Ve401 Probabilistic Methods in Engineering

Spring 2019 — Assigment 2

Date Due: 4:00 PM, Thursday, the 14th of March 2019

This assignment has a total of **(46 Marks)**.

Exercise 2.1 Drawing until First Success in the Hypergeometric Setting

i) Let (X, f_X) be a discrete random variable taking on only positive values, i.e., ran $X \subset \mathbb{N}$. Show that

$$\mathrm{E}[X] = \sum_{x=0}^{\infty} P[X > x].$$

(2 Marks)

ii) A box contains N balls, of which r are red and $N - r$ are black. Balls are drawn from the box until a red ball is drawn. Show that the expected number of draws is

$$\frac{N+1}{r+1}.$$

Hint: $\sum_{x=0}^{r} \binom{N-r+x}{N-r} = \binom{N+1}{N-r+1}$
(4 Marks)

Exercise 2.2 Density of the Poisson Approximation

The density $p_x(t)$, $x \in \mathbb{N}$, of the Poisson distribution is obtained iteratively from the following differential equations:

$$p_0' = -\lambda p_0, \qquad\qquad p_x' + \lambda p_x = \lambda p_{x-1}.$$

Use induction to prove that $p_x(t) = (\lambda t)^x e^{-\lambda t}/x!$. Justify the initial values that you apply in your derivation.
(4 Marks)

Exercise 2.3 Poisson Approximation to the Binomial Distribution

Consider the density f of the binomial distribution,

$$f(x) = \binom{n}{x} p^x (1-p)^{n-x}. \tag{$*$}$$

Let k be fixed so that $np = k$ and set $p = k/n$. Replace p by k/n everywhere in $(*)$ and then let $n \to \infty$. Show that for every x, $f(x) \to (k^x/x!)e^{-k}$, the density of the Poisson distribution with parameter k.
(4 Marks)

Exercise 2.4 Maxwell-Boltzmann Statistics

The distribution function of the speed (modulus of the velocity) V of a gas molecule is described by the Maxwell-Boltzmann law

$$f_V(v) = \begin{cases} \left(\dfrac{2}{\pi}\right)^{1/2} \left(\dfrac{m}{kT}\right)^{3/2} v^2 e^{-\frac{m}{kT}v^2/2} & v > 0 \\ 0 & v \le 0 \end{cases}$$

where $m > 0$ is the mass of the molecule, $T > 0$ is its temperature and $k > 0$ is the Boltzmann constant.

i) Find the mean and variance of V.
(2 Marks)

ii) Find the mean of the kinetic energy $E = mV^2/2$.
(2 Marks)

iii) Find the probability density f_E of E.
(3 Marks)

Exercise 2.5 Half-Integer Values of the Gamma Function
Calculate $\Gamma((2n+1)/2)$, $n \in \mathbb{N}$, where Γ denotes the Euler gamma function.
(3 Marks)

Exercise 2.6 Finding Probabilities with the Normal Distribution
The compressive strength of samples of cement can be modeled by a normal distribution with a mean of 6000 kilograms per square centimeter and a standard deviation of 100 kilograms per square centimeter.

i) What is the probability that a samples strength is less than 6250 kg/cm^2?
(1 Mark)

ii) What is the probability that a samples strength is between 5800 and 5900 kg/cm^2?
(1 Mark)

iii) What strength is exceeded by 95% of the samples?
(2 Marks)

(This exercise appeared in the first midterm exam in the Fall Term of 2012.)

Exercise 2.7 Continuous Uniform Distribution
A continuous random variable X is said to be *uniformly distributed* over an interval (a, b) if its density is given by

$$f(x) = \begin{cases} 1/(b-a) & \text{for } a < x < b, \\ 0 & \text{otherwise.} \end{cases}$$

i) Show that this is a density for a continuous random variable.
(1 Mark)

ii) Sketch the graph of the density and shade the area of the graph hat represents $P[X \le (a+b)/2]$.
(1 Mark)

iii) Find the probability pictured in part ii).
(1 Mark)

iv) Let (c, d) and (e, f) be subintervals of (a, b) of equal length. What is the relationship between $P[c \le X \le d]$ and $P[e \le X \le f]$?
(1 Mark)

v) Find the cumulative distribution function F for a uniformly distributed random variable.
(1 Mark)

vi) Show that $\text{E}[X] = (a+b)/2$ and $\text{Var}\, X = (b-a)^2/12$.
(2 Marks)

Exercise 2.8 A Tricky Question involving the Binomial Distribution
A mathematics textbook has 200 pages on which typographical errors in the equations could occur. Suppose there are in fact five errors randomly dispersed among these 200 pages.

i) What is the probability that a random sample of 50 pages will contain at least one error?
(2 Marks)

ii) How large must the random sample be to assure that at least three errors will be found with 90% probability? (You may use a normal approximation to the binomial distribution.)
(3 Marks)

(This exercise appeared in the first midterm exam in the Fall Term of 2012.)

Exercise 2.9 Reliability of a System
A system consists of two independent components connected in series. The life span (in hours) of the first component follows a Weibull distribution with $\alpha = 0.006$ and $\beta = 0.5$; the second has a lifespan in hours that follows the exponential distribution with $\beta = 25000$.

i) Find the reliability of the system at 2500 hours.
(2 Marks)

ii) Find the probability that the system will fail before 2000 hours.
(2 Marks)

iii) If the two components are connected in parallel, what is the system reliability at 2500 hours?
(2 Marks)