ELECTROMAGNETIC WAVES

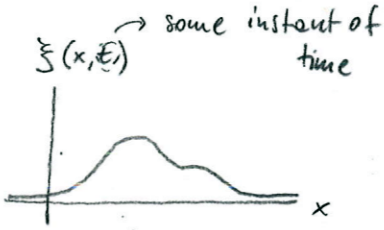
Introduction – waves in general

Recall (summer semester): classical wave equation (in 1D)

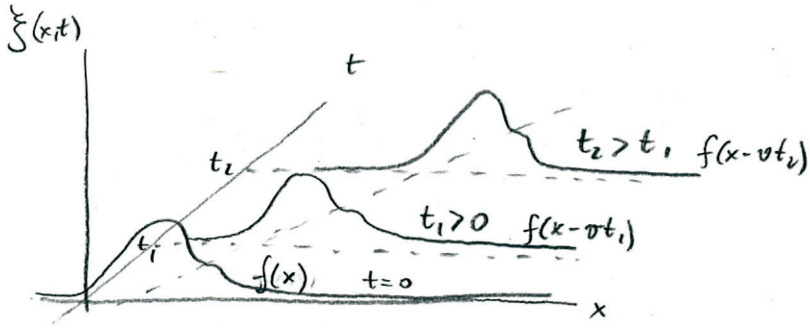
$$\frac{\partial^2 \xi}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 \xi}{\partial t^2}$$

ξ – displacement

v → phase speed



"snapshot" of a wave impulse



impulse travelling to the right with speed v

$\xi(x,t) = f(x - vt)$

↳ (constant shape here)

<u>Note</u>. for an impulse travelling to the left: $\xi_{\leftarrow}(x,t) = f(x + vt)$

<u>Examples</u>: (1) wave on a rope/string

direction of displacement ↕ → direction of propagation — **transverse wave**

(2) sound

⟷ direction of displacement → propagation — **longitudinal wave**

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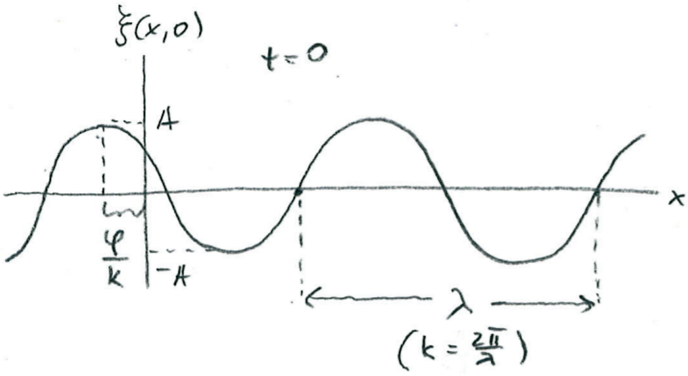
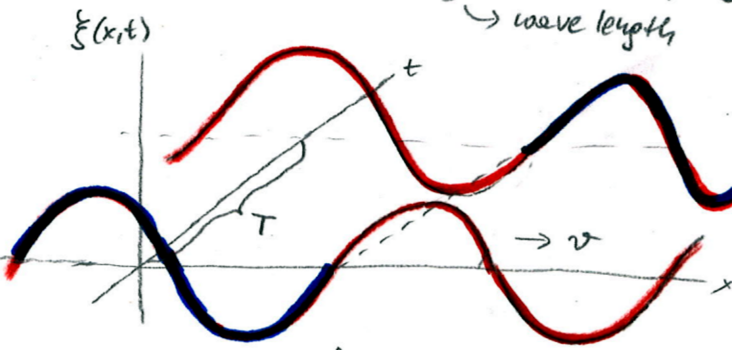
Sinusoidal waves
(harmonic waves)

$$\xi(x,t) = A\cos\left[k(x - vt) + \varphi\right] = A\cos\left\{k\left[(x-vt) + \frac{\varphi}{k}\right]\right\}$$

$$= A\cos\left\{\frac{2\pi}{\lambda}\left[(x - vt) + \frac{\lambda\varphi}{2\pi}\right]\right\}$$

A – amplitude; k – wave number; φ – phase constant; $k(x-vt)+\varphi$ – phase; λ – wave length





