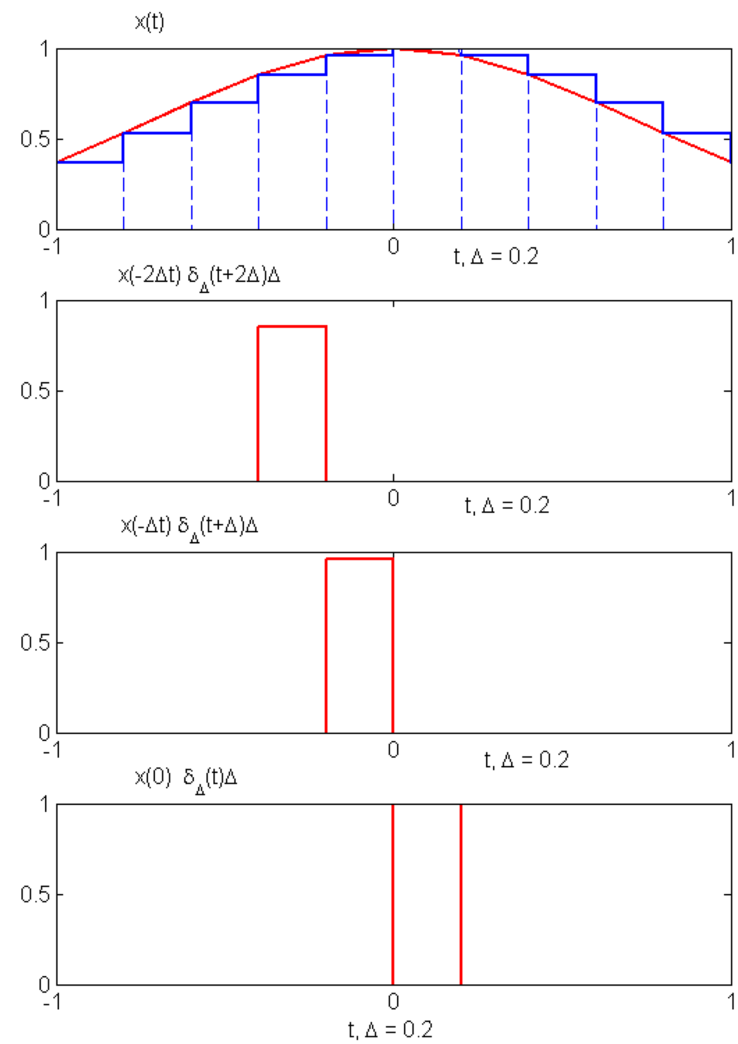
Introduction to Signals and Systems: V216

Lecture #5

Chapter 2: Linear Time-Invariant Systems

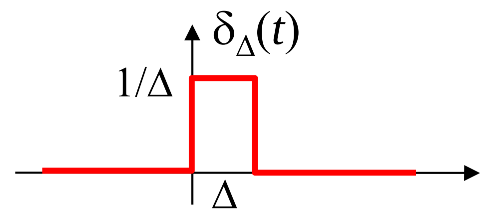
Continuous Convolution

- In this lecture, we're going to understand how the convolution theory can be applied to continuous systems. This is probably most easily introduced by considering the relationship between discrete and continuous systems.
- The convolution sum for discrete systems was based on the **shifting** principle, the input signal can be represented as a superposition (linear combination) of scaled and shifted impulse functions.
- This can be generalised to continuous signals, by thinking of it as the limiting case of arbitrarily thin pulses



Signal "Staircase" Approximation

- As previously shown, any continuous signal can be approximated by a linear combination of thin, delayed pulses:



$$\delta_\Delta(t) = \begin{cases} \frac{1}{\Delta} & 0 \le t < \Delta \\ 0 & \text{otherwise} \end{cases}$$

- Note that this pulse (rectangle) has a unit integral. Then we have:

$$\hat{x}(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)\delta_\Delta(t-k\Delta)\Delta$$

- Only one pulse is non-zero for any value of *t*. Then as $\Delta \to 0$

$$x(t) = \lim_{\Delta \to 0} \sum_{k=-\infty}^{\infty} x(k\Delta)\delta_\Delta(t-k\Delta)\Delta$$

- When $\Delta \to 0$, the summation approaches an integral

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$$

- This is known as the **shifting property** of the continuous-time impulse and there are an infinite number of such impulses $\delta(t-\tau)$

Continuous Time Convolution

- Given that the input signal can be approximated by a sum of scaled, shifted version of the pulse signal, $\delta_\Delta(t\text{-}k\Delta)$

$$\hat{x}(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)\delta_\Delta(t - k\Delta)\Delta$$

- The linear system's output signal $\hat{y}$ is the **superposition** of the responses, $\hat{h}_{k\Delta}(t)$, which is the system response to $\delta_\Delta(t\text{-}k\Delta)$.
- From the discrete-time **convolution**:

$$\hat{y}(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)\hat{h}_{k\Delta}(t)\Delta$$

- What remains is to consider as $\Delta \to 0$. In this case:

$$y(t) = \lim_{\Delta\to 0} \sum_{k=-\infty}^{\infty} x(k\Delta)\hat{h}_{k\Delta}(t)\Delta$$

$$= \int_{-\infty}^{\infty} x(\tau)h_\tau(t)d\tau$$

Example: Discrete to Continuous Time Linear Convolution

- The CT input signal (red) $x(t)$ is approximated (blue) by:

$$\hat{x}(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)\delta_{\Delta}(t-k\Delta)\Delta$$

