On the emergence of Quantum Boltzmann dynamics in boson and fermion gases

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Joint work with
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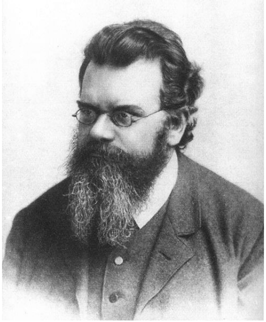
Boltzmann equation



Boltzmann equation

$(\partial_t + v\nabla_x) f_t(x, v) = Q(f_t)$

$f_t(x, v)$ probability density for finding a gas particle at time t in phase space volume $dxdv$ at (x, v).

$Q(f_t) = Q_+(f_t) - Q_-(f_t)$

collision operator, change of f_t due to particles entering (gain) minus particles exiting (loss) phase space volume $dxdv$ at (x, v).

Classical and quantum Boltzmann dynamics

Many systems exhibit collision processes obeying a Boltzmann dynamics under suitable scaling. Examples:

- Rarefied hard-sphere classical gas: Lanford, Bodineau-Gallagher-Saint-Raymond, G-SR-Texier, Pulvirenti-Saffirio-Simonella, ...
- Collisions among lattice vibrations: Spohn
- Quantum gas: Benedetto-Castella-Esposito-Pulvirenti, Paul-Pulvirenti, Erdös-Salmhofer-Yau, Ho-Landau, Spohn
  Obstruction: XChen-Guo

Boltzmann equation

- Anderson model, electrons in semiconductor: Erdös-Yau, Spohn, C, C-Rodnianski, C-Sasaki
- Phenomenology of Boson gases: Alonso-Gamba-Tran, Arkeryd-Nouri, Bijlsma-Zaremba-Stoof, Pomeau-Tran, Reichl-Tran, Soffer-Tran, Spohn, Zaremba-Nikuni-Griffin
- Maxwell molecules: Barbaroux-Hundertmark-Ried-Vugalter, Bobylev-Gamba, Fournier-Mischler, Pavić-Čolić - Tascović, Toscani-Villani
- Biological swarm models: Carlen-Degond-Wennberg et al.
- Kac model: Carlen-Carvalho-Loss et al.
- Opinion formation: Burger-Caffarelli-Markowich

Boltzmann dynamics in Bose gases

Emergence of Boltzmann dynamics in Bose gases
Joint work with Michael Hott

N bosons described by Schrödinger equation

$$i\partial_t \Psi_{N,t} = \mathcal{H}_N \Psi_{N,t} \,,$$

where $\Psi_N \in L^2(\mathbb{R}^3)^{\otimes_{sym} N}$. Mean field Hamiltonian of the system

$$\mathcal{H}_N = \sum_{j=1}^{N} \Big(-\Delta_j + w(x_j) \Big) + \frac{\lambda}{N} \sum_{i<j}^{N} v(x_i - x_j) \,.$$

Assume at time $t = 0$, all N bosons are in the same state ϕ_0

$$\Psi_{N,0}(x_1, \ldots, x_N) = \prod_{j=1}^{N} \phi_0(x_j)$$

This describes a *Bose-Einstein condensate (BEC)*.

Boltzmann dynamics in Bose gases

Mathematical proof of BEC and derivation of Hartree equation:

$$\Psi_{N,t}(x_1, \dots, x_N) = \prod_{j=1}^{N} \phi_t(x_j) + o_N(1)$$

as $N \to \infty$, BEC persists, and

$$i\partial_t \phi_t = -\Delta \phi_t + w\phi_t + \lambda(v * |\phi_t|^2)\phi_t$$

If $v = \delta_{Dirac}$, obtain derivation of NLS.

