Problem 1.

Since central forces conserve angular momentum,

$$L = \bar{r} \times \bar{p} = mr^2\dot{\varphi} = Const$$

$$\frac{d}{d\varphi}\frac{1}{r} = \frac{d}{dt}\frac{1}{r}\frac{dt}{d\varphi} = -\frac{\dot{r}}{r^2\dot{\varphi}} = -\frac{m\dot{r}}{L}$$

$$\frac{d^2}{d\varphi^2}\frac{1}{r} = -\frac{d}{d\varphi}\frac{m\dot{r}}{L} = -\frac{m}{L}\frac{d}{dt}\dot{r}\frac{dt}{d\varphi} = -\frac{m\ddot{r}}{L\dot{\varphi}} = -\frac{m^2\ddot{r}r^2}{L^2}$$

$$\ddot{r} = -\frac{L^2}{m^2r^2}\frac{d^2}{d\varphi^2}\frac{1}{r}$$

$$r\dot{\varphi}^2 = r\frac{L^2}{m^2r^4} = \frac{L^2}{m^2r^3}$$

$$F_r = m(\ddot{r} - r\dot{\varphi}^2) = -\frac{L^2}{mr^2}\left[\frac{d^2}{d\varphi^2}\left(\frac{1}{r}\right) + \frac{1}{r}\right]$$

Problem 2.

$$r = 2R\cos\varphi$$

$$\begin{aligned}
F_r &= -\frac{L^2}{mr^2}\left[\frac{d^2}{d\varphi^2}\left(\frac{1}{2R\cos\varphi}\right) + \frac{1}{r}\right] \\
&= -\frac{L^2}{mr^2}\left(\frac{1}{2R}\frac{2-\cos^2\varphi}{\cos^3\varphi} + \frac{1}{r}\right) \\
&= -\frac{L^2}{mr^2}\left(\frac{8R^2 - r^2}{r^3} + \frac{1}{r}\right) \\
&= -\frac{8L^2R^2}{mr^5}
\end{aligned}$$

Problem 3.

$$\omega = \frac{d\varphi}{dt} = \frac{L}{mr^2} = \frac{L}{2mR^2\cos 2\varphi}$$

$$\frac{2mR^2\cos 2\varphi}{L}d\varphi = dt$$

Do integral on both sides,

$$\int_0^{\varphi}\frac{2mR^2\cos 2\varphi}{L}d\varphi = \int_0^t dt$$

$$\frac{mR^2\sin 2\varphi}{L} = t$$

$$\sin 2\varphi = \frac{Lt}{mR^2}$$

$$\varphi = \frac{1}{2}\arcsin\frac{Lt}{mR^2}$$

Problem 4.

$$I = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix} = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix}$$

$$\begin{aligned}
I_{xx} &= \iiint \rho(y^2+z^2)dydxdz = \iint_0^r \int_0^{2\pi} \rho(r^2 sin^2\theta + z^2) r d\theta dr dz = \int_H \rho \left(\frac{\pi r^4}{4} + \pi r^2 z^2\right) dz \\
I_{yy} &= \iiint \rho(x^2+z^2)dydxdz = \iint_0^r \int_0^{2\pi} \rho(r^2 cos^2\theta + z^2) r d\theta dr dz = \int_H \rho \left(\frac{\pi r^4}{4} + \pi r^2 z^2\right) dz \\
I_{xx} &= \iiint \rho(x^2+y^2)dydxdz = \iint_0^r \int_0^{2\pi} \rho r^3 d\theta dr dz = \int_H \frac{\rho \pi r^4}{2} dz
\end{aligned}$$

Problem 5.

(a)

$$I = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix} = \begin{bmatrix} 2ma^2 & 2ma^2 & 0 \\ 2ma^2 & 2ma^2 & 0 \\ 0 & 0 & 4ma^2 \end{bmatrix}$$

(b)

$$\bar{L} = \begin{bmatrix} I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z \\ I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z \\ I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z \end{bmatrix} = \begin{bmatrix} 2ma^2\omega \\ 2ma^2\omega \\ 0 \end{bmatrix}$$

$$\bar{L} = \bar{r_1} \times (m\bar{\omega_1} \times \bar{r_1}) + \bar{r_2} \times (m\bar{\omega_2} \times \bar{r_2}) = 2ma^2\omega\hat{n_x} + 2ma^2\omega\hat{n_y}$$

$$\bar{L} \text{ is not parallel to } \bar{\omega}$$

(c) According to symmetry arguments, $\tilde{x}$ is along the direction of the rod, $\tilde{y}$ and $\tilde{z}$ is orthogonal to each other.

$$\begin{aligned}
\tilde{x} &= \hat{n_x} - \hat{n_y} \\
\tilde{y} &= \hat{n_x} + a\hat{n_y} + b\hat{n_z} \\
\tilde{z} &= b\hat{n_x} + b\hat{n_y} - (a+1)\hat{n_z}
\end{aligned}$$

where $a, b \in R$
So these axis are not unique because $a, b \in R$.

(d)

$$I = \begin{bmatrix} I_{\tilde{x}\tilde{x}} & -I_{\tilde{x}\tilde{y}} & -I_{\tilde{x}\tilde{z}} \\ -I_{\tilde{y}\tilde{x}} & I_{\tilde{y}\tilde{y}} & -I_{\tilde{y}\tilde{z}} \\ -I_{\tilde{z}\tilde{x}} & -I_{\tilde{z}\tilde{y}} & I_{\tilde{z}\tilde{x}} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 4ma^2 & 0 \\ 0 & 0 & 4ma^2 \end{bmatrix}$$

It's diagonal.

