Methods for Applied Mathematics

Homework 4
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1.(a) Proof: First of all, we will show that $A+B$ is convex.

For any $x = x_1 + x_2, y = y_1 + y_2 \in A + B$, where $x_1, y_1 \in A, x_2, y_2 \in B$. For any $t \in [0,1]$, according to the convexity of A and B, we have

$$tx_1 + (1-t)y_1 \in A, \qquad tx_2 + (1-t)y_2 \in B.$$

So

$$tx + (1-t)y = tx_1 + (1-t)y_1 + tx_2 + (1-t)y_2 \in A + B,$$

which means $A+B$ is convex.

Then we will show $A \bigcap B$ is convexity.

For any $x, y \in A \bigcap B, t \in [0,1]$, according to the convexity of A and B, we have

$$tx + (1-t)y \in A, \qquad tx + (1-t)y \in B,$$

so $tx + (1-t)y \in A \bigcap B$, which means $A \bigcap B$ is convex.

$A \bigcup B$ is not necessarily convex.

Example: Let $X = \mathbb{R}$ and $A = (0,1), B = (2,3)$, then $A \bigcup B = (0,1) \bigcup (2,3)$ which is not a connected set, so it is not convex.

$A \setminus B$ is not necessarily convex.

Example: Let $X = \mathbb{R}$ and $A = (-2,2), B = (-1,1)$ then $A \backslash B = (-2,-1] \bigcup [1,2)$ which is not a connected set, so it is not convex.

(b) Proof: For any $2x \in 2A$, where $x \in A$, we have $2x = x + x \in A + A$. So $2A \subset A + A$.

$2A = A + A$ is not true.

Example: Let $X = \mathbb{R}$, and $A = \mathbb{Z}$ which represents the set of all integers. Then, $2A = 2\mathbb{Z}$ is the set of all even numbers. But $\mathbb{Z} + \mathbb{Z} = \mathbb{Z}$.

4. Proof: Consider the mapping $f : X \to B_r(0)$ with

$$f(x) = \frac{rx}{1 + \|x\|}, \qquad \forall x \in X.$$

Then we will show that f is one to one and onto $B_r(0)$.

If there is $x, y \in X$ such that $f(x) = f(y)$, which means

$$\frac{rx}{1 + \|x\|} = \frac{ry}{1 + \|y\|},$$

then the norms of the two sides are equal to each other, which is

$$\frac{r\|x\|}{1 + \|x\|} = \frac{r\|y\|}{1 + \|y\|},$$

so $\|x\| = \|y\|$, which implies $x = y$ according to $f(x) = f(y)$. Therefore, f is an injection.

For any point $y \in B_r(0)$, since $\|y\| < r$, let $x = \dfrac{y}{r - \|y\|}$, then we will have $f(x) = y$. Therefore, f is a surjection.

So the inverse of f, which is denoted as f^{-1}, exists. That is

$$f^{-1}(y) = \frac{y}{r - \|y\|}, \qquad \forall y \in B_r(0).$$

And it is obvious that both of f and f^{-1} are continuous, which means X is homeomorphic to $B_r(0)$.

5. Proof: To prove any two norms on $\mathbb{R}^d$ are equivalent, we only need to prove that any norm $\|\cdot\|$ on $\mathbb{R}^d$ is equivalent with the 2-norm $\|\cdot\|_2$ on $\mathbb{R}^d$.

$\forall x = (x_1, x_2, \cdots, x_d) \in \mathbb{R}^d$, we have

$$\|x\| = \|\sum_{i=1}^{d} x_i e_i\| \leq \sum_{i=1}^{d} |x_i|\|e_i\| \leq \|x\|_2 \sqrt{\sum_{i=1}^{d} \|e_i\|^2} = M\|x\|_2,$$

where $M = \sqrt{\sum_{i=1}^{d} \|e_i\|^2}$ and e_i represents the unit vector whose ith component is 1.

