Methods of Applied Mathematics

Homework 1
(Due on Friday, Sep 16)

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Exercises 1.4

1. (a) Proof: $\emptyset, X \in T$;
Let $\{U_i\}_{i\in I} \subset T$, then

$$\bigcup_{i\in I} U_i = \left\{ \begin{array}{ll} \emptyset & \text{If all } U_i = \emptyset; \\ X & \text{If there is a } U_i \text{ which is } X. \end{array} \right\} \in T$$

Let $\{V_j\}_{j=1}^n \subset T$, then

$$\bigcap_{j=1}^{n} V_j = \left\{ \begin{array}{ll} \emptyset & \text{If there is a } V_j \text{ which is } \emptyset; \\ X & \text{If all } V_j = X. \end{array} \right\} \in T$$

Therefore, T is a topology on X.

(b) Proof: $\forall x \in X$, $\exists \{x\} \in T_B$ such that $x \in \{x\}$;
If $x \in B_1 \bigcap B_2$, where $B_1, B_2 \in T_B$, then $B_1 = B_2 = \{x\}$. Therefore, there is $\{x\} \in T_B$ such that $x \in \{x\} \subset B_1 \bigcap B_2$. According to proposition 1.2

$$\begin{aligned} \tau &= \{U \subset X : U \text{ is a union of sets in } T_B\} \\ &= 2^X \qquad \text{(The collection of all subsets in } X\text{)} \end{aligned}$$

is a topology on X.

(c) Proof: If X is finite, then any subset in X has finite complements. So $\tau = 2^X$, which is discrete topology on X.

2. Proof: We only need to show that

$$X \text{ is not a Housdorff space.}$$

Since $\tau = \{\emptyset, X, \{a\}\}$, the unique neighborhood of $b \in X$ is X, so there are no disjoint neighborhoods of a and b. Therefore, X is not a Housdorff space.

4. Proof: If X contains only one point x, then it's obvious that $X = \{x\}$ is closed. Let X contains more than one point and $x_0 \in X, y \in \{x_0\}^c$ (That is $y \neq x_0$).

X is Housdorff

$\Rightarrow$ There are disjoint neighborhoods U_1 and U_2 such that $x_0 \in U_1, y \in U_2$.

$\Rightarrow y \in U_2 \subset \{x_0\}^c$.

$\Rightarrow \{x_0\}^c$ open $\Rightarrow \{x_0\}$ closed.

Then we prove: limits of sequences are unique.

If $\{x_n\} \subset X, x_n \to x, x_n \to y$. Suppose $x \neq y$.

X is Housdorff

$\Rightarrow$ There are disjoint neighborhoods U_1 and U_2 such that $x \in U_1, y \in U_2$.

$x_n \to x \Rightarrow \exists N_1, \forall n > N_1$, we have $x_n \in U_1$.

$x_n \to y \Rightarrow \exists N_2, \forall n > N_2$, we have $x_n \in U_2$.

So when $n > \max\{N_1, N_2\}, x_n \in U_1 \bigcap U_2 = \emptyset$, which is a contradiction!

6. Proof: " $\Rightarrow$ ": If $f : X \to Y$ is continuous, let $F \subset Y$ closed, then

$\Rightarrow (f^{-1}(F))^c = f^{-1}(F^c)$ open.

$\Rightarrow f^{-1}(F)$ is a closed set.

" $\Leftarrow$ " : Let $U \subset Y$ open, then $U^c \subset Y$ closed.

$\Rightarrow f^{-1}(U) = (f^{-1}(U^c))^c$ open.

$\Rightarrow f : X \to Y$ is continuous.

7. Proof: Let U be an open neighborhood of $f(x)$. Then

f is continuous $\Rightarrow x \in f^{-1}(U)$ open.

$x_n \to x \Rightarrow \exists N, \forall n > N$, we have $x_n \in f^{-1}(U)$.

$\Rightarrow f(x_n) \in U$ (when $n > N$)

$\Rightarrow f(x_n) \to f(x)$ $(n \to \infty)$.

9. Proof: Let (X, d) be a metric space, $x, y \in X, x \neq y$. Then the two open balls

$$B(x, \frac{d(x,y)}{2}) \text{ and } B(y, \frac{d(x,y)}{2})$$

are disjoint neighborhoods of x and y.

Therefore, (X, d) is Horsdorff.

11. Proof: The infinite open cover of $(0, 1]$ that has no finite subcover:

$$\mathcal{A} = \{I_n = (\frac{1}{n}, 2) : n \in \mathbb{N}\} \qquad\qquad (\mathbb{N} = \{1, 2, 3, \cdots\})$$

$\forall x \in (0, 1], \exists n \in \mathbb{N}$, such that $\frac{1}{n} < x$.
$\Rightarrow x \in (\frac{1}{n}, 2) = I_n \subset \bigcup_{n \in \mathbb{N}} I_n$.
$\Rightarrow (0, 1] \subset \bigcup_{n \in \mathbb{N}} I_n$.
So $\mathcal{A}$ is an open cover of $(0, 1]$.

Suppose $\mathcal{A}$ has a finite subcover of $(0, 1]$:

$$\mathcal{A}' = \{I_{n_k} \in \mathcal{A} : 1 \leq k \leq m\}$$

Let $n_0 = \max_{1 \leq k \leq m} n_k$, then we have

$$\frac{1}{2n_0} \in (0, 1], \text{ but } \frac{1}{2n_0} \bar{\in} \bigcup_{k=1}^{m} I_{n_k}.$$

So $\mathcal{A}$ doesn't have a finite subcover of $(0, 1]$.

The sequence in $(0, 1]$ that doesn't have a convergent subsequence:

$$\{\frac{1}{n}\}_{n=1}^{\infty}$$