UM–SJTU Joint Institute

Physics Laboratory
(VP241)

Laboratory Report

Exercise 5
RC, RL, and RLC Circuits

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Contents

1 Objectives

The objective of this exercise is to understand the physics of alternating-current circuits, in particular the processes of charging/discharging of capacitors, the phenomenon of electromagnetic induction in inductive elements, and other dynamic processes in RC, RL, and RLC series circuits. Moreover, methods for measuring the amplitude-frequency and the phase-frequency characteristics of RC, RL, and RLC series circuits will be studied. The resonance frequency of a RLC circuit as well as the quality factor of the circuit will be found from the amplitude-frequency curve.

2 Theoretical Background

Resistors, capacitors, and inductors are basic elements of electric circuits. Depending on a particular arrangement of these elements, RC, RL, RLC alternating-current (AC) circuits may display various features, including transient, steady state, and resonant behavior.

2.1 Transient Processes in RC, RL, RLC Series Circuits

2.1.1 RC Series Circuits

In a RC circuit, the process of charging or discharging of the capacitor is an example of a transient process. Figure 1 shows a RC series circuit in which a square-wave signal is used as the source signal. In the first half of the cycle, the square-wave voltage is $U(t) = \mathcal{E}$. and it charges the capacitor. In the second half-cycle, the square-wave voltage is zero, and the capacitor discharges through the resistor. The loop equation (Kirchhoff's loop rule) for the charging process is

$$RC\frac{dU_C}{dt} + U_C = \mathcal{E} \tag{1}$$

With the initial condition $U_C(t = 0) = 0$, the solution of Eq. (1) can be found as

$$U_C = \mathcal{E}(1 - e^{-\frac{t}{RC}}) \text{ and } U_R = iR = \mathcal{E}e^{-\frac{t}{RC}}$$

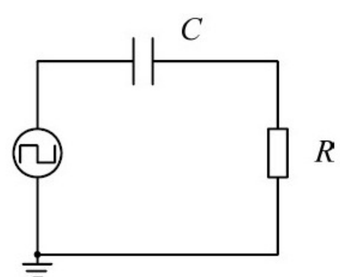


Figure 1: RC series circuit.

Hence the voltage across the capacitor U_C increases exponentially with time t, whereas the voltage on the resistor U_R decreases exponentially with time. The curves $U(t)$, $U_C(t)$, and $U_R(t)$ are shown in Figure 2.

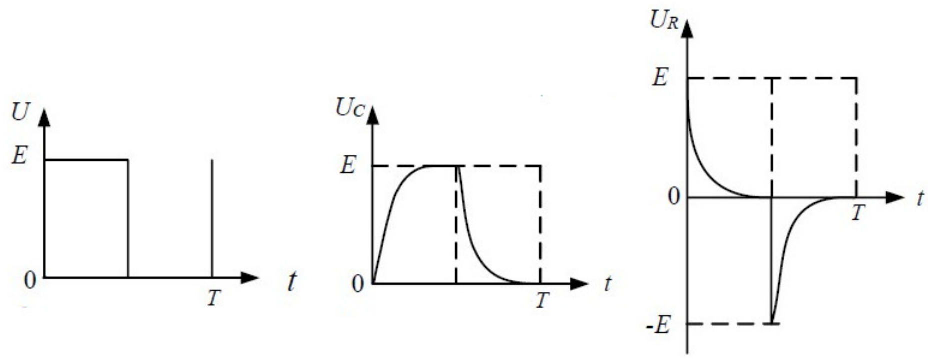


Figure 2: Charging/discharging curves for a RC series circuit.

