



Project: *Numerical solution of Newton's equations of motion*

Due: 8 July 2016, 10 am

1 Introduction

The fundamental problem in Newton's dynamics is to solve the equation of motion of a particle, implied by the second law of dynamics with a given net force \mathbf{F} . Mathematically, we need to find the (unique) solution of the second-order ordinary differential equation (ODE)

$$\frac{d^2\mathbf{r}}{dt^2} = \frac{\mathbf{F}(\mathbf{v}, \mathbf{r}, t)}{m} \quad (1)$$

with the initial conditions¹ for the velocity $\mathbf{v}(0) = \mathbf{v}_0$ and the position $\mathbf{r}(0) = \mathbf{r}_0$. The problem is solved after we have found the position of our particle (and hence its velocity) at any instant of time t .

As we have mentioned in class, the Newton's equation of motion not always can be solved analytically. That is, in some situations, the position of the particle cannot be expressed in terms of elementary functions in a closed form. In that case, numerical methods prove to be particularly useful. The purpose of this project is to learn two simple numerical methods and apply them to solve Newton's equations of motion.

2 Basic numerical methods for solving equations of motion

Although the Newton's equation of motion (1) is a second-order ODE, it can be rewritten as a pair of coupled first-order ODEs by recalling the definitions of the velocity and the acceleration

$$\frac{d\mathbf{v}}{dt} = \frac{\mathbf{F}(\mathbf{v}, \mathbf{r}, t)}{m}, \quad \frac{d\mathbf{r}}{dt} = \mathbf{v}(t). \quad (2)$$

Therefore the problem of solving a second-order ODE can be effectively reduced to the problem of solving two first-order ODEs of the form

$$\frac{df}{dt} = G(f, t) \quad (3)$$

with a given initial condition $f(0) = f_0$.

Suppose that we want to solve the initial value problem (3) numerically, that is using a computer. We need to design a discrete algorithm that will provide us with a solution of the initial value problem. We need to keep in mind that this solution will be approximate, but by carefully choosing the algorithm and its parameters, or by introducing some refinements, in most of the cases, we will be able to get a quite good approximation.

¹In this project we will assume that the initial instant of time is always chosen as $t_0 = 0$.

2.1 Euler method

Let us start with the Taylor expansion of f at the instant of time $t + \Delta t$ and neglect all terms of the order higher than one in Δt . Then

$$f(t + \Delta t) \approx f(t) + \frac{df}{dt} \Delta t \quad (4)$$

$$= f(t) + G(f, t) \Delta t, \quad (5)$$

where we have used Eq. (3) in the last step. Hence, if the value of the function f is known at time t , then its *approximate* value at a later instant $t + \Delta t$ can be found from Eq. (5). Since computers are discrete machines, we need to work on a discrete set of points representing instants of time $t_i = i\Delta t$, with discrete time steps Δt . Then Eq. (5) can be rewritten as

$$f(t_{i+1}) = f(t_i) + G(f(t_i), t_i) \Delta t$$

or, simplifying the notation, as

$$f_{i+1} = f_i + G(f_i, t_i) \Delta t \quad (6)$$

The Euler method simply uses the slope of the tangent to the curve $f = f(t)$ at one end t_i of an interval (t_i, t_{i+1}) to find the (approximate) value of f at the other one (see Fig. 1).

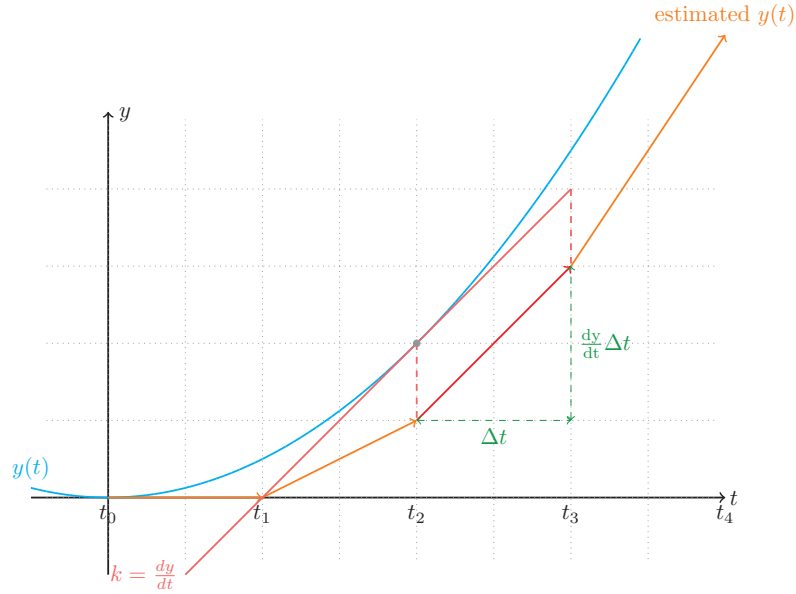


Figure 1: Illustration of the Euler method.

