Spectral analysis of N-body Schrödinger operators by use of Balslev-Combes theory has been subject to intensive study ever since the original paper [3] appeared.

Dilation of the radiation field Hamiltonian H_f requires the choice of a one-photon basis $\{f_i\}$ on $\mathbf{C}_0(\mathbf{R}^3)$ to represent H_f in terms of $a^\dagger(f_i)$, $a(f_j)$

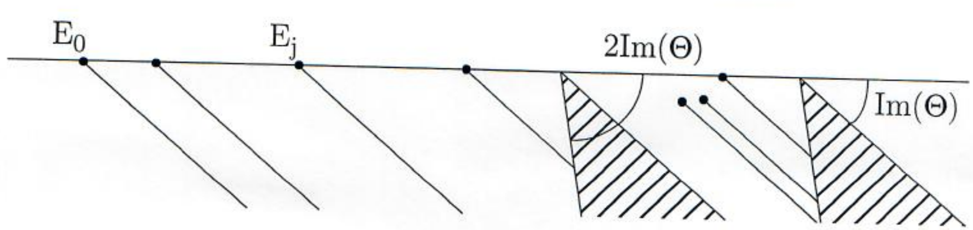
$$H_f = \sum_{i,j} \langle f_i, \omega f_j \rangle \, a^\dagger(f_i) a(f_j) \ .$$

Using $\langle f_i, \omega f_j \rangle = \int d^3\vec{k} \bar{f}_i(\vec{k}) \omega(\vec{k}) f_j(\vec{k})$ and the usual definition of $a^\sharp(f)$, it follows that

$$H_f^{(\theta)} = \sum_{i,j} \langle f_i, \omega_\theta f_j \rangle \, a^\dagger(f_i) a(f_j) \ ,$$

where $\omega_\theta(\vec{k}) := \omega(e^{-\theta}\vec{k})$. For the dispersion relation $\omega(\vec{k}) = |\vec{k}|$, one finds that $H_f^{(\theta)} = e^{-\theta} H_f$. Note that since the modulus function $|\cdot| : \mathbf{R}^3 \to \mathbf{R}^+$ is not analytic, the factor $e^{-\theta}$ must be extracted from $|e^{-\theta}\vec{k}|$ before analytic continuation. The spectrum of H_f consists of a single eigenvalue $\{0\}$ at the bottom of its continuous spectrum, corresponding to the vacuum state Ω_f. There is no separation between the point spectrum and the continuous spectrum of H_f, since photons are massless particles. For $\theta = i\phi \in \mathbf{C}^+$, the continuous spectrum of $H_f^{(i\phi)}$ that emanates from $\{0\}$ is rotated by ϕ into $\mathbf{C}^-$.

The spectrum of the complex dilated, free Hamiltonian $H_{g=0}^{(i\phi)}$ thus consists of branches of continuous spectrum of $H_f^{(i\phi)}$, which emanate from each element of the spectrum of $H_{el}^{(i\phi)}$, from each threshold, and from each resonance, cf. the following figure.



We have introduced the interaction Hamiltonian W_g at the end of section 1.2. The unitarily dilated $W_g^{(\theta)} = gW_1^{(\theta)} + g^2 W_2^{(\theta)}$ is characterized by the coupling functions $G_{m,n}^{(\theta)}$, $1 \le m+n \le 2$, which are defined by

$$\begin{aligned} W_1^{(\theta)} &= \int d^3\vec{k} [G_{10}^{(\theta)}(\vec{k}) \otimes a^\dagger(\vec{k}) \, + \, G_{01}^{(\theta)}(\vec{k}) \otimes a(\vec{k})] \\ W_2^{(\theta)} &= \int d^3\vec{k} d^3\vec{k}' [G_{20}^{(\theta)}(\vec{k}) \otimes a^\dagger(\vec{k}) a^\dagger(\vec{k}') \, + \, G_{02}^{(\theta)}(\vec{k}) \otimes a(\vec{k}) a(\vec{k}') \\ &\quad + G_{11}^{(\theta)}(\vec{k}) \otimes a^\dagger(\vec{k}) a(\vec{k}')] \, . \end{aligned}$$

We have $G_{m,n}^{(\theta)}(\vec{k}) = e^{3\theta/2} G_{m,n}(e^{-\theta}\vec{k})$ for $m+n=1$, and $G_{m,n}^{(\theta)}(\vec{k}, \vec{k}') = e^{3\theta} G_{m,n}(e^{-\theta}\vec{k}, e^{-\theta}\vec{k}')$ for $m+n=2$. Thus, the coupling functions are dilation analytic, and we choose $\theta = i\phi \in \mathbf{C}^+$. The properties of the coupling functions are specified in the following hypothesis.

2 On the structure of the theory, and survey of results

In section 2.1, we will discuss the U(1) gauge invariance of the quantized theory and derive the Ward identities. In section 2.2, we will present the results of this work.

2.1 Gauge invariance and Ward-Takahashi identities

In this section, we will use path integral methods to derive the Ward-Takahashi identities of non-relativistic quantum electrodynamics, which express the $U(1)$ gauge invariance of the theory on the quantum level. We restrict our analysis to one-electron states coupled to the quantized electromagnetic field, i.e. the case $N = 1$. We note that non-relativistic one-electron states can be described with field theoretical methods because U(1) gauge invariance implies particle number conservation in the low-energy limit. The action functional of this system is given by

$$S[\psi^*, \psi, A^\mu_\kappa, \eta^*, \eta, J^\mu] = \int d^4x \, \{L_{Pauli} \, + \, L_C \, + \, L_S\} \quad ,$$

where L, L_C, L_S are defined as follows:

$$\begin{aligned} L_{Pauli} \;&=\; \psi^*(x)\,(i\partial_t \, - \, eA_{\kappa,0}(x))\,\psi(x) \, - \, \psi^*(x)\frac{1}{2m}\left(\frac{1}{i}\vec{\nabla} \, - \, e\vec{A}_\kappa(x)\right)^2 \psi(x) \\ &+ \; \frac{e}{2m}\,\psi^*(x)\,\vec{\sigma}\vec{B}_\kappa(x)\,\psi(x) \, + \, \frac{1}{2}A^\mu_\kappa(x)\left((\partial_t^2 - \Delta)\eta_{\mu\nu} - \partial_\mu\partial_\nu\right)A^\nu_\kappa(x) \end{aligned} \tag{3}$$

is the standard Lagrangian of non-relativistic quantum electrodynamics. $\psi^*(p)$ and $\psi(p)$ are Grassmann field variables which transform like spinors. The bosonic field variables A^μ_κ, $\mu = 0, 1, 2, 3$ account for the ultraviolet cutoff photon field. L_{Pauli} is invariant with respect to the gauge transformations

$$\begin{aligned} \psi(x) \;&\rightarrow\; e^{-ie\chi_\kappa(x)}\psi(x) \, , \\ \psi^*(x) \;&\rightarrow\; e^{ie\chi_\kappa(x)}\psi^*(x) \, , \end{aligned} \tag{4}$$

$$A^\mu_\kappa \;\rightarrow\; A^\mu_\kappa + \partial^\mu\chi_\kappa(x) \; . \tag{5}$$

The function χ_κ is defined by $\chi_\kappa(x) = (\chi * \check{\kappa})(x)$, where χ is a differentiable function satisfying $\sup_x |\chi(x)|$, $\sup_x |\partial_\mu\chi(x)| \;\leq\; 1$, and where κ is the ultraviolet cutoff function defined in section 1.1. These gauge transformations preserve the ultraviolet cutoff imposed on the radiation field, which follows from the momentum space representation of (5), $A^\mu(k)\kappa(k) \rightarrow A^\mu(k)\kappa(k) + i\,k^\mu\,\chi(k)\kappa(k)$. This is necessary to have a finite theory on the perturbative level.

