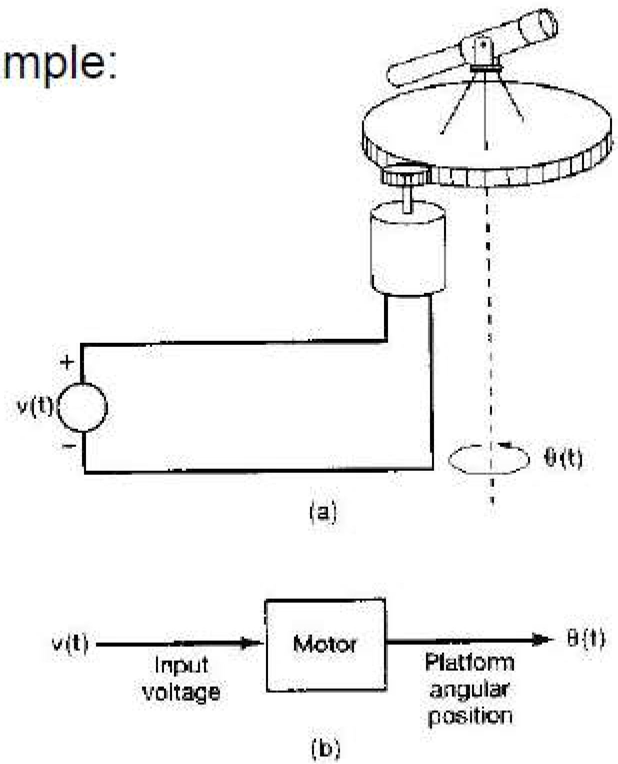
Introduction to Signals and Systems: V216

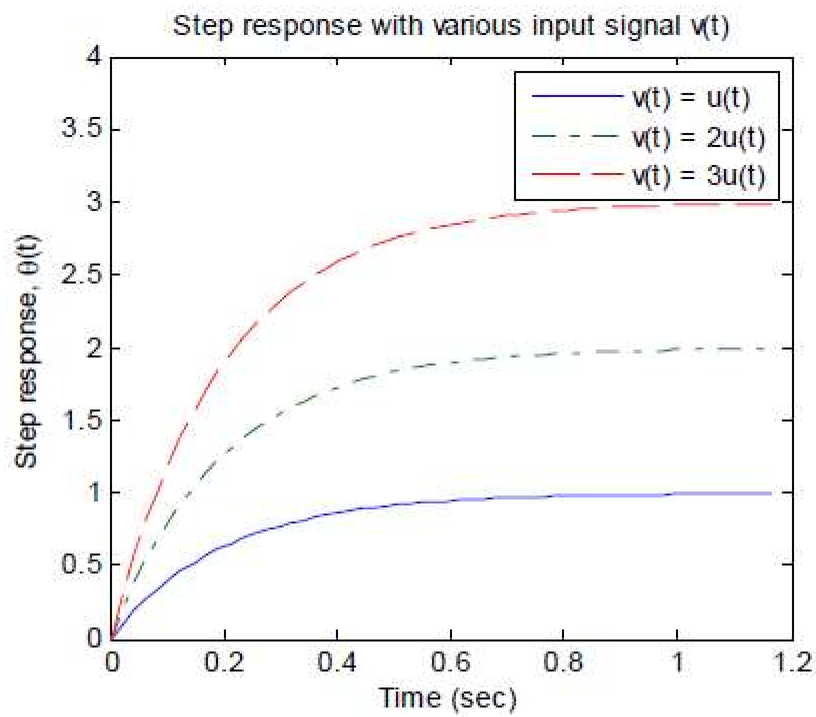
Lecture #17
Chapter 11: Linear Feedback Systems

Introduction

So far, the systems we learnt are referred as open-loop systems. The output of a system is determined by the characteristics of the input signal.

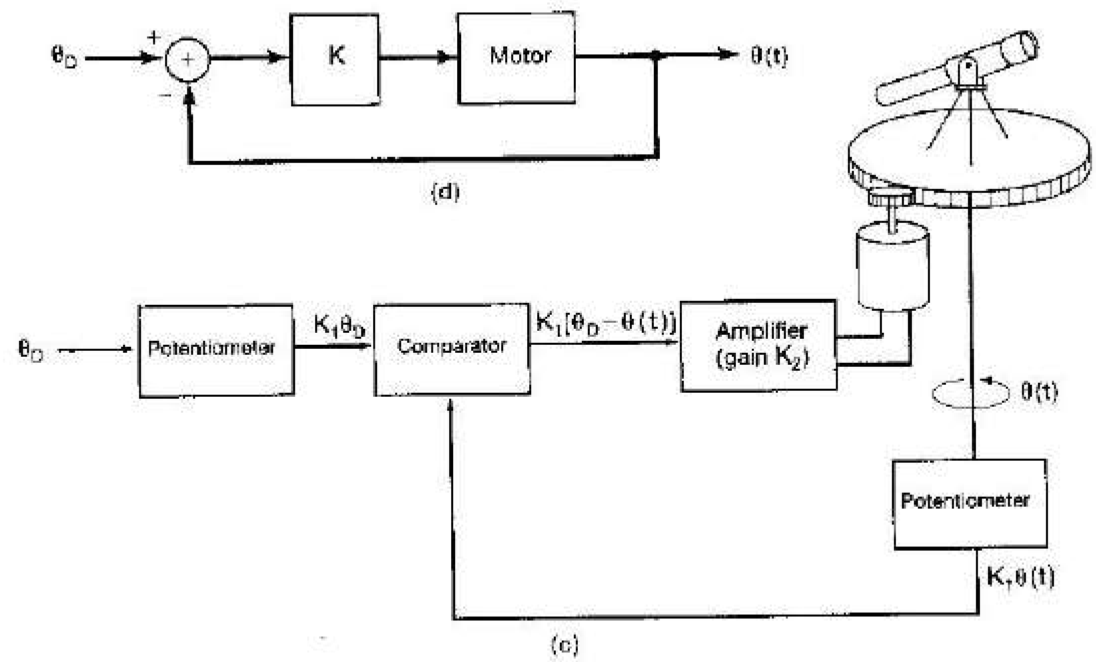
Example:





Introduction

- Instead, we could suggest a different methods for pointing the telescope - the feedback system, by using the output of a system to control or modify the input.
- A feedback system is referred as Closed-loop system.



Introduction

- The closed-loop system have two important advantages:
  - Provide an error-correcting mechanism that can reduce sensitivity to the disturbances and to errors in the modeling of the system
  - Stabilize a system that is inherently unstable

- Typical applications
  - Chemical process control
  - Automotive fuel systems
  - Household heating systems
  - Aerospace systems
  - Stabilizing an inverted pendulum, etc

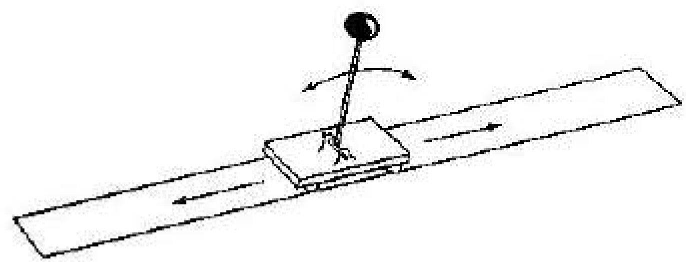
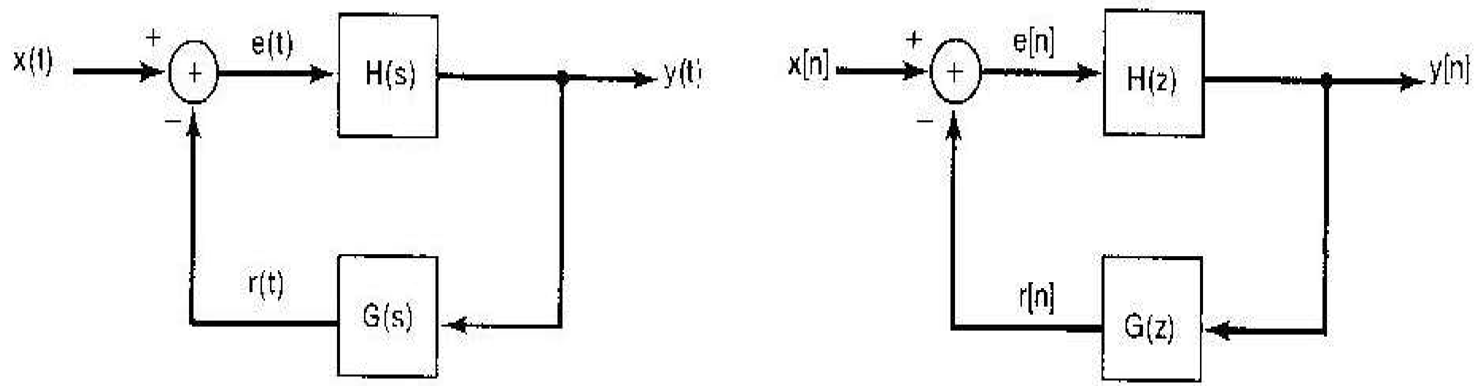


Figure 11.2 An inverted pendulum.

Linear Feedback Systems

- The basic feedback system is configured in



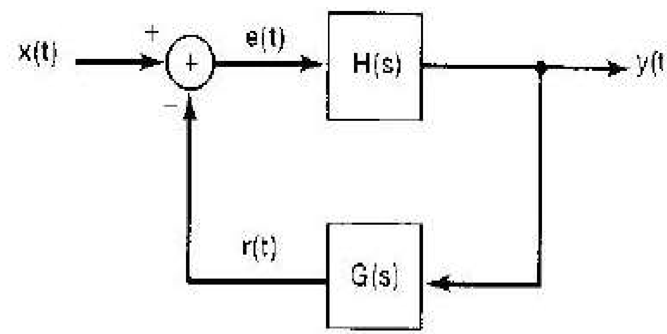
- $H(s)$ (or $H(z)$) is referred as the system function in the forward path
- $G(s)$ (or $G(z)$) is referred as the system function in the feedback path

Linear Feedback Systems

- From the diagram, we obtain the relation

$$Y(s) = H(s)E(s)$$
$$E(s) = X(s) - R(s)$$
$$R(s) = G(s)Y(s)$$

Continuous-Time LTI Feedback System



- These give the closed-loop system function is

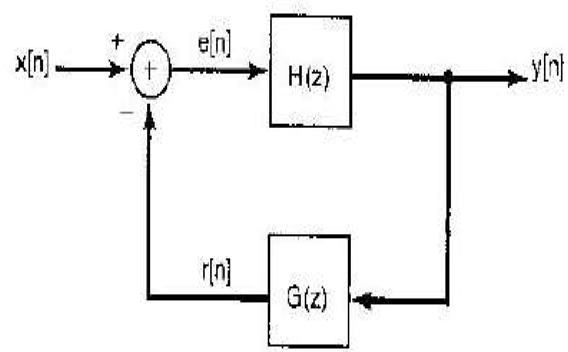
$$\begin{aligned} Y(s) &= H(s)E(s) \\ &= H(s)\left[X(s) - R(s)\right] \\ &= H(s)\left[X(s) - G(s)Y(s)\right] \end{aligned}$$

$$\frac{Y(s)}{X(s)} = Q(s) = \frac{H(s)}{1 + G(s)H(s)}$$

Linear Feedback Systems

- From the diagram, we obtain the relation

Discrete-Time LTI Feedback System



$$Y(z) = H(z)E(z)$$

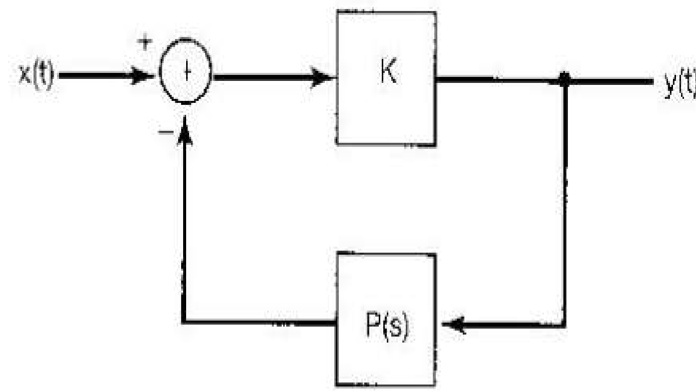
$$E(z) = X(z) - R(z)$$

$$R(z) = G(z)Y(z)$$

- These give the closed-loop system function is

Some applications of feedback

- Section 11.2.1 Inverse System Design
  - Consider a continuous-time LTI system $P(s)$ and is configured as the figure



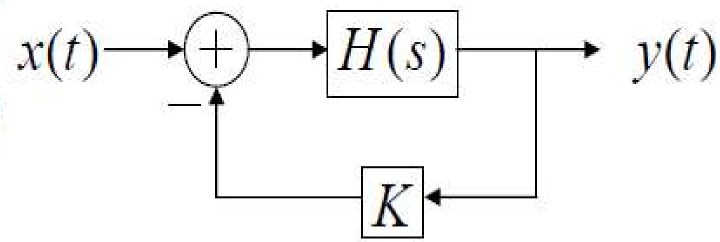
  - The closed-loop system function is

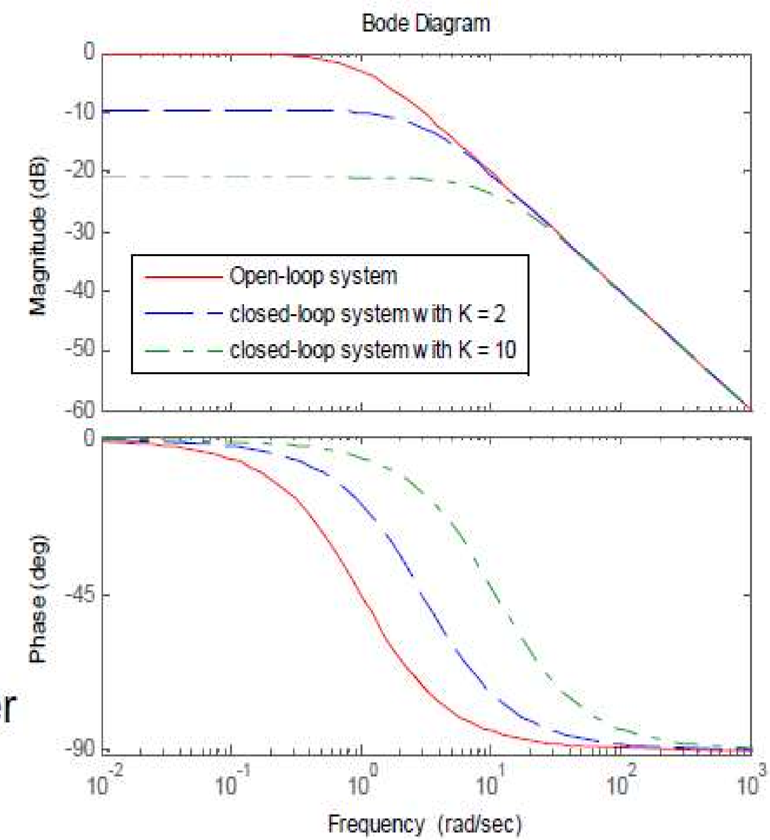
$$\frac{Y(s)}{X(s)} = \frac{K}{1+KP(s)}$$

  - If the gain K is sufficiently large so that $KP(s)$ >> 1, then $\frac{Y(s)}{X(s)} \approx \frac{1}{P(s)}$
  - Applications: Operational amplifiers in feedback systems
    - The implementation of integrators by inserting a capacitor in the feedback path.
    - The implementation of logarithm of input by utilizing the exponential current-voltage characteristics of a diode in feedback path.

