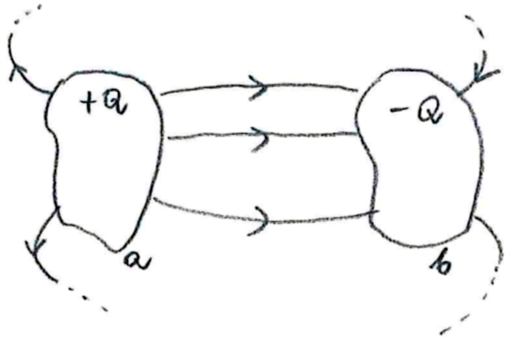
CAPACITANCE AND CAPACITORS



Capacitor – system of two conductors separated by an insulator (or vacuum)

Symbol in diagrams

Capacitor has a charge Q: the conductor at a higher potential has charge $Q>0$, and that at the lower potential has charge $-Q$.

Recall that the surface of a conductor is equipotential, hence we can define

$$V_{ab} = V_a - V_b$$

potential difference across the capacitor (voltage)

Q:
How to charge?

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CAPACITANCE of a capacitor

$$C \overset{\text{def}}{=} \frac{Q}{V_{ab}}$$

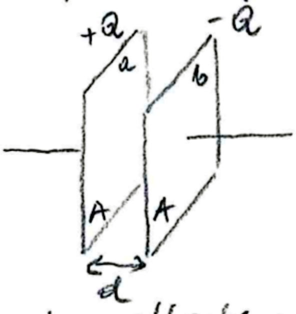
units: $1\,\text{F}$ (farad) $= \frac{1\,\text{Coulomb}}{1\,\text{Volt}}$

Note. 1F is a huge capacitance, usually μF, nF, pF

In almost all cases, C depends on the geometry of the conductors and the nature of the insulator between them.

Examples (vacuum capacitors)

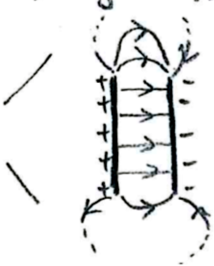
(a) parallel-plate capacitor



$$\sigma = \frac{Q}{A} = \text{const}$$

If $d \ll \sqrt{A}$, edge effects can be ignored

neglect this



Potential difference

$$V_{ab} = V_a - V_b =$$

$$= \int_{a\to\infty} \bar{E}\, d\bar{r} - \int_{b\to\infty} \bar{E}\, d\bar{r} =$$

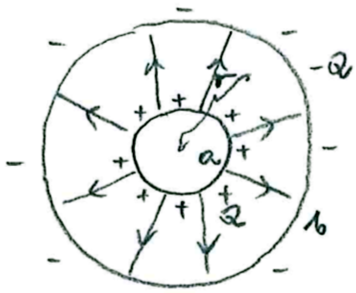
$$\overset{V(\infty)=0}{=} \int_{a\to b} \bar{E}\, d\bar{r} \overset{E=\text{const}}{=}$$

$$= \frac{\sigma}{\varepsilon_0} \int_{a\to b} dr = \frac{\sigma}{\varepsilon_0} d = \frac{Qd}{\varepsilon_0 A}$$

Hence

$$\boxed{C = \frac{Q}{V_{ab}} = \varepsilon_0 \frac{A}{d}}$$

(b) spherical capacitor (inner radius R_a , outer radius R_b)



(cross-section)

$\bar{E}$ between the spheres (magnitude)

$$|\bar{E}| = \frac{Q}{4\pi\varepsilon_0}\frac{1}{r^2}$$

Potential difference

$$V_{ab} = V_a - V_b = \int_{a\to b} \bar{E}\, d\bar{r} = \ldots = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{R_a} - \frac{1}{R_b}\right)$$

$$= \frac{Q}{4\pi\varepsilon_0}\,\frac{R_b - R_a}{R_a \cdot R_b}$$

Hence, the capacitance

$$\boxed{C = \frac{Q}{V_{ab}} = 4\pi\varepsilon_0 \frac{R_a R_b}{R_b - R_a}}$$

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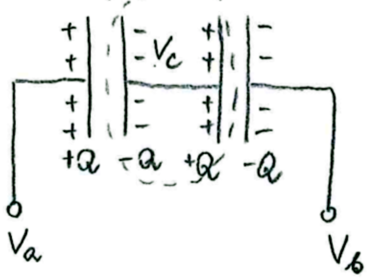
SYSTEMS of CAPACITORS

