EXAMPLE 8–1 A uniform plane wave with $\mathbf{E} = \mathbf{a}_x E_x$ propagates in a lossless simple medium ($\epsilon_r = 4$, $\mu_r = 1$, $\sigma = 0$) in the $+z$-direction. Assume that E_x is sinusoidal with a frequency 100 (MHz) and has a maximum value of $+10^{-4}$ (V/m) at $t = 0$ and $z = \frac{1}{8}$ (m).

a) Write the instantaneous expression for $\mathbf{E}$ for any t and z.
b) Write the instantaneous expression for $\mathbf{H}$.
c) Determine the locations where E_x is a positive maximum when $t = 10^{-8}$ (s).

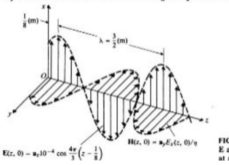

FIGURE 8–2
E and H fields of a uniform plane wave at $t = 0$ (Example 8–1).

Solution First we find k:

$$k = \omega\sqrt{\mu\epsilon} = \frac{\omega}{c}\sqrt{\mu_r \epsilon_r} = \frac{2\pi 10^8}{3 \times 10^8}\sqrt{4} = \frac{4\pi}{3} \quad \text{(rad/m)}.$$

a) Using $\cos \omega t$ as the reference, we find the instantaneous expression for $\mathbf{E}$ to be

$$\mathbf{E}(z, t) = \mathbf{a}_x E_x = \mathbf{a}_x 10^{-4} \cos(2\pi 10^8 t - kz + \psi).$$

Since E_x equals $+10^{-4}$ when the argument of the cosine function equals zero—that is, when

$$2\pi 10^8 t - kz + \psi = 0,$$

we have, at $t = 0$ and $z = \frac{1}{8}$,

$$\psi = kz = \left(\frac{4\pi}{3}\right)\left(\frac{1}{8}\right) = \frac{\pi}{6} \quad \text{(rad)}.$$

Thus,

$$\mathbf{E}(z, t) = \mathbf{a}_x 10^{-4} \cos\left(2\pi 10^8 t - \frac{4\pi}{3} z + \frac{\pi}{6}\right) = \mathbf{a}_x 10^{-4} \cos\left[2\pi 10^8 t - \frac{4\pi}{3}\left(z - \frac{1}{8}\right)\right] \quad \text{(V/m)}.$$

This expression shows a shift of $\frac{1}{8}$ (m) in the $+z$-direction and could have been written down directly from the statement of the problem.

b) The phasor expression for $\mathbf{H}$ is

$$\mathbf{H} = \mathbf{a}_y H_y = \mathbf{a}_y \frac{E_x}{\eta},$$

where

$$\eta = \sqrt{\frac{\mu}{\epsilon}} = \frac{\eta_0}{\sqrt{\epsilon_r}} = 60\pi \quad (\Omega).$$

Hence,

$$\mathbf{H}(z, t) = \mathbf{a}_y \frac{10^{-4}}{60\pi} \cos\left[2\pi 10^8 t - \frac{4\pi}{3}\left(z - \frac{1}{8}\right)\right] \quad \text{(A/m)}.$$

c) At $t = 10^{-8}$, we equate the argument of the cosine function to $+2n\pi$ in order to make E_x a positive maximum:

$$2\pi 10^8(10^{-8}) - \frac{4\pi}{3}\left(z_m - \frac{1}{8}\right) = \pm 2n\pi,$$

from which we get

$$z_m = \frac{13}{8} \pm \frac{3}{2} n \quad \text{(m)}, \qquad n = 0, 1, 2, \ldots; \qquad z_m > 0.$$

Examining this result more closely, we note that the wavelength in the given medium is

$$\lambda = \frac{2\pi}{k} = \frac{3}{2} \quad \text{(m)}.$$

Hence the positive maximum value of E_x occurs at

$$z_m = \frac{13}{8} \pm n\lambda \quad \text{(m)}.$$

The $\mathbf{E}$ and $\mathbf{H}$ fields are shown in Fig. 8–2 as functions of z for the reference time $t = 0$.

EXAMPLE 8–2 If $\mathbf{E}(\mathbf{R})$ of a TEM wave is given, as in Eq. (8–26), $\mathbf{H}(\mathbf{R})$ can be found by using Eq. (8–29). Obtain a relation expressing $\mathbf{E}(\mathbf{R})$ in terms of $\mathbf{H}(\mathbf{R})$.