Some applications of feedback

- Section 11.2.2 Compensation for non-ideal elements
  - Consider an open-loop frequency response $H(j\omega)$ which provides amplification over the specified frequency band, but which is not constant over that range.





  - The closed-loop system function is

$$\frac{Y(j\omega)}{X(j\omega)} = \frac{H(j\omega)}{1 + KH(j\omega)}$$

  - If, over the specified frequency range, $|K\,H(j\omega)| >> 1$, then

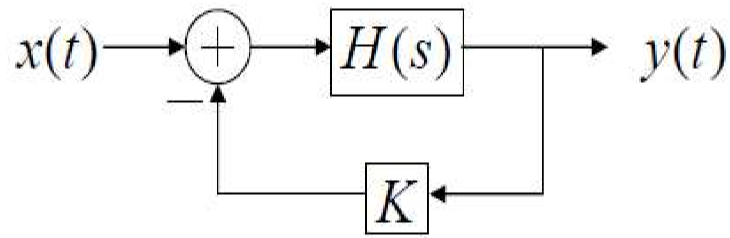
$$\frac{Y(s)}{X(s)} \approx \frac{1}{K}$$

  - Application:
    - Extending the bandwidth of an amplifier
      - Gain x Bandwidth = constant

Some applications of feedback

- Section 11.2.3 Stabilization of Unstable Systems
  - Consider a first-order unstable system with

$$H(s) = \frac{b}{s-a}, \quad a > 0$$

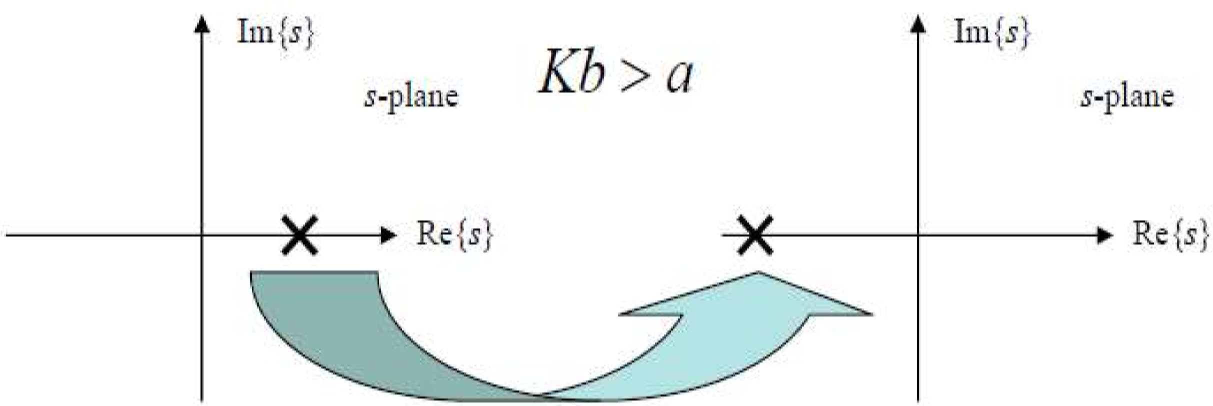


  - The closed-loop system function is

$$\frac{Y(s)}{X(s)} = \frac{H(s)}{1+KH(s)} = \frac{b}{s-a+Kb}$$

  - The closed-loop system will be stable if the pole is moved into the left half of the s-plane. This gives

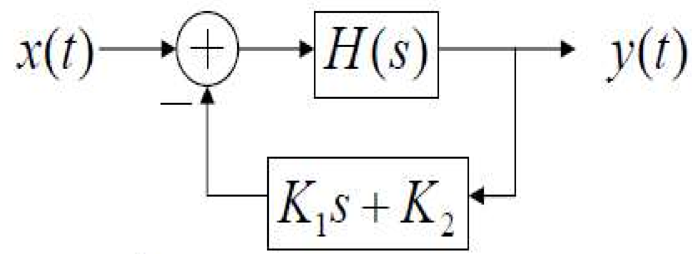
$$Kb > a$$



Some applications of feedback

- Section 11.2.3 Stabilization of Unstable Systems
  - Consider a second-order unstable system with

$$H(s) = \frac{b}{s^2 + a}, \quad a < 0$$

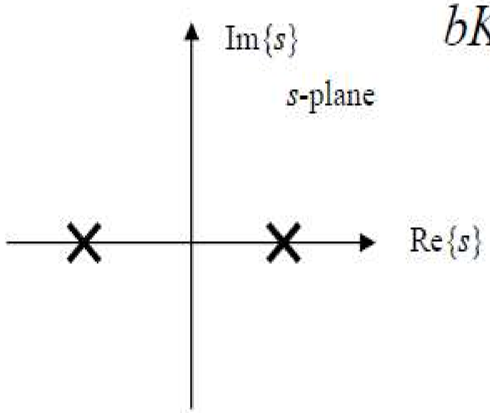


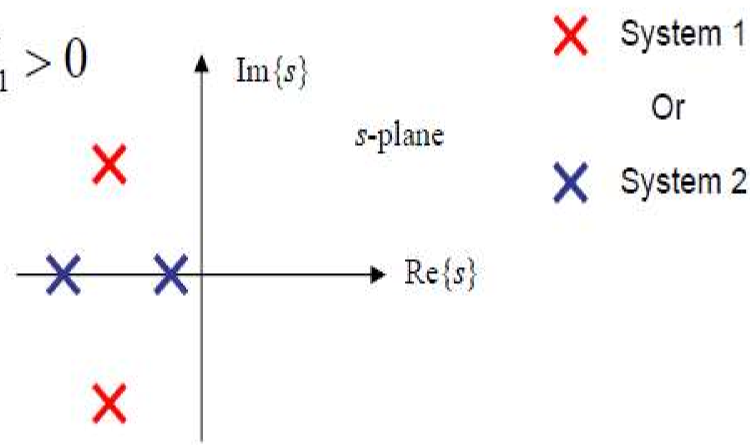
  - The closed-loop system function is

$$\frac{Y(s)}{X(s)} = \frac{H(s)}{1 + (K_1 + K_2 s)H(s)} = \frac{b}{s^2 + bK_2 s + (a + K_1 b)} \qquad \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}.$$

  - The closed-loop system will be stable if the pole is moved into the left half of the s-plane. This gives

$$bK_2 > 0, \quad a + bK_1 > 0$$

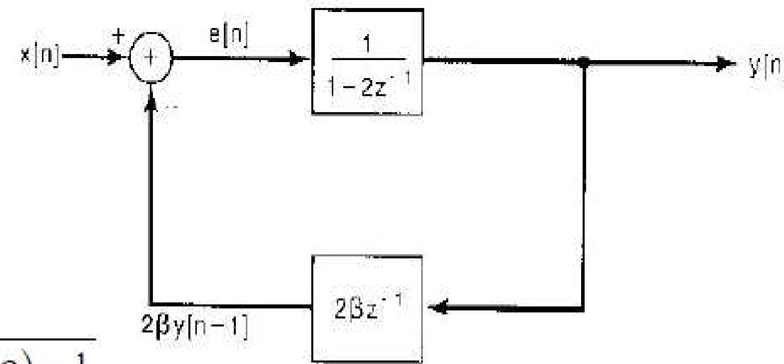




Some applications of feedback

- Section 11.2.3 Stabilization of Unstable Systems
  - Consider a first-order causal but unstable discrete-time system with

$$H(z)=\frac{1}{1-2z^{-1}}$$



  - The closed-loop system function is

$$\frac{Y(z)}{X(z)}=\frac{H(z)}{1+2\beta z^{-1}H(z)}=\frac{1}{1-2(1-\beta)z^{-1}}$$

  - The closed-loop system will be stable if the pole is moved inside the unit circle.

$$|2(1-\beta)|<1$$

$$\Rightarrow -1<2(1-\beta)<1$$

$$\frac{1}{2}<\beta<\frac{3}{2}$$

Some applications of feedback

- Section 11.2.4 Sampled-data feedback systems

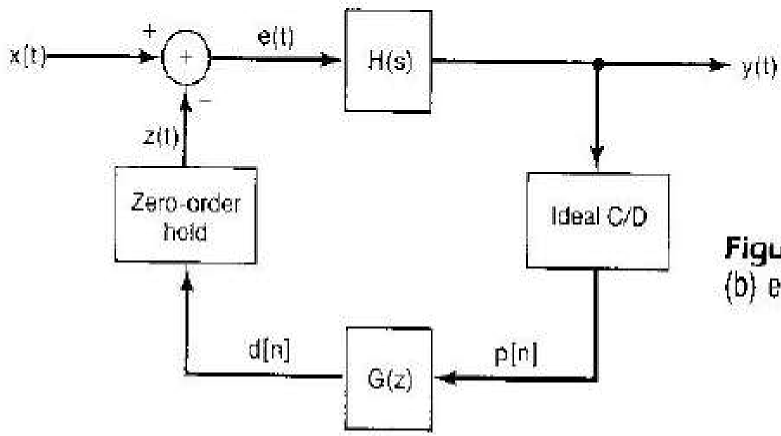
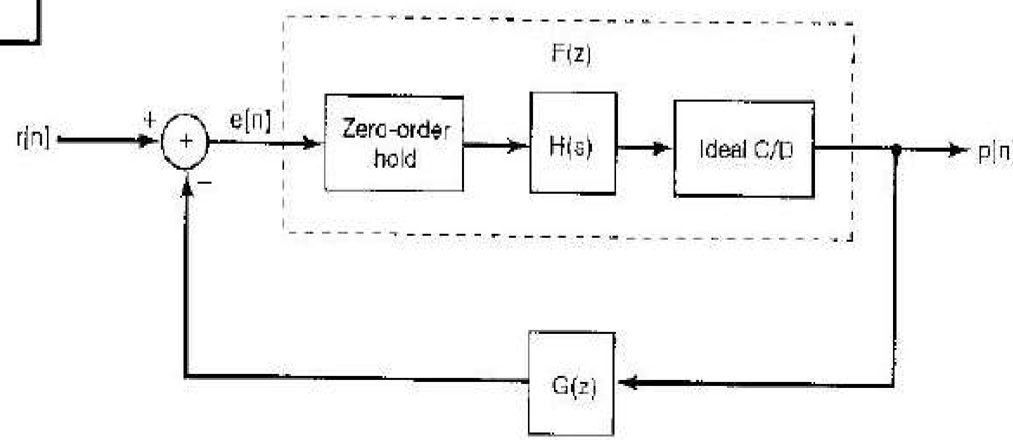


Figure 11.6 (a) A sampled-data feedback system using a zero-order hold; (b) equivalent discrete-time system.

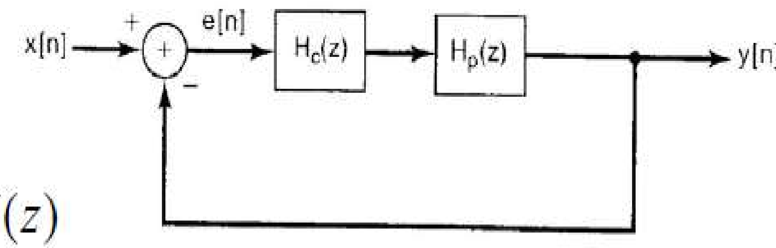


- Noted: $p[n] = y(nT) \qquad x(t) = d[n] \quad for \quad nT \le t < (n+1)T$

$$z(t) = d[n] \quad for \quad nT \le t < (n+1)T$$

Some applications of feedback

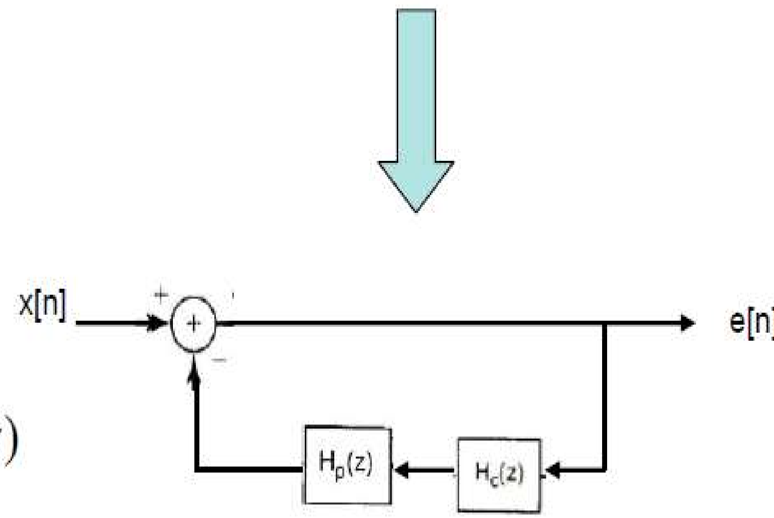
- Section 11.2.5 Tracking systems
  - The output of system can be determined by



$$Y(z)=\frac{H_c(z)H_p(z)}{1+H_c(z)H_p(z)}X(z)$$

  - Also, since $E(z)=Y(z)-X(z)$
  - This gives

$$E(z)=\frac{X(z)}{1+H_c(z)H_p(z)}$$

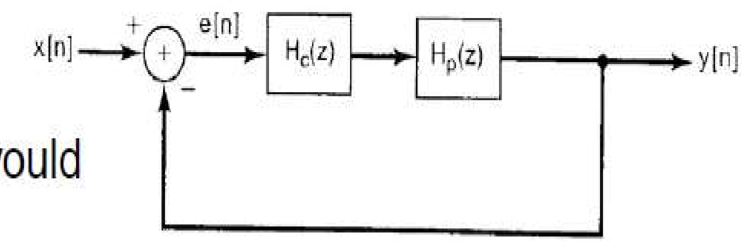


  - Alternative, $Y(z)=H_c(z)H_p(z)E(z)$

$$E(z)=\frac{Y(z)}{H_c(z)H_p(z)}=\frac{X(z)}{1+H_c(z)H_p(z)}$$

Some applications of feedback

- Section 11.2.5 Tracking systems



  - For a good tracking performance, we would like to have a small error (all the time).

$$e[n] \approx 0 \Rightarrow E(e^{j\omega}) = \frac{X(e^{j\omega})}{1 + H_c(e^{j\omega})H_p(e^{j\omega})} \approx 0$$

  - This leads $H_c(e^{j\omega})H_p(e^{j\omega}) >> 1$

    Closed loop system may have trade-offs:
    - the damping ratio is too small
    - may become unstable

