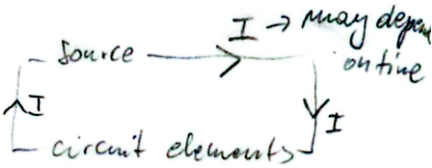
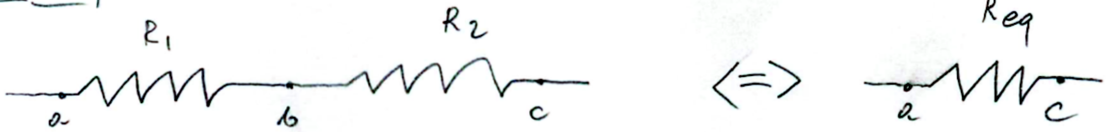
DC circuits

direct-current (no change in direction of the current)



Systems of resistors

SERIES CONNECTION



$$V_{ab} + V_{bc} = V_{ac}$$

$(V = \int \vec{E}\,d\vec{r})$ ↓ additive

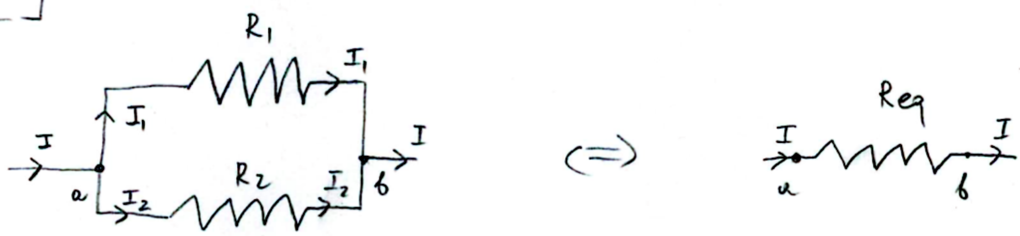
Current through system is I; use $V = IR$ (Ohm's law)

$$IR_1 + IR_2 = IR_{eq}$$

$$\boxed{R_{eq} = R_1 + R_2}$$

can generalize to n resistors $R_{eq} = \sum_{i=1}^{n} R_i$

PARALLEL CONNECTION



$$I_1 + I_2 = I$$ (conservation of charge)

Potential change on both resistors is equal (V_{ab}); use $I = \frac{V}{R}$

$$\frac{V_{ab}}{R_1} + \frac{V_{ab}}{R_2} = \frac{V_{ab}}{R_{eq}}$$

$$\boxed{\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2}}$$

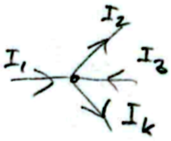
can generalize to n resistors $\frac{1}{R_{eq}} = \sum_{i=1}^{n} \frac{1}{R_i}$

Note. $V_{ab} = I_1 R_1$
$V_{ab} = I_2 R_2$ $\Rightarrow \frac{R_1}{R_2} = \frac{I_2}{I_1}$ most current through resistor of least resistance

Kirchhoff's Rules (mid 19th century)

The algebraic sum of the currents into any junction is zero

$$\boxed{\sum_k I_k = 0}$$



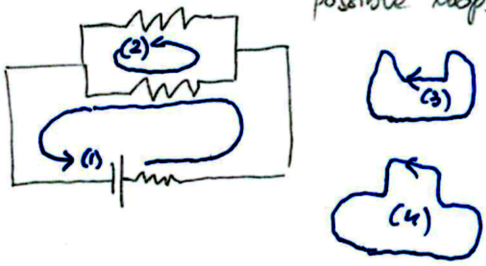
junction rule
(any junction)

The algebraic sum of the potential differences in any loop (including potential differences across emfs and resistors) is zero.

$$\boxed{\sum_k V_k = 0}$$

loop rule
(any loop)

possible loops



Justification

① junction rule – conservation of charge

$$0 = \sum_{i=1}^{n} I_i = \sum_{i=1}^{n} \frac{dQ_i}{dt} = \frac{d}{dt}\left(\sum_{i=1}^{n} Q_i\right) \Rightarrow \sum_{i=1}^{n} Q_i = \text{const}$$