Solution Assuming $\mathbf{H}(\mathbf{R})$ to have the form

$$\mathbf{H}(\mathbf{R}) = \mathbf{H}_0 e^{-jk\mathbf{a}_n \cdot \mathbf{R}}, \tag{8–32}$$

we obtain from Eq. (7–104b)

$$\mathbf{E}(\mathbf{R}) = \frac{1}{j\omega\epsilon} \nabla \times \mathbf{H}(\mathbf{R}) = \frac{1}{j\omega\epsilon}(-jk)\mathbf{a}_n \times \mathbf{H}(\mathbf{R})$$

or

$$\boxed{\mathbf{E}(\mathbf{R}) = -\eta \mathbf{a}_n \times \mathbf{H}(\mathbf{R}) \quad \text{(V/m)}.} \tag{8–33}$$

Alternatively, we can obtain the same result by cross-multiplying both sides of Eq. (8–29) by $\mathbf{a}_n$ and using the back-cab rule in Eq. (2–20).

EXAMPLE 8–3 Prove that a linearly polarized plane wave can be resolved into a right-hand circularly polarized wave and a left-hand circularly polarized wave of equal amplitude.

Solution Consider a linearly polarized plane wave propagating in the $+z$-direction. We can assume, with no loss of generality, that $\mathbf{E}$ is polarized in the x-direction. In phasor notation we have

$$\mathbf{E}(z) = \mathbf{a}_x E_0 e^{-jkz}.$$

But this can be written as

$$\mathbf{E}(z) = \mathbf{E}_{rc}(z) + \mathbf{E}_{lc}(z),$$

where

$$\mathbf{E}_{rc}(z) = \frac{E_0}{2}(\mathbf{a}_x - j\mathbf{a}_y)e^{-jkz} \tag{8–41a}$$

and

$$\mathbf{E}_{lc}(z) = \frac{E_0}{2}(\mathbf{a}_x + j\mathbf{a}_y)e^{-jkz}. \tag{8–41b}$$

From previous discussions we recognize that $\mathbf{E}_{rc}(z)$ in Eq. (8–41a) and $\mathbf{E}_{lc}(z)$ in Eq. (8–41b) represent right-hand and left-hand circularly polarized waves, respectively, each having an amplitude $E_0/2$. The statement of this problem is therefore proved. The converse statement that the sum of two oppositely rotating circularly polarized waves of equal amplitude is a linearly polarized wave is, of course, also true.

EXAMPLE 8–4 The electric field intensity of a linearly polarized uniform plane wave propagating in the $+z$-direction in seawater is $\mathbf{E} = \mathbf{a}_x 100 \cos(10^7 \pi t)$ (V/m) at $z = 0$. The constitutive parameters of seawater are $\epsilon_r = 72$, $\mu_r = 1$, and $\sigma = 4$ (S/m). (a) Determine the attenuation constant, phase constant, intrinsic impedance, phase velocity, wavelength, and skin depth. (b) Find the distance at which the amplitude of $\mathbf{E}$ is 1% of its value at $z = 0$. (c) Write the expressions for $\mathbf{E}(z, t)$ and $\mathbf{H}(z, t)$ at $z = 0.8$ (m) as functions of t.

Solution

$$\omega = 10^7\pi \quad \text{(rad/s)},$$
$$f = \frac{\omega}{2\pi} = 5 \times 10^6 \quad \text{(Hz)},$$
$$\frac{\sigma}{\omega\epsilon} = \frac{\sigma}{\omega\epsilon_0\epsilon_r} = \frac{4}{10^7\pi\left(\frac{1}{36\pi} \times 10^{-9}\right)72} = 200 \gg 1.$$

Hence we can use the formulas for good conductors.

a) Attenuation constant:

$$\alpha = \sqrt{\pi f \mu \sigma} = \sqrt{5\pi 10^6 (4\pi 10^{-7}) 4} = 8.89 \quad \text{(Np/m)}.$$

Phase constant:

$$\beta = \sqrt{\pi f \mu \sigma} = 8.89 \quad \text{(rad/m)}.$$

Intrinsic impedance:

$$\eta_c = (1 + j)\sqrt{\frac{\pi f \mu}{\sigma}} = (1 + j)\sqrt{\frac{\pi(5 \times 10^6)(4\pi \times 10^{-7})}{4}} = \pi e^{j\pi/4} \quad (\Omega)$$

Wavelength:

$$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{8.89} = 0.707 \quad \text{(m)}.$$

Skin depth:

$$\delta = \frac{1}{\alpha} = \frac{1}{8.89} = 0.112 \quad \text{(m)}.$$

b) Distance z_1 at which the amplitude of wave decreases to 1% of its value at $z = 0$:

$$e^{-\alpha z_1} = 0.01 \quad \text{or} \quad e^{\alpha z_1} = \frac{1}{0.01} = 100,$$
$$z_1 = \frac{1}{\alpha} \ln 100 = \frac{4.605}{8.89} = 0.518 \quad \text{(m)}.$$

c) In phasor notation,

$$\mathbf{E}(z) = \mathbf{a}_x 100 e^{-\alpha z} e^{-j\beta z}.$$

The instantaneous expression for $\mathbf{E}$ is

$$\mathbf{E}(z, t) = \mathcal{R}e[\mathbf{E}(z)e^{j\omega t}] = \mathcal{R}e[\mathbf{a}_x 100 e^{-\alpha z} e^{j(\omega t - \beta z)}] = \mathbf{a}_x 100 e^{-\alpha z} \cos(\omega t - \beta z).$$

