ELECTRIC POTENTIAL

Quick review: potential (conservative) vector fields (force fields)

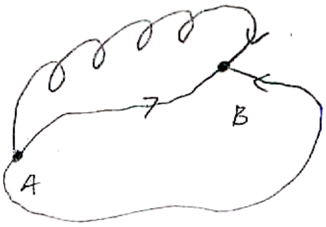
For a potential force field $\bar{F} = \bar{F}(\bar{r})$, the elementary work

$$\delta W = -dU$$

↳ potential energy (scalar!)

Implications:

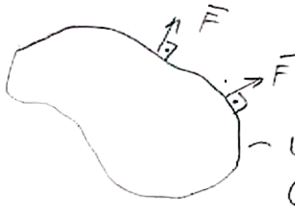
(1) $$W_{A\to B} = \int_{\Gamma_{AB}} \bar{F}\cdot d\bar{r} = U(A) - U(B)$$



(no path dependence)

(2) follows from (1): $\oint_{\Gamma} \bar{F}\cdot d\bar{r} = 0$; Γ - any closed loop

(3) $$\left.\begin{aligned} \delta W &= -dU = -\nabla U\cdot d\bar{r} \\ \delta W &= \bar{F}\cdot d\bar{r} \end{aligned}\right\} \quad \boxed{\bar{F} = -\nabla U = -\operatorname{grad} U}$$



$U(\bar{r}) = \text{const}$ (equipotential surface)

(5) conservation of energy

$$\left.\begin{aligned} \delta W &= -dU \\ \delta W &= dK \end{aligned}\right\} \Rightarrow d(U+K) = 0 \Rightarrow \boxed{U + K = \text{const}}$$

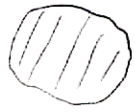
↳ work - k.e. theorem

Criterion

$\bar{F}$ is conservative in a simply-connected region Ω

$\Updownarrow$

$\operatorname{rot} \bar{F} \equiv 0$ throughout Ω

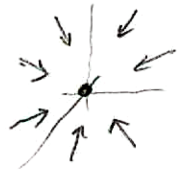


simply-connected

not simply-conn

Central forces - are potential (conservative)

$$\bar{F}(\bar{r}) = f(r)\,\bar{r}$$



inward / outward

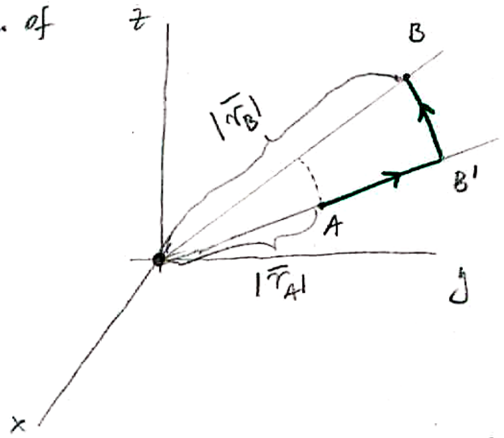
magnitude depends only on the distance from the center (not direction)

Examples: (*) gravitational force $f(r) = -G\dfrac{Mm}{r^3}$

(*) Coulomb force $f(r) = \dfrac{1}{4\pi\varepsilon_0}\dfrac{qQ}{r^3}$

Potential energy due to a central force

Choice of path



Γ_{AB}!

AB' - along radial direction, here $\bar{F} \parallel d\bar{r}$

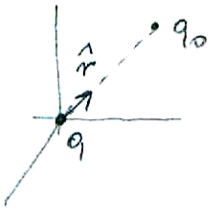
B'B - along an arc of a circle, here $\bar{F} \perp d\bar{r}$

$$U_A - U_B = \int_\Gamma \bar{F}\cdot d\bar{r} = \int_{AB'} \overbrace{\bar{F}(\bar{r})\cdot d\bar{r}}^{f(r)\,r\,dr} + \int_{B'B} \overbrace{\bar{F}(\bar{r})\cdot d\bar{r}}^{=0} = \int_{|\bar{r}_A|}^{|\bar{r}_B|} f(r)\,r\,dr$$

$$\boxed{U_A - U_B = \int_{|\bar{r}_A|}^{|\bar{r}_B|} f(r)\,r\,dr}$$ any central force

Note. Only the difference $U_A - U_B$ is measurable (physical); the potential energy is determined up to an additive constant

