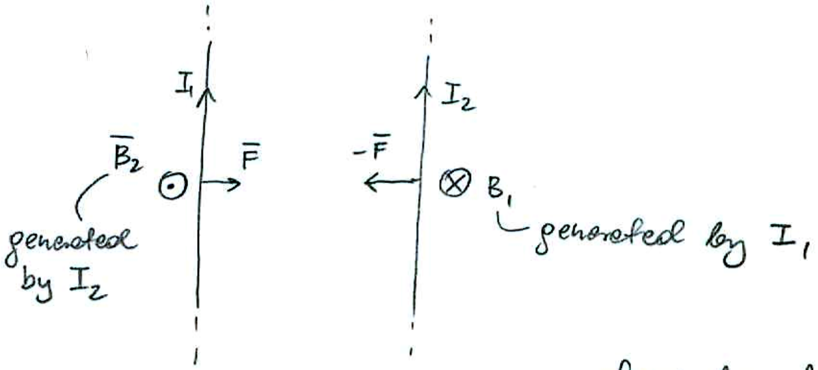
SOURCES OF MAGNETIC FIELD

Introduction – recall the experiment



formula for $\bar{B}$?

Electrostatics and magnetostatics

ELECTROSTATICS stationary charges* $\Longrightarrow$ constant (in time) electric field

MAGNETOSTATICS steady currents $\Longrightarrow$ constant (in time) magnetic field

* actually: enough if ρ does not depend on time

~ o ~

Comment (steady current)

$$I = \oint_{\Sigma} \bar{J}\cdot d\bar{A} \overset{\text{conservation}}{\underset{\text{of charge}}{=\!=}} -\frac{d}{dt} Q_{\Sigma} = -\frac{d}{dt}\int_{\Omega} \rho \, d\tau = -\int_{\Omega} \frac{\partial \rho}{\partial t} d\tau \quad (*)$$

el. of volume

rearrange using G-O theorem $\oint_{\Sigma} \bar{J}\cdot d\bar{A} = \int_{\Omega} \operatorname{div} \bar{J} \, d\tau$ (**)

Compare (*), (**) – rhs

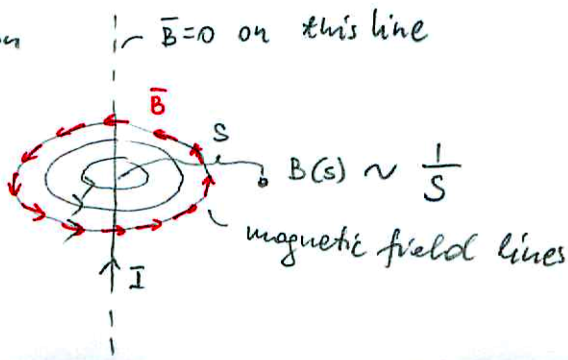
$$\int_{\Omega} \operatorname{div} \bar{J} \, d\tau = -\int_{\Omega} \left(\frac{\partial \rho}{\partial t}\right) d\tau \overset{\Omega\text{-any volume}}{\Longrightarrow} \boxed{\operatorname{div} \bar{J} + \frac{\partial \rho}{\partial t} = 0}$$

Continuity equation (conservation of charge)