Boltzmann dynamics in Bose gases

- Time independent: Lieb-Seiringer-Yngvason, Aizenman-L-S-Solovej-Y, Lewin-Nam-Rougerie, ...
- Dynamical via Fock space: *Hepp, Ginibre-Velo, Rodnianski-Schlein, Grillakis-Machedon-Margetis, Grillakis-Machedon, Schlein-et-al*
- Via BBGKY & resolvent estimates: *Spohn, Erdös-Schlein-Yau, Elgart-E-S-Y, Adami-Bardos-Golse-Teta*.
- Via BBGKY & dispersive nonlinear PDE: *Klainerman-Machedon, Kirkpatrick-Schlein-Staffilani, C-Pavlović, X.Chen, X.C.-Holmer*
- Other approaches: *Fröhlich-Graffi-Schwarz, Fröhlich-Knowles-Pizzo, Golse-Paul, Pickl, Ammari-Liard-Rouffort,...*

Boltzmann dynamics in Bose gases

Thermal fluctuations: Bosonic Fock space with vacuum vector Ω

$$\mathcal{F} = \mathbb{C} \oplus \bigoplus_{n \geq 1} (L^2(\Lambda))^{\otimes_{sym} n} \quad , \quad \Omega = (1, 0, 0, \ldots) \in \mathcal{F}$$

Canonical Commutation relations (CCR)

$$[a_x, a_y^+] = \delta(x - y) \, , \qquad [a_x, a_y] = [a_x^+, a_y^+] = 0 \, .$$
$$a_x \Omega \; = \; 0 \quad \forall \, x \; \in \; \Lambda$$

Particle number and kinetic energy operator

$$\mathcal{N} = \int dx \, a_x^+ a_x \quad , \quad \mathcal{T} = \int dx \, a_x^+ (-\Delta_x) a_x$$

Boltzmann dynamics in Bose gases

Assume initial state near BEC, in Fock space (coherent state):

$$\Psi_{BEC}^{(\phi_0)} = e^{-\frac{1}{2}\|\phi_0\|_2^2}\Big(1, \phi_0(x_1), \frac{1}{2!}\phi_0(x_1)\phi_0(x_2), \dots, \frac{1}{n!}\prod_{j=1}^{n}\phi_0(x_j), \dots\Big)$$

Condensate density N defines large parameter of problem:

$$N|\Lambda| = \Big\langle \Psi_{BEC}^{(\phi_0)}, \mathcal{N}\Psi_{BEC}^{(\phi_0)} \Big\rangle = \sum_{n=0}^{\infty} n\|\Psi_{BEC,n}^{(\phi_0)}\|_{L^2}^2$$

Hamiltonian of the system

$$\mathcal{H}_N := \int_\Lambda dx\, a_x^+\Big(\frac{-\Delta_x}{2} + w(x)\Big)a_x + \frac{\lambda}{2N}\int_{\Lambda^2} dx\, dy\ :\ a_x^+ a_x\, v(x-y)\, a_y^+ a_y\ :$$

We are interested in *thermal quantum fluctuations outside of BEC.*

Boltzmann dynamics in Bose gases

Dynamics relative to BEC

BEC wave function satisfies the Hartree equation

$$i\partial_t\phi_t = (-\Delta/2 + w)\phi_t \, + \, \lambda|\Lambda|(v * |\phi_t|^2)\phi_t \, .$$

Subtract the BEC dynamics using unitary Weyl transform

$$a_x \, \mapsto \, a_x \, - \, \sqrt{N|\Lambda|}\phi_t \;\; , \;\; a_x^+ \, \mapsto \, a_x^+ \, - \, \sqrt{N|\Lambda|} \, \overline{\phi_t}$$

Hamiltonian in orders of N:

$$\mathcal{H}_N \;\; \mapsto \;\; N|\Lambda|E[\phi_t,\overline{\phi_t}] \, + \, \underbrace{\mathcal{H}_{HFB}^{\phi_t}}_{O(1)} \, + \, \underbrace{\mathcal{H}_{cub}^{\phi_t}}_{O(\frac{1}{\sqrt{N}})} \, + \, \underbrace{\mathcal{H}_{quart}}_{O(\frac{1}{N})}$$

