Introduction to Cryptography

Chapter 5: Digital signatures

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Outline

Real life signatures

Middle age:

- Document sealed with a wax imprint of an insignia
- Nobody can reproduce the insignia

Modern time:

- Sign credit card slip
- Compare to the signature at the back of the credit card

Reusing a signature:

- Photocopy
- Cut and paste
- Highly noticeable

Toward digital signatures

Signing an electronic document:

- Digitalize the signature
- Paste it on the electronic document

Reusing a signature:

- Copy and paste on any document
- Anybody can do it
- Signature is not specific to an individual

Toward digital signatures

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- Signature is not specific to an individual

Basic idea for a solution:

- Prevent the signature from being separated from its message
- Signature must be easily verified

Digital signatures

Setup for signatures:

- Message to encrypt is not necessarily secret
- A message might be encrypted after being signed

The signature must be:

- Tied to the signer and to the message being signed
- Easy to verify by anybody
- Hard to forge

Digital signatures

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Conclusion: similar to public key cryptography as public and secret keys are required

Attacks on signatures

Similarly to attacks on encryption scheme (1.11) we define attacks on signature schemes:

- **Key-only attack:** Eve has only access to the public key
- **Known message attack:** Eve has a list of previously signed messages
- **Chosen message attack:** Eve chooses the messages to be signed
- **Selective forgery:** Eve is given a message and is able to sign it without being given the private key
- **Existential forgery:** Eve can find a pair $\langle \mathit{message}, \mathit{signature} \rangle$ without being given the private key

Signatures and hash functions

Drawback:

- Public key cryptography primitives are used
- Signing a whole message m is then slow

Solution: sign the hash of m using a public hash function

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Benefits:

- Faster to generate
- Smaller to store or send
- Conveys the same knowledge as m itself

Given a hash function h, denote the signature of the hash of a message m by $\mathrm{sig}(h(m))$

Signatures and hash functions

- Existential forgery using known message attack:
  1. Get a pair $\langle m, \mathrm{sig}(h(m)) \rangle$
  2. Compute $h(m)$ and attempt to find m' such that $h(m) = h(m')$
  3. Considered impossible if h is second pre-image resistant
- Existential forgery using chosen message attack:
  1. Find two message m and m' such that $h(m) = h(m')$
  2. Persuade the signer to sign m
  3. Attach $\mathrm{sig}(h(m)) = \mathrm{sig}(h(m'))$ to m'
  4. Considered impossible if h is collision resistant
- Existential forgery using key-only attack:
  1. Take a signature scheme, without hash function, which is vulnerable to existential forgery using key-only attack
  2. Compute a signature on $h(m)$ for some unknown m
  3. Determine such an m
  4. Considered impossible if h is pre-image resistant

Signatures and the birthday attack

In the previous chapter we investigated the birthday paradox an illustrated how Eve could use this attack to cheat Alice when signing a contract (4.16).

Such an attack can be conducted as soon as the hash is used in place of the whole document. Therefore Alice should be careful and not sign the document. She should rather slightly alter it, for instance by adding a coma or space.

The document being different from the original its hash will be a totally different value. Hence, Eve cannot append Alice signature to the fraudulent contract.

Eve is then defeated and Alice can enjoy a nice contract.

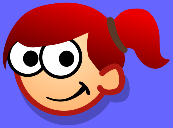
Outline

RSA signatures

Initial setup:

- p, q two primes
- $n = pq$ and $\varphi(n)$
- e, d such that $ed \equiv 1 \bmod \varphi(n)$

- The n from Bob
- The e from Bob



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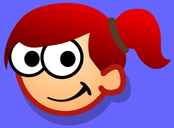

Signature:

- Compute $s \equiv m^d \bmod n$
- Share m and s

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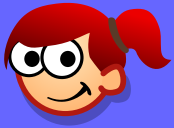

Signature:

- Compute $s \equiv m^d \bmod n$
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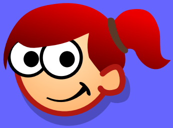
Verification:

