Methods for Applied Mathematics

Homework 6
(Due: Oct 21, 2005)

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CAM program

Exercises 2.9

19. Proof: From **Exercises 2.9 Problem 18**, we know that all the norms $\|\cdot\|_p$ $(1 \le p \le \infty)$ on $X \times Y$ are equivalent, so we only need to show that $X \times Y$ is a Banach space in $\|\cdot\|_\infty$ when X, Y are both Banach spaces.

Let $\{(x_n, y_n)\} \subset X \times Y$ be a Cauchy sequence, when we have

$$\begin{aligned} \|x_n - x_m\|_X &\le \max\{\|x_n - x_m\|_X, \|y_n - y_m\|_Y\} \\ &= \|(x_n - x_m, y_n - y_m)\|_\infty \\ &= \|(x_n, y_n) - (x_m, y_m)\|_\infty \to 0,\ (n, m \to \infty) \end{aligned}$$

which means $\{x_n\}$ is a Cauchy sequence in X. Since X is a Banach space, so there exists $x \in X$ such that $x_n \to x$. In the same way, we know there exists $y \in Y$ such that $y_n \to y$. Therefore, we have

$$\|(x_n, y_n) - (x, y)\|_\infty = \max\{\|x_n - x\|_X, \|y_n - y\|_Y\} \to 0,\ (n \to \infty)$$

which means $(x_n, y_n) \to (x, y)$ in $X \times Y$. So $X \times Y$ is a Banach space.

26. Proof: Let $M^a = \{f \in X^* : f|_M = 0\}$. First, we show that M^a is a linear subspace of X^*.

For any $f, g \in M^a$, $\lambda \in \mathbb{C}$, we have

$$(f + g)|_M = f|_M + g|_M = 0 \text{ and } (\lambda f)|_M = \lambda(f|_M) = 0,$$

which means $f + g, \lambda f \in M^a$. So M^a is closed for addition and scalar multiplication in X^*, which means M^a is a linear subspace of X^*.

Then we will show that $M^a \subset X^*$ is closed.

Let $\{f_n\} \subset M^a$, $f_n \to f \in X^*$, then for any $x \in M$, we have

$$f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} 0 = 0,$$

which means $f|_M = 0$. So $f \in M^a$. So $M^a \subset X^*$ is closed.

And we have

$$X^a = \{f \in X^* : f|_X = 0\} = \{0\}$$

and

$$\{0\}^a = \{f \in X^* : f(0) = 0\} = X^*.$$

27. Proof: We let p be the conjugate exponent of q, which means $\dfrac{1}{p} + \dfrac{1}{q} = 1$ and $p = 1$ if $q = \infty$, $p = \infty$ if $q = 1$. Then we will show that $T \in B(L_p[a,b], L_q[a,b])$. For any $f \in L_p[a,b]$

(i). When $1 < q < \infty$, we have

$$\begin{aligned}
\|Tf\|_q^q &= \int_a^b |\int_a^b K(x,y)f(y)dy|^q dx \\
&\leq \int_a^b \int_a^b |K(x,y)|^q dy (\int_a^b |f(y)^p|dy)^{\frac{q}{p}} dx \\
&= \|K\|_q^q \|f\|_p^q
\end{aligned}$$

So

$$\|Tf\|_q \leq \|K\|_q \|f\|_p,$$

which means $\|T\| \leq \|K\|_q < \infty$. So $T \in B(L_p[a,b], L_q[a,b])$.

(ii). When $q = 1$, $p = \infty$, we have

$$\begin{aligned}
\|Tf\|_1 &= \int_a^b |\int_a^b K(x,y)f(y)dy|dx \\
&\leq \int_a^b \int_a^b |K(x,y)|dy \|f\|_\infty dx \\
&= \|K\|_1 \|f\|_\infty
\end{aligned}$$

So

$$\|Tf\|_1 \leq \|K\|_1 \|f\|_\infty,$$

which means $\|T\| \leq \|K\|_1 < \infty$. So $T \in B(L_\infty[a,b], L_1[a,b])$.

(ii). When $q=\infty$, $p=1$, we have for a.e. $x \in [a,b]$

$$\begin{aligned} |(Tf)(x)| &= |\int_a^b K(x,y)f(y)dy| \\ &\le \int_a^b |K(x,y)f(y)|dy \\ &\le \|K\|_\infty \int_a^b |f(y)|dy = \|K\|_\infty \|f\|_1 \end{aligned}$$

So

$$\|Tf\|_\infty \le \|K\|_\infty \|f\|_1,$$

which means $\|T\| \le \|K\|_\infty < \infty$. So $T \in B(L_1[a,b], L_\infty[a,b])$.

Therefore, $T \in B(L_p[a,b], L_q[a,b])$ when p be the conjugate exponent of q.

Then we will show that if a, b are finite, $K \in L_\infty([a,b] \times [a,b])$, then $T \in B(L_p[a,b], L_p[a,b])$ for all $1 \le p \le \infty$.