Note that the four summands in L_{Pauli} are *individually* gauge invariant. Thus, they will all obtain different renormalizations, as will be shown in the next section.

As described in section 1, only the transverse part of the electromagnetic vector potential will be quantized. Thus, we choose to work in the *Coulomb gauge* $\vec{\nabla}\vec{A}_\kappa(x) = 0$, which is fixed by the Lagrangian

$$L_C \;=\; \frac{1}{2\,\alpha}(\vec{\nabla}\vec{A}_\kappa(x))^2 \; , \tag{6}$$

where α is a finite parameter (not to be confused with the feinstructure constant). Of course, L_C is not gauge invariant. The Faddeev-Popov method cannot be used to fix the Coulomb gauge in abelian theory, because the ghost term in the action is independent of the gauge field and only results in an overall normalization factor of the partition function. The Lagrangian

$$L_S = \eta^*(x)\psi(x) + \psi^*(x)\eta(x) + \vec{A}_\kappa(x)\vec{J}(x)$$

couples the transverse electromagnetic field variable to the external source $\vec{J}(x)$, and the matter field variables to the anticommuting sources $\eta^*(x)$, $\eta(x)$.

The partition function $Z[\eta^*, \eta, \vec{J}]$ is obtained from the functional integral

$$Z[\eta^*, \eta, \vec{J}] = \int D\psi^* \, D\psi \, D\vec{A} \, e^{i\,S[\psi^*, \psi, A_{\kappa,\mu}, \eta^*, \eta, \vec{J}]}$$

over the field variables ψ^*, ψ, $\vec{A}$, and is the generating functional of the n-point functions of the theory. A mathematically rigorous treatment of fermionic functional integrals would require to define the field variables on a countable space. A standard method to achieve this is to introduce box normalization, i.e. periodic boundary conditions, such that momentum space becomes discrete. After performing functional integration in momentum space, the box normalization can be removed by letting the volume of the boxes go to infinity. However, we will restrict ourselves to the use of functional integrals on a formal level.

Gauge invariance of non-relativistic quantum electrodynamics requires the generating functional $Z[\eta^*, \eta, \vec{J}]$ also to be gauge invariant. This *consistency* condition is the source of powerful nonperturbative identities on the quantum level. As a consequence of the fact that the full action of the theory is not gauge invariant, we will obtain a set of constraints inter-relating the n-point functions of the theory, the *Ward-Takahashi identities*. For infinitesimal gauge transformations, (4) reduces to

$$\psi(x) \to \psi(x) - ie\chi_\kappa(x)\psi(x) \ , \quad \psi^*(x) \to \psi^*(x) + ie\chi_\kappa(x)\psi^*(x) .$$

The term $\int d^4x \, L_{Pauli}$ in the action is gauge invariant, but $\int d^4x \, L_C + L_S$ produces an infinitesimal variation of the action functional

$$\delta S := \int d^4x \left\{ \frac{1}{\alpha}(\vec{\nabla} \cdot \vec{A})\Delta\chi_\kappa - \vec{J} \cdot \vec{\nabla}\chi_\kappa - i\,e\,\chi_\kappa(\eta^*\psi - \psi^*\eta) \right\}$$

under gauge transformation. The integrand of the partition function $Z[\eta^*, \eta, \vec{J}]$ picks up an additional factor $\exp(i\,\delta S)$ which is approximately $1 + i\,\delta S$. Gauge invariance of the partition function thus implies that the expectation value of δS vanishes. Using the fact that χ_κ is arbitrary, we obtain

$$\int D\psi^* \, D\psi \, D\vec{A} \left\{ \frac{1}{\alpha}\Delta(\vec{\nabla} \cdot \vec{A}) - \vec{\nabla} \cdot \vec{J} - i\,e\,(\eta^*\psi - \eta\psi^*) \right\} e^{iS} = 0$$

after partial integration in δS. By definition of the partition function, this can be written as

$$-\frac{i}{\alpha}\Delta\left(\vec{\nabla} \cdot \frac{\delta Z}{\delta \vec{J}(x)}\right) - \vec{\nabla} \cdot \vec{J}(x)Z - e\left(\eta^*(x)\frac{\delta Z}{\delta\eta^*(x)} + \frac{\delta Z}{\delta\eta(x)}\eta(x)\right) = 0 \ .$$

The minus sign in the last term is due to the anticommutation relation $\frac{\delta}{\delta\eta_r(x)}\psi_s(y) = -\psi_s(y)\frac{\delta}{\delta\eta_r(x)}$, where r and s are spinor indices. The generating functional of *connected* Feynman graphs is defined by $Z[\eta^*,\eta,\vec{J}] = e^{iW[\eta^*,\eta,\vec{J}]}$, for which

$$\frac{1}{\alpha}\Delta\left(\vec{\nabla}\cdot\frac{\delta W}{\delta\vec{J}(x)}\right) - \vec{\nabla}\cdot\vec{J}(x) - i\,e\left(\eta^*(x)\frac{\delta W}{\delta\eta^*(x)} + \frac{\delta W}{\delta\eta(x)}\eta(x)\right) = 0 \tag{7}$$

holds. The Legendre transform of W is the vertex function Γ, which is the generating functional of the *one-particle irreducible* graphs

$$\Gamma[\psi^*,\psi,\vec{A}] = W[\eta^*,\eta,\vec{J}] - \int d^4x[\eta^*(x)\psi(x) + \psi^*(x)\eta(x) + \vec{A}_\kappa(x)\vec{J}(x)]\,.$$

We substitute the relations

$$\begin{aligned} \tfrac{\delta\Gamma}{\delta\vec{A}(x)} &= -\vec{J}(x)\,, & \frac{\delta W}{\delta\vec{J}(x)} &= \vec{A}(x)\,,\\ \tfrac{\delta\Gamma}{\delta\psi(x)} &= \eta^*(x)\,, & \frac{\delta W}{\delta\eta^*(x)} &= \psi(x)\,,\\ \tfrac{\delta\Gamma}{\delta\psi^*(x)} &= -\eta(x)\,, & \frac{\delta W}{\delta\eta(x)} &= -\psi^*(x) \end{aligned}$$

in Eq. (7) and obtain

$$\frac{1}{\alpha}\Delta\left(\vec{\nabla}\cdot\vec{A}(x)\right) + \vec{\nabla}\cdot\frac{\delta\Gamma}{\delta\vec{A}(x)} - i\,e\frac{\delta\Gamma}{\delta\psi(x)}\psi(x) - i\,e\,\psi^*(x)\frac{\delta\Gamma}{\delta\psi^*(x)} = 0\ . \tag{8}$$

Functional differentiation of this result with respect to $\psi^*(x_1)$ and $\psi(x_2)$, and setting ψ^*, ψ, $\vec{A}$ equal to zero yields

$$-\nabla_x^j\cdot\frac{\delta^3\Gamma[0]}{\delta\psi^*(x_1)\delta\psi(x_2)\delta A^j(x)} = i\,e\,\delta(x-x_1)\,\frac{\delta^2\Gamma[0]}{\delta\psi^*(x)\delta\psi(x_2)} - i\,e\,\delta(x-x_2)\,\frac{\delta^2\Gamma[0]}{\delta\psi^*(x_1)\delta\psi(x)}\,. \tag{9}$$

The proper one-photon vertex function $\Gamma_j^{(1)}(p,q,p')$ of the interacting system is defined by

$$\int d^4xd^4x_1d^4x_2e^{i(p'x_1-px_2-qx)}\frac{\delta^3\Gamma[0]}{\delta\psi^*(x_1)\delta\psi(x_2)\delta A^j(x)} = i\,e\,(2\pi)^4\delta(p'-p-q)\,\Gamma_j^{(1)}(p,q,p')\ ,$$

and the corresponding inverse electron propagator $G_{el}^{-1}(p)$ is given by

$$\int d^4x_1d^4x_2e^{i(p'x_1-px_2)}\frac{\delta^2\Gamma[0]}{\delta\psi^*(x_1)\delta\psi(x_2)} = i\,(2\pi)^4\delta(p'-p-q)\,G_{el}^{-1}(p)\ .$$