- Each pulse signal

$$\delta_{\Delta}(t-k\Delta)$$

- generates a response

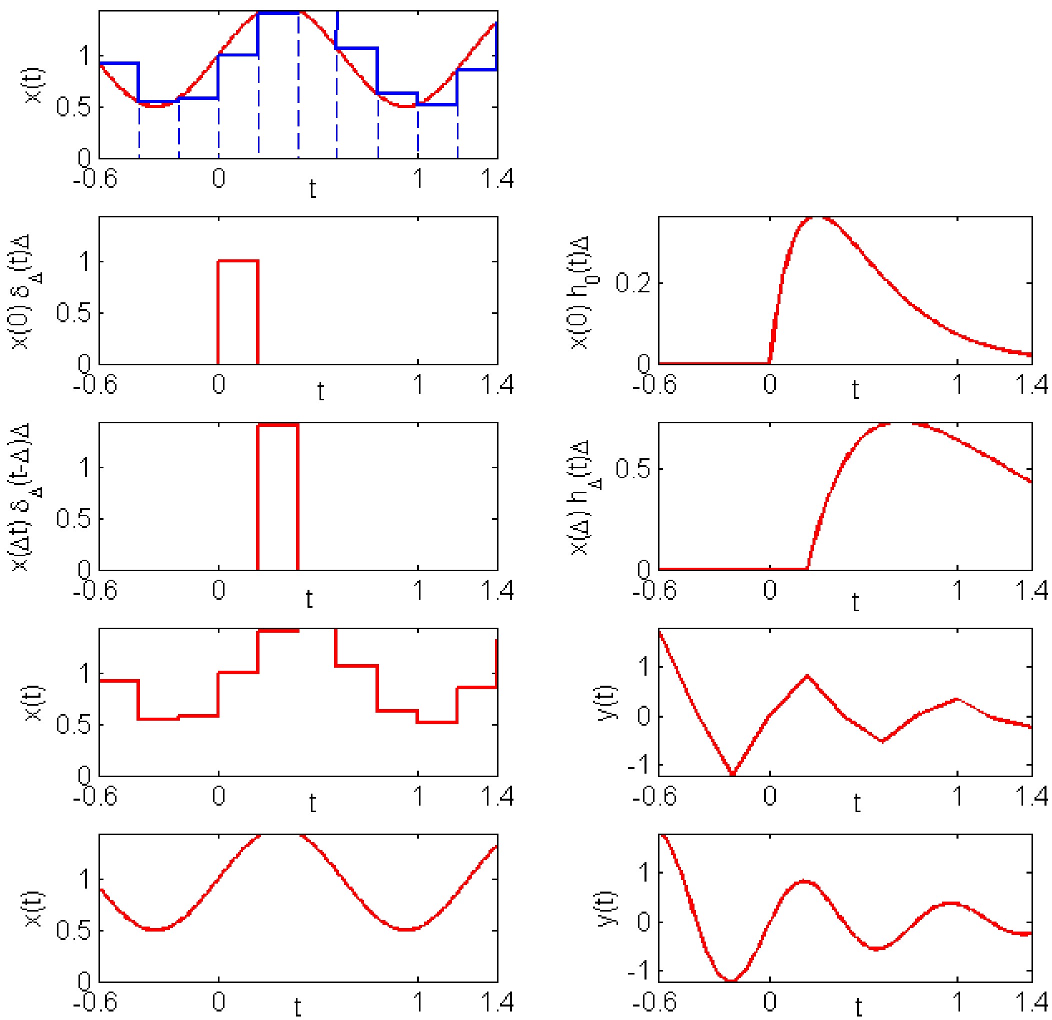
$$\hat{h}_{k\Delta}(t)$$

- Therefore the DT convolution response is

$$\hat{y}(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)\,\hat{h}_{k\Delta}(t)\Delta$$

- Which approximates the CT convolution response

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h_{\tau}(t)d\tau$$



Linear Time Invariant Convolution

- For a linear, time invariant system, all the impulse responses are simply time shifted versions:

$$h_\tau(t) = h(t-\tau)$$

- Therefore, convolution for an LTI system is defined by:

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

- This is known as the **convolution integral** or the **superposition integral**
- Algebraically, it can be written as:

$$y(t) = x(t) * h(t)$$

- To evaluate the integral for a specific value of *t*, obtain the signal $h(t\text{-}\tau)$ and multiply it with $x(\tau)$ and the value $y(t)$ is obtained by integrating over τ from $-\infty$ to ∞.
- Demonstrated in the following examples

Example 1: CT Convolution

- Let $x(t)$ be the input to a LTI system with unit impulse response $h(t)$:

$$x(t) = e^{-at}u(t) \qquad a > 0$$

$$h(t) = u(t)$$

- For t>0:

$$x(\tau)h(t-\tau) = \begin{cases} e^{-a\tau} & 0 < \tau < t \\ 0 & \text{otherwise} \end{cases}$$

