Methods of Applied Mathematics

Homework 2
(Due on Friday, Sep 23)

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CAM Program

Exercise 1.4

10. A topology τ on X makes $F : X \to \mathbb{R}$ continuous iff

$$\{F^{-1}(U) \subset X : U \text{ is an open set in } \mathbb{R}\} \subset \tau.$$

where $\tau_1 = \{F^{-1}(U) \subset X : U$ is an open set in $\mathbb{R}\}$ is the weakest one, and the discrete topology $\tau_2 = \mathcal{P}(X)$ is the strongest one.

12. Proof: Let $f : X \to Y$ be continuous, and $A \subset X$ is compact. If $\{E_\alpha\}_{\alpha \in I}$ is an open cover of $f(A)$
$\Rightarrow f(A) \subset \bigcup_{\alpha \in I} E_\alpha$;
$\Rightarrow A \subset \bigcup_{\alpha \in I} f^{-1}(E_\alpha)$;
$\Rightarrow \exists\ \alpha_1, \alpha_2, \cdots, \alpha_n \in I$ such that $A \subset \bigcup_{i=1}^n f^{-1}(E_{\alpha_i})$ (since A is compact);
$\Rightarrow f(A) \subset \bigcup_{i=1}^n E_{\alpha_i}$;
$\Rightarrow f(A)$ is compact.

14. Proof: According to the conclusion of **Problem 12**, we have $f(X) \subset \mathbb{R}$ is compact
$\Rightarrow f(X) \subset \mathbb{R}$ is a bounded closed set (we know it from Mathematical Analysis);
$\Rightarrow f$ takes on its maximum and minimum values.

16. Proof (a): Let $U \subset \mathbb{R}$ be an open set.
$\Rightarrow g^{-1}(U) \subset \mathbb{R}$ is an open set (since g is continuous);
$\Rightarrow (g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U)) \subset \mathbb{R}^d$ is measurable (since f is measurable);
$\Rightarrow g \circ f$ is measurable.

Proof (b): Let $U \subset \mathbb{R}$ be an open set.
Since $f : \Omega \to \mathbb{R}^d$ is continuous
$\Rightarrow f^{-1}(U) \subset \Omega$ is an open set;
$\Rightarrow \exists$ an open set $V \subset \mathbb{R}^d$ such that $V \bigcap \Omega = f^{-1}(U)$;
$\Rightarrow f^{-1}(U)$ is measurable (since Ω and V are measurable in $\mathbb{R}^d$);
$\Rightarrow f : \Omega \to \mathbb{R}$ is measurable.

17. Proof: First of all, $0 \le d_x \le 1$. We only need to test the countable addition. Let $A_n \in \mathcal{B}$ $(n \in \mathbb{N})$ are disjoint, we have

If all $A_n \bar{\ni} x$, then

$$d_x(\bigcup_{n=1}^{\infty} A_n) = 0 = \sum_{n=1}^{\infty} d_x(A_n);$$

If there is some $A_{n_0} \ni x$, then

$$d_x(\bigcup_{n=1}^{\infty} A_n) = 1 = d_x(A_{n_0}) = \sum_{n=1}^{\infty} d_x(A_n).$$

Any way, we have

$$d_x(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} d_x(A_n),$$

which means that d_x is a measure on $\mathcal{B}$.

Prove LEMMA 1.42 (Chebyshev's Inequality)

Proof: Since $f \ge 0$, we have

$$\int_{\Omega} f(x)dx \ge \int_{\{x\in\Omega: f(x)>\alpha\}} f(x)dx \ge \alpha \int_{\{x\in\Omega: f(x)>\alpha\}} 1dx$$

$$= \alpha\mu(\{x \in \Omega : f(x) > \alpha\})$$

Therefore,

$$\mu(\{x \in \Omega : f(x) > \alpha\}) \le \frac{1}{\alpha} \int_{\Omega} f(x)dx.$$