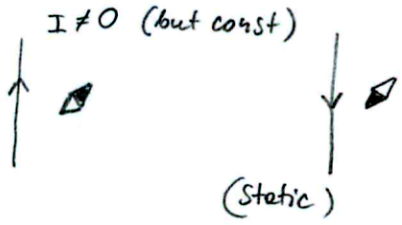
MAGNETISM

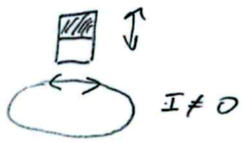
Introduction

* Oersted's experiment
H. Ch. Oersted (1820)



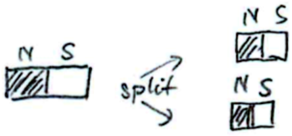
DEMONSTRATION 1

* Faraday's experiment
(M. Faraday, J. Henry)



(dynamic)

* permanent magnets
(magnetic material, Earth)



(no magnetic monopoles!)

* two wires carrying electric current attract / repel each other

MAGNETIC FIELD (vector field)

static → magnetostatics

dynamic → electrodynamics (Maxwell's eqns – four equations unifying electric and magnetic interactions)

Magnetic field vs. electric field

electric field (static)

→ static electric charges ⇒ static electric field

→ electric force $\bar{F} = q\bar{E}$ acts on any other charge placed in the field

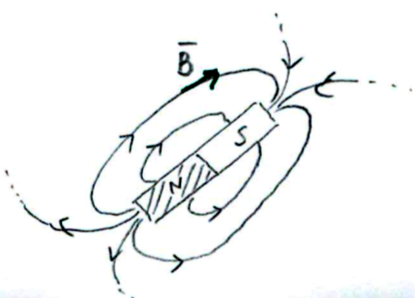
magnetic field

→ moving charges (currents) ⇓ magnetic field

→ magnetic force acts on any other moving charge (current) present in the field

Examples

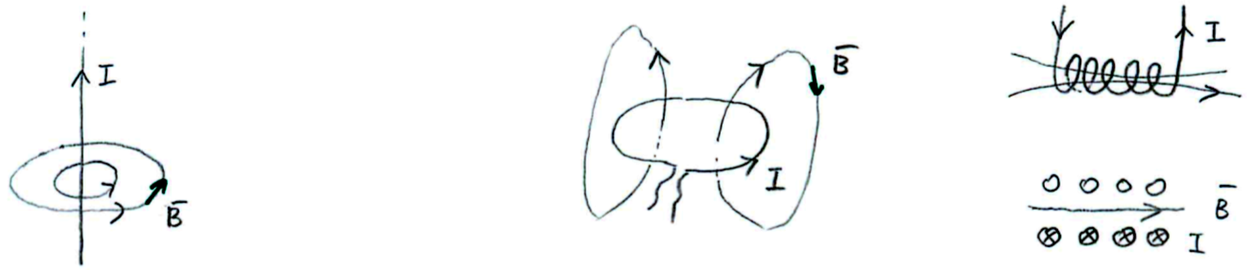
(a) permanent magnet



at any point, the magnetic field vector $\bar{B}$ is tangential to a field line

DEMONSTRATION 2

(b) currents



Straight wire

Current loop

coil (solenoid)

Fig. 1 (27.14)

Observation: Magnetic field lines always form loops!

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Magnetic forces on moving charges

Experimental facts about the force on a charge moving in magnetic field

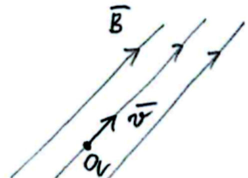
(1) magnitude of the force $\propto$ charge q

(2) — " — — " — — " — $\propto$ magnitude of the magnetic field $|\bar B|$

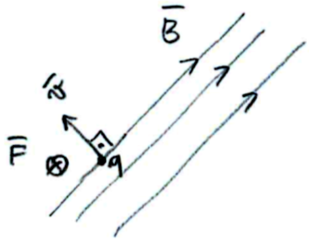
(3) — " — — " — — " — $\propto$ speed of the charge

(4) depends on the direction of motion (w.r.t. $\bar B$)

(here $q>0$)