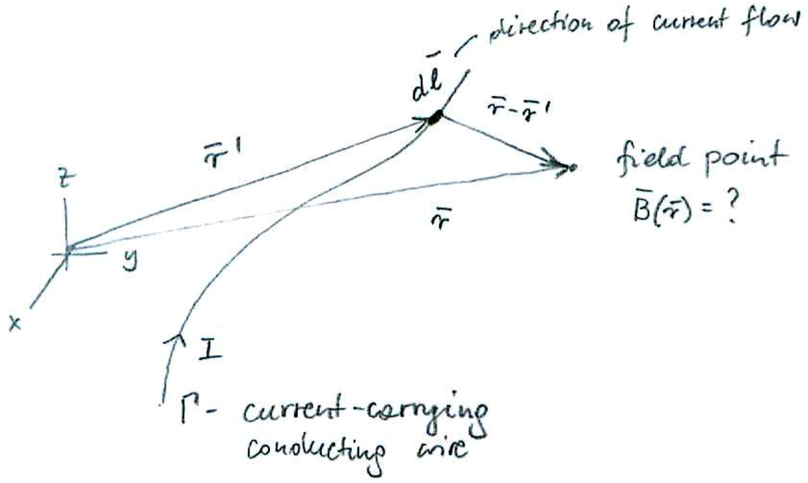
Steady current: $\operatorname{div} \bar{J} = 0 \Rightarrow \frac{\partial \rho}{\partial t} = 0$
(no „pile-up" of charge)

~ o ~

Experimental observation



MAGNETIC FIELD OF A STEADY CURRENT. LAW OF BIOT AND SAVART



$d\bar{l}$, $\bar{r}-\bar{r}'$, $\otimes\ d\bar{B}$

$$\boxed{\bar{B}(\bar{r}) = \frac{\mu_0}{4\pi}\int_\Gamma \frac{I\,d\bar{l}\times(\bar{r}-\bar{r}')}{|\bar{r}-\bar{r}'|^3} = \frac{\mu_0}{4\pi}\int_\Gamma \frac{I\,d\bar{l}\times\frac{\bar{r}-\bar{r}'}{|\bar{r}-\bar{r}'|}}{|\bar{r}-\bar{r}'|^2}}$$

$\frac{\bar{r}-\bar{r}'}{|\bar{r}-\bar{r}'|} \rightarrow$ unit vector

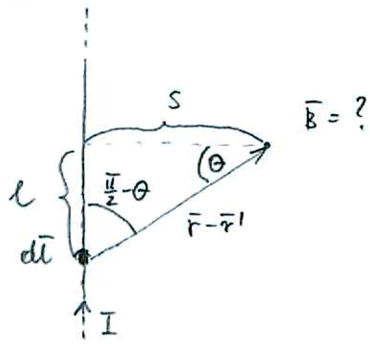
Law of Biot and Savart

$\mu_0 = 4\pi\times10^{-7}\ \frac{N}{A^2}$ - magnetic permeability of free space (magnetic constant)

Note: $c^2 = \frac{1}{\varepsilon_0\mu_0}$ (will be discussed later - e-m waves)
↳ speed of light in vacuum

Examples

(a) straight-line current



$$\bar{B} = \frac{\mu_0}{4\pi}\int_\Gamma \frac{I\,d\bar{l}\times\frac{\bar{r}-\bar{r}'}{|\bar{r}-\bar{r}'|}}{|\bar{r}-\bar{r}'|^2}$$

Observation: All contributions to $\bar{B}$ will be in the direction $\otimes$
enough to find the magnitude $|\bar{B}|$

First find $\left|d\bar{l}\times\frac{\bar{r}-\bar{r}'}{|\bar{r}-\bar{r}'|}\right| = dl\cdot\sin\left(\frac{\pi}{2}-\theta\right) = \cos\theta\,dl$

Also $\frac{l}{s} = \tan\theta \Rightarrow dl = \frac{s}{\cos^2\theta}\,d\theta \Rightarrow \left|d\bar{l}\times\frac{\bar{r}-\bar{r}'}{|\bar{r}-\bar{r}'|}\right| = \frac{s}{\cos\theta}\,d\theta$