For the discharging process, the loop rule gives

$$RC\frac{dU_C}{dt} + U_C = 0 \tag{2}$$

The solution of Eq. (2), with the initial condition $U_C(t = 0) = \mathcal{E}$, is

$$U_C = \mathcal{E}e^{-\frac{t}{RC}}$$

and, consequenctly,

$$U_R = iR = -\mathcal{E}e^{-\frac{t}{RC}}$$

where the magnitudes of both U_C and U_R decrease exponentially with time. Since $RC = \tau$ has the units of time, it is called the time constant of the circuit, and characterizes the dynamics of the transient process. There is another characteristics related to the time constant, easier to measure in experiments, which is called the half-life period $T_{1/2}$. The half-life period is the time needed for U_C to decrease to a half of the initial value (or increase to a half of the terminal value), and may be also used to characterize the dynamics of the transient process. Both quantities, in the process with exponential dynamics discussed above, are related by the equation

$$T_{1/2} = \tau ln2 \approx 0.693\tau$$

2.1.2 RL Series Circuit

A similar analysis can be carried out for a RL series circuit. In this case,

$$\tau = \frac{L}{R} \quad \text{and} \quad T_{1/2} = \frac{L}{R}ln2$$

2.1.3 RLC Series Circuit

For the situation that a power source is suddenly plugged into a RLC circuit, follow the loop rule we have

$$LC\frac{d^2U_C}{dt^2} + RC\frac{dU_C}{dt} + U_C = \mathcal{E} \tag{3}$$

both sides of the equation by LC and introducing the symbols

$$\beta = \frac{R}{2L} \text{ and } \omega_0 = \frac{1}{\sqrt{LC}} \tag{4}$$

Eq. (3) can be rewritten as

$$\frac{d^2U_C}{dt^2} + 2\beta\frac{dU_C}{dt} + \omega_0^2 U_C = \omega_0^2 \mathcal{E} \tag{5}$$

Note that Eq. (5) is an inhomogeneous differential equation and it is mathematically equivalent to the equation of motion of a damped harmonic oscillator with a constant driving force. Therefore, the complementary homogeneous equation is fully analogous to the equation of motion of a damped harmonic oscillator, with β being the damping coefficient, and ω_0 - the natural angular frequency. Moreover, after a specific solution to the inhomogeneous equation is found, a unique solution to the initial value problem consisting of Eq. (5) and the initial conditions

$$U_C(t=0) = 0 \text{ and } \left.\frac{dU_C}{dt}\right|_{t=0} = 0 \tag{6}$$

can be found. Exactly as for mechanical oscillations, depending on the relation between β and ω_0, there are three regimes, as implied by the solution of the complementary homogeneous equation:

- If $\beta^2 - \omega_0^2 < 0$ (weak damping), the system is in the underdamped regime and the solution to the initial value problem is of the form

  $$U_C = \mathcal{E} - \mathcal{E}e^{-\beta t}\left(cos\omega t + \frac{\beta}{\omega}sin\omega t\right)$$

  where $\omega = \sqrt{\omega_0^2 - \beta^2}$
- If $\beta^2 - \omega_0^2 > 0$ (strong damping), the system is in the overdamped regime with the solution of the form

  $$U_C = \mathcal{E} - \frac{\mathcal{E}}{2\gamma}e^{-\beta t}[(\beta+\gamma)e^{\gamma t} - (\beta-\gamma)e^{-\gamma t}]$$

  where $\gamma = \sqrt{\beta^2 - \omega_0^2}$
- If $\beta^2 - \omega_0^2 = 0$ the system is said to be critically damped, and

  $$U_C = \mathcal{E} - \mathcal{E}(1+\beta t)e^{-\beta t} \tag{7}$$

When the circuit reaches a steady state, the power source is suddenly removed. The differential equation for the discharging process is similar to that of the charging process, and there are also three regimes of the process.

The above discussion is valid for an ideal circuit and a step-signal source with zero internal resistance. In the experiment, the ideal system is replaced by a square-wave source with a small internal resistance. The period of the square-signal must be much greater than the time constant of the circuit. Note that, according to the above equations, the voltage across the capacitor U_C will finally reach $\mathcal{E}$ regardless of the regime (Figure 3).