For any $f \in L_p[a,b]$, we have

(i). When $1 < p < \infty$

$$\begin{aligned} \|Tf\|_p^p &= \int_a^b |\int_a^b K(x,y)f(y)dy|^p dx \\ &\le \|K\|_\infty^p \int_a^b (\int_a^b |f(y)|dy)^p dx \\ &= \|K\|_\infty^p (b-a) \|f\|_1^p \\ &\le \|K\|_\infty^p (b-a)(\|1\|_q \|f\|_p)^p \\ &= (b-a)^p \|K\|_\infty^p \|f\|_p^p \end{aligned},$$

which means $\|Tf\|_p \le (b-a)\|K\|_\infty \|f\|_p$. So $\|T\| \le (b-a)\|K\|_\infty < \infty$. So $T \in B(L_p[a,b], L_p[a,b])$.

(ii). When $p = 1$

$$\begin{aligned} \|Tf\|_1 &= \int_a^b |\int_a^b K(x,y)f(y)dy| dx \\ &\le \|K\|_\infty \int_a^b (\int_a^b |f(y)|dy) dx \\ &= \|K\|_\infty (b-a) \|f\|_1 \end{aligned},$$

which means $\|Tf\|_1 \leq (b-a)\|K\|_\infty \|f\|_1$. So $\|T\| \leq (b-a)\|K\|_\infty < \infty$. So $T \in B(L_1[a,b], L_1[a,b])$.

(iii). When $p = \infty$, for a.e. $x \in [a,b]$

$$\begin{aligned} |(Tf)(x)| &= |\int_a^b K(x,y)f(y)dy| \\ &\leq \|K\|_\infty \|f\|_\infty \int_a^b 1dy \ , \\ &= \|K\|_\infty (b-a)\|f\|_\infty \end{aligned}$$

which means $\|Tf\|_\infty \leq (b-a)\|K\|_\infty \|f\|_\infty$. So $\|T\| \leq (b-a)\|K\|_\infty < \infty$. So $T \in B(L_\infty[a,b], L_\infty[a,b])$.

Therefore, $T \in B(L_p[a,b], L_p[a,b])$ for all $1 \leq p \leq \infty$.

28. Proof: Since $\overline{U} = \overline{B(0,r)}$ is a closed, convex and balance set, and $y \in X \backslash \overline{U}$, according to Lemma 2.17 (Mazur Separation Lemma 2), we know there exists $f \in X^*$ such that

$$|f(x)| \leq 1, \ \forall x \in \overline{U} \qquad \text{and} \qquad f(y) > 1.$$

Let $\alpha = \dfrac{1 + f(y)}{2}$, then we have

$$f(x) \leq |f(x)| \leq 1 < \alpha < f(y), \ \forall x \in U$$

which means f separates U from y.

30. Proof: Since X is reflexive, which means $X^{**} \cong X$, so we have

$$(X^*)^{**} = ((X^*)^*)^* = (X^{**})^* \cong X^*,$$

which means X^* is reflexive. (We use " $\cong$ " to represent isometry.)

The converse is also true.

Let X be a Banach space. If X^* is reflexive, then X is reflexive.

Proof: According to the embedding $[\cdot] : X \to X^{**}$, we know the image $[X] \cong X$ is a subspace of X^{**}, where $[X] = \{[x] \in X^{**} : x \in X\}$.

According to the denotation in **Exercises 2.9 Problem 26**, we define a subspace of $(X^{**})^*$:

$$[X]^a = \{F \in (X^{**})^* : F|_{[X]} = 0\}.$$

For any $F \in [X]^a \subset (X^{**})^*$, since X^* is reflexive, so $F \in (X^{**})^* = (X^*)^{**} \cong X^*$. Then there exists $f \in X^*$ such that $F = [f]$, which means

$$F([x]) = [f]([x]) = [x](f), \forall [x] \in X^{**}.$$

Since $F|_{[X]} = 0$, so for any $x \in X$, we have

$$f(x) = [x](f) = [f]([x]) = F([x]) = 0,$$

which means $f = 0 \in X^*$. So $F = [f] = [0] = 0 \in (X^{**})^*$. Since $F \in [X]^a$ is arbitrary, so $[X]^a = \{0\}$.

Suppose $[X] \neq X^{**}$, then $\exists\ \omega \in X^{**} \backslash [X]$. Since X is a Banach space, so $X \cong [X] \subset X^{**}$ is a complete subspace, which means $[X]$ is closed in X^{**}. So we have

$$d(\omega, [X]) = \inf_{[x] \in [X]} \|[x] - \omega\| = d > 0.$$

According to the corollary of Hahn-Banach Theorem, we know $\exists F_0 \in (X^{**})^*$ such that $F_0|_{[X]} = 0$, $F_0(\omega) = d > 0$. So $F_0 \in [X]^a$ and $F_0 \neq 0$, which contradicts $[X]^a = \{0\}$.

Therefore, $X \cong X^{**}$, which means X is reflexive.