T – period $T = \frac{\lambda}{v} = \frac{2\pi}{kv}$

frequency $\nu = \frac{1}{T} = \frac{v}{\lambda} = \frac{kv}{2\pi}$

angular frequency $\omega = 2\pi\nu = kv$

Usual notation

$$\xi(x,t) = A\cos(kx - \omega t + \varphi) \qquad \rightarrow \text{ right-travelling}$$

$$\xi_{\leftarrow}(x,t) = A\cos(kx + \omega t - \varphi) = \qquad \leftarrow \text{ left-travelling}$$

$$= A\cos[-(-kx - \omega t + \varphi)] \overset{\text{even}}{=} A\cos(-kx - \omega t + \varphi)$$

Note. $\xi \overset{k \leftrightarrow -k}{\leftrightarrow} \xi_{\leftarrow}$

Note. In general $k \in \mathbb{R}$, hence $\lambda = \frac{2\pi}{|k|}$ in the general case.

Notation using complex numbers

(easier to manipulate with)

$$\xi(x,t) = Re\,[\overbrace{A e^{i(kx - \omega t + \varphi)}}^{\tilde{\xi}(x,t)}]$$

$$\boxed{\tilde{\xi}(x,t) = \tilde{A}\, e^{i(kx - \omega t)}} \qquad \text{where } \tilde{A} = A e^{i\varphi}$$

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Important comment (why sinusoidal waves are important)

Any (sufficiently smooth) wave shape can be expressed as a linear combination (possibly with infinite number of terms) of sinusoidal (harmonic) waves

$$\tilde{\xi}(x,t) = \int_{-\infty}^{\infty} \hat{A}(k)\, e^{i(kx - \omega t)}\, dk$$

(see math class: Fourier transform

Electromagnetic waves in free space

Recall Maxwell's equations (free space)

$$\nabla\cdot\bar{E} = 0 \qquad \nabla\times\bar{E} = -\frac{\partial\bar{B}}{\partial t}$$

$$\nabla\cdot\bar{B} = 0 \qquad \nabla\times\bar{B} = \mu_0\varepsilon_0\frac{\partial\bar{E}}{\partial t}$$

(system of coupled 1^{st} order PDEs)

$\frac{\partial\bar{B}}{\partial t}\neq 0 \Rightarrow \frac{\partial\bar{E}}{\partial t}\neq 0 \Rightarrow \frac{\partial\bar{B}}{\partial t}\neq 0 \Rightarrow \dots$

Use the identity

$$\nabla\times(\nabla\times\underbrace{\bar{w}}_{\text{any vector}}) = \nabla(\nabla\cdot\bar{w}) - \nabla^2\bar{w}$$

(recall $\bar{a}\times(\bar{b}\times\bar{c}) = \bar{b}(\bar{a}\bar{c}) - \bar{c}(\bar{a}\bar{b})$)

with $\bar{w} = \bar{E}$

$$\nabla\times(\nabla\times\bar{E}) = \nabla(\overbrace{\nabla\cdot\bar{E}}^{=0}) - \nabla^2\bar{E} = -\nabla^2\bar{E}$$

On the other hand

$$\nabla\times(\underbrace{\nabla\times\bar{E}}_{=-\frac{\partial\bar{B}}{\partial t}}) = \nabla\times\left(-\frac{\partial\bar{B}}{\partial t}\right) = -\frac{\partial}{\partial t}(\underbrace{\nabla\times\bar{B}}_{=\mu_0\varepsilon_0\frac{\partial\bar{E}}{\partial t}}) = -\mu_0\varepsilon_0\frac{\partial^2\bar{E}}{\partial t^2}$$

Hence

$$\boxed{\begin{aligned}\nabla^2\bar{E} &= \mu_0\varepsilon_0\frac{\partial^2\bar{E}}{\partial t^2}\\ \nabla^2\bar{B} &= \underbrace{\mu_0\varepsilon_0}_{\frac{1}{(\text{phase speed})^2}}\frac{\partial^2\bar{B}}{\partial t^2}\end{aligned}}$$

analogously for $\bar{B}$

(two decoupled PDEs of 2^{nd} order, in fact 6 equs - 1 for each of the components)

e-m wave equs.

$$v = \frac{1}{\sqrt{\mu_0\varepsilon_0}} \simeq 3\times10^8\ m/s = c \rightarrow \text{speed of light in vacuum}$$

How to generate $\bar{E},\bar{B}$ in the first place?