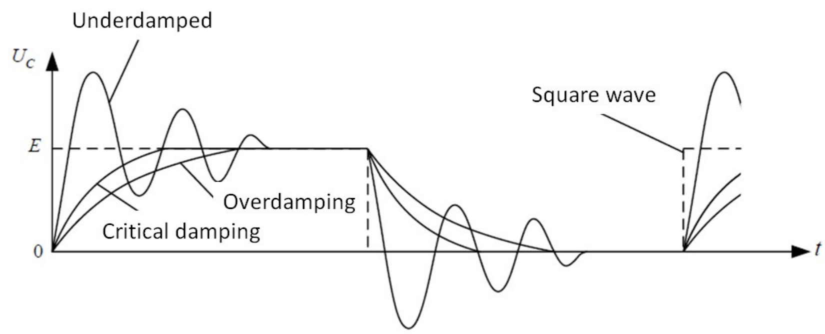


Figure 3: Three different regimes of transient processes in a RLC series circuit.

2.2 RC, RL Steady-State Circuits

When a sinusoidal alternating input voltage is provided to a RC (or RL) series circuit, the amplitude and the phase of the voltage across the capacitor and the resistor will change with the frequency of the input voltage. Then the amplitude vs. frequency relation and the phase vs. frequency relation can be obtained by measuring the voltage across the elements in the circuit for different input signal frequencies

$$\varphi = tan^{-1}\left(\frac{U_L}{U_R}\right) = tan^{-1}\left(\frac{\omega L}{R}\right), \qquad \varphi = tan^{-1}\left(-\frac{U_C}{U_R}\right) = tan^{-1}\left(-\frac{1}{\omega RC}\right)$$

2.3 RLC Resonant Circuit

2.3.1 RLC Series Circuit

A generic RLC series circuit is shown in Figure 4. The impedance and the phase difference in the RLC circuit can be calculated, e.g., by using the phasors technique. Representing the current I by a vector along the horizontal axis, the phase differences between the current and the voltages across the resistor, coil, and capacitor are

$$\varphi_R = 0, \qquad \varphi_L = \frac{\pi}{2}, \qquad \varphi_C = -\frac{\pi}{2}$$

respectively. The corresponding voltage amplitudes across the elements are

$$U_R = IZ = IR, \qquad U_L = IZ_L = I\omega L, \qquad U_C = IZ_C = \frac{I}{\omega C}$$

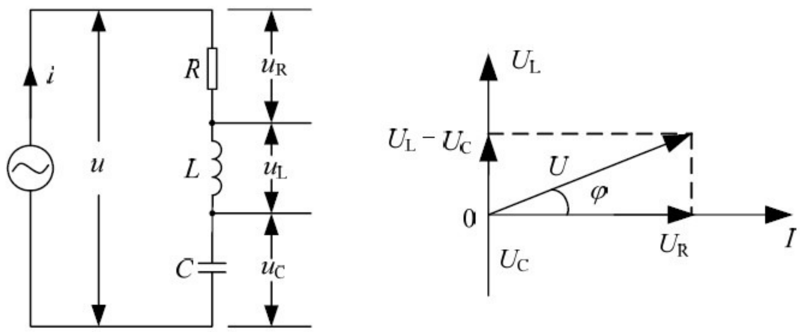


Figure 4: RLC series circuit.

Hence, the voltage amplitude

$$U = \sqrt{U_R^2 + (U_L - U_C)^2} \text{ or } U = I\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2} \tag{8}$$

and the total impedance

$$Z = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2} \tag{9}$$

with the phase difference between the current and the voltage in the circuit

$$\varphi = tan^{-1}\left(\frac{U_L - U_C}{U_R}\right) = tan^{-1}\left(\frac{\omega L - \frac{1}{\omega C}}{R}\right)$$

2.3.2 Resonance

If the frequency of the input signal provided by the source satisfies the condition

$$\omega_0 L = \frac{\omega}{C}, \qquad \text{or, equivalently,} \qquad \omega_0 = \frac{1}{\sqrt{LC}}$$

the total impedance will reach a minimum, $Z_0 = R$. Note that the resistance R in a real circuit includes the internal resistance and all kinds of alternating-current power losses, so its actual value will be greater than the theoretical one.