On the other hand, let $f(x) = \|x\|$, $\forall x = (x_1, x_2, \cdots, x_d)$ and $x^0 = (x_1^0, x_2^0, \cdots, x_d^0) \in \mathbb{R}^d$, we have

$$|f(x) - f(x^0)| = |\|x\| - \|x^0\|| \leq \|x - x^0\| \leq M\|x - x^0\|_2 \to 0 \ (x \to x^0),$$

which means f is continuous on $\mathbb{R}^d$ in the sense of 2-norm. Since the unit sphere $\partial B(0,1) \subset \mathbb{R}^d$ is compact (under 2-norm), so there is an $x_0 \in \partial B(0,1)$ which satisfies

$$f(x_0) = \inf_{\|x\|_2=1} f(x),$$

that is $\forall x \in \partial B(0,1)$, we have

$$\|x\| \geq \|x_0\| > 0 \text{ (since } x_0 \neq 0)$$

Let $\|x_0\| = m$, for any $x \in \mathbb{R}^d, x \neq 0$, we have

$$\|\frac{x}{\|x\|_2}\| \geq m,$$

that is

$$\|x\| \geq m\|x\|_2.$$

And the above inequality is also true for $x = 0$, so

$$\|x\| \geq m\|x\|_2, \qquad \forall x \in \mathbb{R}^d.$$

In the end, we have there is $M, m > 0$ such that

$$m\|x\|_2 \leq \|x\| \leq M\|x\|_2, \qquad \forall x \in \mathbb{R}^d,$$

which means the norm $\|\cdot\|$ and 2-norm $\|\cdot\|_2$ are equivalent. So any two norms on $\mathbb{R}^d$ are equivalent.

6. Proof: Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be NLS over the same field F. Let $\{e_1, e_2, \cdots, e_n\}$ and $\{\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n\}$ be a base of X and Y separately, then we define a linear mapping $T : X \to Y$ such that $Te_i = \varepsilon_i$ for $1 \leq i \leq n$, which is

$$Tx = \sum_{i=1}^n c_i\varepsilon_i, \qquad \forall x = \sum_{i=1}^n c_ie_i \in X, c_i \in F.$$

Then T is continuous on X since X and Y have finite dimension, and obviously, T is one to one onto Y.

On the other hand, we have

$$T^{-1}(y) = \sum_{i=1}^{n} c_i e_i, \qquad \forall y = \sum_{i=1}^{n} c_i \varepsilon_i \in X, c_i \in F.$$

So T^{-1} is also continuous on Y.

Therefore, X and Y are topologically isometric.

7. Proof: Actually, $C[a,b]$ is a vector space with the algebraic structure of pointwise addition and scalar multiplication. And we have $\forall f, g \in C[a,b]$, $\lambda \in \mathbb{R}$

$$\|f\| = \sup_{x\in[a,b]} |f(x)| \geq 0, \text{ and } \|f\| = 0 \text{ iff } f(x) \equiv 0;$$

$$\|\lambda f\| = \sup_{x\in[a,b]} |\lambda f(x)| = \lambda \sup_{x\in[a,b]} |f(x)| = |\lambda| \|f\|;$$

$$\|f+g\| = \sup_{x\in[a,b]} (|f(x)+g(x)|) \leq \sup_{x\in[a,b]} |f(x)| + \sup_{x\in[a,b]} |g(x)| = \|f\| + \|g\|.$$

So $\|\cdot\|$ is a norm on $C[a,b]$.

Then we will show $(C[a,b], \|\cdot\|)$ is complete. Let $\{f_n\}$ is a Cauchy sequence in $C[a,b]$, then we have

$$\sup_{x\in[a,b]} (|f_n(x) - f_m(x)|) = \|f_n - f_m\| \to 0, \ (n, m \to \infty)$$

which means $\{f_n\}$ is a uniform Cauchy function sequence on $[a,b]$, according to the property of uniform Cauchy function sequence in Mathematical Analysis, we know there is an $f \in C[a,b]$ such that

$$f_n(x) \rightrightarrows f(x), \ (n \to \infty, x \in [a,b]),$$

where " $\rightrightarrows$ " means uniform convergence. Therefore,

$$\|f_n - f\| = \sup_{x\in[a,b]} (|f_n(x) - f(x)|) \to 0, \ (n \to \infty).$$

So $\{f_n\}$ is a convergent sequence in $C[a,b]$, which means $(C[a,b], \|\cdot\|)$ is a Banach space.