② loop rule – electric force is conservative

$$\int_{A\to B} \bar{E}\,d\bar{r} \text{ does not depend on path or } \oint \bar{E}\,d\bar{r} = 0$$

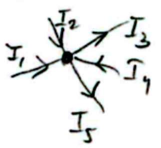
$$0 = \oint_{\text{loop}} \bar{E}\,d\bar{r} = \int_{A_1\to A_2} \bar{E}\,d\bar{r} + \int_{A_2\to A_3} \bar{E}\,d\bar{r} + \ldots + \int_{A_N\to A_1} \bar{E}\,d\bar{r} =$$

$$= V_{A_1A_2} + V_{A_2A_3} + \ldots + V_{A_NA_1}$$

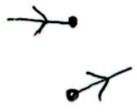
~ o ~

Sign Convention

①



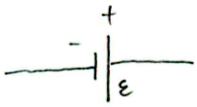
$$I_1 + I_2 - I_3 + I_4 - I_5 = 0$$



"plus"

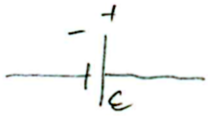
"minus"

② emfs



loop → direction of the loop

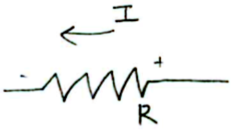
$+\mathcal{E}$



← loop

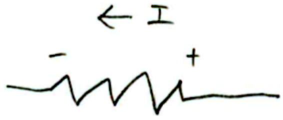
$-\mathcal{E}$

resistors



loop → opposite to current

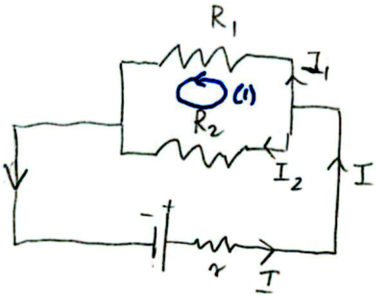
$+IR$



← loop in current direction

$-IR$

Example



loop (1) : $-I_1R_1 + I_2R_2 = 0$

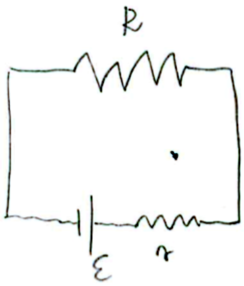
loop (2) : $-Ir - I_2R_2 + \mathcal{E} = 0$

junction rule : $I - I_1 - I_2 = 0$

$\Downarrow$

number of independent equations
$\parallel$
number of unknowns

Example (1)



Find R s.t. the power on ther resistor R is maximum ($\mathcal{E}$, r - given)

loop rule : $-Ir - IR + \mathcal{E} = 0$

$$\Downarrow$$

$$I = \frac{\mathcal{E}}{R+r}$$

Power

$$P = I^2R = \frac{\mathcal{E}^2}{(R+r)^2}R = P(R)$$

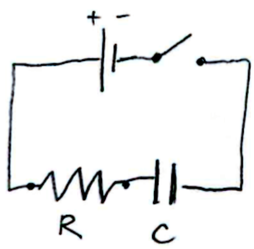
(2) Wheatstone bridge (see problem set)

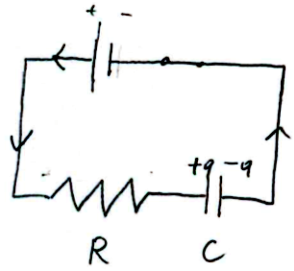
$$\frac{dP}{dR} = \mathcal{E}^2\left[\frac{1}{(R+r)^2} - \frac{2R}{(R+r)^3}\right] = 0 \Rightarrow 2R = R+r$$

$$\boxed{R = r}$$

R-C circuits

Current dependent on time (but still in one direction)





