GAUSS'S LAW

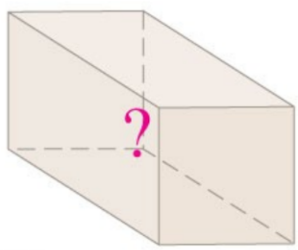
Motivation

charge distribution $\xrightarrow[\text{\& superposition principle}]{\text{Coulomb's law}}$ electric field

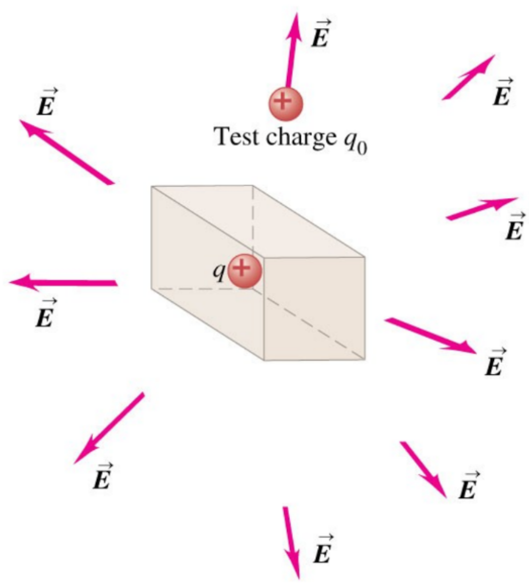
electric field $\xrightarrow{?}$ charge distribution

move test charge to create a „map" of $\bar{E}$

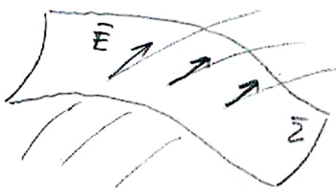
(a) A box containing an unknown amount of charge



(b) Using a test charge outside the box to probe the amount of charge inside the box

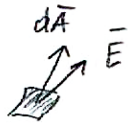


Flux of the electric field (electric flux)



Σ - any surface

Recall: (calculus)

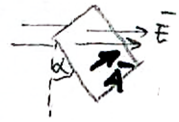


$d\bar{A} = \hat{n} \cdot dA$

points "outwards"

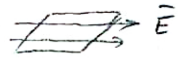
$$\Phi_E \overset{def}{=} \int_{\Sigma} \bar{E} \cdot d\bar{A}$$

Comments: (1) For uniform field through a plane



$\Phi_E = \bar{E} \cdot \bar{A} = EA \cos\alpha$

(if $\alpha = \frac{\pi}{2} \Rightarrow \Phi_E = 0$)



(2) Analogy with water flow

Electric flux through a closed surface

$$\Phi_E = \oint_{\Sigma} \bar{E} \cdot d\bar{A} = \oint_{\Sigma} \bar{E} \cdot \hat{n}\, dA$$

Closed surface

$\boxed{\text{FiG.}}$

Observation (1):

net charge inside	flux
(+)	outward
(−)	inward
0	none

Observation (2): charges outside of the closed surface do not contribute to the flux

net flux depends on net charge enclosed by the surface

Example: Find the electric flux through a sphere with radius R and point charge $q<0$ placed at its center

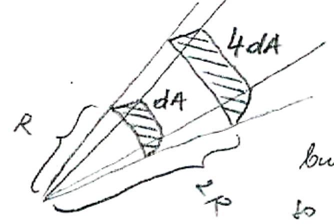
$$\bar{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{R^2}\hat{n} \qquad \text{electric field on the sphere}$$

$$\Phi_E = \oint_\Sigma \bar{E}\,d\bar{A} = \oint_\Sigma \frac{1}{4\pi\varepsilon_0}\frac{q}{R^2}\overbrace{\hat{n}\cdot\hat{n}}^{=1}\,dA = \frac{1}{4\pi\varepsilon_0}\frac{q}{R^2}\underbrace{\oint_\Sigma dA}_{=4\pi R^2} = \frac{q}{\varepsilon_0}$$

$$\boxed{\Phi_E = \frac{q}{\varepsilon_0}} \qquad \text{(single point charge; spherical surface)}$$

Observations: flux does not depend on the radius of the sphere;
flux $\propto$ charge enclosed by the sphere

FiG



but $E(r) \sim \frac{1}{r^2}$
so $E(2R) = \frac{1}{4}E(R)$

Generalization:

Flux through a non-spherical surface (single charge inside)

FACT. We will show that $\oint_{\Sigma_1}\bar{E}\,d\bar{A} = \oint_{\Sigma_2}\bar{E}\,d\bar{A}$

