Problem Set 3

Due: 12 October 2016, 4 p.m.

Problem 1. Three identical charges $+Q$ are placed on the corners of a square of side a.

(a) What is the electric field at the fourth corner (the one missing a charge) due to the first three charges?

(b) What is the electric potential at that corner?

(c) How much work does it take to bring another charge, $+Q$, from infinity and place it at that corner?

(d) How much energy did it take to assemble the configuration of three charges mentioned in the first sentence?

(4 × 1 marks)

Problem 2. A hollow spherical ball is carries charge with density $\rho = k/r$ (where k is a constant) in the region $a \leq r \leq b$ (see the figure below).

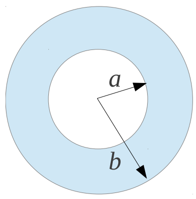
(a) Find the total charge carried by the ball.

(b) Find the electric field $\mathbf{E}$ in the three regions: (i) $r < a$, (ii) $a < r < b$, (iii) $r > b$.

(c) Plot $\mathbf{E}$ as a function of r.

(d) Find the electric potential at any point of space $\mathbf{r}$ and sketch the graph of $V(r)$.

(1 + (1 + 2 + 1) + 1 + 3 marks)



Problem 3. What is the electric potential at a distance s from an infinitely long straight wire charged with uniform density λ? Comment on your choice of the reference point.

Compute the gradient of the potential and check that it yields the correct field (find the field using Gauss's law).

(4 marks)

Problem 4. A conical surface (an empty ice-cream cone) carries a uniform surface charge with density σ. The height of the cone is h, as is the radius of the top. Find the potential difference between the vertex $\mathbf{r}_A$ and the center of the top $\mathbf{r}_B$.

(5 marks)

Problem 5. Because of the Heisenberg uncertainty principle, in quantum mechanics there is no well-defined trajectory of a particle in space, unlike in classical mechanics. Instead, we describe the position of a particle in terms of the so-called wave function that can be used to calculate the probability of finding the particle in a certain region of space.

Consequently, the charge carried, e.g. by the electron in a hydrogen atom, should be considered as continuously distributed in space, with a certain density, rather than being concentrated at a point.

For the lowest-energy state of the hydrogen atom (the ground state) this density is spherically symmetric and at the distance r from the proton, is given by the formula

$$\rho_0(r) = -\frac{e}{\pi a^3} \exp\left(-\frac{2r}{a}\right),$$

where $e > 0$ is the elementary charge and $a \approx 0.532 \times 10^{-10}$ m is the so-called Bohr radius, which turns out to be the most probable distance between the proton and the electron in the ground state.

Find the electric potential $V = V(r)$ and the electric field $\mathbf{E} = E(r)\hat{n}_r$ due to the electron in the hydrogen atom in its ground state. Plot (use a computer) the graphs $E(r)$ and $V(r)$. Compare your results with those for a point charge $-e$ placed at the origin (comment on the limits $r \to 0$ and $r \to \infty$).

(5 marks)

Problem 6. Find the energy stored in a uniformly charged solid sphere of radius R and charge q. (This energy is also often called the "self-energy" of the charge distribution.) Do it in four different ways, using formulas we have derived in class:

(a) Use $U_{\text{conf}} = \frac{1}{2}\int_\Omega \rho V \, d\tau$ (specify ρ and the integration region Ω).

Hint. To find the potential use the integral relation between the potential and the electric field. We have found the electric field due to this charge distribution in class; you may use this result without deriving it again.

(b) Use $U_{\text{conf}} = \frac{\varepsilon_0}{2}\int_{\text{all space}} E^2 \, d\tau$.

(c) Use $U_{\text{conf}} = \frac{\varepsilon_0}{2}\left(\int_\Omega E^2 \, d\tau + \oint_\Sigma V\mathbf{E} \cdot d\mathbf{A}\right)$ taking Σ as a sphere of radius $a > R$ centered at the center of the hollow sphere. Comment on what happens as $a \to \infty$.

(d) Find the amount of work needed to be done to assemble the ball by bringing infinitesimal charges from far away. *Hint.* Use symmetry to chose the infinitesimal charges in a smart way,

(4 × 2 marks)

Problem 7. Use the uniqueness theorem for the Laplace equation to argue that, in electrostatic conditions, the electric potential inside an empty cavity surrounded by a conducting material is constant (and hence the electric field there is zero).

Note. We have proven this fact in class using arguments based on the Gauss's law.

(1 mark)

Problem 8. A point charge q is placed at a distance d from a grounded conducting plane. (a) Find the surface density of charge induced on the plane. (b) Check that the total charge induced is equal to $-q$.

Hint. This exercise is an extension of the problem we have discussed in class to illustrate the method of images. You are allowed to use all results we have derived there.

(2 + 2 marks)

Problem 9. Use the method of images to find the potential due to a point charge q placed at a distance d from the center of a grounded conducting ball with radius $R < d$.

(5 marks)

Problem 10. Use the method of images to find the potential due to a point charge q placed at a distance d from the center of an ungrounded conducting ball with radius $R < d$ charged with charge $-Q$.

(7 marks)

Problem 11. Two semi-infinite grounded conducting planes meet at right angles. In the region between them, there is a point charge q, situated as shown in figure below. (a) Set up the image configuration, and calculate the electric potential in this region. What charges do you need, and where should they be located? (b) What is the force on q? (c) How much work did it take to bring the charge q from infinity? (d) Suppose the planes met as some angle other than $\pi/2$. Would you still be able to solve the problem by the method of images? If not, for what particular angles *does* the method work?

Hints. (a) How many image charges do you need to reproduce the boundary conditions? (d) Where must not be the image charges placed?

(*3 + 1 + 1 + 2 marks*)