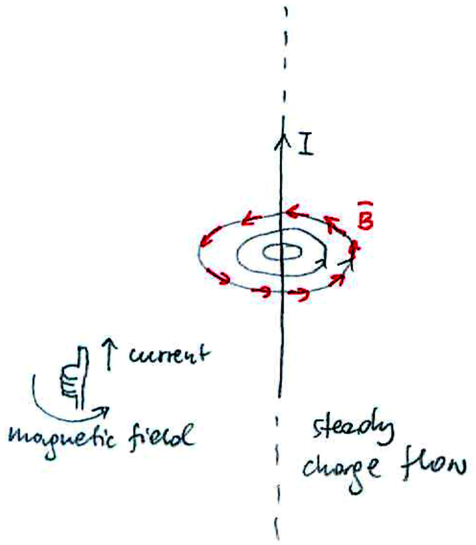
Moreover $\frac{s}{|\bar{r}-\bar{r}'|} = \cos\theta \Rightarrow \frac{1}{|\bar{r}-\bar{r}'|^2} = \frac{\cos^2\theta}{s^2}$

Hence $$B = \frac{\mu_0}{4\pi}I\int_{\theta_1}^{\theta_2}\frac{\frac{s}{\cos\theta}}{\frac{s^2}{\cos^2\theta}}\,d\theta = \frac{\mu_0 I}{4\pi s}\int_{\theta_1}^{\theta_2}\cos\theta\,d\theta = \underline{\underline{\frac{\mu_0 I}{4\pi s}(\sin\theta_2-\sin\theta_1)}}$$

Infinite wire: $\theta_1 = -\frac{\pi}{2}$, $\theta_2 = \frac{\pi}{2}$

$$\boxed{B = \frac{\mu_0 I}{2\pi s}} \sim \frac{1}{s}$$

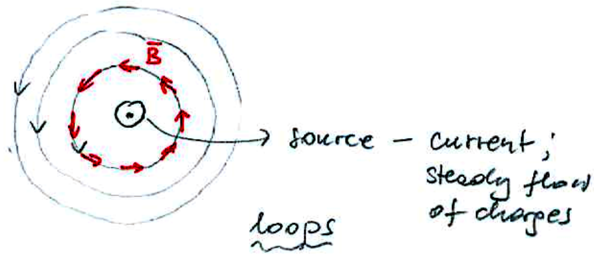
Discussion - magnetic field due to an infinite straight-line current-carrying wire



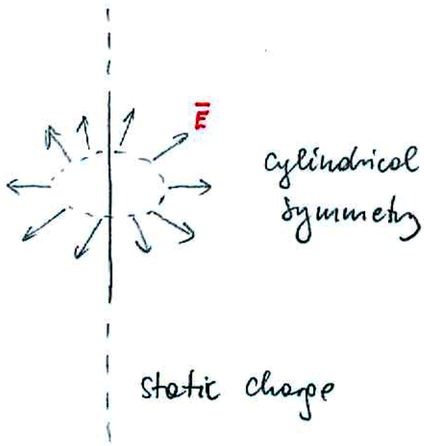
axial symmetry ⇒ magnetic field lines are concentric circles

$$B \sim \frac{I}{\text{distance from the wire}}$$

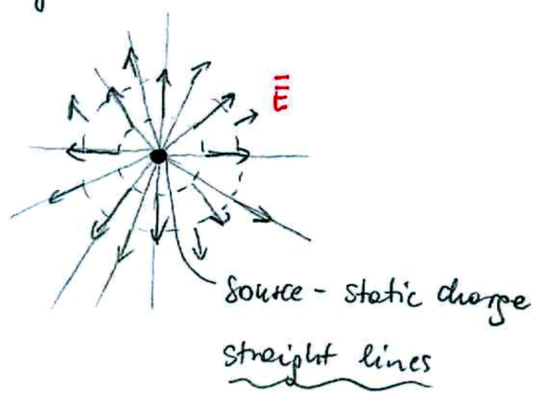
top view



Compare with electric field due to a wire charged uniformly with (positive) charge