The Fourier transform of Eq. (9) with respect to the coordinates x, x_1, x_2 finally yields the *first Ward-Takahashi identity*

$$q^i\Gamma_i^{(1)}(p,q,p+q) = G_{el}^{-1}(p) - G_{el}^{-1}(p+q)\ . \tag{10}$$

which reduces to the *first Ward identity*

$$\Gamma_i^{(1)}(p,0,p) = -\partial_{p_i}G_{el}^{-1}(p).$$

in the limit $q \to 0$. There are two interaction vertices which couple the electron to a single photon line, which originate from the term

$$\begin{aligned} S_I^{(1)} &:= \int d^4x \left[\frac{e}{m}\psi^*(x)\,\vec{A}(x)\,\frac{1}{i}\,\vec{\nabla}\,\psi(x) + \frac{e}{2m}\psi^*(x)\,\vec{\sigma}\,\vec{B}(x)\,\psi(x)\right] \\ &= \int \frac{d^4p}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}\left[\frac{e}{m}\psi^*(p+q)\,\vec{A}(q)\,\vec{p}\,\psi(p) + \frac{e}{2m}\psi^*(p+q)\,\vec{\sigma}\,\vec{B}(q)\,\psi(p)\right] \end{aligned} \quad (11)$$

in the action. Expansion of $e^{iS_I^{(1)}}$ to first order in e shows that the tree level approximation of $ie\Gamma_i^{(1)}(p,q,p+q)$ is the sum of the interaction vertices $i\frac{e}{m}p_k$ and $-\frac{e}{2m}\vec{\sigma}\,\vec{B}$, which we will refer to as the $\vec{p}$-vertex and the $\vec{B}$-vertex for brevity. The $\vec{B}$-vertex can be written in the form $\sigma_{jk}q^jA^k(q)$ with $\vec{B}(q) = i\,\vec{q}\wedge\vec{A}(q)$ and

$$\sigma_{jk} := \frac{i}{2}[\sigma_j,\sigma_k] \ .$$

The renormalized vertices are of the form $f_1(p,q)p_k$ and $f_2(p,q)\sigma_{jk}q^j$, therefore we find

$$ie\,\Gamma_i^{(1)}(p,q,p+q) = f_1(p,q)p_i + f_2(p,q)\sigma_{ji}q^j \ ,$$

where the functions f_1 and f_2 have to be calculated in orders of e^2 using perturbation theory. The lhs of the first Ward-Takahashi identity produces a term $f_2(p,q)\sigma_{ji}q^jq^i$ which vanishes to all orders in e^2 due to the antisymmetry of σ_{ji} in the indices i,j. Hence, we find no implication of the first Ward-Takahashi identity on the correction of the $\vec{B}$-vertex. This is a consequence of the fact that the term $\psi^*(x)\vec{\sigma}\vec{B}(x)\psi(x)$ is by itself gauge invariant. In relativistic quantum electrodynamics, one also finds that the Ward-Takahashi identity does not make any prediction on the correction of the gyromagnetic ratio.

The remaining interaction term of the action is given by

$$\begin{aligned} S_I^{(2)} &:= \int d^4x \left[-\frac{e^2}{2m}\psi^*(x)\,\vec{A}(x)\cdot\vec{A}(x)\,\psi(x)\right] \\ &= \int \frac{d^4p}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}\frac{d^4l}{(2\pi)^4}\left[-\frac{e^2}{2m}\psi^*(p+q)\,\vec{A}(l)\cdot\vec{A}(q+l)\psi(p)\right] \ . \end{aligned} \quad (12)$$

It defines the vertex $-\frac{ie^2}{2m}\delta_{jk}$ which couples the electron propagator to *two* photon lines, and which we will therefore refer to as the "two-photon vertex". The proper two-photon vertex function $\Gamma_j^{(2)}(p,q,l,p')$ is defined by

$$\int d^4xd^4x_1d^4x_2d^4y e^{i(p'x_1-px_2-qx-ly)}\frac{\delta^4\Gamma[0]}{\delta\psi^*(x_1)\delta\psi(x_2)\delta A^j(x)\delta A^k(y)} = \\ i\,e^2\,(2\pi)^4\delta(p'-p-q-l)\,\Gamma_{jk}^{(2)}(p,q,l,p') \ .$$

From functionally differentiating (8) with respect to $\psi^*(x_1)$, $\psi(x_2)$ and $\vec{A}(y)$, and setting ψ^*, ψ, $\vec{A}$ equal to zero, we obtain the *second Ward-Takahashi identity* in momentum space representation

$$q^j\Gamma_{jk}^{(2)}(p,q,l,p+q) = \Gamma_k^{(1)}(p,l,p+l) - \Gamma_k^{(1)}(p+q,l,p+q+l) \ .$$

In contrast to the first Ward-Takahashi identity, the second includes corrections of the $\vec{B}$-vertex. The limit $q \to 0$ determines the second Ward identity

$$\Gamma^{(2)}_{jk}(p, 0, l, p+l) \;=\; -\partial_{p_j}\Gamma^{(1)}_{k}(p, l, p+l) \;\; .$$

At tree level, this result is trivial to check. The tree level approximation of $\Gamma^{(2)}_{jk}(p, q, l, p+q)$ is $2 \cdot (-\frac{1}{2m}\delta_{jk})$. The factor 2 accounts for the two possible choices to attach the labels j and k to the two photon lines. The tree level approximation of $\Gamma^{(1)}_{k}(p, l, p+l)$ is the sum of $\frac{1}{m}p_k$ and $-\frac{1}{2im}\sigma_{jk}l^j$. The $\vec{B}$-vertex is independent of the electron momentum $\vec{p}$, hence it is clear that the second Ward identity holds to lowest order.

2.2 Survey of results

In section 3, we will renormalize the parameters of non-relativistic quantum electrodynamics in perturbation theory. Due to the increased number of interaction vertices as compared to relativistic quantum electrodynamics, we have to consider a much larger number of graphs. But due to gauge invariance of the theory, there are many amplitudes that cancel pairwise. A striking difference between non-relativistic and relativistic QED is the absence of charge renormalization in the non-relativistic case, since there is no positron production in the low-energy limit. In conclusion, we will determine the renormalization of the energy scale, of the electron mass, and of the magnetic momentum. In section 3.3, we will discuss the renormalization group flow of the electron mass.

In section 4, we will rigorously prove Fermi's golden rule for the simplified quantum mechanical model introduced in section 1.2. The main result of this analysis will be that the excited eigenstates of an unperturbed system of bounded electrons become unstable, when the interaction with an external quantized electromagnetic field is turned on. They become *resonances* at the presence of the radiation field. One of the main tools for our analysis is the dilation analyticity of the interacting Hamiltonian H_g. The method is based on Balslev-Combes theory [3] which has a long, successful history in the analysis of Schrödinger operators. By definition, resonances are singularities of the analytically continuated resolvent on the second sheet of the associated Riemann surface.

The location of the resonances will be probed by use of the so-called Feshbach map. It maps H_g to an operator which is isospectral to the analytically continuated resolvent for a piece of the spectrum in a small vicinity of an excited eigenstate of the free system. We will prove a sequence of fairly technical lemmata to show the applicability of the Feshbach map, and that it is invertible on a certain region lying in that small vicinity. The result of section 4 is that all resonances of H_g above the ground state are elements of the second sheet of the Riemann surface, spaced at a distance of order g^2 from the real axis. g is the small coupling constant between the electrons and the radiation field.

