VE216 RC7

Chapter 10 & 11

Some of contents are from professor's lecture slides.

The z-Transform

The (direct) z-transform of a discrete-time signal $x(n)$ is defined as the power series

$$X(z) \triangleq \sum_{n=-\infty}^{\infty} x[n] z^{-n} \qquad (3.1.1)$$

The inverse procedure is called the *inverse* *z-transform*. The inverse z-transform of $X(z)$ is

$$x(n) = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz$$

The z-Transform

We express the relationship with notation

$$x(n) \overset{z}{\leftrightarrow} X(z)$$

Since the z-transform is an infinite series power series, it exists only for those values of z for which this series convergences. The region of convergence (ROC) of $X(z)$ is the set of all values of z for which $X(z)$ attains a finite value. Thus any time we cite a z-transform we should also indicate its ROC.

The z-Transform

Example 3.1.1

$\left\{\underset{\uparrow}{1},2,5,7,0,1\right\} \overset{z}{\leftrightarrow} 1+2z^{-1}+5z^{-2}+7z^{-3}+z^{-5}$, ROC: entire z-plane except $z=0$

$\left\{1,2,5,7,0,\underset{\uparrow}{1}\right\} \overset{z}{\leftrightarrow} z^{5}+2z^{4}+5z^{3}+7z^{2}+1$, ROC: entire z-plane except $z=\infty$

$\left\{1,2,\underset{\uparrow}{5},7,0,1\right\} \overset{z}{\leftrightarrow} z^{2}+2z+5+7z^{-1}+z^{-3}$, ROC: entire z-plane except $z=0$ and $z=\infty$

The z-Transform

TABLE 10.1 PROPERTIES OF THE z-TRANSFORM

Section	Property	Signal	z-Transform	ROC
		$x[n]$	$X(z)$	R
		$x_1[n]$	$X_1(z)$	R_1
		$x_2[n]$	$X_2(z)$	R_2
10.5.1	Linearity	$ax_1[n] + bx_2[n]$	$aX_1(z) + bX_2(z)$	At least the intersection of R_1 and R_2
10.5.2	Time shifting	$x[n - n_0]$	$z^{-n_0}X(z)$	R, except for the possible addition or deletion of the origin
10.5.3	Scaling in the z-domain	$e^{j\omega_0 n}x[n]$	$X(e^{-j\omega_0}z)$	R
		$z_0^n x[n]$	$X\left(\frac{z}{z_0}\right)$	$z_0 R$
		$a^n x[n]$	$X(a^{-1}z)$	Scaled version of R (i.e., $\|a\|R$ = the set of points $\{\|a\|z\}$ for z in R)
10.5.4	Time reversal	$x[-n]$	$X(z^{-1})$	Inverted R (i.e., R^{-1} = the set of points z^{-1}, where z is in R)
10.5.5	Time expansion	$x_{(k)}[n] = \begin{cases} x[r], & n = rk \\ 0, & n \neq rk \end{cases}$ for some integer r	$X(z^k)$	$R^{1/k}$ (i.e., the set of points $z^{1/k}$, where z is in R)

The z-Transform

10.5.6	Conjugation	$x^*[n]$	$X^*(z^*)$	R
10.5.7	Convolution	$x_1[n] * x_2[n]$	$X_1(z)X_2(z)$	At least the intersection of R_1 and R_2
10.5.7	First difference	$x[n] - x[n-1]$	$(1 - z^{-1})X(z)$	At least the intersection of R and $\|z\| > 0$
10.5.7	Accumulation	$\sum_{k=-\infty}^{n} x[k]$	$\frac{1}{1 - z^{-1}} X(z)$	At least the intersection of R and $\|z\| > 1$
10.5.8	Differentiation in the z-domain	$nx[n]$	$-z\frac{dX(z)}{dz}$	R

10.5.9

Initial Value Theorem

If $x[n] = 0$ for $n < 0$, then

$$x[0] = \lim_{z \to \infty} X(z)$$

The z-Transform

TABLE 10.2 SOME COMMON z-TRANSFORM PAIRS

Signal	Transform	ROC
1. $\delta[n]$	1	All z
2. $u[n]$	$\dfrac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
3. $-u[-n-1]$	$\dfrac{1}{1-z^{-1}}$	$\lvert z\rvert < 1$
4. $\delta[n-m]$	z^{-m}	All z, except 0 (if $m > 0$) or ∞ (if $m < 0$)
5. $\alpha^n u[n]$	$\dfrac{1}{1-\alpha z^{-1}}$	$\lvert z\rvert > \lvert\alpha\rvert$
6. $-\alpha^n u[-n-1]$	$\dfrac{1}{1-\alpha z^{-1}}$	$\lvert z\rvert < \lvert\alpha\rvert$
7. $n\alpha^n u[n]$	$\dfrac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$	$\lvert z\rvert > \lvert\alpha\rvert$

The z-Transform

8. $-n\alpha^n u[-n-1]$	$\dfrac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$	$\lvert z\rvert < \lvert\alpha\rvert$
9. $[\cos\omega_0 n]u[n]$	$\dfrac{1-[\cos\omega_0]z^{-1}}{1-[2\cos\omega_0]z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
10. $[\sin\omega_0 n]u[n]$	$\dfrac{[\sin\omega_0]z^{-1}}{1-[2\cos\omega_0]z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
11. $[r^n\cos\omega_0 n]u[n]$	$\dfrac{1-[r\cos\omega_0]z^{-1}}{1-[2r\cos\omega_0]z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
12. $[r^n\sin\omega_0 n]u[n]$	$\dfrac{[r\sin\omega_0]z^{-1}}{1-[2r\cos\omega_0]z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$

Analysis of LTI Systems in the z-Domain

A discrete-time LTI system with rational system function $H(z)$ is causal if and only if:
(1) the ROC is the exterior of a circle outside the outermost pole;
(2) with $H(z)$ expressed as a ratio of polynomials in z, the order of the numerator cannot be greater than the order of the denominator.

An LTI system is stable if and only if:
the ROC of its system function $H(z)$ includes the unit circle, $|z| = 1$.

Example 3.5.2 An LTI system is characterized by

$$H(z)=\frac{3-4z^{-1}}{1-3.5z^{-1}+1.5z^{-2}}=\frac{1}{1-\frac{1}{2}z^{-1}}+\frac{2}{1-3z^{-1}}$$

Specify the ROC of $H(z)$ and determine $h(n)$ for the following conditions:

(a) The system is stable.