Assume ideal emf; $\mathcal{E} = \text{const}$ $r = 0$

Notation: lowercase indicates time dependence $i = i(t)$, $q = q(t)$, $v_{ab} = v_{ab}(t)$

(1) Charging

Initial state $(t=0)$

$q(0) = 0;\ i(0) = I_0$

K. loop law: $\underbrace{-I_0 R}_{v_{ab}} + \mathcal{E} = 0 \Rightarrow I_0 = \frac{\mathcal{E}}{R}$ initial condition

For $t > 0$

K. loop law: loop

$$-iR - \frac{q}{C} + \mathcal{E} = 0 \Rightarrow i = \frac{\mathcal{E}}{R} - \frac{q}{RC}$$

1st order Differential eq. with separable variables $i = \frac{dq}{dt}$

$$\frac{dq}{dt} = \frac{\mathcal{E}}{R} - \frac{q}{RC}$$

$$\int_0^{q(t)} \frac{dq}{q - C\mathcal{E}} = -\int_0^t \frac{dt}{RC}$$

$$\ln \left| \frac{q(t) - C\mathcal{E}}{-C\mathcal{E}} \right| = -\frac{t}{RC}$$

Note $q(t) < C\mathcal{E} = Q_{max}$

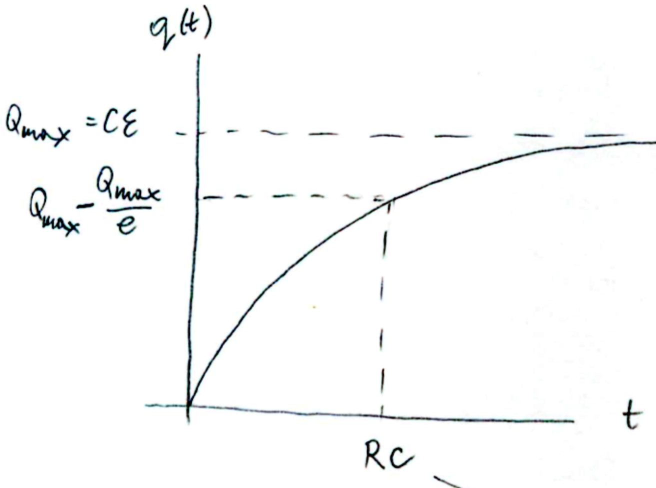
$$\ln \frac{C\mathcal{E} - q(t)}{C\mathcal{E}} = -\frac{t}{RC}$$

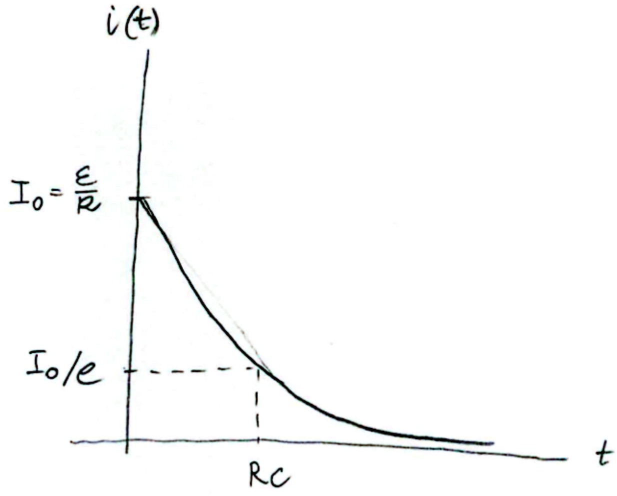
$$C\mathcal{E} - q(t) = C\mathcal{E}\, e^{-t/RC}$$

$$\boxed{q(t) = C\mathcal{E}\left(1 - e^{-t/RC}\right)}$$

Current

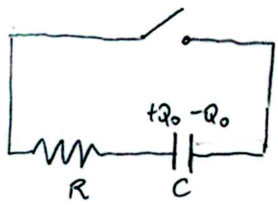
$$\boxed{i = \frac{dq(t)}{dt} = \frac{\mathcal{E}}{R} e^{-t/RC} = I_0 e^{-t/RC}}$$



