Problem 1.

Suppose the direction to the sky to be the positive direction. When the ball goes upwards,

$$a = \frac{dv}{dt} = -g - \frac{k}{m}v$$

$$\frac{dv}{g + \frac{k}{m}v} = -dt$$

Do integral on both side,

$$\int_{v_0}^{v} \frac{1}{g + \frac{k}{m}v} dv = -\int_0^t dt$$

$$\frac{m}{k} ln \frac{mg + kv}{mg + kv_0} = -t$$

When $v = 0$,

$$t_{max} = \frac{m}{k} ln \frac{mg + kv_0}{mg}$$

$$\frac{mg + kv}{mg + kv_0} = e^{-\frac{kt}{m}}$$

$$v = \frac{dh}{dt} = \frac{(mg + kv_0)e^{-\frac{kt}{m}}}{k} - \frac{mg}{k}$$

Do integral on both side,

$$\int_0^h dh = \int_0^t \left(\frac{(mg + kv_0)e^{-\frac{kt}{m}}}{k} - \frac{mg}{k} \right) dt$$

$$h = \left[\frac{-m(mg + kv_0)e^{-\frac{kt}{m}}}{k^2} - \frac{mg}{k}t \right]_0^t = \frac{m(mg + kv_0)}{k^2}(1 - e^{-\frac{kt}{m}}) - \frac{mgt}{k}$$

When $t = t_{max}$,

$$h_{max} = \frac{mv_0}{k} - \frac{m^2 g}{k^2} ln \frac{mg + kv_0}{mg}$$

Similarly, when the ball go downwards,

$$\int_0^v \frac{1}{g + \frac{k}{m}v} dv = -\int_{t_{max}}^t dt$$

$$\frac{m}{k} ln \frac{mg + kv}{mg} = -t + t_{max}$$

$$\frac{mg + kv}{mg} = e^{-\frac{k(t - t_{max})}{m}}$$

$$v = \frac{dh}{dt} = \frac{mge^{-\frac{k(t - t_{max})}{m}}}{k} - \frac{mg}{k}$$

Do integral on both side,

$$\int_{h_{max}}^h dh = \int_{t_{max}}^t \left(\frac{mge^{-\frac{k(t - t_{max})}{m}}}{k} - \frac{mg}{k} \right) dt$$

$$h = h_{max} + \left[\frac{-m^2 ge^{-\frac{k(t - t_{max})}{m}}}{k^2} - \frac{mg}{k}t \right]_{t_{max}}^t = \frac{m^2 g}{k^2}(1 - e^{-\frac{kt}{m}} \frac{mg + kv_0}{mg}) + \frac{m(v_0 - gt)}{k}$$

Problem 2.

Suppose the direction to the sky to be the positive direction.

$$g(h) = \frac{dv}{dt} = \frac{dv}{dh} \cdot \frac{dh}{dt} = -vdv = \frac{g_0 R^2}{(R+h)^2}$$

Do integral on both side,

$$-\int_0^v vdv = \int_H^h \frac{g_0 R^2}{(R+h)^2} dh$$

$$-\frac{1}{2}v^2 = \left[-\frac{g_0 R^2}{R+h}\right]_H^h = g_0 R^2 \left(-\frac{1}{R+h} + \frac{1}{R+H}\right)$$

$$v = \sqrt{\frac{2g_0 R^2 (H-h)}{(R+H)(R+h)}}$$

When $h = 0$,

$$v_m = \sqrt{\frac{2g_0 R^2 (H-h)}{(R+H)(R)}}$$

$$v = -\frac{dh}{dt} = \sqrt{\frac{2g_0 R^2 (H-h)}{(R+H)(R+h)}}$$

$$\sqrt{\frac{(R+H)(R+h)}{2g_0 R^2 (H-h)}} dh = -dt$$

Do integral on both side,

$$\sqrt{\frac{R+H}{2g_0 R^2}} \int_H^0 \sqrt{\frac{R+h}{H-h}} dh = -\int_0^T dt$$

$$\begin{aligned} T &= -\sqrt{\frac{R+H}{2g_0 R^2}} \left[(h-H)\sqrt{\frac{R+h}{H-h}} + (R+H)\arcsin\sqrt{\frac{R+h}{R+H}}\right]_H^0 \\ &= -\sqrt{\frac{R+H}{2g_0 R^2}} \left[-H\sqrt{\frac{R}{H}} + (R+H)\arcsin\sqrt{\frac{R}{R+H}} - (R+H)\frac{\pi}{2}\right] \\ &= \sqrt{\frac{R+H}{2g_0 R^2}} \left[\sqrt{RH} + (R+H)\left(\frac{\pi}{2} - \arcsin\sqrt{\frac{R}{R+H}}\right)\right] \end{aligned}$$

Problem 3.

