Problem Set 5

Due: 24 June 2016, 10 a.m.

Problem 1. Recall that the amplitude of steady-state forced oscillations with a sinusoidal driving force $F_{\mathrm{dr}} = F_0 \cos \omega_{\mathrm{dr}}$ in the presence of linear drag is given by the formula

$$A(\omega_{\mathrm{dr}}) = \frac{F_0}{m\sqrt{\left(\omega_0^2 - \omega_{\mathrm{dr}}^2\right)^2 + \left(\frac{b\omega_{\mathrm{dr}}}{m}\right)^2}}$$

(a) Check that the driving frequency for which the amplitude is maximum, *i.e.* the resonance frequency, is $\omega_{\mathrm{res}} = \sqrt{\omega_0^2 - b^2/2m^2}$.

(b) Suppose that the damping is small and the system is being driven at an angular frequency close to the natural angular frequency ω_0. Show that the amplitude can be then found as

$$A(\omega_{\mathrm{dr}}) \approx \frac{F_0}{2m\omega_0\sqrt{\left(\omega_0 - \omega_{\mathrm{dr}}\right)^2 + \frac{b^2}{4m^2}}}.$$

Plot both the exact and approximate curve $A(\omega_{\mathrm{dr}})$ on the same graph for a few sets of parameters (list the values). Use a computer, attach the graphs to your solution.

(c) Let us call the maximum height of the $A^2(\omega_{\mathrm{dr}})$ curve one unit. Show that the width $\Delta\omega_{\mathrm{dr}}$ of the curve at one half the maximum height, i.e. *the full width at half the maximum* (often abbreviated as FWHM), is $\Delta\omega_{\mathrm{dr}} = b/m$, supposing that b/m is small.

This shows that the resonance is sharper and sharper as the drag effects are made smaller and smaller.

(2 + 2 + 2 marks)

Problem 2. Consider a harmonic oscillator with linear drag, driven by an external force $F_0 \cos \omega_{\mathrm{dr}} t$.

(a) Suppose $b/m = 4\ \mathrm{s}^{-1}$. At what frequency should the force drive the system, to make the phase of steady-state oscillations lag by $\pi/4$ behind the driving force?

(b) For two different driving angular frequencies Ω_1 and Ω_2, the amplitudes of steady-state oscillations are equal. Find the natural frequency of this oscillator.

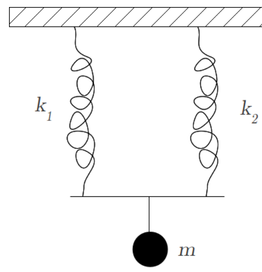
(2 + 2 marks)

Problem 3. Solve Problem 4 from Problem Set 3 in the non-inertial frame of reference of the sliding container.

Make sure that you have clearly distinguished real forces (due to interactions) and inertial forces (kinematic corrections due to the fact that the frame of reference is non-inertial).

(3 marks)

Problem 4. Consider a vertical mass-spring system that is subject to constant gravitational force. The springs are massless, the force needed to change their length depend linearly on the deformation length. The system is placed in an elevator that is (a) at rest, (b) moves upwards with acceleration of magnitude $a < g$, where g is the acceleration due to gravity.



(a) Does the period of oscillations change? What is its value?

(b) Does the equilibrium position change? If so, by what distance?

Clearly indicate the frame of reference you are solving the problem in.

(3/2 + 3/2 marks)

Problem 5. Suppose that we have two observers: one at a pole and one on the equator. Assume that rotational motion of the earth was suddenly stopped over the time of 5 minutes, with constant angular acceleration (what is the direction of the angular acceleration vector?). Describe an effect that one observer would notice whereas the other one would not.

(3 marks)

Problem 6. A small object with mass m is placed on the inner surface of a container with vertical cross section in the shape of the parabola $y = \frac{1}{2}\alpha x^2$, where $\alpha > 0$. The static coefficient of friction between the object and the pot's surface is μ_s. Find all points such that if we place the object there, it remains at rest.

Consider the following two cases

(a) The container is at rest.

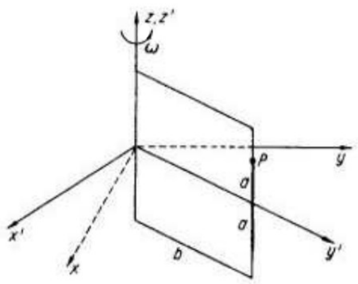
(b) The container rotates with constant angular velocity ω about its axis of symmetry (the object gets in contact with the surface with no relative velocity).

In part (b) clearly identify the frame of reference you are solving the problem in.

(2 + 4 marks)

Problem 7. A particle P moves in simple harmonic motion along a side of a rectangle, rotating with constant angular velocity $\boldsymbol{\omega}$ about the other side (see figure).

(a) Find the position vector $\mathbf{r}'$ of P in the non-inertial frame of reference *x'y'z'* associated with the rotating rectangle. Give its components as functions of time. Find acceleration $\mathbf{a}'$ in this frame of reference.



(b) Consider an inertial frame of reference xyz (see figure). Use the relation between accelerations derived in class and $\mathbf{a}'$ found in part (a) to find acceleration $\mathbf{a}$ of point P in the inertial frame of reference.

(c) Find the position vector $\mathbf{r}$ of P in the inertial frame of reference xyz. Give its components as functions of time. Differentiate them twice with respect to time to check that the result of (b) is reproduced.

(1 + 2 + 1 marks)

Problem 8. This problem will lead you through the solution of the equation of motion for a simple pendulum with length l oscillating on Earth at latitude φ (Foucault's pendulum) in the Earth's frame of reference. As we have seen in class, the Foucault pendulum's oscillation plane rotates. We will find the period of this rotation.

(a) Let us start with assuming that the amplitude of oscillations is small, i.e. the pendulum bob moves in a plane, e.g. $z' = 0$. Write down the equations of motion along the axes x' and y'. Remember to include forces of inertia, but ignore terms of the order of ω^2, where ω is the angular speed of the Earth in its rotational motion.

(b) Rewrite the system of coupled differential equations from part (a) as a single differential equation, by introducing a new (complex) variable $\tilde{\xi} = x' + iy'$. Look for solutions to this equation in the form $\tilde{\xi} = e^{\lambda t}$.

Note that $g/l \gg \omega \sin\varphi$ and show that the general solution is given by

$$\tilde{\xi} = e^{-i\omega t \sin\varphi}\left(Ae^{i\omega_0 t} + Be^{-i\omega_0 t}\right),$$

where $\omega_0 = \sqrt{g/l}$ is the natural angular frequency of the pendulum.

(c) Show (take the real and the imaginary parts) that this complex solution gives

$$\mathbf{r}' = \mathbf{k}R\cos(\omega_0 t),$$

where $\mathbf{k} = [\sin(\omega t \sin\varphi), \cos(\omega t \sin\varphi)]$ is a unit vector rotating about the axis z' with angular velocity $\omega \sin\varphi$, and R stands for the radius of the circle, the pendulum's bob moves within in the $x'y'$ plane.

(d) What is the time needed for the vertical oscillation plane to make a full 360° turn? Calculate the value for Shanghai.

(3/2 + 3 + 3 + 1/2 marks)