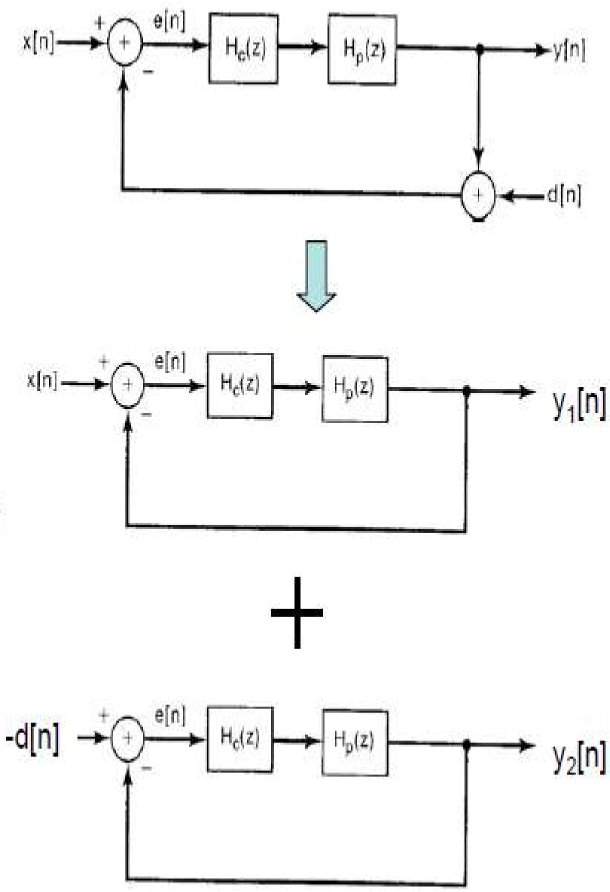
  - In some applications, we are interested in steady state situation.

$$y[n] \to x[n] \quad for \quad n \to \infty \Rightarrow e[n] \to 0$$

Some applications of feedback

- Section 11.2.5 Tracking systems with disturbance
  - Example: with a disturbance $d[n]$ in the feedback path
  - The output of the system is then

$$Y(z)=\underbrace{\frac{H_c(z)H_p(z)}{1+H_c(z)H_p(z)}X(z)}_{\text{Response of input signal}}-\underbrace{\frac{H_c(z)H_p(z)}{1+H_c(z)H_p(z)}D(z)}_{\text{Response of disturbance}}$$



  - We aim to track the system with input and output signals, the tracking error is defined as

$$E(z)=Y(z)-X(z)$$

$$=\frac{1}{1+H_c(z)H_p(z)}X(z)-\underbrace{\frac{H_c(z)H_p(z)}{1+H_c(z)H_p(z)}D(z)}_{}$$

Some applications of feedback

- In general, several typical goals of tracking error
  - Minimize the error due to the disturbance with some designed parameters
    - For instance,

$$\min_{\substack{designed\\parameters}} \sum_{n=-\infty}^{\infty} e_d^2[n] = \min_{\substack{designed\\parameters}} \frac{1}{2\pi}\int_{-\pi}^{\pi} \left\| \frac{H_c(e^{j\omega})H_p(e^{j\omega})D(e^{j\omega})}{1+H_c(e^{j\omega})H_p(e^{j\omega})} \right\|^2 d\omega$$

  - Minimize the error of overall system with some designed parameters
    - For instance,

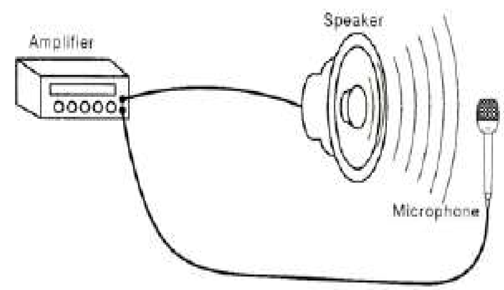
$$\min_{\substack{designed\\parameters}} \sum_{n=-\infty}^{\infty} e^2[n] = \min_{\substack{designed\\parameters}} \frac{1}{2\pi}\int_{-\pi}^{\pi} \left\| \frac{X(e^{j\omega})-H_c(e^{j\omega})H_p(e^{j\omega})D(e^{j\omega})}{1+H_c(e^{j\omega})H_p(e^{j\omega})} \right\|^2 d\omega$$

  - Minimize the steady state error of system with some designed parameters
    - For instance,

$$\min_{\substack{designed\\parameters}} \lim_{n\to\infty} e[n] = \min_{\substack{designed\\parameters}} \lim_{z\to 1}\{Y(z)-X(z)\}$$

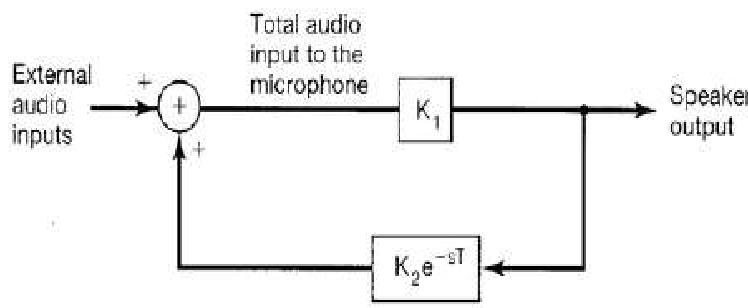
Some applications of feedback

- Section 11.2.6 Destabilization caused by feedback
  - Destabilizing effect of feedback is feedback in audio systems
  - The closed-loop system function is



$$\frac{Y(s)}{X(s)} = \frac{K_1}{1 - K_1 K_2 e^{-sT}}$$

pole:



– The system is unstable if $K_1 K_2 > 1$

– Same result would be obtained by using a technique that in Section 11.3

Root-locus analysis of linear feedback systems

- From the applications we have discussed, a useful type of feedback system is that in which has an adjustable gain K associated with.
- As this gain is varied, it is of interest to examine how the poles of the closed-loop system changes.
  - Broadening the bandwidth (Section 11.2.2)
  - Stabilizing an unstable system (Section 11.2.3)
  - Relocating the poles to improve system performance (Section 11.2.6)

Root-locus analysis of linear feedback systems

- In this section, we discuss a particular method for examining the locus (i.e. the path) in the complex plane of the poles of the closed-loop system as an adjustable gain is varied.

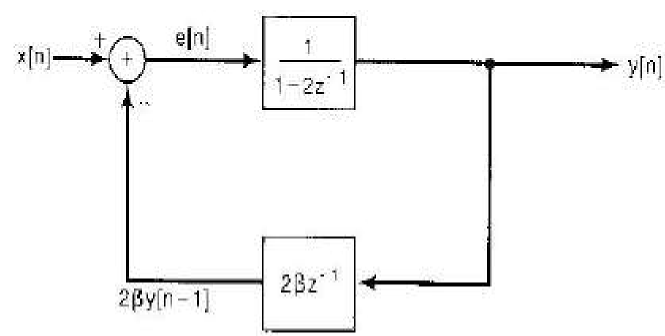
- The procedure, referred to as the root-locus method, is a graphical technique for plotting the closed-loop poles of a rational system function as an adjustable gain is varied.

- The technique works in an identical manner of both continuous-time and discrete-time systems

Root-locus analysis of linear feedback systems

- Example: Reexamine the discrete-time system, we have

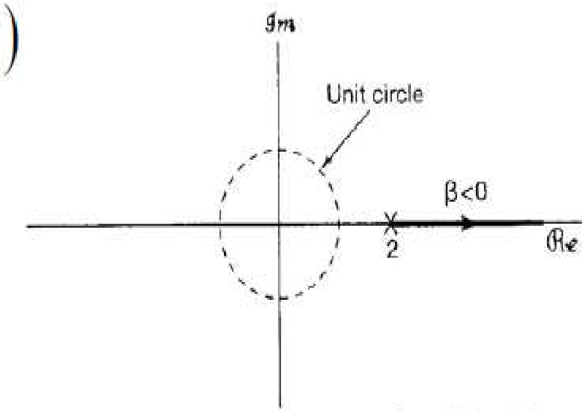
$$H(z) = \frac{1}{1-2z^{-1}} = \frac{z}{z-2} \qquad G(z) = 2\beta z^{-1} = \frac{2\beta}{z}$$

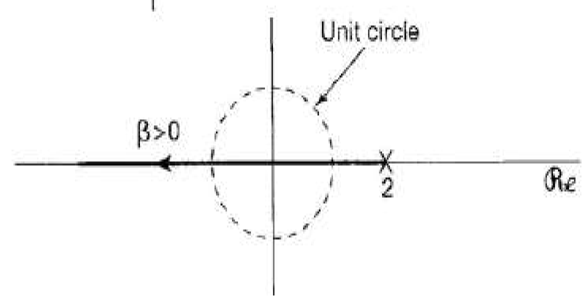


- The closed-loop system function is

$$\frac{Y(z)}{X(z)} = \frac{1}{1-2(1-\beta)z^{-1}} = \frac{z}{z-2(1-\beta)}$$

- The closed-loop zero is
- The closed-loop pole is
- The system is stable if





Root-locus analysis of linear feedback systems

- Example: Consider a continuous-time system with

$$H(s) = \frac{s}{s-2} \qquad G(s) = \frac{2\beta}{s}$$

- The closed-loop system function is $\dfrac{Y(s)}{X(s)} = \dfrac{s}{s-2(1-\beta)}$

- The closed-loop zero is 0 and the closed-loop pole is $2(1-\beta)$

- It is obvious that the locus of the pole as a function of β will be identical to the locus in previous example for discrete-time system.

- The system is stable if

Root-locus analysis of linear feedback systems

- The location of poles are the solutions of the following equations

Continuous-Time LTI Feedback System

$$1 + KG(s)H(s) = 0$$

Discrete-Time LTI Feedback System

$$1 + KG(z)H(z) = 0$$

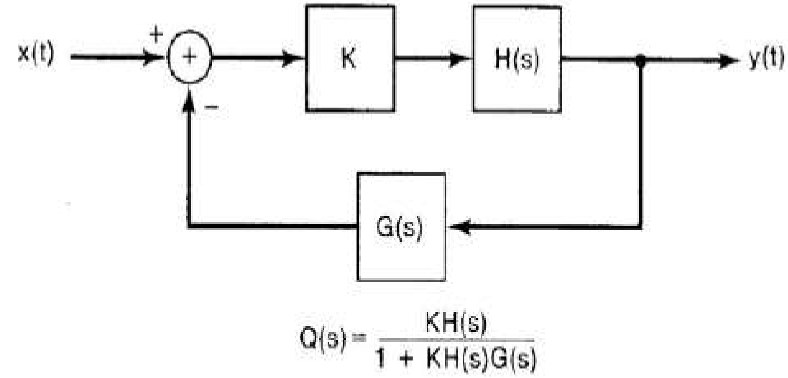
- For a complex system, it is possible to sketch accurately the locus of the poles as the value of gain parameter K is varied from $-\infty$ to ∞, without actually solving for the location of poles for any specific value of the gain.

Root-locus analysis of linear feedback systems

- Consider a modification of the basic feedback system where either $G(s)$ or $H(s)$ is cascaded with an adjustable gain K (real number).

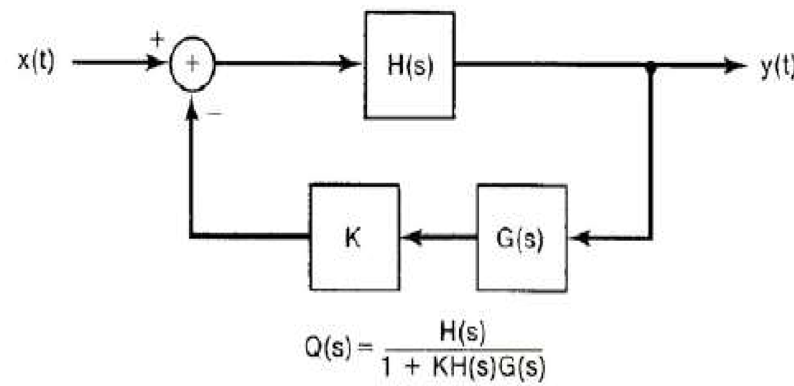
- In either of these cases, the poles of the closed-loop system function is satisfied

$$1 + KG(s)H(s) = 0$$



- Rewrite the equation,

$$G(s)H(s) = \frac{-1}{K}$$



- The technique for plotting the root locus is based on the prosperities of this equation and its solutions.

Root-locus analysis of linear feedback systems

- Usually, we can analyze the root locus with two different regions
  - Root-locus for K is varied from $-\infty$ to 0
  - Root-locus for K is varied from 0 to ∞

- We will phrase our discussion in terms of the Laplace transform variable s, with the understanding that it applies equally well to the discrete-time case.

Root-locus analysis of linear feedback systems

$$G(s)H(s) = \frac{-1}{K}$$

- Criteria 1: End points of the root locus
  - The closed-loop poles s_0 for $K = 0$, and $|K| = +\infty$
  - For $K = 0$, $G(s_0)H(s_0) = \infty$, ➔ closed-loop poles s_0 = poles of $G(s)H(s)$
  - For $|K| = \infty$, $G(s_0)H(s_0) = 0$ ➔ closed-loop poles s_0 = zeroes of $G(s)H(s)$
  - If the order of the numerator of $G(s)H(s)$ is smaller than that of the denominator, then some of zeros, equal in number to the difference in order between the denominator and numerator, will be at infinity.

Root-locus analysis of linear feedback systems

- Criteria 2: The Angle Criterion for the closed-loop pole s_0
  - From $G(s_0)H(s_0)=\frac{-1}{K}$, $G(s_0)H(s_0)$ must be real.
  - The magnitude criterion at closed-loop pole s_0,

$$|G(s_0)H(s_0)| = \left|\frac{1}{K}\right| = \begin{cases} \frac{1}{K}, & K>0 \\ \frac{-1}{K}, & K<0 \end{cases} \Rightarrow$$

The value of K can be evaluated by lengths of Vectors.

  - The angle criterion at closed-loop pole s_0,

$$\angle G(s_0)H(s_0) = q\pi, \quad q \text{ is an integer}$$

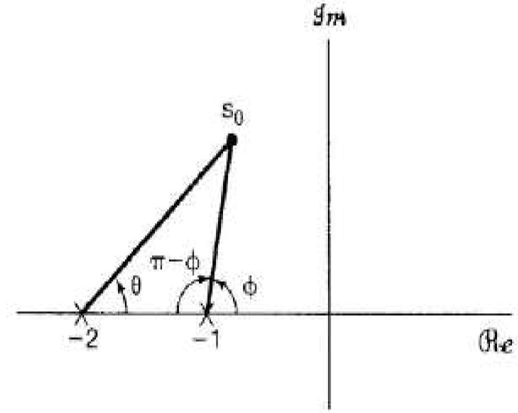
  - In summary, s_0 is the closed-loop pole

$$\textit{if} \quad K>0, \quad G(s_0)H(s_0)<0 \Rightarrow \angle G(s_0)H(s_0) = q\pi, \quad q \text{ is an odd integer}$$

$$\textit{if} \quad K<0, \quad G(s_0)H(s_0)>0 \Rightarrow \angle G(s_0)H(s_0) = q\pi, \quad q \text{ is an even integer}$$

Root-locus analysis of linear feedback systems

- Example 11.1 Consider $G(s)H(s) = \dfrac{1}{(s+1)(s+2)}$



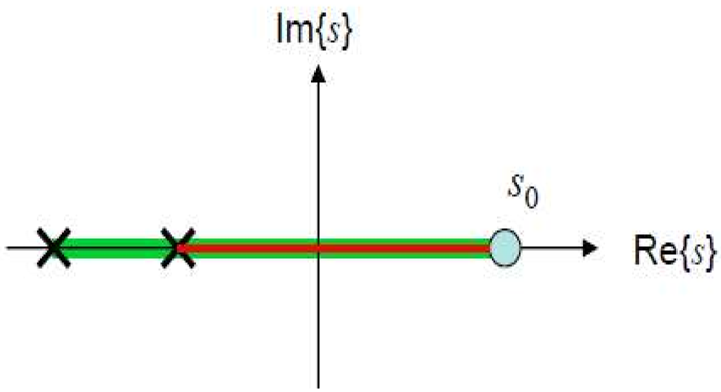
- Examine a point s_0 whether it is a closed-loop pole of closed loop system.

