Problem 1.

$$dM = \frac{Mdl}{L}$$

$$F = \int_x^{L+x} \frac{GMm}{Ll^2} dl = -\frac{GMm}{Ll}\Big|_x^{L+x} = \frac{GMm}{L}\left(\frac{1}{x} - \frac{1}{L+x}\right) = \frac{GMm}{x(L+x)}$$

When $x \gg L$,

$$\frac{x(L+x)}{x(L+x) + \frac{1}{4}L^2} \approx 1$$

$$F \approx \frac{GMm}{x^2 + Lx + \frac{1}{4}L^2} = \frac{GMm}{(x + \frac{1}{2}L)^2}$$

Problem 2.

(a)

$$dm = \frac{Md\theta}{2\pi}$$

$$U = \int_0^{2\pi} -\frac{GMm}{2\pi\sqrt{x^2 + a^2}} d\theta = -\frac{GMm}{\sqrt{x^2 + a^2}}$$

(b) When $x \gg a$,

$$\frac{\sqrt{x^2 + a^2}}{x} \approx 1$$

$$U \approx -\frac{GMm}{x}$$

(c) Since each small force is symmetric at every angle, the net force is pointing the the right.

$$F = \int_0^{2\pi} \frac{GMm}{2\pi(x^2 + a^2)} \frac{x}{\sqrt{x^2 + a^2}} d\theta = \frac{GMmx}{(x^2 + a^2)^{\frac{3}{2}}}$$

When $x \gg a$,

$$\frac{(x^2 + a^2)^{\frac{3}{2}}}{x^3} \approx 1$$

$$F \approx \frac{GMm}{x^2}$$

(d) When $x = 0$,

$$F = \frac{GMmx}{(x^2 + a^2)^{\frac{3}{2}}} = 0$$

Problem 3.

(a)

$$\iint_S E_G \hat{n} dS = \iiint_E div E_G dV = \iiint_E -4\pi G \rho dV = -4\pi G M_{\sum}$$

(b)

$$-4\pi GM = -E_G \cdot 4\pi r^2$$
$$E_G = \frac{GM}{r^2}$$

(c) When $r > R$

$$-4\pi GM = -E_G \cdot 4\pi r^2$$
$$E_G = \frac{GM}{r^2}$$

When $r < R$

$$-4\pi GM \frac{r^3}{R^3} = -E_G \cdot 4\pi r^2$$
$$E_G = \frac{GMr}{R^3}$$

(d) When $r > R$

$$-4\pi GM = -E_G \cdot 4\pi r^2$$
$$E_G = \frac{GM}{r^2}$$

When $r < R$

$$-4\pi GM \cdot 0 = -E_G \cdot 4\pi r^2$$
$$E_G = 0$$

Problem 4.

$$F = -\frac{GMm}{R^3} x$$

It is a harmonic oscillator.

$$k = \frac{GMm}{R^3}$$
$$T = 2\pi \sqrt{\frac{m}{k}} = 2\pi \sqrt{\frac{R^3}{GM}}$$
$$\omega = \frac{2\pi}{T} = \sqrt{\frac{GM}{R^3}}$$
$$x = R \cos\left(\sqrt{\frac{GM}{R^3}} t\right)$$

A satellite orbiting around the planet close to its surface:

$$\frac{GMm}{R^2} = \frac{mv^2}{R}$$

$$v = \sqrt{\frac{GM}{R}}$$

$$T = \frac{2\pi R}{v} = 2\pi\sqrt{\frac{R^3}{GM}}$$

So the time is the same.
If the tunnel is drilled at an angle φ to the diameter,

$$F = -\frac{GMm}{R^3}(\sqrt{R^2 \sin^2 \varphi + x^2})\frac{x}{\sqrt{R^2 \sin^2 \varphi + x^2}} = -\frac{GMm}{R^3}x$$

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{R^3}{GM}}$$

So the answer won't change.