- The message m from Bob
- Compute $m' \equiv s^e \bmod n$
- Compare m' to m



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Reusing a signature:

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Generating a message:

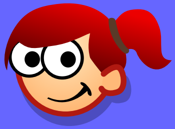
- Given a signature s generate a message $m \equiv s^e \bmod n$
- It is very unlikely that m is meaningful

Elgamal signatures

Initial setup:

- G a group of prime order p
- α a generator of G
- x a secret integer

- G from Bob
- α from Bob
- $\beta \equiv \alpha^x \bmod p$ from Bob



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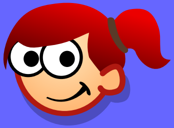

Signature:

- Select a random k, with $\gcd(k, p-1) = 1$
- Compute $r \equiv \alpha^k \bmod p$
- Compute $s \equiv k^{-1}(m - xr) \bmod (p-1)$

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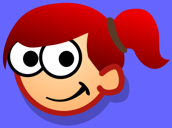

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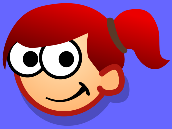
Verification:

- The triple $\langle m, r, s \rangle$
- Compute
  $$v = \beta^r r^s \equiv \alpha^{xr} \alpha^{k \cdot k^{-1}(m-xr)} \equiv \alpha^m \bmod p$$
- The signature is valid only if $v \equiv \alpha^m \bmod p$



Elgamal signatures

Example.
Set $p = 467$, $\alpha = 2$ and $x = 127$. Then $\beta = 2^{127} \equiv 132 \bmod 467$.
The variable x is kept secret, all the others are publicly known.

Signing the message $m = 100$:

- Randomly choose $k = 213$ and keep it since $\gcd(213, 466) = 1$
- Compute $r = 2^{213} \equiv 29 \bmod 467$
- As $k^{-1} \equiv 431 \bmod 466$, $s = 466 \cdot (100 - 127 \cdot 29) \equiv 51 \bmod 466$

To verify the signature $\langle 100, 29, 51 \rangle$, anyone can compute both:

- $132^{29} \cdot 29^{51} \equiv 189 \bmod 467$
- $2^{100} \equiv 189 \bmod 467$

Comments on Elgamal signatures

First we notice that if x is discovered by an attacker, he can signed any document.
Then we observe that given only a message m he can try to

- Find s such that

$$\beta^r r^s \equiv \alpha^m \bmod p. \tag{5.1}$$

  This can be rewritten $r^s \equiv \beta^{-r}\alpha^m \bmod p$, and finding s means solving the DLP.

- Set s and solve eq. (5.1) for r. No feasible solution is known.

- Find r and s simultaneously. It is not known how to do it, but there is no prove that it is impossible to do.

Note that k must remain secret otherwise it is simple to recover x. Indeed if $\gcd(r, p-1) = 1$, then $x \equiv (m - ks)r^{-1} \bmod (p-1)$.

Key-only attack on Elgamal signatures

Goal: generate a message and its signature only knowing the public key

Let i and j be two integers such that $0 \leq i, j \leq p-2$. Define r as $\alpha^i\beta^j \bmod p$. Then α^m can be expressed as

$$\alpha^m \equiv \beta^r \left(\alpha^i\beta^j\right)^s \bmod p.$$

Rearranging the different terms yields $\alpha^{m-is} \equiv \beta^{r+js} \bmod p$. This congruence is clearly true if both $m - is$ and $r + js$ are 0 mod $(p-1)$.

Assuming $\gcd(j, p-1) = 1$, we can determine m and s from the two previous equations. Therefore, by construction the signature

$$\langle m, r, s\rangle = \langle -rij^{-1} \bmod (p-1), \alpha^j\beta^j \bmod p, -rj^{-1} \bmod (p-1)\rangle$$

is a valid signature. Note that m is very unlikely to be meaningful.

