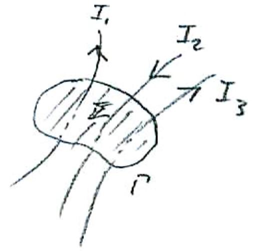
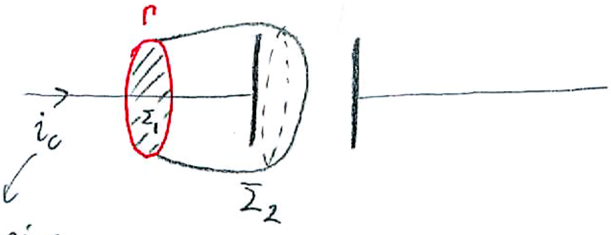
Displacement current. Ampère's law revisited

Recall: $\oint_\Gamma \bar{B}\cdot d\bar{l} = \mu_0 I_{encl}$ (Ampère's law (magnetostatics))



Example

capacitor (parallel-plate)



charging current (conduction current)

Apply Ampère's law with the surfaces chosen as Σ_1, Σ_2

$$I_{encl,\Sigma_1} = i_c$$

$$I_{encl,\Sigma_2} = 0$$

? contradiction

Conclusion: Ampère's law is incomplete

What happens at Σ_2? $|E|$ and Φ_E increase as the capacitor is being charged

Φ_E ↳ flux through Σ_2

$$q = C\cdot v = \varepsilon_0 \frac{A}{d} Ed = \varepsilon_0 AE = \varepsilon_0 \Phi_E$$

$$q = \varepsilon_0 \Phi_E \Rightarrow \dot{q} = \varepsilon_0 \dot{\Phi}_E = i$$

i → corresponding current

Define the displacement current through Σ_2

$$\boxed{i_D \stackrel{def}{=} \varepsilon_0 \frac{d\Phi_E}{dt}}$$

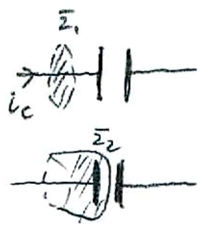
(Maxwell's idea)

Hence the complete Ampère's law

$$\boxed{\oint_\Gamma \bar{B}\cdot d\bar{l} = \mu_0 (i_c + i_D)_{encl}}$$

i_D → displacement current

i_c → conduction current



$$\begin{cases} i_c \neq 0 \\ i_D = 0 \end{cases}$$

$$\begin{cases} i_c = 0 \\ i_D \neq 0 \end{cases}$$

Notes: (1) corresponding displacement current density j_D may be defined, $j_D = i_D/A$

(2) Kirchhof's junction rule is now fixed

$$i_c - i_D = 0$$

How to detect the displacement current?

circular-plate capacitor

Idee:

current $\Rightarrow$ magnetic field

$\Downarrow$

look for magnetic field between the plates

Use Ampère's law with loop Γ (circle with radius r)

1° $r < R$

$$\oint_\Gamma \bar{B}\cdot d\bar{l} \overset{\text{symmetry}}{=} 2\pi r\, B(r) = \mu_0 I_{encl} = \mu_0 \underbrace{\left(\frac{i_D}{\pi R^2}\right)}_{\text{current density}} \pi r^2 = \mu_0 \frac{i_D}{R^2} r^2$$

$$\boxed{B(r) = \frac{\mu_0}{2\pi}\frac{r}{R^2}\, i_D}$$

2° $r > R$

$$\oint_\Gamma \bar{B}\cdot d\bar{l} = 2\pi r\, B(r) = \mu_0 i_D \quad \Rightarrow \quad \boxed{B(r) = \frac{\mu_0}{2\pi r}\, i_D}$$

The displacement current may be detected by a magnetic field measurement ✓.

Maxwell's equations (free space; integral form)

Gauss's law for $\bar{E}$ (1) $\oint_\Sigma \bar{E}\, d\bar{A} = \frac{Q_{encl}}{\varepsilon_0}$ any closed surface Σ

Gauss's law for $\bar{B}$ (2) $\oint_\Sigma \bar{B}\, d\bar{A} = 0$ any closed surface Σ

Ampère's law (3) $\oint_\Gamma \bar{B}\, d\bar{l} = \mu_0 \Big(\underbrace{i}_{\text{conduction current}} + \varepsilon_0 \frac{d\Phi_E}{dt} \Big)_{encl}$ any loop Γ

Faraday's law (4) $\oint_\Gamma \bar{E}\, d\bar{l} = - \frac{d\Phi_B}{dt}$ any stationary loop Γ

Comment on the electric field

$$\bar{E} = \bar{E}_{charge} + \bar{E}_{induced}$$

$\bar{E}_{charge}$: due to charges (conservative!)