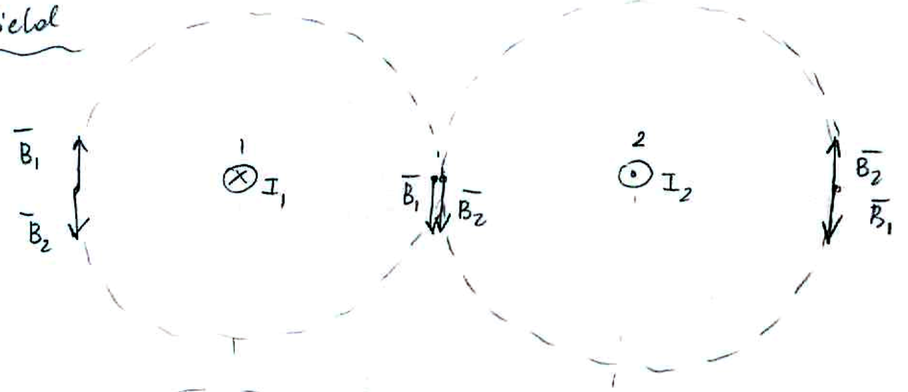
top view



Force between parallel current-carrying conductors (straight line)

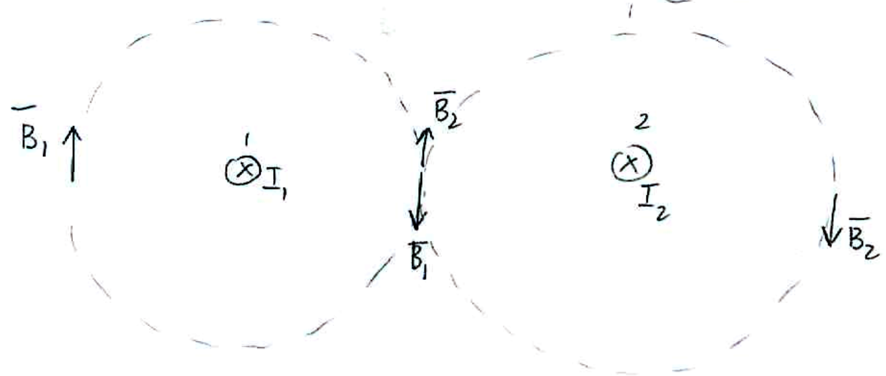
Magnetic field

(1)



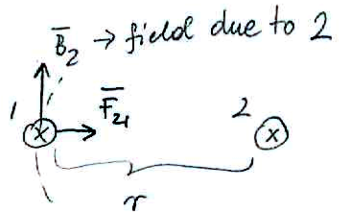
antiparallel currents

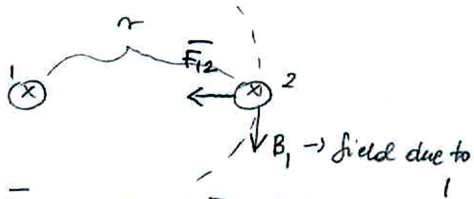
(2)



parallel currents

Force (2 on 1)





$\bar{F}_{21} = I_1 (\bar{\ell} \times \bar{B}_2)$

attraction

$\bar{F}_{12} = I_2 (\bar{\ell} \times \bar{B}_1)$

Magnitude ($\bar{\ell} \perp \bar{B}_{1,2}$)

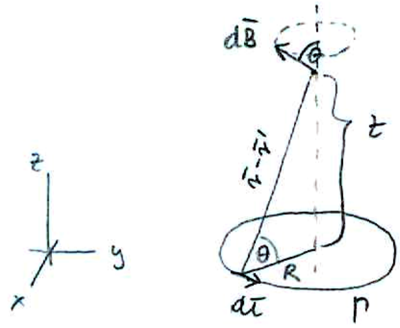
$$|\bar{F}_{21}| = I_1 \ell B_2 = I_1 \ell \frac{\mu_0 I_2}{2\pi r} = I_2 \ell \frac{\mu_0 I_1}{2\pi r} = |\bar{F}_{12}|$$

(3rd law of dynamics)

repulsion

Operational definition of 1 ampere: if $r = 1\,m$, $F_{12} = F_{21} = 2 \times 10^{-7} N$ (force per unit length), equal currents, then the current is 1 A.