Coulomb force (2 point charges)



$$\bar{F}(\bar{r}) = \frac{1}{4\pi\varepsilon_0}\frac{q q_0}{r^2}\hat{r} = \underbrace{\frac{1}{4\pi\varepsilon_0}\frac{q q_0}{r^3}}_{f(r)}\bar{r}$$

$$f(r) = \frac{1}{4\pi\varepsilon_0}\frac{q q_0}{r^3}$$

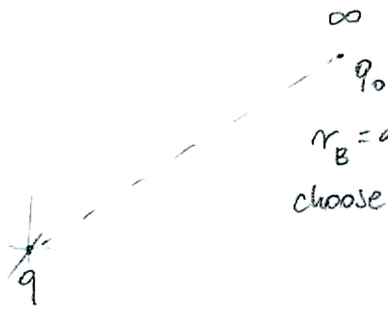
and
$$U_A - U_B = \int_{r_A}^{r_B} \frac{1}{4\pi\varepsilon_0}\frac{q q_0}{r^2}\,dr = \frac{q q_0}{4\pi\varepsilon_0}\left(\frac{1}{r_A} - \frac{1}{r_B}\right)$$

Interpretation (two possible viewpoints)

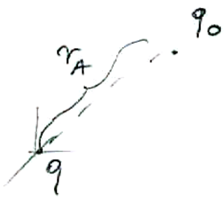
(1) $$U_A - U_B = W_{A\to B} = \int_{A\to B} \bar{F}\,d\bar{r}$$ work done by the electric force on a charge moving from A to B

(2) $$U_A - U_B = \int_{B\to A} \bar{F}_{ext}\cdot d\bar{r} = \int_{B\to A} (-\bar{F})\cdot d\bar{r} = \int_{A\to B} \bar{F}\cdot d\bar{r}$$

work done by an external force on a charge moving from B to A



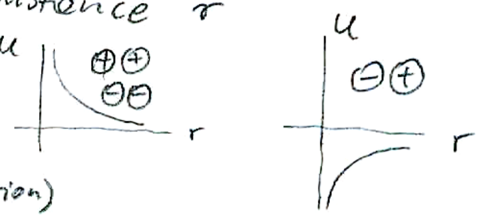
$r_B = \infty$
choose $U(r_B) = 0$



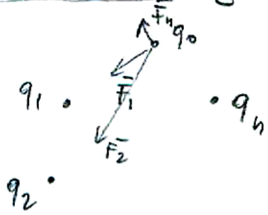
$$U_A = \frac{q q_0}{4\pi\varepsilon_0}\frac{1}{r_A}$$

$$\boxed{U = \frac{1}{4\pi\varepsilon_0}\frac{q q_0}{r}}$$

electric potential energy of two point charges at a distance r



Several point charges



($q_1, \ldots, q_n$ - static charge configuration)

Superposition principle

$$\bar{F} = \bar{F}_1 + \bar{F}_2 + \ldots + \bar{F}_n$$

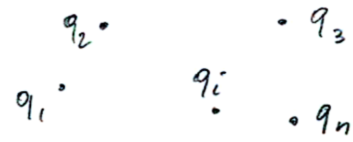
$$U_A - U_B = \int_{A\to B} \bar{F}\cdot d\bar{r} = \int_{A\to B} \bar{F}_1\cdot d\bar{r} + \ldots + \int_{A\to B} \bar{F}_n\cdot d\bar{r}$$

Hence U - additive

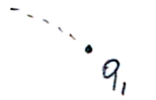
$$\boxed{U = \frac{q_0}{4\pi\varepsilon_0}\sum_{i=1}^{n}\frac{q_i}{r_{0i}}}$$

r_{i0} - distance between q_i and q_0

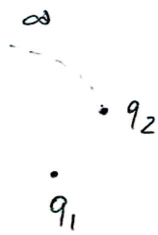
Question: What is the potential energy of the configuration of n charges itself?



start with all charges far apart and bring them together $\longrightarrow$

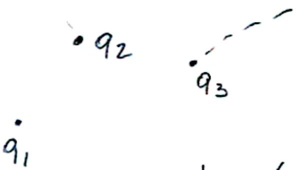


$u_1 = 0$ $\longrightarrow$



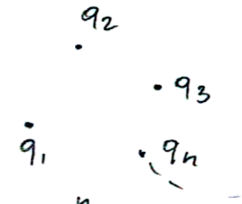
$$u_2 = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r_{12}}$$

