Problem 1.

When A moves upwards, if not considering the movement of B, the left row above A and below A move the same distance and the length of right row remains the origin. However, the middle row above A becomes shorter and the eliminate part moves to the lower part of B. So when A moves x cm, B will move $\frac{x}{2}$ cm. Suppose the upward direction to be positive.

(a)

$$\begin{cases} v_a = -2v_b \\ v_a + v_b = 24cm/s \end{cases} \Rightarrow \begin{cases} v_a = 16cm/s \\ v_b = -8cm/s \end{cases}$$

$$a_a = \frac{v_a}{\delta t} = 2cm/s^2$$
$$a_b = \frac{v_b}{\delta t} = -1cm/s^2$$

(b)

$$v_{bt} = a_b t = -6cm/s$$

$$s_{bt} = \frac{1}{2} a_b t^2 = -18cm$$

Problem 2.

Suppose the direction of v_0 to be the x-axis and the direction to the sky to be the y-axis, the launch point to be the zero point, then

$$\vec{a} = -g\hat{n}_y$$

$$\vec{v} = v_0\hat{n}_x - gt\hat{n}_y$$

$$|\vec{a_t}| = \frac{\vec{a} \cdot \vec{v_t}}{|\vec{v_t}|} = \frac{g^2 t}{\sqrt{v_0^2 + g^2 t^2}}$$

$$|\vec{a_n}| = \sqrt{|\vec{a}| - |\vec{a_t}|^2} = \frac{g v_0}{\sqrt{v_0^2 + g^2 t^2}}$$

Problem 3.

Suppose the east direction to be the x-axis and the north direction to be the y-axis, the start point to be the zero point, then

(a)

$$y = v_1 t$$

$$\vec{v} = v_1\hat{n}_x + v_2\hat{n}_y = v_1\hat{n}_x + v_0 \sin\frac{v_1 \pi t}{L}\hat{n}_y$$

$$|v| = \sqrt{v_1^2 + v_0^2 \sin^2\frac{v_1 \pi t}{L}}$$

(b)

(c)

$$
\begin{aligned}
s_t &= \int_0^t v dt = v_0 \int_0^t \sin \frac{v_1 \pi t}{L} dt \ \hat{n}_x + v_1 t \ \hat{n}_y \\
&= v_0 \left[-\frac{L}{v_1 \pi} \cos \frac{v_1 \pi t}{L} \right]_0^t \ \hat{n}_x + v_1 t \ \hat{n}_y \\
&= \frac{v_0 L}{v_1 \pi} \left(1 - \cos \frac{v_1 \pi t}{L} \right) \ \hat{n}_x + v_1 t \ \hat{n}_y
\end{aligned}
$$

(d) When $t = \frac{L}{v_1}$, the canoe drifted down the river.

$$
s_x = \frac{v_0 L}{v_1 \pi} (1 - \cos \pi) = \frac{2 v_0 L}{v_1 \pi}
$$

Problem 4.

Suppose the rolling direction to be the x-axis and the direction to ground to be the y-axis, the center of the wheel to be the zero point, the origin position of P to be $(R\cos\theta_0,\ R\sin\theta_0)$, then

(a)

$$
\delta\theta = \omega \delta t
$$

$$
P = \left\{ \begin{array}{ll} x & = R\cos(\theta_0 + \omega t) + \omega R t \\ y & = R\sin(\theta_0 + \omega t) \end{array} \right. \quad (t > 0)
$$

(b)

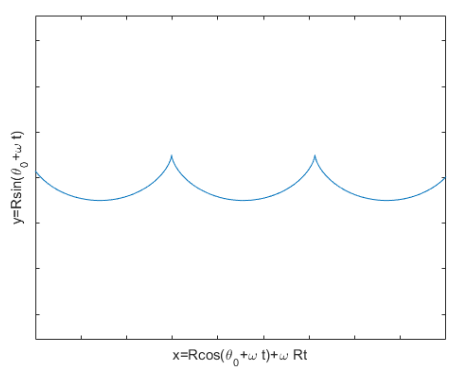


Figure 1: trajectory of P

(c)

$$
\vec{v} = x'(t)\hat{n}_x + y'(t)\hat{n}_y = R\omega[1 - \sin(\theta_0 + \omega t)]\hat{n}_x + R\omega\cos(\theta_0 + \omega t)\hat{n}_y
$$