When the current reaches its maximum, $I_m = U/R$, the circuit is said to be at resonance. The frequency

$$f_0 = \frac{\omega_0}{2\pi} = \frac{1}{2\pi\sqrt{LC}}$$

at which the resonance phenomenon occurs, is called the resonance frequency.

The total impedance Z, the current I, and the phase diference $\varphi = \varphi_u - \varphi_i$ all depend on the frequency, with generic shapes of the three curves shown in Figure 5. According to Eqs. (8) and

(9), when the frequency is low ($f < f_0, i.e. 1/\omega C > \omega L$), then $\varphi < 0$. In this situation the total voltage lags behind the current and the circuit is said to be capacitive.

When the circuit is resonant ($f < f_0, i.e. 1/\omega C > \omega L$), then $\varphi = 0$ and the voltages across the capacitor and the inductor should be equal. The circuit is said to be resistive.

Finally, when the frequency is high ($f > f_0, i.e. 1/\omega C < \omega L$), then $\varphi > 0$. In this situation the total voltage leads the current, and the circuit is said to be inductive.

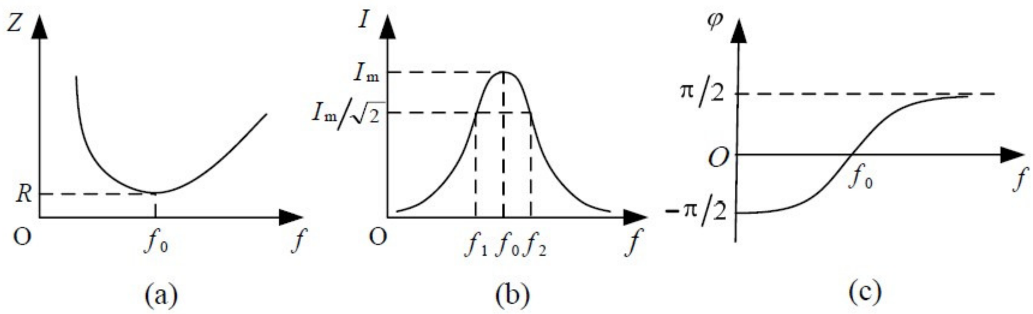


Figure 5: The impedance, the current and the phase difference as functions of the frequency for a RLC series circuit (generic sketches).

2.3.3 Quality Factor in Resonant Circuits

Since Im = U/R, the voltages across the resistor, the inductor, and the capacitor are

$$U_R = I_m R = U$$

$$U_L = I_m Z_L = \frac{U}{R}\omega L$$

respectively. For a circuit driven at the resonance frequency, the ratio of U_L (or U_C) to U is called the quality factor Q of a resonant circuit

$$Q = \frac{U_L}{U} = \frac{\omega_0 L}{R} \text{ or } Q = \frac{U_C}{U} = \frac{1}{\omega_0 RC}$$

When the total voltage is fixed, the greater Q is, the greater U_L and U_C are. The value of Q can be used to quantify the effciency of resonant circuits.

The quality factor can also be found as

$$Q = \frac{f_0}{f_2 - f_1}$$

where f_1 and f_2 are two frequencies such that $I(f_1) = I(f_2) = I_m/\sqrt{2}$ (see Figure 5b).

3 Measurement Setup and Procedure

3.1 Apparatus

The measurement setup consists of the following main elements: a signal generator, an oscilloscope, a digital multimeter, a wiring board, a fixed resistor $100\Omega(2W)$, a variable resistor $2k\Omega(2W)$, two capacitors $0.47\mu F$ and $0.1\mu F$, as well as two inductors (10 mH and 33 mH).

