Problem 1.

(a)

$$(\omega_0^2-\omega_{dr}^2)^2+\left(\frac{b\omega_{dr}}{m}\right)^2=\omega_{dr}^4-\left(2\omega_0^2-\frac{b^2}{m^2}\right)\omega_{dr}^2+\omega_0^4$$

When $\omega_{dr}^2=\omega_0^2-\frac{b^2}{2m^2}$, it get its minimum value

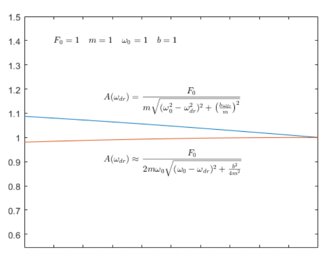
$$\omega_{dr}=\omega_{res}=\sqrt{\omega_0^2-\frac{b^2}{2m^2}}$$

$$A(\omega_{res})=\frac{F_0}{b\sqrt{\omega_0^2-\frac{b^2}{4m^2}}}$$

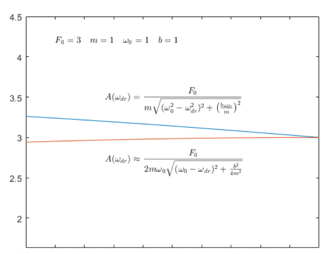
(b)

$$\begin{aligned}(\omega_0^2-\omega_{dr}^2)^2+\left(\frac{b\omega_{dr}}{m}\right)^2&=(\omega_0-\omega_{dr})^2(\omega_0+\omega_{dr})^2+\left(\frac{b\omega_{dr}}{m}\right)^2\\&\approx 4\omega_0^2(\omega_0-\omega_{dr})^2+\omega_0^2\frac{b^2}{m^2}\\&=(2\omega_0)^2\left[(\omega_0-\omega_{dr})^2+\frac{b^2}{4m^2}\right]\end{aligned}$$

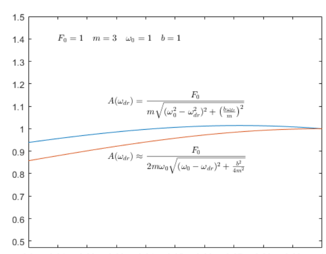
$$A(\omega_{dr})\approx\frac{F_0}{2m\omega_0\sqrt{(\omega_0-\omega_{dr})^2+\frac{b^2}{4m^2}}}$$



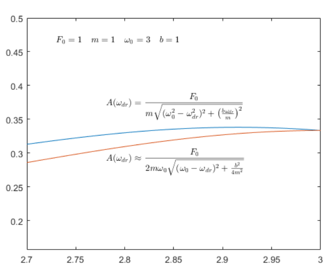
(a) $F_0=1, m=1, \omega_0=1, b=1$



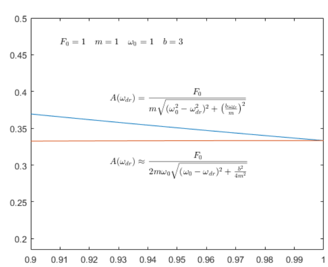
(b) $F_0=3, m=1, \omega_0=1, b=1$



(c) $F_0=1, m=3, \omega_0=1, b=1$



(d) $F_0=1, m=1, \omega_0=3, b=1$



(e) $F_0=1, m=1, \omega_0=1, b=3$

Figure 1: The exact and approximate curve

(c) Since $\Delta\omega_{dr}$ and $\frac{b}{m}$ is small, $\omega_{dr} \approx \omega_{res} \approx \omega_0$

$$A(\omega_{dr}) = \frac{\sqrt{2}}{2}A(\omega_{res}) = \frac{\sqrt{2}F_0}{2b\sqrt{\omega_0^2 - \frac{b^2}{4m^2}}} \approx \frac{F_0}{\sqrt{2}b\omega_0}$$