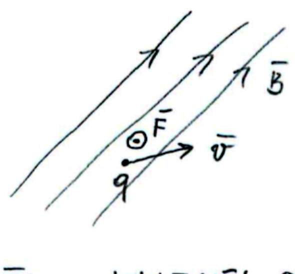


$\bar v \parallel \bar B \Rightarrow \bar F = 0$



$\bar F \perp \bar v$ and $\bar F \perp \bar B$

$\bar v \perp \bar B \Rightarrow |\bar F| = \max = |q|\,|\bar v|\cdot|\bar B|$

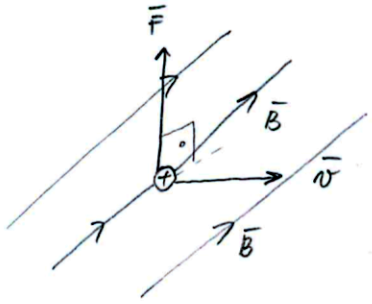


$\bar F \perp \bar v$
$\bar F \perp \bar B$

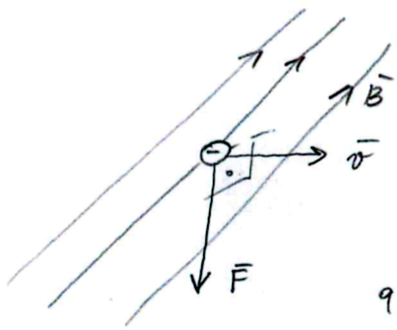
$$|\bar F| = |q|\,|\bar v|\,|\bar B| \sin\sphericalangle(\bar v, \bar B) = |q|\,|\underbrace{\bar v_\perp}_{\text{component } \perp \bar B}|\,|\bar B| = |q|\,|\bar v|\cdot|\underbrace{\bar B_\perp}_{\text{component } \perp \text{ to } \bar v}|$$

$\sphericalangle(\bar v, \bar B)$: angle between $\bar v, \bar B$ (the smaller one)

Direction of $\bar{F}$ depends on the sign of the charge



$q>0$



$q<0$

(left-hand rule for $\bar{v}, \bar{B}, \bar{F}$)

$$\boxed{\bar{F} = q\,\bar{v} \times \bar{B}}$$

magnetic force on a moving charge (Lorentz force)

Units of $\bar{B}$

$$\frac{N}{C \cdot m/s} = \frac{N \cdot \overbrace{s}^{}}{\underbrace{C}_{} \cdot m}^{= A^{-1}} = \frac{N}{A \cdot m} = T \quad (\text{tesla})$$

other units : GAUSS $\quad 1\,G = 10^{-4}\,T$

Note. $\quad \bar{F} = q\,\bar{v} \times \bar{B} \Rightarrow \bar{F} \perp \bar{v} \Rightarrow \bar{F}$ cannot change the magnitude of $\bar{v}$
$(v \ll c)$

Conclusion: Motion of a charged particle acted upon only by the magnetic force is always motion with constant speed

Formally: $\quad \bar{F} \perp \bar{v} \Rightarrow K = \frac{1}{2} m v^2 = \text{const.}$
$(v \ll c)$

$$\frac{dK}{dt} = \frac{d}{dt}\left(\frac{1}{2} m\,\bar{v}\cdot\bar{v}\right) = \frac{1}{2} m \left(\dot{\bar{v}}\cdot\bar{v} + \bar{v}\cdot\dot{\bar{v}}\right) = \underbrace{m\,\dot{\bar{v}}}_{\bar{F}}\cdot\bar{v} = \bar{F}\cdot\bar{v} = 0$$

because $\bar{F} \perp \bar{v}$
$(\bar{F}\cdot\bar{v} = 0)$

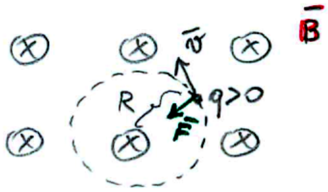
Final comment: If both fields $(\bar{E}, \bar{B})$ are present, then

$$\boxed{\bar{F} = q\left(\bar{E} + \bar{v} \times \bar{B}\right)}$$

Examples

(a) cyclotron motion

$\bar{B} \neq 0$ and $\bar{v} \perp \bar{B}$



trajectory – circle

Lorentz force $\equiv$ centripetal force

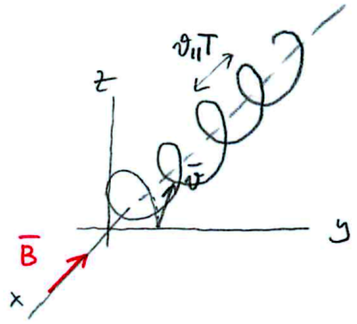
$$\frac{m v^2}{R} = |q| v B \implies v = \frac{|q| B}{m} R = \omega R$$