Σ_1 ↳ any sphere enclosing charge q placed at its center

Σ_2 ↳ any surface enclosing Σ_1

Consider the solid region Ω bounded by $\Sigma = \Sigma_1 \cup \Sigma_2$ (note the direction of $\hat{n}$)

Choose the origin of the coordinate system at q (center of the sphere Σ_1);
find $\operatorname{div}\bar{E}$ inside Ω at any point $P\ (\bar{r}\neq 0)$

$$\operatorname{div}\bar{E} = \nabla\cdot\bar{E} = \nabla\cdot\left(\frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\frac{\bar{r}}{r}\right) = \frac{q}{4\pi\varepsilon_0}\left[\frac{\partial}{\partial x}\left(\frac{x}{r^3}\right)+\frac{\partial}{\partial y}\left(\frac{y}{r^3}\right)+\frac{\partial}{\partial z}\left(\frac{z}{r^3}\right)\right] = 0$$

(see problem set)

Use Gauss - Ostrogradsky theorem (divergence thm) for Ω bounded by Σ

$$\oint_\Sigma \bar{E}\,d\bar{A} = \int_\Omega \underbrace{\operatorname{div}\bar{E}}_{=0}\,dV \Rightarrow \oint_\Sigma \bar{E}\,d\bar{A} = 0, \quad \text{but} \quad \oint_\Sigma \bar{E}\,d\bar{A} = \underbrace{\int_{\Sigma_2}\bar{E}\,\hat{n}_2\,dA}_{\text{flux through }\Sigma_2} + \underbrace{\int_{\Sigma_1}\bar{E}(-\hat{n}_1)\,dA}_{-\text{flux through }\Sigma_1}$$

Hence

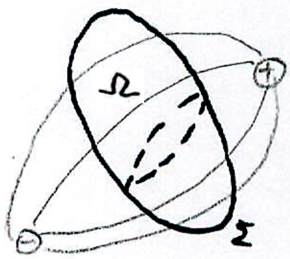
$$\boxed{\oint_{\Sigma_1}\bar{E}\,d\bar{A} = \oint_{\Sigma_2}\bar{E}\,d\bar{A}}$$

Conclusion: (from the fact and the previous example)

Electric flux through any surface Σ enclosing a single point charge q is equal to q/ε_0.

$$\Phi_E = \oint_\Sigma \bar{E}\, d\bar{A} = q/\varepsilon_0$$

Closed surface enclosing no charge



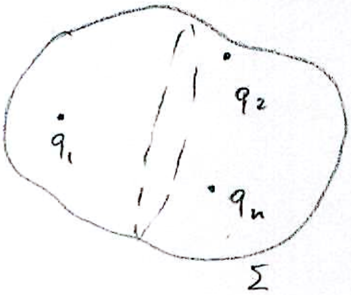
Use G-O thm again

$$\underbrace{\int_\Omega \underbrace{\operatorname{div}\bar{E}}_{=0 \text{ in } \Omega}\, dV}_{} = \oint_\Sigma \bar{E}\, d\bar{A} \Rightarrow \Phi_E = 0$$

Ω bounded by Σ

$$\Phi_E = \oint_\Sigma \bar{E}\cdot d\bar{A} = 0$$

Closed surface enclosing multiple charges



$$\Phi_E = \oint_\Sigma \bar{E}\cdot d\bar{A} = \frac{q_1}{\varepsilon_0} + \frac{q_2}{\varepsilon_0} + \dots + \frac{q_n}{\varepsilon_0} = \frac{q_{encl}}{\varepsilon_0}$$

$\bar{E}$: total field (use superposition principle)

q_{encl}: net charge enclosed by Σ

$$\Phi_E = \frac{q_{encl}}{\varepsilon_0}$$

GAUSS'S LAW

The total electric flux through any closed surface Σ is equal to the net electric charge enclosed by the surface Σ divided by ε_0.

$$\boxed{\oint_{\Sigma} \bar{E} \cdot d\bar{A} = \frac{q_{encl}}{\varepsilon_0}}$$

Note q_{encl} - algebraic sum of all charges enclosed by surface Σ
$\bar{E}$ - total electric field

↳ called "Gaussian surface"

total electric field $\xrightarrow{\text{Gauss's law}}$ charge distribution

Gauss's law in the differential form

Transform lhs using Gauss-Ostrogradsky theorem

$$\oint_{\Sigma} \bar{E}\, d\bar{A} = \int_{\Omega} \operatorname{div} \bar{E}\, dV$$

