Methods for Applied Mathematics

Homework 5
(Due: Oct 14, 2005)

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CAM program

Exercises 2.9

12. Proof: First, we will show the part of "if".

Let $B(x_i, \frac{1}{4}) \subset B(0,1)$, $\forall i \in I$ which is an infinite index set, and

$$B(x_i, \frac{1}{4}) \bigcap B(x_j, \frac{1}{4}) = \emptyset \text{ when } i, j \in I, i \neq j.$$

So we have $\|x_i - x_j\| \geq \frac{1}{2}$ when $i \neq j$. Then we choose a countably infinite subset $\{i_n\}_{n=1}^{\infty} \subset I$, and we get a sequence consisting of the centers of balls $\{x_{i_n}\}_{n=1}^{\infty}$, which has the following two properties:

(i). $\{x_{i_n}\}_{n=1}^{\infty}$ is bounded since $x_{i_n} \in B(0,1)$;

(ii). $\{x_{i_n}\}_{n=1}^{\infty}$ doesn't have convergent subsequences since $\|x_{i_n} - x_{i_m}\| \geq \frac{1}{2}$ when $n \neq m$.

So $\dim X = \infty$. Otherwise, $\{x_{i_n}\}_{n=1}^{\infty}$ will have a convergent subsequence since $B(0,1)$ is compact when $\dim X < \infty$ and X is also a metric space.

Then we will show the part of "only if".

If $\dim X = \infty$, to show that $B(0,1)$ contains an infinite collection of non-overlapping balls of diameter $\dfrac{1}{2}$, we need to use the Riesz's Lemma:

Let X be a NLS and Y a closed proper subspace of X. Then for any $0 < \varepsilon < 1$, there exists $x_0 \in X$ with $\|x_0\| = 1$ such that $\|x_0 - y\| \geq 1 - \varepsilon$ for every $y \in Y$.

(Please refer to the proof of Riesz's Lemma in Appendix 1.)

So for our problem, if $\dim X = \infty$, we choose an $x_1 \in \partial B(0,1)$. Then let Y_1 =span$\{x_1\}$ which is a closed proper subspace of X since $\dim X = \infty$, according to Riesz's Lemma, for $\varepsilon = 1/3$, there exists $x_2 \in X$, $\|x_2\| = 1$ such that

$$\|x_2 - y_1\| \geq 1 - \varepsilon = \frac{2}{3}, \ \forall y_1 \in Y_1.$$

Especially, when $y_1 = x_1 \in Y_1$, we have

$$\|x_2 - x_1\| \geq \frac{2}{3}.$$

In the same way, if we have $x_1, x_2, \cdots, x_k \in \partial B(0,1)$, then $Y_k =$span$\{x_1, x_2, \cdots, x_k\}$ which is a closed proper subspace of X since $\dim X = \infty$, according to Riesz's Lemma, for $\varepsilon = 1/3$, there exists $x_{k+1} \in X$, $\|x_{k+1}\| = 1$ such that

$$\|x_{k+1} - y_k\| \geq 1 - \varepsilon = \frac{2}{3}, \ \forall y_k \in Y_k.$$

Especially, when $y_k = x_i \in Y_k$ $(1 \leq i \leq k)$, we have

$$\|x_{k+1} - x_i\| \geq \frac{2}{3}, \ (1 \leq i \leq k).$$

Repeating this process, we get a sequence $\{x_i\}_{i=1}^{\infty}$ with the properties:

(i). $\|x_i\| = 1, \ \forall i \in \mathbb{N}$;

(ii). $\|x_i - x_j\| \geq \dfrac{2}{3}, \ (i \neq j)$.

So the ball $B(\frac{3}{4}x_i, \frac{1}{4}) \subset B(0,1)$ has diameter $\frac{1}{2}$, and for any $i \neq j$, we have

$$\text{distance between centers} = \|\frac{3}{4}x_i - \frac{3}{4}x_j\| \geq \frac{1}{2} = \text{length of diameter},$$

which means $B(\frac{3}{4}x_i, \frac{1}{4}) \bigcap B(\frac{3}{4}x_j, \frac{1}{4}) = \emptyset$. So we find an infinite collection of non-overlapping balls $\{B(\frac{3}{4}x_i, \frac{1}{4})\}_{i=1}^{\infty}$ contained by $B(0,1)$ with diameter $\frac{1}{2}$.