$\omega = \frac{|q|B}{m}$ cyclotron frequency

(b) motion along a helix

$\bar{B} \neq 0$ and $\bar{v} \not\perp \bar{B}$

$\bar{v} \not\parallel \bar{B}$

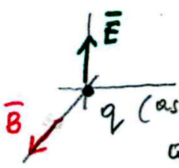


$\bar{v}_{\parallel}$ – component of $\bar{v}$ parallel to $\bar{B}$

(c) cycloid motion

$\left.\begin{array}{l}\bar{B} \neq 0 \\ \bar{E} \neq 0\end{array}\right\}$ both fields are present

$$\bar{E} = (0, 0, \underset{\text{const}}{E}) \qquad \bar{B} = (\underset{\text{const}}{B}, 0, 0)$$



(assume $q>0$)

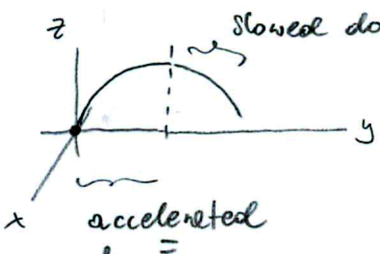
at $t=0$

$\bar{r}(0) = 0$

$\bar{v}(0) = 0$

Qualitative discussion : $t=0 \implies$ only electric force ($\bar{v}=0$); instantaneous acceleration in z-direction

$t>0 \implies \bar{v} \neq 0$; $q(\bar{v} \times \bar{B}) \neq 0 \implies$ deflection



curved by $\bar{B}$

Quantitative analysis : needs to solve equ. of motion + initial cond.

$$\begin{cases} m\bar{a} = q(\bar{E} + \bar{v} \times \bar{B}) \\ \bar{r}(0) = 0 \, ; \quad \bar{v}(0) = 0 \end{cases}$$

Observation : no motion in x-direction ($v_x(0) = 0$ and $a_x \equiv 0$)

$\Downarrow$

$x(t) \equiv 0$

Velocity $\bar{v} = (0, \dot{y}, \dot{z})$

Force $\bar{F} = q(\bar{E} + \bar{v} \times \bar{B}) = qE\hat{n}_z + q \begin{vmatrix} \hat{n}_x & \hat{n}_y & \hat{n}_z \\ 0 & \dot{y} & \dot{z} \\ B & 0 & 0 \end{vmatrix} =$

$= qE\hat{n}_z + q\dot{z}B\hat{n}_y - q\dot{y}B\hat{n}_z =$

$= q\dot{z}B\hat{n}_y + q(E - \dot{y}B)\hat{n}_z$

Equation of motion (y, z - components)

$$\begin{cases} \ddot{y} = \frac{qB}{m}\dot{z} \\ \ddot{z} = \frac{q}{m}(E - \dot{y}B) \end{cases} + \text{ initial conditions } \quad \begin{matrix} y(0) = z(0) = 0 \\ \dot{y}(0) = \dot{z}(0) = 0 \end{matrix}$$

$\hookrightarrow$ system of coupled ODEs (linear); may use standard tools; here use a trick

Rewrite in terms of v_y, v_z

$$\begin{cases} \dot{v}_y = \frac{qB}{m} v_z \\ \dot{v}_z = \frac{q}{m}(E - v_y B) \end{cases} \overset{\omega = \frac{qB}{m}}{\Longleftrightarrow} \begin{cases} \dot{v}_y = \omega v_z \quad (**) \\ \dot{v}_z = \omega\left(\frac{E}{B} - v_y\right) \end{cases}$$

Idea: differentiate the former, plug $\dot{v}_z$ into the latter

$$\ddot{v}_y = \omega \dot{v}_z \quad \text{and} \quad \ddot{v}_y = \omega^2\left(\frac{E}{B} - v_y\right)$$

Hence

$$\boxed{\ddot{v}_y + \omega^2 v_y = \omega^2 \frac{E}{B}} \quad (*)$$

single 2nd order ODE (nonhomogeneous)

Complementary homogeneous eqn.

$$\ddot{v}_y + \omega^2 v_y = 0 \quad \Longrightarrow \quad v_y^{(h)} = \tilde{C}_1 \cos\omega t + \tilde{C}_2 \sin\omega t$$

(general sol. of the homogeneous eqn.)

