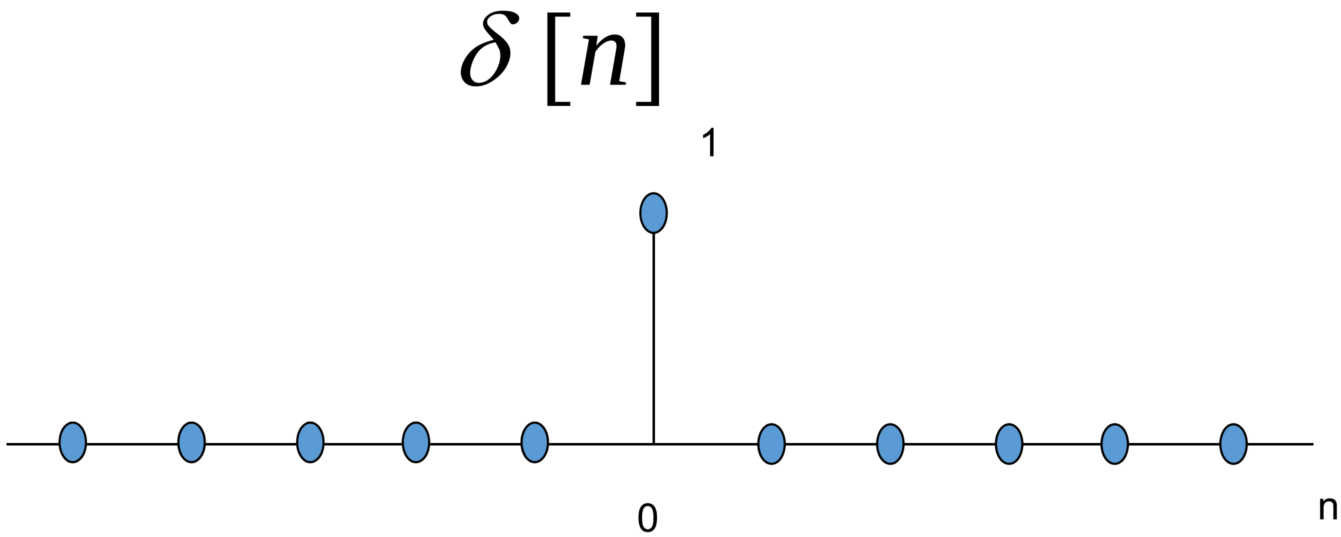
# Introduction to Signals and Systems: V216

Lecture #3

Chapter 1: Signals and Systems

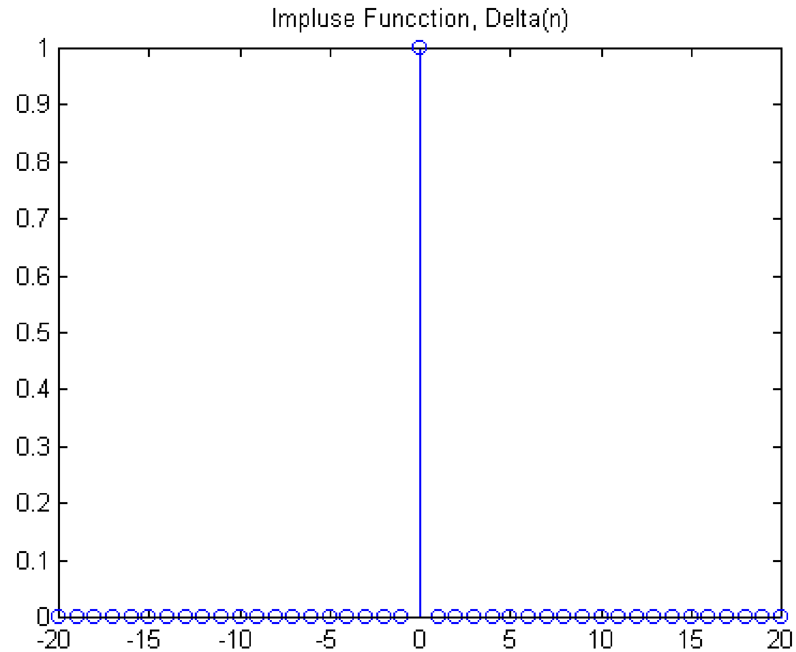
# The Discrete-time Unit Impulse Sequence

$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$



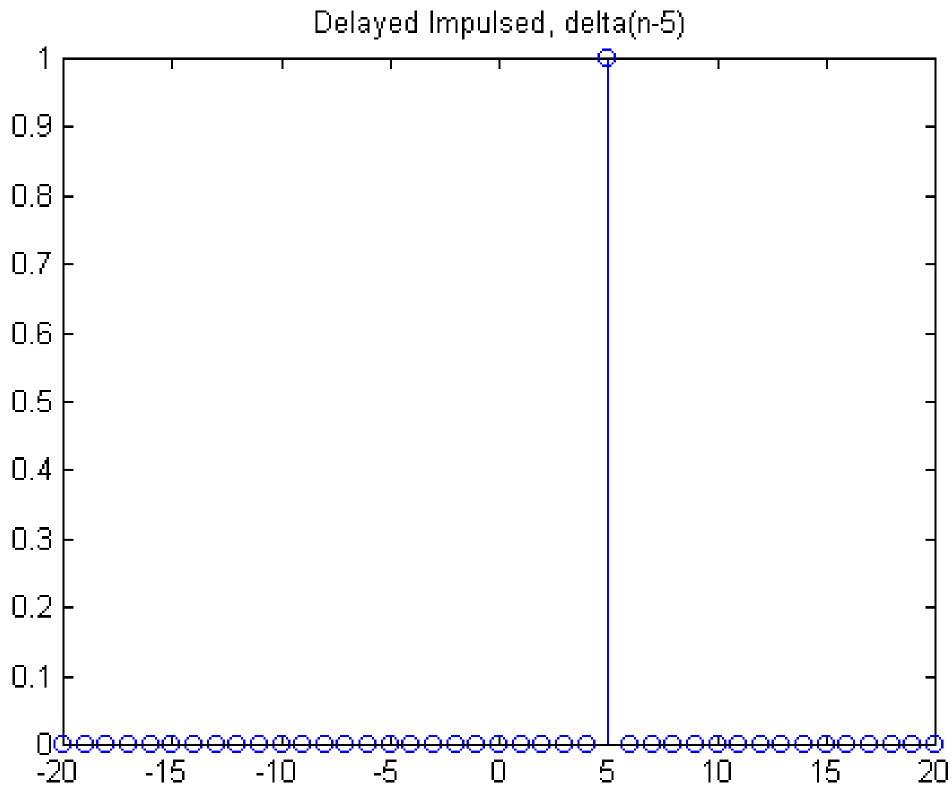
# Impulse Function

volt




n

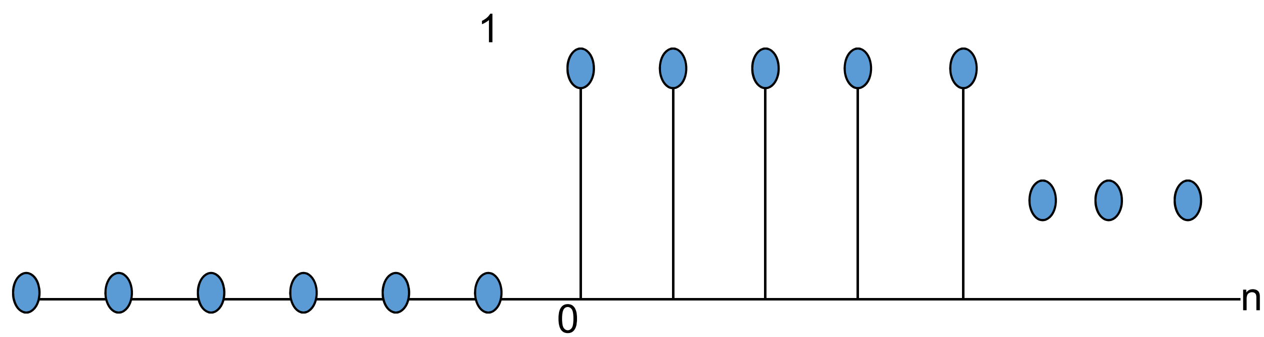
# Delayed Unit Impulse



# The Unit Step Sequence

$$u[n] = 0, n < 0,$$

$$u[n] = 1, n \geq 0.$$



# Relationship between Unit Impulse & Unit Step Sequences

Discrete-time unit impulse is the first difference of the discrete-time unit step.

$$\delta[n] = u[n] - u[n-1].$$

# Relationship between Unit Impulse & Unit Step Sequences

Discrete-time unit step is the running sum of the discrete-time unit impulse or unit sample.