→ series connection

not connected to anything

C_1



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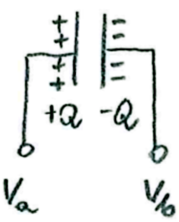
C_{eq} - equivalent capacitance



$$V_{ab} = V_{ac} + V_{cb}$$

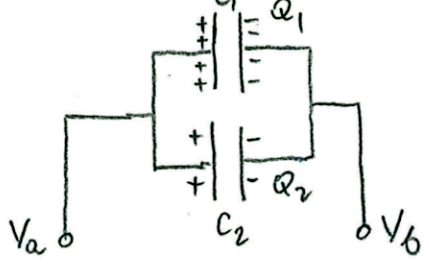
$$V_{ab} = \frac{Q}{C_1} + \frac{Q}{C_2}$$

$$V_{ab} = \frac{Q}{C_{eq}}$$

Hence $\frac{Q}{C_1} + \frac{Q}{C_2} = \frac{Q}{C_{eq}} \Rightarrow \boxed{\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2}}$ series connect

(can generalize to N-capacito

→ parallel connection



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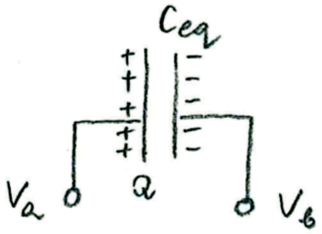


$$Q_1 + Q_2 = Q$$

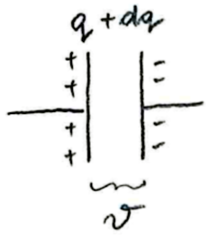
parallel connect

$$C_1 V_{ab} + C_2 V_{ab} = C_{eq} V_{ab} \Rightarrow \boxed{C_{eq} = C_1 + C_2}$$

(again, can be generalized)

Capacitors and energy

energy stored in a capacitor $\equiv$ work needed to charge it



q - changes during the charging process

(convention: lowercase letters denote time dependent quantities)

Elementary work
$$W_{el} = v\,dq = \frac{q}{C}\,dq$$

$Q \rightarrow$ final charge

Total work
$$W = \int_0^Q \frac{q}{C}\,dq = \frac{1}{2}\frac{Q^2}{C}$$

0 ↙ uncharged

Defining the (potential) energy of an uncharged capacitor to be zero, we have

$$\boxed{U = \frac{Q^2}{2C} \quad \left(\text{or } = \frac{1}{2}CV^2 = \frac{1}{2}QV\right)}$$

↳ final voltage across the capacitor

(use $C = \frac{Q}{V}$)

Applications of capacitors

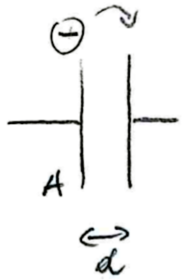
* energy storage & release (flashlight, some electric cars)

* important element in RC, LC, RLC circuits (will be discussed later)

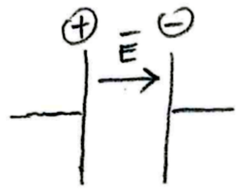
* can act as a „buffer" in circuits, preventing sudden changes of the current

Electric-field energy

E.g. parallel-plate capacitor



charging by moving electrons
⇒ work against the electric field between the plates



Observation:

The energy of a charged capacitor can be regarded as the energy of the electric field in the region between the plates. (outside $\vec{E} = 0$)

Energy density (energy per unit volume)

$$u = \frac{\text{energy}}{\text{volume}} = \frac{\frac{1}{2} C V^2}{A \cdot d}$$

