VP260 PROBLEM SET 3

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Problem 1.

(a)

$$E = \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{a^2}(\hat{n_x}+\hat{n_y})+\frac{1}{2a^2}\frac{\sqrt{2}}{2}(\hat{n_x}+\hat{n_y})\right] = \frac{Q(4+\sqrt{2})}{16\pi\varepsilon_0 a^2}(\hat{n_x}+\hat{n_y})$$

(b)

$$U = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{a}\cdot 2+\frac{1}{\sqrt{2}a}\right) = \frac{Q(4+\sqrt{2})}{8\pi\varepsilon_0 a}$$

(c)

$$W = Q(U-0) = \frac{Q^2(4+\sqrt{2})}{8\pi\varepsilon_0 a}$$

(d)

$$W = 2\cdot\frac{Q^2}{4\pi\varepsilon_0 a}+\frac{Q^2}{4\pi\varepsilon_0\sqrt{2}a} = \frac{Q^2(4+\sqrt{2})}{8\pi\varepsilon_0 a}$$

Problem 2.

(a)

$$q = \int\rho dV = \int_a^b 4\pi r^2\rho dr = \int_a^b 4k\pi r dr = 2k\pi(b^2-a^2)$$

(b) (i)

$$\bar{E} = 0$$

(ii)

$$E\cdot 4\pi r^2 = \frac{2k\pi(r^2-a^2)}{\varepsilon_0}$$

$$E = \frac{k(r^2-a^2)}{2\varepsilon_0 r^2}$$

$$\bar{E} = \frac{k(r^2-a^2)}{2\varepsilon_0 r^3}\bar{r}$$

(iii)

$$E\cdot 4\pi r^2 = \frac{2k\pi(b^2-a^2)}{\varepsilon_0}$$

$$E = \frac{k(b^2-a^2)}{2\varepsilon_0 r^2}$$

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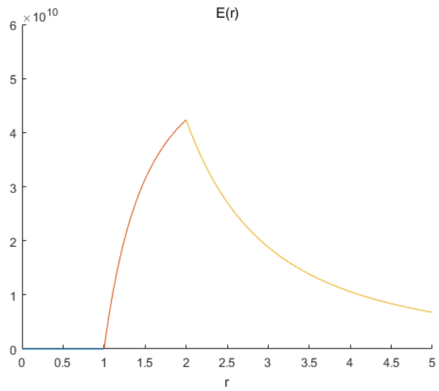
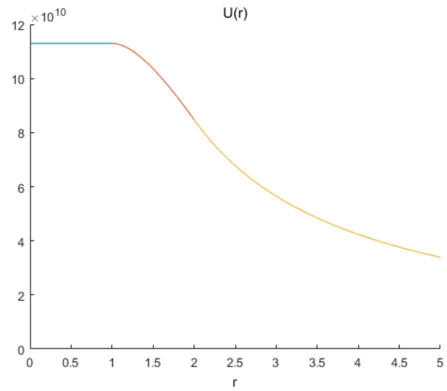
(c)



$E(r)$



$U(r)$

Figure 1: $E(r)$ and $U(r)$

(d) When $r \geqslant b$,

$$U(r) = \int_r^{\infty} \frac{k(b^2 - a^2)}{2\varepsilon_0 r^2} dr = \frac{k}{2\varepsilon_0 r}(b^2 - a^2)$$

When $a \leqslant r < b$,

$$U(r) = \int_r^{b} \frac{k(r^2 - a^2)}{2\varepsilon_0 r^2} dr + U(b) = \frac{k}{2\varepsilon_0 r}(2br - r^2 - a^2)$$

When $r < a$,

$$U(r) = U(a) = \frac{k}{\varepsilon_0}(b - a)$$

Problem 3.

$$E(r) \cdot 2\pi r l = \frac{\lambda l}{\varepsilon_0}$$

$$E(r) = \frac{\lambda}{2\pi r \varepsilon_0}$$

Choose a point of distance R away from the wire as the reference point.

$$U = \int_s^R \frac{\lambda}{2\pi r \varepsilon_0} dr = \frac{\lambda}{2\pi\varepsilon_0} ln\frac{R}{s}$$

Suppose the wire to be x-axis and the line of distance from the wire and the point to be the y-axis.

$$\nabla U = -\frac{\lambda}{2\pi s \varepsilon_0}\hat{n_y}$$

$$\bar{E} = -\nabla U = \frac{\lambda}{2\pi s \varepsilon_0}\hat{n_y}$$

So it yields the correct field.

