Methods for Applied Mathematics

Homework 9
(Due: Nov 11, 2005)

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CAM program

Exercises 3.12

2. Proof (a): Let $R(P) = \{Px \in H : x \in H\}$ and $N(P) = \{x \in H : Px = 0\}$, first we will show that $H = R(P) \oplus N(P)$, where "$\oplus$" means the direct addition.

For any $x \in H$, $x = Px + (x - Px)$, where $Px \in R(P)$ and

$$P(x - Px) = Px - P^2x = Px - Px = 0,$$

which means $x - Px \in N(P)$. So $H = R(P) + N(P)$.

In another hand, for any $y \in R(P) \bigcap N(P)$, we know there is some $x \in H$ such that $Px = y$ and $Py = 0$, so we have

$$y = Px = P^2x = P(Px) = Py = 0,$$

which means $R(P) \bigcap N(P) = \{0\}$. So we have

$$H = R(P) \oplus N(P).$$

To prove that P is an orthogonal projector onto $R(P)$, we only need to show that $R(P) \perp N(P)$.

For any $x \in R(P), y \in N(P), t \in \mathbb{C}$, since $\|P\| = 1$, we have

$$\|x\| = \|Px\| = \|P(x + ty)\| \leq \|P\|\|x + ty\| = \|x + ty\|.$$

So we have

$$\begin{aligned} \|x\|^2 \leq \|x + ty\|^2 &= \|x\|^2 + \|ty\|^2 + (x, ty) + (ty, x) \\ &= \|x\|^2 + |t|^2\|y\|^2 + \bar{t}(x, y) + t(y, x), \end{aligned}$$

which means

$$|t|^2\|y\|^2 + \bar{t}(x, y) + t(y, x) \geq 0, \ \forall x \in R(P), y \in N(P), t \in \mathbb{C}.$$

If $y \neq 0$, we set $t = -\dfrac{(x,y)}{\|y\|^2}$, then we have

$$\frac{|(x,y)|^2}{\|y\|^2} - \frac{|(x,y)|^2}{\|y\|^2} - \frac{|(x,y)|^2}{\|y\|^2} = -\frac{|(x,y)|^2}{\|y\|^2} \geq 0,$$

which means $(x,y) = 0$.

If $y = 0$, obviously, $(x,y) = 0$. Therefore, $R(P) \perp N(P)$, which means P is an orthogonal projector onto $R(P)$.

(b): If P is not bouded, then $\|P\| = \infty \geq 1$.

If $P \in B(H,H)$, since $P^2 = P$, so we have

$$\|P\| = \|P^2\| \leq \|P\|^2.$$

Since $P \neq 0$, so $\|P\| > 0$, so $\|P\| \geq 1$ from the above inequality.

Then we will show by example that if $\dim H \geq 2$, there is a nonorthogonal projector $P : H \longrightarrow H$.

Example: Let $H = \mathbb{R}^2$ with the usual inner product $(\cdot,\cdot)$. We define $P : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ as

$$P(x,y) = (x-y, 0),\ \forall (x,y) \in \mathbb{R}^2.$$

Obviously, P is linear and we have

$$P^2(x,y) = P(x-y,0) = (x-y,0) = P(x,y),\ \forall (x,y) \in \mathbb{R}^2,$$

which means $P^2 = P$, so P is a projector.

Since $(1,0) = P(1,0), P(1,1) = (0,0)$ which means $(1,0) \in R(P), (1,1) \in N(P)$, but $((1,0),(1,1)) = 1 \neq 0$, so P is not an orthogonal projector.

(Please refer to the general proof for part (b) in Appendix.)

4. Proof: We define some subsets of $\mathcal{I}$ as:

$$\mathcal{I}_+ = \{\alpha \in \mathcal{I} : x_\alpha > 0\}$$

and

$$\mathcal{I}_n = \{\alpha \in \mathcal{I} : x_\alpha > \frac{1}{n}\},\ \forall n \in \mathbb{N}.$$

Then we have $\mathcal{I}_+ = \bigcup_{n=1}^{\infty} \mathcal{I}_n$.