$\longrightarrow$



$$u_3 = \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1 q_2}{r_{12}} + \frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}}\right)$$

$\rightarrow \cdots \rightarrow$



$$u_{conf} = \frac{1}{4\pi\varepsilon_0}\sum_{\substack{i,j=1\\ i<j}}^{n}\frac{q_i q_j}{r_{ij}}$$

(all pairs; count once)

$$\boxed{u_{conf} = \frac{1}{4\pi\varepsilon_0}\sum_{\substack{i,j=1\\ i<j}}^{n}\frac{q_i q_j}{r_{ij}} = \frac{1}{8\pi\varepsilon_0}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\frac{q_i q_j}{r_{ij}}}$$

Interpretation: Work needed to be done by an external force to form the system from charges $q_1, q_2, \dots, q_n$ initially far (∞) apart.

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Uniform electric field - potential energy

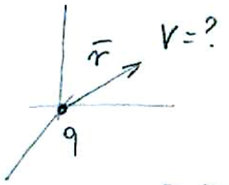
FiG. 1

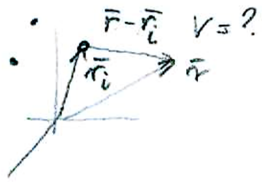
Electric potential

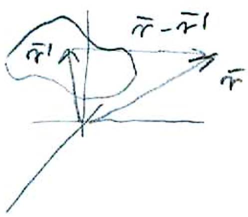
$$V = \frac{U}{q_0} \qquad \left(\begin{matrix}\text{potential energy}\\ \text{per unit charge}\end{matrix}\right)$$

How to find?

1st method (superposition principle)







$V(\bar{r})$	source
$\frac{1}{4\pi\varepsilon_0}\frac{q}{r}$	single point charge
$\frac{1}{4\pi\varepsilon_0}\sum_i \frac{q_i}{\lvert\bar{r}-\bar{r}_i\rvert}$	discrete distribution of charges
$\frac{1}{4\pi\varepsilon_0}\int_{\text{charged object}} \frac{dq}{\lvert\bar{r}-\bar{r}'\rvert}$	Continuous distribution of charges $\quad dq = \rho\, dV$ (3D), $dq = \sigma\, dA$ (2D), $dq = \lambda\, dl$ (1D)

2nd method

$$\bar{F} = -\operatorname{grad} U \quad /:q_0 \qquad \Longleftrightarrow \qquad U_A - U_B = \int_{A\to B} \bar{F}\cdot d\bar{r} \quad /:q_0$$

$$\bar{E} = -\operatorname{grad} V \qquad\qquad V_A - V_B = \int_{A\to B} \bar{E}\cdot d\bar{r}$$

(known $V \Rightarrow$ find $\bar{E}$) $\qquad$ $A\to B$ → any path

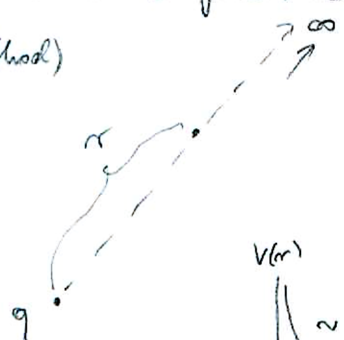
(known $\bar{E} \Rightarrow$ find V)

Choose $V_B = 0$ (B - reference point)

Examples

(a) electric potential of a point charge (choose $V(\infty) = 0$ → ref. point)

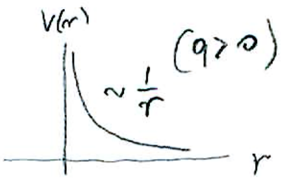
(2nd method)



$$\bar{E}(\bar{r}) = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\frac{\bar{r}}{r}$$