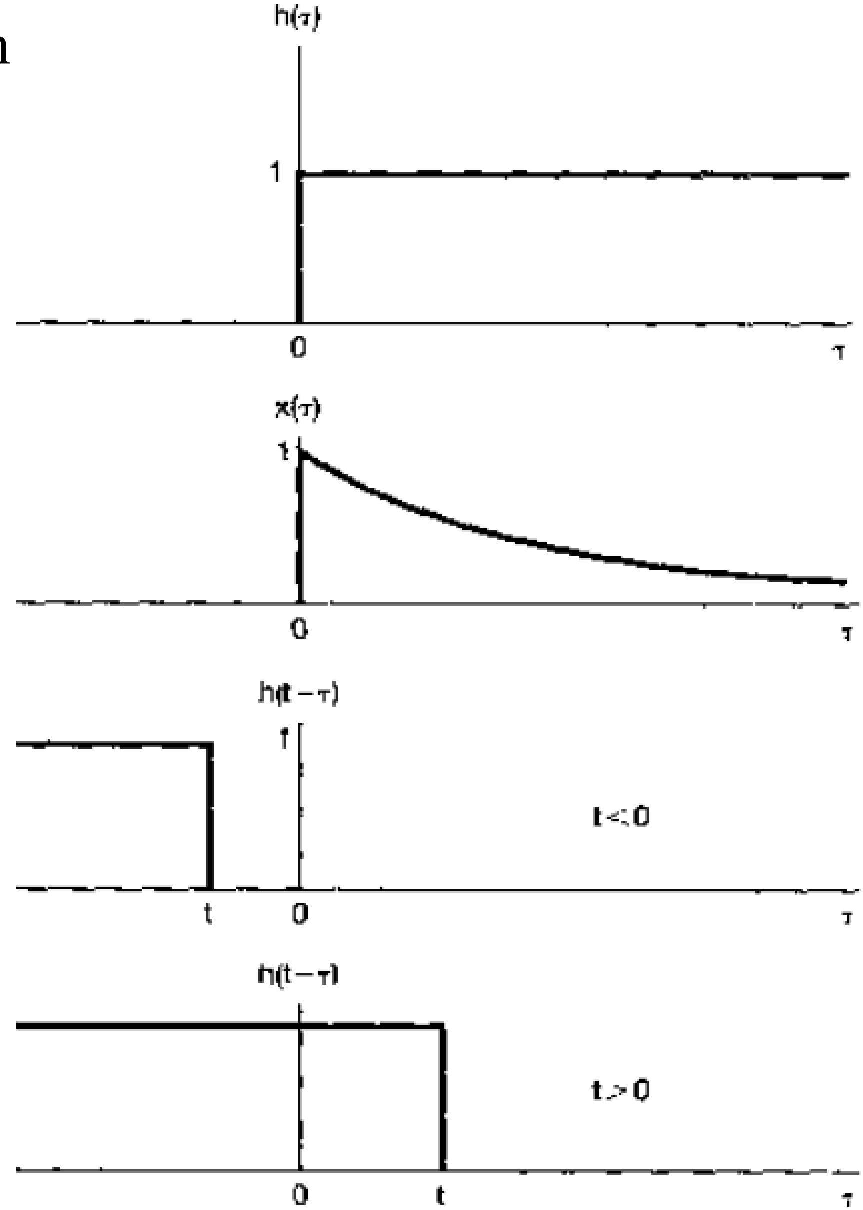
- We can compute $y(t)$ for t>0:

$$y(t) = \int_0^t e^{-a\tau}\,d\tau = -\tfrac{1}{a}e^{-a\tau}\Big|_0^t$$

$$= \tfrac{1}{a}\left(1 - e^{-at}\right)$$

- So for all t:

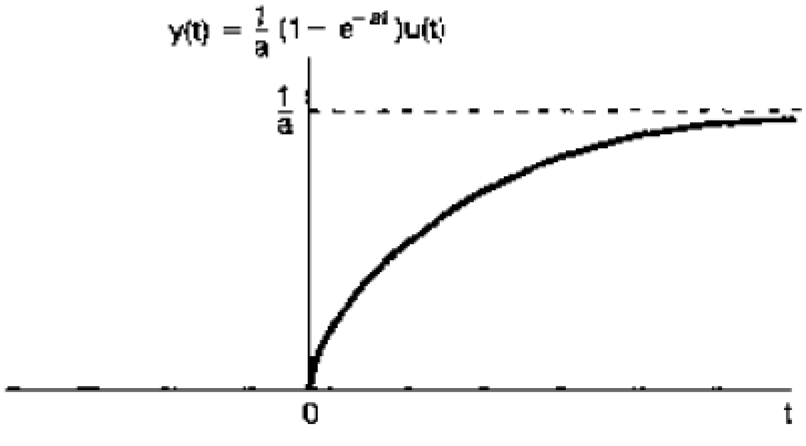
$$y(t) = \tfrac{1}{a}\left(1 - e^{-at}\right)u(t)$$

