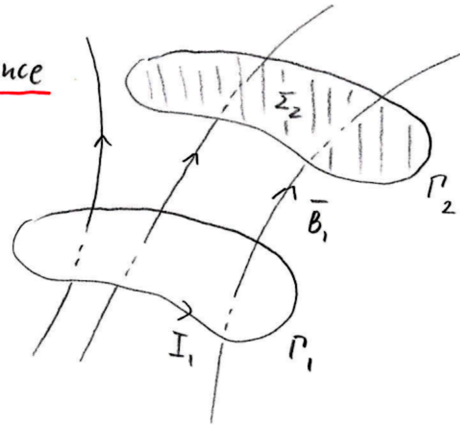
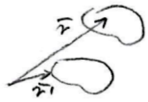
# INDUCTANCE

## Mutual Inductance




$I_1$ in $\Gamma_1$ $\Rightarrow$ $\bar{B}_1$ $\Rightarrow$ $\Phi_{B,2}$ flux through $\Sigma_2$ bounded by $\Gamma_2$

find $\Phi_{B,2}$




Biot and Savart

$$\bar{B}_1 = \frac{\mu_0}{4\pi} \oint_{\Gamma_1} I_1 \frac{d\bar{l}_1 \times (\bar{r}-\bar{r}')}{|\bar{r}-\bar{r}'|^3} \propto I_1$$

$$\Phi_{B,2} = \int_{\Sigma_2} \bar{B}_1 \, d\bar{S} \propto I_1$$

$d\bar{S}$: element of surface

Hence

$$\Phi_{B,2} = M_{21} I_1$$

$M_{21}$: mutual inductance

Note. $\Phi_{B,2} = \int_{\Sigma_2} \bar{B}_1 \cdot d\bar{S} = \int_{\Sigma_2} (\nabla \times \bar{A}_1)\, d\bar{S} \overset{\text{Stokes}}{=} \oint_{\Gamma_2} \bar{A}_1 \cdot d\bar{l}_2$

$\bar{A}_1$: vector potential

But (see problem set 8; recall $dI = \bar{J} \cdot d\bar{S}$)

$$\bar{A}_1 = \frac{\mu_0 I_1}{4\pi} \oint_{\Gamma_1} \frac{d\bar{l}_1}{|\bar{r}-\bar{r}'|}$$

Hence

$$\Phi_{B,2} = \frac{\mu_0 I_1}{4\pi} \oint_{\Gamma_2} \left( \oint_{\Gamma_1} \frac{d\bar{l}_1}{|\bar{r}-\bar{r}'|} \right) d\bar{l}_2 = \underbrace{\frac{\mu_0}{4\pi} \oint_{\Gamma_2} \left( \oint_{\Gamma_1} \frac{d\bar{l}_1}{|\bar{r}-\bar{r}'|} \right) d\bar{l}_2}_{M_{21}} \; I_1$$

$$\boxed{M_{21} = \frac{\mu_0}{4\pi} \oint_{\Gamma_2} \oint_{\Gamma_1} \frac{d\bar{l}_1}{|\bar{r}-\bar{r}'|} \, d\bar{l}_2}$$

**Neumann's formula**

(units of $M_{21}$: henry (H) $\quad H = \frac{Wb}{A} = \frac{V \cdot s}{A} = \frac{J}{A^2}$)

Comments

(1) Not very useful for practical calculations

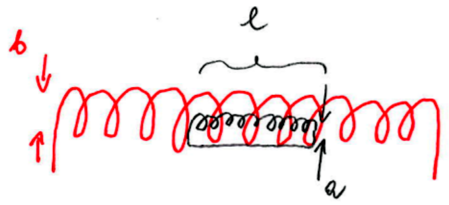
(2) $M_{21}$ depends only on geometry of the system (sizes/shapes) relative positions of the loops)

(3) $M_{21} = M_{12}$ (see Neumann's formula). Hence $M_{21} = M_{12} = M$ ↳ mutual inductance

$\Phi_{B,2} = M_{21} I_1 = M_{12} I_1 = \Phi_{B,1}$ (same current)

**Example** short solenoid (length $l$; radius $a$; $n_1$ turns per unit length) inside a long solenoid (radius $b$; $n_2$ turns per unit length)

Current $I$ flows through the short solenoid. Find the flux through the long one.