$\hookrightarrow$ cf. harmonic oscillator

A particular solution of non-homogeneous eqn — easy to guess

$$v_y^{(p)} = \frac{E}{B}$$

The general solution of $(*)$

$$\boxed{v_y(t) = \tilde{C}_1 \cos\omega t + \tilde{C}_2 \sin\omega t + \frac{E}{B}}$$

Position (integrate v_y)

$$\boxed{y(t) = C_1 \sin\omega t - C_2 \cos\omega t + \frac{E}{B} t + C_3} \qquad \left(\frac{\tilde{C_i}}{\omega} = C_i \right)$$

To find z, use $(**)$, $\dot{v}_y = \omega v_z$

$$\dot{z} = \frac{1}{\omega} \ddot{y} \Rightarrow \dot{z} = -C_1 \omega \sin\omega t + C_2 \omega \cos\omega t$$

Hence

$$\boxed{z(t) = C_1 \cos\omega t + C_2 \sin\omega t + C_4}$$

Apply initial conditions

$$\begin{cases} y(0) = -C_2 + C_3 = 0 \\ z(0) = C_1 + C_4 = 0 \end{cases} \quad \text{and} \quad \begin{cases} \dot{y}(0) = C_1\omega + \frac{E}{B} = 0 \\ \dot{z}(0) = C_2 \omega = 0 \end{cases}$$

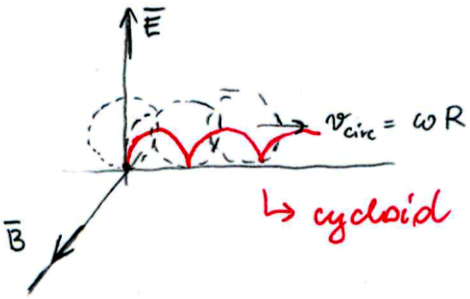
Hence $C_2 = 0 = C_3$, $C_1 = -\frac{E}{\omega B} = -C_4$ and

$$\begin{cases} x(t) \equiv 0 \\ y(t) = \frac{E}{\omega B}(\omega t - \sin\omega t) = R(\omega t - \sin\omega t) \\ z(t) = \frac{E}{\omega B}(1 - \cos\omega t) = R(1 - \cos\omega t) \end{cases}$$

where $R = \frac{E}{\omega B}$. Eliminate $\sin\omega t$, $\cos\omega t$ using $\sin^2\omega t + \cos^2\omega t = 1$

$$(y - R\omega t)^2 + (z - R)^2 = R^2$$

Trajectory of a point on the rim of a circle rolling without slipping down the y-axis with angular velocity ω



→ overall motion of the charge

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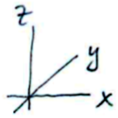
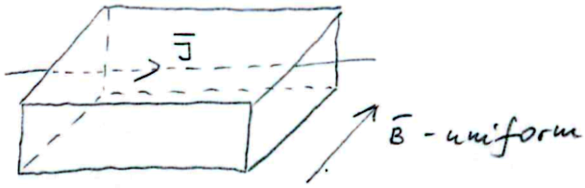
Lorentz force - applications: magnetic trap (fig. 2); bubble chamber (fig. 3); velocity selector (fig. 4); e/m experiment (fig. 5) mass - spectrometer (fig. 6)

The Hall effect

conducting solid





$\bar{J} = (J_x, 0, 0)$

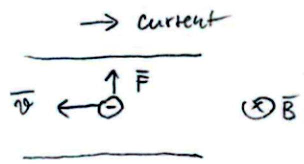
$\bar{B} = (0, B_y, 0)$ $\quad J_x, B_y > 0$

xz cross-section

$q > 0$

$\otimes\ \bar{B}$

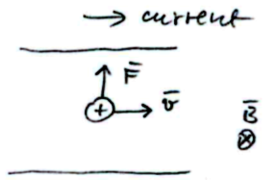
negative charge carriers $\qquad q, v_d > 0 \qquad$ positive charge carriers



$\bar{v} = (-v_d, 0, 0)$

$\bar{F} = -q(\bar{v} \times \bar{B}) =$

$= (0, 0, q v_d B_y)$

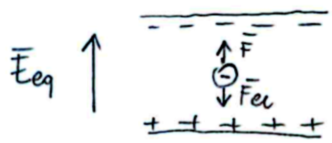


$\bar{v} = (v_d, 0, 0)$

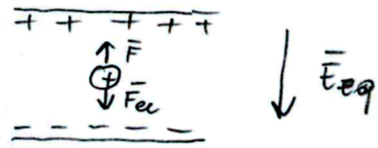
$\bar{F} = q\, \bar{v} \times \bar{B} =$

$= (0, 0, q v_d B_y)$

"equilibrium" free-body diagrams



note the direction of $\bar{E}_{eq}$ in both cases!