For the rhs $\quad \frac{q_{encl}}{\varepsilon_0} = \frac{1}{\varepsilon_0} \int_{\Omega} \rho(\bar{r})\, dV$

↳ bulk density of charge

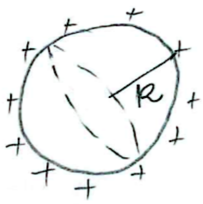
Hence (compare rhs) (!) $\quad \boxed{\operatorname{div} \bar{E}(\bar{r}) = \frac{\rho(\bar{r})}{\varepsilon_0}}$ Gauss's law (differential form)

Examples

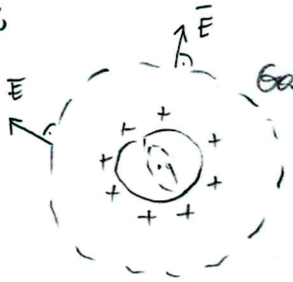
$$\left\{ \Phi_E = \frac{Q_{encl}}{\varepsilon_0} \right\}$$

(1) Conducting sphere – charge resides on surface

$Q > 0$



1° $r > R$



Gaussian surface Σ (sphere of radius r)

$$\Phi_E = \oint_\Sigma \bar{E}\, d\bar{A} = \oint_\Sigma E(r)\, \hat{n}\cdot\hat{n}\, dA = \oint_\Sigma E(r)\, dA =$$

$$= E(r) \oint_\Sigma dA = E(r)\, 4\pi r^2$$

$\bar{E} = E(r)\,\hat{n}$
$\Downarrow$
spherical symmetry

Gauss's law

($Q_{encl} = Q$)

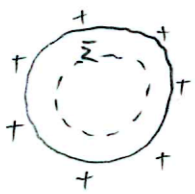
$$E(r)\,4\pi r^2 = \frac{Q}{\varepsilon_0} \Rightarrow E(r) = \frac{Q}{4\pi\varepsilon_0 r^2}$$

$$\boxed{\bar{E}(\bar{r}) = \frac{Q}{4\pi\varepsilon_0}\frac{1}{r^2}\hat{n} = \frac{Q}{4\pi\varepsilon_0}\frac{1}{r^2}\hat{r}}$$ for $r > R$

$\hat{r} = \hat{n}$ (points outward)

(compare with point charge)

2° $r < R$



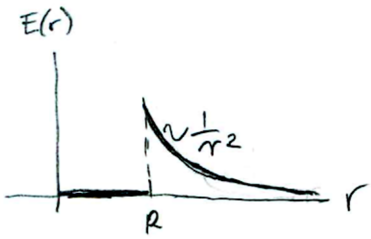
$$\Phi_E = \oint_\Sigma \bar{E}\, d\bar{A} = 4\pi r^2 E(r)$$

as above
$\bar{E}$ has to have spherical symmetry
(may be trivial)

$Q_{encl} = 0$

so

$$\oint_\Sigma \bar{E}\, d\bar{A} = 0 \Longleftrightarrow \boxed{\bar{E} \equiv 0}$$ for $r < R$



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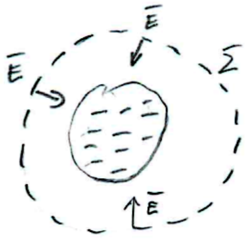
(2) insulating ball, uniformly charged ($Q < 0$)

1° $r > R$



Σ - Gaussian surface (sphere of radius r, center @ center of the insulating sphere)

$\bar{E} = E(r)\,\hat{n} \perp d\bar{A}$ spherical symmetry (here $E(r) < 0$)

$$\Phi_E = \oint_\Sigma \bar{E} \cdot d\bar{A} = \dots \text{ as in ex. (1)} \dots = E(r)\, 4\pi r^2$$

$$Q_{encl} = Q$$

From the Gauss's Law

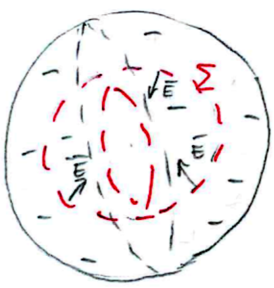
$$E(r) \cdot 4\pi r^2 = \frac{Q}{\varepsilon_0} \Rightarrow E(r) = \frac{1}{4\pi\varepsilon_0} \frac{Q}{r^2} \quad (<0)$$

$$\boxed{\bar{E}(\bar{r}) = \frac{1}{4\pi\varepsilon_0} \frac{Q}{r^2} \hat{r}} \quad \text{for } r > R$$