**Problem:** magnetic field due to the short solenoid – complicated.

**Idea:**

$$\Phi_{B,long} = M_{long,\,short} \underbrace{I}_{\text{through the short one}} = M_{short,\,long} \underbrace{I}_{\text{through the long one}} = \Phi_{B,\,short} = \underbrace{B}_{\text{due to the long one}} \cdot \pi a^2 \cdot \overbrace{n_1 l}^{\#\text{ loops}}$$

Magnetic field inside the long solenoid

$$B = \mu_0 I n_2$$

Hence

$$\Phi_{B,long} = \mu_0 I n_2 \pi a^2 n_1 l = \underbrace{\mu_0 \pi a^2 n_1 n_2 l}_{M} I$$

$$\boxed{M = \mu_0 \pi a^2 n_1 n_2 l} \quad \text{mutual inductance}$$

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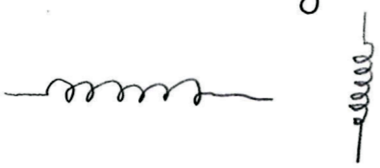
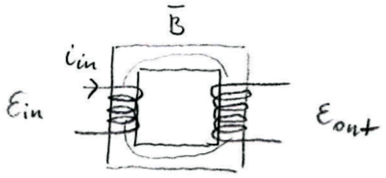
## Mutual inductance in practice

(–) elements of electric circuits may interact ⇒ unwanted effects

Solution:



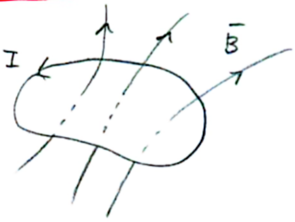
(+) can be used to transform voltage: transformers




if $\dot{i}_{in} \neq 0 \Rightarrow \mathcal{E}_{out} = -M \frac{di_{in}}{dt}$

# Self-inductance (inductance)

single loop

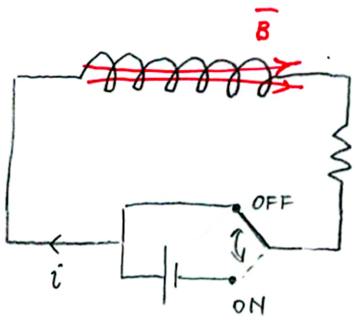



$$\Phi_B \propto I \quad ; \quad \Phi_B = L\, I$$

$\hookrightarrow$ Self-inductance (or simply "inductance") depending on geometry only

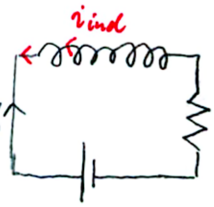
What if $\frac{dI}{dt} \neq 0$? Then $\dot{\Phi}_B \neq 0 \underset{\text{law}}{\overset{\text{Faraday's}}{\Rightarrow}}$ induced emf (direction $\rightarrow$ Lenz's Rule)

## Example

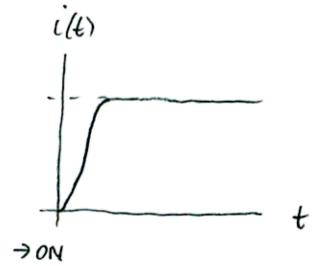



$$\frac{di}{dt} \neq 0 \overset{\dot{B} \neq 0}{\Rightarrow} \frac{d\Phi_B}{dt} \neq 0 \underset{\text{law}}{\overset{\text{Faraday's}}{\Rightarrow}} \mathcal{E}_{induced}$$

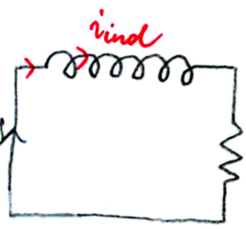
OFF $\rightarrow$ ON



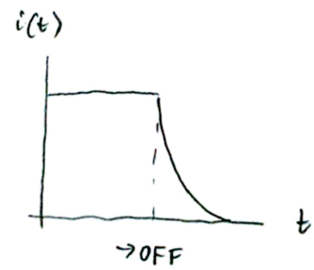

Lenz's rule: $i_{ind}$ opposes the change (increase of $i$)




ON $\rightarrow$ OFF




Lenz's rule: $i_{ind}$ opposes the change (decrease of $i$)

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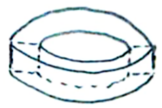
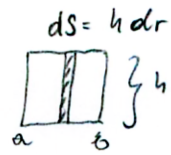
## Example (inductance of a toroidal coil with rectangular cross-section)



cross-section


Magnetic field inside $B = \frac{\mu_0 I N}{2\pi r}$ (see rec. class)