(b) The system is causal.

Answer :

(a) ROC: $\frac{1}{2} < z < 3,\ h(n) = (\frac{1}{2})^n u(n) - 2(3)^n u(-n-1)$

(b) ROC: $z > 3,\ h(n) = (\frac{1}{2})^n u(n) + 2(3)^n u(n)$

Example 4.3.7 Determine the causal signal $x(n)$ having the z-transform

$$X(z) = \frac{1}{(1+z^{-1})(1-z^{-1})^2}$$

Answer :

$$X(z)=\frac{\frac{1}{4}}{1+z^{-1}}+\frac{\frac{3}{4}}{1-z^{-1}}+\frac{\frac{1}{2}z^{-1}}{(1-z^{-1})^{2}}$$

$$x(n)=\frac{1}{4}(-1)^{n}u(n)+\frac{3}{4}u(n)+\frac{1}{2}nu(n)$$

Linear Feedback System

A basic feedback system consists of three components connected together to form a single feedback loop:

- A plant, which acts on an error signal e(t) to produce the output signal $y(t)$;
- A sensor, which measures the output signal $y(t)$ to produce a feedback signal $r(t)$;
- A comparator, which calculates the difference between the externally applied input (reference) signal and the feedback signal $r(t)$ to produce the error signal $e(t)$.

Linear Feedback System

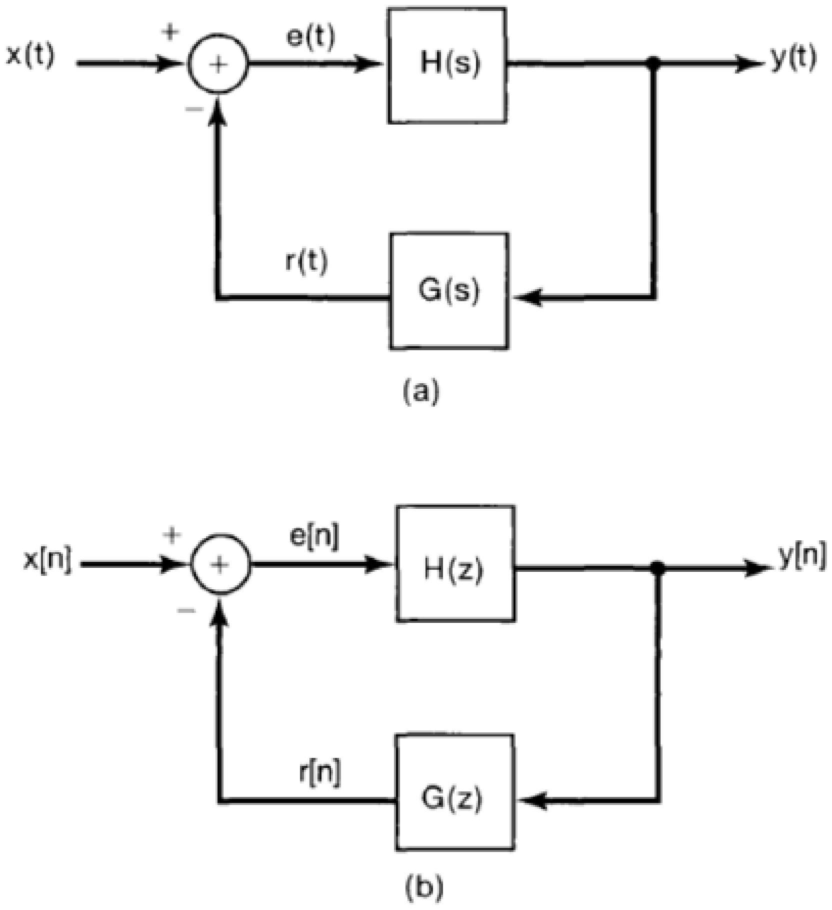


Figure 11.3 Basic feedback system configurations in (a) continuous time and (b) discrete time.

Linear Feedback System

The system function $H(s)$ in Fig. 11.3(a) is referred to as the *system function of the forward path* and $G(s)$ as the *system function of the feedback path*. The system function of the overall system of Fig. 11.3(a) is referred to as the *closed - loop system function* and will be denoted by $Q(s)$. From Fig. 11.3 (a), we have

$$Y(s) = H(s)E(s)$$

$$E(s) = X(s) - R(s)$$

$$R(s) = G(s)Y(s)$$

Linear Feedback System

and

$$Q(s) = \frac{Y(s)}{X(s)} = \frac{H(s)}{1 + G(s)H(s)} \qquad (11.1)$$

Equation (11.1) represents the fundamental equation for the study of LTI feedback systems. In the following sections, we use this equation as the basis for gaining insight into the properties of feedback systems and for developing several tools for their analysis.

Linear Feedback System

Design of linear feedback system:

It's all about arranging the location of the zeros and poles such that the system will meet prescribed specifications.

Linear Feedback System

Let $G(s)$ be the transfer function,

$\boxed{\text{BIBO stable} \Rightarrow \text{poles of } G(s) \text{ cannot be in RHP or on the } j\omega\text{-axis}}$

Proof. We have that

$$G(s) = \mathcal{L}[g(t)] = \int_0^\infty g(t)e^{-st}dt,$$

then

$$|G(s)| = \left|\int_0^\infty g(t)e^{-st}dt\right| \leq \int_0^\infty |g(t)| \cdot |e^{-st}|dt$$

Since $s = \sigma + j\omega$ and $|e^{-st}| = |e^{-\sigma t}|$, we get

$$|G(s)| \leq \int_0^\infty |g(t)| \cdot |e^{-\sigma t}|dt.$$

Some of contents are from Prof. Zhang Jun's lecture slides.

Linear Feedback System

When s assumes a pole of $G(s)$, say $s_1 = \sigma_1 + j\omega_1$, $G(s_1) = \infty$,

$$\Rightarrow \quad \infty \leq \int_0^\infty |g(t)| \cdot |e^{-\sigma_1 t}| dt. \tag{1}$$

If s_1 is in the RHP or on the $j\omega$-axis, $\sigma_1 \geq 0$, then $|e^{-\sigma_1 t}| \leq 1$, thus Eq. (1) becomes

$$\infty \leq \int_0^\infty |g(t)| dt < \infty.$$

The 2nd inequality is from the assumption that the system is BIBO stable. Contradiction! $\square$

Root Locus

$$\frac{Y(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

Characteristic equation:

$$1 + G(s)H(s) = 0$$

Suppose that

$$G(s)H(s) = K\frac{Q(s)}{P(s)},$$

where $Q(s)$ and $P(s)$ are polynomials:

$$Q(s) = s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0,$$
$$P(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0,$$

then

$$1 + G(s)H(s) = 1 + K\frac{Q(s)}{P(s)} = 0.$$

Root Locus

The graph of all possible roots as K changes.