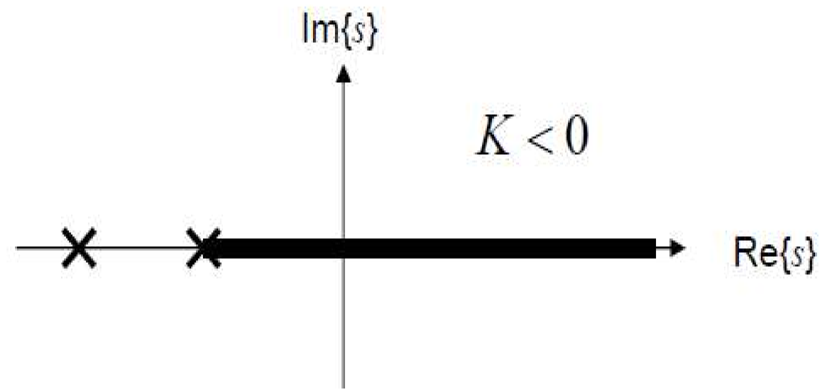
$$G(s_0)H(s_0) = \frac{1}{(s_0+1)(s_0+2)} = \frac{-1}{K}$$

- Consider a point s_0 that lies on real axis of the s-plane
  - Case 1: s_0 is real and s_0 > -1,

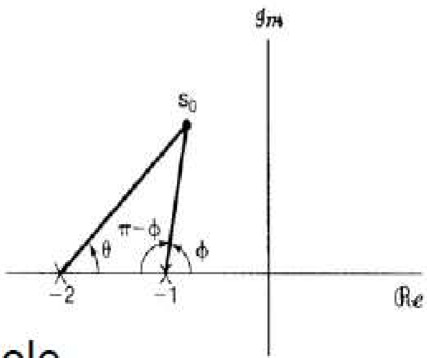
$$\angle G(s_0)H(s_0) = 0 = 0 \cdot \pi$$

It is a closed-loop pole.
➔ Lies on the roots locus with $K < 0$





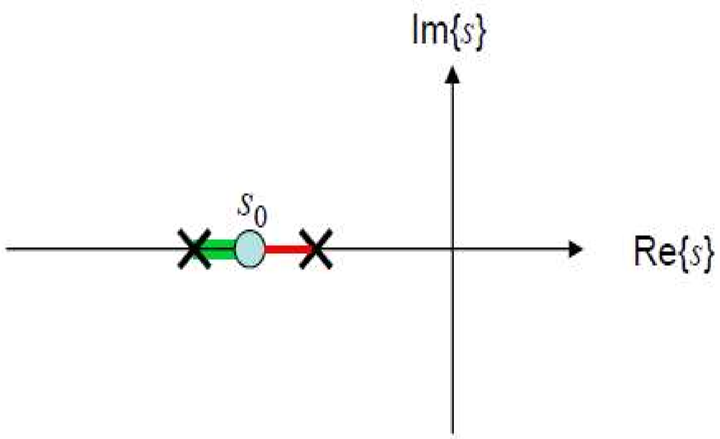
Root-locus analysis of linear feedback systems

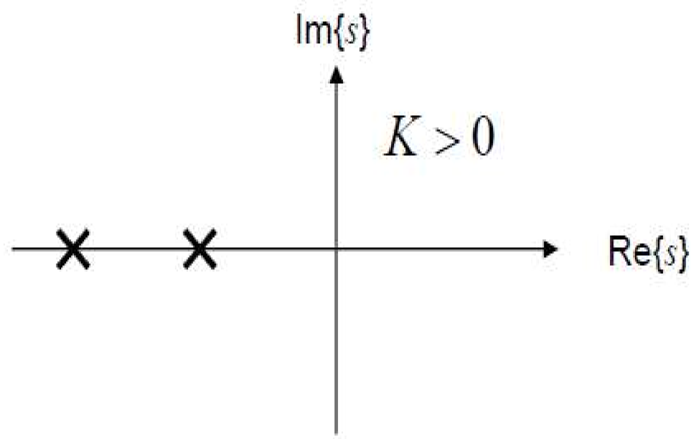


- Example 11.1 (Cont'd)
- Consider a point s_0 that lies on real axis of the s-plane
  - Case 2: s_0 is real and $-2 < s_0 < -1$,

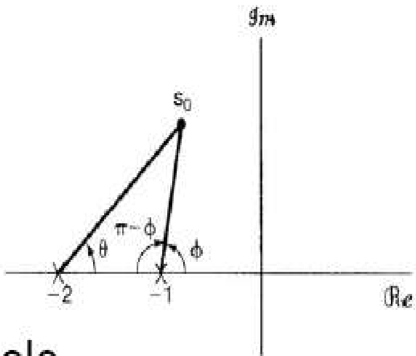
$$\angle G(s_0)H(s_0) = -\pi$$

It is a closed-loop pole.
➔ Lies on the roots locus with $K > 0$





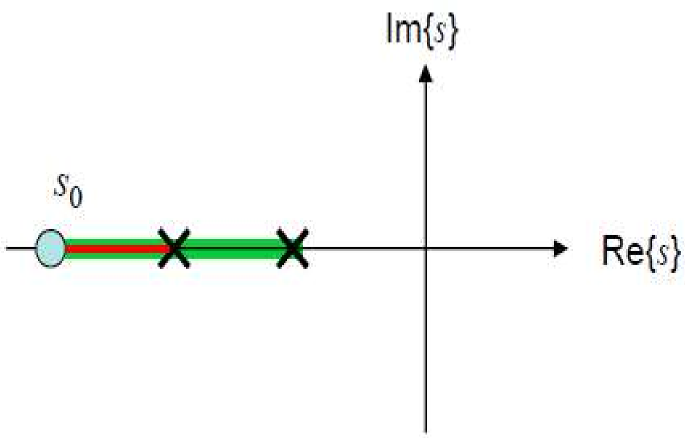
Root-locus analysis of linear feedback systems

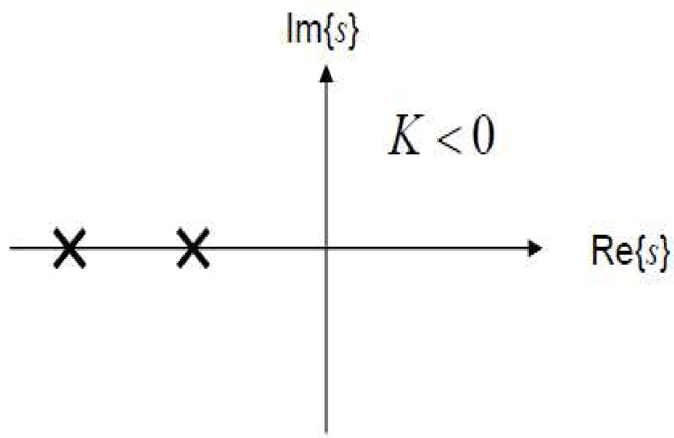


- Example 11.1 (Cont'd)
- Consider a point s_0 that lies on real axis of the s-plane
  - Case 3: s_0 is real and $s_0 < -2$,

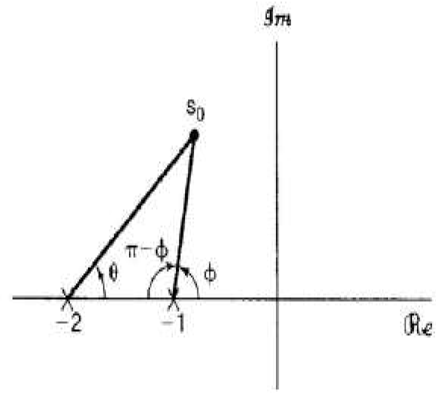
$$\angle G(s_0)H(s_0) = -2\pi$$

It is a closed-loop pole.
➔ Lies on the roots locus with $K < 0$





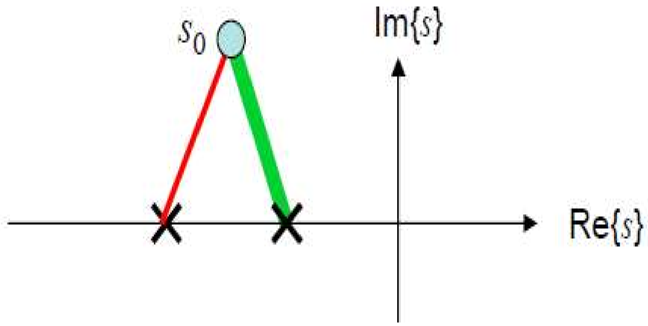
Root-locus analysis of linear feedback systems

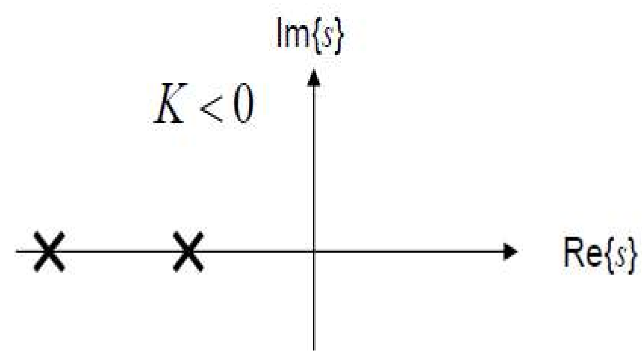


- Example 11.1 (Cont'd)
- Consider a point s_0 that lies on complex plane
  - Case 4: s_0 is complex point in the upper half plane,

$$\angle G(s_0)H(s_0) = -(\theta+\phi)$$

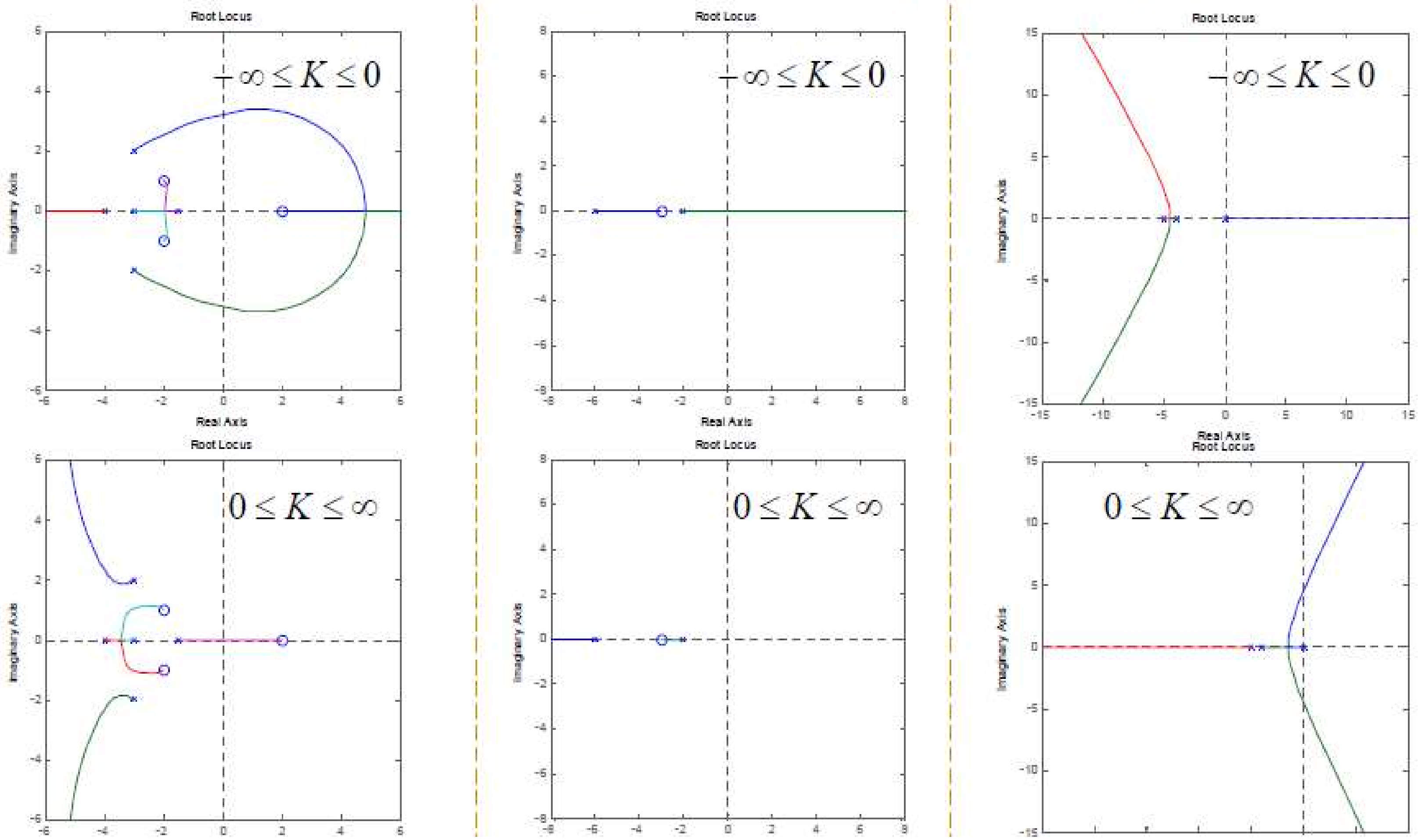
  - As $0<\theta<\pi$ and $0<\phi<\pi$, thus $-2\pi<\theta+\phi<0$
  - There is no point in the upper half-plane can be the locus for $K<0$ as $\angle G(s_0)H(s_0)$ never equals an even multiple of π.
  - In addition, if s_0 is be on the locus for $K>0$, we must have $\angle G(s_0)H(s_0) = -(\theta+\phi) = -\pi$