At $z = 0.8$ (m) we have

$$\mathbf{E}(0.8, t) = \mathbf{a}_x 100 e^{-0.8\alpha} \cos(10^7\pi t - 0.8\beta) = \mathbf{a}_x 0.082 \cos(10^7\pi t - 7.11) \quad \text{(V/m)}.$$

We know that a uniform plane wave is a TEM wave with $\mathbf{E} \perp \mathbf{H}$ and that both are normal to the direction of wave propagation $\mathbf{a}_z$. Thus $\mathbf{H} = \mathbf{a}_y H_y$. To find $\mathbf{H}(z, t)$, the instantaneous expression of $\mathbf{H}$ as a function of t, *we must not make the mistake of writing* $H_y(z, t) = E_x(z, t)/\eta_c$ because this would be mixing real time functions $E_x(z, t)$ and $H_y(z, t)$ with a complex quantity η_c. Phasor quantities $E_x(z)$ and $H_y(z)$ must be used. That is,

$$H_y(z) = \frac{E_x(z)}{\eta_c},$$

from which we obtain the relation between instantaneous quantities

$$H_y(z, t) = \mathcal{R}e\left[\frac{E_x(z)}{\eta_c} e^{j\omega t}\right].$$

For the present problem we have, in phasors,

$$H_y(0.8) = \frac{100e^{-0.8\alpha}e^{-j0.8\beta}}{\pi e^{j\pi/4}} = \frac{0.082e^{-j7.11}}{\pi e^{j\pi/4}} = 0.026e^{-j1.61}.$$

Note that *both* angles must be in radians before combining. The instantaneous expression for $\mathbf{H}$ at $z = 0.8$ (m) is then

$$\mathbf{H}(0.8, t) = \mathbf{a}_y 0.026 \cos(10^7\pi t - 1.61) \quad \text{(A/m)}.$$

We can see that a 5 (MHz) plane wave attenuates very rapidly in seawater and becomes negligibly weak a very short distance from the source. Even at very low frequencies, long-distance radio communication with a submerged submarine is very difficult.

EXAMPLE 8–5 When a spacecraft reenters the earth's atmosphere, its speed and temperature ionize the surrounding atoms and molecules and create a plasma. It has been estimated that the electron density is in the neighborhood of 2×10^8 per (cm³). Discuss the plasma's effect on frequency usage in radio communication between the spacecraft and the mission controllers on earth.

$$f_p = \frac{\omega_p}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{Ne^2}{m\epsilon_0}} \quad \text{(Hz)}. \tag{8–65}$$

Plug in the value of e, m, and ϵ_0

$$f_p \cong 9\sqrt{N} \quad \text{(Hz)}. \tag{8–69}$$

Solution For

$$N = 2 \times 10^8 \text{ per (cm}^3) = 2 \times 10^{14} \text{ per (m}^3),$$

Eq. (8–69) gives $f_p = 9 \times \sqrt{2 \times 10^{14}} = 12.7 \times 10^7$ (Hz), or 127 (MHz). Thus, radio communication cannot be established for frequencies below 127 (MHz).

EXAMPLE 8–6 A narrow-band signal propagates in a lossy dielectric medium which has a loss tangent 0.2 at 550 (kHz), the carrier frequency of the signal. The dielectric constant of the medium is 2.5. (a) Determine α and β. (b) Determine u_p and u_g. Is the medium dispersive?

Solution

a) Since the loss tangent $\epsilon''/\epsilon' = 0.2$ and $\epsilon''^2/8\epsilon'^2 \ll 1$, Eqs. (8–48) and (8–49) can be used to determine α and β respectively. But first we find ϵ'' from the loss tangent:

$$\epsilon'' = 0.2\,\epsilon' = 0.2 \times 2.5\,\epsilon_0 = 4.42 \times 10^{-12} \quad \text{(F/m)}.$$

Thus,

$$\alpha = \frac{\omega\epsilon''}{2}\sqrt{\frac{\mu}{\epsilon'}} = \pi(550 \times 10^3) \times (4.42 \times 10^{-12}) \times \frac{377}{\sqrt{2.5}} = 1.82 \times 10^{-3} \quad \text{(Np/m)};$$

$$\beta = \omega\sqrt{\mu\epsilon'}\left[1 + \frac{1}{8}\left(\frac{\epsilon''}{\epsilon'}\right)^2\right] = 2\pi(550 \times 10^3)\frac{\sqrt{2.5}}{3 \times 10^8}\left[1 + \frac{1}{8}(0.2)^2\right] = 0.0182 \times 1.005 = 0.0183 \quad \text{(rad/m)}.$$

b) Phase velocity (from Eq. 8–51):