14. (a) Proof: First, we will show that $\|\cdot\|_p$ is a norm on l_p when $1 \leq p \leq \infty$.

When $1 \leq p < \infty$, $\|x\|_p = (\sum_{n=1}^{\infty} |x_n|^p)^{\frac{1}{p}}$ for any $x = \{x_n\} \in l_p$, then we will have

(i). $\|x\|_p \geq 0$ and $\|x\|_p = 0$ if and only if every $x_n = 0$, which means $x = 0 \in l_p$;

(ii). For any $\lambda \in \mathbb{C}$, we have

$$\|\lambda x\|_p = (\sum_{n=1}^{\infty} |\lambda x_n|^p)^{\frac{1}{p}} = |\lambda|(\sum_{n=1}^{\infty} |x_n|^p)^{\frac{1}{p}} = |\lambda|\|x\|_p;$$

(iii). For any $x = \{x_n\}, y = \{y_n\} \in l_p$, we can consider $x, y \in L_p(\Omega, \mu)$, where $\Omega = \mathbb{N}, \mu(k) = 1$ for any $k \in \mathbb{N}$. So according to Th 1.51(Minkowski's Inequality), we have

$$\|x + y\|_p \leq \|x\|_p + \|y\|_p.$$

Therefore, $\|\cdot\|_p$ is a norm on l_p when $1 \leq p < \infty$.

When $p = \infty$, $\|x\|_\infty = \sup_{n\in\mathbb{N}} |x_n|$ for any $x = \{x_n\} \in l_\infty$, then we will have

(i). $\|x\|_\infty \geq 0$ and $\|x\|_\infty = 0$ if and only if every $x_n = 0$, which means $x = 0 \in l_\infty$;

(ii). For any $\lambda \in \mathbb{C}$, we have

$$\|\lambda x\|_p = \sup_{n\in\mathbb{N}} |\lambda x_n| = |\lambda| \sup_{n\in\mathbb{N}} |x_n| = |\lambda|\|x\|_\infty;$$

(iii). For any $x = \{x_n\}, y = \{y_n\} \in l_\infty$,

$$\|x + y\|_\infty = \sup_{n\in\mathbb{N}} |x_n + y_n| \leq \sup_{n\in\mathbb{N}} (|x_n| + |y_n|) \leq \|x\|_\infty + \|y\|_\infty.$$

Therefore, $\|\cdot\|_\infty$ is a norm on l_∞.

Then we will show l_p is complete. Let $\{x^{(m)}\}_{m=1}^\infty$ is a Cauchy sequence in l_p, where $x^{(m)} = \{x_i^{(m)}\}_{i=1}^\infty$, then we have

$$|x_i^{(m)} - x_i^{(n)}| \leq \|x^{(m)} - x^{(n)}\|_p \to 0 \qquad \text{when } m, n \to \infty.$$

Since $\mathbb{C}$ is complete, so there is $x_i \in \mathbb{C}$ such that $x_i^{(m)} \to x_i$. And we know this convergence is uniform for $i \in \mathbb{N}$ according to the inequality above.

Let $x = \{x_i\}_{i=1}^\infty$, we will show $x \in l_p$.

When $1 \leq p < \infty$, according to uniform convergence, we have

$$\lim_{m,n\to\infty} \sum_{i=m}^{n} |x_i|^p = \lim_{m,n\to\infty} \sum_{i=m}^{n} |\lim_{k\to\infty} x_i^{(k)}|^p = \lim_{k\to\infty} (\lim_{m,n\to\infty} \sum_{i=m}^{n} |x_i^{(k)}|^p) = \lim_{k\to\infty} 0 = 0,$$

which means $\|x\|_p = (\sum_{n=1}^\infty |x_n|^p)^{\frac{1}{p}} < \infty$, so $x \in l_p$.