In section 5, we will use the results of section 4 to prove the exponential decay of resonances. Again, we will apply Balslev-Combes theory, and we will also take advantage of the analyticity properties of the resolvent.

3 Perturbative renormalization of non-relativistic QED

In section 3.1, the collection of Feynman rules of non-relativistic quantum electrodynamics will be completed. Section 3.2 is devoted to the one-loop renormalization of the parameters of the theory. In section 3.3, the renormalization group flow of the electron mass will be discussed.

3.1 Summary of Feynman rules

The interaction vertices of the theory have already been calculated in section 2.1. We will now derive the Feynman propagators of the free electron, and of the free photon field. The action functional of a free electron is given by

$$S_{0,\,el} = \int \frac{d^4p}{(2\pi)^4} \psi^*(p) \left\{ p_0 - \frac{|\vec{p}|^2}{2m} \right\} \psi(p) \ .$$

To obtain its Feynman propagator $\hat{G}_{el,\,0}$ in momentum space representation, we perform an infinitesimal Wick rotation of the time axis $t \to t\,e^{-i\delta}$, $\delta > 0$. As a consequence, the p_0 axis is infinitesimally rotated to the opposite direction, $p_0 \to p_0\,e^{i\delta}$, and $S_{0,\,el}$ is mapped to $S^{(\delta)}_{0,\,el}$. The electron propagator is obtained from the (formal) functional integral

$$i\hat{G}_{0,el}(p) = Z^{-1}_{0,\,el} \int D\psi^* D\psi\, \psi(p)\psi^*(p)\, e^{iS^{(\delta)}_{0,\,el}} = \frac{i}{p_0 e^{i\delta} - \frac{|\vec{p}|^2}{2m}}$$

with $Z_{0,\,el} := \int D\psi^* D\psi\ e^{iS^{(\delta)}_{0,\,el}}$. The effect of the factor $e^{i\delta}$ is equivalently obtained by a summand $i\epsilon$, yielding

$$i\hat{G}_{0,el}(p) = \frac{i}{p_0 - \frac{|\vec{p}|^2}{2m} + i\,\epsilon} \ . \tag{13}$$

In non-relativistic quantum electrodynamics, the electron propagator has only one pole, not two as opposed to its relativistic counterpart. Thus, there exists only a single time-ordering such that the two-point function $\langle 0|T - \psi(x)\psi^*(y)|0\rangle$ does not vanish, $|0\rangle \in \mathcal{H}_{el}$ being the vacuum of the non-interacting system. As it should be, we obtain no positron production in the non-relativistic limit, and particle number conservation is granted.

The Feynman propagator of the radiation field in the Coulomb gauge is calculated from the action

$$S_{f,0} = \frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \left\{ A^\mu_\kappa(k) \left((k_0^2 - |\vec{k}|^2)\eta_{\mu\nu} - k_\mu k_\nu \right) A^\nu_\kappa(k) + \frac{1}{\alpha}(i\vec{k}\vec{A}_\kappa(k))^2 \right\} \ .$$

Again, one must Wick rotate the time axis $t \to t\,e^{-i\delta}$, $\delta > 0$, which results in $k_0 \to k_0\,e^{i\delta}$. The action $S_{f,0}$ is mapped to $S^{(\delta)}_{f,\,0}$, and the photon propagator is obtained from the (formal) bosonic functional integral

$$i\hat{G}_{0,f}(k) = Z^{-1}_{0,f} \int D\vec{A}\,\vec{A}(k) \otimes \vec{A}(k)\, e^{iS^{(\delta)}_{f,0}} = -i\, \frac{\mathbf{1} + (\alpha - 1)\vec{n} \otimes \vec{n}}{k_0^2 e^{2i\delta} - |\vec{k}|^2}\, \Lambda(\vec{k}) \,,$$

where $Z_{0,f} = \int D\vec{A}\, e^{iS^{(\delta)}_{f,0}}$ and $\vec{n} \equiv \frac{\vec{k}}{|\vec{k}|}$, and where $\Lambda(\vec{k}) := \kappa^2(\vec{k})$ imposes an ultraviolet cutoff. Again, we replace the factor $e^{2i\delta}$ by a term $i\epsilon$, which gives

$$i\hat{G}_{0,f}(k) = -i\,\frac{\mathbf{1} - \vec{n}\otimes\vec{n}}{k_0^2 - |\vec{k}|^2 + i\,\epsilon}\,\Lambda(\vec{k})\,. \tag{14}$$

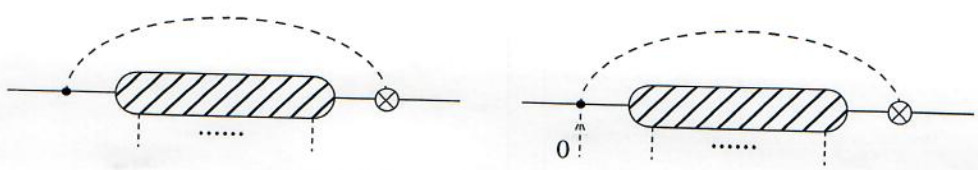
for the choice $\alpha = 0$.

We summarize the Feynman rules of non-relativistic quantum electrodynamics in momentum space representation, before we turn to the perturbative renormalization of the physical parameters of the theory. Note that the raising and lowering of indices is performed by use of δ_{jk}, δ_j^k, and δ^{jk}.

$i\frac{1}{p_0 - \frac{\vec{p}^2}{2m} + i\epsilon}$		$-i\,\frac{1-\vec{n}\otimes\vec{n}}{k_0^2 - \lvert\vec{k}\rvert^2 + i\,\epsilon}\,\Lambda(\vec{k})$			
$i\frac{e}{m}p_j$		$-\frac{e}{2m}\sigma_{jk}q^j$		$-i\frac{e^2}{2m}\delta_{jk}$	

3.2 Perturbative renormalization to one-loop level

In this section, we will calculate the one-loop radiative corrections of the parameters of non-relativistic quantum electrodynamics. The results derived here will only be relevant to quadratic order with respect to the dimensionless electron velocity $\frac{\vec{p}}{m}$, because the Pauli equation approximates the low energy limit of the Dirac equation only with this accuracy. Due to the absence of positron production, all one-loop graphs contain exactly one inner photon line. The Ward (-Takahashi) identities derived in the previous section will be used repeatedly to classify a number of gauge invariant families of graphs. However, we have shown in the last section that this procedure cannot be applied to the $\vec{B}$-vertex, because it is by itself gauge invariant. Instead, gauge invariance induces the pairwise cancellation of a large number of amplitudes, which we will show now. To be more specific, the amplitudes belonging to the following graphs annihilate the ones obtained from exchanging the outermost vertices (where the external momenta always flow from left to right):



Here "0" denotes vanishing external momentum at the outermost external photon leg. The reason for this cancellation is that the signs of the amplitudes reverse under exchange of the vertices adjacent to the inner photon line, which can be easily demonstrated on the level of Rayleigh-Schrödinger theory. For $kx = k_0x^0 - \vec{k}\cdot\vec{x}$, the second quantized transverse electromagnetic vector potential is given by

$$\vec{A}(x) = \sum_\lambda \int \frac{d^4k}{(2\pi)^4} \left\{ a_\lambda(\vec{k})\, \vec{\epsilon}_\lambda(\vec{k})\, e^{ikx} \;+\; a^*_\lambda(\vec{k})\, \vec{\epsilon}_\lambda(\vec{k})\, e^{-ikx} \right\} ,$$

and the $\vec{B}$-field operator is obtained from taking its curl

$$\vec{B}(x) = \vec{\nabla} \wedge \vec{A}(x) = \sum_\lambda \int \frac{d^4k}{(2\pi)^4} \left\{ a_\lambda(\vec{k})\, (-i\vec{k} \wedge \vec{\epsilon}_\lambda(\vec{k}))\, e^{ikx} \;+\; a^*_\lambda(\vec{k})\, (i\vec{k} \wedge \vec{\epsilon}_\lambda(\vec{k}))\, e^{-ikx} \right\} .$$