Key-only attack on Elgamal signatures

Example.
Set $p = 467$, $\alpha = 2$ and $\beta = 132$. Select $i = 99$ and $j = 179$, and then $j^{-1} \equiv 151 \bmod 466$.

The signature is defined by $\langle m, r, s \rangle$ with

$$\begin{cases} r \equiv 2^{99} \cdot 132^{179} & \equiv 117 \bmod 467 \\ s \equiv -117 \cdot 151 & \equiv 41 \bmod 466 \\ m \equiv 99 \cdot 41 & \equiv 331 \bmod 466 \end{cases}$$

The verification is given by

$$132^{117} \cdot 117^{41} \equiv 303 \equiv 2^{331} \bmod 467$$

Known message attack on Elgamal signatures

Given a valid signature $\langle m, r, s \rangle$ an attacker can construct and sign various other messages.

Generate h, i, and j such that $\gcd(hr - js, p - 1) = 1$. Then the triple $\langle m', r', s' \rangle$ defines a valid signature if

$$\begin{cases} r' \equiv r^h \alpha^i \beta^j \bmod p \\ s' \equiv sr'(hr - js)^{-1} \bmod (p-1) \\ m' \equiv r'(hm + is)(hr - js)^{-1} \bmod (p-1) \end{cases}$$

Again this method leads to an existential forgery but cannot be modified into selective forgery. As such those two attacks represent no real threat for Elgamal signatures.

Misuse of Elgamal signatures

Let $\langle m_1, r_1, s_1 \rangle$ and $\langle m_2, r_2, s_2 \rangle$ be the two signatures. If they are generated using a common k, then $r_1 = r_2 = r = \alpha^k \bmod p$ and

$$\begin{cases} \beta^r r^{s_1} & \equiv \alpha^{m_1} \bmod p \\ \beta^r r^{s_2} & \equiv \alpha^{m_2} \bmod p. \end{cases}$$

Thus $\alpha^{m_1 - m_2} \equiv \alpha^{k(s_1 - s_2)} \bmod p$, and from corollary 3.18 we get

$$m_1 - m_2 \equiv k(s_1 - s_2) \bmod (p-1).$$

Since this congruence has $d = \gcd(s_1 - s_2, p-1)$ solutions (lemma 4.10) it is simple to test all of them and recover k. Once k is known x can be recovered as noticed on slide 5.15, and signatures can be forged at will.

Digital Signature Algorithm

Digital Signature Algorithm (DSA):

- Proposed in 1991 by the NSA
- Adopted as a standard in 1994
- Variant of Elgamal signature scheme
- As in Elgamal the hash of the message is signed
- SHA-1 is the historical choice but SHA-2 (SHA-3) is now recommended
- For a given security level DSA defines two lengths l_1 and l_2 for the DLP and the hash to feature a balanced security

Digital Signature Algorithm

Initial setup:

- A prime q, $|q| = l_2$
- A prime p, $|p| = l_1$ and $q \mid (p-1)$
- g a generator of $G = \mathrm{U}(\mathbb{F}_p)$
- $\alpha \equiv g^{(p-1)/q} \bmod p$
- x a secret integer

- p from Bob
- q from Bob
- α from Bob
- $\beta \equiv \alpha^x \bmod p$ from Bob

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Signature:

- Select a random k, $0 < k < q$
- Compute $r \equiv (\alpha^k \bmod p) \bmod q$
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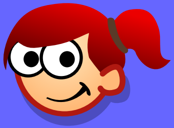
Verification:

- The triple $\langle m, r, s \rangle$
- Compute $v = (\alpha^{s^{-1} m \bmod q} \beta^{s^{-1} r \bmod q} \bmod p) \bmod q$
- The signature is valid only if $v = r$



DSA signature verification

Observe how the verification works:

By definition of s we know that $m \equiv (-xr + ks) \bmod q$. This implies $s^{-1}m \equiv (-xrs^{-1} + k) \bmod q$. Therefore we can write

$$k \equiv s^{-1}m + xrs^{-1} \bmod q.$$

And we finally get

$$\begin{aligned} r &\equiv \alpha^k \bmod p \\ &\equiv \alpha^{s^{-1}m + xrs^{-1} \bmod q} \bmod p \\ &\equiv \alpha^{s^{-1}m \bmod q} \beta^{s^{-1}r \bmod q} \bmod p \\ &= v. \end{aligned}$$

Comments on DSA

Why is DSA different from Elgamal?