$$A(\omega_{dr}) = \frac{F_0}{m\sqrt{(\omega_0^2 - \omega_{dr}^2)^2 + \left(\frac{b\omega_{dr}}{m}\right)^2}} = \frac{F_0}{m\sqrt{(\omega_0 - \omega_{dr})^2(\omega_0 + \omega_{dr})^2 + \frac{b^2\omega_0^2}{m^2}}} \approx \frac{F_0}{m\sqrt{\omega_0^2\Delta\omega_{dr}^2 + \frac{b^2\omega_0^2}{m^2}}}$$

$$2b^2\omega_0^2 = m^2\left(\omega_0^2\Delta\omega_{dr}^2 + \frac{b^2\omega_0^2}{m^2}\right)$$

$$\Delta\omega_0 = \frac{b}{m}$$

Problem 2.

(a)

$$\tan\varphi = \frac{b\omega_{dr}}{m(\omega_{dr}^2 - \omega_0^2)} = -1$$

$$4\omega_{dr} = \omega_0^2 - \omega_{dr}^2$$

$$\omega_{dr} = \sqrt{4 + \omega_0^2} - 2$$

(b)

$$\frac{F_0}{m\sqrt{(\omega_0^2 - \omega_1^2)^2 + \left(\frac{b\Omega_1}{m}\right)^2}} = \frac{F_0}{m\sqrt{(\omega_0^2 - \Omega_2^2)^2 + \left(\frac{b\Omega_2}{m}\right)^2}}$$

$$\omega_0^4 - 2\omega_0^2\Omega_1^2 + \Omega_1^4 + \frac{b^2\Omega_1^2}{m^2} = \omega_0^4 - 2\omega_0^2\Omega_2^2 + \Omega_2^4 + \frac{b^2\Omega_2^2}{m^2}$$

$$\omega_0 = \sqrt{\frac{\Omega_1^2 + \Omega_2^2}{2} + \frac{b^2}{2m^2}}$$

Problem 3.

Solved in a FOR whose x-axis is the perpendicular to the slide and y-axis is opposite to the motion direction.

$$a' = a = g\sin\alpha - \mu g\cos\theta$$

Obviously, in this frame of reference, the surface of the liquid in the container should be perpendicular to the direction of the acceleration.
On the direction of x-axis,

$$a_x = a' - g\sin\theta = -\mu g\cos\theta$$

On the direction of y-axis,

$$a_y = g\cos\theta$$

So the angle between a and y-axis equals $\arctan\mu$, which is the angle that the surface of the liquid in the container forms with the inclined plane.

Problem 4.

1. Solved in the FOR of ground.
   No, it doesn't change.

$$F = -(k_1 + k_2)\Delta l$$
$$T = 2\pi\sqrt{\frac{m}{k_1 + k_2}}$$

2. Solved in the FOR of the elevator.
   Yes, it changes.

$$mg = (k_1 + k_2)(l_1 - l_0)$$
$$m(g - a) = (k_1 + k_2)(l_2 - l_0)$$
$$\Delta l = l_1 - l_2 = \frac{ma}{k_1 + k_2}$$

Problem 5.

The direction of the angular acceleration vector is to the central axis of the rotation of the earth.
The observer on the equator will find himself moving at a speed while another won't.

Problem 6.

Let the horizontal distance between the center and the object be x, all of the points in the domain of x remain at rest

(a)

$$\tan\theta = y' = \alpha x$$
$$\mu_s mg\cos\theta \geqslant mg\sin\theta$$
$$\tan\theta \leqslant \mu_s$$
$$x \leqslant \frac{\mu_s}{\alpha}$$

(b) Solved in the FOR of the rotating container.
Suppose the direction to the center to be the positive direction.