Example (b) Circular current loop (radius R)



symmetry – horizontal (xy-plane) component vanishes

$$\bar{B} = B_z \hat{u}_z$$

$$dB_z = \cos\theta \, dB$$

↳ const for all points on the circle

Hence

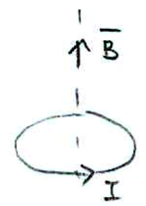
$$B_z = \cos\theta \, \frac{\mu_0}{4\pi} \int_\Gamma \frac{I \left| d\bar{l} \times \frac{\bar{r}-\bar{r}'}{|\bar{r}-\bar{r}'|} \right|}{|\bar{r}-\bar{r}'|^2}$$

But $d\bar{l} \perp \bar{r}-\bar{r}'$, hence $\left| d\bar{l} \times \frac{\bar{r}-\bar{r}'}{|\bar{r}-\bar{r}'|} \right| = dl$, and

$$B_z = \cos\theta \, \frac{\mu_0 I}{4\pi} \int_\Gamma \frac{dl}{z^2+R^2} \underset{\cos\theta = \frac{R}{(z^2+R^2)^{1/2}}}{=} \frac{\mu_0 I}{4\pi} \frac{R}{(z^2+R^2)^{3/2}} \overbrace{\int_\Gamma dl}^{2\pi R} =$$

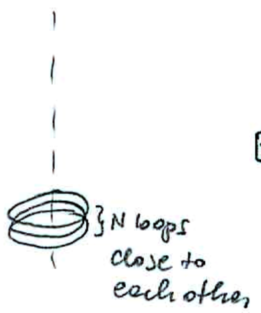
$$= \frac{\mu_0 I}{4\pi} \frac{R}{(z^2+R^2)^{3/2}} \, 2\pi R = \frac{\mu_0 I}{2} \frac{R^2}{(z^2+R^2)^{3/2}}$$

$$\boxed{B = \frac{\mu_0 I}{2} \frac{R^2}{(z^2+R^2)^{3/2}}}$$



(right-hand rule)

Comment: to produce stronger fields – use multiple loops



$$B \cong N \frac{\mu_0 I}{2} \frac{R^2}{(z^2+R^2)^{3/2}}$$

Ampere's law - introduction

electric field $\bar{E}$ ← superposition principle: $\bar{E} = \sum_i \bar{E}_i$

electric field $\bar{E}$ ← Gauss's law

$$\oint_\Sigma \bar{E}\cdot d\bar{A} = \frac{Q_{encl}}{\varepsilon_0}$$

magnetic field (due to moving charges/currents) ← superposition principle (Biot-Savart law)

$$\bar{B} = \sum_i \int_{conductor} \frac{\mu_0}{4\pi}\frac{I_i\, d\bar{l}_i \times \bar{r}}{r^3}$$

magnetic field ← ¿

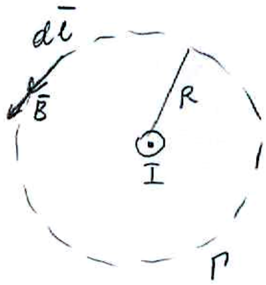
X Gauss's law $\oint_\Sigma \bar{B}\cdot d\bar{A} = 0$ (not very useful to find the field)

~ o ~

Ampere's law

Motivation

(a)



$$|\bar{B}| = \frac{\mu_0 I}{2\pi R}$$

Find the circulation

$$\oint_\Gamma \bar{B}\cdot d\bar{l}$$

↳ circulation of $\bar{B}$ along loop Γ

Recall: Convention for circulation

anticlockwise direction

positive orientation

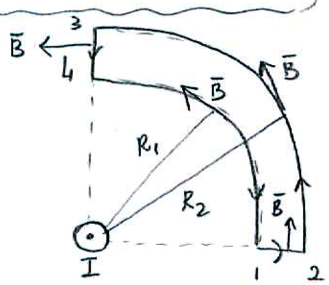
$$\oint_\Gamma \bar{B}\cdot d\bar{l} \overset{\bar{B}\parallel d\bar{l}}{=\!=} \oint_\Gamma B\, dl \overset{|\bar{B}| = const = B \text{ on } \Gamma}{=\!=\!=} B\oint_\Gamma dl = Bl = \frac{\mu_0 I}{2\pi R}\cdot 2\pi R =$$