8. Proof: According to Hölder's inequality, we have

$$\sup_g \int_\Omega |fg| dx = \sup_g \|fg\|_1 \leq \sup_g \|f\|_p \|g\|_q \leq \|f\|_p;$$

On the other hand, when $\|f\|_p = 0$, which means $f = 0$ a.e., we let g be arbitrary function in $\mathcal{L}_q(\Omega)$ which satisfies $\|g\|_p \leq 1$. Then we have

$$\|f\|_p = \sup_g |\int_\Omega fg dx| = 0;$$

When $\|f\|_p > 0$, we let $g = \mathrm{sign}(f)\frac{|f|^{p-1}}{\|f\|_p^{p-1}}$, then

$$\|g\|_q = \|\frac{|f|^{p-1}}{\|f\|_p^{p-1}}\|_q = \frac{1}{\|f\|_p^{p-1}}\||f|^{p-1}\|_q = \frac{1}{\|f\|_p^{p-1}}\|f\|_p^{\frac{p}{q}} = 1,$$

so $g \in \mathcal{L}_q(\Omega)$, and we have

$$\int_\Omega fg dx = \int_\Omega |f|\frac{|f|^{p-1}}{\|f\|_p^{p-1}} dx = \frac{\|f\|_p^p}{\|f\|_p^{p-1}} = \|f\|_p,$$

so for any $f \in \mathcal{L}_p(\Omega)$, we have

$$\sup_g |\int_\Omega fg dx| \geq \|f\|_p.$$

Therefore, we have

$$\|f\|_p \leq \sup_g |\int_\Omega fg dx| \leq \sup_g \int_\Omega |fg| dx \leq \|f\|_p,$$

which means

$$\|f\|_p = \sup_g |\int_\Omega fg dx| = \sup_g \int_\Omega |fg| dx.$$

10. Proof: Since we have

$$\sup_{\|x\|=1} \|Tx\| \leq \sup_{\|x\|\leq 1} \|Tx\| \leq \sup_{0<\|x\|\leq 1} \frac{\|Tx\|}{\|x\|} \leq \sup_{x\neq 0} \frac{\|Tx\|}{\|x\|} = \sup_{\|x\|\neq 0} \|T(\frac{x}{\|x\|})\| = \sup_{\|x\|=1} \|Tx\|,$$

so

$$\sup_{x\neq 0} \frac{\|Tx\|}{\|x\|} = \sup_{\|x\|=1} \|Tx\| = \sup_{\|x\|\leq 1} \|Tx\|.$$

That is

$$\|T\| = \sup_{\|x\|=1} \|Tx\| = \sup_{\|x\|\leq 1} \|Tx\|.$$

Let the set $A = \{M : \|Tx\| \leq M\|x\| \text{ for all } x \in X\}$. According to the definition, we have

$$\|Tx\| \leq \|T\|\|x\| \qquad \forall x \in X.$$

So $\|T\| \in A$, and so $\|T\| \geq \inf A$.

On the other hand, for any $M \in A$, we have

$$\|T\| = \sup_{\|x\|=1} \|Tx\| \leq \sup_{\|x\|=1} M\|x\| = M,$$

so $\|T\| \leq \inf A$. Therefore, $\|T\| = \inf A$.

In the end, we have

$$\|T\| = \sup_{\|x\|=1} \|Tx\| = \sup_{\|x\|\leq 1} \|Tx\| = \inf A,$$

where $A = \{M : \|Tx\| \leq M\|x\| \text{ for all } x \in X\}$.

11.(a) Proof: Let $A \subset X$, we will show co(A) is convex.

For any $x = \sum_{i=1}^n t_i x_i$, $y = \sum_{j=1}^m s_j y_j \in$co$(A)$, where $x_i, y_j \in A, t_i, s_j \in [0,1], \sum_i^n t_i = \sum_{j=1}^m s_j = 1$, and for any $\lambda \in [0,1]$, we have

$$\lambda x + (1-\lambda)y = \sum_{i=1}^n \lambda t_i x_i + \sum_{j=1}^m (1-\lambda) s_j y_j,$$

where $\lambda t_i, (1-\lambda)s_j \in [0,1]$, and

$$\sum_{i=1}^n \lambda t_i + \sum_{j=1}^m (1-\lambda) s_j = \lambda \sum_i^n t_i + (1-\lambda) \sum_{j=1}^m s_j = \lambda + (1-\lambda) = 1.$$

Therefore, $\lambda x + (1-\lambda)y \in$co$(A)$, which means co$(A)$ is convex.

Then we will show that

$$\text{co}(A) = \bigcap_{B \in \mathcal{B}} B,$$

where $\mathcal{B} = \{B \subset X : B \text{ is convex and } A \subset B.\}$

First of all, according to the definition of convex hull, we know $A \subset$co(A), and we have proved that co(A) is convex, so co$(A) \in \mathcal{B}$. Therefore,

$$\bigcap_{B\in\mathcal{B}} B \subset \text{co}(A).$$

On the other hand, $\forall B \in \mathcal{B}$, we will show that co$(A) \subset B$.

