Methods for Applied Mathematics

Homework 7
(Due: Oct 28, 2005)

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CAM program

Exercises 2.9

29. Proof: To prove it, we will use the following two results:

1. Let $1 \le p < \infty$ and q be its conjugate exponent ($q = \infty$ if $p = 1$), then

$$L_p[0,1]^* = L_q[0,1]$$

in the sense that for any $F \in L_p[0,1]^*$, there exists unique $g \in L_q[0,1]$ such that

$$F(f) = \int_0^1 f(x)g(x)dx, \ \forall f \in L_p[0,1].$$

(This is a remark in class.)

2. Let X be a NLS, $x_n, x \in X$ and $x_n \rightharpoonup x$ weakly, then

$$\|x\| \le \liminf_{n\to\infty} \|x_n\|.$$

(Please refer to Appendix 1, 2 to see the proofs.)

Since for any $f \in L_2[0,1]$, according to Hölder's inequality, we have

$$\|f\|_1 \le \|f\|_2 \|1\|_2 = \|f\|_2 < \infty,$$

which means $f \in L_1[0,1]$. So $L_2[0,1] \subset L_1[0,1]$ is a linear subspace.

Let $A_k = \{f \in L_2[0,1] : \|f\|_2 \le k\}$ for all $k \in \mathbb{N}$, then we will have

$$L_2[0,1] = \bigcup_{k=1}^{\infty} A_k.$$

And we will show that: (i) A_k is closed in $L_1[0,1]$; (ii) $A_k^\circ = \emptyset$.

(i). Let $\{f_n\} \subset A_k$ and $f_n \to f \in L_1[0,1]$ in the sense of $L_1[0,1]$, which means $\|f_n - f\|_1 \to 0$. So $\{f_n\}$ is bounded. Since $\{f_n\} \subset A_k \subset L_2[0,1]$

which is a separable and reflexive Banach space (since $L_2[0,1]^{**} = L_2[0,1]^* = L_2[0,1]$), according to **Corollary 2.34**, there is some subsequence $\{f_{n_k}\} \subset \{f_n\}$ and $g \in L_2[0,1]$ such that $f_{n_k} \rightharpoonup g$ weakly in $L_2[0,1]$, which means for any $h \in L_2[0,1]^* \cong L_2[0,1]$, we have

$$\int_0^1 h(x) f_{n_k}(x) dx \longrightarrow \int_0^1 h(x) f(x) dx.$$

Since $L_1[0,1]^* \cong L_\infty[0,1] \subset L_2[0,1]$, so we have

$$\int_0^1 h(x) f_{n_k}(x) dx \longrightarrow \int_0^1 h(x) f(x) dx \qquad \forall h \in L_1[0,1]^*,$$

which means $f_{n_k} \rightharpoonup g$ weakly in $L_1[0,1]$. Since $\|f_n - f\|_1 \to 0$, so we also have $f_{n_k} \rightharpoonup f$ weakly in $L_1[0,1]$. According to the uniqueness of weak limit, we have $f = g$. And since $f_{n_k} \rightharpoonup g$ weakly in $L_2[0,1]$, so we have

$$\|f\|_2 = \|g\|_2 \le \liminf_{k\to\infty} \|f_{n_k}\|_2 \le k,$$

so $f \in A_k$ which means A_k is closed in $L_1[0,1]$.

(ii). For any $f \in A_k$ and any $\varepsilon > 0$, we let $f_\varepsilon(x) = f(x) + \dfrac{\varepsilon}{4\sqrt{x}}$. Then we have

$$\|f_\varepsilon\|_1 \le \|f\|_1 + \int_0^1 \frac{\varepsilon}{4\sqrt{x}} dx = \|f\|_1 + \frac{\varepsilon}{2} < \infty,$$

which means $f_\varepsilon \in L_1[0,1]$. And we have

$$\|f_\varepsilon - f\|_1 = \int_0^1 \frac{\varepsilon}{4\sqrt{x}} dx = \frac{\varepsilon}{2} < \varepsilon,$$

which means $f_\varepsilon \in B(f,\varepsilon)$ in the sense of $L_1[0,1]$. However,

$$\|f_\varepsilon\|_2 \ge \|f_\varepsilon - f\|_2 - \|f\|_2 \ge \left(\int_0^1 \left(\frac{\varepsilon}{4\sqrt{x}}\right)^2 dx\right)^{\frac{1}{2}} - k = \infty,$$

which means $f_\varepsilon \bar{\in} A_k$. So $f \bar{\in} A_k^\circ$. Since $f \in A_k$ is arbitrary, so $A_k^\circ = \emptyset$ in $L_1[0,1]$.