Equilibrium condition

$$\bar{F}_{el} + \bar{F} = 0$$

$$q E_{eq} = q v_d B_y \quad \text{(magnitude)}$$

$$\boxed{E_{eq} = v_d B_y}$$

in the positive z-direction for negative charges

in the negative z-direction for positive charges

Conclusions

Can be used to determine whether (+) or (−) charges carry the current.

Can be used to measure magnetic field (Hall probe); need to know concentration of charges

$$J_x = n q v_d$$

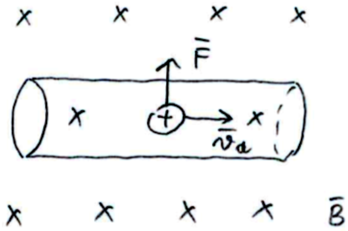
$$nq = \frac{J_x}{v_d} = \frac{J_x}{E_{eq}} B_y \Rightarrow \boxed{B_y = \frac{nq E_{eq}}{J_x}}$$

Magnetic force on a current-carrying conductor

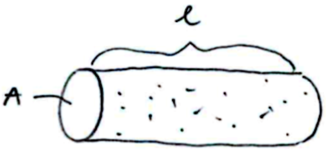
Starting point $\quad \bar{F} = q\,\bar{v} \times \bar{B}$

Assume positive charges, all move with $\bar{v}_d$ along a straight-line conductor

$\bar{B} \perp \bar{J}$
↓
uniform



force on a single charge
↓ add all contributions
total force

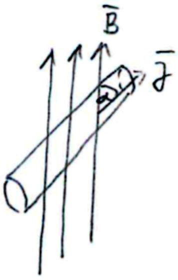


\# of charges $= n \cdot l \cdot A$

total force = # of charges · force on a single charge

$$F = (n \cdot l \cdot A)\, q v_d B = \underbrace{(n q v_d)}_{I} A\, l\, B = I l B$$

$\bar{B} \not\perp \bar{J}$



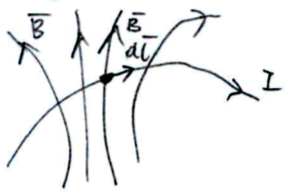
$$F = (nlA)\, q\, v_d\, B \sin\sphericalangle(\bar{J}, \bar{B}) = I l B \sin\sphericalangle(\bar{J}, \bar{B})$$

↙ not a vector

Let $\bar{l}$ point in the current's direction

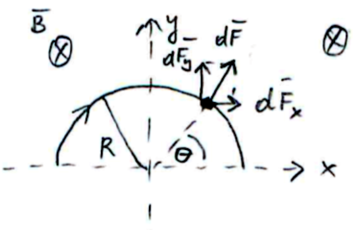
$$\boxed{\bar{F} = I\, \bar{l} \times \bar{B}}$$

If $\bar{B}$ - not uniform or conductor - not a straight line



$$\bar{F} = \int_{\text{Conductor}} I\, d\bar{l} \times \bar{B}$$

Example.



⊗ $\bar{B}$ - uniform

$$dF_x = \cos\theta\, dF$$
$$dF_y = \sin\theta\, dF$$
$$|d\bar{F}| = I\,|d\bar{l} \times \bar{B}| \overset{d\bar{l} \perp \bar{B}}{=\!=} I\,B\,dl$$

$$F_y = \int_{\text{Conductor}} dF_y = \int_{\text{Conductor}} \sin\theta\, I B\, dl = \int_0^{\pi} I R\, B \sin\theta\, d\theta =$$
$$= IBR \cdot 2$$

$$d\theta = \frac{dl}{R}$$
$$dl = R\, d\theta$$

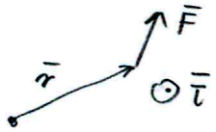
$F_x = 0$ (same method, or symmetry)