$$a' = -\frac{F_r}{m} = \omega^2 x$$
$$N = mg\cos\theta + m\omega^2 x\sin\theta$$

$$\mu_s(mg\cos\theta + m\omega^2 x\sin\theta) \geqslant |mg\sin\theta - m\omega^2 x\cos\theta|$$
$$\mu_s(g + \omega^2 x\tan\theta) \geqslant |g\tan\theta - \omega^2 x|$$
$$\mu_s(g + \omega^2 \alpha x^2) \geqslant |g\alpha x - \omega^2 x|$$

$$\mu_s\omega^2\alpha x^2 - |g\alpha - \omega^2|x + \mu_s g \geqslant 0 \quad (x \geqslant 0)$$
$$\Delta = (g\alpha - \omega^2)^2 - 4\mu_s^2 g\alpha\omega^2$$

I.

$$(g\alpha - \omega^2)^2 - 4\mu_s^2 g\alpha\omega^2 \leqslant 0$$

All of the points on the inner surface of a container remain at rest.

II.

$$(g\alpha - \omega^2)^2 - 4\mu_s^2 g\alpha\omega^2 > 0$$

$$x \in \left[0, \frac{|g\alpha - \omega^2| - \sqrt{(g\alpha - \omega^2)^2 - 4\mu_s^2 g\alpha\omega^2}}{2\mu_s\omega^2\alpha}\right) \bigcup \left(\frac{|g\alpha - \omega^2| + \sqrt{(g\alpha - \omega^2)^2 - 4\mu_s^2 g\alpha\omega^2}}{2\mu_s\omega^2\alpha}, +\infty\right)$$

Problem 7.

(a)

$$\vec{r'} = b\hat{n_y} + a\sin\omega_0 t\hat{n_z}$$

$$\vec{a'} = \frac{d^2\vec{r'}}{dt^2} = -a\omega_0^2\sin\omega_0 t\hat{n_z}$$

(b)

$$\begin{aligned}\vec{a} &= \vec{a_0'} + \vec{a'} + 2\vec{\omega}\times\vec{v'} + \frac{d\vec{\omega}}{dt}\times\vec{r'} + \vec{\omega}\times(\vec{\omega}\times\vec{r'}) \\ &= -b\omega^2\hat{n_y'} - a\omega_0^2\sin\omega_0 t\hat{n_z'}\end{aligned}$$

Since $\hat{n_y'} = \cos\omega t\hat{n_x} + \sin\omega t\hat{n_y}$ and $\hat{n_z'} = \hat{n_z}$

$$\vec{a} = -b\omega^2\cos\omega t\hat{n_x} - b\omega^2\sin\omega t\hat{n_y} - a\omega_0^2\sin\omega_0 t\hat{n_z}$$

(c)

$$\vec{r} = b\cos\omega t\hat{n_x} + b\sin\omega t\hat{n_y} + a\sin\omega_0 t\hat{n_z}$$

$$\vec{a} = \frac{d^2\vec{r}}{dt^2} = -b\omega^2\cos\omega t\hat{n_x} - b\omega^2\sin\omega t\hat{n_y} - a\omega_0^2\sin\omega_0 t\hat{n_z}$$

Problem 8.

Suppose an inertial FOR which is set up with the x axis horizontally due east, the y axis horizontally due north and the z axis vertically upwards.

(a)

$$\vec{a'} = \vec{a_{res}} + \vec{a_{fict}} = \vec{a_{res}} - 2\vec{\omega}\times\vec{v'} - \frac{d\vec{\omega}}{dt}\times\vec{r'} - \vec{\omega}\times(\vec{\omega}\times\vec{r'})$$

$$\vec{\omega} = \omega\hat{n_z} = \omega\cos\varphi\hat{n_y'} + \omega\sin\varphi\hat{n_z'}$$

$$2\vec{\omega}\times\vec{v'} = -2\frac{dy'}{dt}\omega\sin\varphi\hat{n_x'} + 2\frac{dx'}{dt}\omega\sin\varphi\hat{n_y'}$$

$$\vec{\omega}\times(\vec{\omega}\times\vec{r'}) = x'\omega^2\sin\varphi\hat{n_x'} + y'\omega^2\sin\varphi\hat{n_y'} \approx 0$$