Therefore, we have $L_2[0,1] = \bigcup_{k=1}^{\infty} A_k$, where $(\overline{A_k})^\circ = A_k^\circ = \emptyset$, which means A_k is nowhere dense in $L_1[0,1]$, so $L_2[0,1]$ is of the first category in $L_1[0,1]$.

32. Proof: For any $f \in Y^*$, $T^*f \in X^*$, since $x_n \rightharpoonup x$ weakly in X, so we have

$$|f(Tx_n) - f(Tx)| = |(T^*f)(x_n) - (T^*f)(x)| \to 0 \qquad (n \to \infty),$$

which means $Tx_n \rightharpoonup Tx$ weakly in Y.

A weakly sequentially continuous linear operator $T : X \to Y$ **must be bounded.**

Proof: First, we will show that $f \circ T \in X^*$ for any $f \in Y^*$.

Let $x_n, x \in X$, $x_n \to x$, then we also have $x_n \rightharpoonup x$ weakly in X. Since T is weakly sequentially continuous, so $Tx_n \rightharpoonup Tx$ weakly in Y. So $f(Tx_n) \to f(Tx)$, which means $f \circ T$ is continuous on any point $x \in X$. So $f \circ T \in X^*$.

Then we will show that $TB(0,1) \subset Y$ is bounded, where $B(0,1)$ is the unit ball in X.

Since $f \circ T \in X^*$ for any $f \in Y^*$, so $(f \circ T)^{-1}(-1,1)$ is a neighborhood of $0 \in X$, which means $\exists r > 0$ such that $B(0,r) \subset (f \circ T)^{-1}(-1,1)$. So we have $f(TB(0,r)) \subset (-1,1)$. Since both of T and f are linear, we have

$$f(TB(0,1)) \subset (-\frac{1}{r}, \frac{1}{r}).$$

Since Y^* is complete, we can use the principle of uniform boundedness for $\{[Tx]\}_{x \in B(0,1)} \subset Y^{**}$. So we have

$$\sup_{\|x\|<1} |[Tx](f)| = \sup_{\|x\|<1} |f(Tx)| = \sup_{y \in TB(0,1)} |f(y)| \le \frac{1}{r} < \infty, \forall f \in Y^*$$

So

$$\|T\| = \sup_{\|x\|<1} \|Tx\|_Y = \sup_{\|x\|<1} \|[Tx]\|_{Y^{**}} < \infty,$$

which means $T \in B(X,Y)$.

33. Proof: First, we will show that $\forall x \in X$, x has a unique expression: $x = m + n$ where $m \in M, n \in N$. Suppose x has another expression: $x = m' + n'$ where $m' \in M, n' \in N$, then we have $m - m' = n' - n \in M \bigcap N = \{0\}$, so $m = m', n = n'$, which means the expression: $x = m + n$ is unique. So P is well defined.

Then we will show that $P : X \longrightarrow X$ is linear. For any $x_1 = m_1 + n_1, x_2 = m_2 + n_2 \in X$, $\lambda \in \mathbb{R}$ where $m_1, m_2 \in M, n_1, n_2 \in N$, we have

$$P(x_1 + x_2) = P((m_1 + m_2) + (n_1 + n_2)) = m_1 + m_2 = P(x_1) + P(x_2)$$

and

$$P(\lambda x_1) = P(\lambda m_1 + \lambda n_1) = \lambda m_1 = \lambda P(x_1),$$

which means $P : X \longrightarrow X$ is linear.

Then we will show that: $P \in B(X, X) \Longleftrightarrow M, N$ are closed.

" $\Longrightarrow$ ": If $P \in B(X, X)$, let $\{x_n\} \subset M$, and $x_n \to x \in X$, then $x_n = Px_n \to Px \in M$, which means M is closed.

Let $\{y_n\} \subset N$, and $y_n \to y \in X$, then $Py_n = 0$ and so $Py = \lim_{n\to\infty} Py_n = 0$. So $y = y - Py \in N$ which means N is closed.

" $\Longleftarrow$ ": If M, N are closed in X. Since X is a Banach space, according to **Theorem 2.25 (Closed Graph Theorem)**, we only need to show that $P : X \longrightarrow X$ is a closed operator.

Let $x_n, x \in X$, $x_n \to x$, $Px_n \to y \in X$. Then $x = y + (x - y)$,

$$\text{where } y = \lim_{n\to\infty} Px_n \in M \text{ and } x - y = \lim_{n\to\infty}(x_n - Px_n) \in N$$

since $\{Px_n\} \subset M, \{x_n - Px_n\} \subset N$ and M, N is closed. So we have $Px = y$, which means P is a closed operator. So $P \in B(X, X)$.