When $p = \infty$, since $\|x^{(m)} - x^{(n)}\|_\infty \to 0$ when $m, n \to \infty$, so there exists an $N \in \mathbb{N}$ such that $\|x^{(m)} - x^{(n)}\|_\infty < 1$ when $m, n > N$. So $\|x^{(m)}\|_\infty \leq 1 + \|x^{(N+1)}\|_\infty$ when $m > N$. Let

$$M = \max\{\|x^{(1)}\|_\infty, \|x^{(2)}\|_\infty, \cdots, \|x^{(N)}\|_\infty, 1 + \|x^{(N+1)}\|_\infty\},$$

we will have $\|x^{(m)}\|_\infty \leq M$ for every $m \in \mathbb{N}$.

On the other hand, since $x_i^{(m)} \to x_i$ and this convergence is uniform for $i \in \mathbb{N}$, we know there is an $N \in \mathbb{N}$ such that $|x_i^{(m)} - x_i| < 1$ when $m > N$ for every $i \in \mathbb{N}$. So

$$|x_i| < 1 + |x_i^{(m)}| \leq 1 + \|x^{(m)}\|_\infty \leq 1 + M,$$

which means $\|x\|_\infty = \sup_{i \in \mathbb{N}} |x_i| \leq M$, so $x \in l_\infty$.

Then we will show that $x^{(m)} \to x$ when $m \to \infty$. Since the convergence $x_i^{(m)} \to x_i$ is uniform, so we have

When $1 \leq p < \infty$,

$$\lim_{m\to\infty} \|x^{(m)} - x\|_p^p = \lim_{m\to\infty} \sum_{i=1}^{\infty} |x_i^{(m)} - x_i|^p = \sum_{i=1}^{\infty} \lim_{m\to\infty} |x_i^{(m)} - x_i|^p = \sum_{i=1}^{\infty} 0 = 0;$$

When $p = \infty$, for any $\varepsilon > 0, \exists N$, when $m > N$, $|x_i^{(m)} - x_i| < \varepsilon$ for any $i \in \mathbb{N}$, which means

$$\|x^{(m)} - x\|_\infty = \sup_{i \in \mathbb{N}} |x_i^{(m)} - x_i| \leq \varepsilon \qquad \text{when } m > N.$$

So $\lim_{m\to\infty} \|x^{(m)} - x\|_\infty = 0$.

Therefore, $x^{(m)} \to x \in l_p$, which means l_p is a Banach space.

(b) Proof: When $0 < p < 1$, we choose $x = (\frac{1}{2}, 0, 0, \cdots), y = (0, \frac{1}{2}, 0, \cdots) \in l_p$. Then we have

$$\|x + y\|_p = [(\frac{1}{2})^p + (\frac{1}{2})^p]^{\frac{1}{p}} = 2^{\frac{1}{p}-1};$$

and

$$\|x\|_p + \|y\|_p = \frac{1}{2} + \frac{1}{2} = 1.$$

Since $0 < p < 1$, so $2^{\frac{1}{p}-1} > 1$, which means $\|x + y\|_p > \|x\|_p + \|y\|_p$. But this contradicts the triangle inequality of norm, so $\|\cdot\|_p$ is not a norm on l_p when $0 < p < 1$.

18. Proof: Let

$$\|(x, y)\|_\infty = \max\{\|x\|_X, \|y\|_Y\} \text{ and } \|(x, y)\|_p = (\|x\|_X^p + \|y\|_Y^p)^{\frac{1}{p}}.$$

First, we will show that $\|\cdot\|_\infty$ is equivalent to $\|\cdot\|_p$ for any $1 \le p < \infty$. Since $\|x\|_X \le \|(x,y)\|_p$ and $\|y\|_Y \le \|(x,y)\|_p$, so

$$\|(x,y)\|_\infty = \max\{\|x\|_X, \|y\|_Y\} \le \|(x,y)\|_p.$$

On the other hand, we have

$$\|(x,y)\|_p = (\|x\|_X^p + \|y\|_Y^p)^{\frac{1}{p}} \le (2\|(x,y)\|_\infty^p)^{\frac{1}{p}} = 2^{\frac{1}{p}}\|(x,y)\|_\infty.$$

So

$$2^{-\frac{1}{p}}\|(x,y)\|_p \le \|(x,y)\|_\infty \le \|(x,y)\|_p,$$

which means $\|\cdot\|_\infty$ is equivalent to $\|\cdot\|_p$ for any $1 \le p < \infty$.