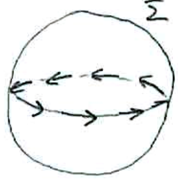
$\bar{E}_{induced}$: due to time-dependent magnetic field (flux) (non-conservative!)

(*) $\oint_\Gamma \bar{E}\, d\bar{l} = \oint_\Gamma (\bar{E}_{charge} + \bar{E}_{induced}) = \underbrace{\oint_\Gamma \bar{E}_{charge}\, d\bar{l}}_{=0 \text{ always!}} + \oint_\Gamma \bar{E}_{induced}\, d\bar{l} =$

$= \oint_\Gamma \bar{E}_{induced}\, d\bar{l}$ only $\bar{E}_{induced}$ may contribute to circulation

(**) $\bar{E}_{induced}$

Σ



here $\oint_\Sigma \bar{E}_{induced}\, d\bar{A} = 0$ (→ flux) (no charge enclosed)

so $\oint_\Sigma \bar{E}\, d\bar{A} = \oint_\Sigma \bar{E}_{charge}\, d\bar{A}$

only $\bar{E}_{charge}$ contributes to flux

Maxwell's equations (free space; differential form)

Apply Gauss-Ostrogradsky theorem and Stokes theorem to (1) - (4)

$$(1')\qquad \operatorname{div} \bar{E} = \frac{\rho}{\varepsilon_0}$$

$$(2')\qquad \operatorname{div} \bar{B} = 0$$

$$(3')\qquad \operatorname{rot} \bar{B} = \mu_0 \bar{J} + \mu_0 \varepsilon_0 \frac{\partial \bar{E}}{\partial t}$$

$$(4')\qquad \operatorname{rot} \bar{E} = - \frac{\partial \bar{B}}{\partial t}$$

Comment (3) ⇒ (3')

$$\oint_\Gamma \bar{B}\, d\bar{l} = \mu_0 \left(i + \varepsilon_0 \frac{d\Phi_E}{dt} \right)_{encl}$$

lhs

$$\oint_\Gamma \bar{B} \cdot d\bar{l} \overset{\text{Stokes}}{=} \underbrace{\int_\Sigma \operatorname{rot} \bar{B}\, d\bar{A}}$$

rhs

$$\mu_0 \left(i + \varepsilon_0 \frac{d\Phi_E}{dt} \right)_{encl} = \mu_0 \left(\int_\Sigma \bar{J}\, d\bar{A} + \varepsilon_0 \frac{d}{dt} \int_\Sigma \bar{E}\, d\bar{A} \right) =$$

$$= \underbrace{\mu_0 \int_\Sigma \left(\bar{J} + \varepsilon_0 \frac{\partial \bar{E}}{\partial t} \right) d\bar{A}}$$

Additional comments

Scalar and vector potentials

The fields $\bar{E}$ and $\bar{B}$ can be described in terms of potentials:

Scalar potential V and vector potential $\bar{A}$, such that

$$\bar{E} = -\text{grad } V - \frac{\partial \bar{A}}{\partial t}$$

$$\bar{B} = \text{rot } \bar{A}$$

Note: $\text{div } \bar{B} = \text{div rot } \bar{A} \equiv 0$

$$\text{rot } \bar{E} = -\underbrace{\text{rot grad } V}_{\equiv 0} - \text{rot } \frac{\partial \bar{A}}{\partial t} = -\frac{\partial}{\partial t} \text{rot } \bar{A} = -\frac{\partial \bar{B}}{\partial t}$$

The potentials are not uniquely determined, i.e. different potentials may result in the same fields - choice of gauge.

Fields are invariant under the following gauge transformation

$$\bar{A}' = \bar{A} + \nabla f \qquad\qquad \bar{B}' = \text{rot}(\bar{A} + \nabla f) = \text{rot } \bar{A} + \overbrace{\text{rot grad} f}^{\equiv 0} =$$

$$V' = V - \frac{\partial f}{\partial t} \quad \Rightarrow \qquad = \bar{B} \quad ✓$$

$$\bar{E}' = -\text{grad } V' - \frac{\partial \bar{A}'}{\partial t} =$$

$$= -\text{grad } V - \cancel{\frac{\partial}{\partial t} \text{grad} f} - \frac{\partial \bar{A}}{\partial t} - \cancel{\frac{\partial}{\partial t} \text{grad} f} =$$

$$= \bar{E} \quad ✓$$

Examples (various ways of choosing the gauge)

(1) Coulomb gauge $\quad \text{div } \bar{A} = 0$

(2) Lorentz gauge $\quad \text{div } \bar{A} + \frac{1}{c^2} \frac{\partial V}{\partial t} = 0 \qquad$ c - speed of light in vacuum

(3) $\bar{A} = -By\, \hat{u}_x \quad$ for uniform magnetic field in z-direction (Landau gauge)

$(\text{or } \bar{A} = Bx\, \hat{u}_y)$