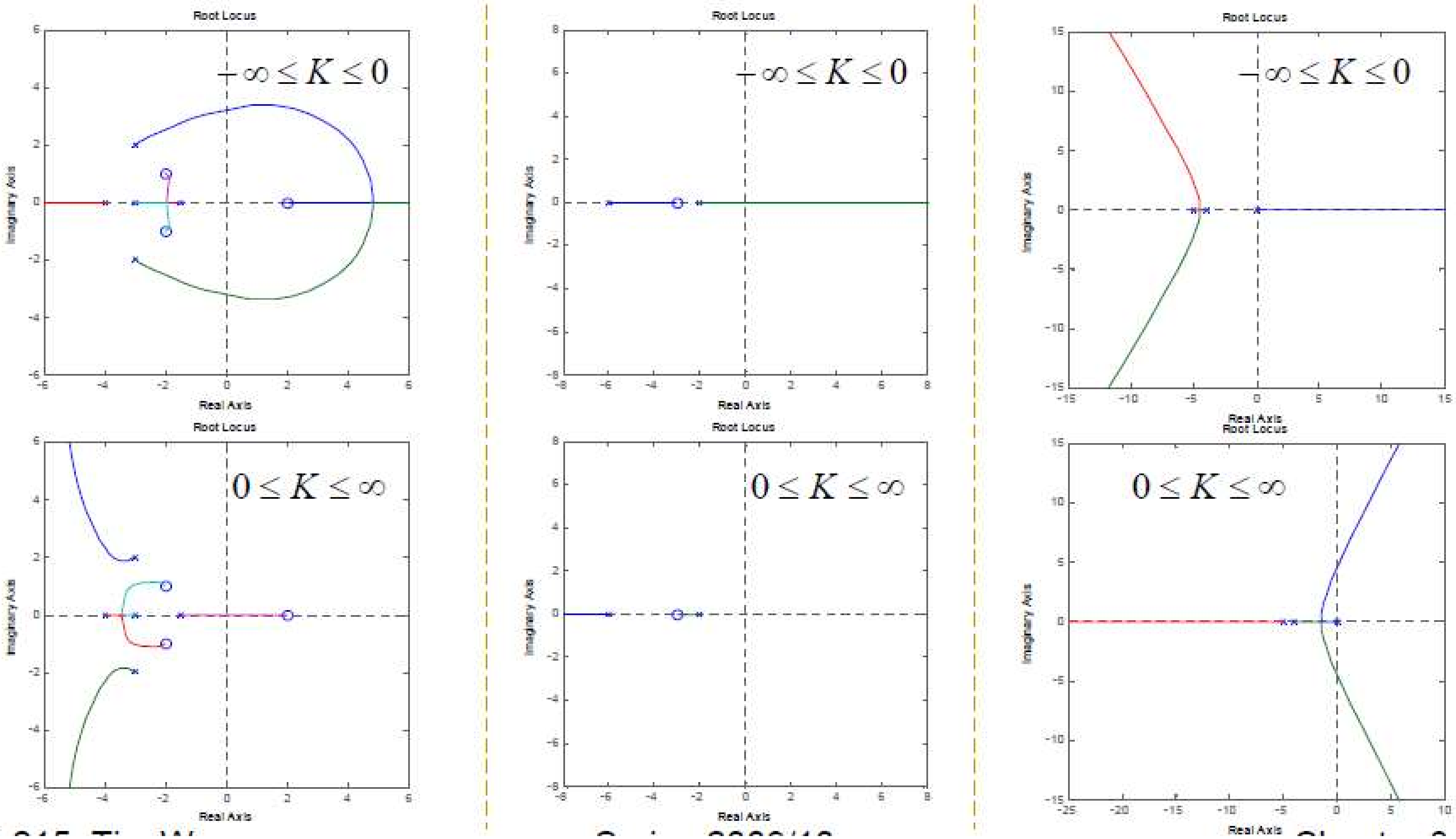
Root-locus analysis of linear feedback systems

- Property 1: For $K = 0$, the solution of $G(s)H(s) = 1/K$ are the poles of $G(s)H(s)$. Since we are assuming n poles, the root locus has n branches, each one starting (for K=0) at a pole of $G(s)H(s)$.



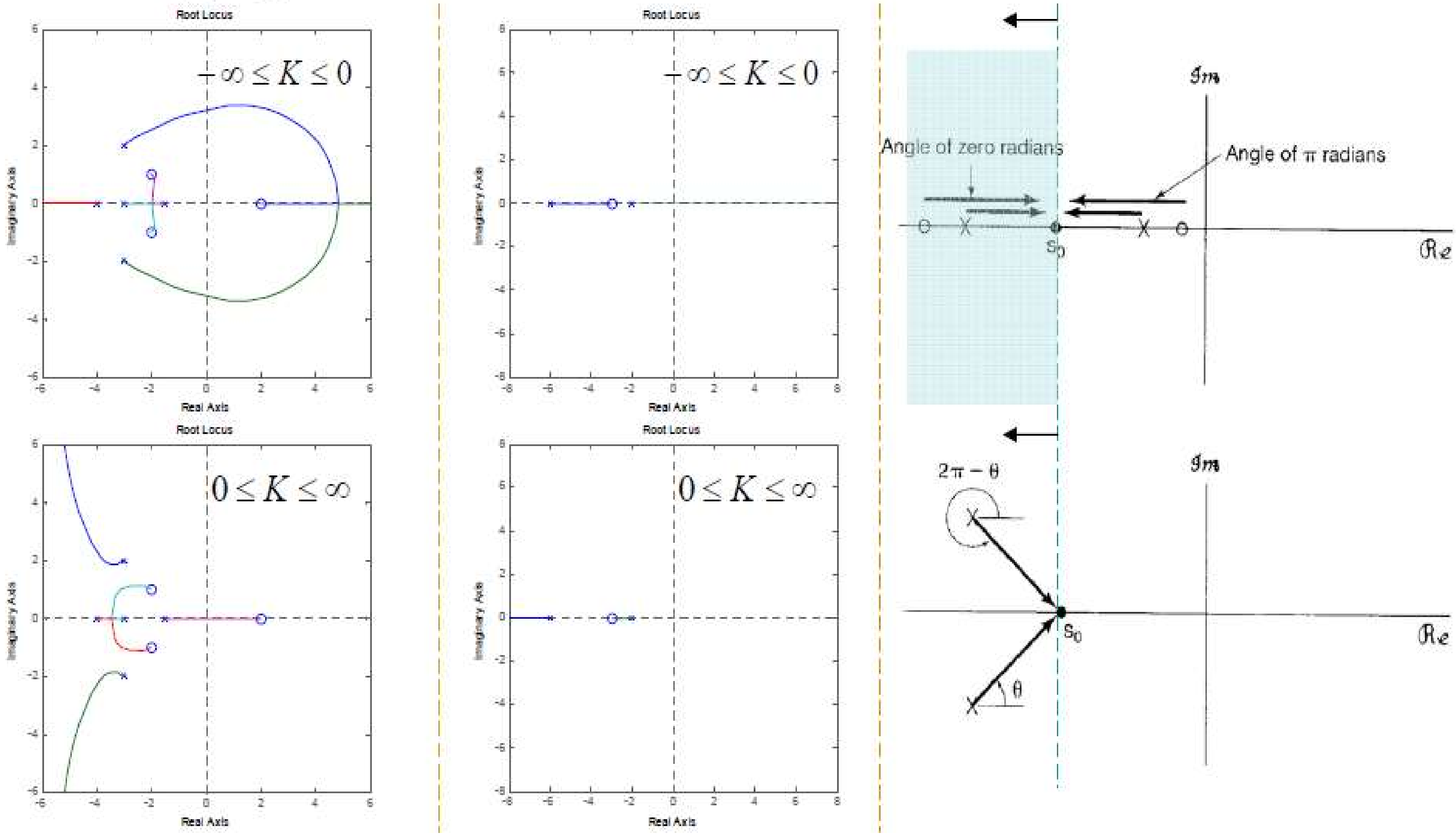
Root-locus analysis of linear feedback systems

- Property 2: As $|K| \rightarrow \infty$, each branch of the root locus approaches a zero of $G(s)H(s)$. Since we are assuming that $m \le n$, n - m of these zeros are at infinity.



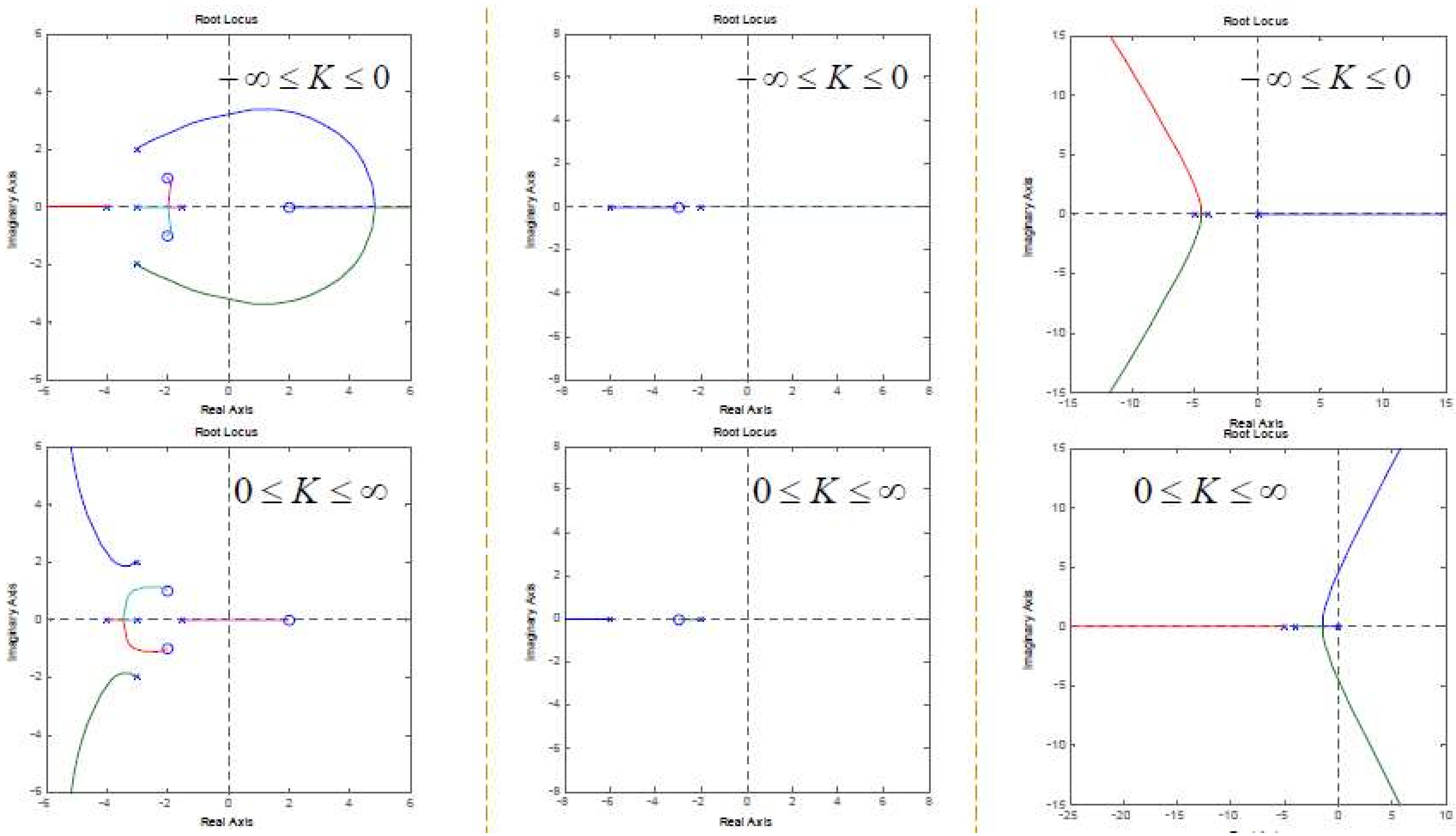
Root-locus analysis of linear feedback systems

- Property 3: Parts of the real s-axis that lie to the left of an odd number real poles and zeros of $G(s)H(s)$ are on the root locus for $K > 0$. Parts of the real s-axis that lie to the left of an even number (possibly zero) real poles and zeros of $G(s)H(s)$ are on the root locus for $K < 0$.



Root-locus analysis of linear feedback systems

- Property 4: Branches of the root locus between two real poles must break off into the complex plane for $|K|$ is large enough.



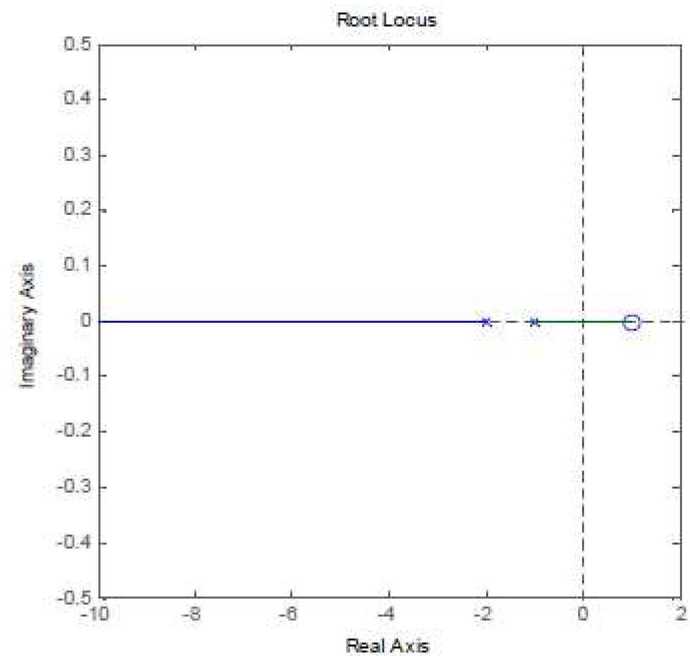
Root-locus analysis of linear feedback systems

- Example 11.2 Consider a continuous-time feedback system with

$$G(s)H(s) = \frac{s-1}{(s+1)(s+2)}$$

- From Properties 1 and 2, the root locus for either K positive or K negative starts at the points s = -1 and s = -2. One branch terminates at the points s = 1 and the other at infinity.
- By property 3,
  - For $K > 0$, the branches are $s < -2$ and $-1 < s < 1$
- For $K > 0$, if K is sufficient large, the system becomes unstable.
  - The root locus passes through $s = 0$

$$K = \left| \frac{1}{G(s)H(s)} \right|_{s=0} =$$



Root-locus analysis of linear feedback systems

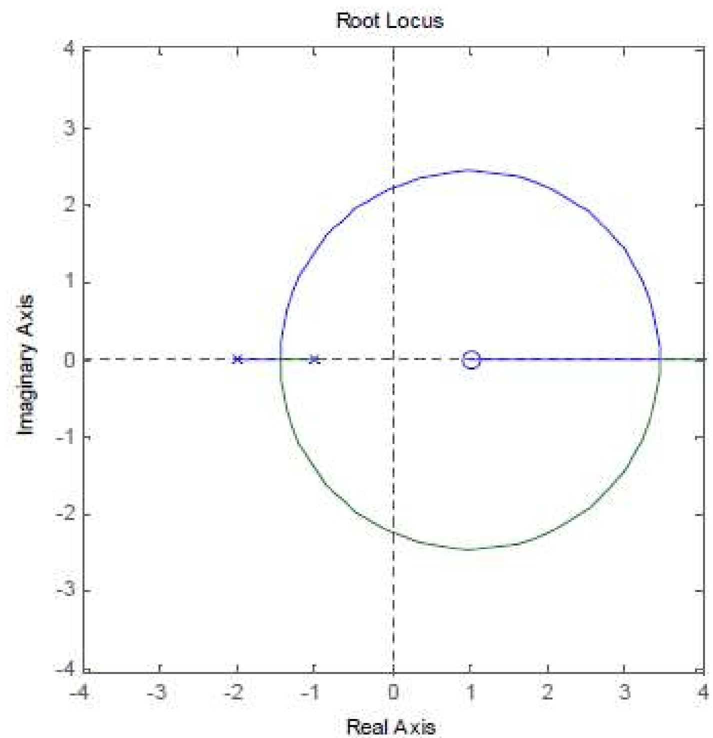
- Example 11.2 (Cont'd)
- By property 3, for $K < 0$, the branches are $-2 < s < -1$ and $s > 1$
- Property 4, the branches break off in between $-2 < s < -1$
- For $K < 0$, if K is sufficient large in magnitude, the system becomes unstable.
  – The root locus passes through $s = j2.2361$

$$K = -\left|\frac{1}{G(s)H(s)}\right|_{s=j2.2361} = -3$$

  – With the use of MATLAB command

    [k, poles]=rlocfind(sys); (just point and click on any desired point on the plot after entering this command)