→ oscillating charge 

$$\underbrace{P}_{\text{rate of energy radiated}} = \frac{q^2|\dot{\bar{v}}|^2}{6\pi\varepsilon_0 c^3}$$

Larmor's formula

→ LC circuits ($\bar{E},\bar{B}$ - time dependent)

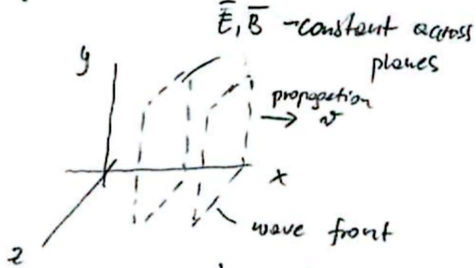
Sinusoidal e-m waves (monochromatic plane waves)

Recall: any wave can be represented as a sum of sinusoidal waves.

Assumption: wave travelling in x-direction ; no z,y-dependence

$\Downarrow$

plane wave



$kx-\omega t = \text{const.}$

$\left(k=\frac{2\pi}{\lambda}\right)$

$\left(\frac{\omega}{k}=c\right)$

$$\tilde{\bar{E}}(x,t) = \tilde{\bar{E}}_0\, e^{i(kx-\omega t)}$$

$$\tilde{\bar{B}}(x,t) = \tilde{\bar{B}}_0\, e^{i(kx-\omega t)} \qquad (*)$$

Exercise: Check that $(*)$ obey the e-m wave equ.

Fig.
e-m spectrum

Note: solution of (free space) Maxwell's equs. $\Rightarrow$ solution of wave equs

$\not\Leftarrow$ extra conditions imposed by Maxwell's equs.

$\rightarrow$ $\nabla\cdot\bar{E}=0 \Rightarrow \tilde{E}_{0x}=0$

$\nabla\cdot\bar{B}=0 \Rightarrow \tilde{B}_{0x}=0$

$\Big\}\Rightarrow$ e-m waves are <u>transverse</u> ($\bar{E}$, $\bar{B}$) $\rightarrow$ propagation

$\rightarrow$ Faraday's law

$$\nabla\times\bar{E} = -\frac{\partial\bar{B}}{\partial t}$$

$$\begin{cases}\tilde{\bar{E}} = (0,\tilde{E}_{0y},\tilde{E}_{0z})\,e^{i(kx-\omega t)}\\ \tilde{\bar{B}} = (0,\tilde{B}_{0y},\tilde{B}_{0z})\,e^{i(kx-\omega t)}\end{cases}$$

$(*)$ $$\nabla\times\tilde{\bar{E}} = \begin{vmatrix}\hat{u}_x & \hat{u}_y & \hat{u}_z\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ 0 & \tilde{E}_{0y}e^{i(kx-\omega t)} & \tilde{E}_{0z}e^{i(kx-\omega t)}\end{vmatrix} =$$

$$= \left(0,\,-ik\tilde{E}_{0z},\,ik\tilde{E}_{0y}\right)e^{i(kx-\omega t)}$$

$(*)$ $$-\frac{\partial\tilde{\bar{B}}}{\partial t} = i\omega\left(0,\tilde{B}_{0y},\tilde{B}_{0z}\right)e^{i(kx-\omega t)}$$

Hence
$$-k\tilde{E}_{0z} = \omega\tilde{B}_{0y}$$
$$k\tilde{E}_{0y} = \omega\tilde{B}_{0z}$$

or in a compact form (exercise!) $\frac{k}{\omega}\left(\hat{u}_x\times\tilde{\bar{E}}_0\right)=\tilde{\bar{B}}_0$

Summary:

$$\tilde{\bar{E}}(x,t) = \tilde{\bar{E}}_0 \, e^{i(kx-\omega t)} \qquad \tilde{E}_{0x} = \tilde{B}_{0x} = 0$$

$$\tilde{\bar{B}}(x,t) = \tilde{\bar{B}}_0 \, e^{i(kx-\omega t)} \qquad \frac{k}{\omega}(\hat{n}_x \times \tilde{\bar{E}}_0) = \tilde{\bar{B}}_0$$