$\forall x = \sum_{i=1}^{n} t_i x_i \in$co$(A)$, where $x_i \in A, t_i \in [0,1], \sum_{i}^{n} t_i = 1$. Since $B \in \mathcal{B}$, so $x_i \in A \subset B$. Then we will show that $x \in B$ by mathematical induction for $n \in \mathbb{N}$.

When $n = 1$, $x = x_1 \in A \subset B$;

When $n = 2$, $x = t_1 x_1 + t_2 x_2$, where $t_2 = 1 - t_1$. Since B is convex, so $x \in B$ according to the definition of convexity;

Suppose "$x = \sum_{i=1}^{n} t_i x_i \in B$" is true for $n = k$;

When $n = k+1$, if $t_{k+1} = 1$, then $x = x_{k+1} \in B$. Otherwise, consider $y = \sum_{i=1}^{k} \dfrac{t_i}{1 - t_{k+1}} x_i$. According to the inductive hypothesis, we know $y \in B$. According to the convexity of B, then we have

$$x = \sum_{i=1}^{k+1} t_i x_i = (1 - t_{k+1}) y + t_{k+1} x_{k+1} \in B.$$

So $x \in B$ for any $n \in \mathbb{N}$, which means co$(A) \subset B$. Since $B \in \mathcal{B}$ is arbitrary, we have

$$\text{co}(A) \subset \bigcap_{B\in\mathcal{B}} B.$$

In the end, we have

$$\text{co}(A) = \bigcap_{B\in\mathcal{B}} B,$$

where $\mathcal{B} = \{B \subset X : B \text{ is convex and } A \subset B.\}$.

(b) Proof: Let $A \subset X$ be an open set, we will show that co(A) is also an open set.

$\forall x = \sum_{i=1}^{n} t_i x_i \in$co$(A)$, where $x_i \in A, t_i \in [0,1], \sum_{i=1}^{n} t_i = 1$ (without losing the generality, we can assume all $t_i > 0$, or just remove the terms of zero), since A is open, $\exists r > 0$ such that $B(x_i, r) \subset A$, $1 \leq i \leq n$.

When $y \in B(x, t_n r)$, it can be expressed as

$$y = t_1 x_1 + t_2 x_2 + \cdots + t_{n-1} x_{n-1} + t_n \tilde{x}_n,$$

where $\tilde{x}_n = \dfrac{1}{t_n}(y - \sum_{i=1}^{n-1} t_i x_i)$, and we have

$$\|\tilde{x}_n - x_n\| = \frac{1}{t_n}\|y - x\| < \frac{t_n r}{t_n} = r,$$

which means $\tilde{x}_n \in B(x_n, r) \subset A$. Therefore, according to the definition of convex hull, we know $y \in$co(A). So $B(x, t_n r) \subset$co(A) is a neighborhood of x. Since $x \in$co(A) is arbitrary, so co(A) is an open set.

(c) The convex hull of a closed set is not necessarily closed.

Example: Let $X = \mathbb{R}^2$, and $A = \{(x,0) : x \in \mathbb{R}\} \bigcup \{(0,1)\}$, then $A \subset \mathbb{R}^2$ is a closed set and its convex hull is

$$\mathrm{co}(A) = \{(x,0) : x \in \mathbb{R}\} \bigcup \{(tx, 1-t) : x \in \mathbb{R}, t \in [0,1]\}.$$

Then we will show that co(A) is not a closed set.

For any $n \in \mathbb{N}$, let $x_n = n$, $t_n = \dfrac{1}{n}$, we will have

$$(t_n x_n, 1 - t_n) = (1, 1 - \frac{1}{n}) \in \mathrm{co}(A).$$

However, $(1, 1 - \dfrac{1}{n}) \to (1,1)$ which is not in co(A). That means co(A) is not a closed set.

(d) Proof: Let $A \subset X$ be a bounded set, which means $\exists M > 0$, such that $\forall x \in A$, $\|x\| \leq M$, then we will show that co(A) is also a bounded set.

$\forall y = \sum_{i=1}^n t_i x_i \in$co$(A)$, where $x_i \in A, t_i \in [0,1], \sum_{i=1}^n t_i = 1$, we have

$$\|y\| \leq \sum_{i=1}^n t_i \|x_i\| \leq M \sum_{i=1}^n t_i = M,$$

which means co(A) is a bounded set.