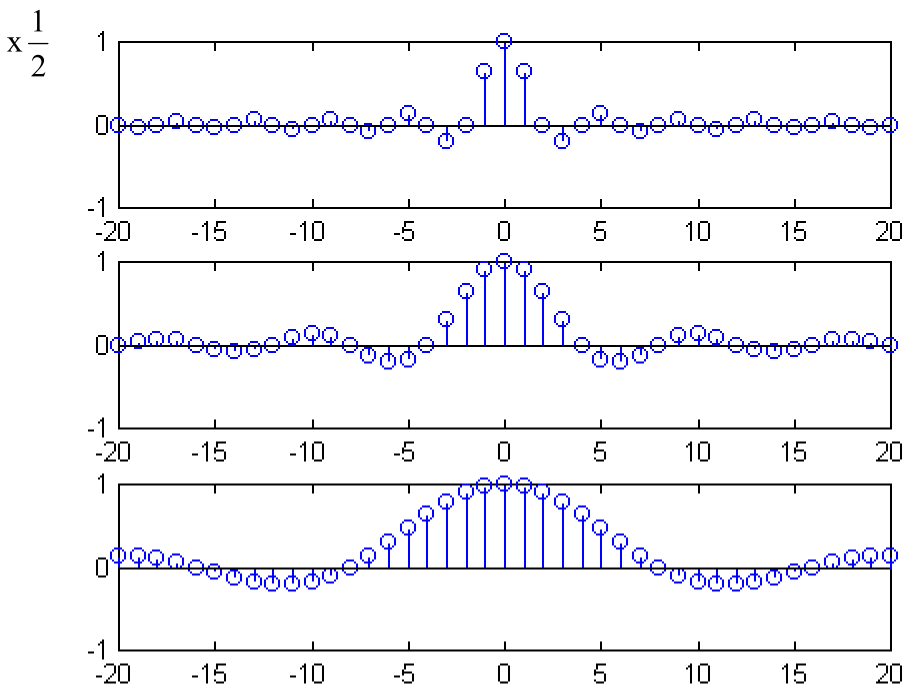
$$\text{From eqn 3.44:- } \mathrm{a_k} = \frac{\sin(\pi\, k/2)}{k\,\pi}, \mathrm{k} \neq 0,$$

$$\therefore a_1 = a_{-1} = \frac{1}{\pi}, \quad a_3 = a_{-3} = -\frac{1}{3\pi}, a_5 = a_{-5} = \frac{1}{5\pi},$$

For even k's the a's are all zeros.

$$\text{and from eqn 3.42 } \mathrm{a_0} = \frac{1}{2}.$$

Plot of the Coefficients with T_1 Fixed and T varied.



a) $T=4T_1$

b) $T=8T_1$

c) $T=16T_1$

Example Determine the FS coefficients for $x(t)$.

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt = \frac{1}{2}\int_0^2 e^{-2t}e^{-jk\pi t}dt$$

$$= \frac{1}{2}\cdot\left.\frac{e^{-(2+jk\pi)t}}{-(2+jk\pi)}\right|_0^2 = \frac{1-e^{-4}}{4+jk2\pi}$$