Leading order yields (scalar) Hartree energy of BEC

$$E[\phi_t,\overline{\phi_t}] \;:= \int_\Lambda dx \, \overline{\phi_t(x)}\Big(\frac{-\Delta_x}{2} \, + \, w(x)\Big)\phi_t(x) \\ + \, \frac{\lambda|\Lambda|}{2}\int_{\Lambda^2} dx \, dy \, v(x-y) \, |\phi_t(x)|^2|\phi_t(y)|^2$$

Boltzmann dynamics in Bose gases

Leading order fluct.: **Hartree-Fock-Bogoliubov** Hamiltonian

$$
\begin{aligned}
\mathcal{H}_{HFB}^{\phi_t} := & \int_\Lambda dx\, a_x^+ \Big(\frac{-\Delta_x}{2} + w(x) \Big) a_x \\
& + \frac{\lambda|\Lambda|}{2} \int_{\Lambda^2} dx\, dy\, v(x-y) \Big(\overline{\phi_t(x)\phi_t(y)} a_y a_x + h.c. \\
& + 2|\phi_t(y)|^2 a_x^+ a_x + 2|\phi_t(x)|^2 a_y^+ a_y \Big)
\end{aligned}
$$

Bilinear in a^+, a.

If taken by itself, generates HFB dynamics:
Bach-Breteaux-C-Fröhlich-Sigal, Grillakis-Machedon, Schlein-et-al, Pickl, Pickl-Pavlović-Soffer, and many others.

Boltzmann dynamics in Bose gases

On torus Λ, static nonlinear ground state solution for $w = -\lambda \hat{v}(0)$

$$\phi_t = \phi_0 = |\Lambda|^{-\frac{1}{2}}$$

Subtract HFB dynamics via unitary conjugation by $e^{-it\mathcal{H}_{HFB}^{\phi_0}}$.

Only interaction operators *cubic and quartic* in a^+, a remain,

$$\mathcal{H}_I(t) = \underbrace{\mathcal{H}_{cub}^{\phi_0}(t)}_{e^{-it\mathcal{H}_{HFB}^{\phi_0}}\mathcal{H}_{cub}^{\phi_0}e^{it\mathcal{H}_{HFB}^{\phi_0}}} + \mathcal{H}_{quart}(t)$$

Dynamics relative to both BEC and HFB:

$$i\partial_t \langle A \rangle_t = \langle [A, \mathcal{H}_I(t)] \rangle_t \quad , \quad \langle A \rangle_0 = \frac{1}{\mathcal{Z}} \mathrm{Tr}\big(e^{-\mathcal{K}} A\big)$$

$$\text{where } \mathcal{K} = \int dp\, K(p) a_p^+ a_p$$

with $K(p) \geq \kappa_0 + \kappa_1 p^2$ suitably regular (hence $\mathcal{K} \geq \beta(\mathcal{T} - \mu\mathcal{N})$).

Boltzmann dynamics in Bose gases

Dynamics of centered correlation functions

Consider kinetic time scale $t = T/\lambda^2$.

Control dynamics of moments $\langle a\rangle_{T\lambda^{-2}}$, $\langle a^+ a\rangle_{T\lambda^{-2}}$, $\langle aa\rangle_{T\lambda^{-2}}$.

Define centered correlation functions:

$$\Psi_T \delta(p) := \langle a_p\rangle_{T\lambda^{-2}}$$
$$F_T(p)|\Lambda| := \langle a_p^+ a_p\rangle_{T\lambda^{-2}} - |\langle a_p\rangle_{T\lambda^{-2}}|^2$$
$$G_T(p)|\Lambda| := \langle a_p a_{-p}\rangle_{T\lambda^{-2}} - \langle a_p\rangle_{T\lambda^{-2}}^2$$

$F_T \sim$ number density of thermal fluctuation beyond leading order.

