Introduction to Signals and Systems: V216

Lecture #14
Chapter 9: Laplace Transform

Introduction to the Laplace Transform

Fourier transforms are extremely useful in the study of many problems of practical importance involving signals and LTI systems.

purely imaginary complex exponentials e^{st}, $s=j\omega$

A large class of signals can be represented as a linear combination of complex exponentials and complex exponentials are eigenfunctions of LTI systems.

However, the **eigenfunction** property applies to any complex number *s*, not just purely imaginary (signals)

This leads to the development of the **Laplace transform** where *s* is an arbitrary complex number.

Laplace and *z*-transforms can be applied to the analysis of un-stable system (signals with infinite energy) and play a role in the analysis of system stability

The Laplace Transform

The response of an LTI system with impulse response $h(t)$ to a complex exponential input, $x(t)=e^{st}$, is

$$y(t) = H(s)e^{st}$$

where s is a complex number and

$$H(s) = \int_{-\infty}^{\infty} h(t)e^{-st}\,dt$$

when s is purely imaginary, this is the **Fourier transform**, $H(j\omega)$

when s is complex, this is the **Laplace transform** of $h(t)$, $H(s)$

The Laplace transform of a general signal $x(t)$ is:

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}\,dt$$

and is usually expressed as:

$$x(t) \overset{L}{\leftrightarrow} X(s)$$

Laplace and Fourier Transform

The Fourier transform is the Laplace transform when s is purely imaginary:

$$X(s)\big|_{s=j\omega} = F\{x(t)\}$$

An alternative way of expressing this is when $s = \sigma + j\omega$

$$\begin{aligned}
X(\sigma + j\omega) &= \int_{-\infty}^{\infty} x(t) e^{-(\sigma + j\omega)t} dt \\
&= \int_{-\infty}^{\infty} \left[x(t) e^{-\sigma t}\right] e^{-j\omega t} dt \\
&= \int_{-\infty}^{\infty} x'(t) e^{-j\omega t} dt \\
&= F\{x'(t)\}
\end{aligned}$$

The Laplace transform is the Fourier transform of the transformed signal $x'(t) = x(t)e^{-\sigma t}$. Depending on whether σ is positive/negative this represents a growing/negative signal

Example 1: Laplace Transform

Consider the signal $x(t) = e^{-at}u(t)$

The Fourier transform $X(j\omega)$ converges for a>0:

$$X(j\omega) = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-j\omega t}dt = \int_{0}^{\infty} e^{-at}e^{-j\omega t}dt = \frac{1}{j\omega + a}, \quad a > 0$$

The Laplace transform is:

$$X(s) = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-st}dt = \int_{0}^{\infty} e^{-(s+a)t}dt$$

$$= \int_{0}^{\infty} e^{-(\sigma+a)t}e^{-j\omega t}dt$$

which is the Fourier Transform of $e^{-(\sigma+a)t}u(t)$

$$X(\sigma + j\omega) = \frac{1}{(\sigma + a) + j\omega}, \quad \sigma + a > 0$$

Or

$$e^{-at}u(t) \overset{L}{\leftrightarrow} X(s) = \frac{1}{s+a}, \quad \mathrm{Re}\{s\} > -a$$

If a is negative or zero, the Laplace Transform still exists

Example 2: Laplace Transform

Consider the signal $x(t) = -e^{-at}u(-t)$

The Laplace transform is:

$$\begin{aligned} X(s) &= -\int_{-\infty}^{\infty} e^{-at} e^{-st} u(-t) dt \\ &= -\int_{-\infty}^{0} e^{-(s+a)t} dt \\ &= \frac{1}{s+a} \end{aligned}$$

Convergence requires that Re{*s*+*a*}<0 or Re{*s*}<-*a*.

The Laplace transform expression is identical to Example 1 (similar but different signals), however the regions of convergence of *s* are mutually exclusive (non-intersecting).

For a Laplace transform, we need both the expression and the Region Of Convergence (ROC).

Example 3: sin(ωt)u(t)

The Laplace transform of the signal $x(t) = \sin(\omega t)u(t)$ is:

$$X(s) = \int_{-\infty}^{\infty} \tfrac{1}{2j}\left(e^{j\omega t} - e^{-j\omega t}\right)u(t)e^{-st}dt$$

$$= \tfrac{1}{2j}\int_{0}^{\infty} e^{-(s-j\omega)t}dt - \tfrac{1}{2j}\int_{0}^{\infty} e^{-(s+j\omega)t}dt \qquad \mathrm{Re}\{s\} > 0$$

$$= \tfrac{1}{2j}\left(\left.\frac{e^{-(s-j\omega)t}}{-(s-j\omega)}\right|_0^{\infty} + \left.\frac{e^{-(s+j\omega)t}}{(s+j\omega)}\right|_0^{\infty}\right)$$

$$= \tfrac{1}{2j}\left(\frac{1}{(s-j\omega)} - \frac{1}{(s+j\omega)}\right)$$

$$= \tfrac{1}{2j}\left(\frac{2j\omega}{s^2+\omega^2}\right)$$

$$= \frac{\omega}{s^2+\omega^2} \qquad \mathrm{Re}\{s\} > 0$$

Fourier Transform does not Converge …

It is worthwhile reflecting that the Fourier transform does not exist for a fairly wide class of signals, such as the response of an unstable, first order system, the Fourier transform does not exist/converge

E.g. $x(t) = e^{at}u(t), \quad a>0$

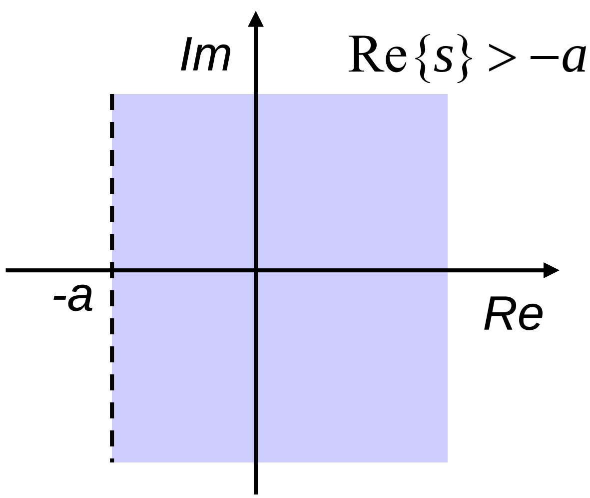
does not exist (is infinite) because the **signal's energy is infinite**

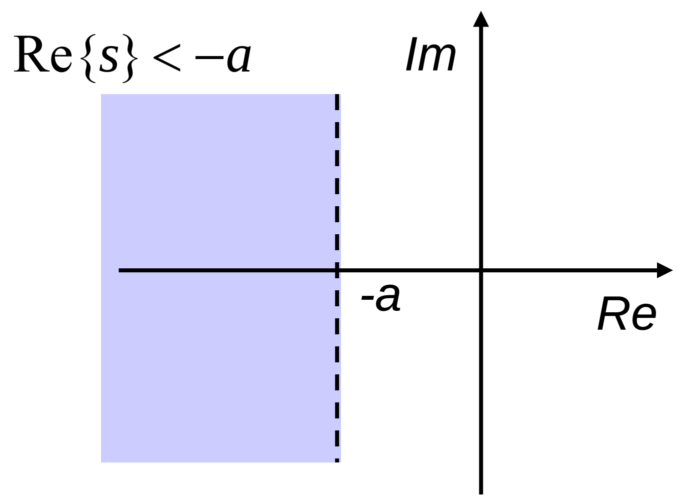
This is because we multiply $x(t)$ by a complex sinusoidal signal which has **unit magnitude** for all t and integrate for all time. Therefore, as the **Dirichlet convergence** conditions say, the Fourier transform exists for most signals with **finite energy**

Region of Convergence

The Region Of Convergence (ROC) of the Laplace transform is the set of values for s ($=\sigma+j\omega$) for which the Fourier transform of $x(t)e^{-\sigma t}$ converges (exists).