But $C = \varepsilon_0 \frac{A}{d}$ and $V = E d$, so

$$\boxed{u = \frac{1}{2} \varepsilon_0 E^2}$$

Total energy stored

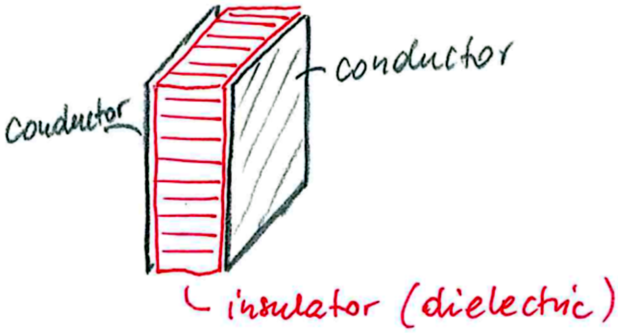
$$\boxed{U = \int_{\substack{\text{space}\\\text{between}\\\text{plates}\\\text{(or whole space)}}} u \, d\tau = \int_{\substack{\text{whole}\\\text{space}}} \frac{1}{2} \varepsilon_0 E^2 \, d\tau}$$

(cf. the value of U_{conf} discussed before)

↳ valid for any electric field configuration in vacuum

Exercise (see rec. class) Find U for a spherical capacitor

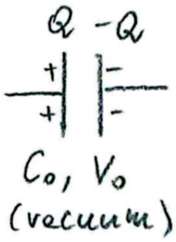
Dielectrics

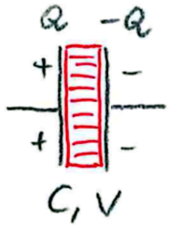


* seperates two conductors (no contact)
* better performance: more energy can be stored

FIG. 1 (24.14)

Dielectric constant





relative permittivity (dielectric constant)

$$\varepsilon_r = C/C_0$$

when detached from power source

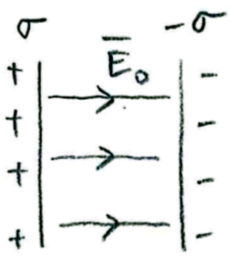
$$Q = \text{const} = C_0 V_0 = CV$$

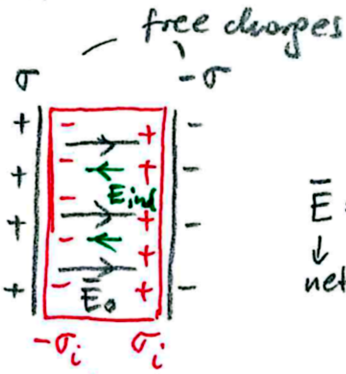
and hence

$$V = \frac{V_0}{\varepsilon_r}$$ → reduction by factor ε_r

TABLE (24.1)

Microscopic mechanism - POLARIZATION & induced charges

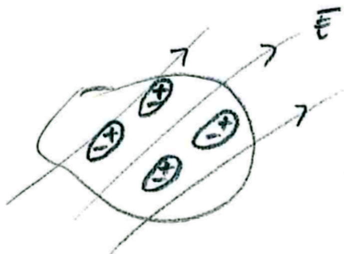




$$\bar{E} = \bar{E}_0 + \bar{E}_{ind}$$

↓ net field

induced charges (bound charges)
induced surface density σ_i



polarization = reorientation due to $\bar{E}$

$$V = \frac{V_0}{\varepsilon_r} \iff E = \frac{E_0}{\varepsilon_r}$$

Comment: For weak fields $\sigma_i \propto$ magnitude of the electric field inside the material

$E_0 = \frac{\sigma}{\varepsilon_0}$

$E = \frac{\sigma - \sigma_i}{\varepsilon_0}$

But $\quad \varepsilon_r = \frac{E_0}{E} = \frac{\sigma}{\sigma - \sigma_i} = \frac{1}{1 - \frac{\sigma_i}{\sigma}} \Rightarrow \boxed{\sigma_i = \sigma\left(1 - \frac{1}{\varepsilon_r}\right)}$

Note. If $\varepsilon_r \to \infty \Rightarrow \sigma_i \to \sigma$ (induced charge density $\to$ free charge density)

Permittivity of dielectric materials

$$E = \frac{E_0}{\varepsilon_r} = \frac{\sigma}{\varepsilon_0 \varepsilon_r} = \frac{\sigma}{\varepsilon}$$

↳ (absolute) permittivity

$$\varepsilon = \varepsilon_r \varepsilon_0$$

↳ relative permittivity (dimensionless)

