Methods for Applied Mathematics

Homework 3
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19. Proof: Since g is bounded, $\exists M > 0$ such that $|g(x)| < M, \forall x \in \Omega$. Then we have

$$\int_{\Omega} |f(x)g(x)|dx \leq M \int_{\Omega} |f(x)|dx < \infty,$$

which means $fg \in \mathcal{L}(\Omega)$.

20. Example: Let $\Omega = \mathbb{R}$ and $f_n(x) = \dfrac{1}{n}\chi_{[0,n]}(x)$ where $\chi_{[0,n]}(x)$ is the characteristic function of $[0, n]$, then $|f_n(x)| \leq \dfrac{1}{n} \to 0$, so $f_n(x) \to 0$ $(n \to \infty)$. So we have

$$\int_{\mathbb{R}} \liminf_{n\to\infty} f_n(x)dx = \int_{\mathbb{R}} 0dx = 0,$$

and

$$\liminf_{n\to\infty} \int_{\mathbb{R}} f_n(x)dx = \liminf_{n\to\infty} 1 = 1.$$

So

$$\int_{\mathbb{R}} \liminf_{n\to\infty} f_n(x)dx < \liminf_{n\to\infty} \int_{\mathbb{R}} f_n(x)dx.$$

21. The example can't be applied to Dominated Convergence Theorem because there is no function $g \in \mathcal{L}(\mathbb{R})$ such that $|f_n(x)| \leq g(x), \forall n \in \mathbb{N}$ and $x \in \mathbb{R}$.

Proof: Suppose not. If there is a function $g \in \mathcal{L}(\mathbb{R})$ such that $|f_n(x)| = \dfrac{1}{n}\chi_{[-n,n]}(x) \leq g(x), \forall n \in \mathbb{N}$ and $x \in \mathbb{R}$, then we have

$$\int_{\mathbb{R}} g(x)dx \geq \sum_{n=1}^{\infty} \int_{[n-1,n]} g(x)dx \geq \sum_{n=1}^{\infty} \int_{[n-1,n]} \frac{1}{n}\chi_{[-n,n]}(x)dx = \sum_{n=1}^{\infty} \frac{1}{n} = \infty,$$

which means $g\bar{\in}\mathcal{L}(\mathbb{R})$. And this contradicts the assumption $g \in \mathcal{L}(\mathbb{R})$.

22. Proof: When $0 \leq y \leq 1$, we have

$$\int_0^\infty f(x,y)dx = \int_0^y (-1)dx + \int_y^{y+1} 1dx = -y+1;$$

when $y \geq 1$, we have

$$\int_0^\infty f(x,y)dx = \int_{y-1}^y (-1)dx + \int_y^{y+1} 1dx = -1+1 = 0.$$

So

$$\int_0^\infty \int_0^\infty f(x,y)dxdy = \int_0^1 (-y+1)dy = -\frac{(-y+1)^2}{2}|_{y=0}^1 = \frac{1}{2}$$

In the same way, when $0 \leq x \leq 1$, we have

$$\int_0^\infty f(x,y)dy = \int_0^x 1dy + \int_x^{x+1} (-1)dy = x-1;$$

when $x \geq 1$, we have

$$\int_0^\infty f(x,y)dy = \int_{x-1}^x 1dy + \int_x^{x+1} (-1)dy = 1-1 = 0.$$

So

$$\int_0^\infty \int_0^\infty f(x,y)dydx = \int_0^1 (x-1)dx = \frac{(x-1)^2}{2}|_{x=0}^1 = -\frac{1}{2}$$

Therefore, we have

$$\int_0^\infty \int_0^\infty f(x,y)dxdy \neq \int_0^\infty \int_0^\infty f(x,y)dydx.$$

This example can't be applied to Fubini's Theorem because

$$f\bar{\in}\mathcal{L}([0,\infty] \times [0,\infty]),$$

which means $\int_{[0,\infty]\times[0,\infty]} |f(x,y)|dxdy = +\infty$.

