VE230 — Electromagnetics I

Homework 8

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— UM-JI (Summer 2019)

P. 7-24

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} = -\mu \frac{\partial \mathbf{H}}{\partial t},$$

$$\nabla \times (\nabla \times \mathbf{E}) = -\mu \frac{\partial}{\partial t}\left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}\right) = -\mu \frac{\partial \mathbf{J}}{\partial t} - \mu\varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2},$$

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = \frac{1}{\varepsilon}\nabla\rho - \nabla^2 \mathbf{E},$$

$$\nabla^2 \mathbf{E} = \frac{1}{\varepsilon}\nabla\rho + \mu\frac{\partial \mathbf{J}}{\partial t} + \mu\varepsilon\frac{\partial^2 \mathbf{E}}{\partial t^2},$$

$$\nabla^2 \mathbf{E} = \frac{1}{\varepsilon}\nabla\rho + \mu j\omega\mathbf{J} + \mu\varepsilon\omega^2\mathbf{E}.$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t},$$

$$\nabla \times (\nabla \times \mathbf{H}) = \nabla \times \mathbf{J} + \varepsilon\frac{\partial}{\partial t}(\nabla \times \mathbf{E}) = \nabla \times \mathbf{J} - \mu\varepsilon\frac{\partial^2 \mathbf{H}}{\partial t^2},$$

$$\nabla \times (\nabla \times \mathbf{H}) = \nabla(\nabla \cdot \mathbf{H}) - \nabla^2 \mathbf{H} = -\nabla^2 \mathbf{H},$$

$$\nabla^2 \mathbf{H} = -\nabla \times \mathbf{J} + \mu\varepsilon\frac{\partial^2 \mathbf{H}}{\partial t^2},$$

$$\nabla^2 \mathbf{H} = -\nabla \times \mathbf{J} + \mu\varepsilon\omega^2\mathbf{H}.$$

P. 7-27

$$\nabla \times \mathbf{E} = \frac{1}{R^2 \sin\theta}\begin{vmatrix} \mathbf{a}_R & \mathbf{a}_\theta R & \mathbf{a}_\phi R\sin\theta \\ \frac{\partial}{\partial R} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ 0 & E_0 \sin\theta e^{-jkR} & 0 \end{vmatrix}$$

$$= \frac{1}{R^2 \sin\theta}\left(\mathbf{a}_R \frac{\partial}{\partial\phi}E_0 \sin\theta e^{-jkR} - \mathbf{a}_\phi R\sin\theta\frac{\partial}{\partial R}E_0 \sin\theta e^{-jkR}\right)$$

$$= \mathbf{a}_\phi \frac{-E_0 jk \sin\theta e^{-jkR}}{R},$$

$$\nabla \times \mathbf{E} = -\mu\frac{\partial \mathbf{H}}{\partial t} = -j\mu\omega\mathbf{H},$$

$$\mathbf{H} = \mathbf{a}_\phi \frac{E_0 k \sin\theta e^{-jkR}}{\mu\omega R}, \quad k = \omega\sqrt{\mu\varepsilon}.$$

$$\mathbf{H}(R) = \mathbf{a}_\phi \frac{E_0 \sqrt{\mu\varepsilon}\sin\theta e^{-j\omega\sqrt{\mu\varepsilon}R}}{\mu R},$$

$$\mathbf{H}(R, t) = \text{Re}[\mathbf{H}(R)e^{j\omega t}] = \mathbf{a}_\phi \frac{E_0\sqrt{\mu\varepsilon}}{\mu R} \sin\theta \cos(\omega t - \omega\sqrt{\mu\varepsilon}R).$$

P. 7-29

a)

$$\nabla \times \mathbf{E} = -j\omega\mu_0\mathbf{H} = \omega^2\mu_0\varepsilon_0\nabla \times \pi_e,$$

$$\nabla \times (\mathbf{E} - \omega^2\mu_0\varepsilon_0\pi_e) = \mathbf{0},$$

$$\mathbf{E} = \omega^2\mu_0\varepsilon_0\pi_e + \mathbf{C}.$$

$$\nabla \times \mathbf{H} = j\omega\mathbf{D} = \omega^2\mu_0\varepsilon_0\nabla \times \pi_e,$$

$$\nabla \times (j\omega\varepsilon_0\nabla \times \pi_e) = j\omega\varepsilon_0(\omega^2\mu_0\varepsilon_0\pi_e + \frac{\mathbf{P}}{\varepsilon_0} + \mathbf{C}),$$

$$\nabla \times (\nabla \times \pi_e) = \nabla(\nabla \cdot \pi_e) - \nabla^2\pi_e = \omega^2\mu_0\varepsilon_0\pi_e + \frac{\mathbf{P}}{\varepsilon_0} + \mathbf{C},$$