    We have K = - 3



Root-locus analysis of linear feedback systems

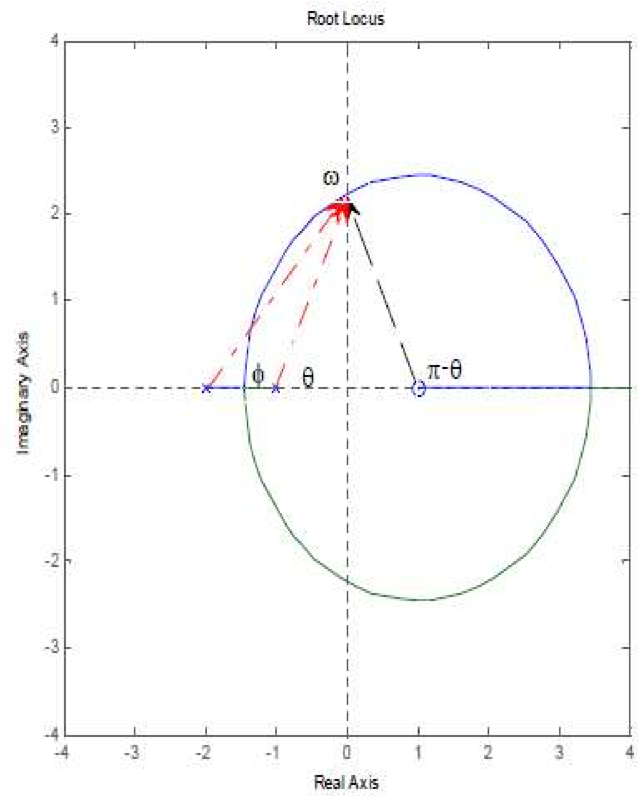
- Example 11.2 (Cont'd)
- We can also find the roots of imaginary axis mathematically.
- The root locus passes through $s = j\omega$ such that $\angle G(j\omega)H(j\omega) = 0$

$$\left(\pi - \tan^{-1}\omega\right) - \tan^{-1}\omega - \tan^{-1}\frac{\omega}{2} = 0$$

$$\tan\left(2\tan^{-1}\omega\right) = \tan\left(\pi - \tan^{-1}\frac{\omega}{2}\right)$$

$$\frac{2\omega}{1-\omega^2} = -\frac{\omega}{2}$$

$$\omega = \pm\sqrt{5} = \pm 2.2361 \quad or \quad 0$$



- This gives

$$K = -\left|\frac{1}{G(s)H(s)}\right|_{s=j2.2361} = -3$$

Routh-Hurwitz Criteria for Stability

- We can find the roots of imaginary axis using Routh-Hurwitz criteria

$1+KG(s)H(s)=0$, $S^2+3S+2+K(S-1)=0$

$S^2+(3+K)S+(2-K)=0$

$$
\begin{array}{c|ll}
S^2 & 1 & (2-K) \\
S^1 & (3+K) & 0 \\
S^0 & \dfrac{(3+K)\,(2-K)+1X0}{(3+K)} &
\end{array}
$$

$3+K=0$, $K=-3$

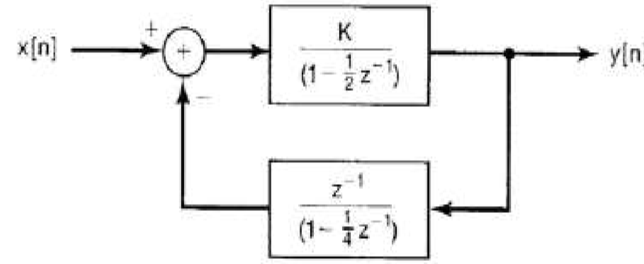
$S^2+(2-K)=0$ $S^2+(2+3)=0$ $S^2+=-5$

$S= \pm j2.2361$

Root-locus analysis of linear feedback systems

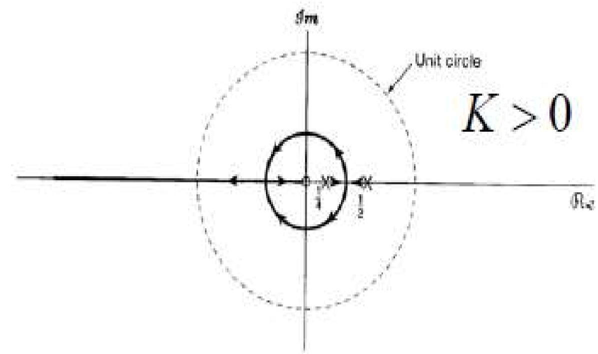
- Example 11.3 Consider a discrete-time feedback system with

$$G(z)H(z)=\frac{z^{-1}}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{4}z^{-1})}=\frac{z}{(z-\frac{1}{2})(z-\frac{1}{4})}$$



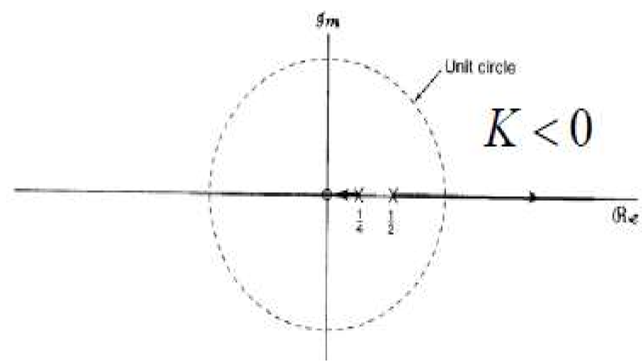
- For $K > 0$, we see that the transition from stability to instability occurs when one of the closed-loop poles is at z = -1.

$$K =$$



- For $K < 0$, we see that the transition from stability to instability occurs when one of the closed-loop poles is at z = 1.

$$K =$$



Root-locus analysis of linear feedback systems

- Other properties of the root locus
  - The root locus is symmetric with respect to the real axis.
  - As $|K| \rightarrow \infty$, n - m each branch of the root locus approaches a zero of asymptotically n - m straight lines (called asymptotes) with angles

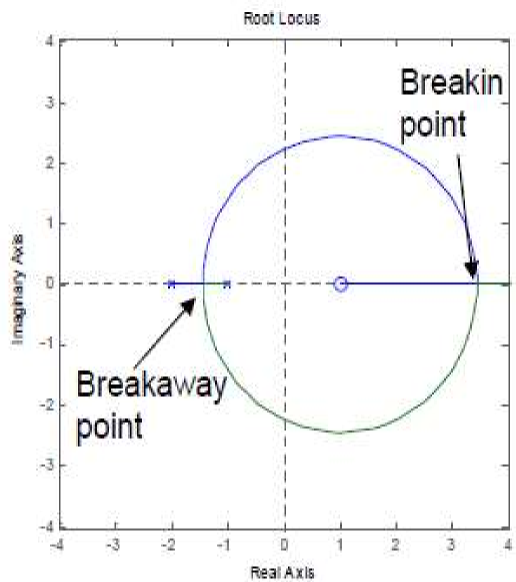
$$\theta = \frac{q\pi}{n-m}, \qquad \begin{cases} q = \text{odd integer}, & K > 0 \\ q = \text{even integer}, & K < 0 \end{cases}$$

and the starting point of all asymptotes is on the real axis at

$$\kappa = \frac{\sum_{i=1}^{n} p_i - \sum_{j=1}^{m} z_j}{n-m} = \frac{\sum poles - \sum zeros}{n-m}$$

  - The breakaway and breakin points are among the roots of equation

$$\frac{d\left[G(s)H(s)\right]}{ds} = 0$$

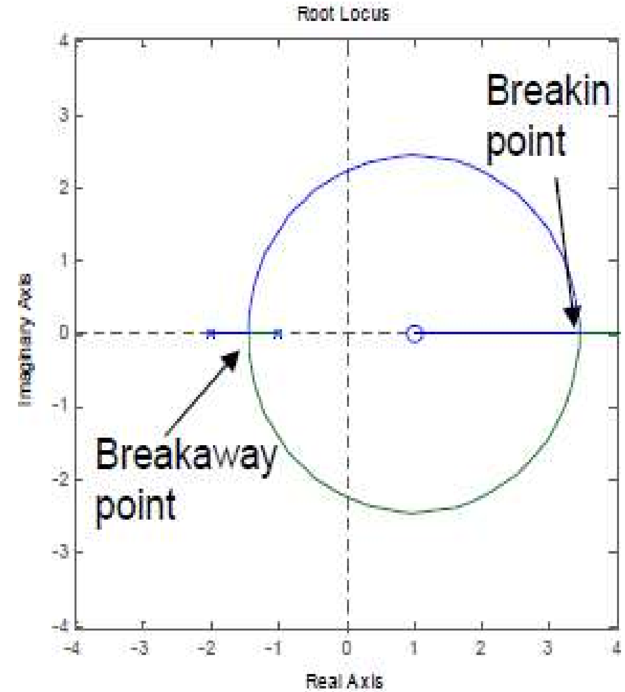


Break away and Break-in Points

1- Mathematical:

- The breakaway and breakin points are among the roots of equation

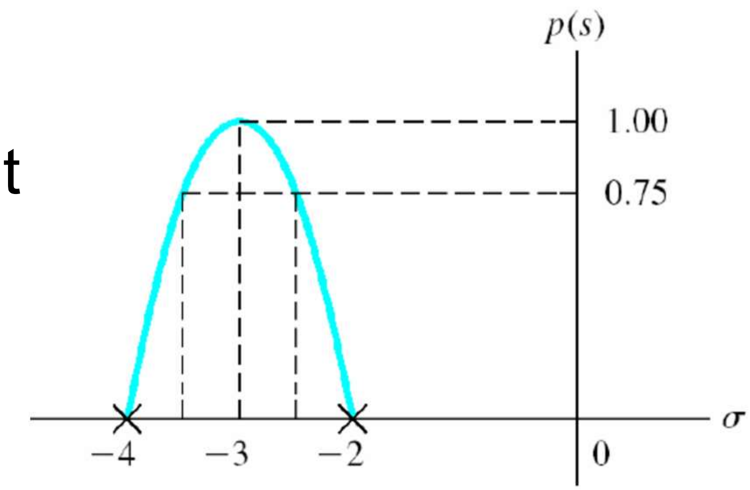
$$\frac{d[G(s)H(s)]}{ds} = 0$$



$$\frac{d}{ds}G(s)H(s) = \frac{d}{ds}\frac{N(s)}{D(s)} = \frac{D(s)\dfrac{dN(s)}{ds} - N(s)\dfrac{dD(s)}{ds}}{D^2(s)} = 0$$

2- Graphical:

Plot G(s)H(s) , then find maximum point



Root-locus analysis of linear feedback systems

- Other properties of the root locus
  - The angle of departure ϕ_k from the pole p_k with order l is given by

$$\phi_k = \frac{\sum_{i=1}^{m} \angle(p_k - z_i) - \sum_{j=1, j\neq k}^{n} \angle(p_k - p_j) \pm q\pi}{l}$$

    where q = 0 for K <0 and q = 1 for K > 0

  - The angle of arrival ψ_k at the zero z_k with order l is given by

$$\psi_k = \frac{-\sum_{i=1, i\neq k}^{m} \angle(z_k - z_i) + \sum_{j=1}^{n} \angle(z_k - p_j) \pm q\pi}{l}$$

    where q = 0 for K <0 and q = 1 for K > 0

Root-locus 7 Steps for K>0

Step	Related Equation or Rule
1. Prepare the root locus sketch.	
(a) Write the characteristic equation so that the parameter of interest, K, appears as a multiplier.	1+KG(s)H(s)=0
(b) Factor $G(s)H(s)$ terms of n poles and M zeros.	$1 + K\dfrac{\prod_{i=1}^{M}(s + z_i)}{\prod_{j=1}^{n}(s + p_j)} = 0.$
(c) Locate the open-loop poles and zeros of $G(s)H(s)$ in the s-plane with selected symbols.	$\times$ = poles, $\bigcirc$ = zeros Locus begins at a pole and ends at a zero.
(d) Determine the number of separate loci, SL.	$SL = n$ when $n \geqslant M$; n = number of finite poles, M = number of finite zeros.
(e) The root loci are symmetrical with respect to the horizontal real axis.	

Root-locus 7 Steps for K>0

2. Locate the segments of the real axis that are root loci.

   Locus lies to the left of an odd number of poles and zeros.

3. The loci proceed to the zeros at infinity along asymptotes centered at σ_A and with angles ϕ_A.

   $$\sigma_A = \frac{\sum(-p_j) - \sum(-z_i)}{n - M}.$$

   $$\phi_A = \frac{2k+1}{n-M}180^\circ,\ k = 0, 1, 2, \ldots (n - M - 1).$$

4. Determine the points at which the locus crosses the imaginary axis (if it does so).

   Use Routh–Hurwitz criterion (see Section 6.2).

5. Determine the breakaway point on the real axis (if any).

   a) Set $K = G(s)H(s)$

   b) Determine roots of dG(s)H(s)/ds=0 or use graphical method to find maximum of $p(s)$.

6. Determine the angle of locus departure from complex poles and the angle of locus arrival at complex zeros, using the phase criterion.

   <G(s)H(s) $= 180^\circ + k360^\circ$ at $s = -p_j$ or $-z_i$.