(d)

$$
\begin{aligned}
l &= \int_0^t \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \\
&= \sqrt{2}R\omega \int_0^t \sqrt{1-\sin(\theta_0+\omega t)} dt \\
&= \sqrt{2}R\omega \int_0^t \left| \sin\left(\frac{\theta_0+\omega t}{2}\right) - \cos\left(\frac{\theta_0+\omega t}{2}\right) \right| dt \\
&= 2R\omega \int_0^t \left| \sin\left(\frac{\omega t}{2} + \frac{2\theta_0-\pi}{4}\right) \right| dt
\end{aligned}
$$

When $t \in \left(\frac{(8k+1)\pi-2\theta_0}{2\omega},\ \frac{(8k+5)\pi-2\theta_0}{2\omega}\right)$, $\sin\left(\frac{\omega t}{2} + \frac{2\theta_0-\pi}{4}\right) > 0,\ k \in R$
When $t \in \left(\frac{(8k+5)\pi-2\theta_0}{2\omega},\ \frac{(8k+9)\pi-2\theta_0}{2\omega}\right)$, $\sin\left(\frac{\omega t}{2} + \frac{2\theta_0-\pi}{4}\right) < 0,\ k \in R$
In half of one period, i.e., $t \in \left(0,\ \frac{2\pi}{\omega}\right)$,

$$
2R\omega \int_0^{\frac{2\pi}{\omega}} \left| \sin\left(\frac{\omega t}{2} + \frac{2\theta_0-\pi}{4}\right) \right| dt = 4R\omega \int_0^{\frac{\pi}{\omega}} \sin\frac{\omega t}{2} dt = 8R\left[-\cos\frac{\omega t}{2}\right]_0^{\frac{\pi}{\omega}} = 8R
$$

$$
l = \begin{cases} 8R\left[\frac{\omega t}{2\pi}\right] + 2R\omega \int_0^t \sin\left(\frac{\omega t}{2} + \frac{2\theta_0-\pi}{4}\right) dt & t \in \left[\frac{(8k+1)\pi-2\theta_0}{2\omega},\ \frac{(8k+5)\pi-2\theta_0}{2\omega}\right) \\ 8R\left[\frac{\omega t}{2\pi}\right] - 2R\omega \int_{\frac{2\pi}{\omega}}^t \sin\left(\frac{\omega t}{2} + \frac{2\theta_0-\pi}{4}\right) dt & t \in \left[\frac{(8k+5)\pi-2\theta_0}{2\omega},\ \frac{(8k+9)\pi-2\theta_0}{2\omega}\right) \end{cases}
$$

$$
l = \begin{cases} 4R\left\{\left[\frac{\omega t}{\pi}\right] + \cos\left(\frac{2\theta_0-\pi}{4}\right) - \cos\left(\frac{\omega t}{2} + \frac{2\theta_0-\pi}{4}\right)\right\} & t \in \left[\frac{(8k+1)\pi-2\theta_0}{2\omega},\ \frac{(8k+5)\pi-2\theta_0}{2\omega}\right) \\ 4R\left\{\left[\frac{\omega t}{\pi}\right] + \cos\left(\frac{2\theta_0-\pi}{4}\right) + \cos\left(\frac{\omega t}{2} + \frac{2\theta_0-\pi}{4}\right)\right\} & t \in \left[\frac{(8k+5)\pi-2\theta_0}{2\omega},\ \frac{(8k+9)\pi-2\theta_0}{2\omega}\right) \end{cases}
$$

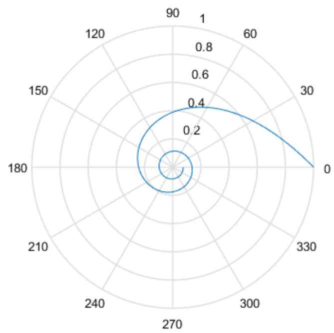
Problem 5.