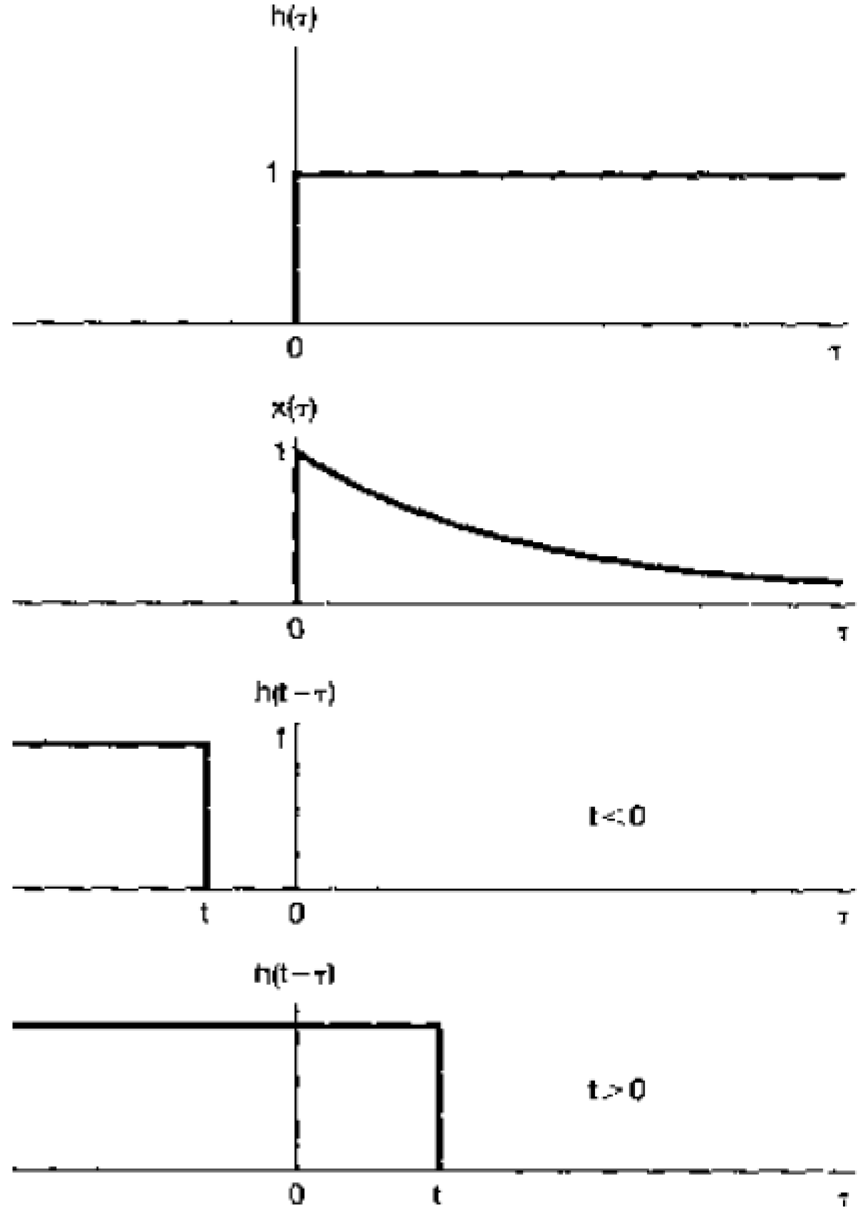


Example 1: CT Convolution

$$y(t) = \int_0^t e^{-a\tau} d\tau = -\tfrac{1}{a} e^{-a\tau}\Big|_0^t$$
$$= \tfrac{1}{a}\left(1 - e^{-at}\right)$$

$$y(t) = \tfrac{1}{a}\left(1 - e^{-at}\right)u(t)$$



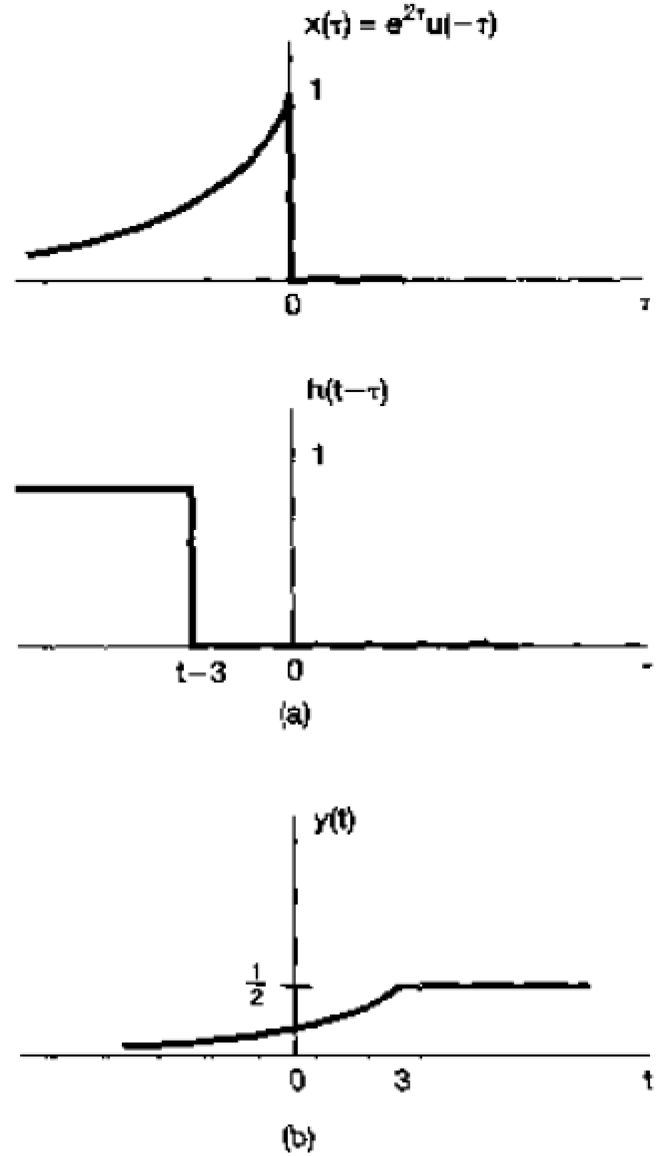


Example 2: CT Convolution

- Calculate the convolution of the following signals
$$x(t) = e^{2t}u(-t)$$
$$h(t) = u(t-3)$$

- The convolution integral for t-3≤ 0, the becomes:
$$y(t) = \int_{-\infty}^{t-3} e^{2\tau} d\tau = \tfrac{1}{2} e^{2(t-3)}$$

- For t-3≥0, the product $x(\tau)h(t-\tau)$ is non-zero for $-\infty<\tau<0$, so the convolution integral becomes:
$$y(t) = \int_{-\infty}^{0} e^{2\tau} d\tau = \tfrac{1}{2}$$



Unit Step Response of Continuous-time LTI System

Similarly, **unit step response** is the running integral of its impulse response.

$$y(t) = \int_{-\infty}^{t} h(\tau) d\tau,$$

The unit **impulse response** is the first derivative of the unit step response:-

$$h(t) = \frac{dy(t)}{dt} = y'(t).$$

Differential and Difference Equations

- Two extremely important classes of **causal LTI** systems:
- 1) CT systems whose input-output response is described by **linear, constant-coefficient, ordinary differential equations** with a forced response

$$\frac{dy(t)}{dt} + ay(t) = bx(t)$$

RC circuit with: $y(t) = v_c(t)$, $x(t) = v_s(t)$, $a = b = 1/RC$.

- 2) DT systems whose input-output response is described by **linear, constant-coefficient, difference equations**
- $$y[n] + ay[n-1] = bx[n]$$

Simple bank account with: $a = -1.01$, $b = 1$.

- Note that to "solve" these equations for $y(t)$ or $y[n]$, we need to know the **initial conditions**
- Examine such systems and relate them to the system properties just described

Continuous-Time Differential Equations

- A general N^{th}-order LTI differential equation is

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

- If the equation involves derivative operators on $y(t)$ ($N>0$) or $x(t)$, it has memory.