Boltzmann dynamics in Bose gases

Theorem[C-Hott '21] Let $T, \kappa_0 > 0$, where $\kappa_0 \approx$ chemical potential, with $|\Lambda| \geq 1$ fixed. Assume

$$\lambda = \log\log(N)/\log(N)$$

Let $E(p) := \frac{|p|^2}{2}$ and $\Omega := \sqrt{E(E+2\lambda\hat{v})}$ Bogoliubov dispersion. Then there exists $\delta > 0$, such that for any test function J,

$$\Big| \int_{\Lambda^*} dp\, J(p) \ \underbrace{\Big(F_T(p) - F_0(p) - \frac{1}{N}\int_0^T dS\, Q_{d;T-S,\lambda}(F_S, F_S, F_S)(p)}_{\text{Quantum Boltzmann}}$$

$$\underbrace{- \frac{1}{N}\int_0^T \int_0^{S_1} dS_1 dS_2 \sum_{j=1}^{2} \lambda^j q^{(j)}_{d,F;S_1,S_2,\lambda}(F_{S_2}, F_{S_2}, F_{S_2})(p)\Big)}_{\text{lattice effect, subdominant as } |\Lambda|\nearrow\infty.\ \text{"\textit{Talbot effect}"}} \Big|$$

$$\leq \frac{C_0 \langle T\rangle^2 \|J\|_\infty}{N^{1+\delta}}$$

Boltzmann dynamics in Bose gases

Cubic quantum Boltzmann collision operator (with $\widetilde{F_S} := 1 + F_S$)

$$
\begin{aligned}
&Q_{d;T-S,\lambda}(F_S, F_S, F_S)(p) \\
&\quad := \int_{(\Lambda^*)^2} dp_1\, dp_2 \, \Big| \hat{v}(p_1) + \hat{v}(p_2) \Big|^2 \\
&\qquad \Big(\delta(p - p_1) + \delta(p - p_2) - \delta(p - p_1 - p_2) \Big) \\
&\qquad \underbrace{\frac{\sin\left(\frac{\Omega(p_1)+\Omega(p_2)-\Omega(p_1+p_2)}{\lambda^2} (T - S) \right)}{\Omega(p_1) + \Omega(p_2) - \Omega(p_1 + p_2)}}_{\sim \text{ mollified delta for energy conservation}} \\
&\qquad \Big(\widetilde{F_S}(p_1) \widetilde{F_S}(p_2) F_S(p_1 + p_2) \; - \; F_S(p_1) F_S(p_2) \widetilde{F_S}(p_1 + p_2) \Big)
\end{aligned}
$$

Mollification due to time-energy Heisenberg uncertainty.

Boltzmann dynamics in Bose gases

Correction of condensate wave function

$$\Big|\Psi_T + \frac{i}{N^{\frac{1}{2}}\lambda}\int_0^T dS \int dp\,\hat{v}(p)F_S(p)\Big| \leq \frac{C_0\langle T\rangle}{N^{\frac{1}{2}+\delta}}$$

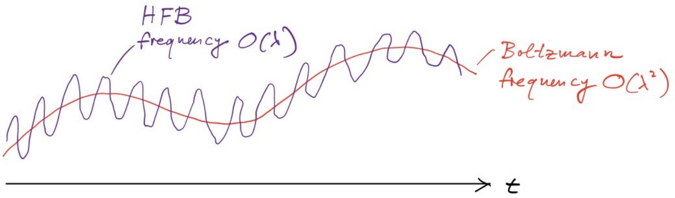
Dynamics of $\langle aa\rangle$ correlation

$$\Big|\int_{\Lambda^*} dp\,J(p)\,\Big(G_T(p) - \frac{1}{N}\mathcal{A}_{d;T,\lambda}(F,F)(p) - \frac{1}{N\lambda^2}\mathcal{Q}_{d,G;T,\lambda}(F,F,F)(p)\Big)\Big| \leq \frac{C_0\langle T\rangle^2\|J\|_\infty}{N^{1+\delta}}\,.$$

Due to choice of initial state, both are driven by F_T.