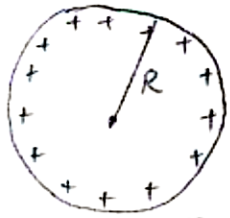
$$V(\bar{r}) = \int_r^\infty \bar{E}(\bar{r})\, d\bar{r} \overset{\bar{E}\parallel d\bar{r}}{=} \int_r^\infty \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\frac{\cancel{r}\,dr}{\cancel{r}} =$$

$$= \int_r^\infty \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\,dr = \frac{1}{4\pi\varepsilon_0}\frac{q}{r}$$



$$\boxed{V(\bar{r}) = \frac{1}{4\pi\varepsilon_0}\frac{q}{r}}$$

Example (6) charged conducting sphere, total charge q

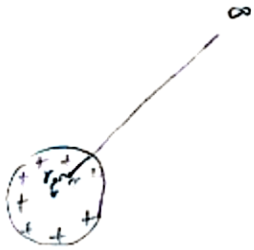


Previous lecture: $\bar{E}(\bar{r}) = \begin{cases} \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\frac{\bar{r}}{r} & \text{if } r>R \\ 0 & \text{if } r<R \end{cases}$

1° $r > R$

$$V(r) - V(\infty) = \int_r^\infty \bar{E}(\bar{r})\,d\bar{r} \overset{\bar{E} \parallel d\bar{r}}{=} \int_r^\infty \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\frac{\cancel{r}\,dr}{\cancel{r}} =$$

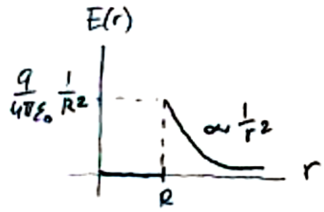
$$= \underline{\frac{1}{4\pi\varepsilon_0}\frac{q}{r}}$$



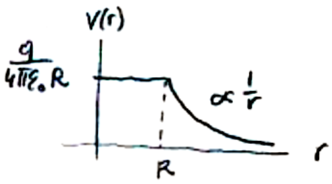
2° $r < R$

$$V(r) - V(\infty) = \int_r^\infty \bar{E}(\bar{r})\,d\bar{r} = \int_r^R \underbrace{\bar{E}(\bar{r})\,d\bar{r}}_{\substack{\equiv 0 \\ \text{(inside)}}} + \int_R^\infty \underbrace{\bar{E}(\bar{r})\,d\bar{r}}_{\text{(outside)}} =$$

$$= \int_R^\infty \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\,dr = \underline{\frac{q}{4\pi\varepsilon_0}\frac{1}{R}}$$

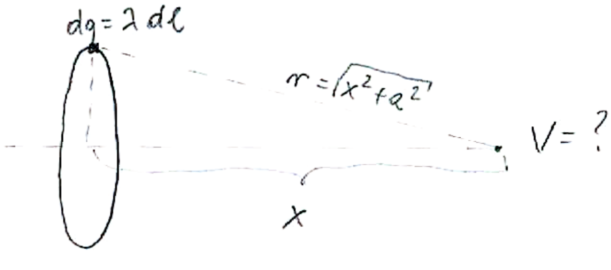


$$V(\bar{r}) = \begin{cases} \frac{1}{4\pi\varepsilon_0}\frac{q}{r} & \text{if } r>R \\ \frac{1}{4\pi\varepsilon_0}\frac{q}{R} & \text{if } r<R \end{cases}$$



(c) charged ring

(1st method)



reference point - infinity

Superposition → add all contributions (treated as point charges)

$$V = \frac{1}{4\pi\varepsilon_0}\int_{ring}\frac{dq}{r} = \frac{1}{4\pi\varepsilon_0}\frac{1}{\sqrt{x^2+a^2}}\int_{ring}\lambda\, dl = \frac{1}{4\pi\varepsilon_0}\frac{\overbrace{(2\pi a\lambda)}^{\text{total charge on the ring}}}{\sqrt{x^2+a^2}}$$

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Potential energy of a (continuous) charge distribution - revisited

Recall: (discrete distribution) $q_1 \cdot\; q_2 \;\ldots\; q_n$

$$U_{conf} = \frac{1}{8\pi\varepsilon_0}\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n}\frac{q_i q_j}{r_{ij}} = \frac{1}{2}\sum_{i=1}^{n} q_i \underbrace{\left(\sum_{\substack{j=1\\ j\neq i}}^{n}\frac{1}{4\pi\varepsilon_0}\frac{q_j}{r_{ij}}\right)}_{V(\bar r_i)}$$