- The system stability depends on the coefficients a_k. For example, a 1st order LTI differential equation with $a_0=1$:

$$\frac{dy(t)}{dt} - a_1 y(t) = 0 \qquad y(t) = Ae^{a_1 t}$$

- If $a_1>0$, the system is unstable as its impulse response represents a growing exponential function of time
- If $a_1<0$ the system is stable as its impulse response corresponds to a decaying exponential function of time

Discrete-Time Difference Equations

- A general N^{th}-order LTI difference equation is
$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$
- If the equation involves difference operators on $y[n]$ (N>0) or $x[n]$, it has memory.
- The system stability depends on the coefficients a_k. For example, a 1st order LTI difference equation with a_0=1:
$$y[n] - a_1 y[n-1] = 0 \quad y[n] = A a_1^n$$
- If a_1>1, the system is unstable as its impulse response represents a growing power function of time
- If a_1<1 the system is stable as its impulse response corresponds to a decaying power function of time

Causal LTI Systems Described By Differential & Difference Equations

For continuous - time LTI systems, the output and the input are related through the differential equations :-

e.g. for a first order DE, $\frac{dy(t)}{dt} + 2y(t) = x(t)$,

where y(t) denotes the output and x(t) the input.

To solve these DEs, we need the initial conditions.

More generally, to solve DE, we must specify one or more intial conditions.

Example 2.14

Consider the input signal as $\mathrm{x(t)} = \mathrm{K}e^{3t}u(t)$,

The complete solution to $\dfrac{dy(t)}{dt} + 2y(t) = x(t)$,

consists of the sum of a particular solution $\mathrm{y_p}(t)$

and a homogeneous solution, $\mathrm{y_h}(t)$ i.e.

$\mathrm{y(t)} = \mathrm{y_p}(t) + \mathrm{y_h}(t)$,

Homogeneous solution is often refer to as the natural response of the system, i.e. a solution where the input is constraint to be zero.

Step 1 : - Particular solution.

Look for a forced response.

i.e. a signal of the same form as input : - $y_p(t) = Ye^{3t}$ for $t > 0$

Subsituting x(t) and y(t) into the DE we have : -

$$3Ye^{3t} + 2Ye^{3t} = Ke^{3t},$$

$$3Y + 2Y = K, \quad Y = \frac{K}{5}, \; y_p(t) = \frac{K}{5}e^{3t}, \; for \; t > 0.$$

Step 2. Homogeneous solution.

Letting $x(t) = 0$ and hypothesising a solution of the form $y_h(t) = Ae^{st}$.

$$Ase^{st} + 2Ae^{st} = Ae^{st}(s+2) = 0 \; i.e. \; s = -2.$$

The complete solution is $y(t) = Ae^{-2t} + \frac{K}{5}e^{3t}$, for $t > 0$.

For causal and LTI systems

the auxiliary condition is the initial rest condition.

x(t) = 0 and y(t) = 0 when t < 0

$$\therefore 0 = A + \frac{K}{5}, \text{ or } A = -\frac{K}{5},$$

$$\text{Thus for } t > 0, \ y(t) = \frac{K}{5}[e^{3t} - e^{-2t}],$$

while for t < 0, y(t) = 0,

$$\text{i.e. } y(t) = \frac{K}{5}[e^{3t} - e^{-2t}]u(t)$$

Example 2.14 with impulse input
(Problem 2.56 (a) Pg 158-159)

Consider the input signal as $x(t) = \delta(t)$,

The complete solution to $\dfrac{dy(t)}{dt} + 2y(t) = x(t)$,

consists of the sum of a particular solution $y_p(t)$

and a homogeneous solution, $y_h(t)$ i.e.

$y(t) = y_p(t) + y_h(t)$,

Homogeneous solution is often refer to as the natural response of the system, i.e. a solution where the input is constraint to be zero.

Step 1: - Particular solution.

Look for a forced response.

i.e. a signal of the same form as input: - $y_p(t) = Y\delta(t)$ for $t > 0$.

i.e $y_p(t) = 0$. and we can ignore the particular solution.

Step 2. Homogeneous solution. This will give us the system impulse response.

Letting $x(t) = 0$ and hypothesising a solution of the form $y_h(t) = Ae^{st}$.

$Ase^{st} + 2Ae^{st} = Ae^{st}(s+2) = 0$ *i.e.* $s = -2$.

The complete solution is $y(t) = Ae^{-2t}$, for $t > 0$.

But what is A???

For causal and LTI systems

the auxiliary condition is the initial rest condition.

$\mathrm{x(t)} = \delta(t) = 0$ and $\mathrm{y(t)} = 0$ when $\mathrm{t} \leq 0^{-}$

$\mathrm{y}(0^{+}) = 1$ when $\mathrm{t} = 0^{+}$

$\therefore \mathrm{A} = 1,$

Thus for $\mathrm{t} > 0$, $\mathrm{y(t)} = e^{-2t}$,

while for $\mathrm{t} < 0$, $\mathrm{y(t)} = 0$,

i.e. $\mathrm{y(t)} = \mathrm{h(t)}$(Impulse response) $= e^{-2t}u(t)$

General Higher N-order DE

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

Similarly the particular solution, homogeneous solution and the auxiliary conditions (initial rest for Causal LTI) will give us the complete solution to these higher order DEs.

Linear Constant-Coefficient Difference equations

The discrete - time counter part of DE, $\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$,

is $\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$,

Similarly the complete solution for y[n] can be written as the sum of a particular solution and the homogeneous solution

$$\sum_{k=0}^{N} a_k y[n-k] = 0.$$

with the auxiliary conditions (initial for Causal LTI systems).

Linear Constant-Coefficient Difference equations

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k],$$

can be written as $\mathrm{y[n]} = \frac{1}{a_0}\{\sum_{k=0}^{M} b_k x[n-k] - \sum_{k=1}^{N} a_k y[n-k]\},$

This equation directly expresses the output at time n in terms of previous values of the input and output. This is a recursive equation.

Special case when N = 0, we have the nonrecursive equation : -

$\mathrm{y[n]} = \sum_{k=0}^{M} \frac{b_k}{a_0} x[n-k],$ this is the convolution sum.