- r only "carries part of the information" on k
  e.g. if $l_1 = 3072$ and $l_2 = 256$, then about 2^{2816} values mod p reduce to a same integer mod q

- From the initial setup (slide 5.21), $\alpha^q \equiv 1 \bmod p$.
  Pohlig-Hellman attack (3.78) does not apply, since q is prime.
  Not even a little piece of information can be recovered.

- Verification step requires only two modular exponentiations vs. three in Elgamal case

Outline

Blind signatures

Basic idea: sign a document without knowing its content

Typical setup: Bob made a new discovery and wants to record it publicly without unveiling it

Strategy: Bob gets his discovery signed by some known authority but without revealing or showing it the content

Danger:

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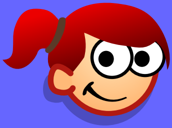
Danger: what is signed?

RSA blind signatures

Initial setup:



- p, q two primes
- $n = pq$ and $\varphi(n)$
- e, d such that $ed \equiv 1 \bmod \varphi(n)$

- n, e from Alice
- A random integer k, $\gcd(k, n) = 1$
- For a message m compute $t \equiv k^e m$

RSA blind signatures

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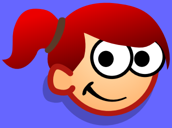

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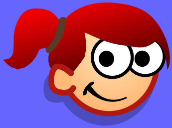

- t from Bob
- Compute $s \equiv t^d \bmod n$
- Send s to Bob

RSA blind signatures

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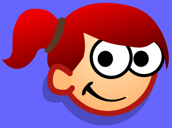

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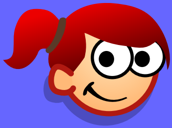

- t from Bob
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- Send s to Bob

Message signature:

- Bob computes

$$\frac{s}{k} \equiv \frac{t^d}{k} \equiv \frac{k^{ed} m^d}{k} \equiv m^d \bmod n$$

- Bob later shares m and e

Comments on RSA blind signatures

- k being random $k^e \bmod n$ is also random and so is $k^e m \bmod n$
- Alice cannot get any information on what she is signing
- The final value is the same as if Bob had gotten his message signed following the standard procedure
- Verification happens as in "regular RSA signatures"
- There is no need to keep d, p and q

Undeniable signatures

Primary goal: design a signature that cannot be verified without the cooperation of the signer

Secondary goals:

- Prevent the signer to disavow a previous signature
- Allow the signer to prove that a forged signature is a forgery

Applications: prevent the illegal distribution of documents without the approval of the author

Structure: composed of three algorithm: signature, verification, and disavowal

Chaum-van Antwerpen signatures

Initial setup:

- p and q two primes $p = 2q + 1$
- G a subgroup of $\mathbb{F}_p^*$ of order q
- α a generator of G
- x a secret integer

- G from Bob
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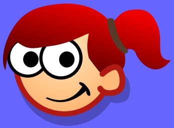

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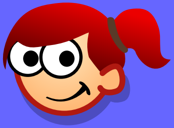

Signature:

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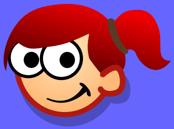
Verification:

3. Compute $t \equiv r^{x^{-1} \bmod q} \bmod p$
4. Share t with Alice

1. Choose random $e_1, e_2 \in \mathbb{F}_q^*$
2. $r \equiv s^{e_1} \beta^{e_2} \bmod p$
5. Valid if and only if $t \equiv m^{e_1} \alpha^{e_2} \bmod p$



Comments on Chaum-van Antwerpen signatures

On a valid signature we have:

$$t \equiv r^{x^{-1}} \bmod p$$
$$\equiv s^{e_1 x^{-1}} \beta^{e_2 x^{-1}} \bmod p$$

Noting that $s \equiv m^x \bmod p$, and $\beta \equiv \alpha^x \bmod p$, we get

$$t \equiv m^{e_1} \alpha^{e_2} \bmod p.$$

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Example.