$V(\bar r_i)$ potential due to all charges (except q_i) at the position of q_i

$$\boxed{U_{conf} = \frac{1}{2}\sum_{i=1}^{n} q_i\, V(\bar r_i)} \quad (*)$$

For a continuous distribution (3D)

$$U_{conf} = \frac{1}{2}\int_\Omega \rho V\, d\tau$$

$d\tau$ ↳ element of volume

Rewrite in terms of $\bar E$ only. Use Gauss's law $\operatorname{div}\bar E = \frac{\rho}{\varepsilon_0}$

$$U_{conf} = \frac{\varepsilon_0}{2}\int_\Omega (\nabla\cdot\bar E)\, V\, d\tau$$

Integrate by parts

$$U_{conf} = \frac{\varepsilon_0}{2}\left[\oint_\Sigma V\bar E\cdot d\bar A - \int_\Omega \bar E\,\nabla V\, d\tau\right] \overset{\nabla V = -\bar E}{=}$$

$$= \frac{\varepsilon_0}{2}\left[\oint_\Sigma V\bar E\cdot d\bar A + \int_\Omega E^2\, d\tau\right]$$

Note that the integration region can be extended to any region that encloses Ω (including extension to $\mathbb{R}^3$) - put $\rho \equiv 0$ outside Ω

The surface integral then vanishes if we integrate over $\mathbb{R}^3$

$$\oint_{\Sigma} V \vec{E} \cdot d\vec{A} \xrightarrow[\text{extended to } \mathbb{R}^3]{\text{integration region}} 0$$

$V \sim \frac{1}{r}$

$E \sim \frac{1}{r^2}$

Surface $\sim r^2$

$$\boxed{U_{\text{conf}} = \frac{\varepsilon_0}{2} \int_{\mathbb{R}^3} E^2 \, d\tau} \qquad (**)$$

$\hookrightarrow$ all space!

Example Find the energy of a uniformly charged spherical shell of total charge q and radius R

Solution I

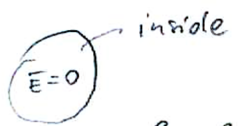
$$U_{\text{conf}} = \frac{1}{2} \int_{\text{sphere}} \sigma V \, dA = \frac{1}{2} \int_{\text{sphere}} \sigma \frac{1}{4\pi\varepsilon_0} \frac{q}{R} \, dA =$$

$$= \frac{1}{8\pi\varepsilon_0} \frac{q}{R} \int_{\text{sphere}} \sigma \, dA = \frac{1}{8\pi\varepsilon_0} \frac{q^2}{R}$$

Solution II

$$U_{\text{conf}} = \frac{\varepsilon_0}{2} \int_{\mathbb{R}^3} E^2 \, d\tau = \frac{\varepsilon_0}{2} \int_{\text{outside}} \frac{q^2}{(4\pi\varepsilon_0)^2 r^4} \, d\tau \overset{\text{spherical coord}}{=}$$

Recall



$\vec{E} = \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2} \hat{r}$

outside

$$= \frac{\varepsilon_0 q^2}{32 \varepsilon_0^2 \pi^2} \underbrace{4\pi}_{\text{over angles}} \int_R^\infty \underbrace{r^2}_{\text{from the Jacobian}} \frac{1}{r^4} \, dr =$$

$$= \frac{1}{8\pi\varepsilon_0} \frac{q^2}{R}$$

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Comments:

Eq. (**) implies $U_{\text{conf}} > 0$, whereas Eq. (*) allows $U_{\text{conf}} < 0$?

$+$ $-$ $U_{\text{conf}} < 0$

Eq. (**): $\hookrightarrow$ total energy stored in a charge configuration

Eq. (*): $\hookrightarrow$ does not take into account the work needed to create the point charges (problem: energy of a point charge is infinite)

Reason:

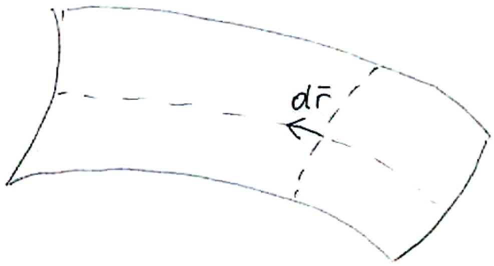
$$U_{\text{conf}} = \frac{1}{2} \sum_{i=1}^{n} q \underbrace{V(\vec{r}_i)}_{\text{due to all charges, except } q_i}$$

$$U_{\text{conf}} = \frac{1}{2} \int_{\Omega} \rho \underbrace{V}_{\text{total potential}} d\tau$$

Equipotential surfaces

Fig. 2

FACT: Electric field lines and equipotential surfaces are always mutually perpendicular.