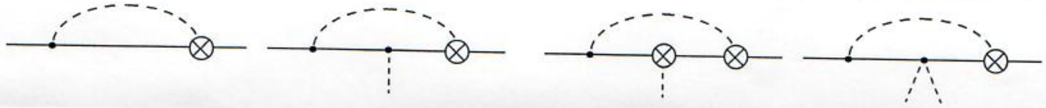
Consider first an arbitrary one-loop graph with the inner photon line adjacing to a $\vec{B}$-vertex and a $\vec{p}$-vertex. The amplitude is determined from summing over a term proportional to one of the expressions

$$\begin{gathered} \langle \vec{p}, \Omega_f \mid \hat{\vec{A}}(k) \cdot \vec{p}_{el} \mid n \rangle \langle\, n \mid \; \vec{\sigma} \cdot \hat{\vec{B}}(k') \mid \vec{p}, \Omega_f \rangle \;, \\ \langle \vec{p}, \Omega_f \mid \vec{\sigma} \cdot \hat{\vec{B}}(k) \;\mid n \rangle \langle\, n \mid \hat{\vec{A}}(k') \cdot \vec{p}_{el} \mid \vec{p}, \Omega_f \rangle \end{gathered}$$

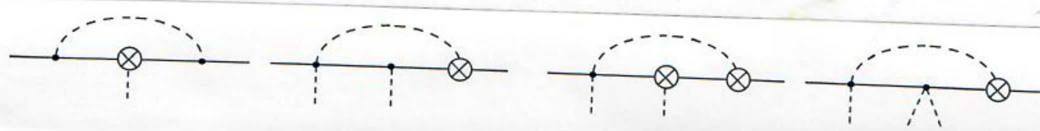
in momentum space, according to the two possible ways to arrange the vertices along the fermion line. $|n\rangle$ denotes an intermediate state in $\mathcal{H}_{el} \otimes \mathcal{H}_f$ with non-zero photon number. $\vec{p}_{el}$ stands for the momentum operator on the electron Hilbert space $\mathcal{H}_{el}$, and the vector $|\vec{p}, \Omega_f\rangle$ is the tensor product of a $\vec{p}_{el}$-eigenstate with the photon vacuum. The only nonzero Fourier components in these expressions are given by the terms

$$\begin{gathered} \langle \vec{p}, \Omega_f \mid a_\lambda(\vec{k})\; (\vec{\epsilon}_\lambda(\vec{k}) \cdot \vec{p}_{el}) \mid n \rangle \langle\, n \mid \vec{\sigma} \cdot (i\vec{k}' \wedge \vec{\epsilon}_\lambda(\vec{k}'))\; a^*_\lambda(\vec{k}') \mid \vec{p}, \Omega_f \rangle \;, \\ \langle \vec{p}, \Omega_f \mid a_\lambda(\vec{k})\; \vec{\sigma} \cdot (-i\vec{k} \wedge \vec{\epsilon}_\lambda(\vec{k})) \mid n \rangle \langle\, n \mid \; (\vec{\epsilon}_\lambda(\vec{k}') \cdot \vec{p}_{el})\; a^*_\lambda(\vec{k}') \mid \vec{p}, \Omega_f \rangle \;, \end{gathered}$$

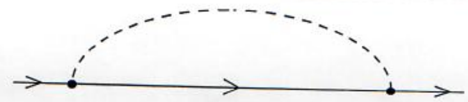
respectively. All other terms vanish since they include annihilation operators that act on the photon vacuum. Considering this last expression, it is obvious that the relative sign between the amplitudes in question is negative. The same argument can be applied if the $\vec{B}$-vertex couples over the inner photon line and the two-photon vertex to an external classical electromagnetic potential with zero momentum. For nonvanishing external momentum $\vec{q}$, gauge invariance implies that the resulting amplitudes must be proportional to $\sigma_{ik}q^i$, as has been explained in the previous section. Thus, they must vanish for momentum $\vec{q} = 0$, which is the explanation of the pairwise cancellations in question. In fact, many of these graphs produce pathological terms proportional to $\sigma_{ik}p^i$, where $\vec{p}$ is the external fermion momentum, which vanish when the amplitudes are grouped in pairs. In conclusion, the following one-loop graphs and their mirror pictures can be discarded.



Furthermore, the following graphs and their eventual mirror pictures can be verified to be negligible for small external electron momentum $\vec{p}$.



From the remaining, dominant graphs, we first consider



which contributes to the renormalization of the electron mass. Using the Feynman rules derived in the previous section, we find

$$i(-i)\left(i\frac{e}{m}\right)^2 \int \frac{d^4k}{(2\pi)^4} \frac{(\delta_{jk} - n_j\, n_k)p^j\, p^k}{(p_0 + k_0 - \frac{|\vec{p}+\vec{k}|^2}{2m} + i\epsilon)(k_0^2 - |\vec{k}|^2 + i\,\epsilon)}\, \Lambda(\vec{k})\ ,$$

where we have used the notation $\vec{n} \equiv \frac{\vec{k}}{k}$. The LSZ reduction formula in scattering theory states that the on-shell transition amplitude between two asymptotic one-particle states is obtained from fixing the external electron lines of the two-point function to the mass shell, yielding $p_0 = \frac{|\vec{p}|^2}{2m}$. It is tempting to do so now, especially in view of the amplitudes that are familiar from Rayleigh-Schrödinger theory. However, note that the Ward (-Takahashi) identities apply to the *general n-point functions*, which are defined for *arbitrary* tuples $(p_0, \vec{p})$. Applying the Ward (-Takahasi) identities to the on-shell amplitudes corresponds to the same error as stating that $\partial_y f(x,y)|_{x=g(y)}$ equals $\partial_y f(g(y), y)$ for arbitrary differentiable functions $f : I\!R^2 \to I\!R$ and $g : I\!R \to I\!R$. Thus, we have to fix the external electron lines to the mass shell *after* application of the Ward (-Takahashi) identities. In relativistic quantum electrodynamics, one is much less tempted to make this error due to relativistic covariance of the theory: One is less likely to treat p_0 any differently from $\vec{p}$.

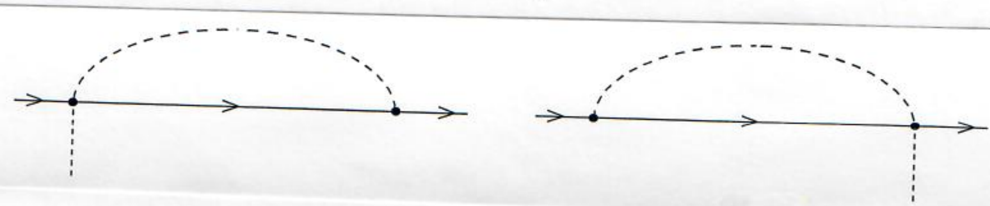
We can now continue our calculation. For simplicity, we assume $\Lambda(\vec{k})$ to be a step function that imposes a sharp ultraviolet cutoff at momentum λ. The integrand has two poles at positive $Re(k_0)$ in the lower half-plane, and one pole at negative $Re(k_0)$ in the upper half-plane. We choose a loop in the upper half-plane for the k_0-integration contour, which yields $\frac{2\pi i}{2|\vec{k}|}$ times the nonsingular factor of the integrand at $k_0 = -|\vec{k}|$. With a slight abuse of notation, where we write k for $|\vec{k}|$, we thus find

$$\frac{ie^2}{m^2} \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{(\delta_{jk} - n_j\, n_k)p^j\, p^k}{2k(p_0 - k - \frac{(\vec{p}+\vec{k})^2}{2m})}\, \Lambda(\vec{k})\ . \tag{15}$$