Let $L(s) = \frac{Q(s)}{P(s)}$, we have the following equivalent forms:

- $P(s) + KQ(s) = 0$;
- $1 + KL(s) = 0$;
- $L(s) = -\frac{1}{K}$.

Root Locus

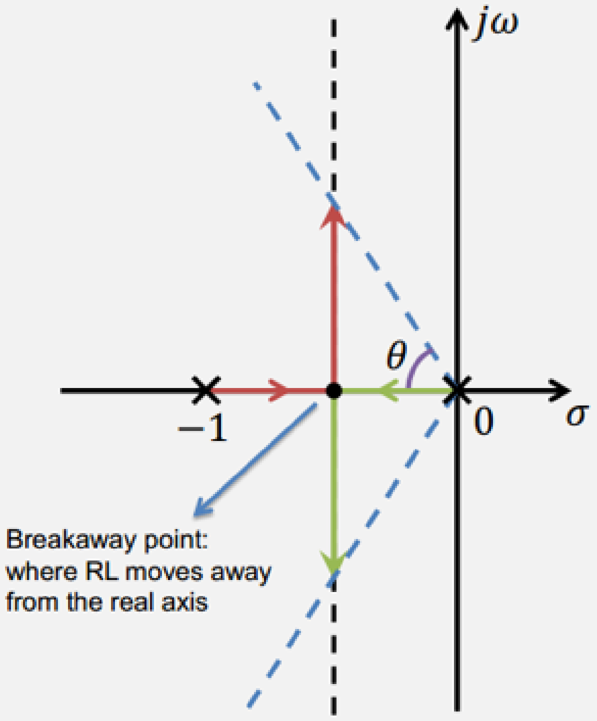
Example (Motor Position Control: Design on Feedback Gain)

Consider $L(s) = \dfrac{1}{s(s+1)}$, $1 + KL(s) = 0$,

$$1 + \frac{K}{s(s+1)} = 0, \quad s^2 + s + K = 0$$

$$\Rightarrow \quad s_{1,2} = \frac{-1 \pm \sqrt{1-4K}}{2}.$$

Root Locus



1. $K = 0$. $s_1 = 0$, $s_2 = -1$.
   $L(s) = -\frac{1}{K} = \infty$,
   $s \longrightarrow$ poles of $L(s)$;
2. $K = \infty$. $s_{1,2} = \infty$.
   $L(s) = -\frac{1}{K} = 0$,
   $s \longrightarrow$ zeros of $L(s)$;
3. $0 \leq K \leq \frac{1}{4}$, two real roots;
4. $K \geq \frac{1}{4}$,
   $s_{1,2} = -\frac{1}{2} \pm \frac{1}{2} j\sqrt{4K-1}$.

Now let's say we want to design K such that $\zeta = \frac{1}{2}$.

$$\cos\theta = \frac{1}{2},\ \theta = \frac{\pi}{3},\ \frac{1}{2}\sqrt{4K-1} = \frac{\sqrt{3}}{2} \ \Rightarrow\ K = 1.$$

Root Locus

$$\frac{Y(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

Characteristic equation:

$$1 + G(s)H(s) = 0$$

Suppose that

$$G(s)H(s) = K\frac{Q(s)}{P(s)},$$

where $Q(s)$ and $P(s)$ are polynomials:

$$Q(s) = s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0,$$
$$P(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0,$$

then

$$1 + G(s)H(s) = 1 + K\frac{Q(s)}{P(s)} = 0.$$

Basic Properties of RL

1. Magnitude condition. Consider $1 + KG_1(s)H_1(s) = 0$. We have that

$$G_1(s)H_1(s) = -\frac{1}{K} \;\Rightarrow\; |G_1(s)H_1(s)| = \frac{1}{|K|}$$

2. Angle condition.

$$\angle G_1(s)H_1(s) = \begin{cases} (2j+1)\pi, & K \geq 0; \\ 2j\pi, & K \leq 0; \end{cases} \quad j = 0, \pm 1, \pm 2, \cdots$$

Basic Properties of RL

Assume that

$$KG_1(s)H_1(s) = \frac{K(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)},$$

then

$$|G_1(s)H_1(s)| = \frac{\prod_{i=1}^{m}|s+z_i|}{\prod_{k=1}^{n}|s+p_k|} = \frac{1}{|K|}.$$

For $K \geq 0$:

$$\angle G_1(s)H_1(s) = \sum_{i=1}^{m}\angle(s+z_i) - \sum_{k=1}^{n}\angle(s+p_k) = (2j+1)\pi;$$

for $K \leq 0$:

$$\angle G_1(s)H_1(s) = \sum_{i=1}^{m}\angle(s+z_i) - \sum_{k=1}^{n}\angle(s+p_k) = 2j\pi.$$