Then we will show that $\|\cdot\|_p$ is equivalent to $\|\cdot\|_q$ for any $1 \le p < q < \infty$. Since

$$\|(x,y)\|_p^p = \|x\|_X^p + \|y\|_Y^p \le (1+1)^{1-\frac{p}{q}}(\|x\|_X^q + \|y\|_Y^q)^{\frac{p}{q}} = 2^{1-\frac{p}{q}}\|(x,y)\|_q^p,$$

that is

$$\|(x,y)\|_p \le 2^{\frac{1}{p}-\frac{1}{q}}\|(x,y)\|_q.$$

On the other hand, we have

$$\|(x,y)\|_q^q = \|x\|_X^q + \|y\|_Y^q = \|x\|_X^{q-p}\|x\|_X^p + \|y\|_Y^{q-p}\|y\|_Y^p$$

$$\le \|(x,y)\|_\infty^{q-p}(\|x\|_X^p + \|y\|_Y^p) \le \|(x,y)\|_p^{q-p}\|(x,y)\|_p^p = \|(x,y)\|_p^q,$$

that is

$$\|(x,y)\|_q \le \|(x,y)\|_p.$$

So

$$\|(x,y)\|_q \le \|(x,y)\|_p \le 2^{\frac{1}{p}-\frac{1}{q}}\|(x,y)\|_q,$$

which means $\|\cdot\|_p$ is equivalent to $\|\cdot\|_q$ for any $1 \le p < q < \infty$.

Therefore, all these norms are equivalent.

20. Proof: First, we will show that $R(T) = \{y \in C^1[0,1] : y(0) = 0\}$. For any $x \in C[0,1]$, $(Tx)(t) = \int_0^t x(\tau)d\tau$, so we have

$$(Tx)'(t) = x(t) \in C[0,1] \text{ and } (Tx)(0) = 0,$$

which means $R(T) \subset \{y \in C^1[0,1] : y(0) = 0\}$.

On the other hand, for any $y \in C^1[0,1]$ and $y(0) = 0$, we have

$$y(t) = y(t) - y(0) = \int_0^t y'(\tau)d\tau = (Ty')(t),$$

which means $\{y \in C^1[0,1] : y(0) = 0\} \subset R(T)$.

Therefore, $R(T) = \{y \in C^1[0,1] : y(0) = 0\}$.

Then we will show that T is invertible on $R(T)$. Since $T : C[0,1] \to R(T)$ is a surjection, so we only need to show it is also an injection.

Let $x, y \in C[0,1]$ such that $Tx = Ty$, which means

$$\int_0^t x(\tau)d\tau = \int_0^t y(\tau)d\tau, \forall t \in [0,1].$$

If we differentiate the both sides of the equality above, we will have

$$x(t) = y(t), \forall t \in [0,1],$$

which means $T : C[0,1] \to R(T)$ is an injection. So T is invertible on $R(T)$.

Since $T^{-1} : R(T) \to C[0,1]$ is $(T^{-1}x)(t) = x'(t)$, so T^{-1} is linear because of the linearity of differentiation.

Then we will show that $T^{-1} : R(T) \to C[0,1]$ is not bounded.

Let $x_n(t) = \arctan(nt)$, where $n \in \mathbb{N}$, then $x_n \in C^1[0,1], x_n(0) = 0$, which means $x_n \in R(T)$. And we have $\|x_n\| \leq \frac{\pi}{2}$ and

$$\|T^{-1}x_n\| = \sup_{t \in [0,1]} |\frac{n}{1 + n^2t^2}| = n \to \infty \text{ when } n \to \infty,$$

which means T^{-1} is not bounded on $R(T)$.