Let $p = 467$, then 2 is a primitive element of $\mathbb{F}_p^*$ and 4 is a generator of the group G of order 233. Taking $x = 101$, $\beta \equiv 4^{101} \equiv 449 \bmod 467$.

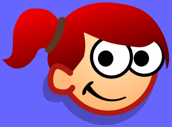
Signing the message $m = 119$ yields $119^{101} \equiv 129 \bmod 467$.

To verify the signature, randomly select $e_1 = 38$ and $e_2 = 397$, then send $r = 13$ while $t = 9$ is replied. Finally test that 9 is congruent to $119^{38} 4^{397} \bmod 467$.

Chaum-van Antwerpen signatures

Disavowal protocol

2-round verification:



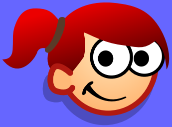
Play the verification protocol using two random values $e_1, e_2 \in \mathbb{F}_q^*$ and expect

$$t_1 \not\equiv m^{e_1}\alpha^{e_2} \bmod p$$

Chaum-van Antwerpen signatures

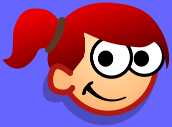
Disavowal protocol

2-round verification:



Play the verification protocol using two random values $e_1, e_2 \in \mathbb{F}_q^*$ and expect

$$t_1 \not\equiv m^{e_1}\alpha^{e_2} \bmod p$$



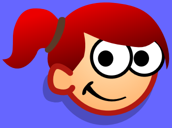
Re-play the verification protocol using two random values $f_1, f_2 \in \mathbb{F}_q^*$ and expect

$$t_2 \not\equiv m^{f_1}\alpha^{f_2} \bmod p$$

Chaum-van Antwerpen signatures

Disavowal protocol

2-round verification:



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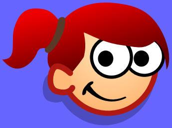

Re-play the verification protocol using two random values $f_1, f_2 \in \mathbb{F}_q^*$ and expect

$$t_2 \not\equiv m^{f_1}\alpha^{f_2} \bmod p$$

Concludes that the signature is a forgery if and only if

$$\left(t_1\alpha^{-e_2}\right)^{f_1} \equiv \left(t_2\alpha^{-f_2}\right)^{e_1} \bmod p$$

The disavowal protocol has two goals:

- Convince Alice that an invalid signature is a forgery
- Prevent Bob from pretending that a valid signature is a forgery

If the signature is invalid then the verification fails. The question is then to know if Bob played a fair game, following the protocol when constructing t_1 and t_2.

The last step, testing the congruence

$$\left(t_1\alpha^{-e_2}\right)^{f_1} \equiv \left(t_2\alpha^{-f_2}\right)^{e_1} \bmod p,$$

ensures Alice that Bob is not trying to disavow a valid signature.

From authentication to signature

As investigated ealier (1.61), zero-knowledge proofs can be used to authenticate. In fact this can also be extended to signatures.

General strategy:

- Send at once all the committed values $C_1, \cdots, C_n$
- For a message m compute H, the hash of $\langle C_1, \cdots, C_n, m \rangle$
- Extract n bits from H to represent the random requests
- Define $H_1, \cdots, H_n$ to be the result of the challenges
- Define the signature of m as $\langle H_1, \cdots, H_n, R_1, \cdots, R_n \rangle$, where R_i is the response to challenge H_i for the committed C_i
- To ensure a proper security level n should be at least 128

Key points

- How to overcome the birthday attack on digital signatures?
- Cite two famous solutions for digital signatures
- What is the reference choice in terms of digital signatures?
- How to transform a zero-knowledge authentication scheme into a signature scheme?

Thank you!