Current loop in a uniform magnetic field

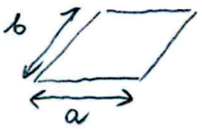
$\boxed{\text{FiG}}$ 29.31

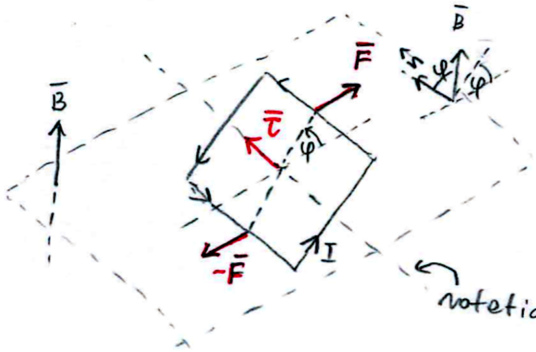
The net force is <u>zero</u>.

Torque $\bar{\tau} = \bar{r} \times \bar{F}$

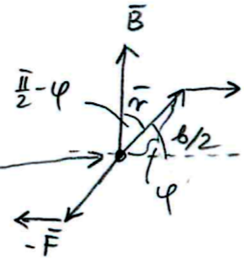


(always with respect to a point/axis!)

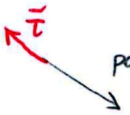




$\varphi = \sphericalangle(\bar{B}, \hat{n})$



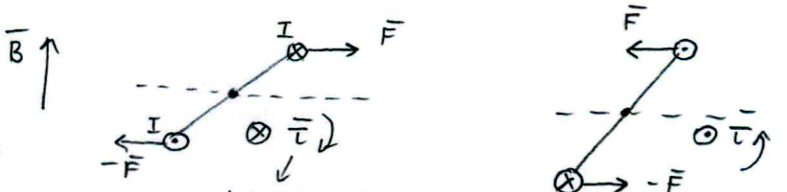
$$\tau = 2 \cdot \frac{b}{2} F \sin\varphi = F b \sin\varphi \overset{F = IaB}{=\!=} IaBb\sin\varphi$$



positive direction of $\bar{\varphi}$

$|\tau| = |I a B \sin\varphi|$

- max for $\varphi = \pi/2, \; \frac{3}{2}\pi$
- min for $\varphi = 0, \pi$
  - $\varphi = 0$: stable equilibrium
  - $\varphi = \pi$: unstable equilibrium



tries to restore the equilibrium $\varphi = 0$

rotates out of $\varphi = \pi$ equilibrium

$$\tau = I \underbrace{(a \cdot b)}_{\text{area } A} B \sin\varphi$$

Define $\mu = IA$ – magnetic dipole moment

$\tau = \mu B \sin\varphi$ or vector form $\boxed{\bar{\tau} = \bar{\mu} \times \bar{B}}$

Right-hand rule for current loop

$\bar{\mu} = (IA)\,\hat{n}$

Compare: electric dipole moment

Potential energy of a magnetic dipole (in a uniform magnetic field)

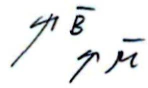
$$U = -\bar{\mu}\bar{B} = -\mu B \cos\sphericalangle(\bar{\mu},\bar{B})$$ → cf. derivation for the electric dipole

$$\boxed{U = -\bar{\mu}\cdot\bar{B}}$$

we have chosen $U(\frac{\pi}{2}) = 0$, ie when $\bar{B} \perp \bar{\mu}$

$$U(\varphi) - \underbrace{U(\tfrac{\pi}{2})}_{=0} = -\mu B \cos\varphi$$

Comments:
- → stable equilibrium (min. of U) if $\varphi = 0$
- → unstable equilibrium (max. of U) if $\varphi = \pi$
- → energetically preferred orientation: $\bar{\mu} \parallel \bar{B}$