$$u_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{\mu\epsilon'}\left[1 + \frac{1}{8}\left(\frac{\epsilon''}{\epsilon'}\right)^2\right]} \cong \frac{1}{\sqrt{\mu\epsilon'}}\left[1 - \frac{1}{8}\left(\frac{\epsilon''}{\epsilon'}\right)^2\right] = \frac{3 \times 10^8}{\sqrt{2.5}}\left[1 - \frac{1}{8}(0.2)^2\right] = 1.888 \times 10^8 \quad \text{(m/s)}.$$

c) Group velocity (from Eq. 8–49):

$$\frac{d\beta}{d\omega} = \sqrt{\mu\epsilon'}\left[1 + \frac{1}{8}\left(\frac{\epsilon''}{\epsilon'}\right)^2\right],$$
$$u_g = \frac{1}{(d\beta/d\omega)} \cong \frac{1}{\sqrt{\mu\epsilon'}} \cong u_p.$$

Thus a low-loss dielectric is nearly nondispersive. Here we have assumed ϵ'' to be independent of frequency. For a high-loss dielectric, ϵ'' will be a function of ω and may have a magnitude comparable to ϵ'. The approximation in Eq. (8–49) will no longer hold, and the medium will be dispersive.

EXAMPLE 8–7 Find the Poynting vector on the surface of a long, straight conducting wire (of radius b and conductivity σ) that carries a direct current I. Verify Poynting's theorem.

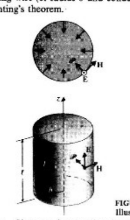

FIGURE 8–8
Illustrating Poynting's theorem (Example 8–7).

Solution Since we have a d-c situation, the current in the wire is uniformly distributed over its cross-sectional area. Let us assume that the axis of the wire coincides with the z-axis. Figure 8–8 shows a segment of length ℓ of the long wire. We have

$$\mathbf{J} = \mathbf{a}_z \frac{I}{\pi b^2}$$

and

$$\mathbf{E} = \frac{\mathbf{J}}{\sigma} = \mathbf{a}_z \frac{I}{\sigma\pi b^2}.$$

On the surface of the wire,

$$\mathbf{H} = \mathbf{a}_\phi \frac{I}{2\pi b}.$$

Thus the Poynting vector at the surface of the wire is

$$\mathscr{P} = \mathbf{E} \times \mathbf{H} = (\mathbf{a}_z \times \mathbf{a}_\phi)\frac{I^2}{2\sigma\pi^2 b^3} = -\mathbf{a}_r \frac{I^2}{2\sigma\pi^2 b^3},$$

which is directed everywhere into the wire surface.

To verify Poynting's theorem, we integrate $\mathscr{P}$ over the wall of the wire segment in Fig. 8–8:

$$-\oint_S \mathscr{P} \cdot d\mathbf{s} = -\oint_S \mathscr{P} \cdot \mathbf{a}_r\, ds = \left(\frac{I^2}{2\sigma\pi^2 b^3}\right)2\pi b\ell = I^2\left(\frac{\ell}{\sigma\pi b^2}\right) = I^2 R,$$

where the formula for the resistance of a straight wire in Eq. (5–27), $R = \ell/\sigma S$, has been used. The above result affirms that the negative surface integral of the Poynting vector is exactly equal to the I^2R ohmic power loss in the conducting wire. Hence Poynting's theorem is verified.

EXAMPLE 8–8 The far field of a short vertical current element $I\,d\ell$ located at the origin of a spherical coordinate system in free space is

$$\mathbf{E}(R, \theta) = \mathbf{a}_\theta E_\theta(R, \theta) = \mathbf{a}_\theta\left(j\frac{60\pi I\,d\ell}{\lambda R}\sin\theta\right)e^{-j\beta R} \quad \text{(V/m)}$$

and

$$\mathbf{H}(R, \theta) = \mathbf{a}_\phi \frac{E_\theta(R, \theta)}{\eta_0} = \mathbf{a}_\phi\left(j\frac{I\,d\ell}{2\lambda R}\sin\theta\right)e^{-j\beta R} \quad \text{(A/m)},$$

Solution

a) We note that $E_\theta/H_\phi = \eta_0 = 120\pi$ (Ω). The instantaneous Poynting vector is

$$\mathscr{P}(R, \theta; t) = \mathcal{R}e[\mathbf{E}(R, \theta)e^{j\omega t}] \times \mathcal{R}e[\mathbf{H}(R, \theta)e^{j\omega t}]$$
$$= (\mathbf{a}_\theta \times \mathbf{a}_\phi)30\pi\left(\frac{I\,d\ell}{\lambda R}\right)^2 \sin^2\theta \sin^2(\omega t - \beta R)$$
$$= \mathbf{a}_R 15\pi\left(\frac{I\,d\ell}{\lambda R}\right)^2 \sin^2\theta[1 - \cos 2(\omega t - \beta R)] \quad \text{(W/m}^2).$$

