Problem Set 9

Due: 22 July 2016, 10 a.m.

Problem 1. Suppose a particle with mass m moves acted upon a central force $\mathbf{F} = F_r \hat{n}_r$. Show that if the trajectory of the particle $r = r(\varphi)$ in polar coordinates is known, the magnitude of the force can be found from

$$F_r = -\frac{L^2}{mr^2}\left[\frac{d^2}{d\varphi^2}\left(\frac{1}{r}\right) + \frac{1}{r}\right],$$

which is known as the Binet's equation.

Hint. Use the fact (proved in class) that central forces conserve angular momentum.

(*6 marks*)

Problem 2. A particle with mass m is acted upon by a central force and moves along a circle with radius R passing through the center of the force field. What is the dependence of the magnitude of this force on the distance from the center?

(*2 marks*)

Problem 3. Solve the previous problem in the case when the particle moves along a lemniscate with equation (in polar coordinates) $r^2 = 2R^2\cos(2\varphi)$, where $R > 0$.

Find the dependence of the angle φ on time, assuming that $\varphi(0) = 0$.

(*4 marks*)

Problem 4. Argue that for solids with cylindrical symmetry, the tensor of inertia about principal axes has only two independent components.

(*3 marks*)

Problem 5. Two particles with mass m each are connected by a massless rod and placed as in figure below (the z axis points out of page). The system rotates about the x axis with constant angular velocity $\boldsymbol{\omega} = (\omega, 0, 0)$

(a) Find the moment of inertia tensor of this rigid body with respect to the axes x, y, z.

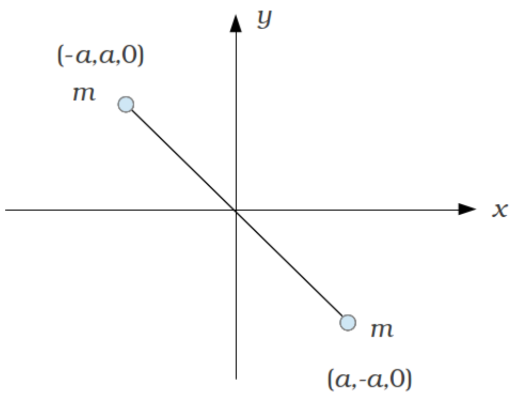
(b) Find the angular momentum $\mathbf{L}$ of this object with respect to the origin by: (1) using the relation $L_\alpha = \sum_\beta I_{\alpha\beta}\omega_\beta$, (2) adding the angular momenta of the individual particles. Is $\mathbf{L} \parallel \boldsymbol{\omega}$?

(c) Use symmetry arguments to find the principal axes $\tilde{x}, \tilde{y}, \tilde{z}$ of the moment of inertia tensor. Are these axes unique? Why or why not?

(d) By directly calculating its components, find the moment of inertia tensor with respect to the axes you have found in the previous step. Check that it is diagonal (it will confirm that the axes you have found are the principal axes).

(e) Check that rotation about any of the principal axes (corresponding to a non-zero moment of inertia) produces angular momentum (with respect to the origin) that is parallel to the angular velocity, i.e. along the axis of rotation.

(f) * (optional, for those of you who are intrigued with it) What is the solution of the eigenproblem for the matrix from (a)? What is the physical interpretation of this result?



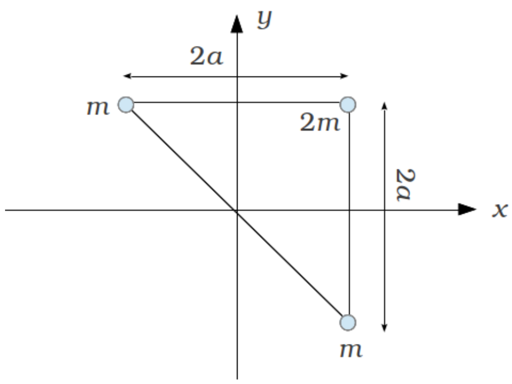
(4 + (1 + 2) + 2 + 3 + 1 + 0 marks)

Problem 6. A right-angle triangle lies in plane $z = 0$ and has masses in its vertices (see figure below).

(a) Find the tensor of inertia with respect to axes parallel to axes x, y, z (and passing through the center of mass).

(b) Solve the eigenproblem for the relevant matrix to find the principal moments of inertia and the directions of principal axes.

Note. You may use a computer to do the algebra.



(4 + 3 marks)

Problem 7. Check that the kinetic energy in rotational motion of a rigid body about an axis of rotation through the center of mass is $K = \frac{1}{2} \sum_{\alpha\beta} I_{\alpha\beta}\omega_\alpha\omega_\beta$, that is $K = \frac{1}{2}\boldsymbol{\omega}^T I \boldsymbol{\omega}$. Assume that there is no translational motion of the body.

Hint. See the derivation of the relation $L_\alpha = \sum_\beta I_{\alpha\beta}\omega_\beta$ we did in class.

(4 marks)