(a)

$$x(t) = \sqrt{B^2 + C^2} \sin\left(\omega_0 t + \arctan\frac{B}{C}\right)$$

$$A = \sqrt{B^2 + C^2}$$

$$\varphi = \arctan\frac{B}{C}$$

(b)

$$x(0) = x_0 = A\cos\varphi$$
$$v(0) = x'(0) = v_0 = -\omega A\sin\varphi$$
$$A^2\cos^2\varphi + A^2\sin^2\varphi = x_0^2 + \frac{v_0^2}{\omega_0^2}$$
$$A = \sqrt{x_0^2 + \frac{v_0^2}{\omega_0^2}}$$
$$\frac{v_0}{x_0} = -\omega_0\tan\varphi$$
$$\varphi = \arctan\left(-\frac{v_0}{\omega_0 x_0}\right)$$

Problem 4.

Suppose the direction to the ground to be the positive direction and the ground to be the frame of reference.

$$h(t) = A\cos(\omega_0 t + \varphi)$$
$$v(t) = h'(t) = -\omega_0 A\sin(\omega_0 t + \varphi)$$
$$a(t) = v'(t) = -\omega_0^2 A\cos(\omega_0 t + \varphi)$$

The block on the platform is always in contact with the platform when $a(t) \leqslant g$ always holds.

$$-\omega_0^2 A\cos(\omega_0 t + \varphi) \leqslant g$$
$$(\omega_0)_{max} = \sqrt{\frac{g}{A}}$$

Problem 5.

$$F = -(k\delta x + \rho g S\delta x) = -(k + \rho g S)\delta x$$
$$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{k+\rho g S}{m}}$$

Suppose that at the equilibrium position, $y = a$

$$mg = k(a - \frac{1}{2}h - l_0) + \rho g S(a + \frac{1}{2}h - H)$$
$$a = \frac{mg + k(\frac{1}{2}h + l_0) + \rho g S(H - \frac{1}{2}h)}{k + \rho g S}$$
$$A = |a - y_0| = \left|\frac{mg + k(\frac{1}{2}h + l_0) + \rho g S(H - \frac{1}{2}h)}{k + \rho g S} - y_0\right|$$
$$y = a + A\sin(\omega_0 + \varphi)$$
$$y = \frac{mg + k(\frac{1}{2}h + l_0) + \rho g S(H - \frac{1}{2}h)}{k + \rho g S} + \left(\frac{mg + k(\frac{1}{2}h + l_0) + \rho g S(H - \frac{1}{2}h)}{k + \rho g S} - y_0\right)\cos(\sqrt{\frac{k+\rho g S}{m}}t)$$

Problem 6.

The net momentum of the system always equal to zero, so the centroid of the system remains the same position.
Suppose the distance between the centroid and m_1, m_2 be l_1, l_2

$$\begin{cases} m_1 l_1 = m_2 l_2 \\ l_1 + l_2 = l_0 \end{cases} \Longrightarrow \begin{cases} l_1 = \frac{m_2}{m_1+m_2} l_0 \\ l_2 = \frac{m_1}{m_1+m_2} l_0 \end{cases}$$

Then the spring can be divided into two separate systems at the centroid point.
On both sides of the centroid, there is a harmonic oscillator, and we know that,

$$kl_0 = k_1 l_1 = k_2 l_2$$

$$k_1 = \frac{m_1 + m_2}{m_2} k, \ k_2 = \frac{m_1 + m_2}{m_1} k$$

$$T_1 = 2\pi \sqrt{\frac{m_1}{k_1}} = 2\pi \sqrt{\frac{m_1 m_2}{k(m_1 + m_2)}}$$

$$T_2 = 2\pi \sqrt{\frac{m_2}{k_2}} = 2\pi \sqrt{\frac{m_1 m_2}{k(m_1 + m_2)}}$$

As $T_1 = T_2$, the whole system is under one harmonic oscillator, and the natural angular frequency,

$$\omega_0 = \frac{2\pi}{T} = \sqrt{\frac{k(m_1 + m_2)}{m_1 m_2}}$$

Problem 7.