Flux through a single turn

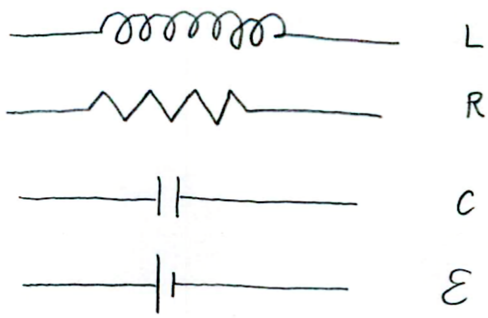



$$\Phi_B^{(1)} = \int \bar{B} \cdot d\bar{S} = \frac{\mu_0 I N}{2\pi} h \int_a^b \frac{dr}{r} = \frac{\mu_0 I N h}{2\pi} \ln\left(\frac{b}{a}\right)$$

Total flux

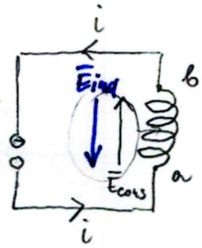
$$\Phi_B = \underset{\#\text{ of turns}}{N} \cdot \Phi_B^{(1)} = \underbrace{\frac{\mu_0 N^2 h}{2\pi} \ln\left(\frac{b}{a}\right)}_{L} I \quad ; \quad \boxed{L = \frac{\mu_0 N^2 h}{2\pi} \ln\left(\frac{b}{a}\right)}$$

# Inductive elements in circuits




## Self inductance and electric circuits

Recall: Electric fields induced by inductive elements are not conservative!




(assume R=0)

$\bar{E}_{total} = \bar{E}_{cons} + \bar{E}_{ind} = 0$

↳ within inductor with R=0

⇓

$\left( \int_a^b \bar{E}_{total} \cdot d\bar{l} = 0 \right)$

Faraday's law

$$\oint \bar{E}_{ind} \, d\bar{l} = -\overbrace{L \frac{di}{dt}}^{N \frac{d\Phi_B}{dt}}$$

↳ only $\bar{E}_{ind}$ contributes to circulation ($\bar{E}_{ind} \neq 0$ only within inductor)

$$\int_a^b \bar{E}_{ind} \cdot d\bar{l} = -L \frac{di}{dt}$$

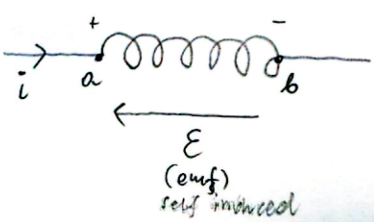
$$\int_a^b \bar{E}_{total} \cdot d\bar{l} = 0 \Rightarrow -\int_a^b \bar{E}_{cons} \, d\bar{l} = -L \frac{di}{dt} \Rightarrow V_{ab} = \int_a^b \bar{E}_{cons} \, dl = +L \frac{di}{dt}$$

↳ due to charge accumulated at the ends (a, b)

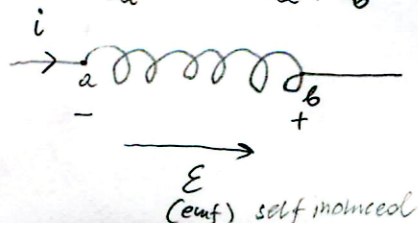
$$\boxed{V_a - V_b = L \frac{di}{dt}}$$

→ contribution of an inductive element to Kirchhoff's loop rule

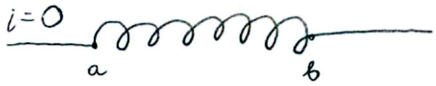
$\frac{di}{dt} > 0 \Rightarrow V_a > V_b$



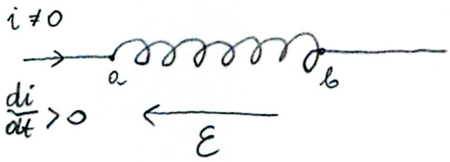

$\frac{di}{dt} < 0 \Rightarrow V_a < V_b$



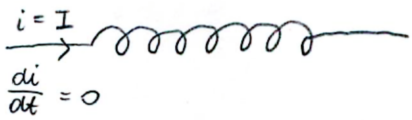
# Energy of the magnetic field

(1) 


$t=0\ ;\ i=0$

(2) 


$t>0\ ;\ i\neq 0\ ;\ \frac{di}{dt}>0$

(3) 


$t>t_f\ ;\ i=I=\text{const}\ ;\ \frac{di}{dt}=0$

Analyze (2)

