Reductions, NP, and NP-Completeness

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CS170 Reviews Session

Time Complexity Classes

Decision problems: problems that only requires a yes/no for an answer.

- Does G has a path from s to t with length less than k? (P)
- Is the boolean formula ϕ satisfiable? (NP-Complete)

P: The set of **decision problems** that can be decided in polynomial time.
NP: The set of **decision problems** with candidate solutions (formally known as certificate/witness/proof) that can be verified in polynomial time.

We only care about decision problems because they're easier to deal with and they're as hard as search and optimization problems.

Quick Digression: Problems (probably) not in NP

UNSAT: boolean formula ϕ is unsatisfiable.
Can't come up with a polynomial size certificate for this problem.

NP-Hard and NP-Complete

NP-Hard: decision problems that are *at least as hard* as every problem in NP.
NP-Completeness: intersection of NP and NP-Hard.

How to prove the notion of "at least as hard"? Reductions!

Reductions

$$A \leq_{\mathrm{P}} B$$

Problem A can be reduced to Problem B in polynomial time if:

- there exists a f such that an input w of problem A is an accepting instance of A if and only if $f(w)$ is an accepting instance of B.
- f can be computed in polynomial time.

If $A \leq_{\mathrm{P}} B$, then A is an "easier" problem than B.

- Intuitively: if I want to lift a feather, I can to use my method to lift a stone to lift this feather. Therefore, the task of lifting feathers is easier than lifting stones.
- If I solved B, then I solved A, so A is the easier problem.

Prove a problem is NP-Complete

Prove A is in NP.

- Design the poly time verifier.

Prove A is NP-Hard.

- Give a poly time reduction from a NP-Complete problem.

Quick Digression

World's first NP-Complete problem: boolean satisfiability problem (Cook-Levin Theorem).
Richard Karp gave 21 NP-Complete problems using tree of reductions from SAT a year after Stephen Cook's paper.

Prove this is NP-Complete

Does a boolean formula has at least two satisfying assignments?

Reduction

Given that the directed graph Rudrata (Hamiltonian) Path problem is NP-Complete, prove that the undirected graph Rudrata Path problem is also NP-Complete.

Reduction

Reduce from 3-SAT to Clique directly to prove it's NP-Complete.