Γ ↳ circle with radius R (oriented positively)

$$= \mu_0 I_{encl}$$

↳ current I enclosed by Γ

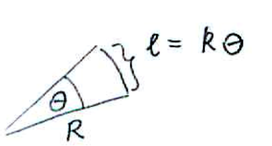
(b)



2,3
4,1 - arcs of circles

$$\oint_\Gamma \bar{B}\cdot d\bar{l} = \int_{12} \overbrace{\bar{B}\cdot d\bar{l}}^{\bar{B}\perp d\bar{l},\ =0} + \int_{23} \overbrace{\bar{B}\cdot d\bar{l}}^{\bar{B}\parallel d\bar{l}} + \int_{34} \overbrace{\bar{B}\cdot d\bar{l}}^{=0} + \int_{41} \underbrace{\bar{B}\cdot d\bar{l}}_{antiparallel} =$$

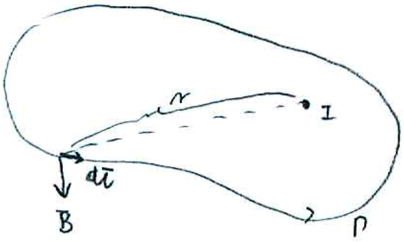
$$\underset{\text{on arcs}}{\overset{|\bar{B}| = const}{=\!=}} 0 + \frac{\mu_0 I}{2\pi R_2}\int_{23} dl + 0 - \frac{\mu_0 I}{2\pi R_1}\int_{41} dl =$$

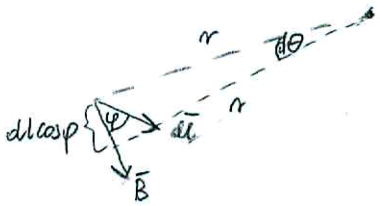


$l = R\Theta$

$$= \frac{\mu_0 I}{2\pi R_2}\Theta R_2 - \frac{\mu_0 I}{2\pi R_1}\Theta R_1 = 0 = \mu\, \underbrace{I_{encl}}_{=0}$$

Arbitrary shape of loop





$$\bar{B}\cdot d\bar{l} = B\cdot \underbrace{dl\cdot\cos\varphi}_{\approx r\,d\theta} = B\,r\,d\theta$$

(a) $I_{encl} \neq 0$ ($= I$)

$$\oint_\Gamma \bar{B}\cdot d\bar{l} = \oint \frac{\mu_0 I}{2\pi r}\, r\,d\theta = \frac{\mu_0 I}{2\pi}\oint d\theta = \frac{\mu_0 I}{2\pi}\cdot 2\pi = \mu_0 I$$

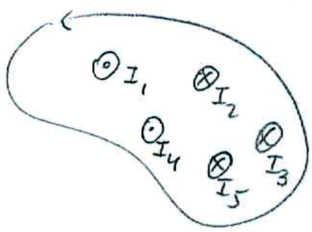
(b) $I_{encl} = 0$

(doesn't make a full turn)

$$\oint_\Gamma \bar{B}\cdot d\bar{l} = 0 \qquad \oint_\Gamma d\theta = \int_{\theta_1}^{\theta_2} d\theta + \int_{\theta_2}^{\theta_1} d\theta = 0$$

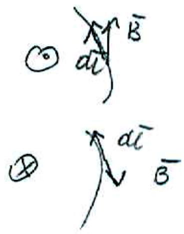
~ o ~

In general (multiple currents)