23. Proof: Actually, we can strengthen the conclusion to show that F is uniformly continuous on $[a,b]$.

Since f is integrable on $[a,b]$, according to Theorem 1.41, we know $\forall \varepsilon > 0, \exists \delta > 0$ such that for any measurable set $A \subset [a,b]$ satisfying $\mu(A) < \delta$, then $\int_A |f(t)|dt < \varepsilon$.

So $\forall x_1, x_2 \in [a,b]$ (say $x_1 < x_2$), when $|x_2 - x_1| < \delta$, we have

$$|F(x_1) - F(x_2)| = |\int_{x_1}^{x_2} f(t)dt| \leq \int_{x_1}^{x_2} |f(t)|dt < \varepsilon,$$

which means F is uniformly continuous on $[a,b]$.

24. (a) Proof: Since $\frac{1}{\frac{q}{q-p}} + \frac{1}{\frac{q}{p}} = 1$ (if $p = q$, we consider $\frac{q}{q-p} = \infty$), according to Hölder's Inequality, we have

$$\|f\|_p^p = \|f^p\|_1 \leq \|1\|_{\frac{q}{q-p}} \|f^p\|_{\frac{q}{p}} = (\mu(\Omega))^{1-\frac{p}{q}} \|f\|_q^p,$$

so

$$\|f\|_p \leq (\mu(\Omega))^{\frac{1}{p}-\frac{1}{q}} \|f\|_q.$$

And thus $f \in \mathcal{L}_p(\Omega)$.

(b) Proof: Since $f \in \mathcal{L}_\infty(\Omega)$, let $\|f\|_\infty = M$, then we have

$$\|f\|_p = (\int_\Omega |f(x)|^p dx)^{\frac{1}{p}} \leq (\int_\Omega M^p dx)^{\frac{1}{p}} = M(\mu(\Omega))^{\frac{1}{p}},$$

so

$$\limsup_{p\to\infty} \|f\|_p \leq \limsup_{p\to\infty} M(\mu(\Omega))^{\frac{1}{p}} = M.$$

On the other hand, for any $\varepsilon > 0$, let

$$A_\varepsilon = \{x \in \Omega : |f(x)| > M - \varepsilon\}.$$

According to the definition of essential supremum, we have $\mu(A_\varepsilon) > 0$. So

$$\|f\|_p = (\int_\Omega |f(x)|^p dx)^{\frac{1}{p}} \geq (\int_{A_\varepsilon} (M-\varepsilon)^p dx)^{\frac{1}{p}} = (M-\varepsilon)(\mu(A_\varepsilon))^{\frac{1}{p}},$$

so

$$\liminf_{p\to\infty} \|f\|_p \geq \liminf_{p\to\infty} (M-\varepsilon)(\mu(A_\varepsilon))^{\frac{1}{p}} = M - \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, so

$$\liminf_{p\to\infty} \|f\|_p \geq M.$$

We have proved

$$M \geq \limsup_{p\to\infty} \|f\|_p \geq \liminf_{p\to\infty} \|f\|_p \geq M,$$

so

$$\lim_{p\to\infty} \|f\|_p = M = \|f\|_\infty.$$

(c) Proof: Suppose not. That is to assume $\|f\|_\infty = \infty$. Then $\forall M > 0$, we let

$$A_M = \{x \in \Omega : |f(x)| > M\},$$

which will mean $\mu(A_M) > 0$ for any $M > 0$ according to the definition of essential supremum. Then we have

$$\|f\|_p = (\int_\Omega |f(x)|^p dx)^{\frac{1}{p}} \geq (\int_{A_M} M^p dx)^{\frac{1}{p}} = M(\mu(A_M))^{\frac{1}{p}}.$$

So

$$\limsup_{p\to\infty} \|f\|_p \geq \limsup_{p\to\infty} M(\mu(A_M))^{\frac{1}{p}} = M.$$

Since $M > 0$ is arbitrary, so

$$\limsup_{p\to\infty} \|f\|_p = \infty,$$

which contradicts $\|f\|_p \leq K$ for all $1 \leq p \leq \infty$. So

$$f \in \mathcal{L}_\infty(\Omega).$$

And according to the conclusion of **24.(b)**, we have

$$\|f\|_\infty = \lim_{p\to\infty} \|f\|_p \leq K.$$