$V = \text{const}$ (defines a surface in space → equipotential surface)

$d\bar{r}$ - infinitesimal displacement on the surface (i.e. locally tangential)

Recall: $\bar{E} = -\operatorname{grad} V \iff dV = -\bar{E} \cdot d\bar{r}$

For $V = \text{const} \Rightarrow dV = 0$. But $dV = -\bar{E} \cdot d\bar{r}$, hence

$$\bar{E} \cdot d\bar{r} = 0 \Rightarrow \boxed{\bar{E} \perp d\bar{r}}$$

Example. Further properties of conductors

(1) When all charges are at rest, the surface of a conductor is always an equipotential surface.

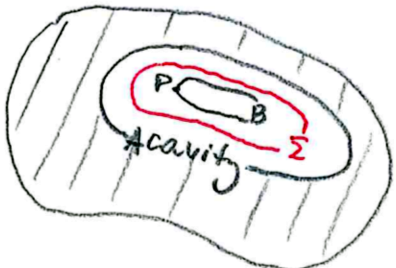
Recall: $\bar{E} \perp$ conductor's surface (cf. discussion of Gauss's law for conductors)

So $q_0 \bar{E} \cdot d\bar{r} = 0$ when moving along the conductor's surface $\Rightarrow dU = 0 \Rightarrow U = \text{const.}$

(2) conductor (may be charged, in general) with an empty cavity

FACT. Under electrostatic conditions, there is no charge anywhere on the surface of the cavity.

Justification: Idea: Show that $\bar{E} =$ inside the cavity $\Rightarrow \sigma = 0$ (on the cavity)



A – cavity's surface

Σ – Gaussian surface

B – surface within Σ; through point P

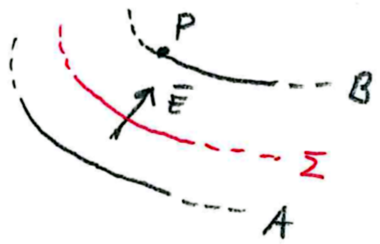
Step 1: Every point in the empty cavity is at the same potential

„Reductio ad absurdum"

(a) suppose P is at a different potential

(b) Construct an equipotential surface B through point P

(c) Consider a Gaussian surface between B and A (cavity's surface)



$V_A \neq V_B$ (from (a))

$\Downarrow$

$\bar{E} \neq 0$

$\Downarrow$

Φ_E through Σ is non-zero

$\Downarrow$ Gauss's law

there is charge inside cavity ✗

CONTRADICTION! We Assumed that the cavity is empty!

Every point in the empty cavity is at the same potential $\Leftarrow$

Step 2: Because of the statement in step 1, the electric field inside the cavity must be zero everywhere.

Step 3: But $|\bar{E}|$ at any point of the conductor's surface is proportional to σ $\Rightarrow$ surface density of charge on the wall of the cavity is zero at every point.

Example. Faraday's cage

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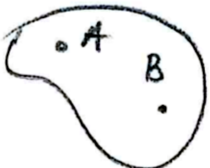
SUMMARY - properties of conductors

(i) $\bar{E} = 0$ inside a conductor.

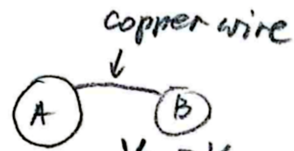
(ii) No net (excess) charge inside a conductor; any net charge resides on the surface.

(iii) A conductor is an equipotential



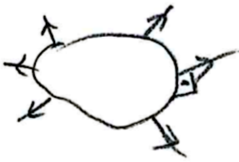
$V_A = V_B$

Example



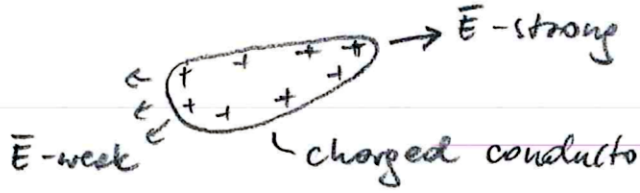
$V_A = V_B$

(iv) $\bar{E}$ is perpendicular to the surface, just outside a conductor.



$$\bar{E} = \frac{\sigma}{\varepsilon_0}\hat{n}$$

Digression:



$\bar{E}$ - depends on curvature

WHY?

$$V_A = V_B \Rightarrow \frac{Q_A}{R_A} = \frac{Q_B}{R_B} \Rightarrow \underbrace{\frac{Q_A}{4\pi R_A^2}}_{}\overset{\sigma_A}{} R_A = \frac{Q_B}{4\pi R_B^2} R_B$$

with $\frac{Q_A}{4\pi R_A^2} = \sigma_A$ and $\frac{Q_B}{4\pi R_B^2} = \sigma_B$.