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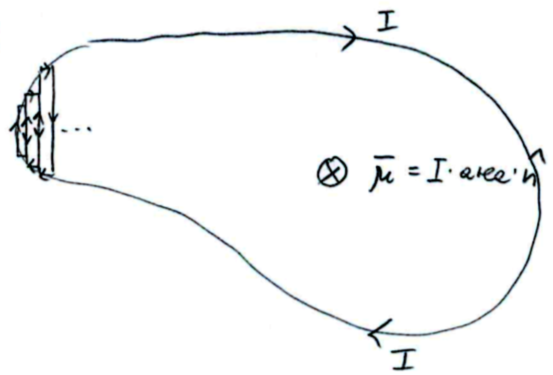
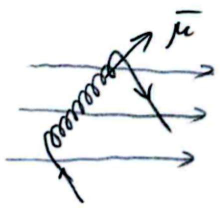
The discussion is also valid for other geometries

arbitrarily-shaped loop



treat the irregular loop as a collection of infinitesimal rectangles (currents through common sides cancel)

solenoid (coil)



collection of N loops of area A

$$\boxed{\bar{\mu} = N \cdot I A \hat{n}}$$ ↳ right-hand rule

~ o ~

Magnetic dipole in a non-uniform field

(1) $\bar{\mu}$ in the direction opposite to the field

net force →

net components

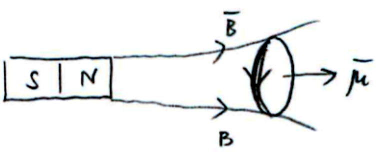
$F_\perp = 0$

$F_\parallel \neq 0$ (to the right)

⇓

repulsion

(2) $\bar{\mu}$ in the direction of the field



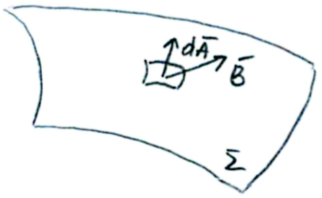
net force ←

$F_\perp = 0$

$F_\parallel \neq 0$ (to the left) ⇒ attraction

MAGNETIC FLUX. GAUSS'S LAW FOR MAGNETISM

$d\bar{A} = \hat{n}\, dA$



Definition of the magnetic flux: fully analogous to electric flux

$$d\Phi_B = \bar{B}\cdot d\bar{A} = B_\perp\, dA = B\, dA \cos\measuredangle(\bar{B},\hat{n})$$

Total flux through surface Σ

$$\boxed{\Phi_B = \int_\Sigma d\Phi_B = \int_\Sigma \bar{B}\cdot d\bar{A}}$$

units: weber

$1\,Wb = 1\,T\cdot 1\,m^2 = \frac{N m^2}{A m} = \frac{Nm}{A}$

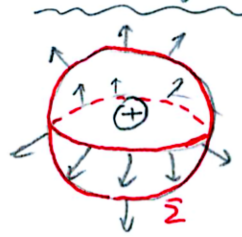
Gauss's law for magnetism (integral form)

$$\boxed{\oint_\Sigma \bar{B}\cdot d\bar{A} = 0}$$

Σ → any closed surface

Important conclusion: NO MAGNETIC MONOPOLES EXIST!

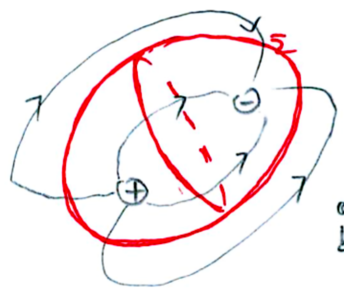
electric field



$\oint_\Sigma \bar{E}\cdot d\bar{A} = \frac{Q_{encl}}{\varepsilon_0}$

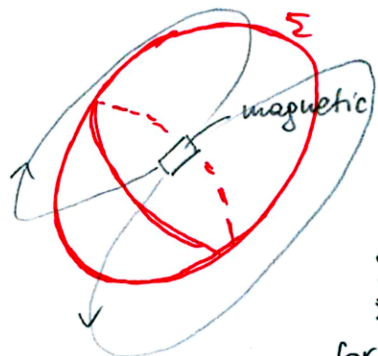
magnetic field

×



$\oint_\Sigma \bar{E}\cdot d\bar{A} = 0$

(for a dipole and Σ enclosing both charges)



$\oint_\Sigma \bar{B}\cdot d\bar{A} = 0$

for any surface

(no magnetic monopoles!)

Electric field lines begin and end on electric charges; magnetic field lines always form closed loops.

Note. The differential form of Gauss's law

$$\underset{\substack{\| \\ 0}}{\oint \bar{B}\cdot d\bar{A}} \overset{G-O}{=} \int_\Omega \operatorname{div}\bar{B}\, d\tau \Rightarrow \boxed{\operatorname{div}\bar{B} = 0}$$

(recall the interpretation of div: there are no magnetic monopoles)