The ROC is generally displayed by drawing separating line/curve in the complex plane, as illustrated below for Examples 1 and 2, respectively.





The shaded regions denote the ROC for the Laplace transform

Example 4: Laplace Transform

Consider a signal that is the sum of two real exponentials:

$$x(t) = 3e^{-2t}u(t) - 2e^{-t}u(t)$$

The Laplace transform is then:

$$X(s) = \int_{-\infty}^{\infty} \left[3e^{-2t}u(t) - 2e^{-t}u(t)\right] e^{-st}\,dt$$

$$= 3\int_{-\infty}^{\infty} e^{-2t}u(t)e^{-st}\,dt - 2\int_{-\infty}^{\infty} e^{-t}u(t)e^{-st}\,dt$$

Using Example 1, each expression can be evaluated as:

$$X(s) = \frac{3}{s+2} - \frac{2}{s+1}$$

The ROC associated with these terms are Re{*s*}>-1 and Re{*s*}>-2. Therefore, both will converge for Re{*s*}>-1, and the Laplace transform:

$$X(s) = \frac{s-1}{s^2+3s+2}$$

Reminder: Laplace Transforms

Equivalent to the Fourier transform when $s = j\omega$

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

$$x(t) \overset{L}{\leftrightarrow} X(s)$$

There is an associated region of convergence for s where the (transformed) signal has finite energy. The Laplace transform is only defined for these values

Laplace transform is linear (easy!)

Examples for the Laplace transforms include

$$e^{-at}u(t) \overset{L}{\leftrightarrow} \frac{1}{s+a}, \quad \mathrm{Re}\{s\} > -a$$

$$3e^{-2t}u(t) - 2e^{-t}u(t) \overset{L}{\leftrightarrow} \frac{s-1}{s^2+3s+2}, \qquad \mathrm{Re}\{s\} > -1$$

Ratio of Polynomials

In each of these examples, the Laplace transform is rational, i.e. it is a ratio of polynomials in the complex variable *s*.

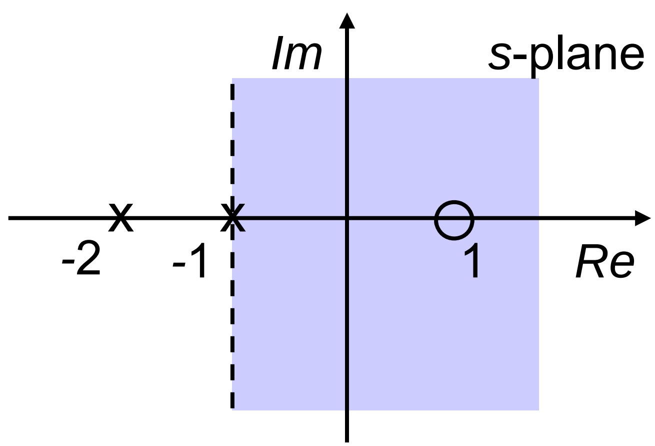
$$X(s) = \frac{N(s)}{D(s)}$$

where *N* and *D* are the numerator and denominator polynomial respectively.

In fact, *X*(*s*) will be rational whenever *x*(*t*) is a linear combination of real or complex exponentials. **Rational transforms** also arise when we consider **LTI systems specified in terms of linear, constant coefficient differential equations**.

We can mark the roots of *N* and *D* in the *s*-plane along with the ROC

Example 3:



○ – roots of *N*(*s*)

x – roots of *D*(*s*)

Poles and Zeros

The roots of $N(s)$ are known as the **zeros**. For these values of s, $X(s)$ is zero.

The roots of $D(s)$ are known as the **poles**. For these values of s, $X(s)$ is infinite, the Region of Convergence for the Laplace transform cannot contain any poles, because the corresponding integral is infinite

The set of poles and zeros completely characterise $X(s)$ to within a scale factor (+ ROC for Laplace transform)

$$X(s) \propto \frac{\prod_i (s - z_i)}{\prod_j (s - p_j)}$$

The graphical representation of $X(s)$ through its poles and zeros in the s-plane is referred to as the **pole-zero** plot of $X(s)$

Example: Poles and Zeros

Consider the signal:

$$x(t) = \delta(t) - \frac{4}{3}e^{-t}u(t) + \frac{1}{3}e^{2t}u(t)$$

By linearity (& last lecture) we can evaluate the second and third terms

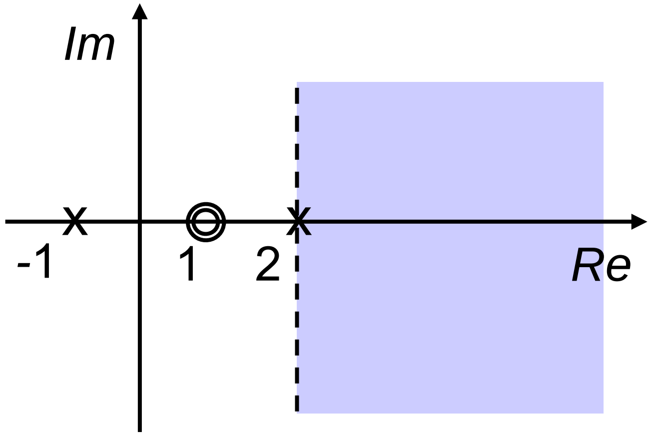
The Laplace transform of the impulse function is:

$$L\{\delta(t)\} = \int_{-\infty}^{\infty} \delta(t)e^{-st}dt = 1$$

which is valid for any *s*. Therefore,

$$X(s) = 1 - \frac{4}{3}\frac{1}{s+1} + \frac{1}{3}\frac{1}{s-2}$$

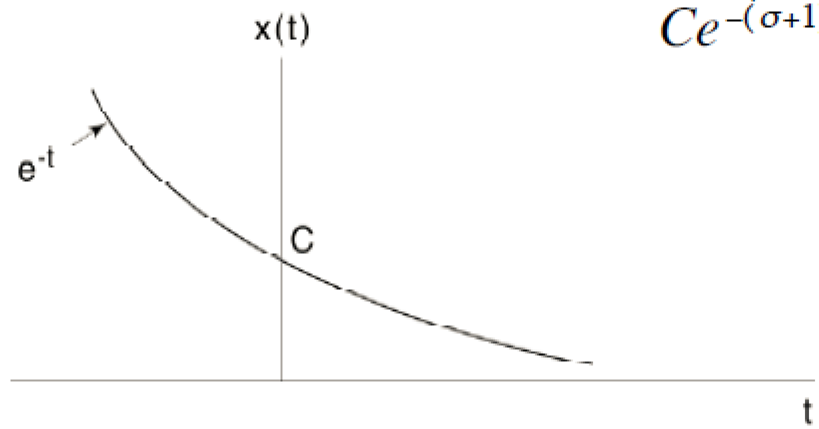
$$= \frac{(s-1)^2}{(s+1)(s-2)}, \qquad \mathrm{Re}\{s\} > 2$$