Hence $\dfrac{\sigma_A}{\sigma_B} = \dfrac{R_B}{R_A} \xRightarrow{R_B < R_A} \sigma_A < \sigma_B$ and $|\bar{E}| \sim \sigma$

Conclusion: The electric field can get very large at sharp edges of a charged conductor

Poisson's equation and Laplace's equation. Method of images

Recall Gauss's law: $\operatorname{div} \bar{E} = \frac{\rho}{\varepsilon_0}$

On the other hand $\bar{E} = -\operatorname{grad} V$. Hence $\operatorname{div} \bar{E} = \operatorname{div}(-\operatorname{grad} V) =$

$$= \nabla \cdot (-\nabla V) = -\nabla^2 V$$

Laplace's operator (Laplace'ian)

Hence

$$\boxed{\nabla^2 V = -\frac{\rho}{\varepsilon_0}}$$ **Poisson's eqn**

(given ρ can find V, have to solve a PDE!)

3rd method

Notes: (i) if $\rho \equiv 0$ in some region, then in that region

$$\boxed{\nabla^2 V = 0}$$ **Laplace's eqn.**

(ii) Poisson's / Laplace's eqns. are PDEs (recall in Cartesian coordinates $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$)

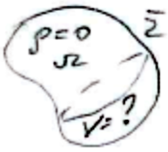
(iii) complete problem

equation in a region Ω + the value of V on Σ (boundary of Ω)

= Boundary Value Problem

(iv) uniqueness thm:

The solution to Laplace's eqn. in a region Ω is uniquely determined if V is specified on Σ (the boundary of Ω)

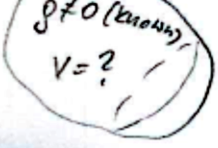


Σ – here V is known

Corollary:

The potential in a region Ω is uniquely determined if (a) the charge density throughout Ω and (b) V at all boundaries are known

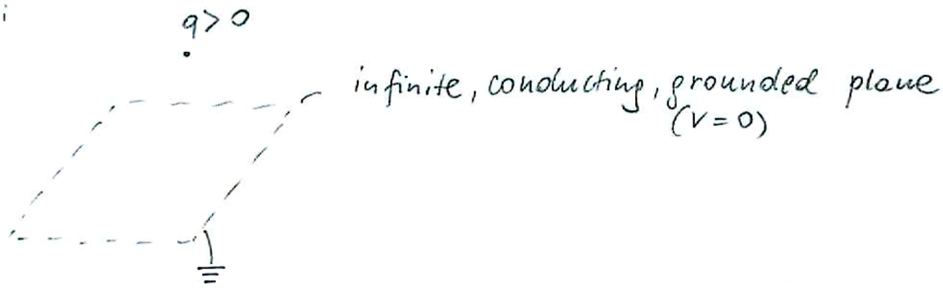
Σ – here V is known



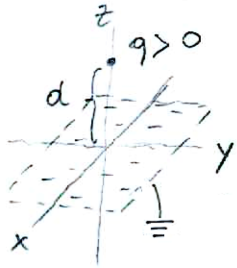
IMPORTANT APPLICATION: METHOD OF IMAGES

METHOD OF IMAGES

Problem:



Find V in the region above the plane and the force of attraction between the charge and the plane

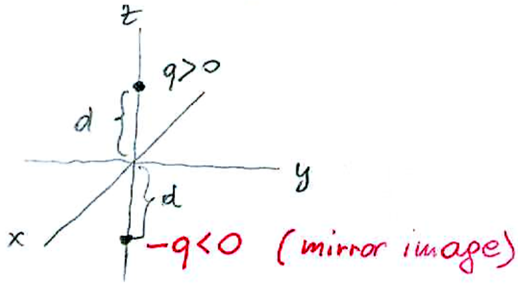


negative charge induced on the plane $z=0$
distribution ?

