Problem 1.

(a)

$$f = 3N - m = 5$$

There is one constraint: the fixed distance between two objects.

(b)

$$f = 3N - m = 6$$

There are no constraints.

(c)

$$f = 3N - m = 6$$

There are three constraints: the fixed distance between every two objects.

(d)

$$f = 3N - m = 9$$

There are no constraints.

Problem 2.

(a) Suppose a point on the incident ray (x_1, y_1), a point on the reflection ray (x_2, y_2) and the speed of rays v.

$$t = \frac{\sqrt{(x_1 - x)^2 + y_1^2}}{v} + \frac{\sqrt{(x_2 - x)^2 + y_2^2}}{v}$$

where $(x, 0)$ is the intersection point of two rays.
When t get the minimum value,

$$\frac{dt}{dx} = \frac{x - x_1}{\sqrt{(x_1 - x)^2 + y_1^2} v} + \frac{x - x_2}{\sqrt{(x_2 - x)^2 + y_2^2} v} = 0$$

$$\frac{\sin\alpha}{v} = \frac{\sin\beta}{v}$$

$$\alpha = \beta$$

where $\alpha,\ \beta$ is the angle between the incident ray, reflection ray and the y-axis.

(b) Suppose a point on the incident ray (x_1, y_1), a point on the refraction ray (x_2, y_2) and the speed of each ray $v_1,\ v_2$.

$$t = \frac{\sqrt{(x_1 - x)^2 + y_1^2}}{v_1} + \frac{\sqrt{(x_2 - x)^2 + y_2^2}}{v_2}$$

where $(x, 0)$ is the intersection point of two rays.
When t get the minimum value,

$$\frac{dt}{dx} = \frac{x - x_1}{\sqrt{(x_1 - x)^2 + y_1^2} v_1} + \frac{x - x_2}{\sqrt{(x_2 - x)^2 + y_2^2} v_2} = 0$$

$$\frac{\sin\alpha}{v_1} = \frac{\sin\beta}{v_2}$$

where $\alpha,\ \beta$ is the angle between the incident ray, refraction ray and the y-axis.

Problem 3.

(a)

$$f = 3N - m = 2$$

The generalized coordinates are the angle θ between the pendulum and z-axis and the rotating angle φ on the xy-plane.

(b) Suppose the center of the sphere to be the zero potential point.

$$L = K - U = \frac{1}{2}mv^2 - (-mgR\cos\theta) = \frac{1}{2}mR^2(\dot{\theta}^2 + \sin^2\theta\dot{\varphi}^2) + mgR\cos\theta$$

(c)

$$\begin{cases} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} &= 0 \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\varphi}}\right) - \frac{\partial L}{\partial \varphi} &= 0 \end{cases} \Longrightarrow \begin{cases} mR^2\ddot{\theta} - mR^2\sin\theta\cos\theta\dot{\varphi}^2 + mgR\sin\theta &= 0 \\ mR^2\sin^2\theta\ddot{\varphi} + 2mR^2\sin\theta\cos\theta\dot{\theta}\dot{\varphi} &= 0 \end{cases}$$

$$\ddot{\theta} = \sin\theta\cos\theta\dot{\varphi}^2 - \frac{g\sin\theta}{R}$$

$$\ddot{\varphi} = -2\cot\theta\dot{\theta}\dot{\varphi}$$

Problem 4.

Suppose M$(x_1, 0)$, m$(x2, H - (x1 - x2)\tan\theta)$, the ground to be the zero potential plane and left to be the positive direction.

$$\begin{aligned} L = K - U &= \frac{1}{2}M\dot{x_1}^2 + \frac{1}{2}m[\dot{x_2}^2 + (\dot{x_1} - \dot{x_2})^2\tan\theta^2] - H + mg((x1 - x2)\tan\theta) \\ &= \frac{1}{2}(M + m\tan^2\theta)\dot{x_1}^2 + \frac{1}{2}(m + m\tan^2\theta)\dot{x_2}^2 - m\tan^2\theta\dot{x_1}\dot{x_2} - H + mg(x_1 - x_2)\tan\theta \end{aligned}$$

$$\begin{cases} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x_1}}\right) - \frac{\partial L}{\partial x_1} &= 0 \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x_2}}\right) - \frac{\partial L}{\partial x_2} &= 0 \end{cases} \Longrightarrow \begin{cases} (M + m\tan^2\theta)\ddot{x_1} - m\tan^2\theta\ddot{x_2} - mg\tan\theta &= 0 \\ (m + m\tan^2\theta)\ddot{x_2} - m\tan^2\theta\ddot{x_1} + mg\tan\theta &= 0 \end{cases}$$

$$\ddot{x_2} = \frac{m\tan^2\theta\ddot{x_1} - mg\tan\theta}{m + m\tan^2\theta} = \frac{\tan^2\theta\ddot{x_1} - g\tan\theta}{1 + \tan^2\theta}$$

$$(M + m\tan^2\theta)\ddot{x_1} - m\tan^2\theta\frac{\tan^2\theta\ddot{x_1} - g\tan\theta}{1 + \tan^2\theta} - mg\tan\theta = 0$$

$$\ddot{x_1} = \frac{-mg\tan^3\theta + (1 + \tan^2\theta)mg\tan\theta}{(1 + \tan^2\theta)(M + m\tan^2\theta) - m\tan^4\theta} = \frac{mg\tan\theta}{M + M\tan^2\theta + m\tan^2\theta}$$

Problem 5.

$$\bar{x} = \frac{\sum_{i=1}^{3} m_i x_i}{\sum_{i=1}^{3} m_i} = \frac{7}{6} cm$$

$$\bar{y} = \frac{\sum_{i=1}^{3} m_i y_i}{\sum_{i=1}^{3} m_i} = 2cm$$

$$\bar{z} = \frac{\sum_{i=1}^{3} m_i z_i}{\sum_{i=1}^{3} m_i} = \frac{17}{6} cm$$

$$a = \frac{F}{m} = \frac{0.05}{0.03} = \frac{5}{3} m/s^2$$

$$\Delta x = \frac{1}{2} a t^2 = \frac{10}{3} m$$

$$x + \Delta x = \frac{669}{2} cm$$

So the position of the mass center is $(\frac{669}{2}, 2, \frac{17}{6})$ cm

Problem 6.