ROC Properties for Laplace Transform

- First of all, some signals do not have Laplace transforms

(a) $x(t) = Ce^{-t}$ for all t since $\int_{-\infty}^{\infty} \underbrace{| x(t)e^{-\sigma t} |}_{Ce^{-(\sigma+1)t}} dt = \infty$ for all σ



(b) $x(t) = e^{j\omega_o t}$ for all t

$$\int_{-\infty}^{+\infty} | x(t)e^{-\sigma t} | dt = \int_{-\infty}^{+\infty} e^{-\sigma t} dt = \infty \quad \text{for all } \sigma$$

$X(s)$ is defined only in ROC, by definition,
$X(s) \neq \infty \Rightarrow$ No $\delta(s)$ is allowed, different from **FT**

ROC Properties for Laplace Transform

Secondly, the ROC can take on only a small number of different forms

1) The ROC consists of a collection of lines parallel to the $j\omega$-axis in the s-plane (i.e. the ROC only depends on σ). Why?

$$\int_{-\infty}^{\infty} | x(t)e^{-st} | \, dt = \int_{-\infty}^{\infty} | x(t)e^{-\sigma t} | \, dt < \infty \text{ depends only on } \sigma = \mathrm{Re}(s)$$

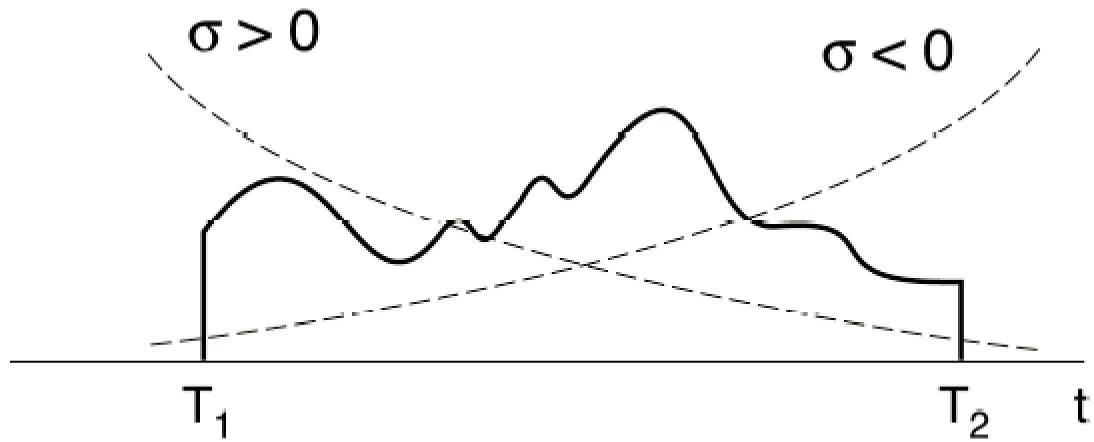
2) If $X(s)$ is rational, then the ROC does not contain any poles. Why?

Poles are places where $D(s) = 0$

$$\Rightarrow X(s) = \frac{N(s)}{D(s)} = \infty \quad \text{Not convergent.}$$

ROC Properties for Laplace Transform

3) If $x(t)$ is of finite duration and is absolutely integrable, then the ROC is the entire s-plane.

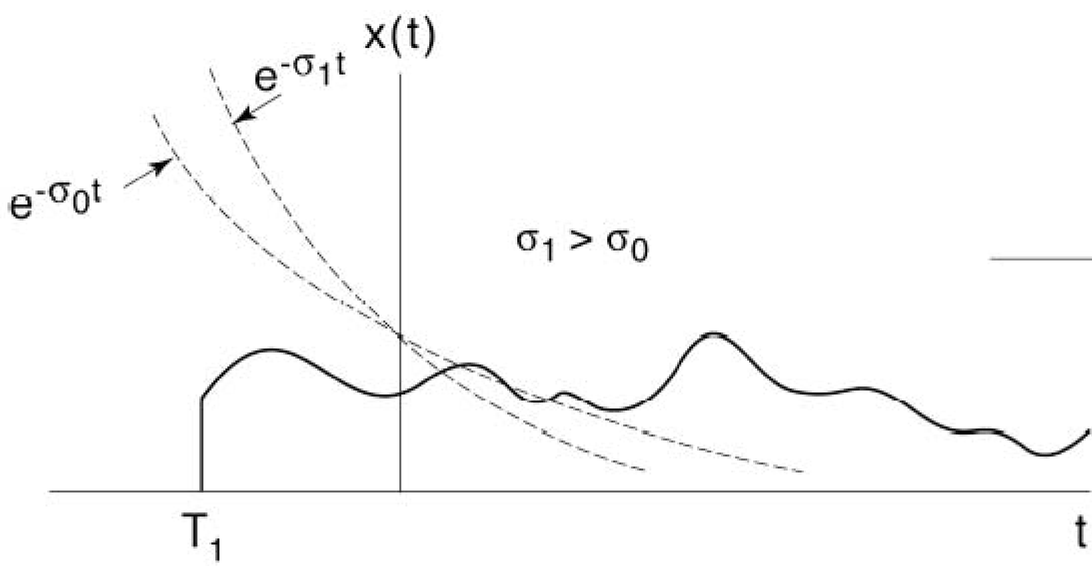


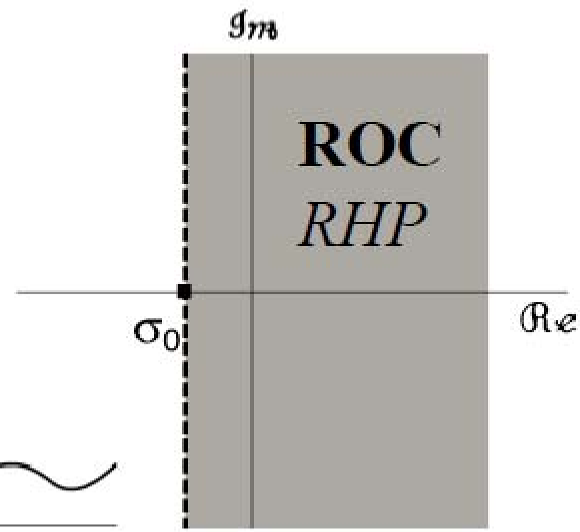
$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt = \underbrace{\int_{T_1}^{T_2} x(t)e^{-st}dt}_{\text{A finite integration}}$$