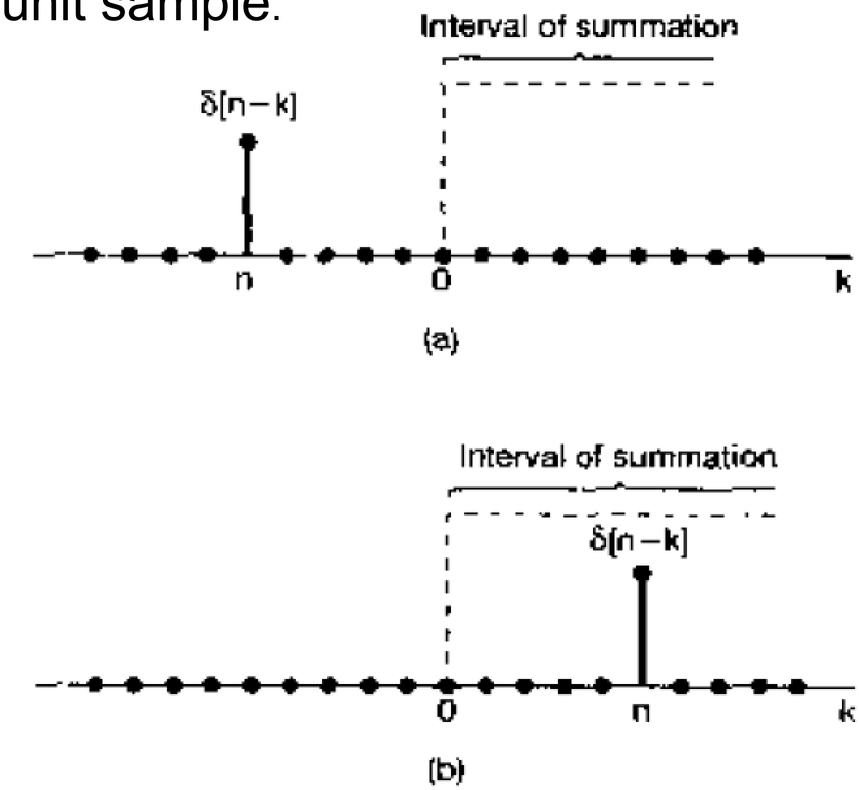



**Figure 1.31** Relationship given in eq. (1.67): (a) $n < 0$; (b) $n > 0$.

# Relationship between Unit Impulse & Unit Step Sequences

Discrete-time unit step is the running sum of the discrete-time unit impulse or unit sample.

$$u[n] = \sum_{k=0}^{\infty} \delta[n-k].$$

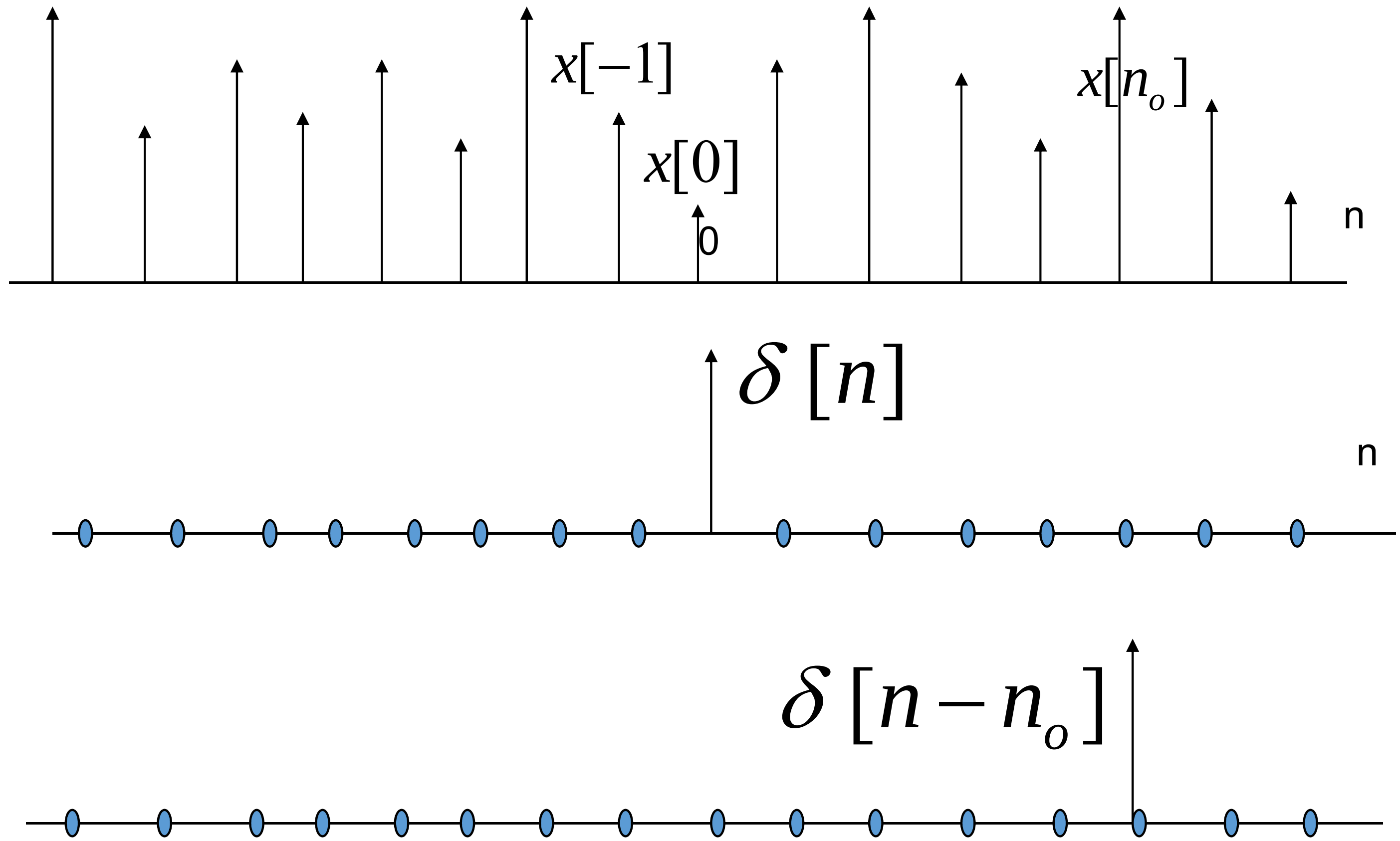
# The Sampling Property of the Unit Impulse.

The unit impulse sequence can be used to sample the value of a signal at n = 0.

Since $\delta[n]$ is nonzero $(=1)$ only for n = 0,

$\therefore x[n]\delta[n] = x[0]\delta[n]$.

It follows that generally since $\delta[n - n_o] = 1$ for $n = n_o$, then $x[n]\delta[n - n_o] = x[n_o]\delta[n - n_o]$.