Mathematical problem: Poisson's eqn (with a single charge q) in $z \geq 0$ + boundary conditions
(i) $V(z=0) \equiv 0$
(ii) $V \to 0$ far away from the charge $r = x^2+y^2+z^2 \gg d$

Idea: Formulate a different problem with the same q and boundary cond. Solve it. The uniqueness thm (corollary) guarantees that the solution is a solution to the original problem

Alternative problem:



no plane!

Solution is easy to find
$$V(\bar{r}) = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{\sqrt{x^2+y^2+(z-d)^2}} - \frac{q}{\sqrt{x^2+y^2+(z+d)^2}}\right]$$

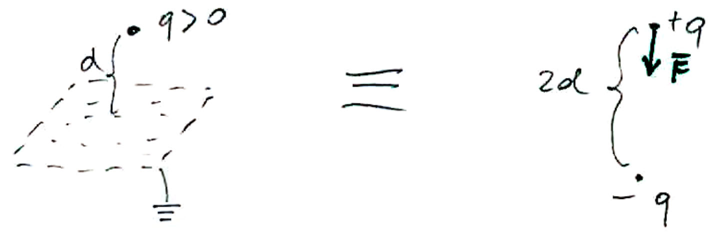
Boundary conditions are compatible with the original problem

(i) $V(z=0) \equiv 0$ (ii) $V \to 0$ for $r \gg d$

Solution of the original problem

$$V(\bar{r}) = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{\sqrt{x^2+y^2+(z-d)^2}} - \frac{1}{\sqrt{x^2+y^2+(z+d)^2}}\right]$$

Force: both problems are equivalent



$$\boxed{\bar{F} = -\frac{1}{4\pi\varepsilon_0}\frac{q^2}{(2d)^2}\,\hat{n}_z}$$

Exercise (see problem set)

Find the distribution of charge on the plane (i.e. surface density $\sigma = \sigma(x,y)$)

and check that

$$q_{ind} = \int_{plane} \sigma(x,y)\, dx dy = -q$$

Hints. (i) For a conductor's surface $\bar{E} = \frac{\sigma}{\varepsilon_0}\hat{n}$.

(ii) use $\bar{E} = -\nabla V$

~ o ~

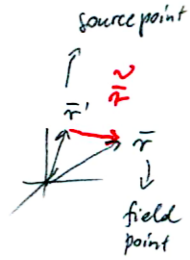
CAUTION: Image charges must not be placed in the region where you are looking for the electric potential!

Final comments – mutual relations between: charge density, electric field, and electric potential

$$V(\bar{r}) = \int_{\text{charge distribution}} \frac{1}{4\pi\varepsilon_0} \frac{\rho(\bar{r}')\, d\tau}{|\tilde{\bar{r}}|} \quad (*)$$

(reference point at ∞)

$$\bar{E}(\bar{r}) = \int_{\text{charge distribution}} \frac{1}{4\pi\varepsilon_0} \frac{\hat{\tilde{\bar{r}}}}{|\tilde{\bar{r}}|} \frac{\rho(\bar{r}')}{|\tilde{\bar{r}}|^2}\, d\tau \quad (*)$$



$$\tilde{\bar{r}} = \bar{r} - \bar{r}'$$

$$\rho \;\; V \;\; \bar{E}$$

$$-\frac{\rho}{\varepsilon_0} = \nabla^2 V$$

$$\frac{\rho}{\varepsilon_0} = \operatorname{div} \bar{E}$$

$$\bar{E} = -\operatorname{grad} V$$

$$V(\bar{r}) - V(\bar{r}_0) = -\int_{\bar{r}_0 \to \bar{r}} \bar{E}(\bar{r})\, d\bar{r}$$

↳ reference point (usually $V(\bar{r}_0) = 0$) → any path

(*) – superposition principle

Note. In many problems in order to find $\bar{E}$ we usually first find V (scalar! easier to add contributions when e.g. using the superposition principle) and then find $\bar{E} = -\operatorname{grad} V$

Units:

ρ	[C/m³ or C/m² or C/m] for 3-, 2- and 1-D	
V	[V]	Volt
$\bar{E}$	[V/m]	or [N/C]