$$\nabla^2\pi_e + \omega^2\mu_0\varepsilon_0\pi_e = \nabla(\nabla \cdot \pi_e) - \frac{\mathbf{P}}{\varepsilon_0} - \mathbf{C},$$

$$\mathbf{C} = \nabla(\nabla \cdot \pi_e),$$

$$\mathbf{E} = \omega^2\mu_0\varepsilon_0\pi_e + \nabla(\nabla \cdot \pi_e).$$

b)

$$k_0^2 = \omega^2\mu_0\varepsilon_0,$$

$$\nabla^2\pi_e + k_0^2\pi_e = -\frac{\mathbf{P}}{\varepsilon_0}.$$

P. 8-7

Let $\phi = \omega t - kz$,

$$\mathbf{E}(\phi) = \mathbf{a}_x E_{10} \sin\phi + \mathbf{a}_y E_{20} \sin(\phi + \psi),$$

$$\frac{E_x}{E_{10}} = \sin\phi,$$

$$\frac{E_y}{E_{20}} = \sin(\phi + \psi) = \sin\phi\cos\psi + \cos\phi\sin\psi = \frac{E_x}{E_{10}}\cos\psi + \sqrt{1 - \left(\frac{E_x}{E_{10}}\right)^2}\sin\psi,$$

$$\left[\sqrt{1 - \left(\frac{E_x}{E_{10}}\right)^2}\sin\psi\right]^2 = \left(\frac{E_y}{E_{20}}\right)^2 + \left(\frac{E_x}{E_{10}}\cos\psi\right)^2 - 2\left(\frac{E_y}{E_{20}}\right)\left(\frac{E_x}{E_{10}}\cos\psi\right),$$

$$\left(\frac{E_x}{E_{10}}\right)^2 + \left(\frac{E_y}{E_{20}}\right)^2 - 2\frac{E_x}{E_{10}}\frac{E_y}{E_{20}}\cos\psi = \sin^2\psi,$$

$$\left(\frac{E_x}{E_{10}\sin\psi}\right)^2 + \left(\frac{E_y}{E_{20}\sin\psi}\right)^2 - 2\frac{E_x}{E_{10}}\frac{E_y}{E_{20}}\frac{\cos\psi}{\sin^2\psi} = 1.$$

P. 8-9

$$\nabla^2\mathbf{E} + k_c^2\mathbf{E} = 0,$$

$$k_c = \omega\sqrt{\mu\varepsilon} = \beta - j\alpha,$$

$$\omega^2 \mu \varepsilon = \omega^2 \mu(\varepsilon - j\sigma/\omega) = \beta^2 - \alpha^2 - 2j\alpha\beta,$$

$$\begin{cases} \beta^2 + \alpha^2 = \omega^2 \mu \sqrt{\varepsilon^2 + \sigma^2/\omega^2} \\ \beta^2 - \alpha^2 = \omega^2 \mu \varepsilon \end{cases} \implies \begin{cases} \alpha = \omega \sqrt{\dfrac{\mu\varepsilon}{2}} \left[\sqrt{1 + \left(\dfrac{\sigma^2}{\omega\varepsilon}\right)^2} - 1 \right]^{1/2} \\ \beta = \omega \sqrt{\dfrac{\mu\varepsilon}{2}} \left[\sqrt{1 + \left(\dfrac{\sigma^2}{\omega\varepsilon}\right)^2} + 1 \right]^{1/2} \end{cases}.$$

P. 8-14

$$f_{max} = \frac{1}{2\pi} \sqrt{\frac{N_{max} e^2}{m\varepsilon_0}},$$

$$\varepsilon_{min} = \varepsilon_0 \left[1 - \left(\frac{f_{max}}{f}\right)^2 \right],$$

$$\frac{1}{\sin\theta_i} = \sqrt{\frac{\varepsilon_0}{\varepsilon_{min}}},$$

$$\varepsilon_{min} = \varepsilon_0 \sin^2\theta_i = \varepsilon_0 \left[1 - \left(\frac{f_{max}}{f}\right)^2 \right],$$

$$f = \frac{f_{max}}{\cos\theta_i} = \frac{1}{2\pi \cos\theta_i} \sqrt{\frac{N_{max} e^2}{m\varepsilon_0}}.$$

P. 8-15

a)

$$u_g = \frac{dw}{d\beta} = \frac{d}{d\beta}(\beta u_p) = u_p + \beta \frac{du_p}{d\beta}.$$

b)

$$\lambda = \frac{2\pi}{\beta},$$

$$\frac{d\lambda}{d\beta} = \frac{d}{d\beta}\left(\frac{2\pi}{\beta}\right) = -\frac{2\pi}{\beta^2} = -\frac{\lambda}{\beta},$$

$$u_g = u_p + \beta \frac{du_p}{d\beta} = u_p + \beta \frac{du_p}{d\lambda} \cdot \frac{d\lambda}{d\beta} = u_p - \lambda \frac{du_p}{d\lambda}.$$