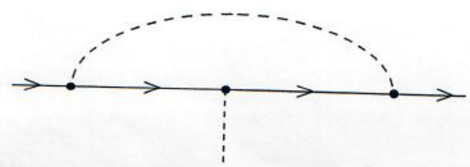
All amplitudes that will be computed in this section include the same k_0-contour integral that has just been demonstrated. Thus, we will always start directly at this step of the calculation in what follows. We can now apply the first Ward identity do derive the corresponding one- and two-photon amplitudes for zero external photon momentum. This is achieved by taking the derivative of (15) with respect to the external electron momentum $\vec{p}$, and multiplying the result by a factor $(-e)$. We obtain a sum of two terms

$$\frac{ie^3}{m^2}\int \frac{d^3\vec{k}}{(2\pi)^3}\,\frac{2\,(\delta_{jk}\,-\,n_j\,n_k)\,p^k}{2k(p_0-k-\frac{(\vec{p}+\vec{k})^2}{2m})}\,\Lambda(\vec{k}) \;+\; \frac{-ie^3}{m^3}\int \frac{d^3\vec{k}}{(2\pi)^3}\,\frac{(\delta_{jk}\,-\,n_j\,n_k)p^j\,p^k\;\;(\vec{p}+\vec{k})}{2k(p_0-k-\frac{(\vec{p}+\vec{k})^2}{2m})^2}\,\Lambda(\vec{k})\;.(16)$$

The first term is identical to the sum of the amplitudes of



and the second term is the amplitude of

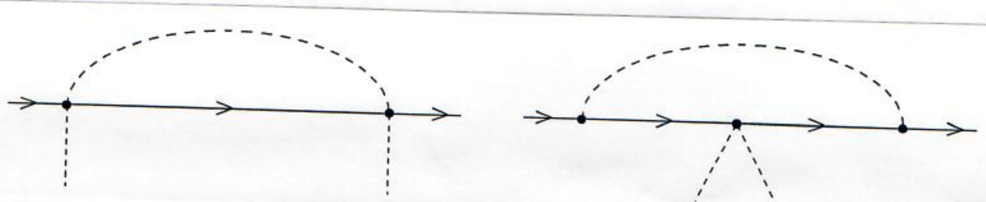


as can be checked by direct calculation, where a factor 2 must be accounted for each two-photon vertex, since there are two possibilities of labelling the adjacing photon lines. Because time reflection invariance is broken in the nonrelativistic limit, the flux direction of the incoming photon momentum (which we assume to be zero wherever we apply the Ward identities) is uniquely defined, and indicated by the arrows. From power counting, we conclude that the first term in (16) is logarithmically divergent, whereas the second one is finite, and of higer order in $\frac{\vec{p}}{m}$.

Using the second Ward identity for zero external photon momentum (i.e. deriving (16) with respect to $\vec{p}$, and multiplying by $(-e)$), we obtain four additional amplitudes, two of which can be discarded, since they correspond to higher order corrections in e. The two remaining terms

$$\frac{-ie^2}{m^2}\int \frac{d^3\vec{k}}{(2\pi)^3}\,\frac{2\,(\delta_{jk}\,-\,n_j\,n_k)}{2k(p_0-k-\frac{(\vec{p}+\vec{k})^2}{2m})}\,\Lambda(\vec{k}) \;+\; \frac{ie^2}{m^3}\int \frac{d^3\vec{k}}{(2\pi)^3}\,\frac{(\delta_{jk}\,-\,n_j\,n_k)p^j\,p^k}{2k(p_0-k-\frac{(\vec{p}+\vec{k})^2}{2m})^2}\,\Lambda(\vec{k})$$

are associated to the graphs



respectively. Again, the first term is logarithmically divergent, and the second term is finite.

We can now evaluate (15) for small external fermion momentum, setting $\vec{p}$ and $p_0 = \frac{|\vec{p}|^2}{2m}$ equal to zero, where the latter is now fixed to the mass shell. In this limit, the inverse mass of the electron is renormalized by the term

$$\frac{-i|\,\vec{p}\,|^2}{2m}\,\frac{2e^2}{3\pi^2}\int_0^{\frac{\lambda}{2m}}\frac{d\kappa}{1+\kappa} = \frac{-i|\,\vec{p}\,|^2}{2m}\,\frac{2e^2}{3\pi^2}\,log\left(1+\frac{\lambda}{2m}\right)\,, \tag{17}$$

which must be added to the inverse electron propagator $-ip_0 + i\frac{|\vec{p}|^2}{2m}$. In this calculation, we have substituted the new variable $\kappa := \frac{k}{2m} \in [0, \frac{\lambda}{2}]$, and have used the formula

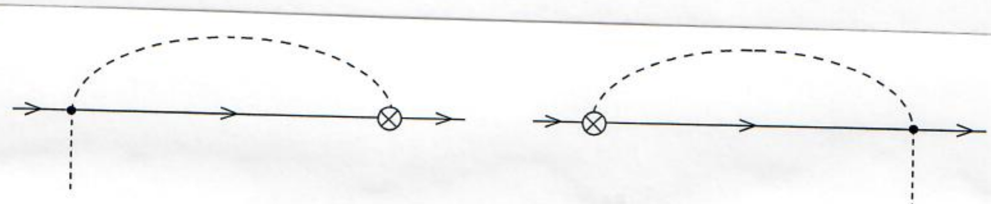
$$\int_{S^2} d\Omega_{\vec{k}}\,(\delta_{jk} - n_j\,n_k) \;=\; \frac{8\pi\delta_{jk}}{3}$$

in the numerator of the integrand. S^2 is the two dimensional unit sphere, $d\Omega_{\vec{n}}$ is the integration measure on S^2, and the subscript corresponds to the point located at the tip of $\vec{n}$. It can be shown in the same manner that the corresponding radiative corrections of the $\vec{p}$- and the two-photon vertex are given by

$$\frac{-ie\vec{p}}{m}\,\frac{2e^2}{3\pi^2}\,log\left(1+\frac{\lambda}{2m}\right)\quad,\qquad \frac{ie^2\delta_{jk}}{2m}\,\frac{2e^2}{3\pi^2}\,log\left(1+\frac{\lambda}{2m}\right)\,, \tag{18}$$

for small $\vec{p}$, which have to be added to the tree level vertices $\frac{ie\vec{p}}{m}$ and $\frac{-ie^2\delta_{jk}}{2m}$ respectively.

As explained in the previous section, it is not possible to derive the amplitudes associated to the graphs



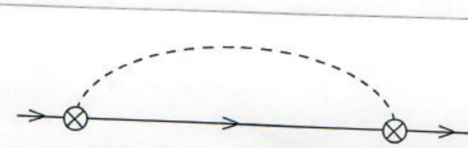
from the Ward (-Takahashi) identities. In order to calculate them directly from the Feynman rules, it is appropriate to consider pairs of graphs with opposite relative positions of the interaction vertices adjacing to the inner photon line, to obtain the desired radiative corrections proportional to $\sigma_{ik}q^i$. Moreover, a factor 2 arises due to the two-photon vertices, as explained above. Consequently, the total amplitude associated to these graphs is given by

$$2\times\frac{-e^3}{4m^2}\int\frac{d^3\vec{k}}{(2\pi)^3}\left\{\frac{-\vec{\sigma}\cdot(\vec{k}\wedge\vec{A}_{cl})}{2k(p_0-k-\frac{(\vec{p}+\vec{k})^2}{2m})} + \frac{\vec{\sigma}\cdot(\vec{k}\wedge\vec{A}_{cl})}{2k(p_0-k-\frac{(\vec{p}+\vec{k}+\vec{q})^2}{2m})}\right\}\Lambda(\vec{k})\,.$$