For any $n \in \mathbb{N}$, we have

$$\infty > \sum_{\alpha\in\mathcal{I}} x_\alpha \geq \sum_{\alpha\in\mathcal{I}_n} \frac{1}{n} = \frac{|\mathcal{I}_n|}{n},$$

where $|\mathcal{I}_n|$ represents the number of elements in $\mathcal{I}_n$. So $|\mathcal{I}_n| < \infty$, which means $\mathcal{I}_+ = \bigcup_{n=1}^{\infty} \mathcal{I}_n$ is an at most countable set.

5. Proof: Since $\{u_\alpha\}_{\alpha\in\mathcal{I}}$ is a maximal ON set in H, by **Theorem 3.18**, we know $\text{span}\{u_\alpha\}_{\alpha\in\mathcal{I}}$ is dense in H, and we have

$$\sum_{\alpha\in\mathcal{I}} |(x, u_\alpha)|^2 = \|x\|_H^2 < \infty.$$

Since $|(x, u_\alpha)|^2 \geq$, by the result of Problem 4, we know that there exists at most countably many $\alpha_i \in \mathcal{I}$ such that $|(x, u_{\alpha_i})|^2 > 0$ and $(x, u_\alpha) = 0, \forall\alpha \in \mathcal{I} \setminus \{\alpha_i\}_{i=1}^{\infty}$.

Let $x_n = \sum_{i=1}^{n}(x, u_{\alpha_i})u_{\alpha_i}$, since $\sum_{i=1}^{\infty} |(x, u_{\alpha_i})|^2 = \|x\|_H^2 < \infty$, then for $m > n$, we have

$$\|x_m - x_n\|^2 = \| \sum_{i=n+1}^{m} (x, u_{\alpha_i})u_{\alpha_i}\|^2 = \sum_{i=n+1}^{m} |(x, u_{\alpha_i})|^2 \to 0,\ (m > n \to \infty)$$

which means $\{x_n\}$ is a Cauchy sequence in H. Since H is complete, so $y = \sum_{i=1}^{\infty}(x, u_{\alpha_i})u_{\alpha_i} \in H$ is well defined, and $x_n \to y$ in H. By the continuity of inner product, we have

$$(y, u_{\alpha_j}) = \lim_{n\to\infty}(x_n, u_{\alpha_j}) = (x, u_{\alpha_j}),\ \forall j \in \mathbb{N}.$$

By the linearity of inner product, we have

$$(y, u) = (x, u),\ \forall u \in \text{span}\{u_\alpha\}_{\alpha\in\mathcal{I}} \text{ which is dense in } H,$$

which means $(y, u) = (x, u)$ for any $u \in H$ by the continuity of inner product. So we have

$$x = y = \sum_{i=1}^{\infty}(x, u_{\alpha_i})u_{\alpha_i}.$$

6. Proof: By the definition of $l_2(\mathcal{I})$ with the inner product given by

$$(f,g) = \sum_{\alpha\in\mathcal{I}} f(\alpha)\overline{g(\alpha)},$$

to show $l_2(\mathcal{I})$ is a Hilbert space, we only need to show the completeness.

Let $\{f_n\} \subset l_2(\mathcal{I})$ be a Cauchy sequence, then we have for any $n \in \mathbb{N}$, $\|f_n\|^2 = \sum_{\alpha\in\mathcal{I}} |f_n(\alpha)|^2 < \infty$. By the result of Problem 4, we know that there exists at most countably many indexes $\alpha_i^n \in \mathcal{I}$ such that $f_n(\alpha_i^n) \neq 0$ and $f_n(\alpha) = 0, \forall \alpha \in \mathcal{I} \setminus \{\alpha_i^n\}_{i=1}^{\infty}$. Then we know $\mathcal{I}_+ = \bigcup_{i,n=1}^{\infty}\{\alpha_i^n\}$ is still at most countable, and so we can label $\mathcal{I}_+ = \{\alpha_i\}_{i=1}^{\infty}$, and $f_n(\alpha) = 0$, $\forall \alpha \in \mathcal{I}\setminus\mathcal{I}_+$, $\forall n \in \mathbb{N}$.