Conclusions

(1) $\bar{E}$ in phase with $\bar{B}$

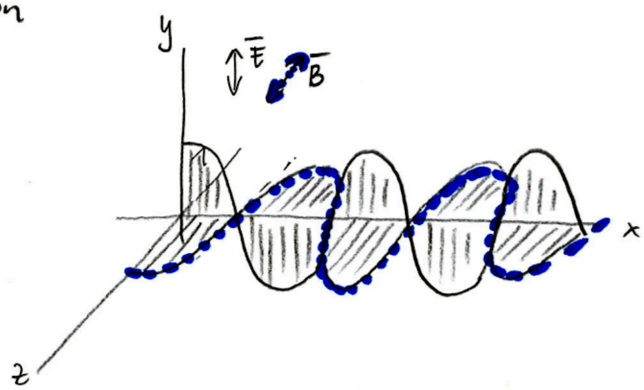
(2) $\bar{E} \perp \bar{B}$

(3) $\bar{E}, \bar{B} \perp \hat{n}_x \Rightarrow$ transverse wave
↳ propagation direction

(4) amplitudes

$$B_0 = \frac{k}{\omega} E_0 = \frac{1}{c} E_0$$

Example - illustration



wave polarized in y-direction
↳ (direction of $\bar{E}$)

$$\tilde{\bar{E}}(x,t) = \tilde{E}_0 \, e^{i(kx-\omega t)} \, \hat{n}_y \qquad \xrightarrow[\text{(physical meaning)}]{\text{real parts}} \qquad E(x,t) = E_0 \cos(kx-\omega t + \varphi) \, \hat{n}_y$$

$$\tilde{\bar{B}}(x,t) = \frac{1}{c} \tilde{E}_0 \, e^{i(kx-\omega t)} \, \hat{n}_z \qquad\qquad B(x,t) = \frac{1}{c} E_0 \cos(kx-\omega t+\varphi) \, \hat{n}_z$$

In general (polarized in $\hat{n}$-direction, propagates in $\hat{k}$-direction)

$$\tilde{\bar{E}}(\bar{r},t) = \tilde{E}_0 \, e^{i(\bar{k}\bar{r}-\omega t)} \, \hat{n} \qquad |\bar{k}| = \frac{2\pi}{\lambda}$$

$$\tilde{\bar{B}}(\bar{r},t) = \frac{1}{c} \tilde{E}_0 \, e^{i(\bar{k}\bar{r}-\omega t)} \, (\hat{k} \times \hat{n})$$

↳ unit vectors
(Note: $\hat{n} \perp \hat{k}$)

Energy and momentum in E-M waves

Poynting's theorem (see Griffiths 8.1.2) - valid in general configuration (with charges/currents)

Recall: energy density (both fields present)

$$u = \frac{1}{2}\varepsilon_0 E^2 + \frac{1}{2\mu_0} B^2$$

$$U = \int u \, d\tau = \frac{1}{2}\int \left(\varepsilon_0 E^2 + \frac{1}{\mu_0} B^2\right) d\tau \quad \rightarrow \text{energy stored in the fields}$$

↳ element of volume

Problem:

Question: Work δW done by electromagnetic forces acting on the charges in the interval dt?

Consider charge q_{ei}:

$$\bar{F}\cdot d\bar{r} = q_{ei}\left(\bar{E} + \bar{v}\times\bar{B}\right) \underbrace{d\bar{r}}_{\bar{v}\cdot dt} = q_{ei}\bar{E}\,\bar{v}\cdot dt$$

But $q_{ei} = \rho \, d\tau$ and $\rho\bar{v} = \bar{J}$, hence

$$\frac{\delta W}{dt} = \int_\Omega \bar{E}\cdot\bar{J}\, d\tau$$

rate of work done on all charges in volume Ω { power delivered to volume

Use Ampère-Maxwell equation to express $\bar{J}$ in terms of $\bar{E}, \bar{B}$ $\left(\nabla\times\bar{B} = \mu_0\bar{J} + \mu_0\varepsilon_0 \frac{\partial \bar{E}}{\partial t}\right)$

$$\bar{E}\cdot\bar{J} = \frac{1}{\mu_0}\bar{E}\cdot(\nabla\times\bar{B}) - \bar{E}\cdot\varepsilon_0\frac{\partial\bar{E}}{\partial t}$$

vector identity: $\nabla\cdot(\bar{E}\times\bar{B}) = \bar{B}\cdot(\nabla\times\bar{E}) - \bar{E}(\nabla\times\bar{B})$

+ Faraday's law: $\nabla\times\bar{E} = -\frac{\partial\bar{B}}{\partial t}$

⇓

$$\bar{E}(\nabla\times\bar{B}) = -\bar{B}\frac{\partial\bar{B}}{\partial t} - \nabla\cdot(\bar{E}\times\bar{B})$$

Also $\bar{B}\cdot\frac{\partial\bar{B}}{\partial t} = \frac{1}{2}\frac{\partial}{\partial t}(B^2)$ and $\bar{E}\frac{\partial\bar{E}}{\partial t} = \frac{1}{2}\frac{\partial}{\partial t}(E^2)$.

