VG401 — Probabilistic Methods in Engineering

Assigment 1

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Group 37 — UM-JI (Spring 2019)

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Exercise 1.1 Elementary Probability

First, do not consider my friend, the probability that I will be chosen is

$$P[\text{me}] = \frac{120}{2000}.$$

Then, if I'm already been chosen, the probability that my friend will also be chosen is

$$P[\text{friend}] = \frac{119}{1999}.$$

So the probability that I and my friend will both be chosen is

$$P[\text{both}] = P[\text{me}]P[\text{friend}] = \frac{597}{99950}.$$

Exercise 1.2 Some Routine Calculations

i) Let $A^C = B \setminus A$, then $A \cup A^C = B$, $A \cap A^C = \emptyset$, according to the axioms of probability,

$$P[B] = P[A \cup A^C] = P[A] + P[A^C].$$

Since $P[A^C] \geqslant 0$, we can get

$$P[A] \leqslant P[B].$$

ii) Since A and B are independent, according to the axioms of probability,

$$P[A]P[B] = P[A \cap B] > 0.$$

However, if A and B are not mutually exclusive, $A \cap B = \emptyset$, then

$$P[A \cap B] = P[\emptyset] = 0,$$

which is a contradiction, so A and B are not mutually exclusive.

iii) Let $X = A \setminus (A \cap B)$, $Y = B \setminus (A \cap B)$, then

$$A \cup B = X \cup Y \cup (A \cap B),$$

$$X \cap Y = X \cap (A \cap B) = Y \cap (A \cap B) = \emptyset.$$

According to the axioms of probability,

$$P[A \cup B] = P[X \cup Y \cup (A \cap B)] = P[X] + P[Y] + P[A \cap B],$$

$$P[A] = P[X \cup (A \cap B)] = P[X] + P[A \cap B],$$

$$P[B] = P[Y \cup (A \cap B)] = P[Y] + P[A \cap B].$$

So

$$P[A \cup B] = P[X] + P[Y] - P[A \cap B].$$

Exercise 1.3 D'Alembert's Coins

i) Suppose it is possible to have a coin biased in the way that the probability of head is p, thus the probability of tail should be $1 - p$. Record the head as h and the tail as t, then

$$\begin{cases} P[hh] = P[h]^2 = p^2 & = \dfrac{1}{3}, \\ P[ht] + P[th] = 2P[h]P[t] = p(1-p) & = \dfrac{1}{3}, \\ P[tt] = P[t]^2 = (1-p)^2 & = \dfrac{1}{3}. \end{cases}$$

If $p^2 = (1-p)^2$, then $p = \dfrac{1}{2}$, but $p^2 = \dfrac{1}{3}$, so the system of equations can't be solved, it's impossible.

ii) Suppose it is possible to have the head of first coin with the probability of p_1, and that of the second coin with the probability of p_2, thus the probability of tails should be $1 - p_1$ and $1 - p_2$. Use the same notation h and t, then

$$\begin{cases} P[hh] = P[h_1]P[h_2] = p_1 p_2 & = \dfrac{1}{3}, \\ P[ht] + P[th] = P[h_1]P[t_2] + P[t_1]P[h_2] = p_1(1-p_2) + p_2(1-p_1) & = \dfrac{1}{3}, \\ P[tt] = P[t_1]P[t_2] = (1-p_1)(1-p_2) = 1 - p_1 - p_2 + p_1 p_2 & = \dfrac{1}{3}. \end{cases}$$

We can simply get $p_1 + p_2 = 1$, and $p_1^2 + (1-p_1)^2 = \dfrac{1}{3}$, but this equation can't be solved in $\mathbb{R}$, so it's impossible.

Exercise 1.4 Independence

Suppose event A be "the first question concerning the barn is asked", and event B be "claiming to have seen the non-existing barn in the second question", $P[A] = 0.5$.

i)

$$P[B] = P[B|A]P[A] + P[B|\neg A]P[\neg A] = 0.17 \cdot 0.5 + 0.03 \cdot 0.5 = 0.1.$$

ii)

$$P[A|B] = \frac{P[B|A]P[A]}{P[B]} = \frac{0.17 \cdot 0.5}{0.1} = 0.85 \neq P[A].$$

So claiming to see the barn is not independent of being asked the first question about the barn.

Exercise 1.5 This one may need a little thinking about...

Suppose event A be "the chip is stolen", and event B be "the chip is defective", then

$$P[B|A] = 0.5, \quad P[B|\neg A] = 0.05, \quad P[A] = 0.01, \quad P[\neg A] = 0.99.$$

According to Bayes's Theorem,

$$P[A|B] = \frac{P[B|A]P[A]}{P[B|A]P[A] + P[B|\neg A]P[\neg A]} = \frac{0.5 \cdot 0.01}{0.5 \cdot 0.01 + 0.05 \cdot 0.99} = \frac{10}{109}.$$

Exercise 1.6 Monty Hall in Prison?