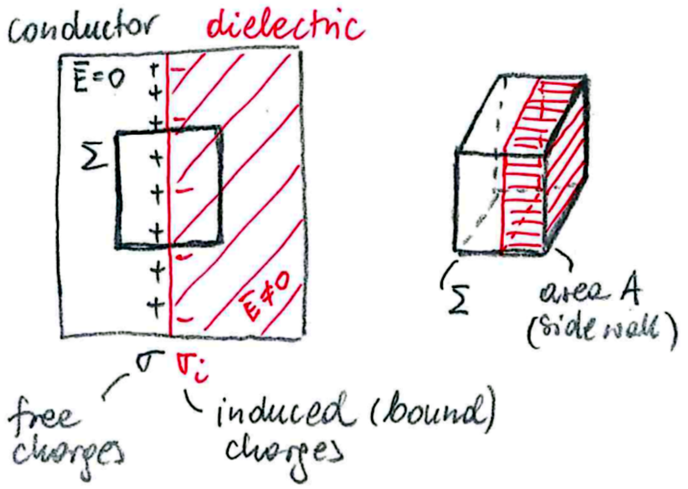
Other formulas for capacitors with dielectrics ($\varepsilon_0 \leftrightarrow \varepsilon_r \varepsilon_0$)

E.g.

$$C = \varepsilon_r \varepsilon_0 \frac{A}{d} = \varepsilon \frac{A}{d}$$

$$u = \frac{1}{2} \varepsilon_r \varepsilon_0 E^2 = \frac{1}{2} \varepsilon E^2$$

GAUSS'S LAW IN DIELECTRICS



Gauss's law $\displaystyle \oint_\Sigma \bar E \, d\bar A = \frac{Q_{encl}}{\varepsilon_0}$

$$\Downarrow$$

$$EA = \frac{\overbrace{(\sigma - \sigma_i)A}^{Q_{encl}}}{\varepsilon_0}$$

But $\displaystyle \sigma_i = \sigma\left(1 - \frac{1}{\varepsilon_r}\right)$

$$\Updownarrow$$

$$\sigma - \sigma_i = \frac{\sigma}{\varepsilon_r}$$

Hence $\displaystyle EA = \frac{\sigma A}{\varepsilon_r \varepsilon_0}$

or

$$\boxed{\varepsilon_r E A = \frac{\sigma A}{\varepsilon_0}}$$

flux of $\varepsilon_r \bar E$ through Σ

free charge

Gauss's law for dielectrics

$$\boxed{\oint_\Sigma \varepsilon_r \bar E \, d\bar A = \frac{Q_{free-enclosed}}{\varepsilon_0}}$$

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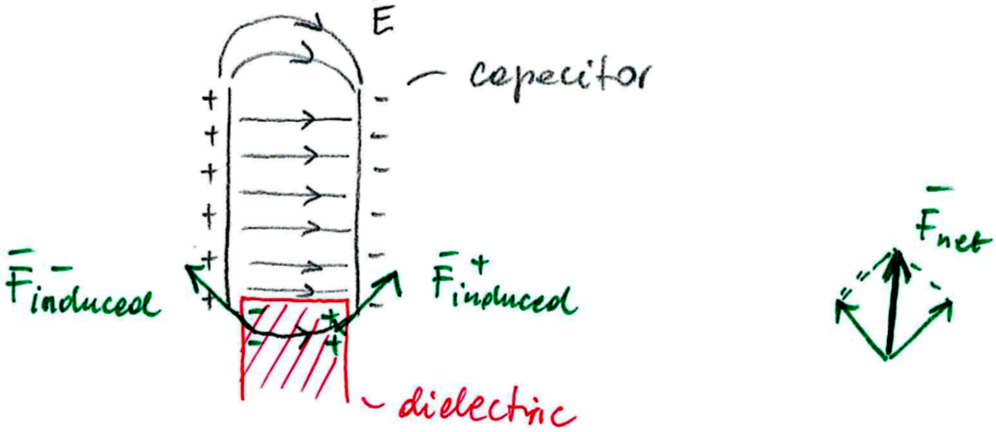
Final remark – FRINGING EFFECT (also, see problem set 4)



There is a net pull on the dielectric due to the fringing field