3.2 Measurement Procedure

3.2.1 RC, RL Series Circuit

1. Choose a capacitor and an inductor to assemble a circuit with the fixed-resistance 100 resistor. Adjust the output frequency of the square-wave signal provided by the signal generator. Observe the change of the waveform (curve shown on the oscilloscope screen) when the time constant is smaller or greater than the half-period of the square wave. Use the PRINT function of the oscilloscope to record the waveforms.

2. Adjust display parameters of the oscilloscope and measure $T_{1/2}$ for the studied circuits. Then, calculate the time constant and compare it with the theoretical value.In order to find the time constant accurately, only one period should be displayed on the oscilloscope screen.

3.2.2 RLC Series Circuit

1. Choose a capacitor and an inductor to assemble a RLC series circuit with the variable resistor. Observe the waveform of the capacitor voltage in the underdamped, critically damped, and overdamped regimes. Use the PRINT function of the oscilloscope to store the waveforms.

2. Adjust the variable resistor to the critically damped regime. According to the definition of the half-life period $T_{1/2}$, we have $T_{1/2} = 1.68$. By finding the value of $T_{1/2}$, the time constant can be found as $\tau = 1/\beta = T_{1/2}/1.68$. Compare your result with the theoretical value.

3.2.3 RLC Resonant Circuit

Apply a sinusoidal input voltage U_i to the RLC series circuit, change the frequency, then observe the change of the voltage U_R for a fixed resistor R, as well as the phase difference between U_R and U_i. Measure how U_R changes with U_i and calculate the phase difference according to Figure 4. Plot the graphs f/f_0 vs. I/I_m and f/f_0 vs.ϕ. Estimate the resonance frequency and calculate the quality factor Q.

4 Results

4.1 RC, RL Series Circuit

The measurement of a RC series circuit was shown in Table 1.

$$\tau_{theorem} = RC = 9.933 \pm 0.0014\,\mu\text{s}$$

R [Ω] ± 0.01 [Ω]	96.95
f [Hz] ± 0.001 [Hz]	1000
ε [Vpp] ± 0.001 [Vpp]	4
C [nF] ± 0.01 [nF]	102.45
$T_{1/2}$ [μs] ± 0.001 [μs]	6.800

Table 1: $T_{1/2}$ measurement data for a RC series circuit.

$$\tau_{experiment} = \frac{T_{1/2}}{\ln 2} = 9.810 \pm 0.0014\,\mu\text{s}$$

The images printed in the experiment was shown in Figure 6.

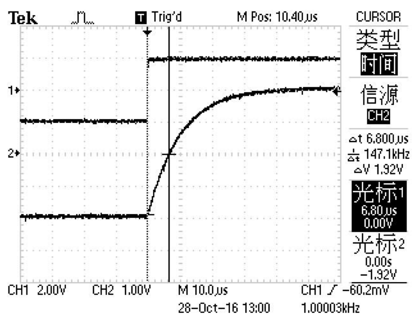


Figure 6: RC circuit.

The measurement of a RL series circuit was shown in Table 2.

R [Ω] ± 0.01 [Ω]	96.95
f [Hz] ± 0.001 [Hz]	1000
ε [Vpp] ± 0.001 [Vpp]	4
L [H] ± 0 [H]	0.01
$T_{1/2}$ [μs] ± 0.01 [μs]	68.00

Table 2: $T_{1/2}$ measurement data for a RL series circuit.

$$\tau_{theorem} = \frac{L}{R} = 103.15 \pm 0.01\,\mu\text{s}$$

$$\tau_{experiment} = \frac{T_{1/2}}{\ln 2} = 98.10 \pm 0.014\,\mu\text{s}$$

The images printed in the experiment was shown in Figure 7.

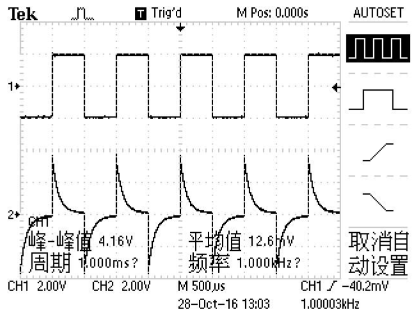


Figure 7: RL circuit.

