Problem 1.

$$\begin{cases} I = m_1 r_1^2 + m_2 r_2^2 \\ m_1 r_1 = m_2 r_2 \end{cases} \implies \begin{cases} r_1 &= \sqrt{\frac{m_2 I}{m_1^2 + m_1 m_2}} \\ r_2 &= \sqrt{\frac{m_1 I}{m_2^2 + m_1 m_2}} \end{cases}$$

$$H = r_1 + r_2 = \sqrt{\frac{I}{m_1 + m_2}} \left(\sqrt{\frac{m_2}{m_1}} + \sqrt{\frac{m_1}{m_2}} \right) = 9.29 \times 10^{-11} m$$

Problem 2.

(a)

$$I = \int_M r^2 dm = \int_{R_1}^{R_2} 2\alpha\pi H r^4 dr = \frac{2}{5}\alpha\pi H (R_2^2 - R_1^2)$$

(b)

$$I = \int_M r^2 dm = \int_0^h \frac{\sigma a r^2 (h - r)}{h} dr = \frac{1}{12}\sigma a h^3$$

(c)

$$I = \int_M r^2 dm = \int_0^h \frac{\sigma a r^3}{h} dr = \frac{1}{4}\sigma a h^3$$

Problem 3.

$$\begin{aligned} I_x &= \int_M (x^2 + z^2) dm \approx \int_M x^2 dm \\ I_y &= \int_M (y^2 + z^2) dm \approx \int_M y^2 dm \\ I_z &= \int_M (x^2 + y^2) dm = I_x + I_y \end{aligned}$$

Problem 4.

Suppose the equation of the parabola is

$$z = ar^2 + C$$

$$V_0 = \int_V dV = \int_0^R 2\pi r (ar^2 + C) dr = \frac{1}{2} a\pi R^4 + C\pi R^2$$

Considering a point on the parabola,

$$mg \tan\theta = m\omega^2$$

$$\tan\theta = \frac{\omega^2}{g} = z' = 2a$$

$$a = \frac{\omega^2}{2g}$$

$$C = \frac{V_0 - \frac{1}{2}a\pi R^4}{\pi R^2} = \frac{V_0}{\pi R^2} - \frac{\omega^2 R^2}{4g}$$
$$z = \frac{\omega^2}{2g}r^2 + \frac{V_0}{\pi R^2} - \frac{\omega^2 R^2}{4g}$$

$$\begin{aligned}
I_w &= \int_M r^2 dm = \int_0^R 2\rho\pi r^3(ar^2 + C)dr = 2\rho\pi\left(\frac{1}{6}aR^6 + \frac{1}{4}CR^4\right) \\
&= 2\rho\pi\left[\frac{1}{6}\frac{\omega^2}{2g}R^6 + \frac{1}{4}\left(\frac{V_0}{\pi R^2} - \frac{\omega^2 R^2}{4g}\right)R^4\right] \\
&= \frac{1}{24}\rho\pi\omega^2 R^6 + \frac{1}{2}\rho V_0 R^2
\end{aligned}$$

$$E_k = \frac{1}{2}(I_0 + I_w)\omega^2 = \frac{1}{2}I_0\omega^2 + \frac{1}{48}\rho\pi\omega^4 R^6 + \frac{1}{4}\rho V_0\omega^2 R^2$$

Problem 5.

$$\frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 + \frac{1}{2}kx^2 = C$$
$$\omega = \frac{v}{R}$$
$$\frac{1}{2}mv^2 + \frac{1}{2}I\frac{v^2}{R^2} + \frac{1}{2}kx^2 = C$$
$$\frac{1}{2}\left(m + \frac{I}{R^2}\right)v^2 + \frac{1}{2}kx^2 = C$$

According to the equation of harmonic oscillation,

$$T = 2\pi\sqrt{\frac{m + \frac{I}{R^2}}{k}} = 2\pi\sqrt{\frac{mR^2 + I}{kR^2}}$$

Problem 6.

