PROBLEM SET 10

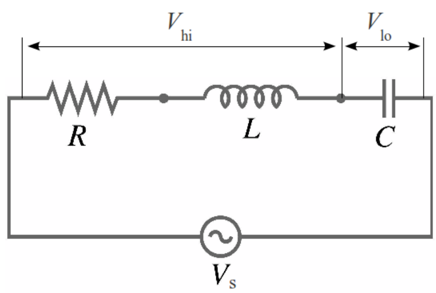
Due: 1 December 2016, 2 p.m.

Problem 1. One application of LRC series circuits is to high-pass or low-pass filters, which filter out either the low- or high-frequency components of a signal.

(a) In a high-pass filter the output voltage is taken across the LR combination (see the figure below). Derive an expression for $V_{\text{hi}}/V_{\text{s}}$, the ratio of the output and source amplitudes as a function of the angular frequency ω of the source. Show that when ω is small, this ratio is proportional to ω and thus is small, and show that the ratio approaches unity in the limit of large frequency.

(b) In a low-pass filter the output voltage is taken across the capacitor in an LRC circuit. Derive an expression for $V_{\text{lo}}/V_{\text{s}}$, the ratio of the output and source amplitudes as a function of the angular frequency ω of the source. Show that when ω is large this ratio is proportional to ω^{-2} and thus is small, and show that the ratio approaches unity in the limit of small frequency.

(*3 + 3 marks*)



Problem 2. A resistor, inductor, and capacitor are connected in parallel to an AC source with voltage amplitude V and angular frequency ω. Let the source voltage be given by $v(t) = V \cos \omega t$.

(a) Argue that the instantaneous voltages v_R, v_L, and v_C at any instant are each equal to v and that $i = i_R + i_L + i_C$, where i is the current through the source and i_R, i_L, and i_C, are the currents through the resistor, the inductor, and the capacitor, respectively.

(b) What are the phases of i_R, i_L, and i_C with respect to v? Draw the corresponding diagram on the complex plane.

(c) Show that the current amplitude I for the current i throught the source is given by $I = \sqrt{I_R^2 + (I_C - I_L)^2}$ and this result can be written as $I = V/Z$, with $Z^{-1} = \sqrt{1/R^2 + (\omega C - 1/\omega L)^2}$.

(d) Show that at the angular frequency $\omega_0 = 1/\sqrt{LC}$, $I_C = I_L$ and I is a minimum. Since I is a minimum at resonance, is it to correct to say that the power delivered to the resistor is also a minimum at $\omega = \omega_0$? Explain.

(e) At resonance, what is the phase angle of the source current with respect to source voltage? How does this compare to the phase angle for an LRC series circuit at resonance?

(*1 + 1 + 2 + 1 + 1 marks*)

Problem 3. Electromagnetic radiation is emitted by accelerating charges. As we briefly mentioned in class, the rate at which energy E is emitted by an accelerating charge q is given by the Larmor formula (in SI units) $\frac{dE}{dt} = \frac{q^2 a^2}{6\pi\varepsilon_0 c^3}$, where a is acceleration of the charge and c is the speed of light.

(a) If a proton with a kinetic energy of 6 MeV is traveling in a particle accelerator in a circular orbit of radius 0.75 m, what fraction of its energy does it radiate per second?

(b) Consider an electron orbiting with the same speed and radius. What fraction of its energy does it radiate per second?

(c) (the "classical hydrogen atom") Consider the electron in a hydrogen atom as a particle moving in a circular orbit of radius 0.0539 nm, with kinetic energy of 13.6 eV. If the electron behaved classically, how much energy would it radiate per second? What does it tell you about the use of classical physics in describing the atom within this model?

(1 + 1 + 2 marks)

Problem 4. Let $f_1(x,t) = Ae^{-k(x-vt)^2}$, $f_2(x,t) = A\sin[k(x-vt)]$, $f_3(x,t) = \frac{A}{k(x-vt)^2+1}$, and $f_4(x,t) = Ae^{-k(kx^2+vt)}$, $f_5(x,t) = A\sin(kx)\cos(kvt)^3$. Check that the functions f_1, f_2, and f_3 satisfy the 1D classical wave equation, but f_4 and f_5 do not.

(5 × 1 marks)

Problem 5. Show that (a) the *standing wave* $\xi(x,t) = A\sin(kx)\cos(\omega t)$ satisfies the wave equation and (b) express it as a sum of a wave traveling to the left and a wave traveling to the right.

(1 + 1 marks)

Problem 6. Rewrite the classical wave equation $\frac{\partial^2 \xi(x,t)}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 \xi(x,t)}{\partial t^2} = 0$ using new variables $\alpha = x + vt$, $\beta = x - vt$ and show that any solution of this equation may be expressed as a sum of a wave traveling to the left and a wave traveling to the right, *i.e.* $\xi(x,t) = \xi_1(x+vt) + \xi_2(x-vt)$.

(2 marks)

Problem 7. Electromagnetic waves propagate much differently in conductors than they do in dielectrics or in vacuum. If the resistivity of the conductor is sufficiently low (that is, if it is a sufficiently good conductor), the oscillating electric field of the wave gives rise to an oscillating conduction current that is much larger than the displacement current. In this case, the wave equation for an electric field $\mathbf{E}(x,t) = (0, E_y(x,t), 0)$ propagating in the $+x$ direction within a conductor is

$$\frac{\partial^2 E_y(x,t)}{\partial x^2} = \frac{\mu}{\rho}\frac{\partial E_y(x,t)}{\partial t},$$

where μ is the permeability of the conductor and ρ is its resistivity.

(a) Check that

$$E_y(x,t) = E_0 e^{-k_C x}\sin(k_C x - \omega t),$$

where $k_C = \sqrt{\omega\mu/2\rho}$, is a solution of the above wave equation.

(b) The exponential term shows that the electric field decreases in amplitude as it propagates. Explain why this happens.

Hint. The field does work to move charges within the conductor. The current of these moving charges causes heating within conductor. Where does this energy come from?

(c) Note that the electric-field amplitude decreases by a factor of $1/e$ in a distance $1/k_C$ and calculate this distance for a radio wave with frequency $\nu = 1$ MHz in copper (resistivity $1.72 \times 10^{-8}\ \Omega \cdot \mathrm{m}$, relative permeability $\mu_r = 1$). Since the distance is so short, electromagnetic waves of this frequency can hardly propagate at all into copper. Instead they are reflected at the surface of the metal. This is why radio waves cannot penetrate through copper or other metals.

(2 + 3 + 1 marks)