(points inwards)

Same as point charge Q put @ center of sphere

2° $r < R$



Σ - Gaussian surface ($r < R$)

again $\bar{E}$ is spherically symmetric

$$\bar{E} = E(r)\,\hat{n} \quad (\hat{n} = \hat{r})$$

$$\Phi_E = \oint_\Sigma \bar{E}\, d\bar{A} = \dots = E(r)\, 4\pi r^2$$

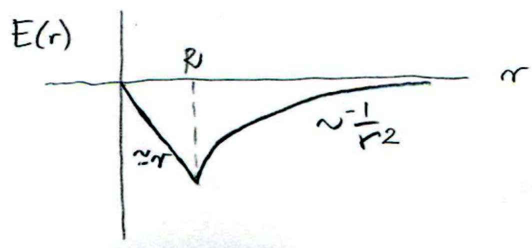
but now Σ encloses only a fraction of charge

$$Q_{encl} = \frac{\frac{4}{3}\pi r^3}{\frac{4}{3}\pi R^3} Q = \frac{r^3}{R^3} Q$$

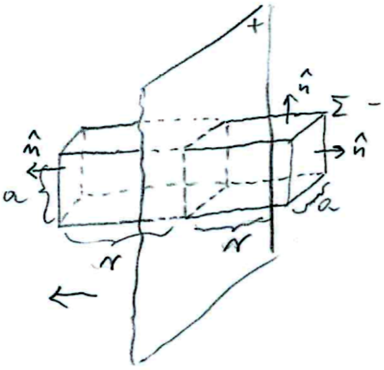
Gauss's law

$$E(r)\, 4\pi r^2 = \frac{Q}{\varepsilon_0} \frac{r^3}{R^3}$$

$$E(r) = \frac{1}{4\pi\varepsilon_0} \frac{Q}{R^3} r \Rightarrow \boxed{\bar{E}(\bar{r}) = \frac{1}{4\pi\varepsilon_0} \frac{Q}{R^3} r \hat{r}}$$



(3) infinite charged plane $(\sigma > 0)$ $\quad \sigma = \text{const}$



Σ - Gaussian surface

$\rightarrow$ $\bar{E}$ perpendicular to the plane (symmetry)

Σ = left base $\cup$ side walls $\cup$ right base

$Q_{encl} = \sigma \cdot a^2$

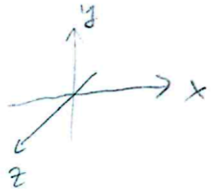
$$\Phi_E = \oint_{\Sigma} \bar{E}\, d\bar{A} = \int_{\substack{\text{left}\\ \text{base}}} \bar{E}\, d\bar{A} + \underbrace{\int_{\substack{\text{side}\\ \text{walls}}} \bar{E}\, d\bar{A}}_{=0,\ \bar{E} \perp \hat{n}} + \int_{\substack{\text{right}\\ \text{base}}} \bar{E}\, d\bar{A}$$

$$= \dots = E(r) \int_{\text{l. base}} dA + 0 + E(r) \int_{\text{r. base}} dA =$$

$$= 2 \cdot E(r)\, a^2$$

Gauss's Law

$$\Phi_E = \frac{Q_{encl}}{\varepsilon_0} \Rightarrow 2E(r)\, a^2 = \frac{\sigma a^2}{\varepsilon_0} \Rightarrow E(r) = \frac{\sigma}{2\varepsilon_0}$$

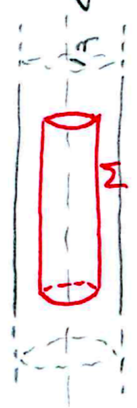


$$\boxed{\bar{E}(\bar{r}) = \begin{cases} \dfrac{\sigma}{2\varepsilon_0}\, \hat{n}_x & \text{for } x > 0 \\ -\dfrac{\sigma}{2\varepsilon_0}\, \hat{n}_x & \text{for } x < 0 \end{cases}}$$

cf. with the result obtained from superposition principle

(4) nonuniformly charged objects – see problem set

$\rho = \rho(r)$
$\downarrow$
cylindrical symmetry



e.g. inside

$Q_{encl} = \int_{\Omega} \rho(r)\, dV$

$\Omega \hookrightarrow$ solid enclosed by surface Σ

$\oint_{\Sigma} \bar{E}\, d\bar{A}$ – only the lateral surface contributes

Comment: The symmetry of the Gaussian surface should coincide with the symmetry of the electric field (determined by charge distribution). Then the flux can be easily found.