b) The average power density vector is, from Eq. (8–96),

$$\mathscr{P}_{av}(R, \theta) = \mathbf{a}_R 15\pi\left(\frac{I\,d\ell}{\lambda R}\right)^2 \sin^2\theta,$$

which is seen to equal the time-average value of $\mathscr{P}(R, \theta; t)$ given in part (a) of this solution. The total average power radiated is obtained by integrating $\mathscr{P}_{av}(R, \theta)$ over the surface of the sphere of radius R:

$$\text{Total } P_{av} = \oint_S \mathscr{P}_{av}(R, \theta) \cdot d\mathbf{s} = \int_0^{2\pi}\int_0^{\pi}\left[15\pi\left(\frac{I\,d\ell}{\lambda R}\right)^2 \sin^2\theta\right]R^2 \sin\theta\, d\theta\, d\phi$$
$$= 40\pi^2\left(\frac{d\ell}{\lambda}\right)^2 I^2 \quad \text{(W)},$$

where I is the amplitude ($\sqrt{2}$ times the effective value) of the sinusoidal current in $d\ell$.

EXAMPLE 8–9 A y-polarized uniform plane wave $(\mathbf{E}_i, \mathbf{H}_i)$ with a frequency 100 (MHz) propagates in air in the $+x$ direction and impinges normally on a perfectly conducting plane at $x = 0$. Assuming the amplitude of $\mathbf{E}_i$ to be 6 (mV/m), write the phasor and instantaneous expressions for (a) $\mathbf{E}_i$ and $\mathbf{H}_i$ of the incident wave; (b) $\mathbf{E}_r$ and $\mathbf{H}_r$ of the reflected wave; and (c) $\mathbf{E}_1$ and $\mathbf{H}_1$ of the total wave in air. (d) Determine the location nearest to the conducting plane where $\mathbf{E}_1$ is zero.

Solution At the given frequency 100 (MHz),

$$\omega = 2\pi f = 2\pi \times 10^8 \quad \text{(rad/s)},$$
$$\beta_1 = k_0 = \frac{\omega}{c} = \frac{2\pi \times 10^8}{3 \times 10^8} = \frac{2\pi}{3} \quad \text{(rad/m)},$$
$$\eta_1 = \eta_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} = 120\pi \quad (\Omega).$$

a) *For the incident wave (a traveling wave):*

i) Phasor expressions:

$$\mathbf{E}_i(x) = \mathbf{a}_y 6 \times 10^{-3} e^{-j2\pi x/3} \quad \text{(V/m)},$$
$$\mathbf{H}_i(x) = \frac{1}{\eta_1}\mathbf{a}_x \times \mathbf{E}_i(x) = \mathbf{a}_z \frac{10^{-4}}{2\pi} e^{-j2\pi x/3} \quad \text{(A/m)}.$$

ii) Instantaneous expressions:

$$\mathbf{E}_i(x, t) = \mathcal{R}e[\mathbf{E}_i(x)e^{j\omega t}] = \mathbf{a}_y 6 \times 10^{-3} \cos\left(2\pi \times 10^8 t - \frac{2\pi}{3}x\right) \quad \text{(V/m)},$$
$$\mathbf{H}_i(x, t) = \mathbf{a}_z \frac{10^{-4}}{2\pi} \cos\left(2\pi \times 10^8 t - \frac{2\pi}{3}x\right) \quad \text{(A/m)}.$$

b) *For the reflected wave (a traveling wave):*

i) Phasor expressions:

$$\mathbf{E}_r(x) = -\mathbf{a}_y 6 \times 10^{-3} e^{j2\pi x/3} \quad \text{(V/m)},$$
$$\mathbf{H}_r(x) = \frac{1}{\eta_1}(-\mathbf{a}_x) \times \mathbf{E}_r(x) = \mathbf{a}_z \frac{10^{-4}}{2\pi} e^{j2\pi x/3} \quad \text{(A/m)}$$

ii) Instantaneous expressions:

$$\mathbf{E}_r(x, t) = \mathcal{R}e[\mathbf{E}_r(x)e^{j\omega t}] = -\mathbf{a}_y 6 \times 10^{-3} \cos\left(2\pi \times 10^8 t + \frac{2\pi}{3}x\right) \quad \text{(V/m)},$$
$$\mathbf{H}_r(x, t) = \frac{10^{-4}}{2\pi} \cos\left(2\pi \times 10^8 t + \frac{2\pi}{3}x\right) \quad \text{(A/m)}.$$

c) *For the total wave (a standing wave):*

i) Phasor expressions:

$$\mathbf{E}_1(x) = \mathbf{E}_i(x) + \mathbf{E}_r(x) = -\mathbf{a}_y j12 \times 10^{-3} \sin\left(\frac{2\pi}{3}x\right) \quad \text{(V/m)},$$
$$\mathbf{H}_1(x) = \mathbf{H}_i(x) + \mathbf{H}_r(x) = \mathbf{a}_z \frac{10^{-4}}{\pi} \cos\left(\frac{2\pi}{3}x\right) \quad \text{(A/m)}.$$