Problem 4.

$$U_A = \int_0^h \frac{1}{4\pi\varepsilon_0} \cdot \frac{\sigma \cdot 2\pi r\sqrt{2}dr}{\sqrt{2}r} = \frac{\sigma h}{2\varepsilon_0}$$

$$\begin{aligned}
U_B &= \int_0^h \frac{1}{4\pi\varepsilon_0} \cdot \frac{\sigma \cdot 2\pi r\sqrt{2}dr}{\sqrt{(h-r)^2+r^2}} \\
&= \frac{\sqrt{2}\sigma}{2\varepsilon_0}\left(\frac{1}{2}\sqrt{(h-r)^2+r^2} + \frac{h}{2\sqrt{2}} ln|4r - 2h + 2\sqrt{2}\sqrt{(h-r)^2+r^2}|\right)\Bigg|_0^h \\
&= \frac{\sqrt{2}\sigma}{2\varepsilon_0}\left(\frac{h-h}{2} + \frac{h}{2\sqrt{2}} ln\left|\frac{2h+2\sqrt{2}h}{-2h+2\sqrt{2}h}\right|\right) \\
&= \frac{\sigma h}{2\varepsilon_0} ln(1+\sqrt{2})
\end{aligned}$$

$$\Delta U = U_B - U_A = \frac{\sigma h}{2\varepsilon_0}[ln(1+\sqrt{2}) - 1]$$

Problem 5.

$$\begin{aligned}
q &= \int \rho dV = \int_0^r 4\pi r^2 \rho dr = \int_0^r -\frac{4\mathbf{e}r^2}{a^3} e^{-\frac{2r}{a}} dr = \frac{e}{2}\int_0^{-\frac{2r}{a}} u^2 e^u du \\
&= \frac{\mathbf{e}}{2}(u^2 - 2u + 2)e^u\Big|_0^{-\frac{2r}{a}} = \frac{\mathbf{e}}{2}\left[\left(\frac{4r^2}{a^2} + \frac{4r}{a} + 2\right)e^{-\frac{2r}{a}} - 2\right] \\
&= \mathbf{e}\left[\left(\frac{2r^2}{a^2} + \frac{2r}{a} + 1\right)e^{-\frac{2r}{a}} - 1\right]
\end{aligned}$$

$$E \cdot 4\pi r^2 = \frac{q}{\varepsilon_0}$$

$$\bar{E} = \frac{\mathbf{e}}{4\pi\varepsilon_0}\left[\left(\frac{2}{a^2} + \frac{2}{ar} + \frac{1}{r^2}\right)e^{-\frac{2r}{a}} - \frac{1}{r^2}\right]\hat{n_r}$$

$$V = \int_r^\infty E dr = \frac{\mathbf{e}}{4\pi\varepsilon_0}\left(-\frac{a+r}{ar}e^{-\frac{2r}{a}} + \frac{1}{r}\right)\Bigg|_r^\infty = \frac{\mathbf{e}}{4\pi r\varepsilon_0}\left(\frac{a+r}{a}e^{-\frac{2r}{a}} - 1\right)$$

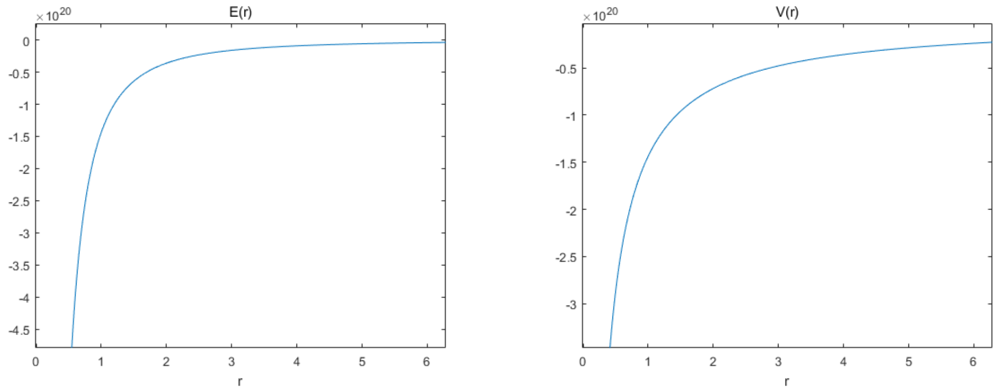
When $r \to 0$, $E \to -\infty$, $V \to -\infty$
When $r \to \infty$, $E \to 0$, $V \to 0$
They are similar to a point charge $-e$ placed at the origin.

