Methods for Applied Mathematics

Homework 8
(Due: Nov 7, 2005)

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CAM program

Exercises 2.9

42. Proof: Let $\{x_n\} \subset X$ be a weak Cauchy sequence, then for any $f \in X^*$, we have $\{f(x_n)\}$ is a Cauchy sequence in $\mathbb{C}$ which is complete, so $\lim_{n\to\infty} f(x_n)$ exists. So $\{f(x_n)\}$ is bounded in $\mathbb{C}$, then we have

$$\sup_{n\in\mathbb{N}} |[x_n](f)| = \sup_{n\in\mathbb{N}} |f(x_n)| < \infty, \ \forall f \in X^*.$$

Since X^* is complete, by the principle of uniform boundedness, we have

$$\sup_{n\in\mathbb{N}} \|x_n\| = \sup_{n\in\mathbb{N}} \|[x_n]\| < \infty,$$

which means $\exists M > 0$ such that $\|x_n\| < M, \forall n \in \mathbb{N}$. Then we define a map $F : X^* \longrightarrow \mathbb{C}$ as

$$F(f) = \lim_{n\to\infty} f(x_n), \forall f \in X^*.$$

Obviously, F is linear, and we have

$$|F(f)| = \lim_{n\to\infty} |f(x_n)| \le \liminf_{n\to\infty} \|f\|\|x_n\| \le M\|f\|,$$

which means $F \in X^{**}$. Since X is reflexive, so there is some $x \in X$ such that $F = [x]$, so we have

$$\lim_{n\to\infty} f(x_n) = F(f) = [x](f) = f(x), \ \forall f \in X^*,$$

which means $x_n \rightharpoonup x$ weakly. So X is weakly complete.

43. Proof: Let X be a vector space over $\mathbb{C}$ and $\dim X = n$, $\{e_i\}_{i=1}^n$ be a basis of X. Let $f_i \in X^*$ such that $f_i(e_j) = \delta_{ij}$. Then we will show that $\{f_i\}_{i=1}^n$ is a basis of X^*.

If there are $a_1, a_2, \cdots, a_n \in \mathbb{C}$ such that

$$\sum_{i=1}^{n} a_i f_i(x) = 0, \ \forall x \in X,$$

we set $x = e_j$ $(j = 1, 2, \cdots, n)$, so

$$\sum_{i=1}^{n} a_i f_i(e_j) = \sum_{i=1}^{n} a_i \delta_{ij} = a_j = 0,$$

which means $\{f_i\}_{i=1}^n$ is linearly independent.

In the other hand, for any $f \in X^*$, let $f(e_i) = c_i$ $(i = 1, 2, \cdots, n)$, for any $x = \sum_{i=1}^n a_i e_i \in X$, we have

$$f(x) = \sum_{i=1}^{n} a_i f(e_i) = \sum_{i=1}^{n} a_i c_i = \sum_{j=1}^{n}\sum_{i=1}^{n} a_i c_j f_j(e_i) = \sum_{j=1}^{n} c_j f_j(\sum_{i=1}^{n} a_i e_i) = (\sum_{j=1}^{n} c_j f_j)(x),$$

which means $f = \sum_{j=1}^n c_j f_j \in$span$\{f_i\}_{i=1}^n$. So $\{f_i\}_{i=1}^n$ is a basis of X^*. So dimX^* =dim$X = n$.

In the same way, we can have dimX^{**} =dimX^* =dim$X$ $= n$, so the embedding $[\cdot] : X \longrightarrow X^{**}$ must be a surjection, which means X is reflexive.

45. Proof "If": If there is some $M > 0$ such that for any $T \in E$, $\|T\| < M$, then we have for any $x_1, x_2 \in X$

$$\|Tx_1 - Tx_2\| = \|T(x_1 - x_2)\| \leq \|T\|\|x_1 - x_2\| \leq M\|x_1 - x_2\|, \ \forall T \in E,$$

which means E is equicontinuous.

"Only if": If $E \subset B(X, Y)$ is equicontinuous, which means $\forall \varepsilon > 0, \exists \delta > 0$ such that for all $x_1, x_2 \in X$, when $\|x_1 - x_2\| < \delta$, we have $\|Tx_1 - Tx_2\| < \varepsilon, \ \forall T \in E$. So especially, for $\varepsilon_0 = 1$, there is some $\delta_0 > 0$ such that

$$\|Tx\| < 1, \ \forall T \in E, x \in B(0, \delta_0).$$

So for any $x \in X, x \neq 0$, we have

$$\sup_{T \in E} \|Tx\| = \frac{2\|x\|}{\delta_0} \sup_{T \in E} \|T(\frac{\delta_0 x}{2\|x\|})\| \leq \frac{2\|x\|}{\delta_0}.$$

So for any $x \in X$, the inequality above is also true. Since X is a Banach space, by the principle of uniform boundedness, we have

$$\sup_{T \in E} \|T\| < \infty,$$

which means there is some $M > 0$ such that $\|T\| < M,\ \forall T \in E$.

Exercises 3.12

1. Proof: Let $(H, (\cdot, \cdot))$ be a Hilbert space and $x, y \in H$, then we have

$$\begin{aligned}
\|x+y\|^2 + \|x-y\|^2 &= (x+y, x+y) + (x-y, x-y) \\
&= (x,x) + (y,y) + (x,y) + (y,x) + (x,x) + (y,y) - (x,y) - (y,x) \\
&= 2(x,x) + 2(y,y) \\
&= 2(\|x\|^2 + \|y\|^2).
\end{aligned}$$