ii) Instantaneous expressions:

$$\mathbf{E}_1(x, t) = \mathcal{R}e[\mathbf{E}_1(x)e^{j\omega t}] = \mathbf{a}_y 12 \times 10^{-3} \sin\left(\frac{2\pi}{3}x\right)\sin(2\pi \times 10^8 t) \quad \text{(V/m)},$$
$$\mathbf{H}_1(x, t) = \mathbf{a}_z \frac{10^{-4}}{\pi} \cos\left(\frac{2\pi}{3}x\right)\cos(2\pi \times 10^8 t) \quad \text{(A/m)}.$$

d) The electric field vanishes at the surface of the conducting plane at $x = 0$. In medium 1 the first null occurs at

$$x = -\frac{\lambda_1}{2} = -\frac{\pi}{\beta_1} = -\frac{3}{2} \quad \text{(m)}.$$

EXAMPLE 8–10 A uniform plane wave $(\mathbf{E}_i, \mathbf{H}_i)$ of an angular frequency ω is incident from air on a very large, perfectly conducting wall at an angle of incidence θ_i with perpendicular polarization. Find (a) the current induced on the wall surface, and (b) the time-average Poynting vector in medium 1.

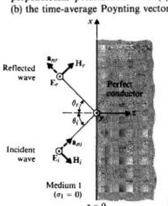

FIGURE 8–11
Plane wave incident obliquely on a plane conducting boundary (perpendicular polarization).

a) The conditions of this problem are exactly those we have just discussed; hence we could use the formulas directly. Let $z = 0$ be the plane representing the surface of the perfectly conducting wall, and let $\mathbf{E}_i$ be polarized in the y direction, as was shown in Fig. 8–11. At $z = 0$, $\mathbf{E}_1(x, 0) = 0$, and $\mathbf{H}_1(x, 0)$ can be obtained from Eq. (8–114):

$$\mathbf{H}_1(x, 0) = -\frac{E_{i0}}{\eta_0}(\mathbf{a}_x 2\cos\theta_i)e^{-j\beta_0 x \sin\theta_i}. \tag{8–121}$$

Inside the perfectly conducting wall, both $\mathbf{E}_2$ and $\mathbf{H}_2$ must vanish. There is then a discontinuity in the magnetic field. The amount of discontinuity is equal to the surface current. From Eq. (7–68b) we have

$$\mathbf{J}_s(x) = \mathbf{a}_{n2} \times \mathbf{H}_1(x, 0)$$
$$= (-\mathbf{a}_z) \times (-\mathbf{a}_x)\frac{E_{i0}}{\eta_0}(2\cos\theta_i)e^{-j\beta_0 x \sin\theta_i}$$
$$= \mathbf{a}_y \frac{E_{i0}}{60\pi}(\cos\theta_i)e^{-j(\omega/c)x\sin\theta_i}.$$

The instantaneous expression for the surface current is

$$\mathbf{J}_s(x, t) = \mathbf{a}_y \frac{E_{i0}}{60\pi}\cos\theta_i \cos\omega\left(t - \frac{x}{c}\sin\theta_i\right) \quad \text{(A/m)}. \tag{8–122}$$

It is this induced current on the wall surface that gives rise to the reflected wave in medium 1 and cancels the incident wave in the conducting wall.

b) The time-average Poynting vector in medium 1 is found by using Eqs. (8–113) and (8–114) in Eq. (8–96). Since E_{1y} and H_{1x} are in time quadrature, $\mathscr{P}_{av}$ will only have a nonvanishing x-component arising from E_{1y} and H_{1z}:

$$\mathscr{P}_{av_1} = \frac{1}{2}\mathcal{R}e[E_1(x, z) \times \mathbf{H}_1^*(x, z)]$$
$$= \mathbf{a}_x \frac{2E_{i0}^2}{\eta_1}\sin\theta_i \sin^2\beta_{1z}z, \tag{8–123}$$

where $\beta_{1z} = \beta_1\cos\theta_i$. The time-average Poynting vector in medium 2 (a perfect conductor) is, of course, zero.

EXAMPLE 8–11 A uniform plane wave in a lossless medium with intrinsic impedance η_1 is incident normally onto another lossless medium with intrinsic impedance η_2 through a plane boundary. Obtain the expressions for the time-average power densities in both media.