34. Proof: First, we will show that $\{\|T_n\|\}$ is bounded.

Since for any $x \in X$, $\{T_n x\}$ is a Cauchy sequence in Y, so there exists $N_x \in \mathbb{N}$ such that

$$\|T_n x - T_m x\| < 1 \text{ when } n, m > N_x.$$

Let $M_x = \max\{\|T_1 x\|, \|T_2 x\|, \cdots, \|T_{N_x} x\|, \|T_{N_x+1} x\| + 1\}$, then we have $\|T_n x\| \le M_x$ for all $n \in \mathbb{N}$. So $\sup_{n\in\mathbb{N}} \|T_n x\| \le M_x < \infty$. Since X is a Banach space, according to the principle of uniform boundedness, we have $\sup_{n\in\mathbb{N}} \|T_n\| < \infty$, which means $\{\|T_n\|\}$ is bounded.

If Y is also a Banach space, there exists $Tx \in Y$ such that $T_n x \to Tx, \forall x \in X$. So for any $x, y \in X, \lambda \in \mathbb{R}$, we have

$$T(x+y) = \lim_{n\to\infty} T_n(x+y) = \lim_{n\to\infty} T_n x + \lim_{n\to\infty} T_n y = Tx + Ty,$$

and

$$T(\lambda x) = \lim_{n\to\infty} T_n(\lambda x) = \lambda \lim_{n\to\infty} T_n x = \lambda Tx,$$

which means $T : X \longrightarrow Y$ is linear. And

$$\|Tx\| = \|\lim_{n\to\infty} T_n x\| = \lim_{n\to\infty} \|T_n x\| \leq \liminf_{n\to\infty} \|T_n\|\|x\|,$$

which means $\|T\| \leq \liminf_{n\to\infty} \|T_n\| < \infty$. So $T \in B(X, Y)$.

40. Proof: (1). Show that if $f_n \rightharpoonup f$ weakly in $C[a, b]$, then $\{f_n\}$ is pointwise convergent.

For any $t \in [a, b]$, we define $F_t : C[a, b] \to \mathbb{R}$ as $F_t(x) = x(t)$, $\forall x \in C[a, b]$. Obviously, F_t is linear, and we have

$$|F_t(x)| = |x(t)| \leq \|x\|_C,$$

which means $F_t \in C[a, b]^*$, $\forall t \in [a, b]$. Since $f_n \rightharpoonup f$ weakly in $C[a, b]$, so we have

$$f_n(t) = F_t(f_n) \to F_t(f) = f(t),$$

which means $\{f_n\}$ is pointwise convergent.

(2). Show that if $f_n \rightharpoonup f$ weakly in $C^1[a, b]$, then $\{f_n\}$ is convergent in $C[a, b]$.

For any $t \in [a, b]$, we define $F_t : C^1[a, b] \to \mathbb{R}$ as $F_t(x) = x(t)$, $\forall x \in C^1[a, b]$. Obviously, F_t is linear, and we have

$$|F_t(x)| = |x(t)| \leq \|x\|_{C^1},$$

which means $F_t \in C^1[a, b]^*$, $\forall t \in [a, b]$. Since $f_n \rightharpoonup f$ weakly in $C^1[a, b]$, so we have

$$f_n(t) = F_t(f_n) \to F_t(f) = f(t),$$

and $\{\|f_n\|_{C^1}\}$ is bounded since $\{f_n\}$ is weakly convergent, which means there exists $M > 0$ such that

$$\sup_{t\in[a,b]} |f_n'(t)| \leq \|f_n\|_{C^1} < M, \forall n \in \mathbb{N}.$$

Since $f \in C^1[a,b]$, so f is uniformly continuous on $[a,b]$, which means $\forall \varepsilon > 0, \exists \delta_1 > 0$, such that

$$|f(t') - f(t'')| < \frac{\varepsilon}{3},\ \forall t', t'' \in [a,b] \text{ when } |t' - t''| < \delta_1.$$

Let $\delta = \min\{\frac{\varepsilon}{3M}, \delta_1\}$. We can construct a partition $\{t_i\}_{i=1}^m$ of $[a,b]$ which is

$$a = t_0 < t_1 < t_2 < \cdots < t_m = b,$$

such that $|t_i - t_{i-1}| < \delta,\ \forall i = 1, 2, \cdots, m$.