Potential difference across inductive element: $V_{ab}=L\frac{di}{dt}$

Rate at which energy is delivered to inductor: $P=V_{ab}\,i$
(instanteneous)

$$P=\frac{dU}{dt}=L\frac{di}{dt}i \quad\Rightarrow\quad dU=L\,i\,di$$

Total energy delivered from $t=0$ to $t=t_f$ ((1) to (3))

$$U=\int_0^I L\,i\,di=\frac{1}{2}LI^2$$

## Comments:

(1) If $i=I=\text{const}$ the inductor (with $R=0$) acts as a perfect conductor, but there is energy accumulated during the process of current increase (from 0 to $I$) that is stored in the inductor

(2) The inductor is „energetically active" only if $\frac{di}{dt}\neq 0$

Compare: Resistor dissipates energy (at rate $i^2R$) whenever there is current through it ($i\neq 0$)

**Example.** Energy stored in a toroidal solenoid. (Assume $b-a \ll a$)

$$L = \frac{\mu_0 N^2 A}{2\pi R}$$

$R = \frac{a+b}{2}$

$2\pi R \times A$

$A = (b-a) \times h$

$$U = \frac{1}{2} L I^2 = \frac{1}{2} \frac{\mu_0 N^2 A}{2\pi R} I^2$$

Volume of the torroidal coil: $V = 2\pi R A$.

Magnetic energy density (energy/unit volume)

$$u = \frac{U}{V} = \frac{1}{2}\mu_0 \frac{N^2 I^2}{(2\pi R)^2}$$

Recall: $B = \frac{\mu_0 I N}{2\pi R}$, so $\frac{N^2 I^2}{(2\pi R)^2} = \frac{B^2}{\mu_0^2}$

Hence

$$\boxed{u = \frac{B^2}{2\mu_0}}$$

(vacuum)

$\rightarrow$ also valid for other geometries

**Comment:** Compare with the energy stored in the electric field between plates of a capacitor: $u = \frac{1}{2}\varepsilon_0 E^2$

**Note.** When there is some material (core) in the torus: $\mu = \mu_r \mu_0$

$\downarrow$

relative magnetic permeability

# CIRCUITS WITH INDUCTIVE ELEMENTS

① ⇒ apply Kirchhof's rules (valid at any instant of time)

(1) junction rule $\sum i = 0$

(2) loop rule $\sum V = 0$

potential difference accros inductor


② ⇒ step 1 yields a system of (differential) equs.

③ ⇒ solve

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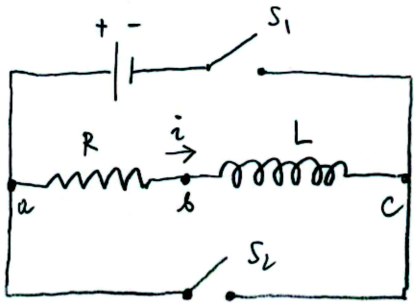
## RL circuit




R – the only resistive element

1° $S_1, S_2$ – open ; closing switch $S_1$

Initial condition $i(0) = 0$

Let $t > 0$, then $i = i(t)$ – current increasing ($\frac{di}{dt} > 0$)

$$v_{ab} = iR$$

$$v_{bc} = L\frac{di}{dt} > 0$$

$$(v_a > v_b > v_c)$$

Loop rule ( ) $\quad -iR - L\frac{di}{dt} + \mathcal{E} = 0 \quad$ ($1^{st}$ order ODE, separable)

$$\frac{di}{dt} = \frac{\mathcal{E} - iR}{L}$$

at $t=0$ : $\left.\frac{di}{dt}\right|_{t=0} = \frac{\mathcal{E}}{L}$

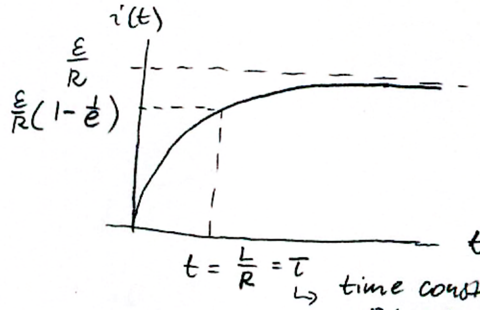
the larger L, the slower $i$ increases

$$\frac{di}{dt} = -\frac{R}{L}\left(i - \frac{\mathcal{E}}{R}\right)$$

$$\int_0^{i(t)} \frac{di}{i - \frac{\mathcal{E}}{R}} = -\frac{R}{L}\int_0^t dt$$

$$\ln\left|i - \frac{\mathcal{E}}{R}\right|_0^{i(t)} = -\frac{R}{L}t \quad \xRightarrow{i(t) < \frac{\mathcal{E}}{R}} \quad \ln\frac{-i(t) + \frac{\mathcal{E}}{R}}{\frac{\mathcal{E}}{R}} = -\frac{R}{L}t$$

$$\boxed{i(t) = \frac{\mathcal{E}}{R}\left(1 - e^{-\frac{R}{L}t}\right)}$$




↳ time constant for a RL circuit

Limiting cases:

(*) $t = 0 \Rightarrow i(0) = 0, \quad \frac{di}{dt}\Big|_{t=0} = \frac{\mathcal{E}}{L}$

(*) $t \to \infty \Rightarrow i(\infty) = \frac{\mathcal{E}}{R}, \quad \frac{di}{dt}\Big|_{t\to\infty} = 0$

Energy:

$\mathcal{E} i\, dt$ – delivered by source

$-i^2 R\, dt$ – dissipated in resistor

$-Li\, di$ – stored in inductor

$$\left.\begin{array}{l} \\ \\ \end{array}\right\} \mathcal{E} i\, dt - i^2 R\, dt - Li\, di = 0$$

Exercise: Calculate the total energy delivered, dissipated & stored (see calculations for RC circuit)

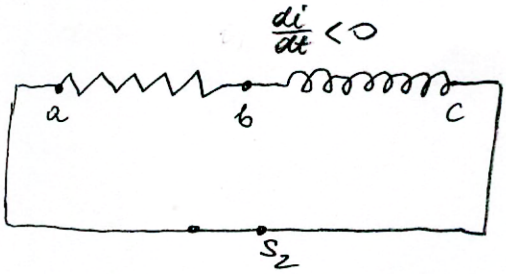
2° $S_1$ closed, $S_2$ open $\longrightarrow$ $S_1$ open, $S_2$ closed

initial condition $i(0) = I_0$

Now for $t > 0$, $i = i(t)$

$v_{ab} = iR$

$v_{bc} = L\frac{di}{dt} < 0$

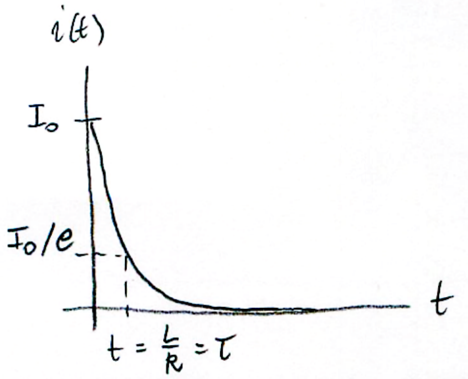



Loop rule ↻

$$-iR - L\underbrace{\frac{di}{dt}}_{<0} = 0 \Rightarrow \frac{di}{dt} + \frac{R}{L}i = 0$$

Solution:

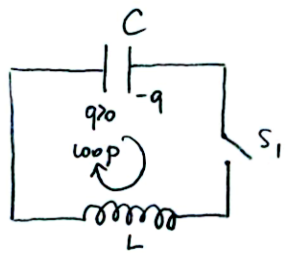
$$i(t) = I_0 e^{-\frac{R}{L}t}$$

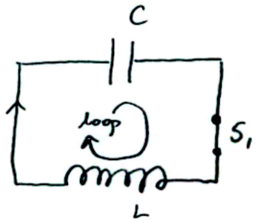



cf. Discussion of the discharging process in a RC circuit (however, here: energy stored in inductor's magnetic field)

# LC circuit

$\boxed{\text{FIG. 1}}$ (30,14)




At $t=0$ close switch $S_1$: $\left.\begin{array}{l} i(0)=0 \\ q(0)=Q_0 \end{array}\right\}$ initial conditions

Kirchhof's loop rule (assume positive direction of the current)

$$-\frac{q}{C}-L\frac{di}{dt}=0$$

But $i=\frac{dq}{dt}$

$$\frac{d^2q}{dt}+\frac{q}{LC}=0$$

($2^{nd}$ order linear homogeneous ODE with constant coeffs.)