Problem 6.

(a)

$$V = \int_r^R \frac{q}{4\pi r^2\varepsilon_0}\frac{r^3}{R^3}dr + \int_R^\infty \frac{q}{4\pi r^2\varepsilon_0}dr = \frac{q}{4\pi\varepsilon_0}\left[\frac{\frac{1}{2}(R^2-r^2)}{R^3} + \frac{1}{R}\right] = \frac{q}{8\pi R^3\varepsilon_0}(3R^2 - r^2)$$

$$U_{conf} = \frac{1}{2}\int_\Omega \frac{q}{\frac{4}{3}\pi R^3}\frac{q}{8\pi R^3\varepsilon_0}(3R^2 - r^2)d\tau = \frac{3q^2}{8\pi R^6\varepsilon_0}\int_0^R r^2(3R^2 - r^2)dr = \frac{3q^2}{20\pi R\varepsilon_0}$$



(a) $E(r)$ (b) $V(r)$

Figure 2: $E(r)$ and $V(r)$

(b)

$$
\begin{aligned}
U_{conf} &= \frac{\varepsilon_0}{2}\left(\int_{\Omega_1} E_1^2 d\tau + \int_{\Omega_2} E_2^2 d\tau\right) \\
&= \frac{\varepsilon_0}{2}\left[\int_0^R \left(\frac{q}{4\pi r^2\varepsilon_0}\frac{r^3}{R^3}\right)^2 4\pi r^2 dr + \int_R^\infty \left(\frac{q}{4\pi r^2\varepsilon_0}\right)^2 4\pi r^2 dr\right] \\
&= \frac{q^2}{8\pi\varepsilon_0}\left(\int_0^R \frac{r^4}{R^6} dr + \int_R^\infty \frac{1}{r^2} dr\right) \\
&= \frac{3q^2}{20\pi R\varepsilon_0}
\end{aligned}
$$

(c)

$$
\begin{aligned}
U_{conf} &= \frac{\varepsilon_0}{2}\left(\int_{\Omega} E^2 d\tau + \oint_{\Sigma} VEdA\right) \\
&= \frac{\varepsilon_0}{2}\left[\int_0^R \left(\frac{q}{4\pi r^2\varepsilon_0}\frac{r^3}{R^3}\right)^2 4\pi r^2 dr + \int_R^\infty \frac{q}{4\pi r^2\varepsilon_0} dr \frac{q}{4\pi a^2\varepsilon_0}\cdot 4\pi a^2\right] \\
&= \frac{q^2}{8\pi\varepsilon_0}\left(\int_0^R \frac{r^4}{R^6} dr + \int_R^\infty \frac{1}{r^2} dr\right) \\
&= \frac{3q^2}{20\pi R\varepsilon_0}
\end{aligned}
$$

When $a \to \infty$, $\oint_\Sigma VEdA \to 0$, $U_{conf} = \frac{\varepsilon_0}{2}\int_{all\ space} E^2 d\tau$

(d)

$$Q = \frac{qr^3}{R^3}$$

$$dQ = \frac{3qr^2 dr}{R^3}$$

$$\int \frac{Q}{4\pi r\varepsilon} dQ = \int_0^R \frac{3q^2r^4}{4\pi R^6\varepsilon_0} dr = \frac{3q^2}{20\pi R\varepsilon_0}$$

Problem 7.

$$q = 0$$
$$\nabla^2 V = 0$$

Since V is specified on the boundary of Ω, the solution to the equation is unique.
Only when V is a constant, the solution is unique.
So the electric potential inside is constant, which means $E = 0$ there.

Problem 8.