I_1, I_4 - positive contribution

I_2, I_3, I_5 - negative contribution



$$\boxed{\oint_\Gamma \bar{B}\cdot d\bar{l} = \mu_0 I_{encl} \quad \text{for any } \Gamma}$$

Ampère's law (magnetostatics)

Note:

* electric field $\oint_\Gamma \bar{E}\cdot d\bar{l} = 0 \;/\cdot q$

$$\oint_\Gamma q\bar{E}\cdot d\bar{l} = 0 \Rightarrow \oint_\Gamma \bar{F}\cdot d\bar{l} = 0 \Rightarrow$$ $\bar{E}$ is a potential field

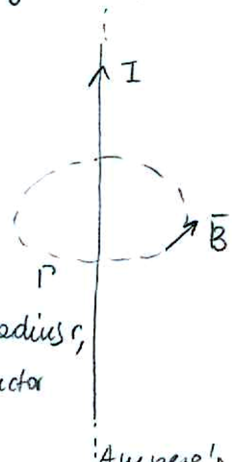
* magnetic field

$\oint_\Gamma \bar{B}\cdot d\bar{l} \rightarrow$ not related to work

$\bar{F} = q(\bar{v}\times\bar{B}) \not\parallel \bar{B}$ (but force F is not conservative anyway)

Examples

(a) long (infinite) straight current-carrying conductor



Γ: circle of radius r, axis - conductor

(1) symmetry: $B(\bar{r}) = B(r)$ ↳ depends on distance, not direction

(2) $\bar{B}$ has to be tangent to concentric circles (otherwise violation of Gauss's law - magnetic monopoles!)

Ampere's law $\quad \oint_\Gamma \bar{B}\cdot d\bar{l} = \mu_0 I_{encl}$

circulation: $\oint_\Gamma \bar{B}\cdot d\bar{l} \overset{\bar{B}\parallel d\bar{l}}{=} \int_\Gamma B(r)\, dl \overset{B=\text{const on }\Gamma}{=} B(r)\int_\Gamma dl = B(r)\, 2\pi r$

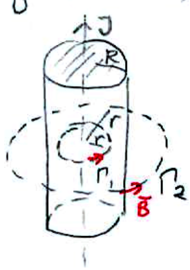
current enclosed:

$$I_{encl} = I$$

Hence

$$B(r)\, 2\pi r = \mu_0 I \Rightarrow \boxed{B(r) = \frac{\mu_0 I}{2\pi r}}$$

(b) long (infinite) solid cylinder carrying current with uniform density J



J = const

$I = J\cdot\pi R^2$

(1), (2) as in example (a)

$$\oint_\Gamma \bar{B}\cdot d\bar{l} = \mu_0 I_{encl}$$

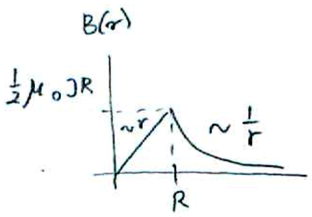
1° $r < R$

circulation $\oint_{\Gamma_1} \bar{B}\cdot d\bar{l} = \underset{\text{as in (a)}}{\dots} = B(r)\, 2\pi r$

current enclosed $I_{encl} = J\cdot\pi r^2$

Ampere's law $\quad B(r)\, 2\pi r = \mu_0 J \pi r^2$

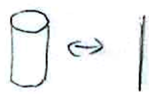
$$\boxed{B(r) = \frac{\mu_0 J}{2} r \quad \text{or} \quad B(r) = \frac{\mu_0 I}{2\pi R^2} r}$$