Applications of Gauss's law – some properties of conductors

GENERAL FACT: Under electrostatic conditions, the electric field at every point inside a conductor is zero.

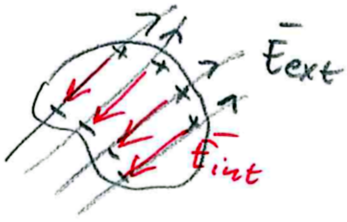
Justification („reductio ad absurdum")

assume $\bar{E} \neq 0$ inside $\xrightarrow[\text{in the conductor}]{\text{mobile charges}}$ moving charges $\Rightarrow$ NO ELECTROSTATICS

$\times$ CONTRADICTION

Illustration

neutral (not charged) conductor in an external electric field



$$\bar{E}_{ext} + \bar{E}_{int} = 0$$

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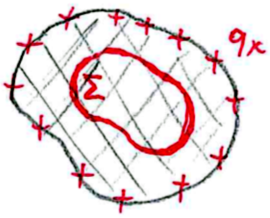
Applications of Gauss's law

(1) Solid conductor

FACT: Excess charge (q_x) placed on a solid conductor, in electrostatic conditions, resides entirely on the conductor's surface.



$\bar{E}$ inside is zero, hence Φ_E through Σ is zero

$\Downarrow$ Gauss's law

charge enclosed by Σ is zero (Σ - any closed surface inside the conductor)

$\Downarrow$ shrink Σ to a point

no charge anywhere inside

(2) Conductor with a cavity

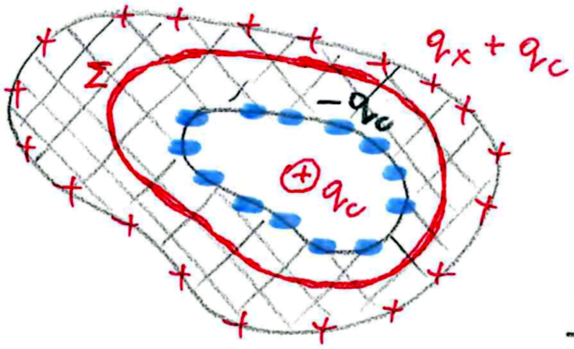
(A) excess charge & empty cavity

$E = 0$ on the Gaussian surface, hence $\Phi_E = 0$

$\Downarrow$ Gauss's law; shrink Σ to cavity's shap

FACT: Net charge on the empty cavity's surface is zero

(B) excess charge + charge in the cavity



Again $\bar{E}$ on the Gaussian surface is zero, hence $\Phi_E = 0$

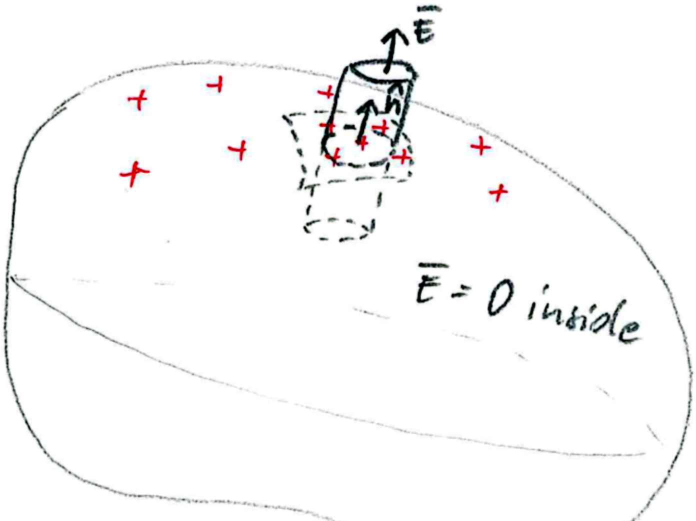
$\Downarrow$ Gauss's law; shrink Σ to cavity's shape

FACT: There is charge $-q_c$ appearing on the surface of the cavity; and the charge on the conductor's outer surface is $q_x + q_c$

Example: Faraday's icepail experiment

(FIGURE)

(3) Electric field at the surface of a charged conductor



FACT 1: E has to be normal to the surface, otherwise the charges will be moving.

FACT 2: Use Gauss law with Σ

$$\underbrace{AE}_{\Phi_E} = \underbrace{\frac{\sigma A}{\varepsilon_0}}_{\frac{Q_{encl}}{\varepsilon_0}} \Rightarrow \boxed{\bar{E} = \frac{\sigma}{\varepsilon_0}\hat{n}}$$