Boltzmann dynamics in Bose gases

- The dynamics of F is hidden in three types of noises: Correction of BEC Ψ, HFB correction terms of F, and G.
- The Boltzmann dynamics is much *slower* than the HFB dynamics. It modulates the HFB fluctuations at a slow rate.



Boltzmann dynamics in Bose gases

$$
\begin{aligned}
\langle A\rangle_{\frac{T}{\lambda^2}} = & \langle A\rangle_0 + \int_0^{\frac{T}{\lambda^2}} ds \langle [A, \mathcal{H}_I(s)]\rangle_0 \\
& + \int_0^{\frac{T}{\lambda^2}} ds_1 \int_0^{s_1} ds_2 \, \langle [[A, \mathcal{H}_I(s_1)], \mathcal{H}_I(s_2)]\rangle_0 \\
& + \int_0^{\frac{T}{\lambda^2}} ds_1 \int_0^{s_1} ds_2 \int_0^{s_2} ds_3 \langle [[[A, \mathcal{H}_I(s_1)], \mathcal{H}_I(s_2)], \mathcal{H}_I(s_3)]\rangle_{s_3}
\end{aligned}
$$

Kinetic time scale $t = \frac{T}{\lambda^2}$ to extract B eq from $O(\lambda^2)$-term.

- Control tail in Duhamel expansion
- Propagation of approximate restricted quasi-freeness

Related preliminary works for fermions:
Erdös-Salmhofer-Yau '04: B eq assuming restricted quasi-freeness.
Benedetto-Castella-Esposito-Pulvirenti '04,'07,'08: Extraction of B eq from Feynman graph expansions, no error control.

Boltzmann dynamics in Bose gases

Our initial state $\nu_0 = \langle \cdot \rangle_0$ satisfies $\langle a \rangle_0 = \langle aa \rangle_0 = 0$.

$\langle \cdot \rangle_0$ *quasi-free* $\iff$ satisfies Wick Theorem

$$\langle a^{\#_1} a^{\#_2} \dots a^{\#_{2n}} \rangle_0 = a^{\#_1} a^{\#_2} \dots a^{\#_n} + \text{all ordered combinations}$$

$$\langle a^{\#_1} a^{\#_2} \dots a^{\#_{2n-1}} \rangle_0 = 0$$

where $\overbracket{a^{\#_j} a^{\#_k}} = \langle a^{\#_j} a^{\#_k} \rangle_0$.

However, for evolution generated by $H_I(t)$, time-dependent state $\nu_t = \langle \cdot \rangle_t$ is *not* quasi-free, and $\langle a \rangle_t, \langle aa \rangle_t \neq 0$.

Boltzmann dynamics in Bose gases

We prove a main technical result:

State $\nu_t = \langle \cdot \rangle_t$ is approximately restricted quasi-free

$\Longleftrightarrow$ approximate restricted Wick Theorem

$$\langle a^{\#_1} a^{\#_2} \dots a^{\#_{2n}} \rangle_t = a^{\#_1} a^{\#_2} \dots a^{\#_{2n}} + \text{all ordered combinations} \\ + l.o.t.$$

$$\langle a^{\#_1} a^{\#_2} \dots a^{\#_{2n-1}} \rangle_t = l.o.t.$$

up to $n \leq n_0$, with $\overbracket{a^{\#_j} a^{\#_k}} = \langle a^{\#_j} a^{\#_k} \rangle_t$

Boltzmann dynamics in Fermi gas

Emergence of Quantum Boltzmann dynamics in Fermi gas
Joint work with Esteban Cárdenas

Gas of N fermions in box $\Lambda = [-L/2, L/2]^d$.
Fermion Fock space, Canonical Anticommutation Relations (CAR)

$$\{a_p^+, a_q\} = |\Lambda|\delta_{p,q} \quad , \quad \{a_p^\sharp, a_q^\sharp\} = 0 \quad , \quad \forall p, q \in \Lambda^*$$