$$\vec{a_{res}} = -\frac{gx'}{l}\hat{n_x'} - \frac{gy'}{l}\hat{n_y'}$$

$$\frac{d^2x'}{dt^2} = -\frac{gx'}{l} + 2\frac{dy'}{dt}\omega\sin\varphi$$

$$\frac{d^2y'}{dt^2} = -\frac{gy'}{l} - 2\frac{dx'}{dt}\omega\sin\varphi$$

(b)

$$\frac{d^2x'}{dt^2}+i\frac{d^2y'}{dt^2}=-\omega_0^2(x'+iy')+2\left(\frac{dy'}{dt}-i\frac{dx'}{dt}\right)\omega\sin\varphi$$

$$\frac{d^2\tilde{\xi}}{dt^2}=-\omega_0^2\tilde{\xi}-2i\omega\sin\varphi\frac{d\tilde{\xi}}{dt}$$

$$\frac{d^2\tilde{\xi}}{dt^2}+2i\omega\sin\varphi\frac{d\tilde{\xi}}{dt}+\omega_0^2\tilde{\xi}=0$$

Let $\tilde{\xi}=e^{rt}$, then $\frac{d\tilde{\xi}}{dt}=re^{rt}$, $\frac{d^2\tilde{\xi}}{dt^2}=r^2e^{rt}$

$$r^2+2i\omega\sin\varphi r+\omega_0^2=0$$

$$\Delta=-4\omega^2\sin^2\varphi-4\omega_0^2\approx-4\omega_0^2$$

$$r_{1,2}=-i\omega\sin\varphi\pm i\omega_0$$

$$\tilde{\xi}=e^{-i\omega\sin\varphi t}\left(Ae^{i\omega_0 t}+Be^{-i\omega_0 t}\right)$$

(c)

$$\begin{aligned}
\tilde{\xi}&=Ae^{i(\omega_0-\omega\sin\varphi)t}+Be^{i(-\omega_0-\omega\sin\varphi)t}\\
&=A\cos(\omega_0t-\omega\sin\varphi t)+Bi\sin(\omega_0t-\omega\sin\varphi t)+A\cos(-\omega_0t-\omega\sin\varphi t)+Bi\sin(-\omega_0t-\omega\sin\varphi t)\\
&=A\cos(\omega_0t)\cos(\omega\sin\varphi t)+A\sin(\omega_0t)\sin(\omega\sin\varphi t)+Bi\sin(\omega_0t)\cos(\omega\sin\varphi t)-Bi\cos(\omega_0t)\sin(\omega\sin\varphi t)\\
&+A\cos(\omega_0t)\cos(\omega\sin\varphi t)-A\sin(\omega_0t)\sin(\omega\sin\varphi t)-Bi\sin(\omega_0t)\cos(\omega\sin\varphi t)-Bi\cos(\omega_0t)\sin(\omega\sin\varphi t)\\
&=2A\cos(\omega_0t)\cos(\omega\sin\varphi t)-2Bi\cos(\omega_0t)\sin(\omega\sin\varphi t)
\end{aligned}$$

When $A=B=\frac{iR}{2}$,

$$\tilde{\xi}=R\cos(\omega_0t)\sin(\omega\sin\varphi t)+iR\cos(\omega_0t)\cos(\omega\sin\varphi t)$$

$$x'=R\cos(\omega_0t)\sin(\omega\sin\varphi t),\ \ y'=R\cos(\omega_0t)\cos(\omega\sin\varphi t)$$

$$r'=kR\cos(\omega_0t)$$

where $k=[\sin(\omega\sin\varphi t),\cos(\omega\sin\varphi t)]$

(d)

$$T=\frac{2\pi}{\omega\sin\varphi}=\frac{2\pi}{\frac{\pi}{12}\sin 31^\circ}\approx 46.60h$$