The impulse response of this system is when $\mathrm{x[n]} = \delta[\mathrm{n}]$

i.e. $\mathrm{y[n]} = \mathrm{h[n]} = \sum_{k=0}^{M} \frac{b_k}{a_0} \delta[n-k] = \frac{b_n}{a_0}, 0 \le n \le M, h[n] = 0$ otherwise.

This is often called a finite impulse response (FIR) system.

Example 2.15 : - $y[n] - \frac{1}{2}y[n-1] = x[n],\ y[n] = x[n] + \frac{1}{2}y[n-1]$,

we need previous value of output to get at the present output.

Consider input $x[n] = K\delta[n]$, and initial rest condition $y[n] = 0$ for $n \le 0$, we have $y[-1] = 0$,

$$\therefore\ y[0] = x[0] + \frac{1}{2}y[-1] = K,$$

$$y[1] = x[1] + \frac{1}{2}y[0] = \frac{1}{2}K,$$

$$y[2] = x[2] + \frac{1}{2}y[1] = (\frac{1}{2})^2 K,$$

$$y[n] = x[n] + \frac{1}{2}y[n-1] = (\frac{1}{2})^n K,$$

taking $K = 1$, we have the impulse response as

$h[n] = (\frac{1}{2})^n u[n]$. which is infinite. Such systems are commonly referred as infinite inpulse response (IIR) systems.

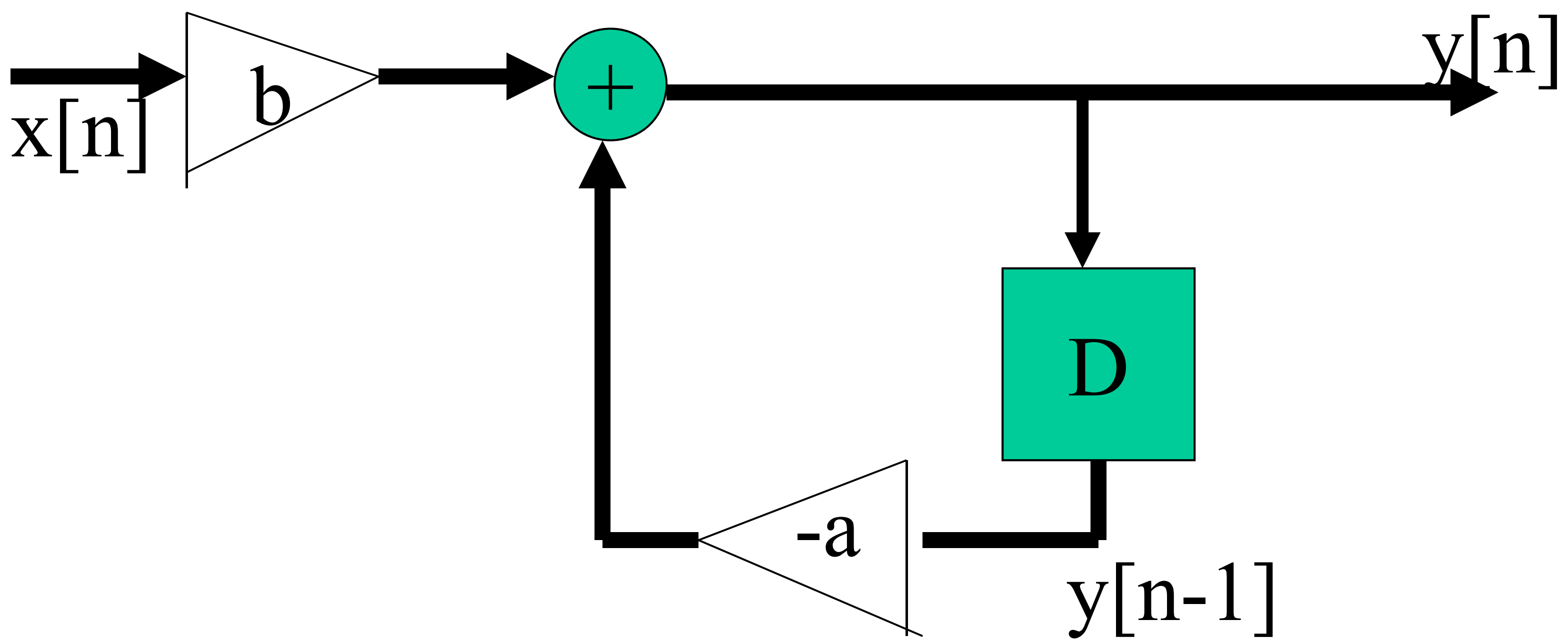
Block Diagram Representations of First- Order Systems.

- Provides a pictorial representation which can add to our understanding of the behavior and properties of these systems.
- Simulation or implementation of the systems.
- Basis for analog computer simulation of systems described by DE.
- Digital simulation & Digital Hardware implementations

First-Order Recursive Discrete-time System.