The unit of c is s^{-1} and the unit of r_0 is m.

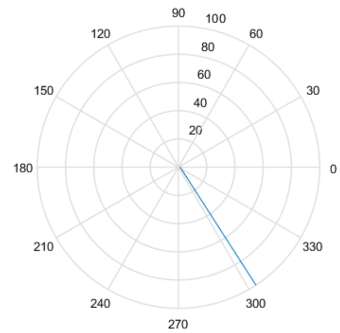
(a)

$$
r(t) = r_0(1-ct),\ \varphi(t) + 1 = \frac{1}{1-ct}
$$

$$
r(t) + r(t)\varphi(t) = r_0
$$



(a) $c > 0$



(b) $c < 0$

Figure 2: $r(\varphi) = \frac{r_0}{\varphi+1}$

(b)

$$\vec{v_r} = \frac{dr}{dt}\hat{n_r} = -cr_0\hat{n_r}$$
$$\vec{v_\varphi} = r\frac{d\varphi}{dt}\hat{n_\varphi} = \frac{cr_0}{1-ct}\hat{n_\varphi}$$
$$|v| = \sqrt{\vec{v_r}^2 + \vec{v_\varphi}^2} = \frac{|cr_0|\sqrt{2-2ct+c^2t^2}}{1-ct}$$

(c)

$$\vec{a_r} = \left[\frac{d^2r}{dt^2} - r\left(\frac{d\varphi}{dt}\right)^2\right]\hat{n_r} = -\frac{c^2r_0}{(1-ct)^3}\hat{n_r}$$
$$\vec{a_\varphi} = \left(r\frac{d^2\varphi}{dt^2} + 2\frac{dr}{dt}\frac{d\varphi}{dt}\right)\hat{n_\varphi} = \left(-\frac{2c^2r_0}{(1-ct)^2} + \frac{2c^2r_0}{(1-ct)^2}\right)\hat{n_\varphi} = \vec{0}$$

(d) When $c > 0$, $\varphi \in (-\infty, -1) \cup (0, +\infty)$, $r \in (0, r_0)$
When $c < 0$, $\varphi \in (-1, 0)$, $r \in (r_0, +\infty)$

Problem 6.

Suppose right to to be the x-axis and the direction to the sky to be the y-axis, point A to be the zero point, θ to be the angle between rod AB and the x-axis, then

(a)

$$\theta(t) = 2(\varphi_0 + \dot{\varphi}t)$$
$$x = b\cos\theta,\ y = b\cos\theta$$
$$v_x = -2\dot{\varphi}b\sin\theta,\ v_y = 2\dot{\varphi}b\cos\theta$$
$$a_x = -4\dot{\varphi}^2b\cos\theta,\ a_y = -4\dot{\varphi}^2b\sin\theta$$

$$|\vec{a}| = \sqrt{a_x^2 + a_y^2} = 4\dot{\varphi}^2 b$$

So the acceleration of pin B is of constant magnitude.

(b)

$$\vec{a} = -4\dot{\varphi}^2 b\cos\theta\hat{n_x} - 4\dot{\varphi}^2 b\sin\theta\hat{n_y} = -4\dot{\varphi}^2 b(\cos\theta\hat{n_x} + \sin\theta\hat{n_y})$$

the direction of the acceleration of pin B is the direction from pin B to Point A

Problem 7.

(a)

$$\vec{v} = \frac{dr}{dt}\hat{n_r} + r\frac{d\varphi}{dt}\hat{n_\varphi}$$

$$\tan\alpha = \frac{|\vec{v_\varphi}|}{|\vec{v_r}|} = \frac{\left|r\frac{d\varphi}{dt}\hat{n_\varphi}\right|}{\left|\frac{dr}{dt}\hat{n_r}\right|} = \frac{rd\varphi}{dr}$$