Given the initial condition $f(0) = f_0$, Eq. (6) can be iterated to find the (numerical) value of the function f in the required time interval, that is to numerically solve the first-order ODE (3) with that initial condition.

In particular, applying the Euler algorithm to the pair of ODEs (2), in the case of 1D motion, we get

$$v_{x,i+1} = v_{x,i} + \frac{F_x(v_i, x_i, t_i)}{m} \Delta t, \quad (7)$$

$$x_{i+1} = x_i + v_{x,i} \Delta t. \quad (8)$$

$$t_{i+1} = i \Delta t \quad (9)$$

Now, starting with $i = 0$ and the initial values of the position and the velocity, and running the algorithm in a loop, we are able to find both the position and the velocity of the particle at any later instant of time.

2.2 Second-order Runge-Kutta method

Recall that the Euler algorithm uses the slope of the function f at one end of an interval to find the value of f at the other one. We may refine this method by using the slope evaluated at the center of the interval $(t_i, t_i + \Delta t_i)$ rather than at its left end. This modification is known as the Runge-Kutta method or, precisely, the second-order Runge-Kutta method.

Let us again consider Eq. (3). By implementing the described idea, we can find the set of Runge-Kutta equations used to numerically solve this differential equation in the following form

$$k_1 = G(f_i, t_i)\Delta t, \quad (10)$$

$$k_2 = G(f_i + k_1/2, t_i + \Delta t/2)\Delta t, \quad (11)$$

$$f_{i+1} = f_i + k_2. \quad (12)$$

In the first equation the Runge-Kutta method uses the Euler method to find the difference between $f(t + \Delta t)$ and $f(t)$. The second equation uses the Euler estimate of the tangent's slope at the midpoint of the interval. Finally, the third equation generates the value of f at the next instant of time (see Fig. 2).

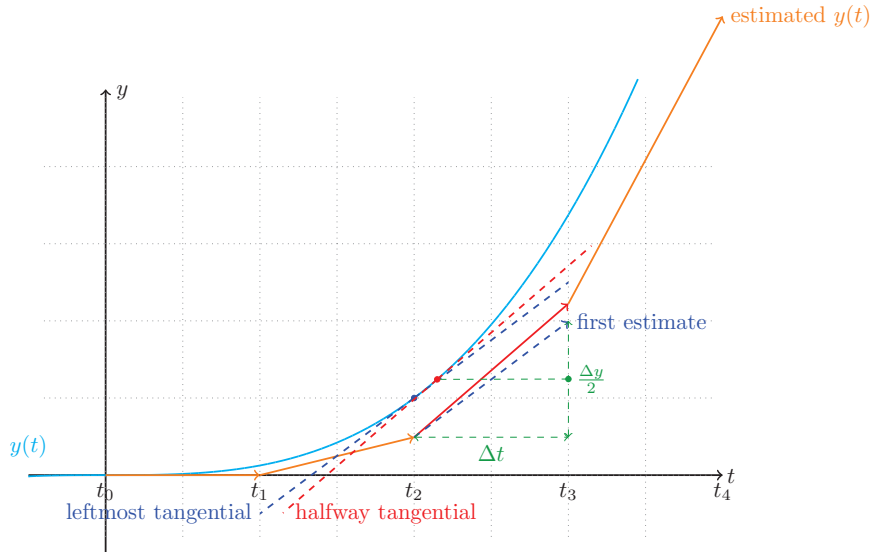


Figure 2: Illustration of the second-order Runge-Kutta method.

3 Problems and tasks

3.1 Projectile motion with air drag

First consider the problem of a 2D projectile of mass $m = 1$ kg moving with a linear drag, moving close to the Earth's surface. The equation of motion in this problem is

$$\ddot{\mathbf{r}} = -\mathbf{g} - \kappa \mathbf{v},$$

where $\kappa = k/m$ and k is the drag coefficient. We assume that the projectile starts out at the origin with velocity $\mathbf{v}_0 = 90$ m/s at an angle α to the horizontal.

1. Use the Euler method to write down two pairs of recursive equations for velocity and position of the projectile (both for x and y components).
2. For fixed initial conditions examine whether the numerical result is sensitive to the choice of the step Δt . To do so, choose three different values of Δt , plot the trajectory and compare with that obtained from the analytical formulas.

Hint. You may visualize the difference by calculating $\mathbf{r}_{\text{analytical}} - \mathbf{r}_{\text{numerical}}$.

3. Based on the results from the previous point choose an appropriate value of the step. With the initial speed of the projectile fixed, solve the problem numerically for five different values of the angle α .
 - (a) On one graph plot the trajectories and comment on their shape (give qualitative comments on the range and the maximum height).
 - (b) On one graph plot the time dependence of the speed of the particle. Comment on the result.
4. With the angle α fixed, solve the problem numerically for five different values of the drag coefficient k .
 - (a) On one graph, plot the trajectories and comment on their shape (give qualitative comments on the range and the maximum height).
 - (b) On one graph, plot the time dependence of the speed of the particle. Comment on the result.