Let $\int dp \equiv \frac{1}{|\Lambda|} \sum_{p \in \Lambda^*}$. Hamiltonian

$$\mathcal{H} = \int dp\, |p|^2 a_p^+ a_p + \lambda \int_{(\Lambda^*)^3} dp\, dq\, dk\, \widehat{V}(k)\, a_{p+k}^+ a_{p-k}^+ a_p a_q$$

$diam(supp(\widehat{V})) < r$, $\widehat{V}(p) = \widehat{V}(-p)$, $\widehat{V}(0) = 0$, $\sup_\Lambda \|\widehat{V}\|_{L^1(\Lambda^*)} < \infty$.

Boltzmann dynamics in Fermi gas

Fermi Ball (for $L = 2\pi$), radius = Fermi momentum

$$\mathcal{B} = \{p \in \mathbb{Z}^d \mid |p| \leq p_F = N^{1/d}\}$$

Ground state of non-interacting system ($\lambda = 0$):

$$\Psi_0 = \prod_{p \in \mathcal{B}} a_p^+ \Omega$$

Fermi shell

$$\mathcal{S} = \{p \in \mathbb{Z}^d \mid \big| \, p_F - 3r \leq |p| \leq p_F + 3r\}$$

Interacting model: Benedikter-Nam-Porta-Schlein-Seiringer, Benedikter-Porta-Saffirio-Schlein, Christiansen-Hainzl-Nam, Fröhlich-Knowles, Petrat-Pickl, Elgart-Erdös-Schlein-Yau, Giacomelli, ...

Boltzmann dynamics in Fermi gas

Translation-invariant, quasi-free initial state ν satisfying

$$\nu\Big(\prod_{i=1}^{k} a_{p_i}^{+} \prod_{j=1}^{k'} a_{q_j}\Big) = \delta_{k,k'}(-1)^{\frac{k(k-1)}{2}} det\Big[\nu(a_{p_i}^{+} a_{q_j})\Big]$$

Charge neutrality (# holes in $\mathcal{B}$ = # particles outside $\mathcal{B}$)

$$\int_{\mathcal{B}} dp \, \nu(a_p^{+} a_p) = \int_{\mathcal{B}^c} dp \, \nu(a_p^{+} a_p)$$

Low particle occupation in $\mathcal{S}$

$$\int_{\mathcal{S}} dp \, \nu(a_p^{+} a_p) < C(\lambda|\Lambda| p_f^{d-1})^2$$

Boltzmann dynamics in Fermi gas

Particle-hole transformation

$$\mathcal{R}^* a_p^+ \mathcal{R} = \chi^\perp(p) a_p^+ + \chi(p) a_p$$

Creates particle in $\mathcal{B}^c$ and hole in $\mathcal{B}$.

$$\mathfrak{h} := \mathcal{R}^* \mathcal{H} \mathcal{R}$$

with dispersion relation (where $\chi \equiv \mathbf{1}_{\mathcal{B}}$)

$$E(p) = -\chi(p)\Big(\frac{|p|^2}{2} + \lambda(\widehat{V} * \chi^\perp)(p)\Big) + \chi^\perp(p)\Big(\frac{|p|^2}{2} + \lambda(\widehat{V} * \chi)(p)\Big)$$

Main object of study: Dynamics of pair correlation

$$f_t(p) := \frac{\nu\big(e^{it\mathfrak{h}} a_p^+ a_p e^{-it\mathfrak{h}}\big)}{|\Lambda|}$$

density of particles in $\mathcal{B}^c$ and holes in $\mathcal{B}$.