Solution ($\omega_0=\sqrt{1/LC}$)

$$q(t)=A\sin\omega_0 t+B\cos\omega_0 t=$$

$$=C\cos(\omega_0 t+\varphi) \rightarrow \text{general solution}$$

to be found from initial conditions

Initial conditions:

$$q(0)=C\cos\varphi=Q_0 \quad \Rightarrow C=Q_0$$

$$i(0)=-\omega_0 C\sin\varphi=0 \quad \Rightarrow \varphi=0$$

Particular solution (satisfying the initial conditions)

$$\boxed{q(t)=Q_0\cos\omega_0 t}$$

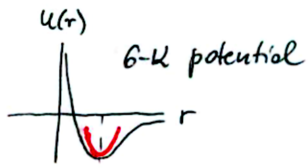
Compare: $m\ddot{x}+kx=0$

harmonic oscillator


$\omega_0=\sqrt{k/m}$


(small angles)

$\omega_0=\sqrt{l/g}$




$q(t) \leftrightarrow x(t)=C\cos(\omega_0 t+\varphi)$

$i(t) \leftrightarrow v(t)=\dot{x}(t)$

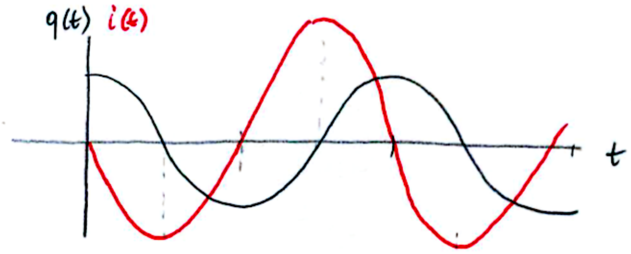
Any shape: equilibrium $=0$

$$u(x)=u(x_0)+u'(x_0)(x-x_0)+\frac{1}{2}u''(x_0)(x-x_0)^2+\ldots$$

$$\approx u(x_0)+\frac{1}{2}u''(x_0)(x-x_0)^2$$

Results:

$$q(t) = Q_0 \cos \omega_0 t$$
$$i(t) = \dot{q}(t) = -Q_0 \omega_0 \sin \omega_0 t$$




Energy (any instant $t$)

$$\frac{1}{2} L i^2 + \frac{q^2}{2C} = \frac{L}{2}(Q_0 \omega_0)^2 \sin^2 \omega_0 t + \frac{1}{2C} Q_0^2 \cos^2 \omega_0 t \qquad \underline{\underline{\omega_0 = (LC)^{-1/2}}}$$
$$= \frac{L}{2} Q_0^2 \frac{1}{LC} \sin^2 \omega_0 t + \frac{1}{2C} Q_0^2 \cos^2 \omega_0 t =$$
$$= \frac{Q_0^2}{2L}(\sin^2 \omega_0 t + \cos^2 \omega_0 t) = \frac{Q_0^2}{2L} = \text{const.}$$

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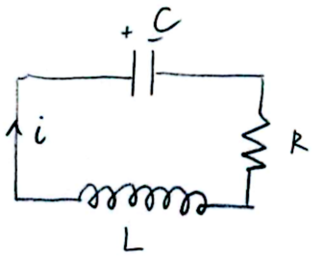
LRC circuit




Initial conditions:
e.g. $q(0) = Q_0$
$i(0) = 0$

loop rule
$$-iR - L\frac{di}{dt} - \frac{q}{C} = 0$$

Again $i = \dot{q}$
$$\ddot{q} + \frac{R}{L}\dot{q} + \frac{1}{LC} q = 0$$
2nd order linear ODE homogeneous with constant coeffs

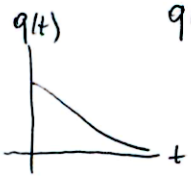
Solution

(1) underdamped regime $\left(R < \sqrt{4L/C}\right)$ [roots of char. eqn. complex]

$$q(t) = A e^{-\frac{R}{2L}t} \cos\left(\sqrt{\omega_0^2 - \frac{R^2}{4L^2}}\, t + \varphi\right)$$


(2) overdamped regime $\left(R > \sqrt{4L/C}\right)$ [roots of char. eqn. real & different]

$$q(t) = A e^{-\left(\frac{R}{2L} + \sqrt{\frac{R^2}{4L^2} - \omega_0^2}\right)t} + B e^{-\left(\frac{R}{2L} - \sqrt{\frac{R^2}{4L^2} - \omega_0^2}\right)t}$$




aperiodic

(3) critical damping $\left(R = \sqrt{4L/C}\right)$ $\quad q(t) = A e^{-\frac{R}{2L}t} + B t e^{-\frac{R}{2L}t}$ [roots real & equal]