$y[n]+ay[n-1]=bx[n]$

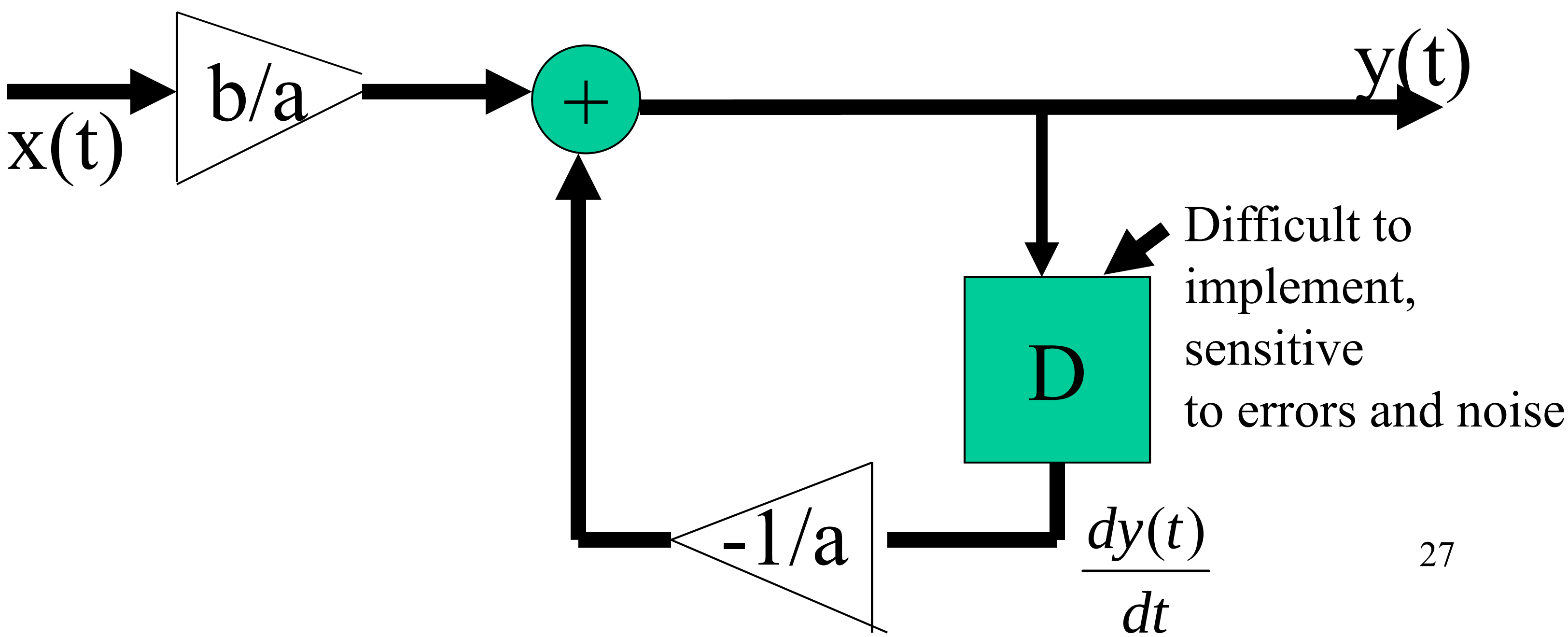
$y[n]=-ay[n-1]+bx[n]$



First-Order Continuous-time System Described By Differential Equation

$$\frac{dy(t)}{dt} + ay(t) = bx(t)$$

$$\text{rewriting } \mathrm{y(t)} = -\frac{1}{\mathrm{a}}\frac{dy(t)}{dt} + \frac{b}{a}x(t)$$

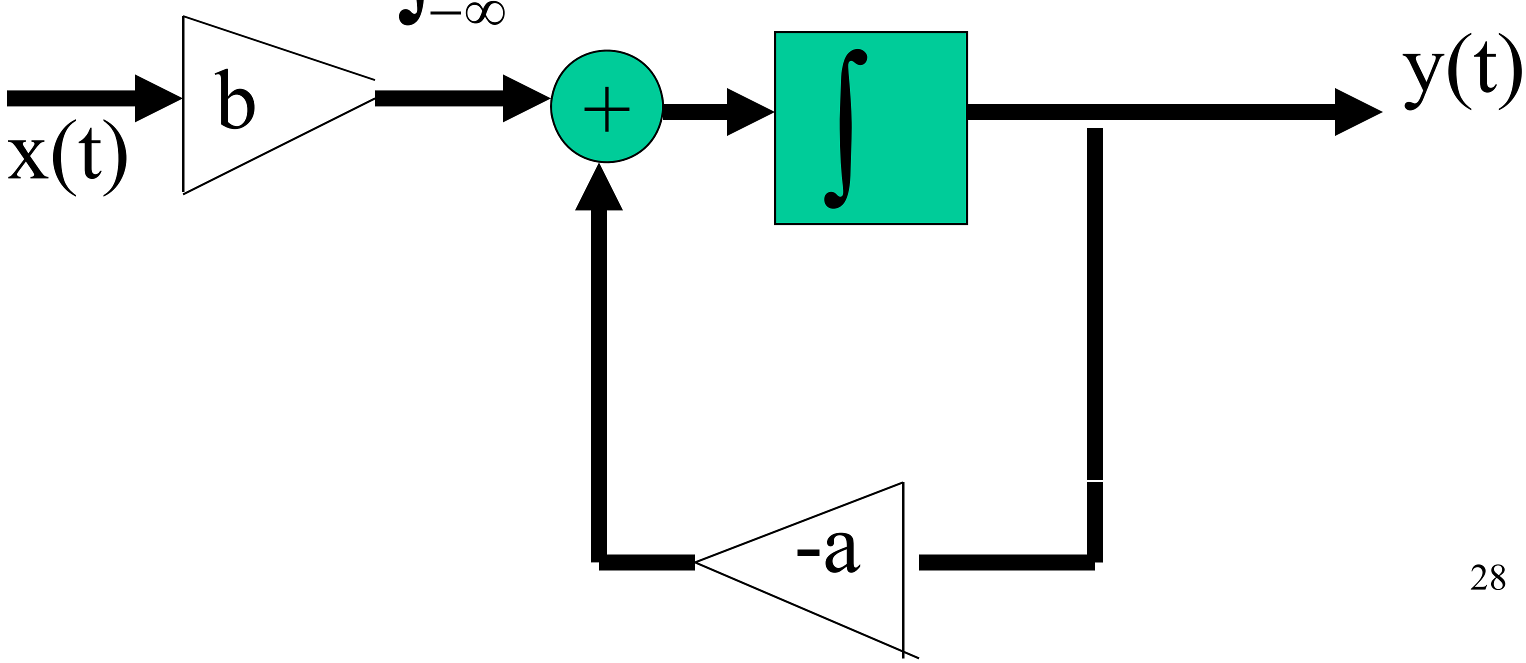


First-Order Continuous-time System Described By Differential Equation Alternative Block Diagram.