After some algebra, and setting p_0 and $\vec{p}$ in the denominator equal to zero, we arrive at

$$\frac{e}{2m}\,\vec{\sigma}\cdot(\vec{q}\wedge\vec{A}_{cl})\,\frac{e^2}{3\pi^2}\int_0^{\frac{\lambda}{2m}}\frac{d\kappa\;\kappa}{(1+\kappa)^2}\,. \tag{19}$$

The second dominant family of one-loop graphs is characterized by an inner photon line that joins two $\vec{B}$-vertices. The corresponding radiative correction of the electron propagator is given by



and is associated to the amplitude

$$i(-i)[-\frac{e}{2m}]^2 i \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{\sigma_{ri} k^r (\delta^{ij} - n^i n^j) \sigma_{sj} k^s}{2k(p_0 - k - \frac{(\vec{p}+\vec{k})^2}{2m})} \Lambda(\vec{k}) \, .$$

The expression in the numerator reduces to $\sigma_{ri} k^r (\delta^{ij} - n^i n^j) \sigma_{sj} k^s = -2k^2$, which simplifies the integral to

$$\frac{-ie^2}{4m^2} \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{2\, k^2}{2k(p_0 - k - \frac{(\vec{p}+\vec{k})^2}{2m})} \Lambda(\vec{k}) \, . \tag{20}$$

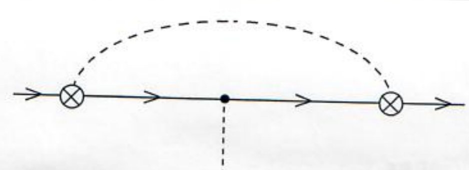
From power counting, we conclude that the result diverges *quadratically.* In the limit of small $\vec{p}$, where $\vec{p}$ and p_0 are set equal to zero in the denominator, we see immediately that the quadratically divergent term is independent of $\vec{p}$. Its value in this limit is given by $\frac{-ime^2}{\pi^2}\left(\frac{\lambda}{2m}\right)^2$, which is a shift of the electron energy scale denoted by p_0. However, Taylor expansion of (20) up to second order with respect to $\frac{\vec{p}}{m}$ generates a logarithmically divergent term,

$$\frac{i|\vec{p}|^2}{2m} \frac{e^2}{4m^2} \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{2\, k^2}{2k(p_0 - k - \frac{(\vec{p}+\vec{k})^2}{2m})^2} \Lambda(\vec{k}) \, ,$$

which contributes to the radiative correction of the inverse electron mass. Applying the first Ward identity, i.e. taking the derivative of (20) with respect to $\vec{p}$, and multiplying by $(-e)$, we find

$$\frac{ie^3}{4m^3} \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{2\, k^2\ (\vec{p}+\vec{k})}{2k(p_0 - k - \frac{(\vec{p}+\vec{k})^2}{2m})^2} \Lambda(\vec{k}) \, ,$$

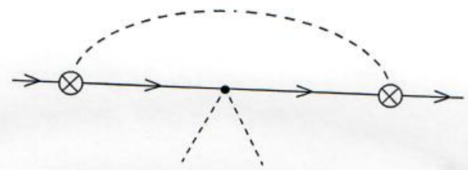
which is the amplitude of the graph



Application of the second Ward identity yields

$$\frac{-ie^4\delta_{jk}}{4m^3} \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{2\, k^2}{2k(p_0 - k - \frac{(\vec{p}+\vec{k})^2}{2m})^2} \Lambda(\vec{k}) \, ,$$

which is the amplitude of

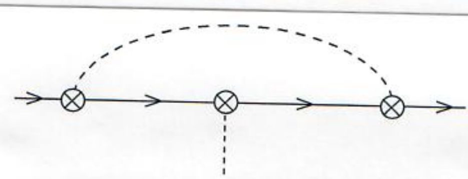


The second term that arises from the $\vec{p}$-derivative corresponds to a contribution of higher order in e, and has been dropped. After setting the external electron momentum equal to zero, and fixing its energy to the mass shell, we obtain the radiative corrections of the mass, the $\vec{p}$-vertex, and the two-photon vertex

$$\frac{i|\vec{p}\,|^2}{2m}\frac{e^2}{2\pi^2}\int_0^{\frac{\lambda}{2m}}\frac{d\kappa\ \kappa}{(1+\kappa)^2}\ ,\quad \frac{ie\vec{p}}{m}\frac{e^2}{2\pi^2}\int_0^{\frac{\lambda}{2m}}\frac{d\kappa\ \kappa}{(1+\kappa)^2}\ ,\quad \frac{-ie^2\delta_{jk}}{2m}\frac{e^2}{2\pi^2}\int_0^{\frac{\lambda}{2m}}\frac{d\kappa\ \kappa}{(1+\kappa)^2}\ ,$$

respectively.

The correction of the $\vec{B}$-vertex belonging to this family of Feynman graphs



is "invisible" in regard of the Ward-Takahashi identities. Its amplitude is given by

$$\frac{-e^3\sigma_{jk}q^j}{8m^3}\int\frac{d^3\vec{k}}{(2\pi)^3}\frac{2\,k^2}{2k(p_0-k-\frac{(\vec{p}+\vec{k})^2}{2m})^2}\Lambda(\vec{k})\ \rightarrow\ \frac{-e\sigma_{jk}q^j}{2m}\frac{e^2}{2\pi^2}\int_0^{\frac{\lambda}{2m}}\frac{d\kappa\ \kappa}{(1+\kappa)^2} \tag{21}$$

in the limit of small external electron momentum. In contrast to (19), this correction of the $\vec{B}$-vertex is given by the *same* logarithmically divergent factor as the other graphs in its family. Thus, all logarithmically divergent one-loop corrections with two $\vec{B}$-vertices adjacing to the inner photon line contribute solely to the renormalization of the inverse electron mass $\frac{1}{m}$, yielding the contribution

$$\frac{1}{m}\frac{e^2}{2\pi^2}\int_0^{\frac{\lambda}{2m}}\frac{d\kappa\ \kappa}{(1+\kappa)^2}\ =\ \frac{1}{m}\frac{e^2}{2\pi^2}\left(log\left(1+\frac{\lambda}{2m}\right)-\frac{1}{1+\frac{2m}{\lambda}}\right)\ .$$

For the total renormalization of the inverse electron mass, we thus obtain from this and (17)

$$\begin{aligned}\frac{1}{m}\ &\rightarrow\ \frac{1}{m}\left\{1\ -\ \frac{2e^2}{3\pi^2}\left[log\left(1+\frac{\lambda}{2m}\right)\right]\ +\ \frac{e^2}{2\pi^2}\left[log\left(1+\frac{\lambda}{2m}\right)-\frac{1}{1+\frac{2m}{\lambda}}\right]\right\}\\ &=\ \frac{1}{m}\left\{1\ -\ \frac{e^2}{6\pi^2}log\left(1+\frac{\lambda}{2m}\right)-\frac{e^2}{2\pi^2}\frac{1}{1+\frac{2m}{\lambda}}\right\}\ .\end{aligned} \tag{22}$$