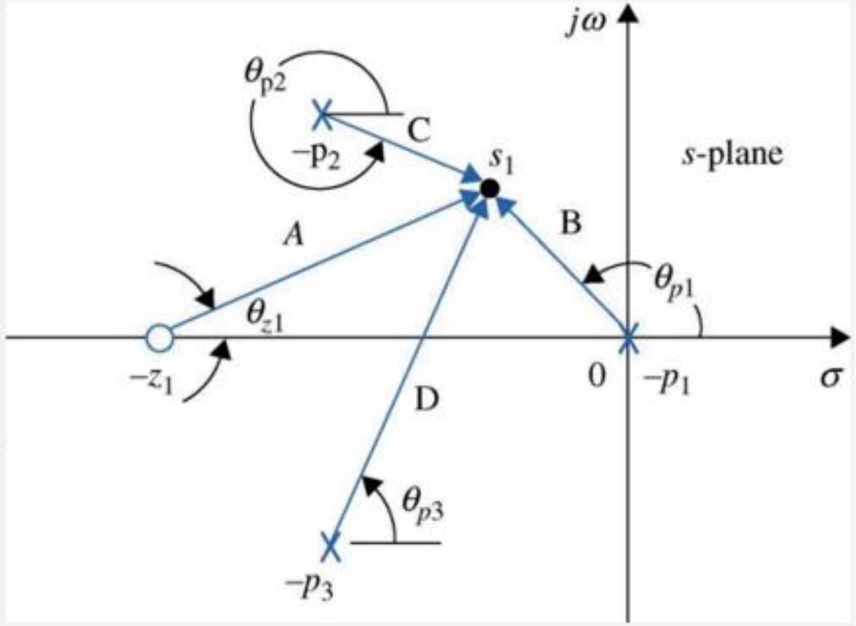
Basic Properties of RL

Example

Consider

$$G(s)H(s) = \frac{K(s+z_1)}{s(s+p_2)(s+p_3)}.$$

Select a trial point s_1. For $K \geq 0$, we have



$$\angle(s_1+z_1) - \angle(s_1+p_2) - \angle(s_1+p_3) - \angle s_1$$
$$= \theta_{z_1} - \theta_{p_2} - \theta_{p_3} - \theta_{p_1} = (2j+1)\pi.$$

$$\frac{|s_1+z_1|}{|s_1(s_1+p_2)(s_1+p_3)|} = \frac{1}{K} \quad \Rightarrow \quad \frac{A}{BCD} = \frac{1}{K}$$

Basic Properties of RL

3. Symmetry: root locus is symmetric w.r.t. real axis.
4. $K = 0$ and $K = \pm\infty$. From

$$G(s)H(s) = -\frac{1}{K},$$

when $K = 0$,

$$G(s)H(s) = -\infty,$$

s is a pole of $G(s)H(s)$;
when $K = \infty$,

$$G(s)H(s) = 0,$$

s is a zero of $G(s)H(s)$.

5. Consider $K \geq 0$. Root locus has n branches starting from poles, m branches ending at zeros:

$$G(s)H(s) = \frac{Q(s)}{P(s)}, \quad \begin{array}{c} m = \deg Q(s) \\ n = \deg P(s) \end{array}.$$

Basic Properties of RL

Consider an example $s(s+2)(s+3) + K(s+1) = 0$.

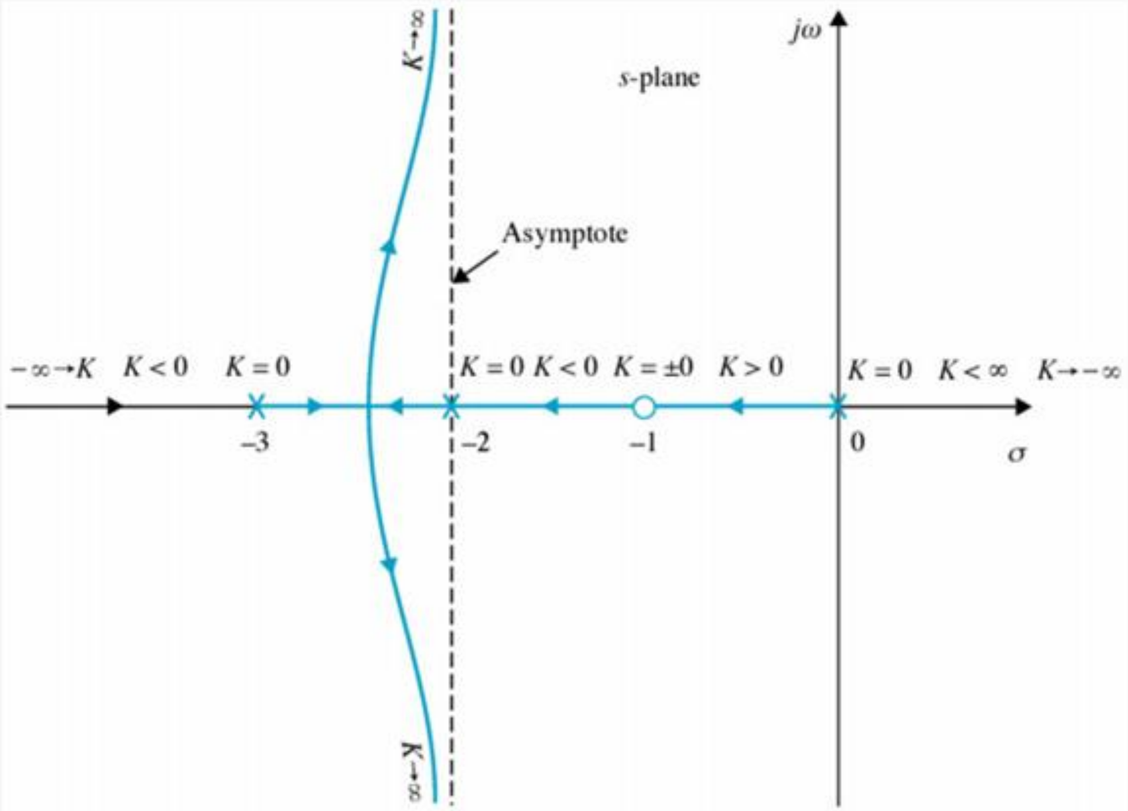


Figure: Root locus and asymptotes

Basic Properties of RL

6. RL on the real axis for $K \geq 0$ are found to the left of an odd number of real poles and real zeros.

   For a trial point s_1 on the real axis, if s_1 is to the left of $-p$ or $-z$,

   $$\angle(s_1 + p) = \pi, \quad \angle(s_1 + z) = \pi.$$

   If s_1 is to the right of $-p$ or $-z$,

   $$\angle(s_1 + p) = 0, \quad \angle(s_1 + z) = 0;$$

   So they do not have contribution in

   $$\sum_i \angle(s + z_i) - \sum_k \angle(s + p_k) = (2j + 1)\pi.$$

Basic Properties of RL

6. RL on the real axis for $K \geq 0$ are found to the left of an odd number of real poles and real zeros.

   For a trial point s_1 on the real axis, if s_1 is to the left of $-p$ or $-z$,

   $$\angle(s_1 + p) = \pi, \quad \angle(s_1 + z) = \pi.$$

   If s_1 is to the right of $-p$ or $-z$,

   $$\angle(s_1 + p) = 0, \quad \angle(s_1 + z) = 0;$$

   So they do not have contribution in

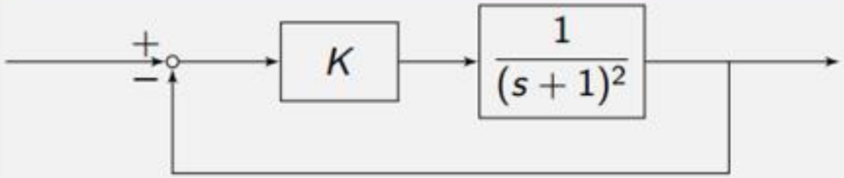
   $$\sum_i \angle(s + z_i) - \sum_k \angle(s + p_k) = (2j + 1)\pi.$$

Nyquist Stability Criterion

- Nyquist plot of the loop transfer function $G(s)H(s)$ (or $L(s)$) is a plot of $\mathrm{Im}[L(j\omega)]$ vs $\mathrm{Re}[L(j\omega)]$ as $\omega : 0 \rightarrow \infty$.
- Use $L(s)$ to study CL system.