In the next part, suppose that the same particle is subject to a quadratic drag. That is, its equation of motion reads

$$\ddot{\mathbf{r}} = -\mathbf{g} - \beta |\mathbf{v}| \mathbf{v},$$

where $\beta = b/m$ and b is the drag coefficient. We assume again that the projectile starts out at the origin with velocity $\mathbf{v}_0 = 90$ m/s at an angle α to the horizontal.

5. Use the Euler method to write down the recursive equations for velocity and position of the projectile.
6. For fixed initial conditions examine whether the numerical result is sensitive to the choice of the step Δt . To do so, choose three different values of Δt and plot the trajectory. Comment on the result.

7. Based on the results from the previous point choose an appropriate value of the step. With the initial speed of the projectile fixed, solve the problem numerically for five different values of the angle α .
 - (a) On one graph, plot the trajectories and comment on their shape (give qualitative comments on the range and the maximum height).
 - (b) On one graph, plot the time dependence of the speed of the particle. Comment on the result.
8. With the angle α fixed, solve the problem numerically for five different values of the drag coefficient k .
 - (a) On one graph, plot the trajectories and comment on their shape (give qualitative comments on the range and the maximum height).
 - (b) On one graph, plot the time dependence of the speed of the particle. Comment on the result.
9. Choose one set of the initial conditions and on one graph plot the trajectories of a projectile moving (a) without air drag, (b) with a linear drag, and (c) with a quadratic drag. For the drag coefficient choose the values that give the same terminal speed.

3.2 Simple harmonic oscillator

In this part you are asked to numerically solve the equation of motion of the 1D simple harmonic oscillator $\ddot{x} = -\omega_0^2 x$, with the initial conditions $x(0) = 0.5$ m and $v_x(0) = 1$ m/s. Assume the natural angular frequency $\omega_0 = 1.5$ s⁻¹.

1. Use the Euler method to write down the recursive equations for velocity and position of the oscillating particle.
2. Examine whether the numerical result is sensitive to the choice of the step Δt . To do so, choose three different values of Δt , plot the trajectory in both configuration and phase space and compare with those obtained from the analytical formulas.
3. At every step calculate $E(t) = \frac{1}{2}m\omega_0^2 x^2 + \frac{1}{2}mv_x^2$ and plot the graph $E(t)$. Comment on the result. Does the effect you observe have anything to do with physics?
4. Write down the recursive second-order Runge–Kutta equations (10)–(12) for v_i and x_i for the simple harmonic oscillator.
5. Choose the worst-performing step size Δt from point 2 and use the second-order Runge–Kutta method with this step size to solve the problem again.

Plot the trajectory of the particle both in configuration and phase space. Plot the Euler, the second-order Runge–Kutta, and the analytical results together on one graph. Comment on the plot.

At every step calculate $E(t) = \frac{1}{2}m\omega_0^2 x^2 + \frac{1}{2}mv_x^2$ for both the Euler and second-order Runge–Kutta method, and plot the graph $E(t)$. Comment on the result.

Note. In points 2, 3, and 5 make sure that the time interval covers several periods.

4 Remarks on the accuracy of numerical methods

Let us emphasize that the solutions we find by applying numerical methods are approximate. This is clearly visible in the case of the harmonic oscillator. There are a few factors limiting the accuracy of numerical methods. First of all, the algorithms are discrete, therefore approximations are inevitable. However, here we can control the accuracy to some extent, by constructing more sophisticated algorithms and properly choosing the time step. The accuracy of numerical methods is also limited by the precision of computer's floating-point arithmetic, although in our case it does not significantly affect the numerical solution.

Exercise (*optional, but you may include the solution in your report*)

Check that the Runge–Kutta method discussed in this project is a second-order method, that is it is accurate to second order in Δt . What is the order of the Euler method?

5 Deliverables

After completing all tasks you need to write a self-contained report presenting your results. It should be starting with a short introduction, similar to the opening sections of this document. The report should be typed (L^AT_EX is recommended, but not compulsory) and submitted in a printed form, including graphs, with all pages stapled together.

You will need to implement the two numerical methods studied in this project. There are no restrictions imposed on the programming environment (C++, Fortran, Matlab, Mathematica, Octave, Maxima,... are all allowed). Please keep your code clear, simple, and concise. The source code files should be packed into a single ZIP file and uploaded into the Sakai system by the date given.

In this project, you will be working in groups of three. You need to make sure that workload is distributed uniformly among all group members. On the title page please include the following statement: *"We state that each of us has contributed equally to this project."* and sign it. The code should be also labelled with the names of group members. Each group should submit a single report and a single ZIP file. The Honor Code applies to the project in the same way it does to homework.