$$< \infty \ \text{ if } \int_{T_1}^{T_2} |x(t)|\,dt < \infty$$

ROC Properties for Laplace Transform

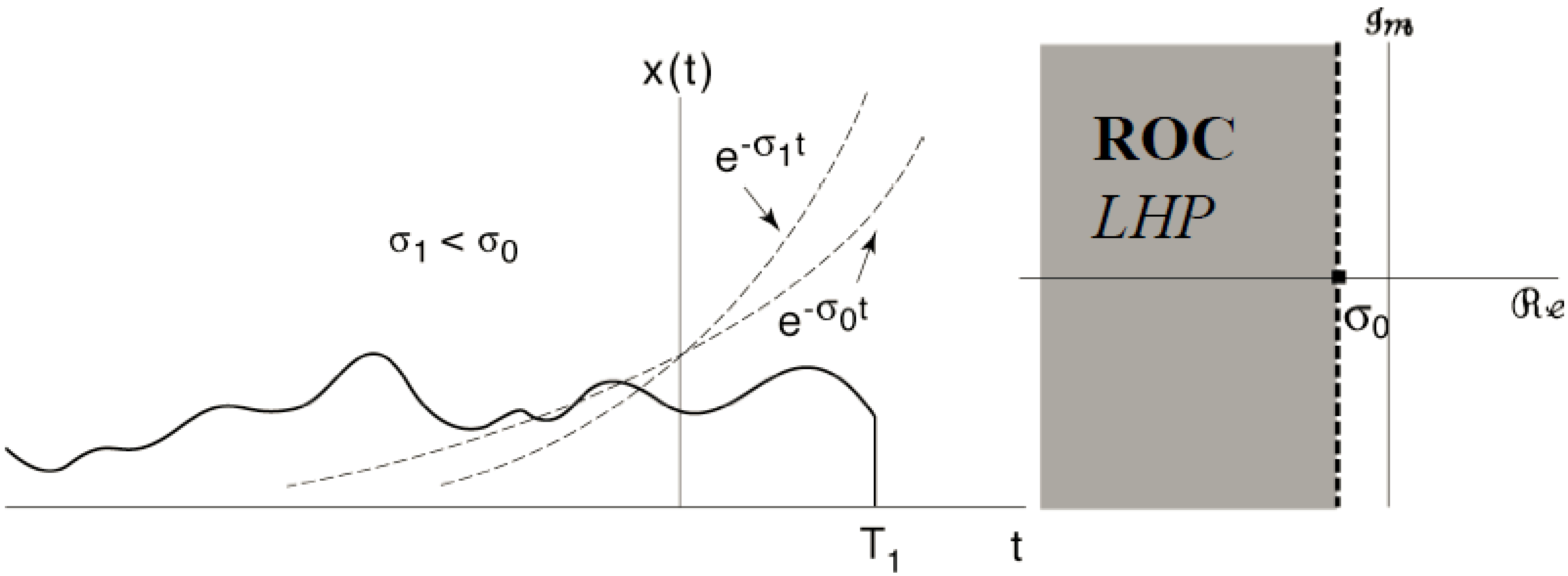
4) If $x(t)$ is right-sided (i.e. if it is zero *before* some time), and if $Re(s) = \sigma_o$ is in the ROC, then all values of s on the right side of the vertical line $Re(s) = \sigma_o$ are also in the ROC.





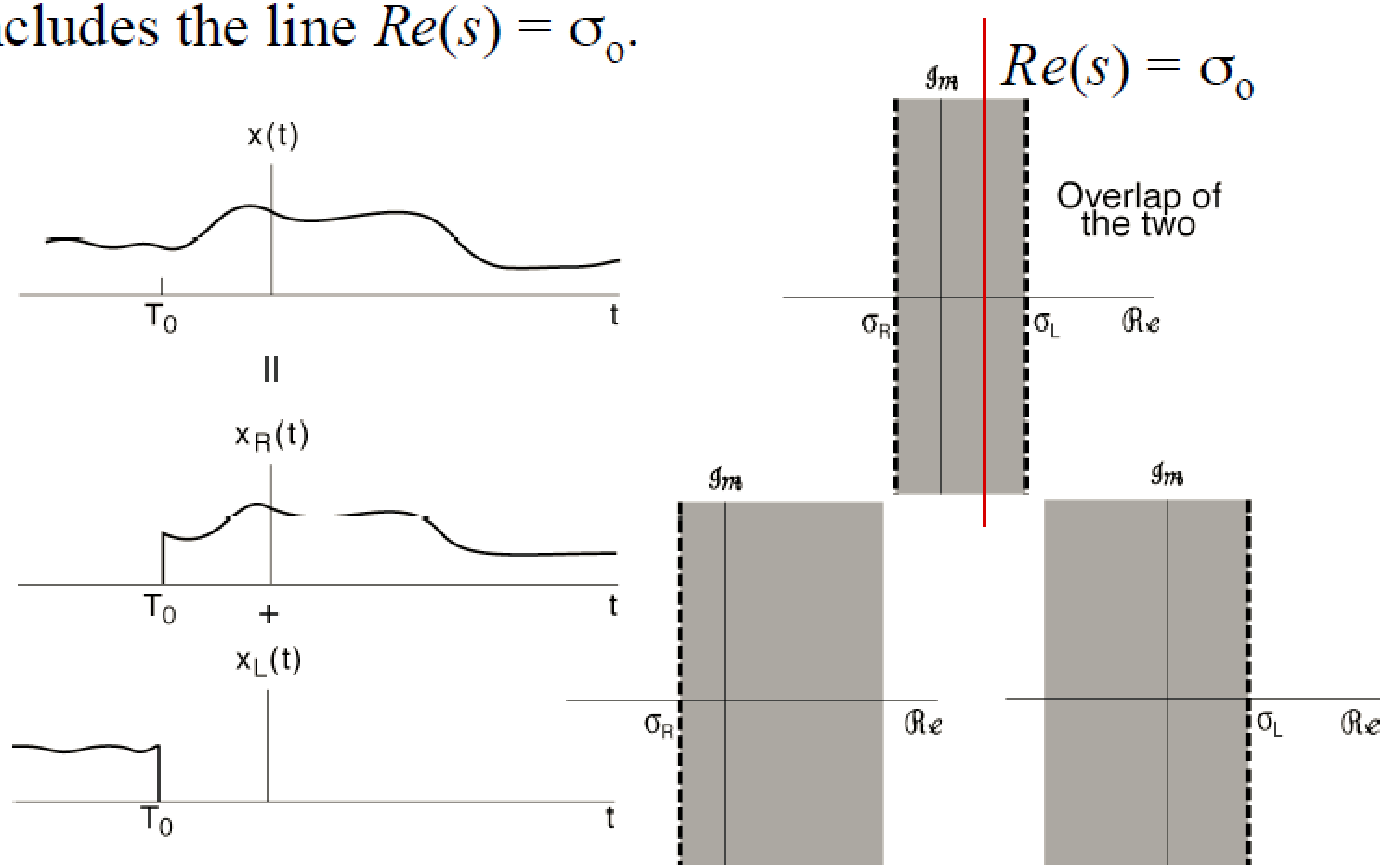
ROC Properties for Laplace Transform

5) If $x(t)$ is left-sided (i.e. if it is zero *after* some time), and if $Re(s) = \sigma_o$ is in the ROC, then all values of s on the left side of the vertical line $Re(s) = \sigma_o$ are also in the ROC.