$$\begin{cases} \delta h_1 = \delta l \sin\alpha \\ \delta h_2 = \delta l \sin\beta \end{cases} \Longrightarrow \delta h = \delta l(\sin\alpha + \sin\beta)$$

$$P = \rho g \delta h = \rho g(\sin\alpha + \sin\beta)\delta l$$

Suppose the radius of the tube to be r,

$$F_{tube} = -PS_1 \sin\alpha = -PS_2 \sin\beta = -P\pi r^2 = -\rho g(\sin\alpha + \sin\beta)\pi r^2 \delta l$$

$$k = \rho g(\sin\alpha + \sin\beta)\pi r^2$$

$$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{\rho g(\sin\alpha + \sin\beta)\pi r^2}{\rho\pi r^2 l}} = \sqrt{\frac{g(\sin\alpha + \sin\beta)}{l}}$$

Problem 8.

$$F = q(E + v \times B)$$

$$a = \frac{F}{m} = -\frac{qE_0}{m}\hat{n_x} + \frac{qB_0}{m} v_z \hat{n_y} - \frac{qB_0}{m} v_y \hat{n_z}$$

$$\frac{a_y}{a_z} = \frac{\frac{dv_y}{dt}}{\frac{dv_z}{dt}} = \frac{dv_y}{dv_z} = -\frac{v_z}{v_y}$$

$$v_y dv_y = -v_z dv_z$$

Do integral on both side,

$$\int v_y dv_y = -\int v_z dv_z$$

$$\frac{1}{2}v_y^2 = -\frac{1}{2}v_z^2 + C$$

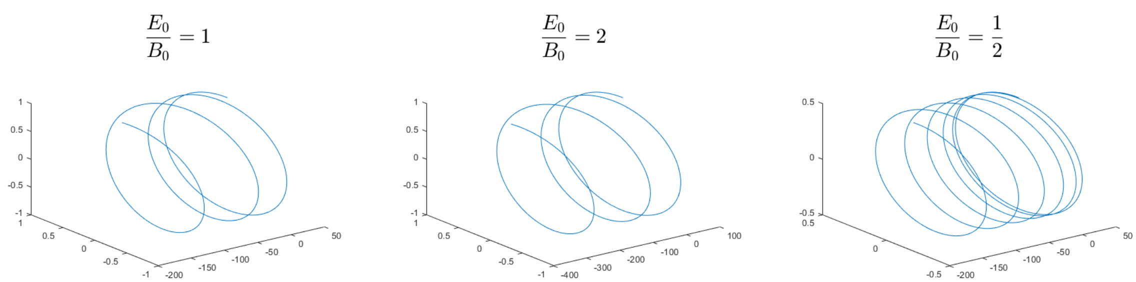
$$v_y^2 + v_z^2 = C = v_{0y}^2$$

$$v_y = v_{0y}\cos\frac{qB_0}{m}t,\ \ a_y = \frac{qB_0}{m}v_{0y}\sin\frac{qB_0}{m}t$$

$$v_z = -v_{0y}\sin\frac{qB_0}{m}t,\ \ a_z = -\frac{qB_0}{m}v_{0y}\cos\frac{qB_0}{m}t$$

$$v(t) = \left(v_{0x} - \frac{qE_0}{m}t\right)\hat{n_x} + \left(v_{0y}\cos\frac{qB_0}{m}t\right)\hat{n_y} - \left(v_{0y}\sin\frac{qB_0}{m}t\right)\hat{n_z}$$

$$r(t) = \left(v_{0x}t - \frac{qE_0}{2m}t^2\right)\hat{n_x} + \left(\frac{m}{qB_0}v_{0y}\sin\frac{qB_0}{m}t\right)\hat{n_y} + \left(\frac{m}{qB_0}v_{0y}(\cos\frac{qB_0}{m}t - 1)\right)\hat{n_z}$$



(a) (b) (c)

Figure 1: $r(t) = (t - \frac{1}{2}t^2)\hat{n_x} + \frac{\sin B_0 t}{B_0}\hat{n_y} + \frac{\cos B_0 t}{B_0}\hat{n_z}$

Problem 9.

$$x(t) = D_1 e^{-\frac{b}{2m}t} + D_2 t e^{-\frac{b}{2m}t}$$

$$x'(t) = \left(D_2 - \frac{b(D_1 + D_2 t)}{2m}\right)e^{-\frac{b}{2m}t}$$

As $e^{-\frac{b}{2m}t} > 0$, $x'(t) = 0$ have only one solution $t = \frac{2mD_2 - bD_1}{bD_2}$
So $x(t)$ have only one critical point, which means $x(t) = 0$ have at most one solution.
So the oscillating mass can pass through the equilibrium position at most once, regardless of initial conditions.