Eventually

$$\bar{E}\cdot\bar{J} = -\frac{1}{2}\frac{\partial}{\partial t}\left(\varepsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) - \frac{1}{\mu_0}\nabla(\bar{E}\times\bar{B})$$

$$\frac{\delta W}{dt} = \int_\Omega \left(-\frac{1}{2}\frac{\partial}{\partial t}\left(\varepsilon_0 E^2 + \frac{1}{\mu_0}B^2\right)\right) d\tau - \frac{1}{\mu_0}\int_\Omega \nabla\cdot(\bar{E}\times\bar{B})\, d\tau \overset{G-O}{=}$$

$$= -\frac{d}{dt}\int_\Omega \frac{1}{2}\left(\varepsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) d\tau - \frac{1}{\mu_0}\oint_\Sigma (\bar{E}\times\bar{B})\, d\bar{S}$$

Poynting's Theorem

$$\boxed{\frac{dW}{dt} = -\frac{d}{dt}\int_{\Omega}\frac{1}{2}\left(\varepsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) d\tau \;-\frac{1}{\mu_0}\oint_{\Sigma}(\bar{E}\times\bar{B})\, d\bar{A}}$$

decrease of energy stored in fields

flow of energy through boundary

$$\boxed{\bar{S} \overset{def}{=} \frac{1}{\mu_0}(\bar{E}\times\bar{B})}$$

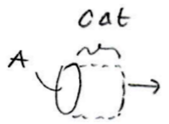
↳ Poynting vector

Interpretation: $|\bar{S}|$ = energy per unit time, per unit area transported by the fields

Example. Plane (monochromatic) wave propagating in x-direction

$$\bar{S} = \frac{1}{\mu_0}(\bar{E}\times\bar{B}) = \frac{1}{\mu_0}E_0 B_0 \cos^2(kx-\omega t+\delta)\hat{n}_x = c\varepsilon_0 E_0^2\cos^2(kx-\omega t+\delta)\hat{n}_x =$$

$$= c\cdot u\,\hat{n}_x = \underbrace{u}_{\text{energy density}}\cdot\underbrace{(c\,\hat{n}_x)}_{\text{phase velocity}}$$



$u\,Ac\Delta t = (uc)\,A\Delta t$

↳ energy flow per unit area, unit time

Momentum transport

The fields also carry momentum

$$\bar{\pi} = \frac{1}{c^2}\bar{S}$$

↳ momentum density

Recall: Special theory of relativity

$$E = \sqrt{p^2c^2 + m_0^2c^4}$$

for $m_0 = 0$ (photons) $\Rightarrow E = pc$

Example. Plane wave (monochromatic)

$$\bar{\pi} = \frac{1}{c}\varepsilon_0 E_0^2\cos^2(kx-\omega t+\delta)\hat{n}_x = \frac{1}{c}u\,\hat{n}_x$$

For light ($\lambda \sim 5\times10^{-7}$ m, $T\sim10^{-15}$ s) any macroscopic measurement extends over many periods $\Rightarrow$ average values are measured

$$\langle u\rangle = \frac{1}{2}\varepsilon_0 E_0^2 \qquad \left(\langle\cos^2\theta\rangle = \tfrac{1}{2}\right)$$

↳ over a full period

$$\langle\bar{S}\rangle = \frac{1}{2}c\varepsilon_0 E_0^2\,\hat{n}_x$$

$$\langle\bar{\pi}\rangle = \frac{1}{2c}\varepsilon_0 E_0^2\,\hat{n}_x$$

Intensity - average power per unit area $\quad I \overset{def}{=} \langle|\bar{S}|\rangle = \frac{1}{2}c\varepsilon_0 E_0^2$

