Problem Set 8

Due: 12 July 2016, 10 a.m.

Problem 1. What is the number of degrees of freedom for the following systems: (a) two particles connected by a rigid massless rod, (b) two particles connected by a massless spring, (c) three particles connected with each other by massless rigid rods, (d) three particles connected with each other by massless springs? Explain your answers.

(*1/2 + 1/2 + 1 + 1 marks*)

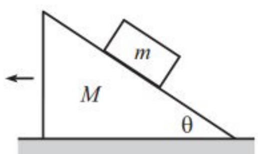
Problem 2. In class we mentioned that in the case of a free particle, the Hamilton's least action principle implies that a particle moves between two points along a trajectory that minimizes the travel time. Using this corollary for light rays, which is known in optics as the Fermat's principle, derive the two fundamental laws of geometric optics: (a) the law of reflection and (b) the law of refraction (also known as the Snell's law).

(*3/2 + 5/2 marks*)

Problem 3. (Spherical pendulum) A particle of mass m, that can move on a sphere of radius R, is subject to gravitational force. (a) Determine the number of degrees of freedom and find the generalized coordinates. (b) Find the Lagrangian and (c) write down the Euler–Lagrange equations of motion (Lagrange equations of the second kind).

(*1 + 5/2 + 3/2 marks*)

Problem 4. A block of mass m is held motionless on a frictionless plane of mass M and angle of inclination theta (see the attached figure). The plane rests on a frictionless horizontal surface. The block is released. What is the horizontal acceleration of the plane? Solve the problem using the Lagrange's formalism. (Note that we discussed this problem in class a few weeks ago using Newton's formalism.)



Hint. This week's recitation class.

(*5 marks*)

Problem 5. Consider a system of three particles with masses $m_1 = 5$ g, $m_2 = 10$ g, $m_3 = 15$ g. At the initial instant of time $t = 0$ the particles were placed at points (3,4,5), (2,4,6), and (0,0,0), respectively (the coordinates are given in centimeters). The particles are acted upon by external forces whose sum is a vector of magnitude $F = 0,05$ N directed along the x axis. Find the position of the center of mass of the system at $t = 2$ s.

(*2 marks*)

Problem 6. Consider a rod of length l, mass m, and cross-sectional area A. Let us set the origin of the coordinate system at one end of the rod with the positive x-axis along the symmetry axis of the rod. (a) If the bulk density $\rho = m/Al$ of the rod is constant, show that the x-coordinate of the center of mass of the rod is at its half of its length. (b) If the density of the rod varies linearly with x — that is, $\rho = \alpha x$, where α is a positive constant — calculate the x-coordinate of the rod's center of mass.

(*1 + 2 marks*)

Problem 7. Find the position of the center of mass for: (a) a uniform half-circle with radius R and constant linear mass density λ, (b) a half-cylinder with constant mass density ρ, height a, and radius R.

(2 + 2 marks)

Problem 8. A fisherman (with mass m) walks from one end of his boat of mass M and length l to the other. Find the distance the boat has moved with respect to the bank.

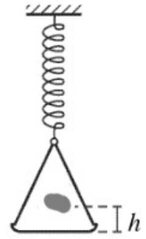
(2 marks)

Problem 9. Three boats, each with mass M, move in a straight line one after another with the same speed v. At some instant, two bags of sand, each with mass m, are thrown from the middle boat with speed u (relative to the boat): one into the boat ahead and another into that in the back.

What is the speed of the three boats?

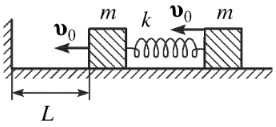
(2 marks)

Problem 10. A frame with mass M when suspended from a coil spring obeying the Hooke's law, stretches the spring by l. A lump of putty with mass m is dropped from rest onto the frame from a height h. Find the amplitude of the resulting oscillations.



(4 marks)

Problem 11. Two identical blocks with mass m, connected by a light spring (neither compressed nor stretched), slide with speed v_0 on a frictionless surface towards a wall (see figure below). At $t = 0$ the left block is at distance L from the wall. Find the time after which the center of mass of this system will be again at the position it was when $t = 0$. Assume that the block collides with the wall elastically, and the duration of the collision is negligibly short. The force constant of the spring is k.



(4 marks)

Problem 12. A toy-rocket with initial mass m_0 is placed at rest on a frictionless horizontal surface some distance from a wall. At $t = 0$, the engine is turned on and the rocket starts moving in the direction perpendicular to the wall colliding with it after time T. The collision is elastic and the rocket does not change its orientation with respect to the wall. How much time after the collision does it take until the (instantaneous) velocity of the rocket is zero?

Assume that the burned fuel is ejected from the rocket at a constant speed (with respect to the rocket) and the mass of the rocket decreases with time at a constant rate α.

What is the numerical answer if $m = 1$ kg, $\alpha = 0.01$ kg/s, and $T = 10$ s?

(5 marks)