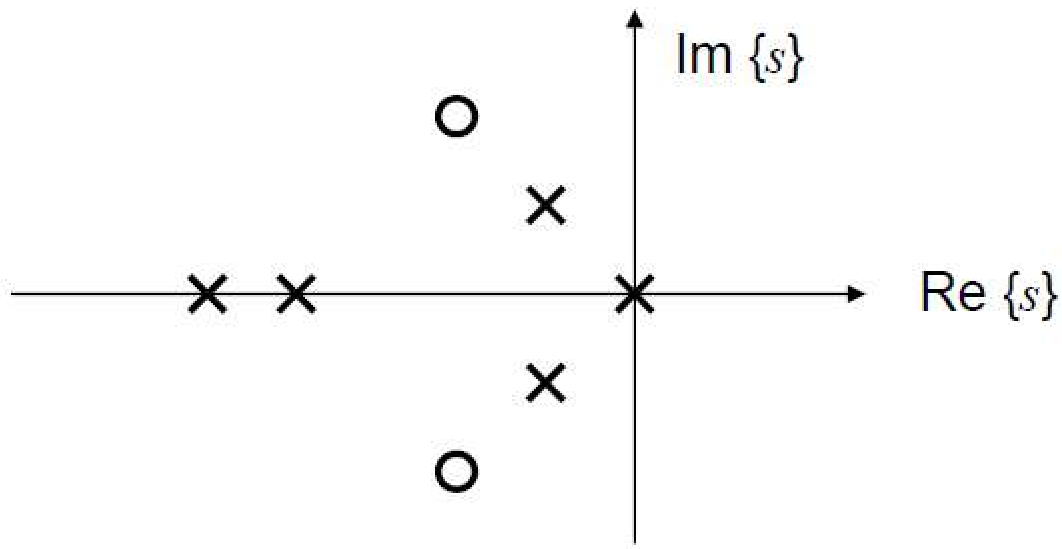
7. Complete the root locus sketch.

Root-locus analysis of linear feedback systems

- Sketch the root locus of a continuous-time feedback system with

$$G(s)H(s) = \frac{(s^2+4s+8)}{s(s+4)(s+5)(s^2+2s+2)}$$

  and $K > 0$.
- poles: 0, -4, -5 , -1 + j and -1 - j zeros: -2 + 2 j and -2 – 2 j
- From properties 1 and 2, three branches terminate at infinity.
- Property 3, the root locus on the real axis are shown in red color.



Root-locus analysis of linear feedback systems

The branches break off in between -4 < s < 0 with

$$\frac{d}{ds}G(s)H(s)=\frac{d}{ds}\frac{N(s)}{D(s)}=\frac{D(s)\dfrac{dN(s)}{ds}-N(s)\dfrac{dD(s)}{ds}}{D^2(s)}=0$$

Command Window

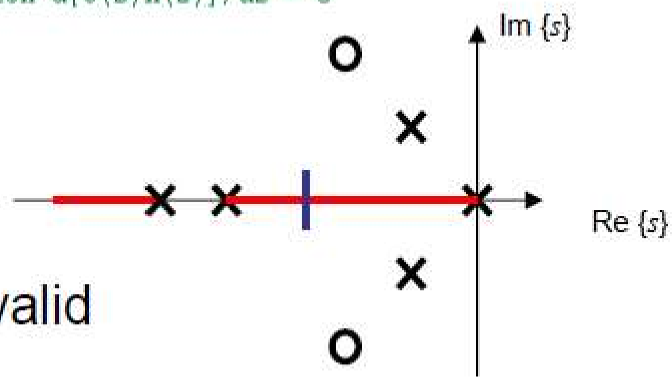
New to MATLAB? Watch this Video, see Demos, or read Getting Started.

```
>> num=[1 4 8]; % vector for numerator of G(s)H(s)
den=conv(poly([0 -4 -5]), [1 2 2]); % vector for denominator of G(s)H(s)
diff_num=num(1:2).*[2 1];  % vector for differentiation of numerator of G(s)H(s
diff_den=den(1:5).*[5:-1:1]; % vector for differentiation of denominator of G(s)H(s)
new_Num=conv(den,diff_num)-conv(num,diff_den); % vector for new numerator of d[G(s)H(s)]/ds
roots(new_Num)  % the roots of the equation d[G(s)H(s)]/ds = 0

ans =

  -4.5322
  -2.1060 + 2.9299i
  -2.1060 - 2.9299i
  -2.5149
  -0.7038 + 0.4728i
  -0.7038 - 0.4728i
```

valid



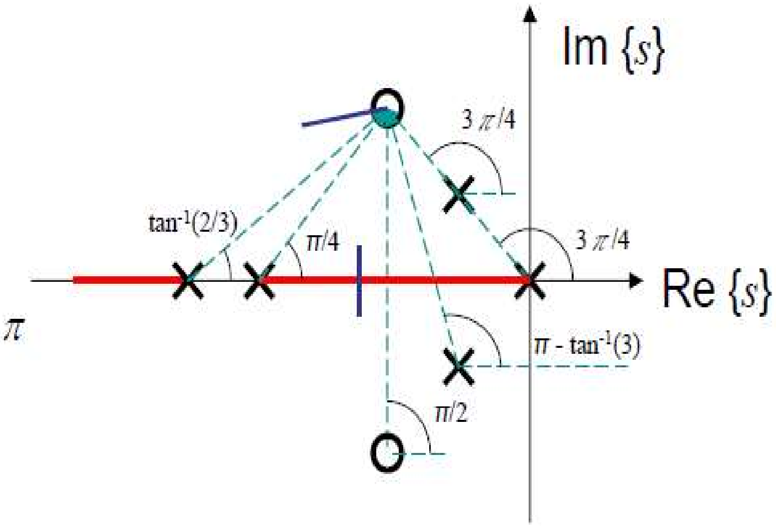
Root-locus analysis of linear feedback systems

- Angle of arrival at the zero -2 + 2 j

$$\psi_k = -\sum_{i=1, i\neq k}^{m} \angle(z_k - z_i) + \sum_{j=1}^{n} \angle(z_k - p_j) \pm q\pi$$

$$= -\frac{\pi}{2} + \left(\frac{3\pi}{4} + \frac{3\pi}{4} + \pi - \tan^{-1} 3 + \frac{\pi}{4} + \tan^{-1}\frac{2}{3} \right) - \pi$$

$$= 1.0396\pi$$

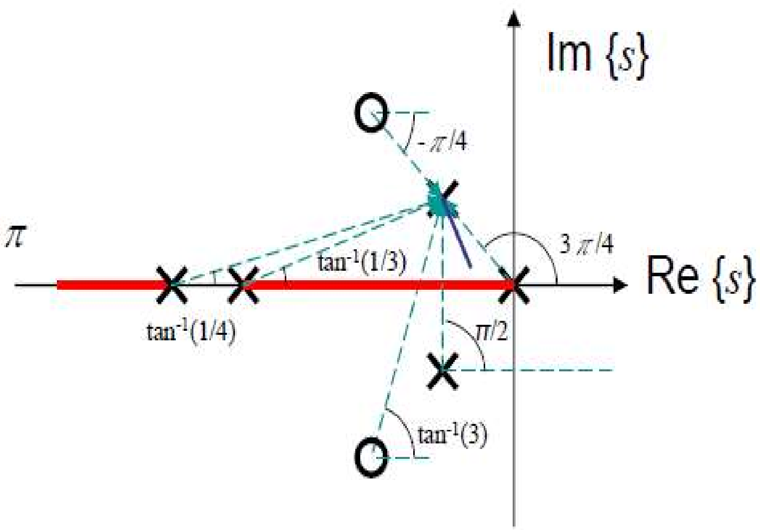


- Angle of departure at the pole -1 + j

$$\phi_k = \sum_{i=1}^{m} \angle(p_k - z_i) - \sum_{j=1, j\neq k}^{n} \angle(p_k - p_j) \pm q\pi$$

$$= \left(-\frac{\pi}{4} + \tan^{-1} 3 \right) - \left(\frac{3\pi}{4} + \frac{\pi}{2} + \tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{4} \right) + \pi$$

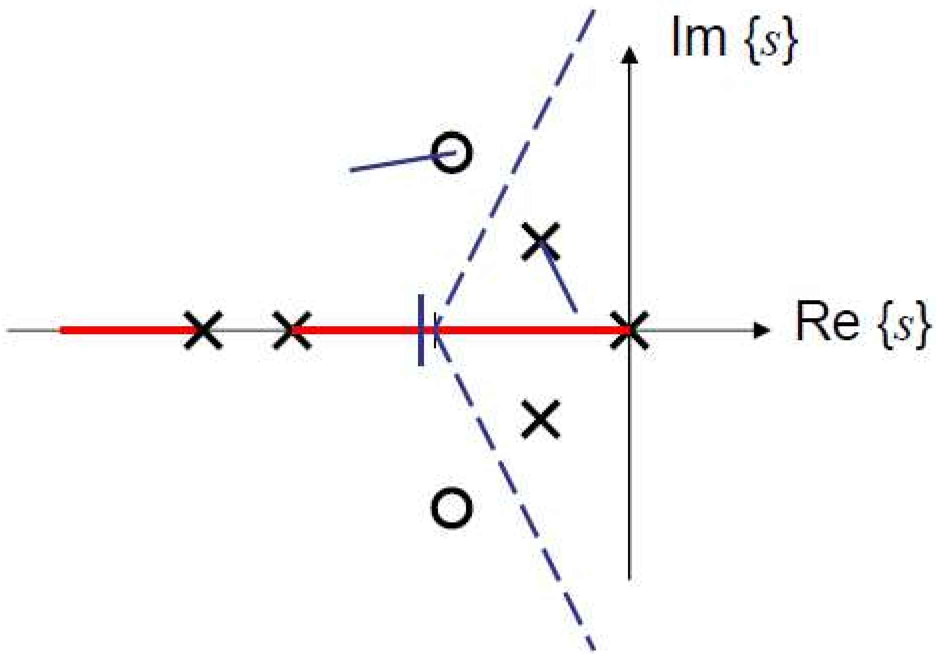
$$= -0.2828\pi$$

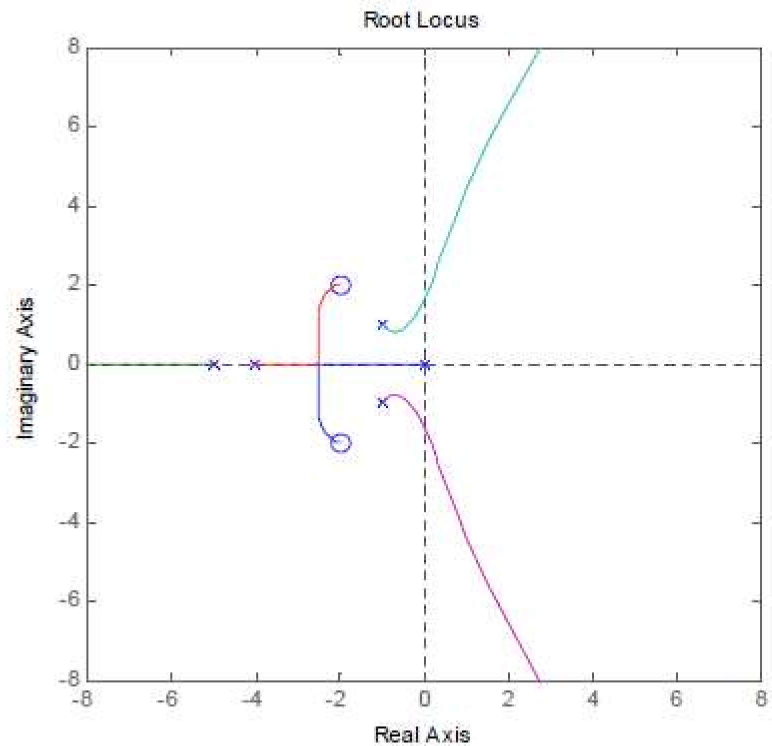


Root-locus analysis of linear feedback systems

- The 3 asymptotes with angles $\theta = \dfrac{q\pi}{3} = \left\{ \dfrac{-\pi}{3}, \ \dfrac{\pi}{3}, \ \pi \right\}$
- Starting point of all asymptotes is

$$\kappa = \frac{\sum_{i=1}^{n} p_i - \sum_{j=1}^{m} z_j}{n-m} = \frac{(0-4-5-1+j-1-j)-(-2+j-2-j)}{3} = \frac{-7}{3}$$





Gain and Phase Margins

- In this section, we introduce and examine the concept of the margin of stability in a feedback system.
- Assume a typical feedback system, as shown in Figure 11.25, is stable.

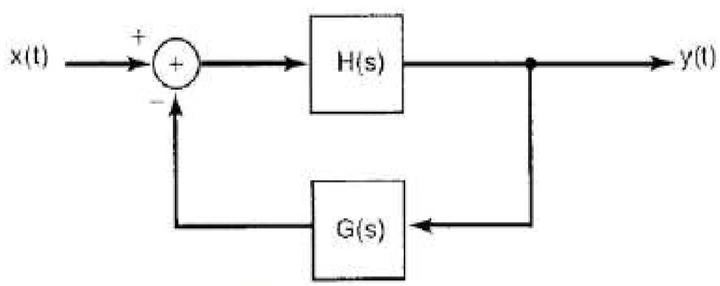


Figure 11.25

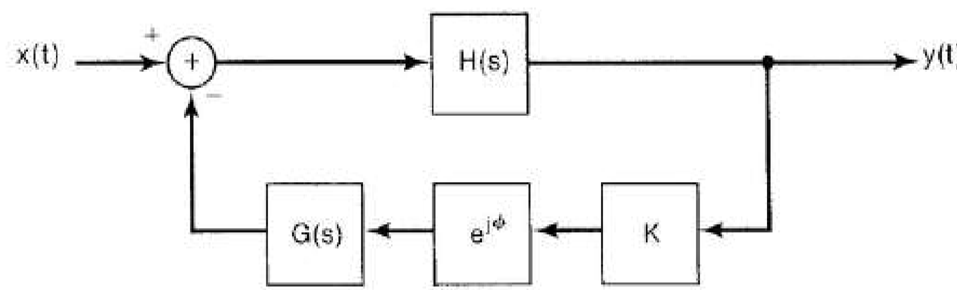


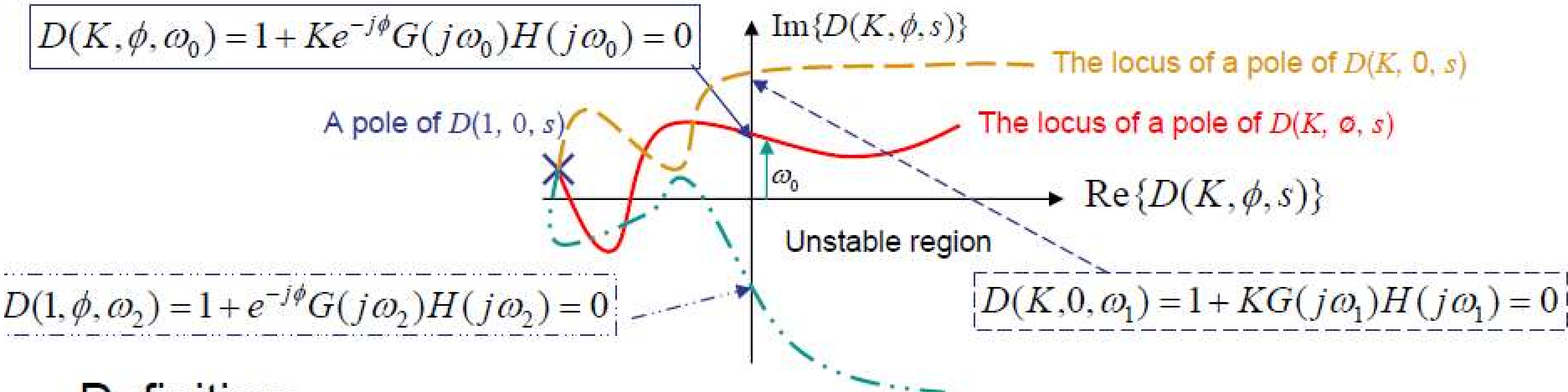
Figure 11.26

- To asses the margin of stability, we introduce two building blocks for possibility of a gain K and phase shift ø in the feedback path as shown in Figure 11.26.