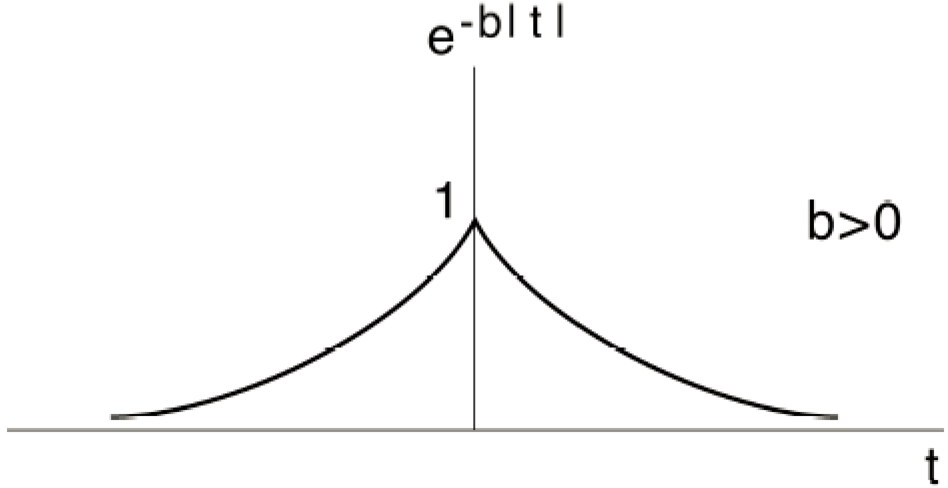
ROC Properties for Laplace Transform

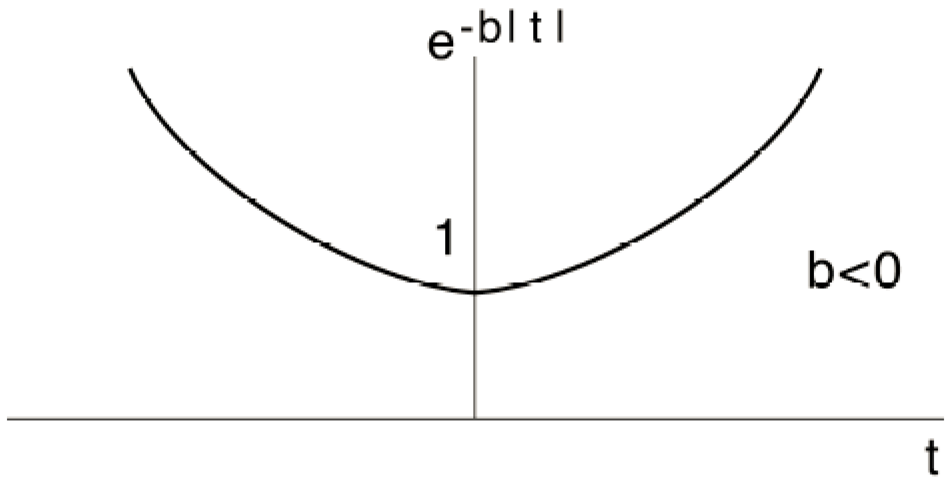
6) If $x(t)$ is two-sided and if the line $Re(s) = \sigma_o$ is in the ROC, then the ROC consists of a strip in the s-plane that includes the line $Re(s) = \sigma_o$.



ROC Properties for Laplace Transform

$$x(t) = e^{-b|t|}$$





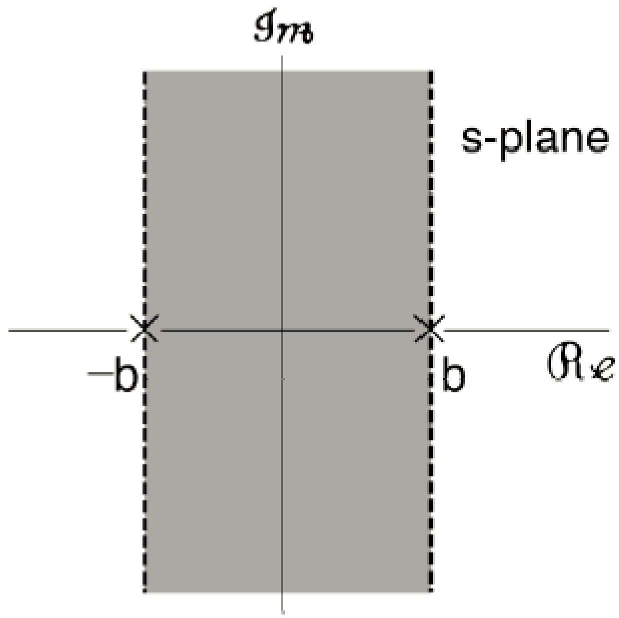
ROC Properties for Laplace Transform

$$x(t) \quad = \quad e^{bt}u(-t) \quad + \quad e^{-bt}u(t)$$

$$\updownarrow \qquad\qquad \updownarrow$$

$$-\frac{1}{s-b}\ ,\ \mathrm{Re}\{s\} < b \qquad \frac{1}{s+b}\ ,\ \mathrm{Re}\{s\} > -b$$