4.2 RLC Series Circuit

The measurement of a RLC series circuit was shown in Table 3.

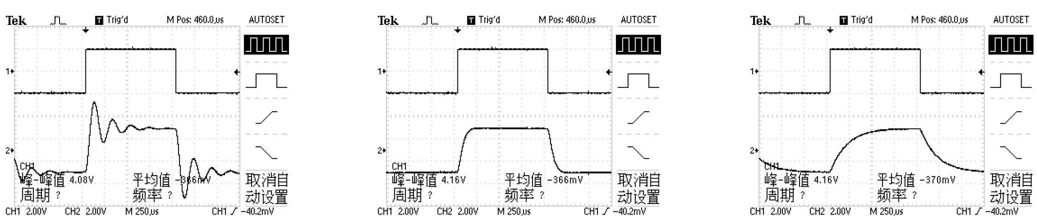
$L\ [H] \pm 0\ [H]$	0.01
$C\ [nF] \pm 0.01\ [nF]$	102.45
$f\ [Hz] \pm 0.001\ [Hz]$	500
$\varepsilon\ [Vpp] \pm 0.001\ [Vpp]$	4
$T_{1/2}\ [\mu s] \pm 0.01\ [\mu s]$	56.00

Table 3: $T_{1/2}$ measurement data for a critically damped RLC series circuit.

$$\tau_{theorem} = \frac{1}{\beta} = \sqrt{LC} = 32.078 \pm 0.006\,\mu\text{s}$$

$$\tau_{experiment} = \frac{T_{1/2}}{1.68} = 33.33 \pm 0.006\,\mu\text{s}$$

The images printed in the experiment were shown in Figure 8.



(a) Underdamped (b) Critical Damped (c) Overdamped

Figure 8: RLC circuit.

4.3 RLC Resonant Circuit

The measurement of a RLC resonant circuit was shown in Table 4.

R [Ω] $\pm$ 0.01 [Ω]		96.95	
L [H] $\pm$ 0 [H]		0.01	
C [nF] $\pm$ 0.01 [nF]		102.45	
f_0 [Hz] $\pm$ 0.001 [Hz]		5032	
ε [Vpp] $\pm$ 0.001 [Vpp]		4	
	U_R [mV] $\pm$ 10 [mV]	f [Hz] $\pm$ 0.001 [Hz]	ϕ [rad]
1	132	500.000	-1.539 ± 0.000
2	268	1000.000	-1.506 ± 0.000
3	416	1500.000	-1.468 ± 0.000
4	608	2000.000	-1.423 ± 0.000
5	816	2500.000	-1.365 ± 0.000
6	1160	3000.000	-1.285 ± 0.000
7	1580	3500.000	-1.162 ± 0.000
8	2320	4000.000	-0.955 ± 0.000
9	2680	4250.000	-0.793 ± 0.000
10	3160	4500.000	-0.572 ± 0.000
11	3600	4750.000	-0.287 ± 0.000
12	3800	5000.000	0.036 ± 0.000
13	3680	5250.000	0.337 ± 0.000
14	3360	5500.000	0.577 ± 0.000
15	2920	5750.000	0.754 ± 0.000
16	2680	6000.000	0.883 ± 0.000
17	2080	6500.000	1.051 ± 0.000
18	1680	7000.000	1.152 ± 0.000
19	1440	7500.000	1.219 ± 0.000
20	1280	8000.000	1.266 ± 0.000
21	1140	8500.000	1.302 ± 0.000
22	992	9000.000	1.329 ± 0.000

Table 4: Measurement data for the U_R vs. f dependence for a RLC resonant circuit.

The images analyzed were plotted in MATLAB and shown in Figure 9. and Figure 10. below.

The uncertainty of the two plots are too small so that it can't be displayed.