The $\vec{p}$- and the two-photon vertex are renormalized by the same value, which is in agreement with the Ward (-Takahashi) identities. For this reason, the term $\frac{1}{2m}(\vec{p}-e\vec{A})^2$ in the Hamiltonian is renormalized by replacing $\frac{1}{m}$ with (22). The radiative correction of the gyromagnetic ratio of the electron is implicit in the renormalization of the $\vec{B}$-vertex, given by

$$\begin{aligned}\frac{-e\sigma_{jk}q^j}{2m} \quad &\rightarrow \quad \frac{-e\sigma_{jk}q^j}{2m}\left\{1 - \frac{e^2}{3\pi^2}\int_0^{\frac{\lambda}{2m}} \frac{d\kappa\ \kappa}{(1+\kappa)^2} + \frac{e^2}{2\pi^2}\int_0^{\frac{\lambda}{2m}} \frac{d\kappa\ \kappa}{(1+\kappa)^2}\right\} \\ &= \quad \frac{-e\sigma_{jk}q^j}{2m}\left\{1 + \frac{e^2}{6\pi^2}log\left(1+\frac{\lambda}{2m}\right) - \frac{e^2}{6\pi^2}\frac{1}{1+\frac{2m}{\lambda}}\right\},\end{aligned}$$

due to (19) and (21). We observe that the difference between the renormalization of the $\vec{B}$-vertex and the inverse electron mass is a *finite* additive term

$$\frac{e^2}{3\pi^2}\frac{1}{1+\frac{2m}{\lambda}},$$

which originates from the renormalization of the gyromagnetic ratio of the electron. If we choose the photon energy cutoff to be given by the electron rest energy, $\lambda = m$, as Bethe did in his original derivation of the Lamb shift, we obtain the value

$$\frac{e^2}{9\pi^2}.$$

This is a fairly good approximation of the famous relativistic result derived by Schwinger, $\frac{\alpha}{2\pi} = \frac{e^2}{8\pi^2}$. It shows, as Bethe's calculation of the Lamb shift also does, that the seemingly ad-hoc choice of the cutoff $\lambda = m$ does have physical significance in non-relativistic quantum electrodynamics. The usual argument for choosing this value is that the relevant part of the energy spectrum in the non-relativistic limit is located close to the point on the mass shell which corresponds to the electron rest mass. Thus, all energies involved in the theory can exceed the electron rest energy only by a small fraction, even in case of *virtual* states, since otherwise, the non-relativistic limit would simply not be valid. The cutoff imposed due to this heuristic argument leads to results which are in very good agreement with experimental data, i.e. the Lamb shift and the anomalous magnetic moment.

3.3 Renormalization group flow of the electron mass

Using the results derived in the previous section, we will now analyze the renormalization group flow of the electron mass. Note that due to the absence of charge renormalization in the non-relativistic limit, it is the mass renormalization that accounts for the qualitative behaviour of the interaction for varying energy scales. According to the standard renormalization procedure, the renormalized mass (22) is given the (constant) physical value m_R of the electron mass, hence

$$\frac{1}{m_R} = \frac{1}{m_B(\lambda)}\left\{1 - \frac{3\alpha}{4\pi}\log\left(1+\frac{\lambda}{2m_B(\lambda)}\right)\right\}. \tag{23}$$

This equation implicitly defines the dependency of the *bare mass* $m_B(\lambda)$ on the cutoff frequency λ. Put more precisely, the invariance of this equation with respect to redefinitions of the renormalization point λ induces the *renormalization group flow* of the bare mass. We obtain the Callan-Symanzik equation for the flow of the bare mass by applying the operator ∂_λ to both sides of (23), which yields

$$0 = -\frac{\partial_\lambda m_B(\lambda)}{m_B(\lambda)} \left\{ 1 - \frac{3\alpha}{4\pi} log \left(1 + \frac{\lambda}{2m_B(\lambda)} \right) - \frac{3\alpha}{4\pi} \frac{1}{2m_B(\lambda) + \lambda} \right\} - \frac{3\alpha}{4\pi} \frac{1}{2m_B(\lambda) + \lambda}$$

after some rearrangement of the summands. Since (23) implies that $m_B(\lambda) = m_R + O(\alpha)$, we observe that this equality is of the form

$$0 = -\frac{\partial_\lambda m_B(\lambda)}{m_B(\lambda)} (1 + O(\alpha)) - \frac{3\alpha}{4\pi} \left(\frac{1}{2m_R + \lambda} + O(\alpha) \right) .$$

Only keeping the leading order terms of order $O(1 = \alpha^0)$ in the brackets, we obtain the approximate flow equation for the mass

$$\frac{\partial_\lambda m_B(\lambda)}{m_B(\lambda)} = -\frac{3\alpha}{4\pi} \frac{1}{2m_R + \lambda} .$$

Integration with respect to λ results in

$$log(m_B(\lambda)) = -\frac{3\alpha}{4\pi} log\,(2m_R + \lambda) + const ,$$

which finally yields

$$m_B(\lambda) = m_R \left(\frac{3m_R}{2m_R + \lambda} \right)^{\frac{3\alpha}{4\pi}}$$

for the initial condition $m_B(\lambda = m_R) := m_R$. The renormalization group equation for the bare mass is a consequence of the invariance of the theory with respect to redefinitions of the renormalization point. The *physical* implication is that it also expresses the scale transformation invariance of the theory. This is because a change of the momentum scale by a factor s results in a shift of the renormalization point by s. Since m_R is invariant with respect to this scale transformation and to the corresponding shift of the renormalization point, one recovers the Callan-Symanzik equation derived above. Assuming that a certain value has been attributed to $m_B(\lambda_0)$ at some fixed renormalization point λ_0, the *running mass* $m(s) \equiv m_B(s\lambda_0)$ shows how the electron mass of the interacting system varies effectively for a change of the momentum scale. In our case, $\lambda_0 = m_R$ implies that $s \in [0, 1]$.

The scaling limit of $m(s)$ for $s \to 0$ is given by

$$m(0) \; = \; m_R \left(\frac{3}{2} \right)^{\frac{3\alpha}{4\pi}} \approx m_R \left\{ 1 + \frac{3\alpha}{4\pi} log \left(\frac{3}{2} \right) \right\} , \tag{24}$$

which is (23), solved for $m_B(\lambda)$ with the substitution $\lambda \to m_R = m_B(\lambda = m_R)$. The limit $s \to 0$ in momentum space is equivalent to the scaling limit $\frac{1}{s}\vec{x} \to \infty$ in the Euclidean $\mathbf{R}^3$, i.e. the limit of *macroscopic* dimensions. Thus, (24) corresponds to the *macroscopically* observed, effective mass of an electron that is spatially restricted to a *microscopic* system. As an example, consider bound state electrons confined to atomic orbits. The reference length scale at which $m(s=1) = m_R$ is fixed, is the Bohr radius, whereas the macroscopic experimenter makes his observation at a (relative) scale $s \to 0$. Therefore, the experimentally observed effective mass in this case is the one given in (24), and not the classical electron mass m_R, as is exemplified by the Lamb shift.

As a contrasting example, measurement of the Compton process involves free electrons upon which no intrinsic microscopic length scale is imposed, in the contrary, the electrons follow paths of macroscopic extension. Accordingly, the experimenter does not measure any electron mass that differs from its classical value (in the Compton experiment, the momentum and the velocity of the electron are both measurable, hence the mass is known).

In conclusion, we summarize some phenomenological issues on the observation of quantum processes and the renormalization group. The prescription of attributing physical values to the renormalized parameters of a quantum field theory shows that the relevant scale of the quantum process of interest defines an *intrinsic* scale of the problem. The correspondence to the length scale that is intrinsic to the observer is induced by the renormalization group flow of these parameters. Similar to active and passive transformations of coordinate systems, the renormalization group equations express the physical self-similarity of a quantum field theory under "active" scale transformations, as well as the invariance of the theory with respect to "passive" changes of the renormalization point. Here one finds a "principle of relative scales" which is very similar to the principle of relativistic covariance. In special relativity, the only sensible way to specify the intrinsic length of an object or the intrinsic duration of a process is to do so with respect to the *rest frame* of the given system. In all other frames, the observed values vary with the ratio of relative velocities. In analogy, the parameters of a quantum field theoretic system are given the classical, *"physical"* value at the intrinsic or "proper" scale of the problem. The observed values are functions of the relative scale difference between the quantum system and the observer system. In case of large discrepancies between these scales, the observed parameter values are *scaling limits* under the renormalization group flow.