Overlap only if $b > 0$ $\Rightarrow X(s) = \dfrac{-2b}{s^2 - b^2}$, with ROC

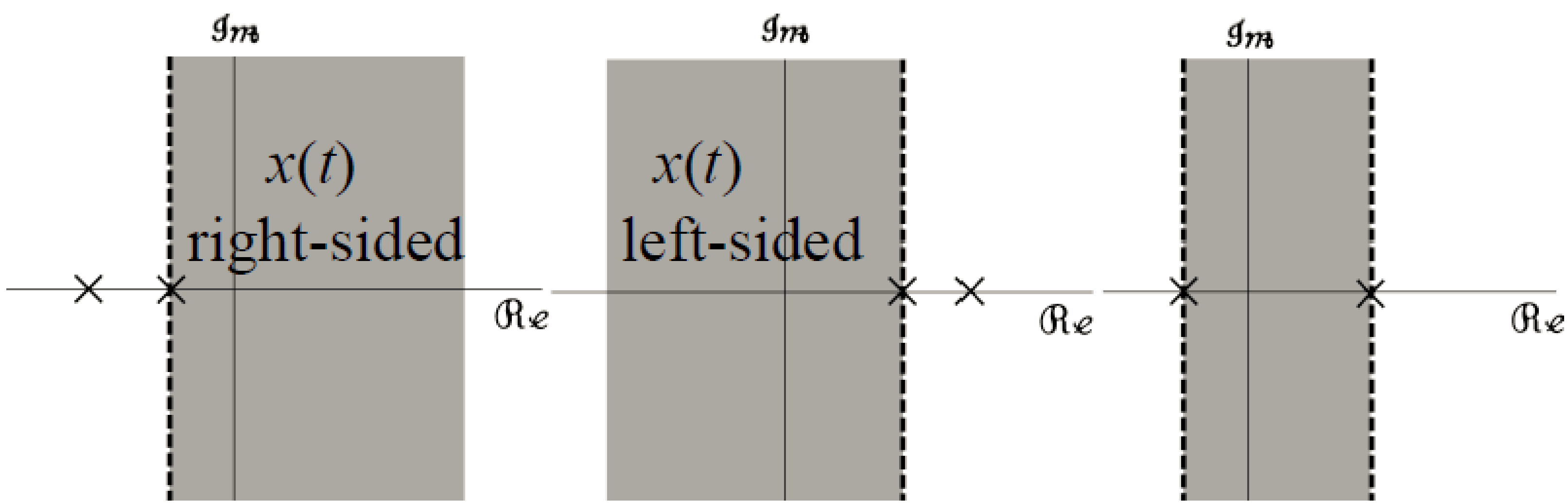


What if $b < 0$? $\Rightarrow$ No overlap $\Rightarrow$ No Laplace Transform

ROC Properties for Laplace Transform

7) If $X(s)$ is rational, then its ROC is bounded by poles or extends to infinity. In addition, no poles of $X(s)$ are contained in the ROC.

8) Suppose $X(s)$ is rational, then
   (a) If $x(t)$ is right-sided, the ROC is to the right of the rightmost pole.
   (b) If $x(t)$ is left-sided, the ROC is to the left of the leftmost pole.



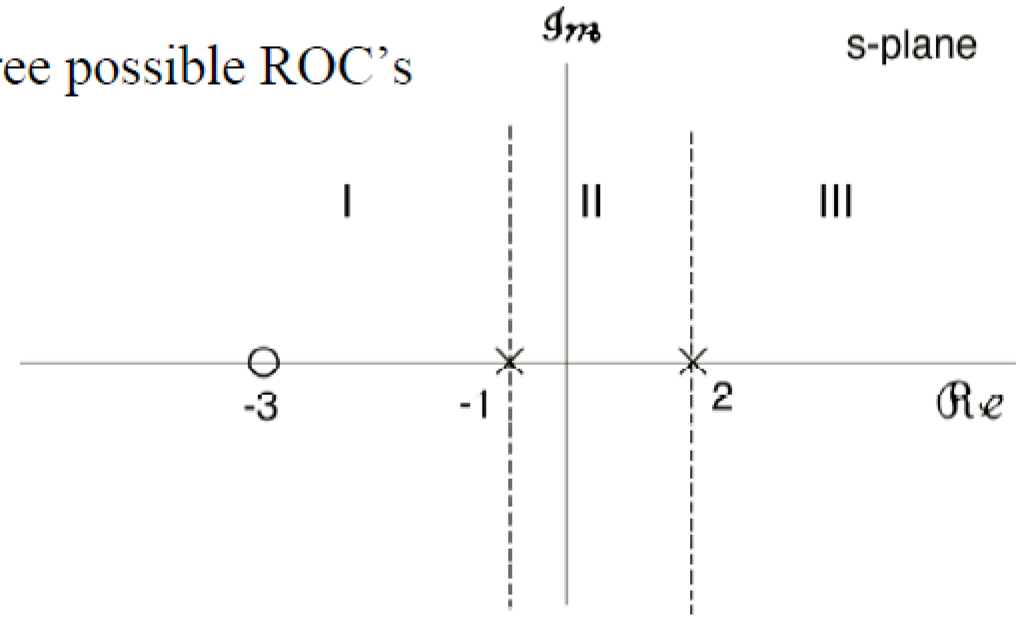
(a) (b) (c)

ROC Properties for Laplace Transform

9) If ROC of $X(s)$ includes the $j\omega$-axis, then *FT* of $x(t)$ exists.

Example: $$X(s) = \frac{(s+3)}{(s+1)(s-2)}$$

Two poles $\Rightarrow$ three possible ROC's



$x(t)$ is right-sided $\Rightarrow$ ROC: III

$x(t)$ is left-sided $\Rightarrow$ ROC: I

$x(t)$ has a Fourier transform $\Rightarrow$ ROC: II

Inverse Laplace Transform

The Laplace transform of a signal $x(t)$ is:

$$X(\sigma + j\omega) = F\{x(t)e^{-\sigma t}\} = \int_{-\infty}^{\infty} x(t)e^{-\sigma t}e^{-j\omega t}dt$$

We can invert this relationship using the inverse Fourier transform

$$x(t)e^{-\sigma t} = F^{-1}\{X(\sigma + j\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\sigma + j\omega)e^{j\omega t}d\omega$$

Multiplying both sides by $e^{\sigma t}$:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\sigma + j\omega)e^{(\sigma + j\omega)t}d\omega$$

Therefore, we can recover $x(t)$ from $X(s)$, where the real component is fixed and we integrate over the imaginary part, noting that $ds = jd\omega$

$$x(t) = \frac{1}{2\pi j}\int_{\sigma - j\infty}^{\sigma + j\infty} X(s)e^{st}ds$$

Inverse Laplace Transform Interpretation

$$x(t) = \frac{1}{2\pi j}\int_{\sigma - j\infty}^{\sigma + j\infty} X(s)e^{st}\,ds$$

Just about all real-valued signals, $x(t)$, can be represented as a weighted, $X(s)$, integral of complex exponentials, e^{st}.

The contour of integration is a straight line (in the complex plane) from σ-$j\infty$ to σ+$j\infty$ (**we won't be explicitly evaluating this, just spotting known transformations**)

We can choose any σ for this integration line, as long as the integral converges

For the class of **rational Laplace transforms**, we can express $X(s)$ as **partial fractions** to determine the **inverse Fourier transform**.

$$X(s) = \sum_{i=1}^{M} \frac{A_i}{s + a_i}$$

$$L^{-1}\{A_i/(s+a_i)\} \rightarrow \begin{cases} A_i e^{-a_i t}u(t) & \mathrm{Re}\{s\} > -a_i \\ -A_i e^{-a_i t}u(-t) & \mathrm{Re}\{s\} < -a_i \end{cases}$$

Example 1: Inverting the Laplace Transform

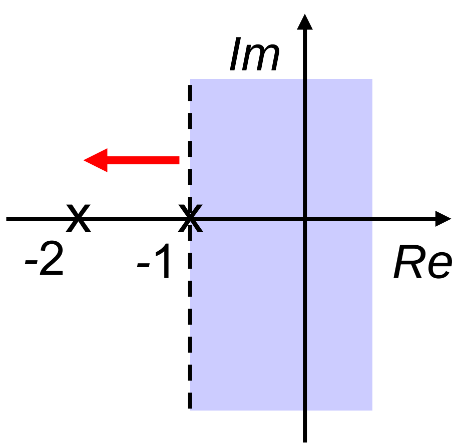
Consider when

$$X(s) = \frac{1}{(s+1)(s+2)} \qquad \Re(s) > -1$$

Like the inverse Fourier transform, expand as partial fractions

$$X(s) = \frac{1}{(s+1)(s+2)} = \frac{A}{(s+1)} + \frac{B}{(s+2)} = \frac{1}{(s+1)} - \frac{1}{(s+2)}$$