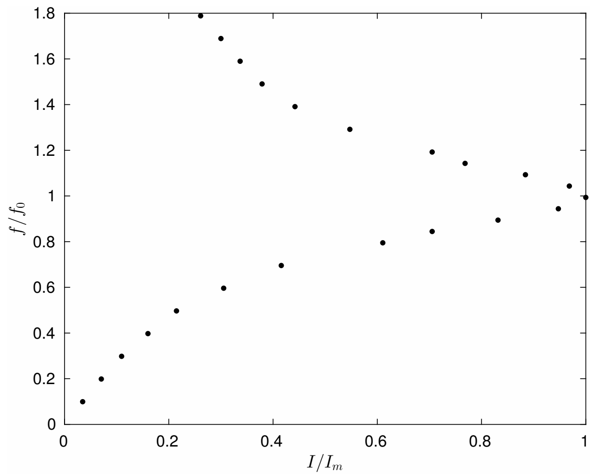


Figure 9: f/f_0 vs. I/I_m.

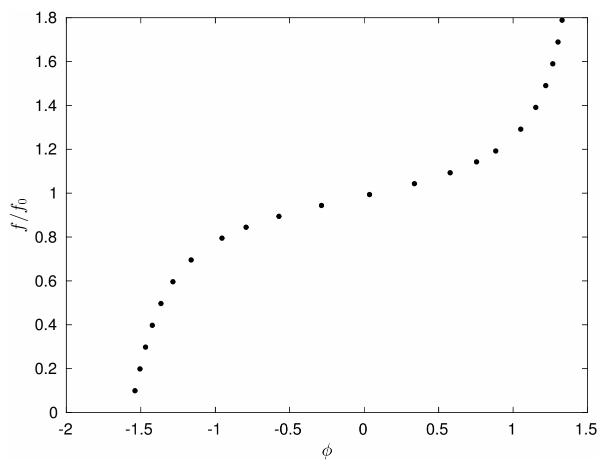


Figure 10: f/f_0 vs. ϕ.

$$f_0 = 5032\,\mathrm{Hz}$$
$$Q = \frac{\omega_0 L}{R} = 3.26 \pm 0.00$$

5 Measurement uncertainty analysis

5.1 theorem value of τ

The theorem value of τ in a RC series circuit can be calculated by the equation $\tau = RC$. Therefore its uncertainty u_τ is found by applying the uncertainty propagation formula

$$
\begin{aligned}
u_\tau &= \sqrt{\left(\frac{\partial \tau}{\partial R}\right)^2 u_R^2 + \left(\frac{\partial \tau}{\partial C}\right)^2 u_C^2} \\
&= \sqrt{C^2 u_R^2 + R^2 u_C^2} \\
&= 1.41 \times 10^{-3}\,\mu\text{s}
\end{aligned}
$$

The theorem value of τ in a RL series circuit can be calculated by the equation $\tau = \frac{L}{R}$. Therefore its uncertainty u_τ is found by applying the uncertainty propagation formula

$$
\begin{aligned}
u_\tau &= \sqrt{\left(\frac{\partial \tau}{\partial L}\right)^2 u_L^2 + \left(\frac{\partial \tau}{\partial R}\right)^2 u_R^2} \\
&= \sqrt{\left(\frac{1}{R}\right)^2 u_L^2 + \left(\frac{L}{R^2}\right)^2 u_R^2} \\
&= 1.064 \times 10^{-2}\,\mu\text{s}
\end{aligned}
$$

The theorem value of τ in a RLC series circuit can be calculated by the equation $\tau = \sqrt{LC}$. Therefore its uncertainty u_τ is found by applying the uncertainty propagation formula

$$
\begin{aligned}
u_\tau &= \sqrt{\left(\frac{\partial \tau}{\partial L}\right)^2 u_L^2 + \left(\frac{\partial \tau}{\partial C}\right)^2 u_C^2} \\
&= \sqrt{\left(\sqrt{\frac{C}{4L}}\right)^2 u_L^2 + \left(\sqrt{\frac{L}{4C}}\right)^2 u_C^2} \\
&= 6.248 \times 10^{-3}\,\mu\text{s}
\end{aligned}
$$