Nyquist Stability Criterion

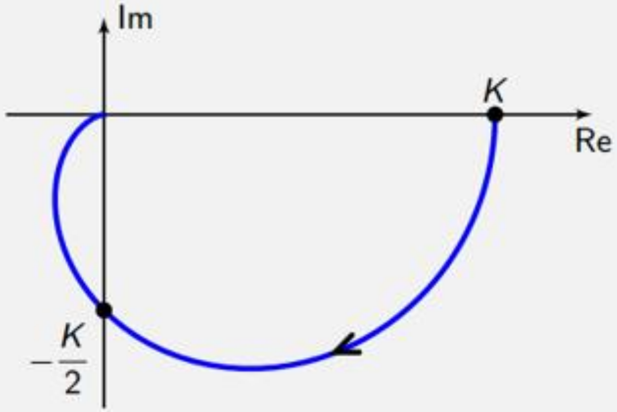
Example



CLTF: $\dfrac{K\frac{1}{(s+1)^2}}{1+K\frac{1}{(s+1)^2}}$

$$1 + L(s) = 1 + \frac{K}{(s+1)^2} = 0.$$

Let $s = j\omega$, $L(j\omega) = K\dfrac{1}{(j\omega+1)^2} = \dfrac{K}{1-\omega^2+2j\omega}$.



When $\omega = 0$, $L(j\omega) = K$;
When $\omega = 1$, $L(j\omega) = -\frac{K}{2}j$;
When $\omega = \infty$, $|L(j\omega)| = 0$

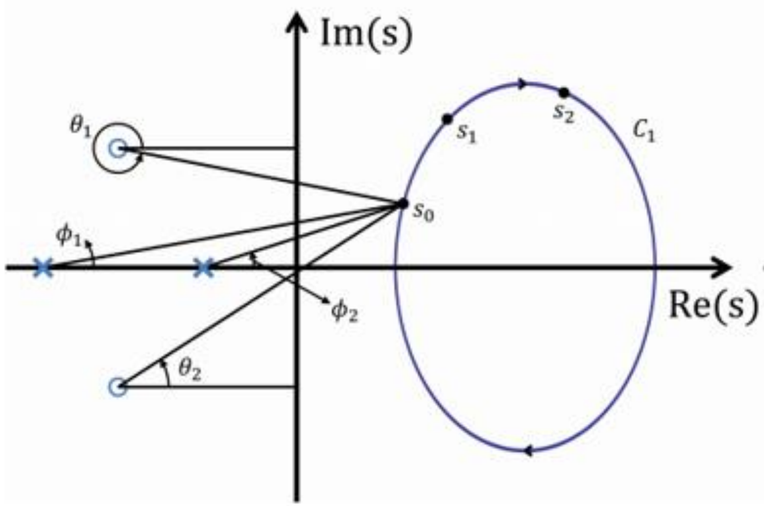
$$\angle L(j\omega) = \angle\frac{K}{(j\omega)^2} = \angle\frac{K}{-\omega^2}$$
$$= \pi.$$

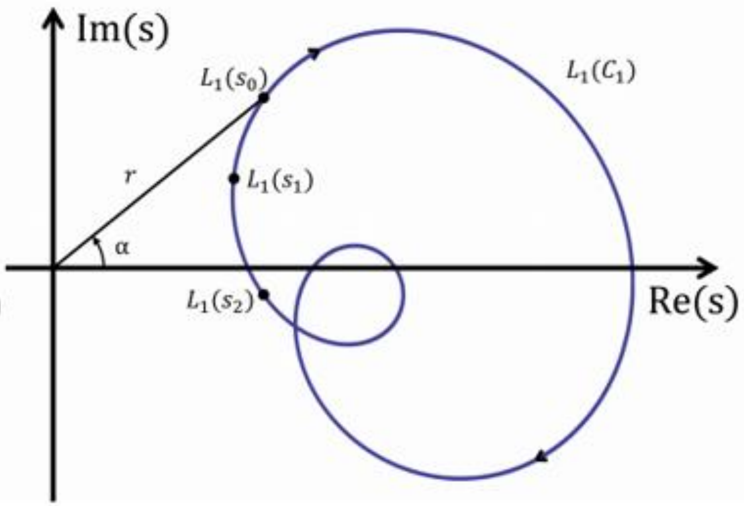
Argument Principle

Consider a loop TF $L_1(s)$: zeros & poles are indicated as below. We want to evaluate $L_1(s)$ for values of s on clockwise path C_1. Take a point s_0 on C_1, $L_1(s_0) = re^{i\alpha}$,

$$\alpha = \theta_1 + \theta_2 - \phi_1 - \phi_2.$$

As s travels along C_1, α will change as well.