Pole-zero plots and ROC for combined & individual terms



$$e^{-t}u(t) \overset{L}{\leftrightarrow} \frac{1}{s+1}, \quad \mathrm{Re}\{s\} > -1$$

$$e^{-2t}u(t) \overset{L}{\leftrightarrow} \frac{1}{s+2}, \quad \mathrm{Re}\{s\} > -2$$

$$x(t) = (e^{-t} - e^{-2t})u(t) \overset{L}{\leftrightarrow} \frac{1}{(s+1)(s+2)}, \quad \mathrm{Re}\{s\} > -1$$

Example 2

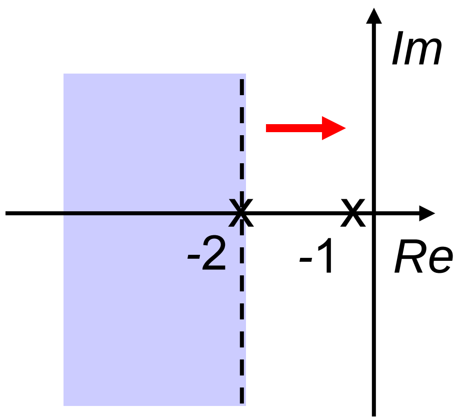
Consider when

$$X(s) = \frac{1}{(s+1)(s+2)} \qquad \mathrm{Re}\{s\} < -2$$

Like the inverse Fourier transform, expand as partial fractions

$$X(s) = \frac{1}{(s+1)(s+2)} = \frac{A}{(s+1)} + \frac{B}{(s+2)} = \frac{1}{(s+1)} - \frac{1}{(s+2)}$$

Pole-zero plots and ROC for combined & individual terms



$$-e^{-t}u(-t) \overset{L}{\leftrightarrow} \frac{1}{s+1}, \qquad \mathrm{Re}\{s\} < -1$$

$$-e^{-2t}u(-t) \overset{L}{\leftrightarrow} \frac{1}{s+2}, \qquad \mathrm{Re}\{s\} < -2$$

$$x(t) = (-e^{-t} + e^{-2t})u(-t) \overset{L}{\leftrightarrow} \frac{1}{(s+1)(s+2)}, \qquad \mathrm{Re}\{s\} < -2$$

Laplace Transform Properties

Linearity of the Laplace Transform

If $\quad x_1(t) \stackrel{L}{\leftrightarrow} X_1(s) \quad$ ROC=R_1

and $\quad x_2(t) \stackrel{L}{\leftrightarrow} X_2(s) \quad$ ROC=R_2

Then $\quad ax_1(t) + bx_2(t) \stackrel{L}{\leftrightarrow} aX_1(s) + bX_2(s) \quad$ ROC= $R_1 \cap R_2$

This follows directly from the definition of the Laplace transform (as the integral operator is linear). It is easily extended to a linear combination of an arbitrary number of signals

Time Shifting & Laplace Transforms

If $\quad x(t) \overset{L}{\leftrightarrow} X(s) \quad$ ROC=R

Then $\quad x(t-t_0) \overset{L}{\leftrightarrow} e^{-st_0} X(s) \quad$ ROC=R

Proof $\quad x(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} X(s) e^{st} ds$

Now replacing t by t-t_0

$$x(t-t_0) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} X(s) e^{s(t-t_0)} ds$$

$$= \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} (e^{-st_0} X(s)) e^{st} ds$$

Recognising this as

$$L\{x(t-t_0)\} = e^{-st_0} X(s)$$

A signal which is shifted in time may have both the **magnitude and the phase** of the Laplace transform altered.

Example: Linear and Time Shift

Consider the signal (linear sum of two time shifted sinusoids)

$$x(t) = 2x_1(t-2.5) - 0.5x_1(t-4)$$

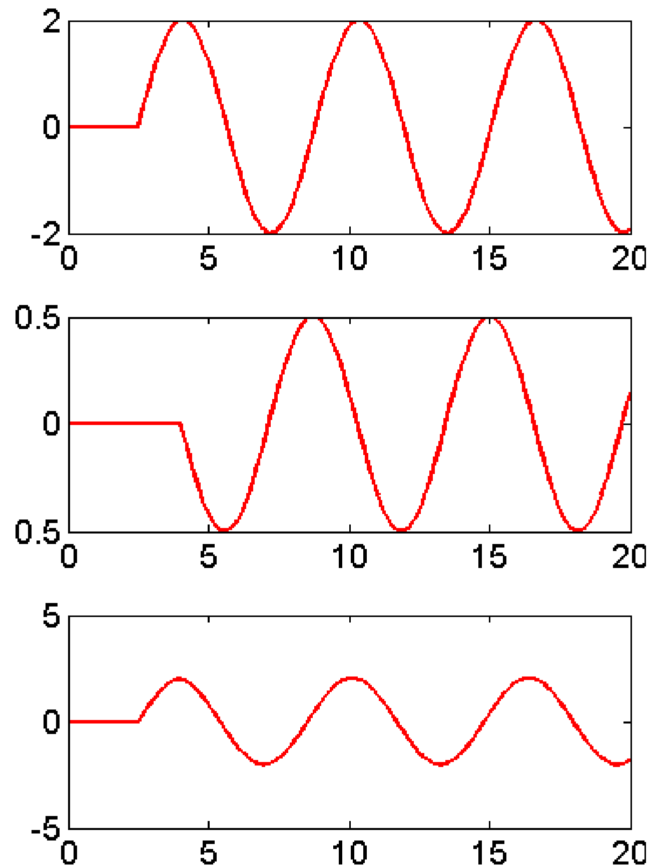
where $x_1(t) = \sin(\omega_0 t)u(t)$.