$x[-1]$
$x[0]$
$x[n_o]$
0
n
$\delta[n]$
n
$\delta[n-n_o]$

# The Continuous-time Unit Step

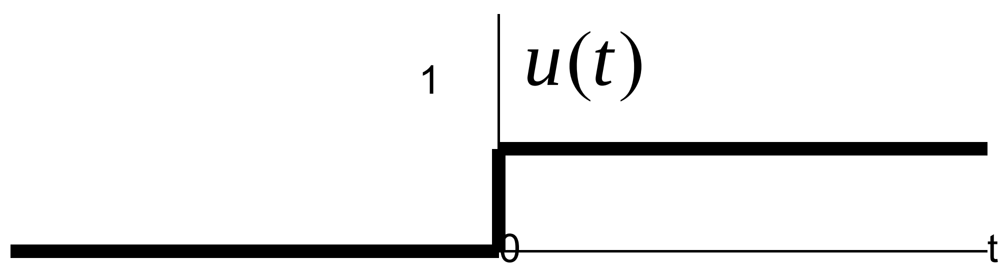
$$Defination\ of\ \ unit\ step\ \ function:-$$

$$u(t) = 0, t < 0,$$

$$u(t) = 1, t > 0.$$

$$This\ \ function\ is\ discontinuous$$

$$at\ t = 0.$$

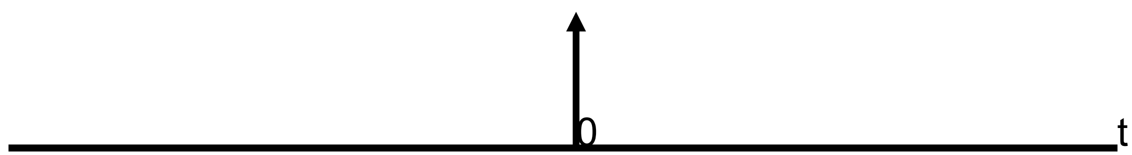


The Continuous-time Unit Impulse Function.

$$\text{Defination : -}$$

$$\delta\,(\mathrm{t}) = 0,\ \mathrm{t} \neq 0,$$

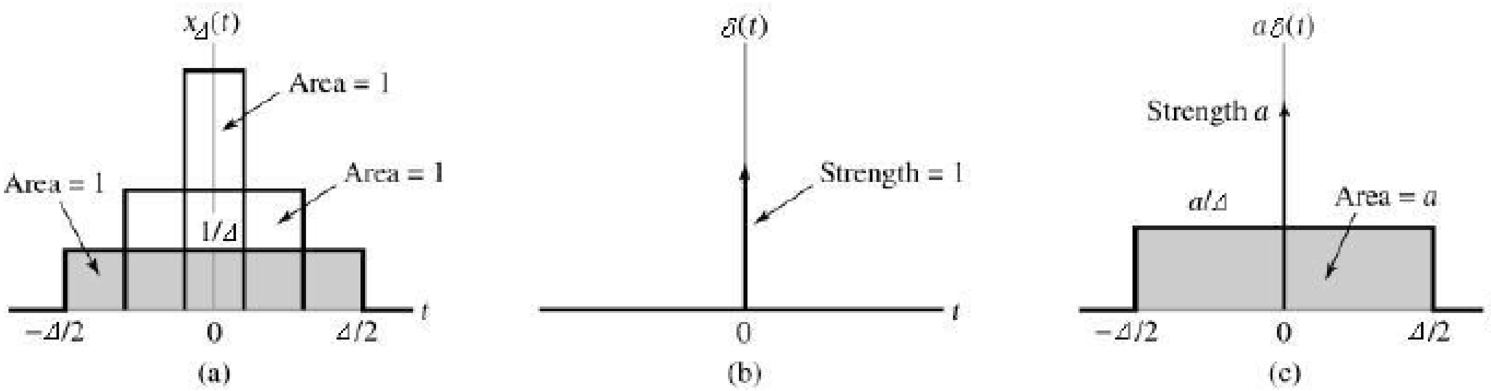
$$\delta\,(\mathrm{t}) = 1(area),\ \mathrm{t} = 0,$$



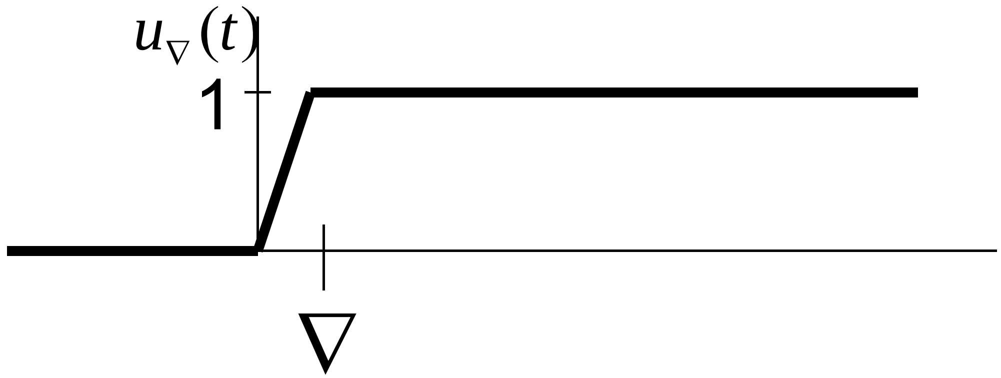
# The Continuous-time Unit Impulse Function.

$$\int_{-\infty}^{\infty} \delta(t)dt = 1$$

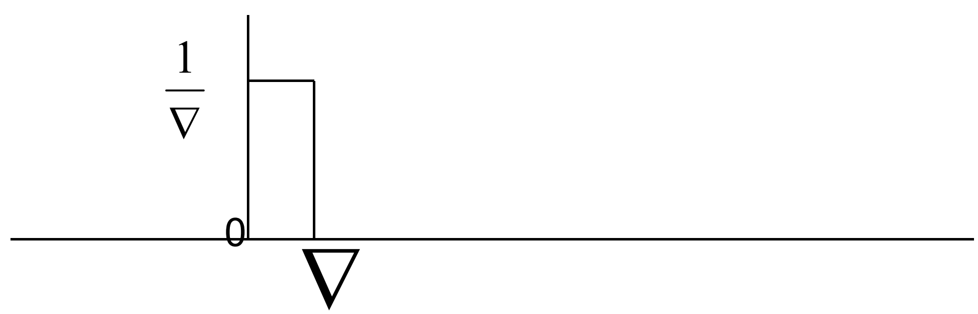
(a) Evolution of a rectangular pulse of unit area into an impulse of unit strength (i.e., unit impulse). (b) Graphical symbol for unit impulse.
(c) Representation of an impulse of strength *a* that results from allowing the duration Δ of a rectangular pulse of area *a* to approach zero.



# Discontinuity at t=0, poses problem of differentiation.




$$\delta(t) = \lim_{\Delta\to 0} x_\Delta(t) = \lim_{\Delta\to 0} \frac{u(t \qquad) - u(t-\Delta \quad)}{\Delta} = \frac{d}{dt}u(t)$$