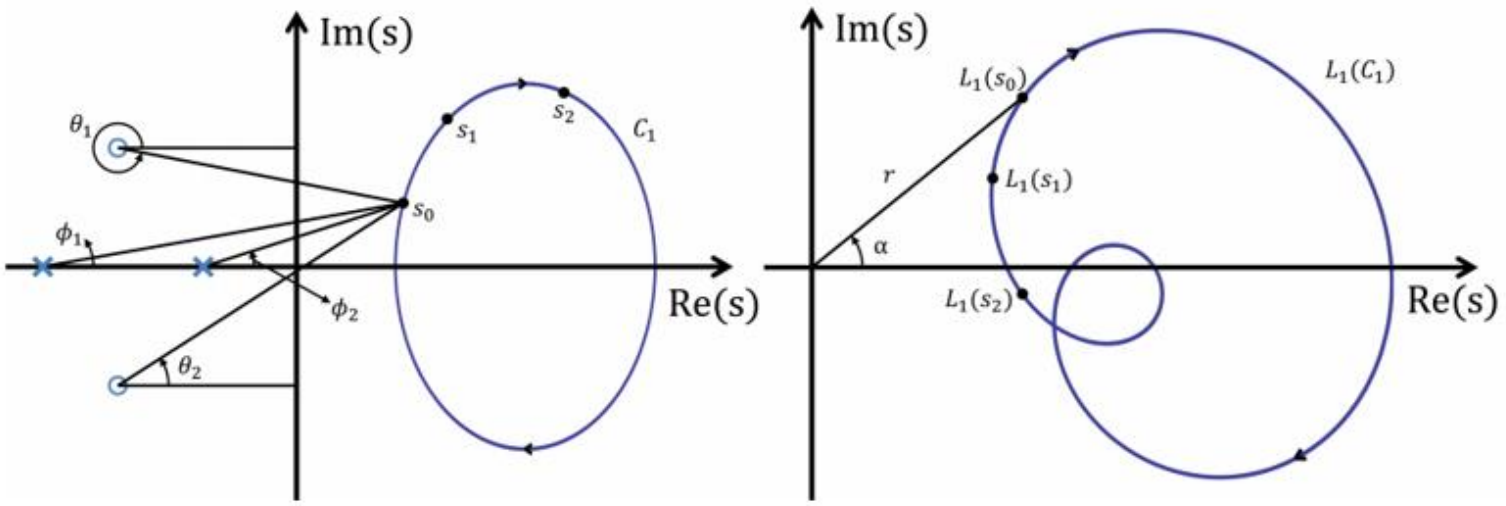




Argument Principle

Note that if C_1 contains no zeros or poles of $L_1(s)$, α will not undergo a net change of 2π.

This is because none of θ_1, θ_2, ϕ_1, ϕ_2 goes through a net revolution, and then $L_1(C_1)$ will not encircle the origin.

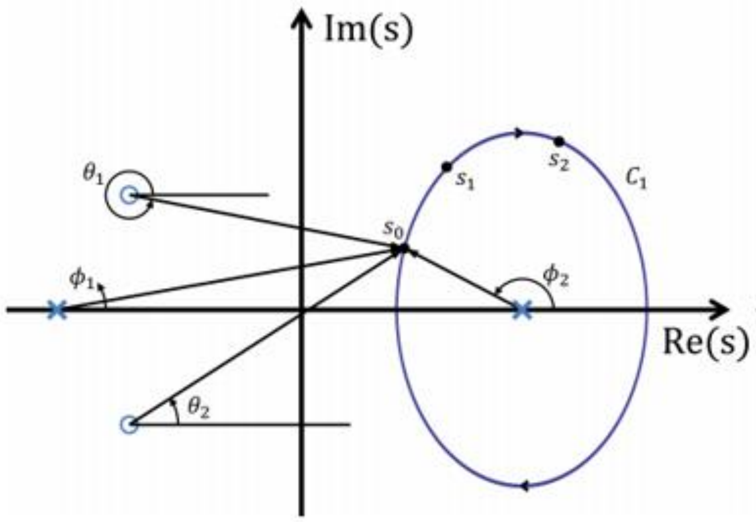


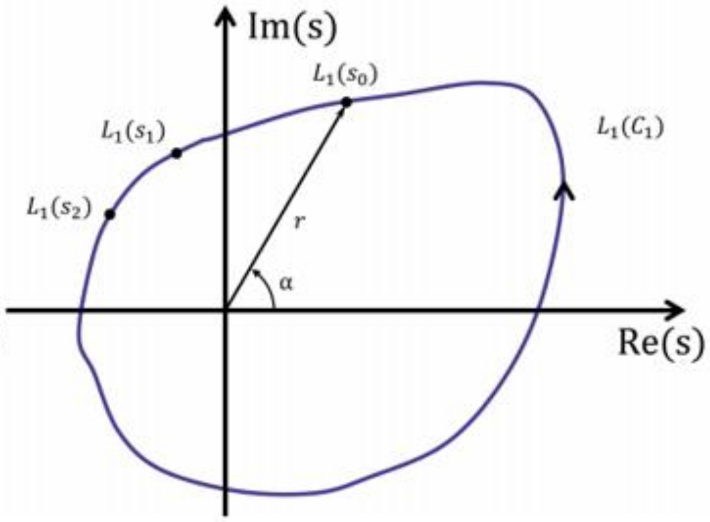
Argument Principle

Consider another $L_1(s)$ as below:

- θ_1, θ_2, ϕ_1 change, but they return to original values;
- ϕ_2 undergoes a net change of -2π;

Therefore, α undergoes a net change of 2π.
$\Rightarrow L_1(C_1)$ encircles the origin in the CCW direction.





Argument Principle

Theorem (Argument Principle)

$L_1(C_1)$ will encircle the origin $Z - P$ times, where

- Z is the number of zeros of $L_1(s)$ inside C_1; and
- P is the number of poles of $L_1(s)$ inside C_1.

In the previous example, we have $Z = 0$ and $P = 2$. Therefore, $L_1(C_1)$ encircles the origin -2 times, where the negative sign indicates the counter clockwise direction.

Nyquist Stability Criterion

If $L(s) = \frac{b(s)}{a(s)}$, we have

$$1 + KL(s) = 1 + K\frac{b(s)}{a(s)} = \frac{a(s) + Kb(s)}{a(s)}$$

- poles of $1 + KL(s)$ are also poles of $L(s)$;
  $\rightarrow$ open-loop poles, easy to get.
- zeros of $1 + KL(s)$ are closed-loop poles.