Boltzmann dynamics in Fermi gas

Theorem [Cárdenas-C] Let $R \equiv |\Lambda| p_f^{d-1} \sim L N^{\frac{d-1}{d}}$ and

$$n = |\Lambda| \int_{\Lambda^*} dp \; f_0(p) \;\; \leq \;\; C R^{1/2}$$

total number of particles in $\mathcal{B}^c$ and holes in $\mathcal{B}$. Then,

$$\left\| f_t - f_0 - \lambda^2 t \; Q_t[f_0] - \lambda^2 t \; B_t[f_0] \right\|_{\ell^\infty_m} \leq C \lambda^2 t \Big(\theta_1 t \langle t \rangle + \theta_2 t \Big) \; e^{C \lambda R \langle t \rangle}$$

where $\|f\|_{\ell^\infty_m} := \sup_p |f(p)| + \sup_{p \in \mathcal{S}} \langle p \rangle^{-m} |f(p)|$ and

$$\theta_1 := \lambda R^2 (R^{\frac{1}{2}} + n^2) \;\; \text{and} \;\; \theta_2 := \frac{R^3}{p_f^m}$$

Q_t energy-mollified Boltzmann collision operator
B_t energy-mollified interaction with bosonized particle-holes in $\mathcal{S}$

Boltzmann dynamics in Fermi gas

Example parameters where lhs dominates over error: For $T > 0$,

$$t = N^{\frac{1}{3}} T \quad \text{and} \quad \lambda = \frac{1}{N^2} \quad \text{in } d = 3$$

Boltzmann collision operator (where $\widetilde{f} := 1 - f$)

$$Q_t[f](p) = \pi \int_{(\Lambda^*)^4} d\vec{p}\, \sigma(\vec{p}) \Big[\delta(p - p_1) + \delta(p - p_2) - \delta(p - p_3) - \delta(p - p_4) \Big]$$

$$\delta_t \Big(E_{p_1} + E_{p_2} - E_{p_3} - E_{p_4} \Big) \Big(f(p_1) f(p_2) \widetilde{f}(p_3) \widetilde{f}(p_4) - \widetilde{f}(p_1) \widetilde{f}(p_2) f(p_3) f(p_4) \Big)$$

where $\delta_t(x) = t\delta_1(tx)$ with $\delta_1(x) \equiv \frac{2}{\pi} \frac{\sin^2(\frac{x}{2})}{x^2}$.

Collision kernel $\sigma \Rightarrow$ Momentum conservation, dependence on $\widehat{V}$.

Boltzmann dynamics in Fermi gas

Interaction with bosonized particle-holes in $\mathcal{S}$:

$$B_t = B_t^{(H)} + B_t^{(P)}$$

$$B_t^{(H)}[f](h) := 2\pi \int_{\Lambda^*} dk |\widehat{V}(k)|^2 \Big(\alpha_t^H(h-k,k) f(h-k) \widetilde{f}(h) - \alpha_t^H(h,k) f(h) \widetilde{f}(h+k) \Big)$$

$$\alpha_t^H(h,k) := \underbrace{\chi(h)\chi(h+k)}_{\text{holes in } \mathcal{B}} \int_{\Lambda^*} dr\, \chi(r)\chi^{\perp}(r+k)\delta_t\Big(E_{p_1}+E_{p_2}-E_{p_3}-E_{p_4}\Big)$$

and similarly, define $B_t^{(H)}$, α_t^P for particles.

Boltzmann dynamics in Fermi gas

Origin of terms: Let $k \in supp(\widehat{V})$ (resp. $k \in \Lambda^*$).

Define D-operators in the bulk

$$D_k := \int_{\Lambda^*} dp \, \underbrace{\chi^\perp(p)\chi^\perp(p-k)a^+_{p-k}a_p}_{\text{particle momentum shift by } -k} - \int_{\Lambda^*} dh \, \underbrace{\chi(h)\chi(h+k)a^+_{h+k}a_h}_{\text{hole momentum shift by } k}$$

and b-operators at the boundary $\mathcal{S}$

$$b^+_k := \int_{\Lambda^*} dp \, \underbrace{\chi^\perp(p)\chi(p-k)}_{\text{supp}(\dots) \subset \mathcal{S}} a^+_{p-k}a^+_p$$

creates bosonized particle-hole pair; particle at p and hole at $p-k$.