2° $r > R$

circulation as in 1° $\oint_{\Gamma_2} \bar{B}\cdot d\bar{l} = 2\pi r\, B(r)$

current enclosed $I_{encl} = J\cdot\pi R^2$

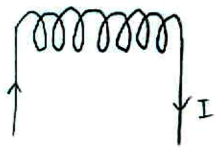


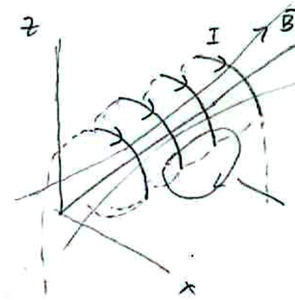
Ampere's law $\quad B(r)\, 2\pi r = \mu_0 J \pi R^2$

$$\boxed{B(r) = \frac{\mu_0 J R^2}{2r} \quad \text{or} \quad B(r) = \frac{\mu_0 I}{2\pi r}}$$

as for a line (if outside of the cylinder)

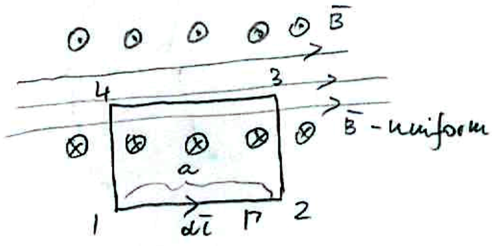
(c) solenoid (long, # of coils per unit length : n)





neglect $\bar{B}$ here (corresponds to $l \to \infty$)

cross-section (zy plane)



$$\oint_\Gamma \bar{B}\cdot d\bar{l} = \mu_0 I_{encl}$$

Circulation

$$\oint_\Gamma \bar{B}\cdot d\bar{l} = \underbrace{\int_{12} \bar{B}\cdot d\bar{l}}_{\substack{=0 \\ (\bar{B}=0)}} + \underbrace{\int_{23} \bar{B}\cdot d\bar{l}} + \int_{34} \bar{B}\cdot d\bar{l} + \underbrace{\int_{41} \bar{B}\cdot d\bar{l}} =$$

$= 0$ ($\bar{B} = 0$ or $\bar{B} \perp d\bar{l}$)

$$\overset{\bar{B} \parallel -d\bar{l}}{=} -\int_{34} B\,dl = -B \underbrace{\int_{34} dl}_{=a} = -Ba$$

Current enclosed

$$I_{encl} = -\underbrace{n.a}_{\#\text{ of coils}}\, I$$

(negative - opposite to $\hat{n}$ defined by loop orientation)

Ampere's law

$$-Ba = -\mu_0 n a I \Rightarrow \boxed{B = \mu_0 n \cdot I}$$

SUMMARY (of what we know so far)

electrostatics

$$\begin{cases} \oint_\Sigma \bar{E} \cdot d\bar{A} = \dfrac{Q_{encl}}{\varepsilon_0} \\ \oint_\Gamma \bar{E} \cdot d\bar{l} = 0 \end{cases} \qquad \begin{cases} \operatorname{div} \bar{E} = \dfrac{\rho}{\varepsilon_0} \\ \operatorname{rot} \bar{E} = 0 \end{cases}$$

$\rightarrow$ point sources of $\bar{E}$ exist

magnetostatics

$$\begin{cases} \oint_\Sigma \bar{B} \cdot d\bar{A} = 0 \\ \oint_\Gamma \bar{B} \cdot d\bar{l} = \mu_0 I_{encl} \end{cases} \qquad \begin{cases} \operatorname{div} \bar{B} = 0 \\ \operatorname{rot} \bar{B} = \mu_0 \bar{J} \end{cases}$$

$\rightarrow$ point sources of $\bar{B}$ do not exist

+ force on a charge $\quad \bar{F} = q\bar{E} + q(\bar{v} \times \bar{B})$

Comment (differential form of Ampère's law)

Γ - boundary of surface Σ

$$\left.\begin{aligned} \oint_\Gamma \bar{B}\, d\bar{l} &\overset{\text{Stokes}}{=} \int_\Sigma \operatorname{rot} \bar{B}\, d\bar{A} \\ \mu_0 I_{encl} &= \mu_0 \int_\Sigma \bar{J}\, d\bar{A} \end{aligned}\right\} \operatorname{rot} \bar{B} = \mu_0 \bar{J}$$