(e)

$$L_{\tilde{x}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 4ma^2 & 0 \\ 0 & 0 & 4ma^2 \end{bmatrix} \begin{bmatrix} \omega \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$L_{\tilde{y}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 4ma^2 & 0 \\ 0 & 0 & 4ma^2 \end{bmatrix} \begin{bmatrix} 0 \\ \omega \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 4ma^2\omega \\ 0 \end{bmatrix}$$

$$L_{\tilde{z}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 4ma^2 & 0 \\ 0 & 0 & 4ma^2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \omega \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 4ma^2\omega \end{bmatrix}$$

They are parallel to the angular velocity.

(f)

$$det(I - \lambda E) = \begin{vmatrix} 2ma^2 - \lambda & 2ma^2 & 0 \\ 2ma^2 & 2ma^2 - \lambda & 0 \\ 0 & 0 & 4ma^2 - \lambda \end{vmatrix} = 0$$

$$\begin{gathered} (4ma^2 - \lambda)[(2ma^2 - \lambda)^2 - (2ma^2)^2] = 0 \\ \lambda(4ma^2 - \lambda)^2 = 0 \\ \lambda_1 = 0 \quad \lambda_2 = \lambda_3 = 4ma^2 \end{gathered}$$

Since two of the eigenvalues are the same, two axis can be selected arbitrarily.

Problem 6.

(a)

$$\begin{aligned} \bar{x} &= \frac{-am + 2am + am}{4m} = \frac{1}{2}a \\ \bar{y} &= \frac{am + 2am - am}{4m} = \frac{1}{2}a \\ \bar{z} &= 0 \end{aligned}$$

$$\begin{aligned} I &= \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix} \\ &= \begin{bmatrix} \frac{9}{4}ma^2 + \frac{3}{4}ma^2 & -(-\frac{3}{4}ma^2 + \frac{1}{2}ma^2 - \frac{3}{4}ma^2) & 0 \\ -(-\frac{3}{4}ma^2 + \frac{1}{2}ma^2 - -\frac{3}{4}ma^2) & \frac{9}{4}ma^2 + \frac{3}{4}ma^2 & 0 \\ 0 & 0 & 2\left(\frac{9}{4}ma^2 + \frac{3}{4}ma^2\right) \end{bmatrix} \\ &= \begin{bmatrix} 3ma^2 & ma^2 & 0 \\ ma^2 & 3ma^2 & 0 \\ 0 & 0 & 6ma^2 \end{bmatrix} \end{aligned}$$

(b)

$$det(I - \lambda E) = \begin{vmatrix} 3ma^2 - \lambda & ma^2 & 0 \\ ma^2 & 3ma^2 - \lambda & 0 \\ 0 & 0 & 6ma^2 - \lambda \end{vmatrix} = 0$$

$$(6ma^2 - \lambda)[(3ma^2 - \lambda)^2 - (ma^2)^2] = 0$$
$$(6ma^2 - \lambda)(4ma^2 - \lambda)(2ma^2 - \lambda) = 0$$
$$\lambda_1 = 6ma^2 \quad \lambda_2 = 4ma^2 \quad \lambda_3 = 2ma^2$$

(1) $\lambda = 6ma^2$

$$\begin{bmatrix} -3ma^2 & ma^2 & 0 \\ ma^2 & -3ma^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Longrightarrow \tilde{x} = \hat{n_z}$$

(2) $\lambda = 4ma^2$

$$\begin{bmatrix} -ma^2 & ma^2 & 0 \\ ma^2 & -ma^2 & 0 \\ 0 & 0 & 2ma^2 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Longrightarrow \tilde{y} = \frac{\sqrt{2}}{2}\hat{n_x} + \frac{\sqrt{2}}{2}\hat{n_y}$$

(3) $\lambda = 2ma^2$

$$\begin{bmatrix} a^2 & ma^2 & 0 \\ ma^2 & ma^2 & 0 \\ 0 & 0 & 4ma^2 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Longrightarrow \tilde{z} = \frac{\sqrt{2}}{2}\hat{n_x} - \frac{\sqrt{2}}{2}\hat{n_y}$$

Problem 7.

$$\begin{aligned} K &= \frac{1}{2}\sum m_i|\bar{\omega} \times \bar{r_i}|^2 = \frac{1}{2}\sum m_i[(\omega_y z - \omega_z y)^2 + (\omega_z x - \omega_x z)^2 + (\omega_x y - \omega_y x)^2] \\ &= \frac{1}{2}\sum m_i[(y^2 + z^2)\omega_x^2 + (x^2 + z^2)\omega_y^2 + (x^2 + y^2)\omega_z^2 - 2yz\omega_y\omega_z - 2xz\omega_x\omega_z - 2xy\omega_x\omega_y] \\ &= \frac{1}{2}(I_{xx}\omega_x^2 + I_{yy}\omega_y^2 + I_{zz}\omega_z^2 - 2I_{xy}\omega_x\omega_y - 2I_{xz}\omega_x\omega_z - 2I_{yz}\omega_y\omega_z) \\ &= \frac{1}{2}\begin{bmatrix} \omega_x & \omega_y & \omega_z \end{bmatrix} \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \\ &= \frac{1}{2}\sum_{\alpha\beta} I_{\alpha\beta}\omega_\alpha\omega_\beta \end{aligned}$$