(a)

$$\begin{aligned}
V &= \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{\sqrt{x^2+y^2+(z-d)^2}} - \frac{1}{\sqrt{x^2+y^2+(z+d)^2}}\right] \\
\sigma &= -\varepsilon_0 \frac{\partial V}{\partial z}\bigg|_{z=0} \\
&= -\frac{q}{4\pi}\{(2z+d)[x^2+y^2+(z+d)^2]^{-\frac{3}{2}} - (2z-d)[x^2+y^2+(z-d)^2]^{-\frac{3}{2}}\}\bigg|_{z=0} \\
&= -\frac{qd}{2\pi}(r^2+d^2)^{-\frac{3}{2}}
\end{aligned}$$

(b)

$$q_i = \int_0^{2\pi}\int_0^{\infty} \sigma r dr d\theta = -qd\int_0^{\infty} \frac{r}{(r^2+d^2)^{\frac{3}{2}}} dr = qd\frac{1}{\sqrt{r^2+d^2}}\bigg|_0^{\infty} = -q$$

Problem 9.

Suppose the line from the center of the ball to the point charge to be the z-axis and the plane perpendicular to be the xy-plane.

$$r = \sqrt{x^2+y^2+(z-d)^2}$$
$$r' = \sqrt{x^2+y^2+(z-d')^2}$$
$$\frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r} + \frac{q'}{r'}\right) = 0$$

$$\left\{\begin{array}{cc} (R^2+d^2)q'^2 - (R^2+d'^2)q^2 & = 0 \\ 2R(dq'^2 - d'q^2) & = 0 \end{array}\right. \Rightarrow \left\{\begin{array}{ll} q' & = -\frac{R}{d}q \\ d' & = \frac{R^2}{d} \end{array}\right.$$

$$V(x,y,z) = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{\sqrt{x^2+y^2+(z-d)^2}} - \frac{R}{d\sqrt{x^2+y^2+(z+\frac{R^2}{d})^2}}\right]$$

Problem 10.

Suppose the line from the center of the ball to the point charge to be the z-axis and the plane perpendicular to be the xy-plane.
According to Problem 9,

$$\begin{cases} q' &= -\frac{R}{d}q \\ d' &= \frac{R^2}{d} \end{cases}$$

Since the ball in ungrounded, we need another q'' to cancel out electric potential on the ball.

$$q'' = -Q - q' = -Q + \frac{R}{d}q$$

$$r'' = \sqrt{x^2 + y^2 + z^2}$$

$$\begin{aligned} V(x, y, z) &= \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r} + \frac{q'}{r'} + \frac{q''}{r''}\right) \\ &= \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{\sqrt{x^2 + y^2 + (z-d)^2}} - \frac{qR}{d\sqrt{x^2 + y^2 + (z + \frac{R^2}{d})^2}} + \frac{-Q + \frac{R}{d}q}{\sqrt{x^2 + y^2 + z^2}}\right] \end{aligned}$$

Problem 11.

(a) We need three charges.
One point charge q at $(-b, -a)$
Two point charge $-q$ at $(-b, a)$ and $(b, -a)$

(b)

$$\begin{aligned} F &= \frac{q^2}{4\pi\varepsilon_0}\left(-\frac{1}{4b^2}\hat{n_x} - \frac{1}{4a^2}\hat{n_y} + \frac{1}{4a^2 + 4b^2}\frac{b\hat{n_x} + a\hat{n_y}}{\sqrt{a^2 + b^2}}\right) \\ &= \frac{q^2}{4\pi\varepsilon_0}\left[\left(\frac{b}{4(a^2 + b^2)^{\frac{3}{2}}} - \frac{1}{4b^2}\right)\hat{n_x} + \left(\frac{a}{4(a^2 + b^2)^{\frac{3}{2}}} - \frac{1}{4a^2}\right)\hat{n_y}\right] \end{aligned}$$

(c)

$$V = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{2\sqrt{a^2 + b^2}} - \frac{1}{2a} - \frac{1}{2b}\right)$$

$$W = \frac{q^2}{8\pi\varepsilon_0}\left(\frac{1}{\sqrt{a^2 + b^2}} - \frac{1}{a} - \frac{1}{b}\right)$$

(d) No, the method works only when the angle $\theta = \frac{\pi}{k}, k \geq 2, k \in Z$
It is because the point charge should be imaged $k + 2$ times and then the next image is the charge itself. If k is not a normal number, it doesn't work.