VE230 — Electromagnetics I

Homework 4

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— UM-JI (Summer 2019)

P. 4-1

Let x be the up direction.

$$D_1 = D_2,$$

$$\varepsilon_0 E_1 = \varepsilon_0 \varepsilon_r E_2,$$

$$E_1 = 6E_2.$$

Let V_1 be the upper potential of the dielectric slab,

$$\begin{cases} V_0 = E_1 \cdot 0.2d + E_2 \cdot 0.8d \\ V_1 = E_2 \cdot 0.8d \end{cases} \Longrightarrow V_1 = 0.4V_0.$$

Then

$$\begin{cases} V_0 = C_1 d + C_2 \\ V_1 = C_1 \cdot 0.8d + C_2 \\ V_1 = C_3 \cdot 0.8d + C_4 \\ \ 0 = C_3 \cdot 0 + C_4 \end{cases} \Longrightarrow \begin{cases} C_1 = \dfrac{3V_0}{d} \\ C_2 = -2V_0 \\ C_3 = \dfrac{V_0}{2d} \\ C_4 = 0 \end{cases}.$$

a) In the dielectric slab, $0 \leqslant x \leqslant 0.8d$,

$$V = \frac{V_0}{2d}x, \quad \mathbf{E} = -\frac{V_0}{2d}\mathbf{a_x}.$$

b) In the air space between the dielectric slab and the upper plate, $0.8d \leqslant x \leqslant d$,

$$V = \frac{3V_0}{d}x - 2V_0, \quad \mathbf{E} = -\frac{3V_0}{d}\mathbf{a_x}.$$

c) On the upper plate,

$$\rho_s = \varepsilon_0 |\mathbf{E}| = \frac{3V_0\varepsilon_0}{d}.$$

On the lower plate,

$$\rho_s = -\varepsilon_0\varepsilon_r |\mathbf{E}| = -\frac{3V_0\varepsilon_0}{d}.$$

d) If there is no dielectric slab,

$$V = \frac{V_0}{d}x, \quad \mathbf{E} = -\frac{V_0}{d}\mathbf{a_x}.$$

P. 4-5

a)

$$R_+ = [x^2 + (y-d)^2 + z^2]^{1/2},$$

$$R_- = [x^2 + (y+d)^2 + z^2]^{1/2},$$

$$\frac{\partial R_+}{\partial x} = \frac{1}{2R_+} \cdot 2x = \frac{x}{R_+}, \frac{\partial R_-}{\partial x} = \frac{1}{2R_-} \cdot 2x = \frac{x}{R_-},$$

$$\frac{\partial V}{\partial x} = \frac{Q}{4\pi\epsilon_0} \left(\frac{\partial R_+^{-1}}{\partial R_+} \cdot \frac{\partial R_+}{\partial x} - \frac{\partial R_+^{-1}}{\partial R_-} \cdot \frac{\partial R_-}{\partial x} \right) = \frac{Q}{4\pi\epsilon_0}(-R_+^{-3}x + R_-^{-3}x),$$

$$\frac{\partial^2 V}{\partial x^2} = \frac{Q}{4\pi\epsilon_0} \left(-\frac{\partial R_+^{-3}x}{\partial R_+} \cdot \frac{\partial R_+}{\partial x} + \frac{\partial R_-^{-3}x}{\partial R_-} \cdot \frac{\partial R_-}{\partial x} \right) = \frac{Q}{4\pi\epsilon_0}(3R_+^{-5}x^2 - R_+^{-3} - 3R_-^{-5}x^2 + R_-^{-3}).$$

Similarly,

$$\frac{\partial^2 V}{\partial y^2} = \frac{Q}{4\pi\epsilon_0}[3R_+^{-5}(y-d)^2 - R_+^{-3} - 3R_-^{-5}(y+d)^2 + R_-^{-3}],$$

$$\frac{\partial^2 V}{\partial z^2} = \frac{Q}{4\pi\epsilon_0}(3R_+^{-5}z^2 - R_+^{-3} - 3R_-^{-5}z^2 + R_-^{-3}),$$

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = \frac{Q}{4\pi\epsilon_0}\{3R_+^{-5}[x^2+(y-d)^2+z^2] - 3R_+^{-3} - 3R_-^{-5}[x^2+(y+d)^2+z^2] + 3R_-^{-3}\} = 0.$$

b)

$$V(x,0,z) = \frac{Q}{4\pi\epsilon_0}\left[\frac{1}{(x^2+d^2+z^2)^{1/2}} - \frac{1}{(x^2+d^2+z^2)^{1/2}}\right] + k = V_0,$$

$$V(x,y,z) = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{R_+} - \frac{1}{R_-}\right) + V_0.$$

c)

$$F = \frac{1}{4\pi\epsilon_0} \cdot \frac{Q^2}{4d^2} = \frac{Q^2}{16\pi\epsilon_0 d^2}.$$