Now from Argument Principle,

$$\begin{aligned}&\#\{(1 + KL(s)) \text{ encircling the origin}\}\\ =&\#\{\text{zeros of } (1 + KL(s)) \text{ in RHP}\} - \#\{\text{poles of } (1 + KL(s)) \text{ in RHP}\}.\end{aligned}$$

Hence,

$$\begin{aligned}\#\{\text{closed-loop poles in RHP}\} =&\#\{\text{open-loop poles in RHP}\}\\ &+\#\{(1 + KL(s)) \text{ encircling the origin}\}\end{aligned}$$

$$\text{If LHS} \neq 0 \quad \Rightarrow \quad \text{unstable (minimum phase)}$$

Nyquist Stability Criterion

How to apply Nyquist stability criterion

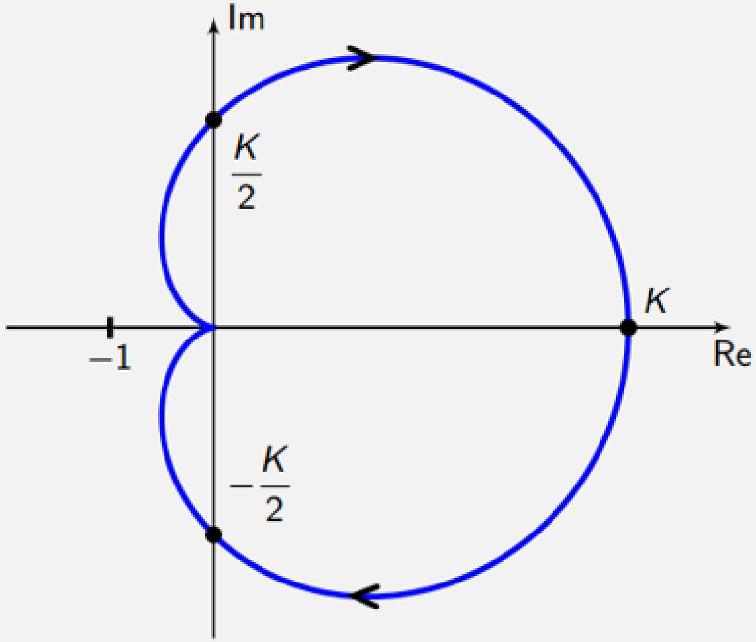
1. Plot $KL(s)$, $-j\infty \leq s \leq j\infty$
2. Get N: $\#\{KL(s)$ encircling $(-1,0)$, CW:$+$, CCW:$-\}$
3. Get P: $\#\{$RHP poles of $L(s)\}$
4. $Z = N + P \rightarrow$ unstable closed-loop poles

We wish to have $Z = 0$.

Nyquist Stability Criterion

Example 1

$$KL(s) = \frac{K}{(s+1)^2}.$$



$$P = 0, N = 0$$

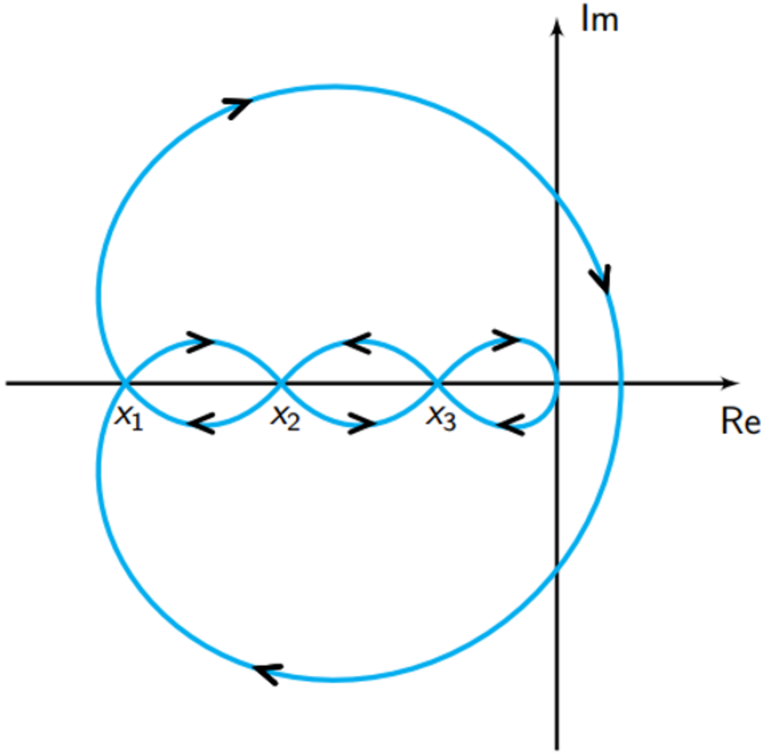
$$\Rightarrow \quad Z = N + P = 0.$$

Therefore, it is stable.

Nyquist Stability Criterion

Example 2

$$KL(s) = \frac{K(0.1s+1)^2}{(s+1)^3(0.01s+1)^2}.$$



Nyquist Stability Criterion

Answer:

1. $-\frac{1}{K} < x_1 \quad \Rightarrow \quad K < 19.2$, stable;
2. $x_1 \leq -\frac{1}{K} \leq x_2 \quad \Rightarrow \quad 19.2 \leq K \leq 334.11$, unstable;
3. $x_2 < -\frac{1}{K} < x_3 \quad \Rightarrow \quad 334.11 < K < 13233$, stable;
4. $x_3 < -\frac{1}{K} \quad \Rightarrow \quad K \geq 13233$, unstable.

Gain and Phase Margins

In this section, we introduce and examine the concept of the *margin of stability* in a feedback system. It is often of interest not only to know whether a feedback system is stable, but also to determine how much the gain in the system can be perturbed and how much additional phase shift can be added to the system before it becomes unstable.

Gain and Phase Margins

To assess the margin of stability in our feedback system, suppose that the actual system is as depicted in Fig. 11.26, where we have allowed for the possibility of a gain K and phase shift ϕ in the feedback path.

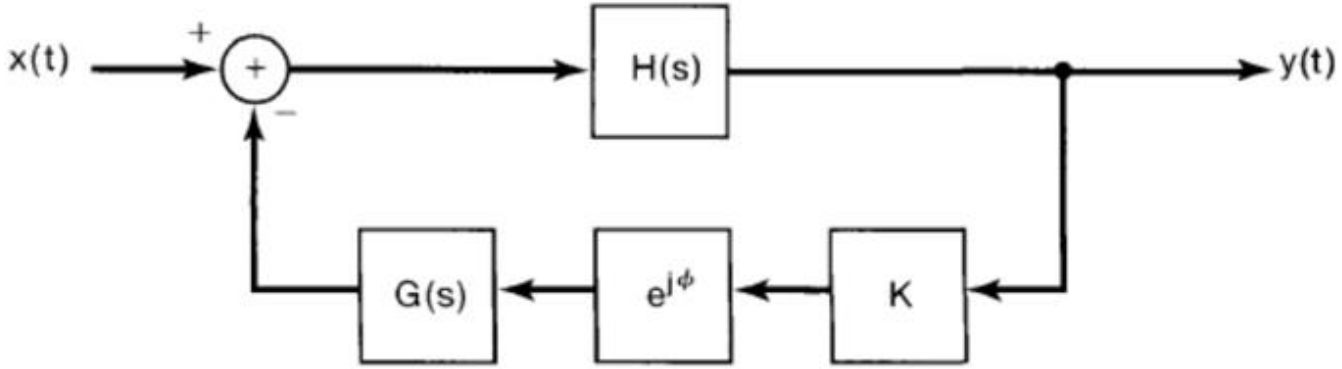


Figure 11.26 Feedback system containing possible gain and phase deviations from the nominal description depicted in Figure 11.25.