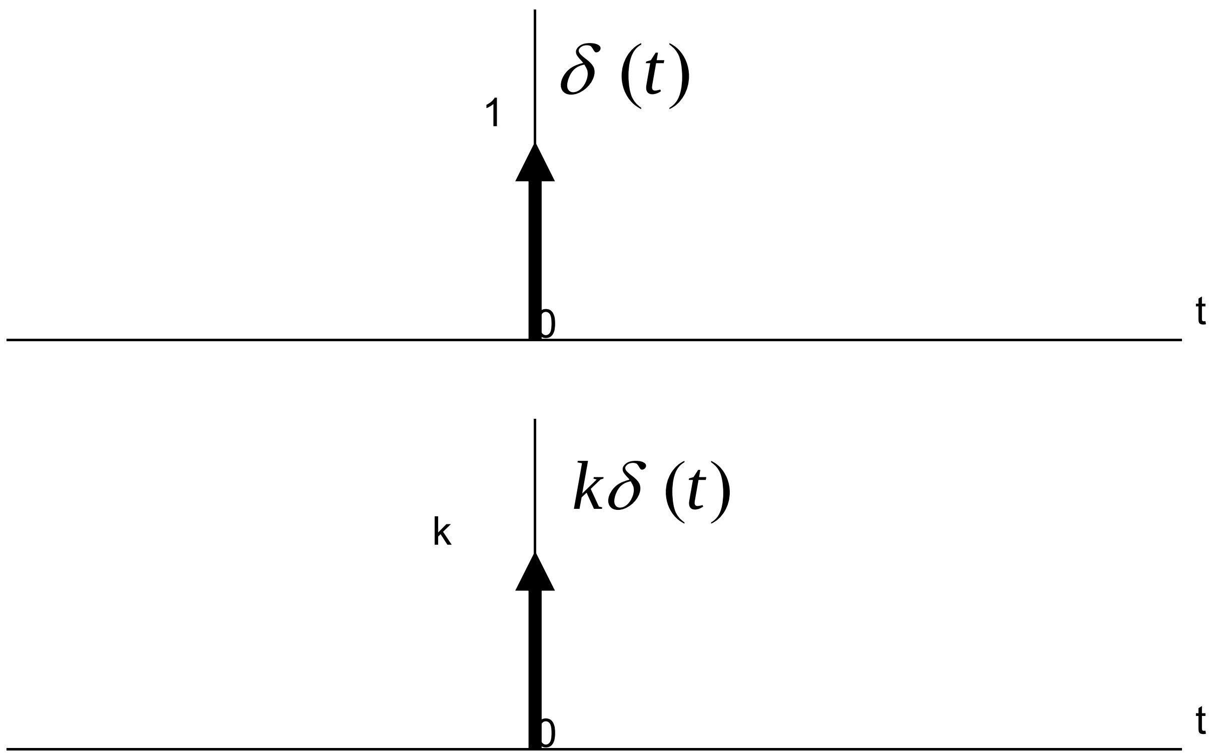


# Relationship between time unit step and unit impulse.

$$u(t) = \int_{-\infty}^{t} \delta(\tau)d\tau.$$

$$\delta(t) = \frac{du(t)}{dt.}$$

# Unit and scaled Impulse



# Scaled *impulse and relationship of unit step & impulse.*

$$\int_{-\infty}^{t} k\delta(\tau)d\tau = ku(t).$$

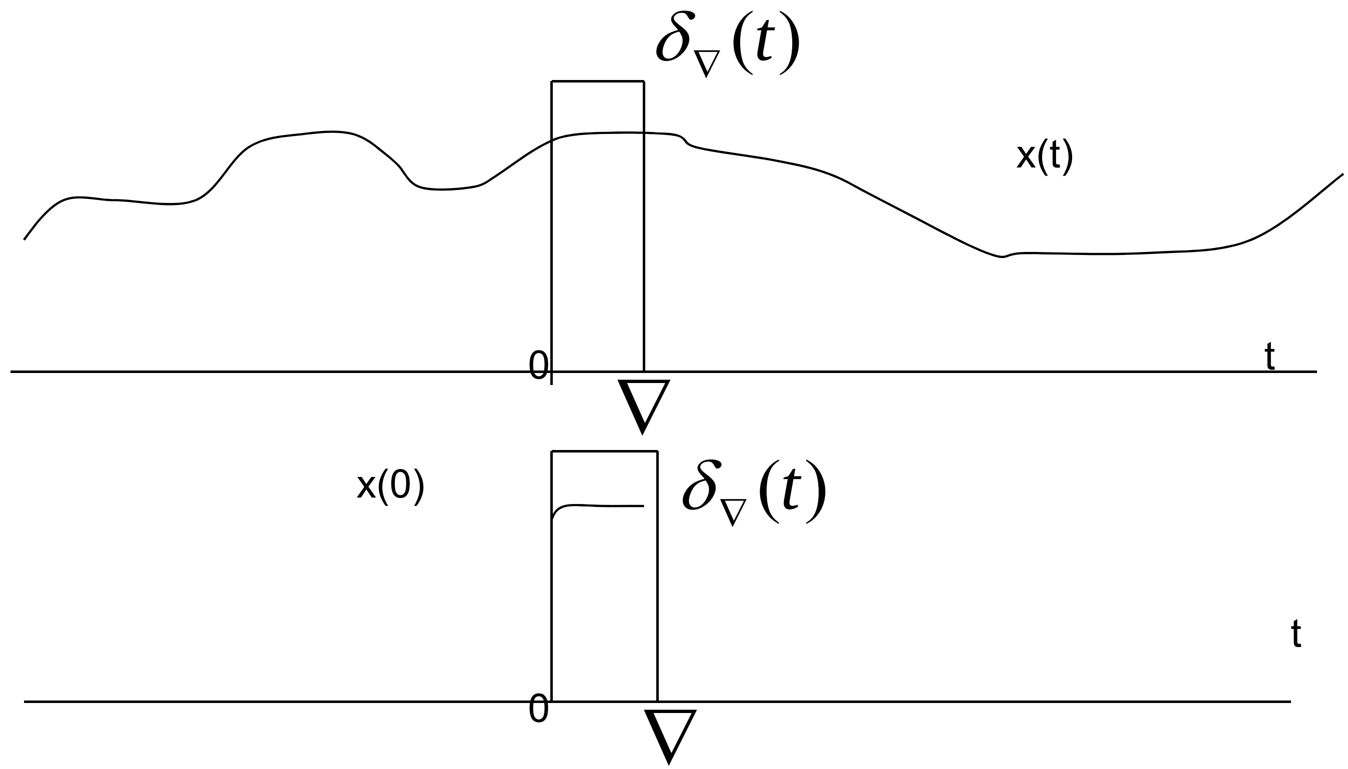
$$Letting\ \sigma = t - \tau,\ and\ scale\ k = 1,$$

$$u(t) = \int_{-\infty}^{t} \delta(\tau)d\tau = \int_{\infty}^{0} \delta(t - \sigma)(-d\sigma),$$

$$or\ equivalently:-$$

$$u(t) = \int_{0}^{\infty} \delta(t - \sigma)d\sigma.$$

$$\text{Consider } \mathrm{x}_1(t) = x(t)\delta_{\nabla}(t).$$



For $\nabla$ sufficiently small, x(t) is constant over interval $\nabla$,

$\therefore \mathrm{x(t)}\delta_{\nabla}(t) \approx x(0)\delta_{\nabla}(t)$.

*Since* $\delta(\mathrm{t}) = \delta_{\nabla}(t)$ as $\nabla \rightarrow 0$.

$\therefore \mathrm{x(t)}\delta(\mathrm{t}) = \mathrm{x(0)}\delta(\mathrm{t})$.

Similarly by same argument,

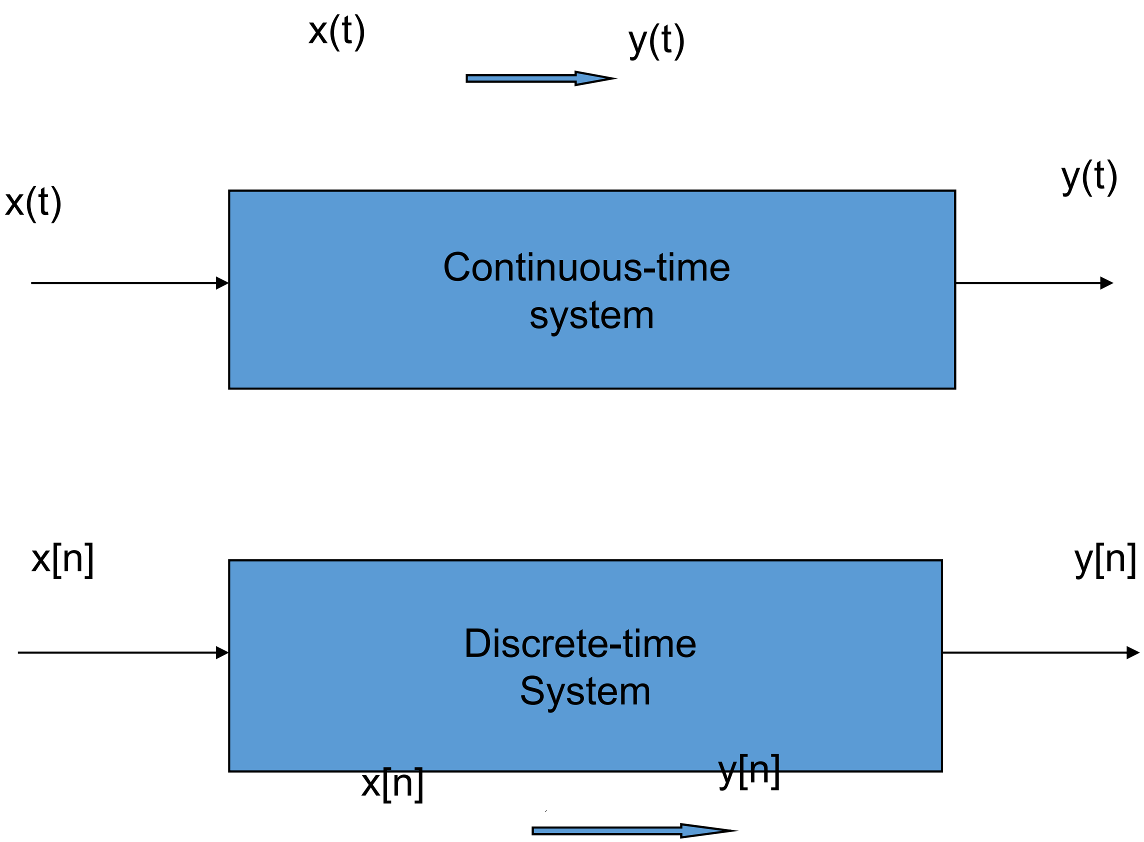
$\mathrm{x(t)}\delta(\mathrm{t - t_o}) = x(t_o)\delta(t - t_o)$