Solution Equation (8–96) provides the formula for computing the time-average power density, or time-average Poynting vector:

$$\mathscr{P}_{av} = \frac{1}{2}\mathcal{R}e(\mathbf{E} \times \mathbf{H}^*).$$

In medium 1 we use Eqs. (8–144) and (8–149):

$$(\mathscr{P}_{av})_1 = \mathbf{a}_z \frac{E_{i0}^2}{2\eta_1}\mathcal{R}e[(1 + \Gamma e^{j2\beta_1 z})(1 - \Gamma e^{-j2\beta_1 z})]$$
$$= \mathbf{a}_z \frac{E_{i0}^2}{2\eta_1}\mathcal{R}e[(1 - \Gamma^2) + \Gamma(e^{j2\beta_1 z} - e^{-j2\beta_1 z})]$$
$$= \mathbf{a}_z \frac{E_{i0}^2}{2\eta_1}\mathcal{R}e[(1 - \Gamma^2) + j2\Gamma \sin 2\beta_1 z]$$
$$= \mathbf{a}_z \frac{E_{i0}^2}{2\eta_1}(1 - \Gamma^2), \tag{8–152}$$

where Γ is a real number because both media are lossless.

In medium 2 we use Eqs. (8–150) and (8–151) to obtain

$$(\mathscr{P}_{av})_2 = \mathbf{a}_z \frac{E_{i0}^2}{2\eta_2}\tau^2. \tag{8–153}$$

Since we are dealing with lossless media, the power flow in medium 1 must equal that in medium 2; that is,

$$(\mathscr{P}_{av})_1 = (\mathscr{P}_{av})_2, \tag{8–154}$$

or

$$\boxed{1 - \Gamma^2 = \frac{\eta_1}{\eta_2}\tau^2.} \tag{8–155}$$

That Eq. (8–155) is true can be readily verified by using Eqs. (8–140) and (8–141).

EXAMPLE 8–12 A dielectric layer of thickness d and intrinsic impedance η_2 is placed between media 1 and 3 having intrinsic impedances η_1 and η_3, respectively. Determine d and η_2 such that no reflection occurs when a uniform plane wave in medium 1 impinges normally on the interface with medium 2.

Solution With the dielectric layer interposed between media 1 and 3 as shown in Fig. 8–15, the condition of no reflection at interface $z = 0$ requires $\Gamma_0 = 0$, or $Z_2(0) = \eta_1$. From Eq. (8–173) we have

$$\eta_2(\eta_3 \cos\beta_2 d + j\eta_2 \sin\beta_2 d) = \eta_1(\eta_2 \cos\beta_2 d + j\eta_3 \sin\beta_2 d). \tag{8–175}$$

Equating the real and imaginary parts separately, we require

$$\eta_3 \cos\beta_2 d = \eta_1 \cos\beta_2 d \tag{8–176}$$

and

$$\eta_2^2 \sin\beta_2 d = \eta_1\eta_3 \sin\beta_2 d. \tag{8–177}$$

Equation (8–176) is satisfied if *either*

$$\eta_3 = \eta_1 \tag{8–178}$$

or

$$\cos\beta_2 d = 0. \tag{8–179}$$

which implies that

$$\beta_2 d = (2n + 1)\frac{\pi}{2},$$

or

$$d = (2n + 1)\frac{\lambda_2}{4}, \qquad n = 0, 1, 2, \ldots \tag{8–180}$$

On the one hand, if condition (8–178) holds, Eq. (8–177) can be satisfied when either (a) $\eta_2 = \eta_3 = \eta_1$, which is the trivial case of no discontinuities at all, or (b) $\sin\beta_2 d = 0$, or $d = n\lambda_2/2$.

On the other hand, if relation (8–179) or (8–180) holds, $\sin\beta_2 d$ does not vanish, and Eq. (8–177) can be satisfied when $\eta_2 = \sqrt{\eta_1\eta_3}$. We have then two possibilities for the condition of no reflection.

1. *When $\eta_3 = \eta_1$, we require*

$$d = n\frac{\lambda_2}{2}, \qquad n = 0, 1, 2, \ldots, \tag{8–181}$$

that is, that the thickness of the dielectric layer be a multiple of a half-wavelength in the dielectric at the operating frequency. Such a dielectric layer is referred to as a ***half-wave dielectric window***. Since $\lambda_2 = u_{p2}/f = 1/f\sqrt{\mu_2\epsilon_2}$, where f is the operating frequency, a half-wave dielectric window is a narrow-band device.

2. *When $\eta_3 \neq \eta_1$, we require*

$$\eta_2 = \sqrt{\eta_1\eta_3} \tag{8–182a}$$

and

$$d = (2n + 1)\frac{\lambda_2}{4}, \qquad n = 0, 1, 2, \ldots. \tag{8–182b}$$

When media 1 and 3 are different, η_2 should be the geometric mean of η_1 and η_3, and d should be an odd multiple of a quarter wavelength in the dielectric layer at the operating frequency in order to eliminate reflection. Under these conditions the dielectric layer (medium 2) acts like a ***quarter-wave impedance transformer***. We will refer to this term again when we study analogous transmission-line problems in Chapter 9.

EXAMPLE 8–13 The permittivity of water at optical frequencies is $1.75\epsilon_0$. It is found that an isotropic light source at a distance d under water yields an illuminated circular area of a radius 5 (m). Determine d.