$$\frac{dy(t)}{dt} = bx(t) - ay(t)$$

Integrating from - ∞ to t,

$$\mathrm{y(t)} = \int_{-\infty}^{t} | bx(\tau) - ay(\tau) | \, d\tau$$



The Operation Definition of the Unit Impulse

$\delta(t)$ — idealization of a unit-area pulse that is so short that, for any physical systems of interest to us, the system responds only to the area of the pulse and is insensitive to its duration

Operationally: The unit impulse is the signal which when applied to any LTI system results in an output equal to the impulse response of the system. That is,

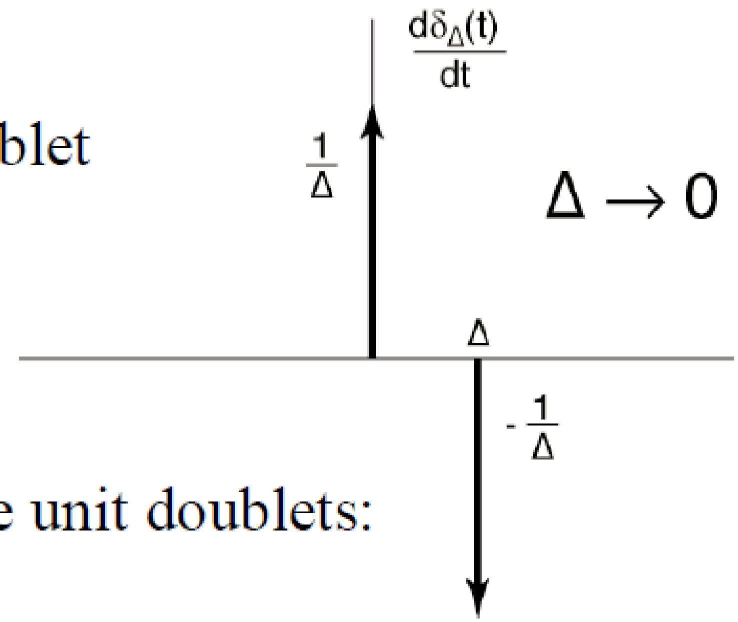
$$\delta(t) * h(t) = h(t) \qquad \text{for all} \quad h(t)$$

The Unit Doublet — Differentiator

$$x(t) \longrightarrow \boxed{\frac{d}{dt}} \longrightarrow y(t) = \frac{dx(t)}{dt}$$

Impulse response = unit doublet

$$u_1(t) = \frac{d\delta(t)}{dt}$$



The operational definitions of the unit doublets:

$$x(t) * u_1(t) = \frac{dx(t)}{dt}$$

Triplets and beyond!

$n > 0$

$$u_n(t) = \underbrace{u_1(t) * \cdots * u_1(t)}_{n\ times}$$

Operational definitions:

$$x(t) * u_n(t) = \frac{d^n x(t)}{dt^n} \qquad (n > 0)$$

Integrators

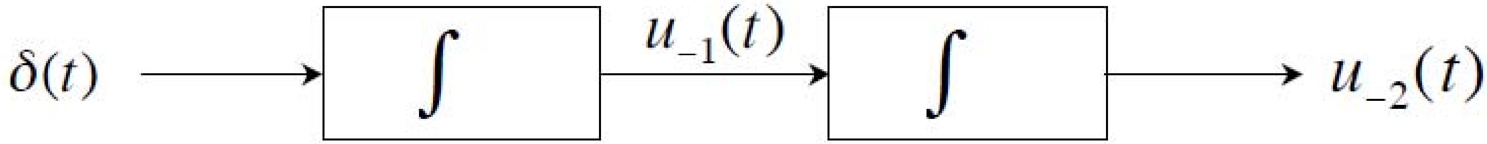
$$x(t) \longrightarrow \boxed{\int} \longrightarrow y(t) = \int_{-\infty}^{t} x(\tau)\,d\tau$$

Impulse response: $\quad u_{-1}(t) \equiv u(t)$

Operational definition: $\quad x(t) * u_{-1}(t) = \int_{-\infty}^{t} x(\tau)\,d\tau$

Cascade of n integrators:

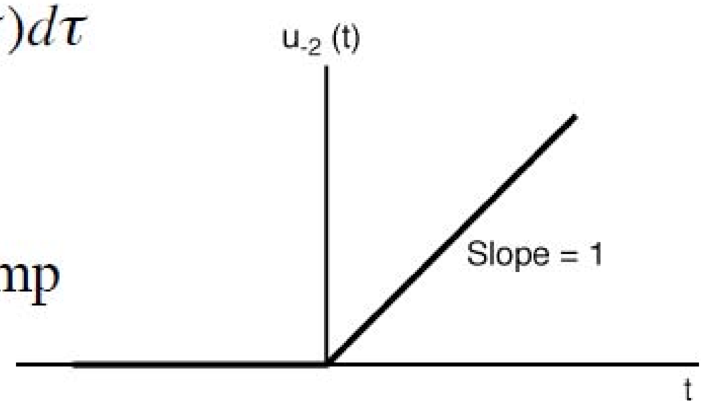
$$u_{-n}(t) = \underbrace{u_{-1}(t) * \cdots * u_{-1}(t)}_{n\ times} \qquad (n > 0)$$



$$u_{-2}(t) = \int_{-\infty}^{t} u_{-1}(\tau)d\tau = \int_{-\infty}^{t} u(\tau)d\tau$$

$$= u(t)\int_{0}^{t} d\tau$$

$$= t \cdot u(t) \qquad \text{the unit ramp}$$



More generally, for $n > 0$

$$u_{-n}(t) = \frac{t^{(n-1)}}{(n-1)!}u(t)$$

Notation

Define $$u_0(t) = \delta(t)$$

Then $$u_n(t) * u_m(t) = u_{n+m}(t)$$

n and m can be $\pm$

E.g. $$u_1(t) * u_{-1}(t) = u_0(t)$$

$$||$$

$$\left(\frac{d}{dt}u(t)\right) = \delta(t)$$