22. According to the definition of f, for any $x \in C[-1,1]$ we have

$$|f(x)| \leq \int_{-1}^0 |x(t)|dt + \int_0^1 |x(t)|dt = \int_{-1}^1 |x(t)|dt \leq \|x\| \int_{-1}^1 dt = 2\|x\|,$$

which means $\|f\| \leq 2$.

On the other hand, we let $\{x_n\} \subset C[-1,1]$, $(n \in \mathbb{N})$ where

$$x_n(t) = \left\{ \begin{array}{cc} 1, & t \in [-1, -\frac{1}{n}]; \\ \\ -nt, & t \in [-\frac{1}{n}, \frac{1}{n}]; \\ \\ -1, & t \in [\frac{1}{n}, 1]. \end{array} \right\}$$

Then $\|x_n\| = 1$, and we have

$$\begin{aligned} |f(x_n)| &= \int_{-1}^{-\frac{1}{n}} dt - n\int_{-\frac{1}{n}}^{0} tdt + \int_{\frac{1}{n}}^{1} dt + n\int_{0}^{\frac{1}{n}} tdt \\ &= (1 - \frac{1}{n}) + \frac{1}{2n} + (1 - \frac{1}{n}) + \frac{1}{2n} \\ &= 2 - \frac{1}{n} \to 2 \qquad (n \to \infty) \end{aligned}$$

Then we have

$$\|f\| = \sup_{\|x\|=1} |f(x)| \geq 2.$$

Therefore, $\|f\| = 2$.

24.(a) Proof: According to the definition of $\|\cdot\|$, we know
(i) For any $x \in C^1[a,b]$, we have

$$\|x\| \geq 0 \text{ and } \|x\| = 0 \text{ iff } x(t) = x'(t) = 0, \forall t \in [a,b],$$

which means $x = 0 \in C^1[a,b]$.
(ii) For any $x \in C^1[a,b], \lambda \in \mathbb{R}$, we have

$$\|\lambda x\| = \sup_{t\in[a,b]} |\lambda x(t)| + \sup_{t\in[a,b]} |\lambda x'(t)| = |\lambda|(\sup_{t\in[a,b]} |x(t)| + \sup_{t\in[a,b]} |x'(t)|) = |\lambda|\|x\|.$$

(iii) For any $x, y \in C^1[a,b]$, we have

$$\|x+y\| = \sup_{t\in[a,b]} |x(t)+y(t)| + \sup_{t\in[a,b]} |x'(t)+y'(t)|$$

$$\leq \sup_{t\in[a,b]} (|x(t)| + |y(t)|) + \sup_{t\in[a,b]} (|x'(t)| + |y'(t)|) \leq \|x\| + \|y\|.$$

Therefore, $\|\cdot\|$ is indeed a norm.

(b) Proof: According to the linearity of differentiation, we know the function $f(x) = x'(\frac{a+b}{2})$ is linear. Then we only need to show it is continuous.

Let $x, x_n \in C^1[a,b]$, if $x_n \to x$, then we have

$$|f(x_n)-f(x)| = |x_n'(\frac{a+b}{2})-x'(\frac{a+b}{2})| \le \sup_{t\in[a,b]} |x_n'(t)-x'(t)| \le \|x_n-x\| \to 0 \ (n\to\infty),$$

which means f is continuous. Therefore, f defines a continuous linear functional on $C^1[a,b]$.

(c) Proof: Let $y_n(t) = \arctan[n(x-\frac{a+b}{2})]$, then $y_n \in C^1[a,b]$, and $\|y_n\|_C \le \frac{\pi}{2}$. However,

$$|f(y_n)| = |y_n'(\frac{a+b}{2})| = n \to \infty \ (n\to\infty),$$

which means f is not bounded on the subspace of $C[a,b]$ consisting of all functions in $C^1[a,b]$ with the norm inherited from $C[a,b]$.