Both of them are not correct.
Suppose event X be "prisoner X will be executed the next morning", event X^* be "the warden told that X is not to be executed", then

$$P[A] = P[B] = P[C] = \frac{1}{3},$$

$$P[B^*|A] = \frac{1}{2}, \quad P[B^*|B] = 0, \quad P[B^*|C] = 1.$$

$$P[A|B^*] = \frac{P[B^*|A]P[A]}{P[B^*|A]P[A] + P[B^*|B]P[B] + P[B^*|C]P[C]} = \frac{\frac{1}{2} \cdot \frac{1}{3}}{\frac{1}{2} \cdot \frac{1}{3} + 1 \cdot \frac{1}{3}} = \frac{1}{3}.$$

According to symmetric, if we use C^*, the result will be the same, so prisoner A still have $\frac{1}{3}$ chance of dying, the wander won't give A any information. However, if the wander told A that B is not to be executed, then he will give information about B and C, since $P[B|B^*] = 0$ and $P[C|B^*] = \frac{2}{3}$.

Exercise 1.7 Two Children Paradox - Birthday Party!

Since the birthday party is in July, we can conclude that if there are two boys, at least one boy was born in July; if there is one boy and one girl, the boy must be born in July.

$$P[\text{two boys, at least one born in July}] = \frac{1}{4} \cdot \left(1 - \frac{11^2}{12^2}\right) = \frac{23}{576}.$$

$$P[\text{one boy born in July and one girl}] = \frac{1}{2} \cdot \frac{1}{12} = \frac{1}{24}.$$

So the probability that the lady's other child is a girl is

$$P = \frac{P[\text{one boy born in July and one girl}]}{P[\text{two boys, at least one born in July}] + P[\text{one boy born in July and one girl}]} = \frac{24}{47}.$$

Exercise 1.8 Discrete Uniform Distribution

i)

$$m_X(t) = E[e^{tX}] = \sum_{k=1}^{n} \frac{1}{n} e^{kt} = \frac{e^t(1 - e^{nt})}{n(1 - e^t)}.$$

ii)

$$E[X] = \left.\frac{dm_X(t)}{dt}\right|_{t=0} = \left.\sum_{k=1}^{n} \frac{k}{n} e^{kt}\right|_{t=0} = \frac{n(n+1)}{2n} = \frac{n+1}{2}.$$

$$E[X^2] = \left.\frac{d^2 m_X(t)}{dt^2}\right|_{t=0} = \left.\sum_{k=1}^{n} \frac{k^2}{n} e^{kt}\right|_{t=0} = \frac{n(n+1)(2n+1)}{6n} = \frac{(n+1)(2n+1)}{6}.$$

$$Var[X] = E[X^2] - E[X]^2 = \frac{(n+1)(2n+1)}{6} - \frac{(n+1)^2}{4} = \frac{(n-1)(n+1)}{12}.$$

Exercise 1.9 Uniqueness of Moment Generating Functions - Simple Case

$$m_X(t) = E[e^{tX}] = \sum_{k=0}^{n} f_X e^{kt},$$

$$m_Y(t) = E[e^{tY}] = \sum_{k=0}^{n} f_Y e^{kt}.$$

For all $t \in (-\varepsilon, \varepsilon)$, $\varepsilon > 0$,

$$m_X(t) - m_Y(t) = \sum_{k=0}^{n} (f_X(k) - f_Y(k)) e^{kt} = 0.$$

Let series $a_k = f_X(k) - f_Y(k)$ and $x = e^t$, where $x \in (e^{-\varepsilon}, e^{\varepsilon})$,

$$m_X(t) - m_Y(t) = \sum_{k=0}^{n} a_k x^k = 0.$$

It becomes a polynomial of x of at most degree n with coefficients $\{a_k\}$ if a_k is not always zero, and the value of this polynomial is always zero. Then we can write it into a polynomial equation of infinite number of roots $x \in (e^{-\varepsilon}, e^{\varepsilon})$,

$$F(x) = \sum_{k=0}^{n} a_k x^k = 0.$$

However, according to the fundamental theorem of algebra, a polynomial of degree n can have at most n (complex) roots, which makes a contradiction. So $F(x)$ can't be a polynomial, meaning that $a_k = 0$ for all $k = 0, \dots, n$. Then we can deducted that

$$f_X(x) = f_Y(x) \quad \text{for} \quad x = 0, \dots, n.$$