Gain and Phase Margins

- Consider the equation for the poles of closed-loop system is

$$D(K,\phi,s) = 1 + Ke^{-j\phi}G(s)H(s) = 0$$



- Definition:
  - Gain Margin of the feedback system, $K > 1$
    - The minimum amount of additional gain K with $ø = 0$,that is required the closed-loop system becomes unstable.
  - Phase Margin of the feedback system, $ø > 0$
    - The minimum amount of additional phase ø with $K = 1$,that is required the closed-loop system becomes unstable.

Gain and Phase Margins

- Example 11.9 Consider a system function

$$G(s)H(s) = \frac{4(\frac{1}{2}s+1)}{s(2s+1)[1+0.05s+(0.125s)^2]}$$

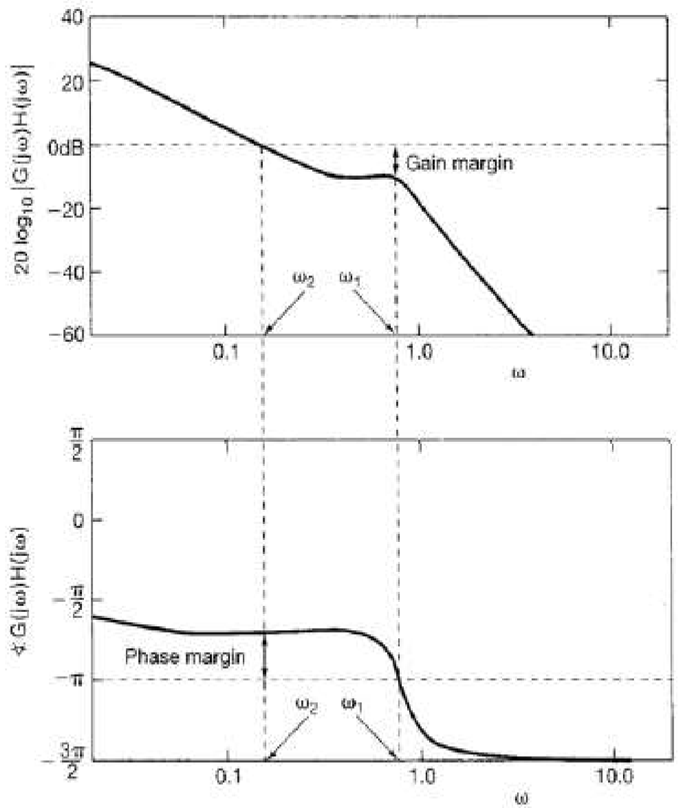


Figure 11.27 Use of Bode plots to calculate gain and phase margins for the system of Example 11.9.

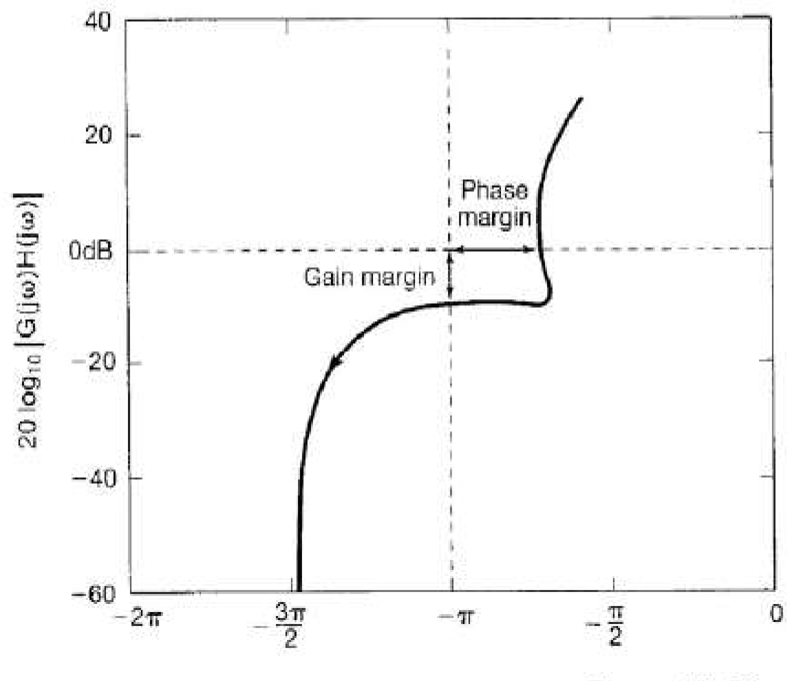


Figure 11.28 Log magnitude-phase plot for the system of Example 11.9.

Gain and Phase Margins

- Example 11.10 Consider a system function

$$G(s)H(s)=\frac{1}{\tau s+1}, \qquad \tau>0$$

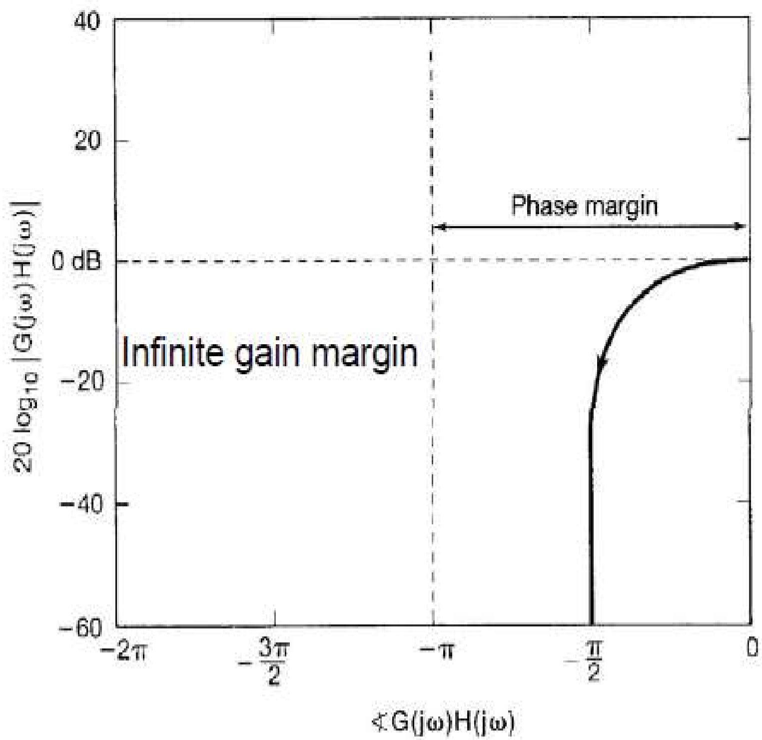


Figure 11.29 Log magnitude-phase plot for the first-order system of Example 11.10.

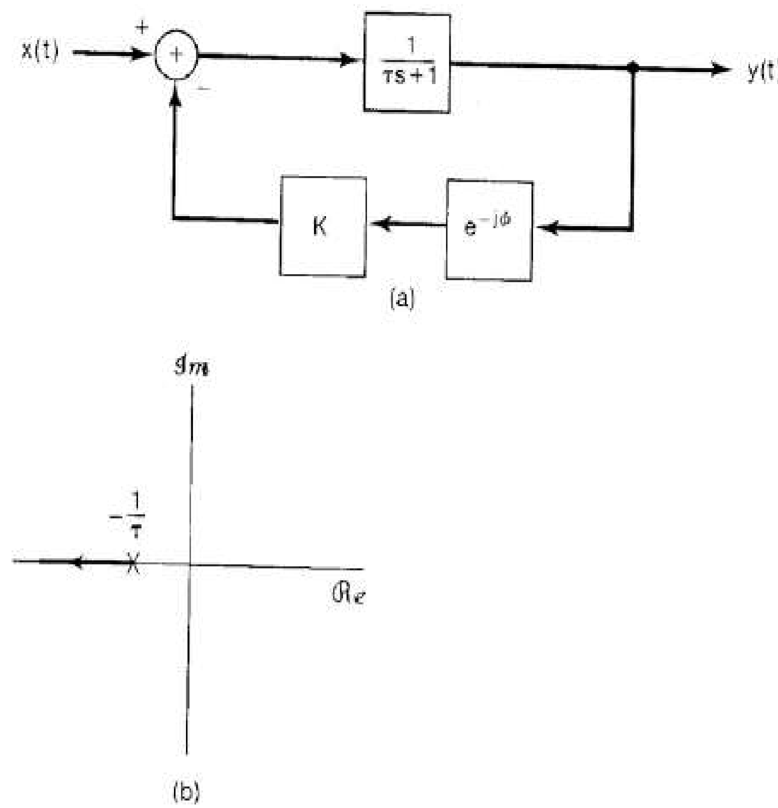
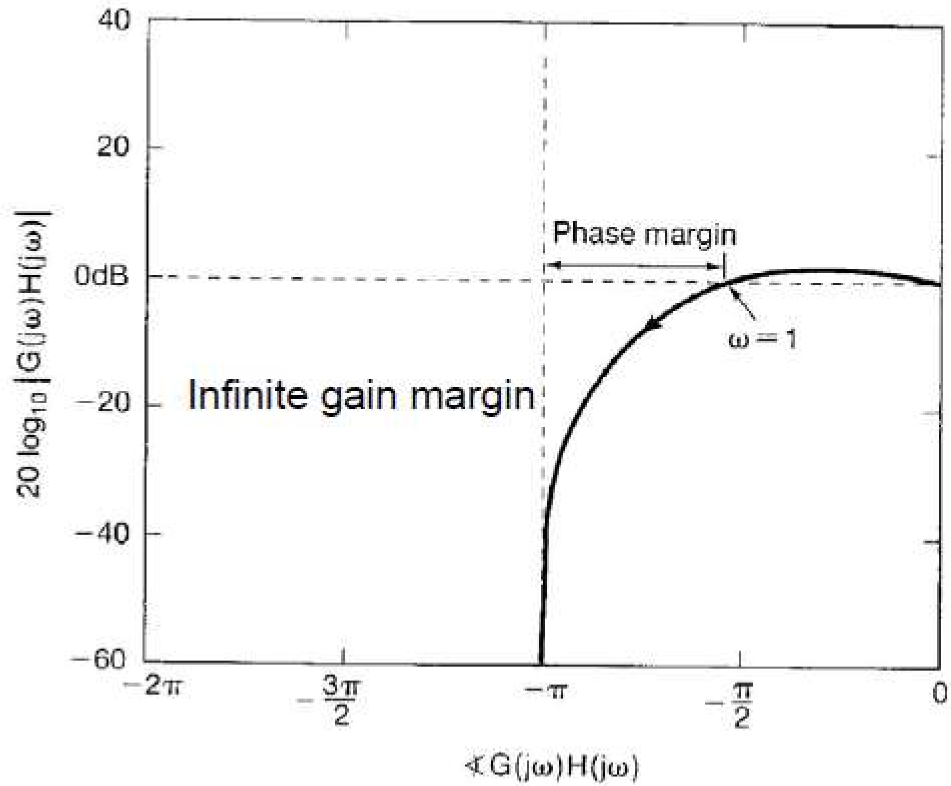


Figure 11.30 (a) First-order feedback system with possible gain and phase variations in the feedback path; (b) root locus for this system with $\phi=0$, $K>0$.

Gain and Phase Margins

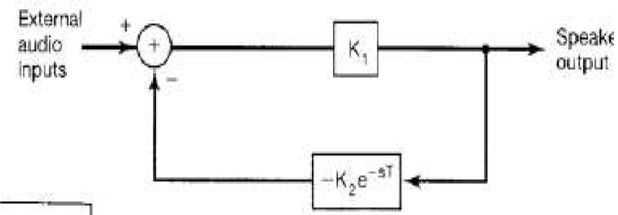
- Example 11.11 Consider the system

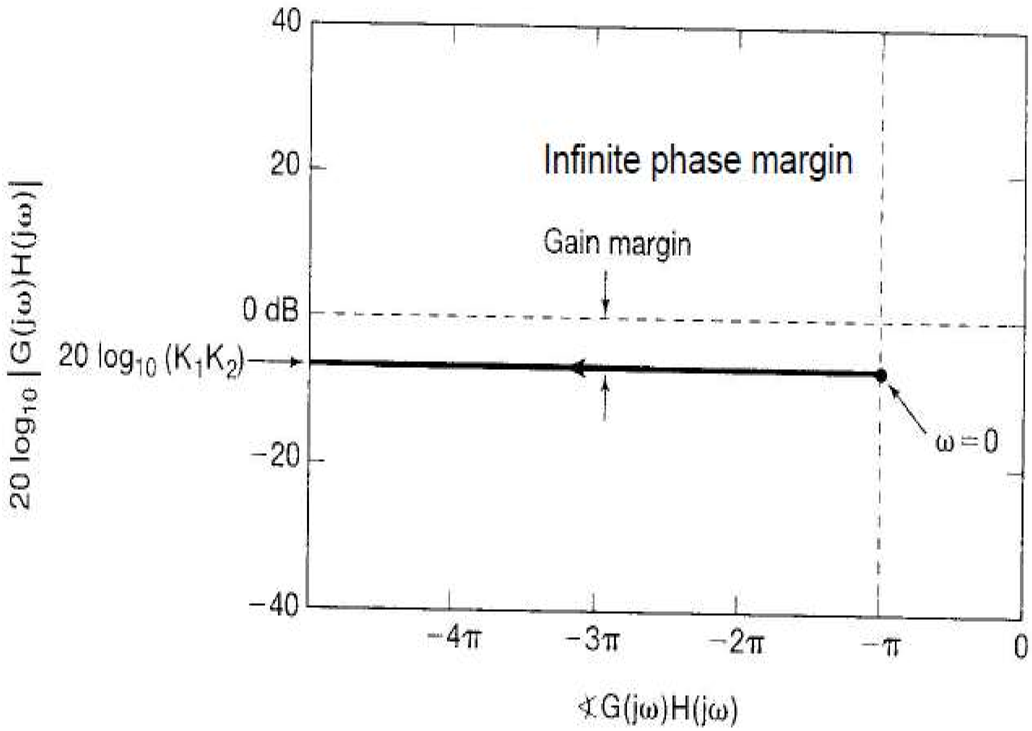
$$H(s)=\frac{1}{s^2+s+1}, \qquad G(s)=1$$



Gain and Phase Margins

- Example 11.12 Reexamine the acoustic feedback system.
- We assume the system has been designed with $K_1K_2 < 1$, so that the closed-loop system is stable.





Gain and Phase Margins

- Example 11.13 Consider a discrete-time open-loop system

$$G(z)H(z)=\frac{\frac{7\sqrt{2}}{4}z^{-1}}{1-\frac{7\sqrt{2}}{8}z^{-1}+\frac{49}{64}z^{-2}}$$