25. Proof: Let $\bar{X}$ represent its algebraic dual, which is the set of all linear functions on X. Then we will show that

$$\dim X < \infty \text{ iff } \bar{X} = X^*.$$

If $\dim X < \infty$, let $\dim X = n$. And we take a unit basis of X: $\{e_i\}_{i=1}^n$. For any $x \in X$, let

$$x = \sum_{i=1}^n x_i e_i.$$

Since any two norms on finitely dimensional space are equivalent (which had been proved in last homework), we only need to show that $\forall f \in \bar{X}$ is bounded in the ∞-norm, which is

$$\|x\|_\infty = \max_{1\le i\le n} |x_i|.$$

For any $f \in \bar{X}$, let $c_i = f(e_i)$ $(1 \le i \le n)$. For any $x \in X$, we have

$$|f(x)| = |f(\sum_{i=1}^n x_i e_i)| = |\sum_{i=1}^n x_i f(e_i)| \le \sum_{i=1}^n |x_i||c_i| \le \sum_{i=1}^n |c_i| \|x\|_\infty,$$

which means f is bounded in ∞-norm. So $f \in X^*$. Since $f \in \bar{X}$ is arbitrary, so $\bar{X} \subset X^*$. But it's always true that $X^* \subset \bar{X}$, so $X^* = \bar{X}$.

If $X^* = \bar{X}$, we will show that $\dim X < \infty$.

Suppose not, that is if $\dim X = \infty$, we will find $f \in \bar{X}$, but $f \bar{\in} X^*$. To prove this, we need to use the following theorem:

Every vector space has a Hamel Basis. That is let X be a vector space over any field $\mathbb{F}$, then there is a linearly independent subset $\{e_i\}_{i \in I}$ of X, for any $x \in X$, there exists unique finite index $i_1, i_2, \cdots, i_n \in I$, and elements $x_{i_1}, x_{i_2}, \cdots, x_{i_n} \in \mathbb{F}$ such that

$$x = \sum_{k=1}^{n} x_{i_k} e_{i_k}.$$

(Please refer to the proof of this theorem in Appendix 2.)

So for our problem, let $\{e_i\}_{i \in I}$ be a Hamel Basis of X. Without losing generality, we let $\|e_i\| = 1, \ \forall i \in I$. Since we suppose $\dim X = \infty$, so I is an infinite index set. We choose $\{\alpha_k\}_{k=1}^{\infty} \subset I$ to be a countably infinite subset.

Then we define $f \in \bar{X}$ such that

$$f(e_i) = \left\{ \begin{array}{ll} k, & i = \alpha_k; \\ 0, & i \in I \setminus \{\alpha_k\}_{k=1}^{\infty}. \end{array} \right\}$$

Then f is well defined on X, that is for any $x \in X$, which can be uniquely represented as $x = \sum_{i \in I} x_i e_i$ (actually there are only finite terms in the summation), so according to the linearity of $f \in \bar{X}$, we have

$$f(x) = \sum_{k=1}^{\infty} k x_{\alpha_k}.$$

Since $\|e_{\alpha_k}\| = 1$, then $\{e_{\alpha_k}\}_{k=1}^{\infty} \subset X$ is a bounded set, but $\{f(e_{\alpha_k})\}_{k=1}^{\infty} = \{k\}_{k=1}^{\infty} = \mathbb{N}$ is not bounded. So $f \bar{\in} X^*$. Therefore, $X^* \subsetneqq \bar{X}$, which contradicts $X^* = \bar{X}$.

Therefore, $\dim X < \infty$ when $X^* = \bar{X}$.

Appendix

1 Riesz's Lemma:

Let X be a NLS and Y a closed proper subspace of X. Then for any $0 < \varepsilon < 1$, there exists $x_0 \in X$ with $\|x_0\| = 1$ such that $\|x_0 - y\| \geq 1 - \varepsilon$ for every $y \in Y$.