(a)

$$m(x) = \rho A$$

$$\bar{x} = \frac{\int_0^l x m(x) dx}{\int_0^l m(x) dx} = \frac{\frac{1}{2}\rho A l^2}{\rho A l} = \frac{1}{2} l$$

(b)

$$m(x) = \rho A = \alpha A x$$

$$\bar{x} = \frac{\int_0^l x m(x) dx}{\int_0^l m(x) dx} = \frac{\frac{1}{3}\alpha A l^3}{\frac{1}{2}\alpha A l^2} = \frac{2}{3} l$$

Problem 7.

(a) Since the half-circle is symmetrical about the y-axis,

$$\bar{x} = 0$$

According to Pappus centroid theorem, $S = Lr$

$$4\pi R^2 = \pi R \cdot 2\pi \bar{y}$$

$$\bar{y} = \frac{2R}{\pi}$$

(b) Since the half-cylinder is symmetrical about the y-axis,

$$\bar{x} = 0$$

Since it is a half-cylinder,

$$\bar{z} = \frac{1}{2}a$$

According to Pappus centroid theorem, $V = Sr$

$$\frac{4}{3}\pi R^3 = \frac{1}{2}\pi R^2 \cdot 2\pi\bar{y}$$

$$\bar{y} = \frac{4R}{3\pi}$$

Problem 8.

The mass center of the system won't move.
Suppose the direction he moved to be the positive direction, the displacement of the fisherman and the boat to be x_1, x_2

$$\begin{cases} x_1 - x_2 & = l \\ mx_1 + Mx_2 & = 0 \end{cases} \Longrightarrow x_2 = -\frac{ml}{M+m}$$

So the distance the boat has moved with respect to the bank is $\frac{ml}{M+m}$

Problem 9.

Suppose the speed direction to be positive.
the change momentum of the middle boat is

$$\Delta P = mu - mu = 0$$

So the speed of the middle boat won't change.
In the system of the first boat and the thrown object,

$$Mv + m(v+u) = (M+m)v_1$$

$$v_1 = \frac{Mv + m(v+u)}{M+m} = v + \frac{mu}{M+m}$$

In the system of the last boat and the thrown object,

$$Mv + m(v-u) = (M+m)v_2$$

$$v_2 = \frac{Mv + m(v-u)}{M+m} = v - \frac{mu}{M+m}$$

Problem 10.

After the collision, the equilibrium point will change by Δl

$$Mg = kl$$

$$(M+m)g = k(l+\Delta l)$$

$$\Delta l = \frac{mg}{k}$$

The speed of the system will be v'

$$v = \sqrt{2gh}$$

$$mv = (M+m)v'$$

$$v' = \frac{m\sqrt{2gh}}{M+m}$$

According to the law of conservation of energy,

$$\frac{1}{2}kA^2 = \frac{1}{2}(M+m)v'^2 + \frac{1}{2}k\Delta l^2$$

$$A = \sqrt{\frac{2m^g h}{k(M+m)} + \frac{m^2 g^2}{k}} = \sqrt{\frac{m^2 l}{M}\left(\frac{2h}{M+m} + \frac{l}{M}\right)}$$

Problem 11.

At first, two blocks move left together until the left block collides with the wall.

$$t_1 = \frac{L}{v_0}$$

Then the left block collides with the wall so that it turns right at the same speed v_0. There happens a harmonic oscillation between two blacks until the left block collides with the wall again.

$$t_2 = \frac{1}{2}T = \pi\sqrt{\frac{m}{2k}}$$

After that the left block turns right again and the right block also moves right until they return to the origin position.

$$t_3 = \frac{L}{v_0}$$

$$t = t_1 + t_2 + t_3 = \frac{2L}{v_0} + \pi\sqrt{\frac{m}{2k}}$$

Problem 12.

Suppose left to be the positive direction

$$mv = (m - dm)(v + dv) + (v + u)dm$$

$$mdv + udm = 0$$
$$dv = -\frac{u}{m}dm$$

Do integral on both sides,

$$\int_0^v dv = -u\int_m^{m-\alpha T}\frac{1}{m}dm$$
$$v = uln\frac{m}{m - \alpha T}$$

After the collision, we can also find

$$mv = (m - dm)(v + dv) + (v + u)dm$$

$$v = uln\frac{m - \alpha T}{m - \alpha(T + t)}$$

$$\frac{m}{m - \alpha T} = \frac{m - \alpha T}{m - \alpha(T + t)}$$

$$m - \alpha(T + t) = \frac{(m - \alpha T)^2}{m}$$

$$t = \frac{1}{\alpha}\left[m - \frac{(m - \alpha T)^2}{m}\right] - T$$

When $m = 1kg$, $\alpha = 0.01kg/s$, $T = 10s$,

$$t = 9s$$