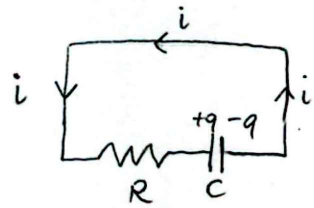


$RC = \tau$ time constant for R-C circuit [unit 1 s]

(2) Discharging



$\longrightarrow$



loop rule

$$-iR - \frac{q}{C} = 0$$

$$i = \frac{-q}{RC}$$

Initial state ($t=0$)

$$q(0) = Q_0 \; ; \quad i(0) = -\frac{Q_0}{RC} = I_0$$

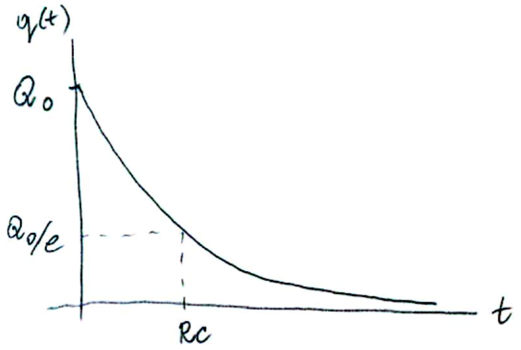
For $t > 0$

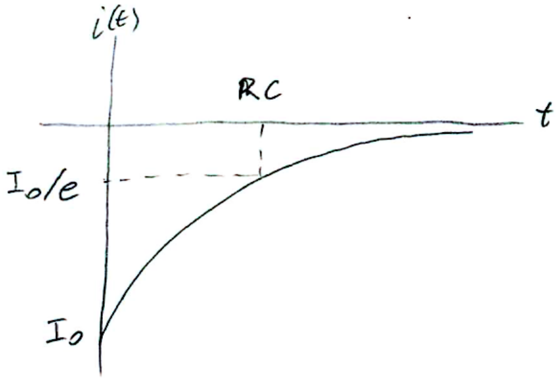
$$i = -\frac{q}{RC} \Rightarrow \frac{dq}{dt} = -\frac{q}{RC} \Rightarrow \int_{Q_0}^{q(t)} \frac{dq}{q} = -\frac{dt}{RC}$$

$$\ln \frac{q(t)}{Q_0} = -\frac{t}{RC} \Rightarrow \boxed{q(t) = Q_0 e^{-t/RC}}$$

Current

$$\boxed{i = -\frac{q}{RC} = -\frac{Q_0}{RC} e^{-t/RC} = I_0 e^{-t/RC}}$$





Energy in the charging process

$$-iR - \frac{q}{C} + \mathcal{E} = 0 \quad / \cdot i$$

$$-i^2R - \frac{iq}{C} + \mathcal{E}i = 0$$

power dissipated in resistor — part stored in capacitor — power supplied by battery

Total energy:

* supplied by battery

$$E_{batt} = \int_0^\infty \mathcal{E} i \, dt = I_0 \mathcal{E} \int_0^\infty e^{-t/RC} \, dt = I_0 RC \mathcal{E} \overset{I_0 = \frac{\mathcal{E}}{R}}{=} C\mathcal{E}^2$$

* dissipated in resistor

$$E_{res} = \int_0^\infty i^2 R \, dt = I_0^2 R \int_0^\infty e^{-2t/RC} \, dt = \frac{C\mathcal{E}^2}{2} \qquad (50\%)$$

* stored in capacitor

$$E_{cap} = \int_0^\infty \frac{iq}{C} = I_0 C\mathcal{E} \int_0^\infty \left(e^{-t/RC} - e^{-2t/RC}\right) = \frac{C\mathcal{E}^2}{2} \qquad (50\%)$$