$$\tan\alpha\frac{dr}{r} = d\varphi$$

Do integral on both side,

$$\tan\alpha\ lnr = \varphi + C$$

As $\varphi(0) = 0$ and $r(0) = r_0$,

$$C = \tan\alpha\ lnr_0$$

$$\tan\alpha\ lnr = \varphi + \tan\alpha\ lnr_0$$

$$ln\left(\frac{r}{r_0}\right)\sin\alpha = \varphi\cos\alpha$$

(b)

$$ln\left(\frac{r}{r_0}\right) = \frac{\varphi}{\tan\alpha}$$

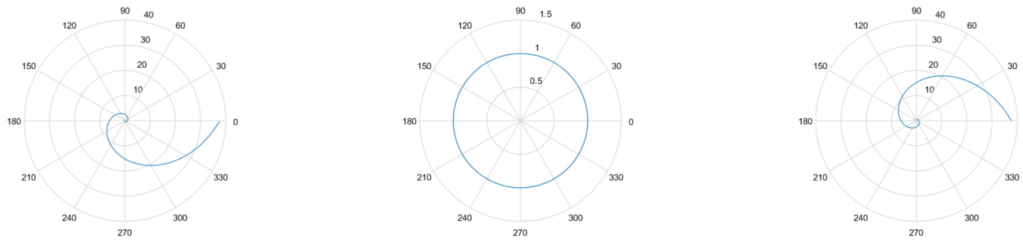
$$r(\varphi) = r_0 e^{\frac{\varphi}{\tan\alpha}}$$

$$r'(\varphi) = \frac{1}{\tan\alpha}r_0 e^{\frac{\varphi}{\tan\alpha}}$$

$$\int_0^\varphi \sqrt{r(\varphi)^2 + r'(\varphi)^2}d\varphi = \frac{r_0}{\sin\alpha}\int_0^\varphi e^{\frac{\varphi}{\tan\alpha}}d\varphi = \frac{r_0}{\sin\alpha}\left(\tan\alpha e^{\frac{\varphi}{\tan\alpha}} - 1\right)$$

$$l = \begin{cases} \frac{r_0}{\sin\alpha}\left(\tan\alpha e^{\frac{\varphi}{\tan\alpha}} - 1\right) & \alpha \in (0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi) \\ not\ defined & \alpha = 0\ or\ \pi \\ 2\pi r_0 & \alpha = \frac{\pi}{2} \end{cases}$$

(c)



(a) $\alpha \in (0, \frac{\pi}{2})$ (b) $\alpha = \frac{\pi}{2}$ (c) $\alpha \in (\frac{\pi}{2}, \pi)$

Figure 3: $r(\varphi) = r_0 e^{\frac{\varphi}{\tan\alpha}}$

(d) When $\alpha \in (0, \frac{\pi}{2})$, the solution is a helix as shown in Figure 3(a)
When $\alpha = \frac{\pi}{2}$, the solution is a circle as shown in Figure 3(b)
When $\alpha \in (\frac{\pi}{2}, \pi)$, the solution is a helix as shown in Figure 3(c)
When $\alpha = 0$ or π, the direction of $\vec{v}$ and $\vec{r}$ are same or opposite, so the trajectory is a line but the equation and length can't be calculated because of lack of conditions.

Problem 8.

$$r = r_0 - ct$$

$$\vec{v} = \frac{dr}{dt}\hat{n_r} + r\frac{d\varphi}{dt}\hat{n_\varphi}$$

$$v^2 = c^2 + r^2\left(\frac{d\varphi}{dt}\right)^2$$

$$\frac{d\varphi}{dt} = \frac{\sqrt{v^2 - c^2}}{r_0 - ct}$$

$$\frac{dt}{r_0 - ct}\sqrt{v^2 - c^2} = d\varphi$$

Do integral on both side,

$$-\frac{\sqrt{v^2 - c^2}}{c} ln\left(\frac{1}{r_0 - ct}\right) = \varphi + C$$

As $\varphi(0) = 0$ and $r(0) = r_0$,

$$-\frac{\sqrt{v^2 - c^2}}{c} ln\left(\frac{r}{r_0}\right) = \varphi$$

$$r(\varphi) = r_0 e^{-\frac{c\varphi}{\sqrt{v^2 - c^2}}}$$