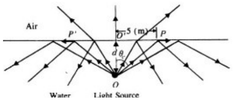

FIGURE 8–18
An underwater light source (Example 8–13).

Solution The index of refraction of water is $n_w = \sqrt{1.75} = 1.32$. Refer to Fig. 8–18. The radius of illuminated area, $O'P = 5$ (m), corresponds to the critical angle

$$\theta_c = \sin^{-1}\left(\frac{1}{n_w}\right) = \sin^{-1}\left(\frac{1}{1.32}\right) = 49.2°.$$

Hence,

$$d = \frac{\overline{O'P}}{\tan\theta_c} = \frac{5}{\tan 49.2°} = 4.32 \quad \text{(m)}.$$

As illustrated in Fig. 8–18, an incident ray with $\theta_i = \theta_c$ at P results in a reflected ray and a tangential refracted ray. Incident waves for $\theta_i < \theta_c$ are partially reflected back into the water and partially refracted into the air above, and those for $\theta_i > \theta_c$ are totally reflected (the evanescent surface waves are not shown).

EXAMPLE 8–14 A dielectric rod or fiber of a transparent material can be used to guide light or an electromagnetic wave under the conditions of total internal reflection. Determine the minimum dielectric constant of the guiding medium so that a wave incident on one end at any angle will be confined within the rod until it emerges from the other end.

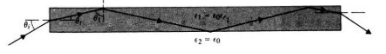

FIGURE 8–19
Dielectric rod or fiber guiding electromagnetic wave by total internal reflection.

Solution Refer to Fig. 8–19. For total internal reflection, θ_1 must be greater than or equal to θ_c for the guiding dielectric medium; that is,

$$\sin\theta_1 \geq \sin\theta_c$$

or, since $\theta_1 = \pi/2 - \theta_t$,

$$\cos\theta_t \geq \sin\theta_c. \tag{8–193}$$

From Snell's law of refraction, Eq. (8–186), we have

$$\sin\theta_t = \frac{1}{\sqrt{\epsilon_{r1}}}\sin\theta_i. \tag{8–194}$$

It is important to note here that the dielectric medium has been designated as medium 1 (the denser medium) in order to be consistent with the notation of this subsection. Combining Eqs. (8–193), (8–194), and (8–187), we obtain

$$\sqrt{1 - \frac{1}{\epsilon_{r1}}\sin^2\theta_i} \geq \sqrt{\frac{\epsilon_0}{\epsilon_1}} = \frac{1}{\sqrt{\epsilon_{r1}}},$$

which requires

$$\epsilon_{r1} \geq 1 + \sin^2\theta_i. \tag{8–195}$$

Since the largest value of the right side of (8–195) is reached when $\theta_i = \pi/2$, we require the dielectric constant of the guiding medium to be at least 2, which corresponds to an index of refraction $n_1 = \sqrt{2}$. This requirement is satisfied by glass and quartz.

EXAMPLE 8–15 The dielectric constant of pure water is 80. (a) Determine the Brewster angle for parallel polarization, $\theta_{B\parallel}$, and the corresponding angle of transmission. (b) A plane wave with perpendicular polarization is incident from air on water surface at $\theta_i = \theta_{B\parallel}$. Find the reflection and transmission coefficients.

Solution

a) The Brewster angle of no reflection for parallel polarization can be obtained directly from Eq. (8–226):

$$\theta_{B\parallel} = \sin^{-1}\frac{1}{\sqrt{1 + (1/\epsilon_{r2})}} = \sin^{-1}\frac{1}{\sqrt{1 + (1/80)}} = 81.0°.$$

The corresponding angle of transmission is, from Eq. (8–186),

$$\theta_t = \sin^{-1}\left(\frac{\sin\theta_{B\parallel}}{\sqrt{\epsilon_{r2}}}\right) = \sin^{-1}\left(\frac{1}{\sqrt{\epsilon_{r2} + 1}}\right) = \sin^{-1}\left(\frac{1}{\sqrt{81}}\right) = 6.38°.$$

b) For an incident wave with perpendicular polarization, we use Eqs. (8–206) and (8–207) to find $\Gamma_\perp$ and $\tau_\perp$ at $\theta_i = 81.0°$ and $\theta_t = 6.38°$:

$$\eta_1 = 377 \quad (\Omega), \qquad \eta_1/\cos\theta_i = 2410 \quad (\Omega),$$
$$\eta_2 = \frac{377}{\sqrt{\epsilon_{r2}}} = 40.1 \quad (\Omega), \qquad \eta_2/\cos\theta_t = 40.4 \quad (\Omega).$$

Thus,

$$\Gamma_\perp = \frac{40.4 - 2410}{40.4 + 2410} = -0.967,$$
$$\tau_\perp = \frac{2 \times 40.4}{40.4 + 2410} = 0.033.$$

We note that the relation between $\Gamma_\perp$ and $\tau_\perp$ given in Eq. (8–208) is satisfied.