5.2 experimental value of τ

The uncertainty of experimental value of τ can be simply calculated by directed divide a coefficient.

u_τ in a RC circuit is $0.001/\ln 2 = 0.0014\,\mu$s

u_τ in a RL circuit is $0.01/\ln 2 = 0.014\,\mu$s

u_τ in a RLC circuit is $0.01/1.68 = 0.006\,\mu$s

5.3 the phase ϕ

The phase ϕ in a RLC resonant circuit can be calculated by the equation $\phi = tan^{-1}\left(\frac{\omega L - \frac{1}{\omega C}}{R}\right)$. Therefore its uncertainty u_ϕ is found by applying the uncertainty propagation formula

$$
\begin{aligned}
u_\phi &= \sqrt{\left(\frac{\partial \phi}{\partial L}\right)^2 u_L^2 + \left(\frac{\partial \phi}{\partial C}\right)^2 u_C^2 + \left(\frac{\partial \phi}{\partial R}\right)^2 u_R^2 + \left(\frac{\partial \phi}{\partial \omega}\right)^2 u_\omega^2} \\
&= \frac{1}{1+\left(\frac{\omega L - \frac{1}{\omega C}}{R}\right)^2} \sqrt{\left(\frac{\omega}{R}\right)^2 u_L^2 + \left(\frac{1}{\omega R C^2}\right)^2 u_C^2 + \left(\frac{\omega L - \frac{1}{\omega C}}{R^2}\right)^2 u_R^2 + \left(\frac{L}{R} - \frac{1}{\omega^2 R C}\right)^2 u_\omega^2}
\end{aligned}
$$

5.4 The quality factor Q

The theorem value of Q in a RL series circuit can be calculated by the equation $Q = \frac{\omega_0 L}{R}$. Therefore its uncertainty u_Q is found by applying the uncertainty propagation formula

$$
\begin{aligned}
u_Q &= \sqrt{\left(\frac{\partial Q}{\partial \omega_0}\right)^2 u_{\omega_0}^2 + \left(\frac{\partial Q}{\partial L}\right)^2 u_L^2 + \left(\frac{\partial Q}{\partial R}\right)^2 u_R^2} \\
&= \sqrt{\left(\frac{L}{R}\right)^2 u_{\omega_0}^2 + \left(\frac{\omega_0}{R}\right)^2 u_L^2 + \left(\frac{\omega_0 L}{R^2}\right)^2 u_R^2} \\
&= 3.36 \times 10^{-4}
\end{aligned}
$$

6 Conclusion

In this experiment, we understand the physics of alternating-current circuits, in particular the processes of charging/discharging of capacitors, the phenomenon of electromagnetic induction in inductive elements, and other dynamic processes in RC, RL, and RLC series circuits. Moreover, methods for measuring the amplitude-frequency and the phase-frequency characteristics of RC, RL, and RLC series circuits will be studied. The resonance frequency of a RLC circuit as well as the quality factor of the circuit will be found from the amplitude-frequency curve.

In a RC series circuit, we can find

$$\tau = RC$$

In a RL series circuit, we can find

$$\tau = \frac{L}{R}$$

In a RLC series circuit critical damped , we can find

$$\tau = \sqrt{LC}$$

Then we discovered the RLC resonant circuit, which is similar to a simple harmonic oscillation. We plotted the experiment results in certain method so that we can find the property of it. We also calculated the The quality factor Q within the theorem.

7 Reference

(a) Qin Tian, Feng Yaming, Mateusz Krzyzosiak, VP241 Exercise 5, RC, RL, and RLC Circuits, based on materials provided by the Department of Physics, Shanghai Jiaotong University.

8 Data sheet

The Data sheet is attached at the end of the report.