# Continuous-time & Discrete-time Systems.

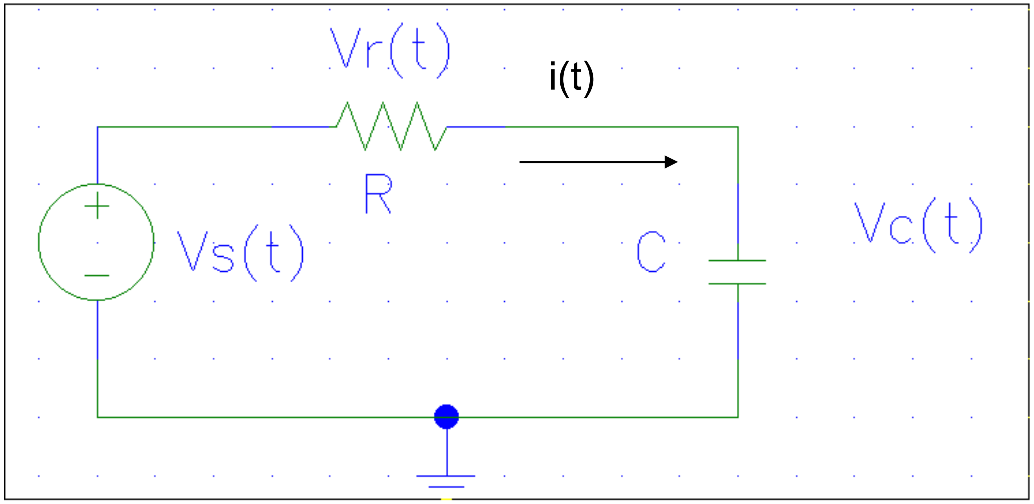
- Physical Systems are interconnection of
  - components, devices, or subsystems.
- System can be viewed as a process in which
  - input signals are transformed by the system or
  - cause the system to response in some way,

  resulting in other signals as outputs.

# Continuous-time & Discrete-time  Systems.



# Examples Of Systems




From Ohms's Law :- $i(t) = \dfrac{v_s(t) - v_c(t)}{R}$,

Relationship of current and voltage for a capasitor :-

$i(t) = C\dfrac{dv_c(t)}{dt}$, and substituting this into the above equation :-

We have the differential equation describing the relationship

between the input $v_s(t)$ and the output $v_c(t)$ :–

$$\frac{dv_c(t)}{dt} + \frac{1}{RC}v_c(t) = \frac{1}{RC}v_s(t)$$

# Example of Discrete-time System

Simple Model for Monthly Bank Balance

$y[n]$=present current balance.

$x[n]$=net deposit(deposits-withdrawals).

Accrue 1% interest on monthly past balance.

$y[n]=1.01y[n-1]+x[n]$.

or $y[n]-1.01y[n-1]=x[n]$.

Digital Simulation of Differential Equation Through Difference Equation.

$$\frac{dv(t)}{dt}+\frac{\rho}{m}v(t)=\frac{1}{m}f(t).$$

By first backward difference: $\frac{\mathrm{dv(t)}}{\mathrm{dt}}=\frac{\mathrm{v[n\Delta]-v[n-1]\Delta}}{\Delta}$

The differential equation can be expresses as :-

$$\frac{\mathrm{v[n\Delta]-v[n-1]\Delta}}{\Delta}+\frac{\rho}{\mathrm{m}}v[n\Delta]=\frac{1}{m}f[n\Delta].$$

$$v[n\Delta](\frac{1}{\Delta}+\frac{\rho}{\mathrm{m}})-\frac{v[n-1]\Delta}{\Delta}=\frac{1}{m}f[n\Delta].$$

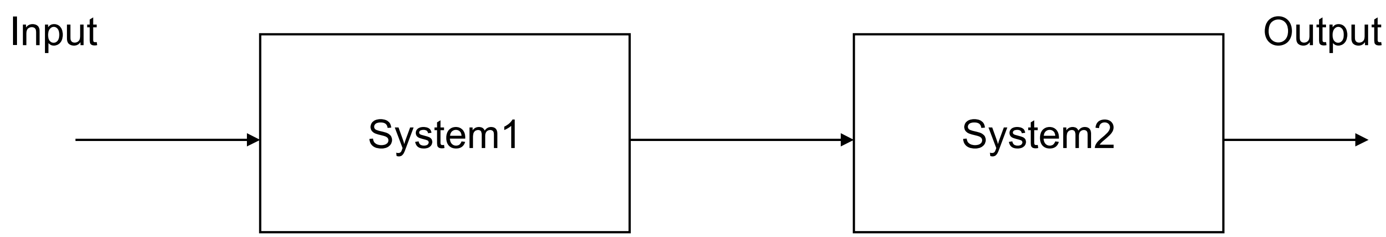
$$v[n\Delta]-\frac{m}{m+\rho\Delta}v[n-1]\Delta=\frac{\Delta}{m+\rho\Delta}f[n\Delta],$$

Letting, $\mathrm{v[n]}=v[n\Delta]$ and $\mathrm{f[n]=f[n\Delta]}$.

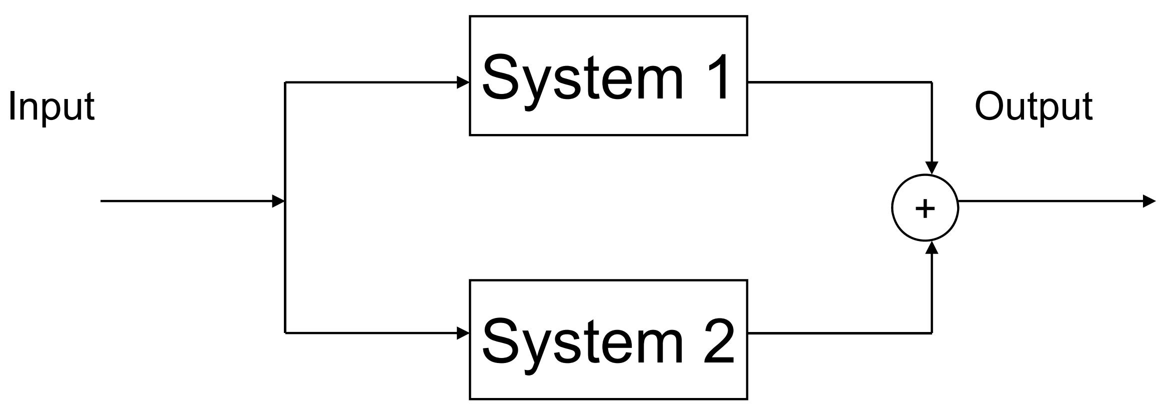
$$v[n]-\frac{m}{m+\rho\Delta}v[n-1]=\frac{\Delta}{m+\rho\Delta}f[n],$$