Using the sin() Laplace transform example

$$X_1(s) = \frac{\omega_0}{s^2 + \omega_0^2} \qquad \mathrm{Re}\{s\} > 0$$

Then using the **linearity** and **time shift** Laplace transform properties

$$X(s) = \left(2e^{-2.5s} - 0.5e^{-4s}\right)\frac{\omega_0}{s^2 + \omega_0^2} \qquad \mathrm{Re}\{s\} > 0$$



Convolution

The Laplace transform also has the multiplication property, i.e.

$$x(t) \overset{L}{\leftrightarrow} X(s) \qquad \text{ROC}=R_1$$

$$h(t) \overset{L}{\leftrightarrow} H(s) \qquad \text{ROC}=R_2$$

$$x(t) * h(t) \overset{L}{\leftrightarrow} X(s)H(s) \qquad \text{ROC} \supseteq R_1 \cap R_2$$

Proof is “identical” to the Fourier transform convolution

Note that pole-zero cancellation may occur between $H(s)$ and $X(s)$ which extends the ROC

$$X(s) = \frac{s+1}{s+2} \qquad \Re\{s\} > -2$$

$$H(s) = \frac{s+2}{s+1} \qquad \Re\{s\} > -1$$

$$X(s)H(s) = 1 \qquad -\infty < \Re\{s\} < \infty$$

Example 1: 1^{st} Order Input & First Order System Impulse Response

Consider the Laplace transform of the output of a first order system when the input is an exponential (decay?)

$$x(t) = e^{-at}u(t)$$

$$h(t) = e^{-bt}u(t)$$

Solved with Fourier transforms when a,b>0

Taking Laplace transforms

$$X(s) = \frac{1}{s+a} \qquad \mathrm{Re}\{s\} > -a$$

$$H(s) = \frac{1}{s+b}, \quad \mathrm{Re}\{s\} > -b$$

Laplace transform of the output is

$$Y(s) = \frac{1}{s+a}\frac{1}{s+b} \qquad \mathrm{Re}\{s\} > \max\{-a, -b\}$$

Example 1: Continued …

Splitting into partial fractions

$$Y(s) = \left(\frac{1}{b-a}\right)\left(\frac{1}{s+a} - \frac{1}{s+b}\right) \qquad \mathrm{Re}\{s\} > \max\{-a,-b\}$$

and using the inverse Laplace transform

$$y(t) = \tfrac{1}{b-a}\left(e^{-at}u(t) - e^{-bt}u(t)\right)$$

Note that this is the same as was obtained earlier, expect it is **valid for all** *a* & *b*, i.e. we can use the Laplace transforms to solve ODEs of LTI systems, using the system's impulse response

$$h(t) \overset{L}{\leftrightarrow} H(s)$$

Example 2: Sinusoidal Input

Consider the 1st order (possible unstable) system response with input $x(t)$

$$h(t) = e^{-at}u(t)$$

$$x(t) = \cos(\omega_0 t)u(t)$$

Taking Laplace transforms

$$H(s) = \frac{1}{s+a} \qquad \mathrm{Re}\{s\} > -a$$

$$X(s) = \frac{s}{s^2+\omega_0^2} \qquad \mathrm{Re}\{s\} > 0$$

The Laplace transform of the output of the system is therefore

$$Y(s) = \frac{s}{s^2+\omega_0^2}\frac{1}{s+a} \qquad \mathrm{Re}\{s\} > \max\{0,-a\}$$

$$= \left(\frac{1}{a^2+\omega_0^2}\right)\frac{as+\omega_0^2}{s^2+\omega_0^2} + \left(\frac{-a}{a^2+\omega_0^2}\right)\frac{1}{s+a}$$

and the inverse Laplace transform is

$$y(t) = \frac{u(t)}{a^2+\omega_0^2}\left(a\sin(\omega_0 t) + \omega_0\cos(\omega_0 t) - ae^{-at}\right)$$

PROPERTIES OF THE LAPLACE TRANSFORM

Property	Signal	Laplace Transform	ROC
	$x(t)$	$X(s)$	R
	$x_1(t)$	$X_1(s)$	R_1
	$x_2(t)$	$X_2(s)$	R_2
Linearity	$ax_1(t) + bx_2(t)$	$aX_1(s) + bX_2(s)$	At least $R_1 \cap R_2$
Time shifting	$x(t - t_0)$	$e^{-st_0}X(s)$	R
Shifting in the s-Domain	$e^{s_0 t}x(t)$	$X(s - s_0)$	Shifted version of R (i.e., s is in the ROC if $s - s_0$ is in R)
Time scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{s}{a}\right)$	Scaled ROC (i.e., s is in the ROC if s/a is in R)
Conjugation	$x^*(t)$	$X^*(s^*)$	R
Convolution	$x_1(t) * x_2(t)$	$X_1(s)X_2(s)$	At least $R_1 \cap R_2$
Differentiation in the Time Domain	$\frac{d}{dt}x(t)$	$sX(s)$	At least R
Differentiation in the s-Domain	$-tx(t)$	$\frac{d}{ds}X(s)$	R
Integration in the Time Domain	$\int_{-\infty}^{t} x(\tau)d(\tau)$	$\frac{1}{s}X(s)$	At least $R \cap \{\mathcal{R}e\{s\} > 0\}$

PROPERTIES OF THE LAPLACE TRANSFORM

Initial- and Final-Value Theorems

9.5.10 If $x(t) = 0$ for $t < 0$ and $x(t)$ contains no impulses or higher-order singularities at $t = 0$, then

$$x(0^+) = \lim_{s \to \infty} sX(s)$$

If $x(t) = 0$ for $t < 0$ and $x(t)$ has a finite limit as $t \longrightarrow \infty$, then

$$\lim_{t \to \infty} x(t) = \lim_{s \to \infty} sX(s)$$

SOME LAPLACE TRANSFORM PAIRS

Signal	Transform	ROC
$\delta(t)$	1	All s
$u(t)$	$\frac{1}{s}$	$\Re e\{s\} > 0$
$-u(-t)$	$\frac{1}{s}$	$\Re e\{s\} < 0$
$\frac{t^{n-1}}{(n-1)!}u(t)$	$\frac{1}{s^n}$	$\Re e\{s\} > 0$
$-\frac{t^{n-1}}{(n-1)!}u(-t)$	$\frac{1}{s^n}$	$\Re e\{s\} < 0$
$e^{-\alpha t}u(t)$	$\frac{1}{s+\alpha}$	$\Re e\{s\} > -\alpha$
$-e^{-\alpha t}u(-t)$	$\frac{1}{s+\alpha}$	$\Re e\{s\} < -\alpha$
$\frac{t^{n-1}}{(n-1)!}e^{-\alpha t}u(t)$	$\frac{1}{(s+\alpha)^n}$	$\Re e\{s\} > -\alpha$
$-\frac{t^{n-1}}{(n-1)!}e^{-\alpha t}u(-t)$	$\frac{1}{(s+\alpha)^n}$	$\Re e\{s\} < -\alpha$

SOME LAPLACE TRANSFORM PAIRS

$\delta(t-T)$	e^{-sT}	All s
$[\cos\omega_0 t]u(t)$	$\dfrac{s}{s^2+\omega_0^2}$	$\Re e\{s\} > 0$
$[\sin\omega_0 t]u(t)$	$\dfrac{\omega_0}{s^2+\omega_0^2}$	$\Re e\{s\} > 0$
$[e^{-\alpha t}\cos\omega_0 t]u(t)$	$\dfrac{s+\alpha}{(s+\alpha)^2+\omega_0^2}$	$\Re e\{s\} > -\alpha$
$[e^{-\alpha t}\sin\omega_0 t]u(t)$	$\dfrac{\omega_0}{(s+\alpha)^2+\omega_0^2}$	$\Re e\{s\} > -\alpha$
$u_n(t) = \dfrac{d^n\delta(t)}{dt^n}$	s^n	All s
$u_{-n}(t) = \underbrace{u(t) * \cdots * u(t)}_{n \text{ times}}$	$\dfrac{1}{s^n}$	$\Re e\{s\} > 0$