(a) the left one:
The moment of inertia about the axis perpendicular to the square and passing through the center of the square is

$$I_c = \frac{1}{6}mL^2$$

The moment of inertia about the axis perpendicular to the hanging point is

$$I = I_c + md^2 = \frac{1}{6}mL^2 + \frac{9}{4}mL^2 = \frac{29}{12}mL^2$$

$$T = 2\pi\sqrt{\frac{I}{mgd}} = 2\pi\sqrt{\frac{29L}{18g}}$$

(b) the right one:
The edge of the square won't rotate, so

$$F = -mg\tan\theta = -\frac{mg}{L}x$$

$$T = 2\pi\sqrt{\frac{k}{m}} = 2\pi\sqrt{\frac{L}{g}}$$

Problem 7.

Suppose the moving direction to be the positive direction and the rolling direction to be the positive angular direction.

(a)

$$a_x = \frac{F}{M} = \frac{\mu_k Mg}{M} = \mu_k g$$

$$\varepsilon_z = -\frac{RF}{I} = -\frac{\mu_k MgR}{\frac{1}{2}MR^2} = -\frac{2\mu_k g}{R}$$

(b)

$$\mu_k gt = \left(\omega_0 - \frac{2\mu_k g}{R}t\right)R$$

$$t = \frac{\omega_0 R}{3\mu_k g}$$

$$x = \frac{1}{2}a_x t^2 = \frac{\omega_0^2 R^2}{18\mu_k g}$$

(c)

$$x' = v_0 t - \frac{1}{2}at^2 = \omega_0 Rt - \frac{1}{2}\varepsilon_z Rt^2 = \frac{2\omega_0^2 R^2}{9\mu_k g}$$

$$W = -fx' = -\frac{2}{9}M\omega_0^2 R^2$$

Problem 8.

$$F = ma = f - mg\sin\theta$$

$$fR = I\varepsilon$$

$$a = R\varepsilon = \frac{fR^2}{I} = \frac{m(g\sin\theta - a)R^2}{I}$$

$$a = \frac{mgR^2\sin\theta}{mR^2 + I}$$

$$I_b = \frac{2}{5}mR^2 \quad a_b = \frac{5}{7}g\sin\theta$$

$$I_r = mR^2 \quad a_r = \frac{1}{2}g\sin\theta$$

$$\frac{1}{2}a_b t^2 = v_0 t + \frac{1}{2}a_l t^2$$

$$v_0 = \frac{1}{2}t(a_b - a_l) = \frac{3}{28}gt\sin\theta$$

Problem 9.

$$m_1 = \rho a\pi r^2$$

$$m_2 = \rho a\pi R^2$$

$$I = \frac{1}{2}m_1 r^2 + m_2 R^2 = \frac{1}{2}\rho a\pi r^4 + \rho a\pi R^4 = \rho a\pi\left(\frac{1}{2}r^4 + R^4\right)$$

$$m = m_1 + 2m_2 = \rho a\pi(r^2 + 2R^2)$$

$$F = ma = mg - f$$

$$a = R\varepsilon = \frac{fr^2}{I} = \frac{m(a-g)r^2}{I}$$

$$a = \frac{mgr^2}{mr^2 + I} = \frac{\rho a\pi(r^2 + 2R^2)gr^2}{\rho a\pi(r^2 + 2R^2)r^2 + \rho a\pi\left(\frac{1}{2}r^4 + R^4\right)} = \frac{3r^2(r^2 + 2R^2)}{3r^4 + 4R^2r^2 + 2R^4}$$

Problem 10.

$$F = ma = mg\sin\alpha - f - \mu mg\cos\alpha$$

$$fr - \mu mgR\cos\alpha = I_0\varepsilon$$

$$a = r\varepsilon = \frac{r(fr - \mu mgR\cos\alpha)}{I_0} = \frac{(mg\sin\alpha - ma - \mu mg\cos\alpha)r^2 - \mu mgrR\cos\alpha}{I_0}$$

$$a = \frac{mg[r^2\sin\alpha - \mu(r^2 + R^2)\cos\alpha]}{mr^2 + I_0}$$