Since $\{f_n\} \subset l_2(\mathcal{I})$ is a Cauchy sequence, so we have

$$|\|f_n\| - \|f_m\|| \leq \|f_n - f_m\| \to 0,\ (n,m\to\infty),$$

which means $\{\|f_n\|\}$ is a Cauchy sequence in $\mathbb{R}$, and so $\lim_{n\to\infty}\|f_n\|$ exists.

We also have

$$\sup_{i\in\mathbb{N}} |f_n(\alpha_i) - f_m(\alpha_i)|^2 \leq \|f_n - f_m\|^2 \to 0,\ (n,m\to\infty)$$

which means $\{f_n(\alpha_i)\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathbb{C}$ uniformly for $i \in \mathbb{N}$. So by the completeness of $\mathbb{C}$, we know there exists some $x_{\alpha_i} \in \mathbb{C}$ such that

$$f_n(\alpha_i) \rightrightarrows x_{\alpha_i},\ (n\to\infty,\ \text{uniform for } i\in\mathbb{N}).$$

Let $f : \mathcal{I} \longrightarrow \mathbb{C}$ with

$$f(\alpha) = \left\{\begin{array}{ll} x_{\alpha_i}, & \alpha = \alpha_i \text{ for some } i\in\mathbb{N} \\ 0, & \alpha\in\mathcal{I}\setminus\mathcal{I}_+. \end{array}\right\}$$

By the property of uniform convergence, we have

$$\|f\| = \sqrt{\sum_{i=1}^{\infty}|f(\alpha_i)|^2} = \sqrt{\sum_{i=1}^{\infty}|\lim_{n\to\infty} f_n(\alpha_i)|^2} = \lim_{n\to\infty}\sqrt{\sum_{i=1}^{\infty}|f_n(\alpha_i)|^2} = \lim_{n\to\infty}\|f_n\| < \infty,$$

which means $f \in l_2(\mathcal{I})$.

And we have

$$\lim_{n\to\infty}\|f_n - f\|^2 = \lim_{n\to\infty}\sum_{i=1}^{\infty}|f_n(\alpha_i) - f(\alpha_i)|^2 = \sum_{i=1}^{\infty}\lim_{n\to\infty}|f_n(\alpha_i) - f(\alpha_i)|^2 = \sum_{i=1}^{\infty} 0 = 0,$$

which means $f_n \to f$ in $l_2(\mathcal{I})$. So $l_2(\mathcal{I})$ is a Hilbert space.

Appendix

General proof for Problem 2, part (b):

If dim$H \geq 2$, there is a nonorthogonal projector $P : H \longrightarrow H$.

Proof: Since dim$H \geq 2$, we can choose two linearly independent vector $x, y \in H$ such that $\|x\| = \|y\| = 1, (x, y) \neq 0$. Since span$\{x, y\}$ is finite dimensional, so is a complete subspace of H, we have the following decomposition of H by direct addition:

$$\begin{aligned} H &= \mathrm{span}\{x, y\} \oplus \{x, y\}^{\perp} \\ &= \mathrm{span}\{x\} \oplus \mathrm{span}\{y\} \oplus \{x, y\}^{\perp} \\ &= \mathrm{span}\{x\} \oplus (\mathrm{span}\{y\} \oplus \{x, y\}^{\perp}) \end{aligned}$$

where $\{x, y\}^{\perp} = \{z \in H : (z, x) = (z, y) = 0\}$.

So for any $z \in H$, there is a unique expression $z = z_1 + z_2$ where $z_1 \in \mathrm{span}\{x\}, z_2 \in \mathrm{span}\{y\} \oplus \{x, y\}^{\perp}$. We can define $P : H \longrightarrow H$ as $Pz = z_1$. Then we have $R(P) = \mathrm{span}\{x\}, N(P) = \mathrm{span}\{y\} \oplus \{x, y\}^{\perp}$, and

$$P^2 z = P z_1 = z_1 = P z, \ \forall z \in H,$$

which means $P^2 = P$, so P is a projector.

For $x \in R(P), y \in N(P)$, we have $(x, y) \neq 0$, which means P is a nonorthogonal projector.