# Interconnections of Systems

Series or Cascade Form

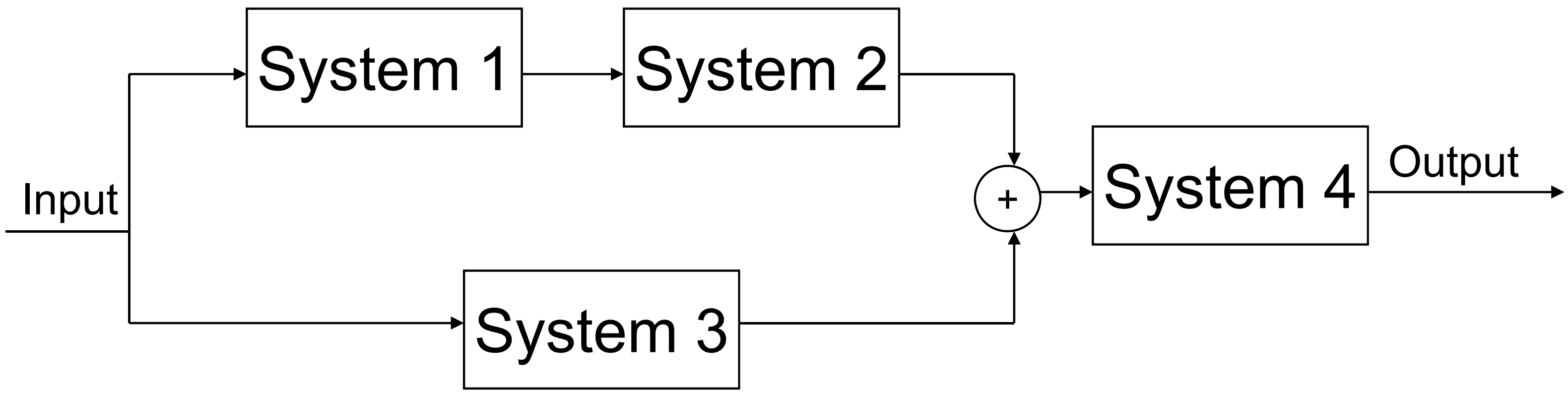



Parallel Form

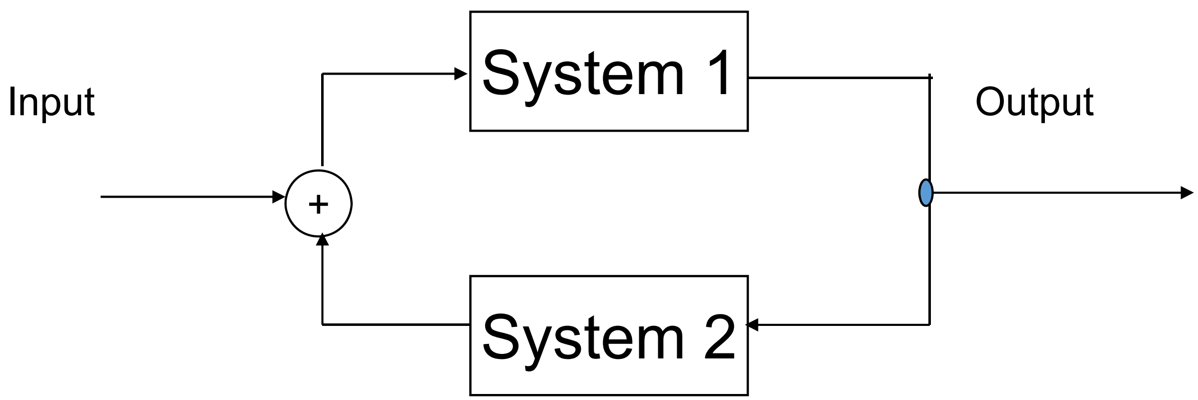


# Interconnections of Systems

Series - Parallel Form




Feedback Form



# Basic System Properties

- Systems with and without memory.
- Invertibility and Inverse Systems.
- Causality.
- Stability
- Time Invariance
- Linearity.

# Systems without memory.

- System is memoryless if its output at any one time depends only on the input at the same time.
- E.g. $y[n]=(2x[n]-x^2[n])^2$
- A resistor is memoryless because $y(t)=Rx(t)$
- So too an identity system is memoryless because $y(t)=x(t)$, $y[n]=x[n]$.

# Systems with memory.

- System with memory depicts its output at any one time that is dependent not only on the present input but also past (future) values of input and output.
- E.g. accumulator/summer

$$y[n] = \sum_{k=-\infty}^{n} x[k],$$

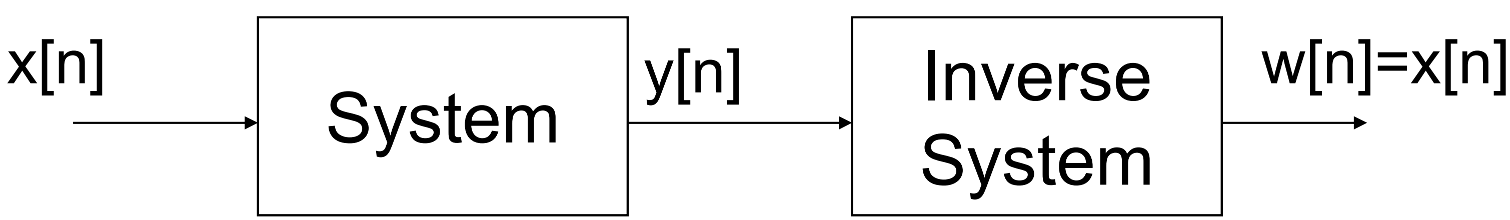
$$\text{Delay, } y[n] = x[n-1].$$

A capacitor is a memory analog device,

$$y(t) = \frac{1}{C}\int_{-\infty}^{t} x(\tau)\,d\tau,$$

# Invertible System

- **Systems whereby distinct inputs lead to distinct outputs**
- As such an inverse system exits that, when cascaded with the original system, yields an output w[n] equal to the input x[n] to the first system.



# Invertible System

- A system is said to be invertible if the input of the system can be recovered from the output.

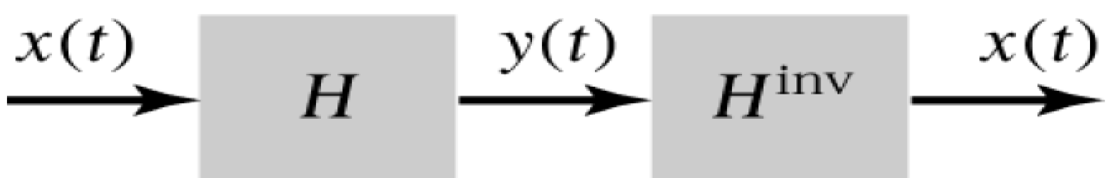
$$y(t) = H\{x(t)\}$$

$$x(t) = H^{inv}\{y(t)\}$$

$$= H^{inv}\{H\{x(t)\}\}$$

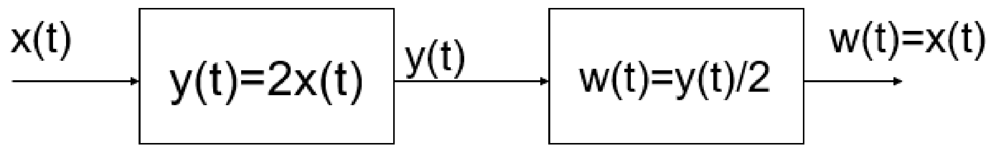
$$= H^{inv}H\{x(t)\}$$

$$\Rightarrow H^{inv}H = I$$



# Example 1: invertible continuous-time system.

$y(t)=2x(t)$ for which the inverse system is $w(t)=y(t)/2$.