Proof: We choose a $v_0 \in X \backslash Y$. Since Y is closed, so $d = \inf_{y \in Y} \|v_0 - y\| > 0$. $\forall 0 < \varepsilon < 1, \exists y_\varepsilon \in Y$, such that

$$d \leq \|v_0 - y_\varepsilon\| \leq d + \frac{d\varepsilon}{1-\varepsilon} = \frac{d}{1-\varepsilon}.$$

Let $x_0 = \dfrac{v_0 - y_\varepsilon}{\|v_0 - y_\varepsilon\|}$, then $\|x_0\| = 1$ and for every $y \in Y$, we have

$$\|x_0 - y\| = \frac{\|v_0 - y'\|}{\|v_0 - y_\varepsilon\|} \geq \frac{d}{\frac{d}{1-\varepsilon}} = 1 - \varepsilon,$$

where $y' = y_\varepsilon + \|v_0 - y_\varepsilon\| y \in Y$.

2 Hamel Basis

Theorem: Every vector space has a Hamel Basis.

That is let X be a vector space over any field $\mathbb{F}$, then there exists a linearly independent subset $\{e_i\}_{i \in I}$ of X, such that for any $x \in X$, there exists unique finite index $i_1, i_2, \cdots, i_n \in I$, and elements $x_{i_1}, x_{i_2}, \cdots, x_{i_n} \in \mathbb{F}$ such that

$$x = \sum_{k=1}^{n} x_{i_k} e_{i_k}.$$

Proof: We will use Zorn's Lemma to prove it. Let $e_1 \in X$ be a nonzero vector, and denote $L = \{e_1\}$. Then we let

$$S = \{A \subseteq X : \text{the vectors in } A \text{ are linearly independent and } L \subseteq A.\}$$

Obviously, $L \in S$, so $S \neq \emptyset$, and S is partially ordered by inclusion relation of sets, which means $\forall A, B \in S$, we define $A \preceq B$ iff $A \subseteq B$.

Then for each chain $C \subseteq S$, let $\tilde{C} = \bigcup_{A \in C} A$, then $L \subseteq \tilde{C}$. Next we will show $\tilde{C} \in S$, which means we need to show any finite collection of vectors $V = \{v_1, v_2, \cdots, v_n\} \subseteq \tilde{C}$ is linearly independent.

Since $v_i \in \tilde{C} = \bigcup_{A \in C} A$, then there exist sets $A_i \in C$ such that $v_i \in A_i$ for all $1 \leq i \leq n$. Since $\tilde{C}$ is a chain, there is a k with $1 \leq k \leq n$ such that $A_k = \bigcup_{i=1}^{n} A_i$ and thus $V \subseteq \bigcup_{i=1}^{n} A_i = A_k$, which means V is linearly independent. Therefore, $\tilde{C} \in S$, which is an upper bound of C.

According to Zorn's Lemma, S has a maximal element $M \in S$, which we will show is a Hamel Basis of X. Let span$M$ be all finitely linear combination of vectors in M over $\mathbb{F}$, we need to show span$M = X$.

Suppose not, that is if $\exists x_0 \in X \backslash \text{span} M$, let $\{x_1, x_2, \cdots, x_n\} \subset M$ be any finite collection of vectors. If there is $a_0, a_1, \cdots, a_n \in \mathbb{F}$ such that

$$a_0 x_0 + a_1 x_1 + \cdots + a_n x_n = 0.$$

If $a \neq 0$, we will have

$$x_0 = -\frac{a_1}{a_0} x_1 - \frac{a_2}{a_0} x_2 - \cdots - \frac{a_n}{a_0} x_n \in \ \text{span} M,$$

which contradicts $x_0 \in X \backslash \text{span} M$. So $a = 0$, that is

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = 0.$$

Since $x_1, x_2, \cdots, x_n \in M$, and $M \in S$ whose vectors are linearly independent, so $a_1 = a_2 = \cdots = a_n = 0$, which means any finite collection $\{x_0, x_1, \cdots, x_n\} \subset M \bigcup \{x_0\}$ is linearly independent, and $L \subseteq M \subset M \bigcup \{x_0\}$. So $M \bigcup \{x_0\} \in S$ and obviously $M \bigcup \{x_0\} \succeq M, M \bigcup \{x_0\} \neq M$, which contradicts M is a maximal element of S.

Therefore, span$M = X$, and thus M is a Hamel Basis of X.