Radiation pressure

$\bar{p}\rightarrow$ | absorption

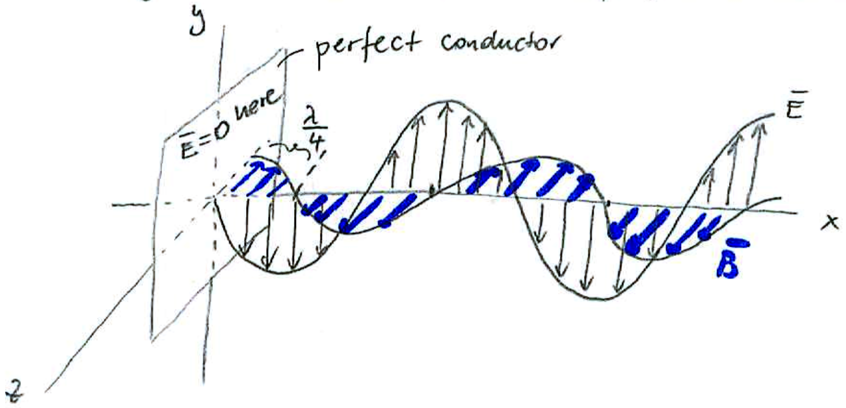
$$\text{pressure} = \frac{1}{A}\frac{\Delta p}{\Delta t} = \frac{1}{A}\frac{|\langle\bar{\pi}\rangle|\,Ac\Delta t}{\Delta t} = \frac{1}{2}\varepsilon_0 E_0^2 = \frac{I}{c}$$

reflection

$\bar{p}\rightarrow$, $\bar{p}\leftarrow$ | pressure $= \frac{2I}{c}$

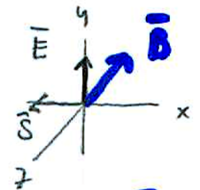
Standing e-m waves

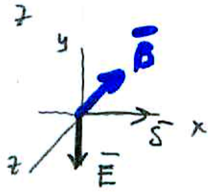
Reflection at a single metallic surface (perfect conductor)



(wave polarized in y-direction)

No electric field at the surface: two waves $\leftrightarrows$ phase-shifted by π





$$\tilde{\bar{E}}(x,t) = \underbrace{\tilde{E}_0 e^{i(kx+\omega t)} \hat{n}_y}_{\leftarrow \text{ incident}} + \underbrace{(-\tilde{E}_0) e^{i(kx-\omega t)} \hat{n}_y}_{\rightarrow \text{ reflected}}$$

$$= \tilde{E}_0 \left(e^{i(kx+\omega t)} - e^{i(kx-\omega t)} \right) \hat{n}_y$$

$$= \tilde{E}_0 e^{ikx} \underbrace{\left(e^{i\omega t} - e^{-i\omega t} \right)}_{2i \sin \omega t} \hat{n}_y = 2i \tilde{E}_0 e^{ikx} (\sin \omega t) \hat{n}_y$$

Real part (with $\varphi = 0$)

$$\boxed{\bar{E}(x,t) = -2 E_0 \hat{n}_y \sin(kx) \sin(\omega t)}$$

For $\bar{B}$: $\tilde{\bar{B}}(x,t) = -\tilde{B}_0 e^{i(kx+\omega t)} \hat{n}_z - \tilde{B}_0 e^{i(kx-\omega t)} \hat{n}_z$

$$\boxed{\bar{B}(x,t) = -2 B_0 \cos(kx) \cos(\omega t)}$$

Conclusion:

(1) $\bar{E}$ and $\bar{B}$ are phase-shifted (unlike in free-space travelling waves)

(2) "frozen" nodes of $\bar{E}$ and $\bar{B}$

$x_n = n \frac{\lambda}{2}$ $(n = 0,1,2,\dots)$ $\qquad$ $x_n = (2n+1) \frac{\lambda}{4}$ $(n = 0,1,2,\dots)$

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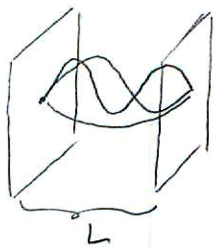
Reflection (two metallic surfaces)



$L = n \cdot \frac{\lambda}{2}$ $(n = 1,2,\dots)$ or $\lambda = \frac{2L}{n} = \lambda_n$

Corresponding frequencies

$\nu_n = \frac{c}{\lambda_n} = n \frac{c}{2L}$

Eq. in a microwave oven $\frac{\lambda}{2} \approx$ single cm