Boltzmann dynamics in Fermi gas

$$\mathfrak{h} - \mu_1 \mathbf{1} - \mu_2 \mathcal{Q} = \mathfrak{h}_0 + \mathcal{V}$$

$$\mathfrak{h}_0 = \int_{\Lambda^*} dp\, E(p)\, a_p^+ a_p \quad , \quad \mathcal{V} = V_F + V_{FB} + V_B$$

with charge operator $\mathcal{Q} = D_0$ and

$$V_F = \frac{1}{2} \int_{\Lambda^*} dk\, \widehat{V}(k) D_k^+ D_k \quad ,$$

$$V_{FB} = \int_{\Lambda^*} dk\, \widehat{V}(k)\, D_k^+ \big[b_k + b_{-k}^+ \big]$$

$$V_B = \int_{\Lambda^*} dk\, \widehat{V}(k)\, \Big[b_k^+ b_k + \frac{1}{2} b_k^+ b_{-k}^+ + \frac{1}{2} b_{-k} b_k \Big]$$

Let

$$V_\alpha(t) := e^{it\mathfrak{h}_0} V_\alpha e^{-it\mathfrak{h}_0}$$

for $\alpha \in \{F, FB, B\}$.

Boltzmann dynamics in Fermi gas

$$f_t - f_0 = -\frac{\lambda^2}{|\Lambda^2|} \sum_{\alpha,\beta\in\{F,FB,B\}} T_{\alpha,\beta}(t)$$

where

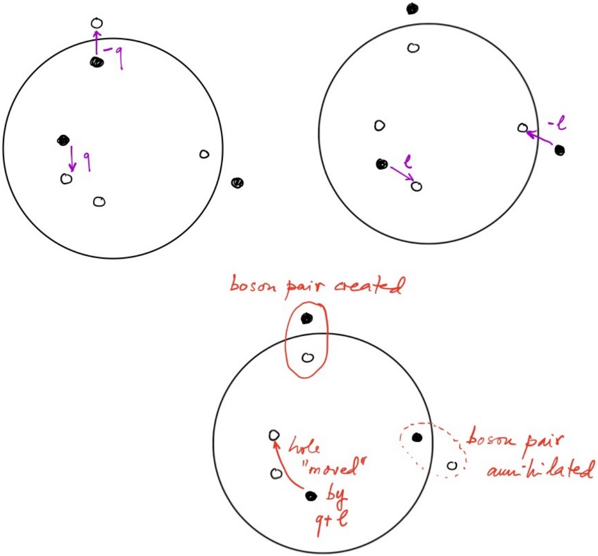
$$T_{\alpha,\beta}(t,p) := \int_0^t \int_0^{t_1} dt_1 dt_2 \; \nu_{t_2}\Big([[a_p^+ a_p, V_\alpha(t_1)], V_\beta(t_2)] \Big)$$

Boltzmann term Q_t stems from $T_{F,F}$.
Particle-hole interaction term B_t stems from $T_{FB,FB}$, main term

$$\int_0^t \int_0^{t_1} \int_{(\Lambda^*)^2} dt_1 dt_2 dk d\ell \; \nu_{t_2}\Big(D_k^+(t_1) \underbrace{[b_k(t_1), b_\ell^+(t_2)]}_{\Rightarrow \text{boson propagator}} D_\ell(t_2) \Big)$$

All other $T_{\alpha,\beta}$ contribute to remainder term (by choice of ν !).

Boltzmann dynamics in Fermi gas



Boltzmann dynamics in Fermi gas

- Boltzmann dynamics emerges in the bulk of $\mathcal{B}$ for *holes*, and in $\mathcal{B}^c$ for *particles*.
- Additional phenomenon: Interactions between bulk holes / excited particles via mediation of virtual bosonized particle-hole pairs at the boundary $\mathcal{S}$.
  These interactions are smaller than Boltzmann term at center of $\mathcal{B}$ and become dominant near $\mathcal{S}$.

Boltzmann dynamics in quantum gases

Thank you for your attention.