P. 4-11

P. 4-14

a)

$$C_1 = \frac{1}{2D}(a_2^2 - a_1^2 - D^2), \quad C_2 = \frac{1}{2D}(a^2 - a_1^2 + D^2), \quad b^2 = c_1^2 - a_1^2.$$

$$V_1 = \frac{\rho_l}{2\pi\epsilon_0} \ln \frac{b + (c_1 - a_1)}{b - (c_1 - a_1)},$$

$$V_2 = \frac{\rho_l}{2\pi\epsilon_0} \ln \frac{b + (c_2 - a_2)}{b - (c_2 - a_2)},$$

$$C = \frac{\rho_l}{V_1 - V_2} = 2\pi\epsilon_0 \ln\left[\frac{b + (c_1 - a_1)}{b - (c_1 - a_1)} \cdot \frac{b - (c_1 - a_1)}{b + (c_1 - a_1)}\right]^{-1}.$$

b)

$$F = \frac{1}{2\pi\epsilon_0} \cdot \frac{\rho_l^2}{4b^2}.$$

P. 4-17

a)

$$V_1(x,y,z) = \frac{1}{4\pi\epsilon_0}\left(\frac{Q}{R_+} - \frac{Q_1}{R_-}\right),$$

$$R_+ = [(x-d)^2 + y^2 + z^2]^{1/2},$$

$$R_- = [(x+d)^2 + y^2 + z^2]^{1/2}.$$

Similar to P. 4-5,

$$\nabla^2 V_1 = \frac{Q}{4\pi\epsilon_0}\{3R_+^{-5}[(x-d)^2+y^2+z^2] - 3R_+^{-3}\} - \frac{Q_1}{4\pi\epsilon_0}\{3R_-^{-5}[(x+d)^2+y^2+z^2] - 3R_-^{-3}\} = 0.$$

b)

$$V_2(x,y,z) = \frac{1}{4\pi\epsilon_0}\left(\frac{Q}{R_+} + \frac{Q_2}{R_+}\right)$$

Similar to P. 4-5,

$$\nabla^2 V_2 = \frac{Q+Q_2}{4\pi\epsilon_0}\{3R_+^{-5}[(x-d)^2+y^2+z^2] - 3R_+^{-3}\} = 0.$$

c) When $x=0$, $R_+ = R_- = R = [d^2+y^2+z^2]^{1/2}$,

$$\frac{\partial V_1}{\partial x} = \frac{1}{4\pi\epsilon_0}[-QR_+^{-3}(x-d) + Q_1R_-^{-3}(x+d)] = \frac{d}{4\pi\epsilon_0 R^3}(Q+Q_1),$$

$$\frac{\partial V_2}{\partial x} = \frac{1}{4\pi\epsilon_0}[-QR_+^{-3}(x-d) - Q_2R_+^{-3}(x-d)] = \frac{d}{4\pi\epsilon_0 R^3}(Q+Q_2),$$

$$\frac{\partial V_1}{\partial y} = \frac{1}{4\pi\epsilon_0}[-QR_+^{-3}y + Q_1R_-^{-3}y] = \frac{1}{4\pi\epsilon_0 R^3}(-Q+Q_1),$$

$$\frac{\partial V_2}{\partial y} = \frac{1}{4\pi\epsilon_0}[-QR_+^{-3}y - Q_2R_+^{-3}y] = \frac{1}{4\pi\epsilon_0 R^3}(-Q-Q_2).$$

Since

$$\frac{\partial V_1}{\partial x} = \frac{\partial V_2}{\partial x}, \quad \epsilon_1\frac{\partial V_1}{\partial y} = \epsilon_2\frac{\partial V_1}{\partial y},$$

$$Q_1 = Q_2 = \frac{\epsilon_2 - \epsilon_1}{\epsilon_2 + \epsilon_1}Q.$$

P. 4-23

$$\nabla^2 V = \frac{1}{r^2}\cdot\frac{\partial^2 V}{\partial\phi^2} = 0.$$

a) When $\phi = 0$, $V = 0$; when $\phi = \alpha$, $V = V_0$,

$$V = C_1\phi + C_2 = \frac{V_0}{\alpha}\phi.$$

b) When $\phi = 2pi$, $V = 0$; when $\phi = \alpha$, $V = V_0$,

$$V = C_1\phi + C_2 = \frac{V_0}{\alpha - 2\pi}(\phi - 2\pi).$$

P. 4-28

a)

$$V(b,\theta) = V_0,$$

$$V(R,\theta) = B_0R^{-1} + (B_1R^{-2} - E_0R)\cos\theta + \sum_{n=2}^{\infty} B_nR^{-n-1}P_n\cos\theta, \quad R \geqslant b.$$

Since the sphere is charged, $B_0 = bV_0$, we can obtain $B_1 = E_0 b^3$ and $B_n = 0$ for $n \geqslant 2$.

$$V(R, \theta) = \frac{bV_0}{R} - E_0 \left[1 - \left(\frac{b}{R}\right)^3\right] R\cos\theta, \quad R \geqslant b.$$

b)

$$\mathbf{E}(R, \theta) = \mathbf{a}_R E_R + \mathbf{a}_\theta E_\theta,$$

where

$$E_R = -\frac{\partial V}{\partial R} = -\frac{bV_0}{R^2} + E_0 \left[1 + 2\left(\frac{b}{R}\right)^3\right] \cos\theta, \quad R \geqslant b,$$

$$E_\theta = -\frac{\partial V}{R\partial\theta} = -E_0 \left[1 - \left(\frac{b}{R}\right)^3\right] R\sin\theta, \quad R \geqslant b.$$