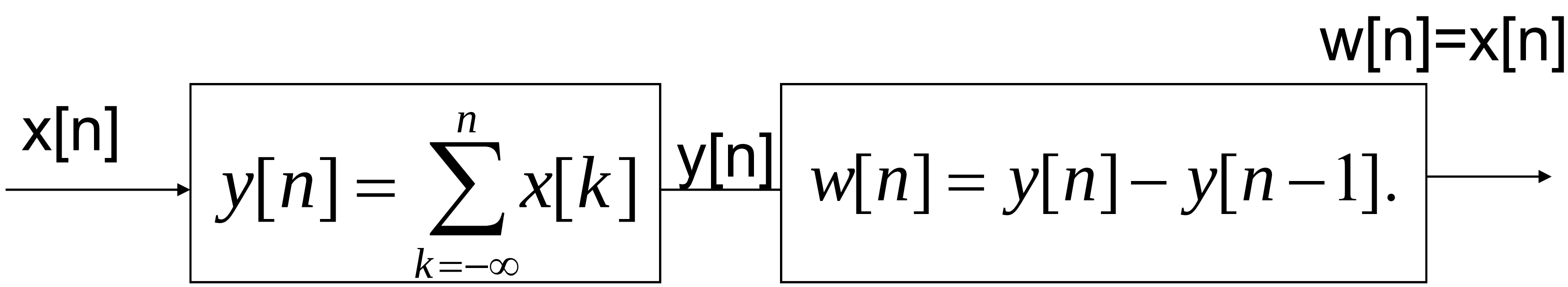
# Example 2: invertible discrete -time system.

$$y[n] = \sum_{k=-\infty}^{n} x[k],\ y[n] = \sum_{k=-\infty}^{n-1} x[k] + x[n],$$

$$y[n] = y[n-1] + x[n],\ x[n] = y[n] - y[n-1].$$

$$\therefore \text{the inverse system is : -}$$

$$w[n] = x[n] = y[n] - y[n-1].$$



# Examples of Noninvertible Systems

1- $y[n]=0$.

The output is always zero for any value of input $x[n]$.

2- $y(t)=x^2(t)$. The sign for the input $x(t)$ cannot be determined for a certain value of output $y(t)$

For both cases the values of the output is not distinct for distinct values of input.

# Causality

- A system is causal because its output depends only on **present and past** values of the input
- Such a system does not anticipate **future** values of input.

- y[n]=x[n]-x[n+1] non-causal systems.
- y(t)=x(t+1) non-causal systems.

**Memoryless system is a causal system**

# Causality

Examples.

1. $y[n] = x[n-1]$ is causal, because $y[n]$ depends on the past sample $x[n-1]$.

2. $y[n] = x[n] + x[n+1]$ is not causal, because $x[n+1]$ is a future sample.

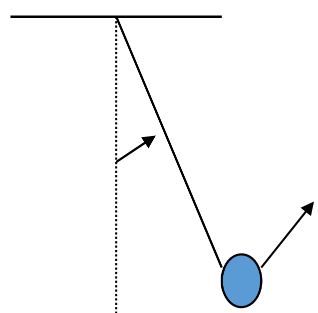
3. $y(t) = \int_{-\infty}^{t} x(\tau)d\tau$ is causal, because the integral evaluates $\tau$ from $-\infty$ to $t$ (which are all in the past).

4. $y[n] = x[-n]$ is not causal, because $y[-1] = x[1]$, which means the output at $n = -1$ depends an input in the future.

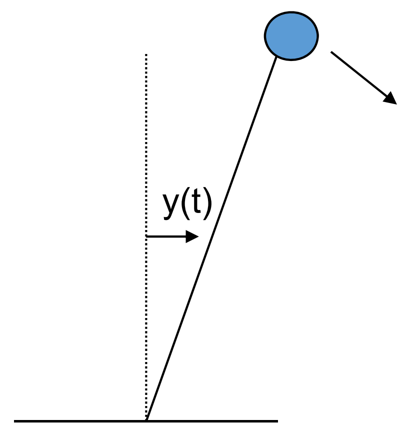
5. $y(t) = x(t)\cos(t+1)$ causal (and memoryless), because $\cos(t+1)$ is a constant with respect to $x(t)$.

# Stability

- A stable system is one in which small inputs lead to response that do not diverge.



stable pendulum




unstable pendulum

# Stability

A system is said to be bounded-input, bounded-output (BIBO) stable if and only if every bounded input results in a bounded output.

$$|x(t)| \le M_x < \infty \text{ for all } t$$

$$\Downarrow$$

$$|y(t)| \le M_y < \infty \text{ for all } t$$

# Stability

**Example 1.**
The system $y(t) = 2x^2(t-1) + x(3t)$ is stable.

*Proof.* To show the system is stable, let us consider a bounded signal $x(t)$, that is, $|x(t)| \leq B$ for some $B < \infty$. Then

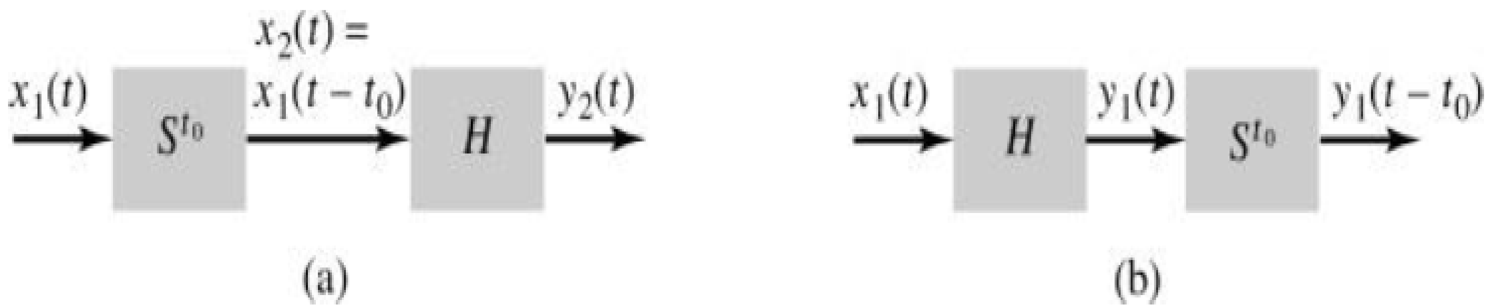
$$\begin{aligned}
|y(t)| &= |2x^2(t-1) + x(3t)| \\
&\leq |2x^2(t-1)| + |x(3t)| \qquad \text{, by Triangle Inequality} \\
&\leq 2|x^2(t-1)| + |x(3t)| \\
&\leq 2B^2 + B < \infty.
\end{aligned}$$

Therefore, for any bounded input $x(t)$, the output $y(t)$ is always bounded. Hence the system is stable. $\square$

# Time Invariance

- System is time invariant if the behavior and characteristics of the system are fixed over time.
- E.g. the RC circuitry where the values of the parameter of the components R and C do not changed with time i.e. constant.
- <u>A system is time invariant if a time shift in the input signal results in an identical time shift in the output signal.</u>

# Time Invariance




The notion of time invariance. (a) Time-shift operator $S^{t0}$ preceding operator $H$. (b) Time-shift operator $S^{t0}$ following operator $H$. These two situations are equivalent, provided that $H$ is time invariant.