Gain and Phase Margins

In the nominal system K is unity and ϕ is zero, but in the actual system either or both may have a different value. Therefore, it is of interest to know how much variation can be tolerated in these quantities without losing closed-loop system stability. In particular, the *gain margin* of the feedback system is defined as the minimum amount of additional gain K, with $\phi = 0$, that is required so that the closed-loop system becomes unstable.

Gain and Phase Margins

Similarly, the *phase margin* is the additional amount of phase shift, with $K = 1$, that is required for the system to be unstable. By convention, the phase margin is expressed as a positive quantity; that is, it equals the magnitude of the additional negative phase shift at which the feedback system becomes unstable.

Gain and Phase Margins

Since the closed-loop system of Fig. 11.25 is stable, the system of Fig. 11.26 can become unstable if, as K and ϕ are varied, at least one pole of the closed-loop system crosses the $j\omega$-axis. If a pole of the closed-loop system is on the $j\omega$-axis at, say, $\omega = \omega_0$, then at this frequency

$$1 + Ke^{-j\phi}G(j\omega_0)H(j\omega_0) = 0 \qquad (11.99)$$

or

$$Ke^{-j\phi}G(j\omega_0)H(j\omega_0) = -1 \qquad (11.100)$$

Gain and Phase Margins

The gain margin of this system is the minimum value of $K > 1$ for which eq. (11.100) has a solution for some ω_0 with $\phi = 0$. That is, the gain margin is the smallest value of K for which the equation

$$KG(j\omega_0)H(j\omega_0) = -1 \qquad (11.101)$$

has a solution. Similarly, the phase margin is the smallest value of ϕ for which the equation

$$e^{-j\phi}G(j\omega_0)H(j\omega_0) = -1 \qquad (11.102)$$

has a solution.

Gain and Phase Margins

Example 11.9 Determine the gain and phase margins of the system with

$$G(s)H(s) = \frac{4(1+\frac{1}{2}s)}{s(1+2s)[1+0.05s+(0.125s)^2]}$$

Solution : The Bode plot for this example is shown in Fig. 11.27. By inspection, the gain margin K and the phase margin ϕ are

$$K = \frac{1}{|G(j\omega_1)H(j\omega_1)|} \text{ and } \phi = \pi + \measuredangle G(j\omega_2)H(j\omega_2)$$

respectively.

Gain and Phase Margins

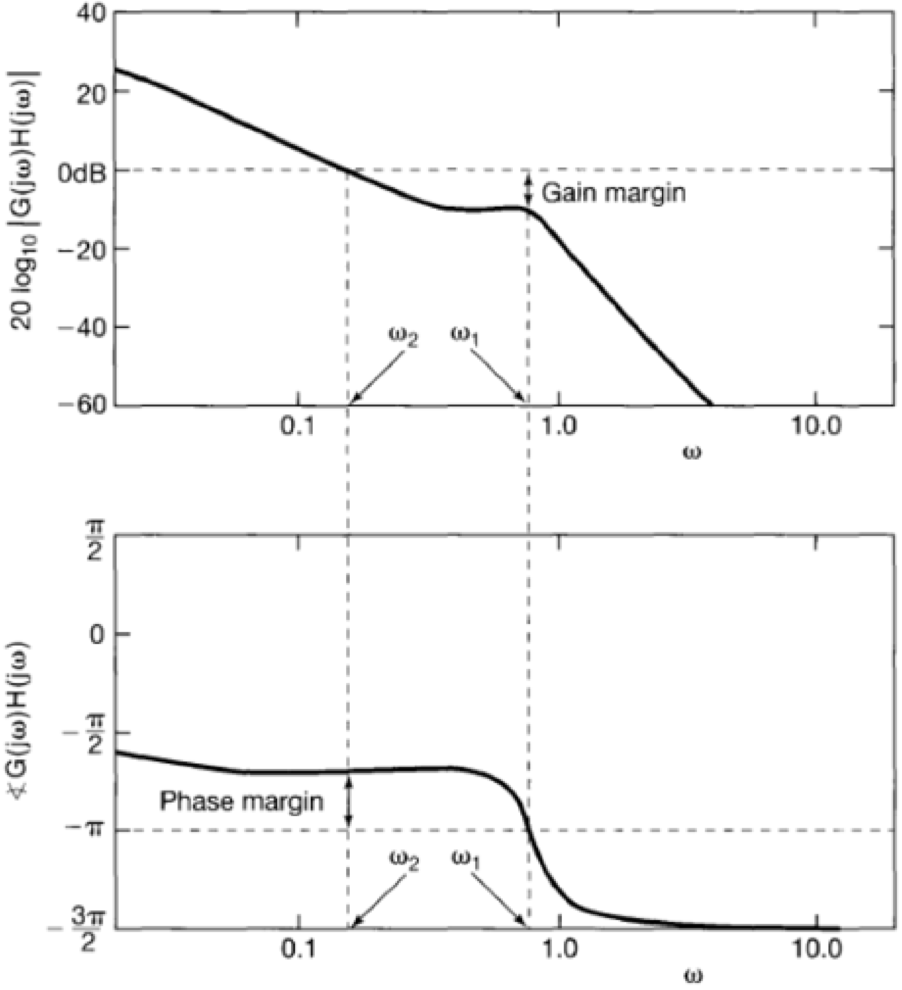


Figure 11.27 Use of Bode plots to calculate gain and phase margins for the system of Example 11.9.

Gain and Phase Margins

Example:

$$G(s)H(s) = \frac{4(1+\frac{1}{2}s)}{s(1+2s)[1+0.05s+(0.125s)^2]}$$