# Time Invariance

**Example 1.**
The system $y(t) = \sin[x(t)]$ is time-invariant.

*Proof.* Let us consider a time-shifted signal $x_1(t) = x(t - t_0)$. Correspondingly, we let $y_1(t)$ be the output of $x_1(t)$. Therefore,

$$y_1(t) = \sin[x_1(t)] = \sin[x(t - t_0)].$$

Now, we have to check whether $y_1(t) = y(t - t_0)$. To show this, we note that

$$y(t - t_0) = \sin[x(t - t_0)],$$

which is the same as $y_1(t)$. Therefore, the system is time-invariant. $\square$

# Time Invariance

**Example 2.**
The system $y[n] = nx[n]$ is not time-invariant.

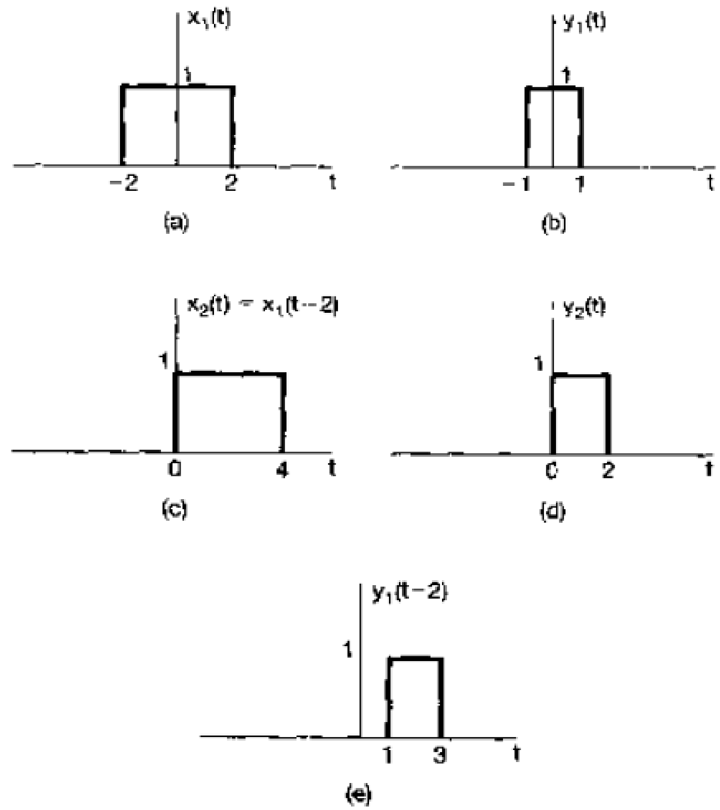
*Proof.* To show that the system in not time-invariant, we can construct a counter example. Let $x[n] = \delta[n]$, then $y[n] = n\delta[n] = 0$, $\forall n$ (Why?). Now, let $x_1[n] = x[n-1] = \delta[n-1]$. If $y_1[n]$ is the output produced by $x_1[n]$, it is easy to show that

$$\begin{aligned} y_1[n] &= nx_1[n] \\ &= n\delta[n-1] \\ &= \delta[n-1]. \qquad \text{(Why?)} \end{aligned}$$

However, $y[n-1] = (n-1)x[n-1] = (n-1)\delta[n-1] = 0$ for all $n$. So $y_1[n] \neq y[n-1]$. In other words, we have constructed an example such that $y[n-1]$ is not the output of $x[n-1]$. $\square$

# Time Invariance

$$y(t) = x(2t)$$




Not Time Invariance

**Figure 1.47** (a) The input $x_1(t)$ to the system in Example 1.16; (b) the output $y_1(t)$ corresponding to $x_1(t)$; (c) the shifted input $x_2(t) = x_1(t-2)$; (d) the output $y_2(t)$ corresponding to $x_2(t)$; (e) the shifted signal $y_1(t-2)$. Note that $y_2(t) \neq y_1(t-2)$, showing that the system is not time invariant.

# Linearity

- The system is linear if it possesses the superposition property.

Let $y_1(t)$ be the response of a continuous - time system to an input $x_1(t)$,
and let $y_2(t)$ be the output corresponding to the input $x_2(t)$.

System is linear if : -

1) The response to $x_1(t) + x_2(t)$ is $y_1(t) + y_2(t)$.

2) The response to $ax_1(t)$ is $ay_1(t)$, where "a" is any complex constant.

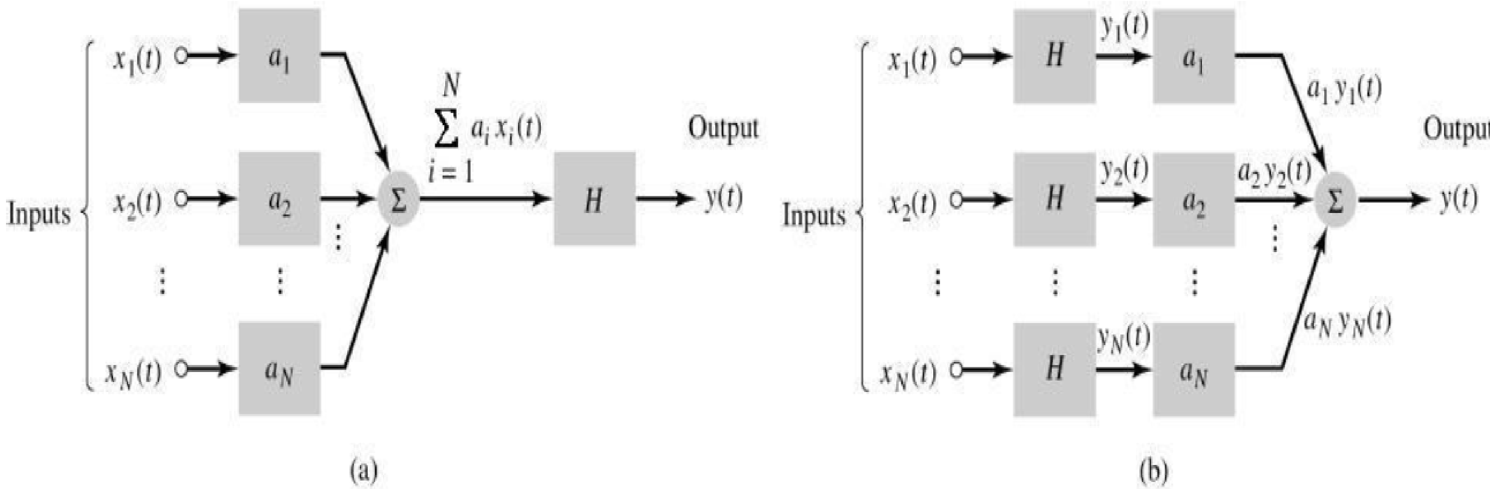
# Linearity

Combining the two property for linearity.

$$ax_1(t)+bx_2(t)\Rightarrow ay_1(t)+by_2(t)$$

$$ax_1[n]+bx_2[n]\Rightarrow ay_1[n]+by_2[n].$$

# Linearity




The linearity property of a system. (a) The combined operation of amplitude scaling and summation precedes the operator *H* for multiple inputs. (b) The operator *H* precedes amplitude scaling for each input; the resulting outputs are summed to produce the overall output $y(t)$. If these two configurations produce the same output $y(t)$, the operator *H* is linear.

# Linearity

**Example 1.**

The system $y(t) = 2\pi x(t)$ is linear. To see this, let's consider a signal

$$x(t) = ax_1(t) + bx_2(t),$$

where $y_1(t) = 2\pi x_1(t)$ and $y_2(t) = 2\pi x_2(t)$. Then

$$\begin{aligned} ay_1(t) + by_2(t) &= a\left(2\pi x_1(t)\right) + b\left(2\pi x_2(t)\right) \\ &= 2\pi\left[ax_1(t) + bx_2(t)\right] = 2\pi x(t) = y(t). \end{aligned}$$

# Linearity

**Example 2.**
The system $y[n] = (x[2n])^2$ is not linear. To see this, let's consider the signal

$$x[n] = ax_1[n] + bx_2[n],$$

where $y_1[n] = (x_1[2n])^2$ and $y_2[n] = (x_2[2n])^2$. We want to see whether $y[n] = ay_1[n] + by_2[n]$. It holds that

$$ay_1[n] + by_2[n] = a\left(x_1[2n]\right)^2 + b\left(x_2[2n]\right)^2.$$

However,

$$y[n] = (x[2n])^2 = (ax_1[2n] + bx_2[2n])^2 = a^2(x_1[2n])^2 + b^2(x_2[2n])^2 + 2abx_1[n]x_2[n].$$