Since $f_n(t) \to f(t),\ \forall t \in [a,b]$, and the partition $\{t_i\}_{i=1}^m$ has finite points, so there exists $N \in \mathbb{N}$, when $n > N$, we have

$$|f_n(t_i) - f(t_i)| < \frac{\varepsilon}{3},\ \forall i = 0, 1, \cdots, m.$$

Then when $n > N$, for any $t \in [a,b]$, $\exists i \in \{1, 2, \cdots, m\}$ such that $t \in [t_{i-1}, t_i]$. According to Lagrangian mean value theorem, we have

$$\begin{aligned} |f_n(t) - f(t)| &\leq |f_n(t) - f_n(t_i)| + |f_n(t_i) - f(t_i)| + |f(t_i) - f(t)| \\ &\leq |f_n'(\xi_t)| \cdot |t - t_i| + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \qquad (\text{where } \xi_t \in [t, t_i]) \\ &\leq M \cdot \frac{\varepsilon}{3M} + \frac{2\varepsilon}{3} = \varepsilon \end{aligned}$$

which means $f_n(t) \rightrightarrows f(t)$ on $[a,b]$, where " $\rightrightarrows$ " means uniform convergence. So $\sup_{t\in[a,b]} |f_n(x) - f(x)| = \|f_n - f\|_C \to 0$, which means $f_n \to f$ in $C[a,b]$.

This conclusion will not be true if $[a,b]$ is replaced by $\mathbb{R}$, which is:

there exists a sequence $\{f_n\} \subset C^1(\mathbb{R})$, $f_n \rightharpoonup f \in C^1(\mathbb{R})$ weakly, but $f_n \nrightarrow f$ in $C(\mathbb{R})$.

Counterexample: Let

$$f_n(x) = g(\frac{x}{n} - n) = e^{-(\frac{x}{n}-n)^2}, \text{ where } g(x) = e^{-x^2}, \ n \in \mathbb{N},$$

then $g, f_n \in C^1(\mathbb{R})$, $\sup_{x\in\mathbb{R}} |g'(x)| = \sup_{x\in\mathbb{R}} |2xe^{-x^2}| < \infty$, and we have

$$f_n(x) = e^{-(\frac{x}{n}-n)^2} \longrightarrow 0, \ (n \to \infty, \forall x \in \mathbb{R})$$

and

$$\sup_{x\in\mathbb{R}} |f_n'(x)| = \sup_{x\in\mathbb{R}} |\frac{1}{n} g'(\frac{x}{n} - n)| = \frac{1}{n} \sup_{x\in\mathbb{R}} |g'(x)| \to 0 \ (n \to \infty).$$

So $f_n \rightharpoonup 0 \in C^1(\mathbb{R})$ weakly, but

$$\|f_n\|_C = \sup_{x\in\mathbb{R}} |f_n(x)| = \sup_{x\in\mathbb{R}} e^{-(\frac{x}{n}-n)^2} = 1,$$

which means $f_n \nrightarrow 0$ in $C(\mathbb{R})$.

Appendix

1. Let $1 \leq p < \infty$ and q be its conjugate exponent ($q = \infty$ if $p = 1$), then

$$L_p[0,1]^* = L_q[0,1];$$

Proof: (1)When $1 < p < \infty$.

For any $g \in L_q[0,1]$, we define $F_g : L_p[0,1] \to \mathbb{R}$ as

$$F_g(f) = \int_0^1 f(x)g(x)dx, \ \forall f \in L_p[0,1].$$

Obviously, $F_g : L_p[0,1] \to \mathbb{R}$ is linear and we have

$$|F_g(f)| = |\int_0^1 f(x)g(x)dx| \leq \|f\|_p \|g\|_q,$$

which means $\|F_g\| \leq \|g\|_q < \infty$. So $F_g \in L_p[0,1]^*$. Since $g \in L_q[0,1]$ is arbitrary, we have $L_q[0,1] \subset L_p[0,1]^*$.

On the other hand, for any $F \in L_p[0,1]^*$, there exists some $g \in L_q[0,1]$ such that $F = F_g$ as defined above.

Since $\chi_{[0,t)} \in L_p[0,1]$, which is the characteristic function of $[0,t)$, let $u(t) = F(\chi_{[0,t)})$. Then we will show that $u(t)$ is an absolutely continuous function on $[0,1]$.

Let $\{[a_i,b_i]\}_{i=1}^n$ is a finite collection of disjoint intervals in $[0,1]$, and put $\varepsilon_i =$sign$(u(b_i) - u(a_i))$, then we have

$$\begin{aligned}
\textstyle\sum_{i=1}^n |u(b_i) - u(a_i)| &= \textstyle\sum_{i=1}^n \varepsilon_i(u(b_i) - u(a_i)) = F(\sum_{i=1}^n \varepsilon_i(\chi_{[0,b_i)} - \chi_{[0,a_i)})) \\
&\leq \|F\| \|\textstyle\sum_{i=1}^n \varepsilon_i(\chi_{[0,b_i)} - \chi_{[0,a_i)})\|_p \\
&\leq \|F\| \textstyle\sum_{i=1}^n \|\chi_{[a_i,b_i)}\|_p \\
&= \|F\| \textstyle\sum_{i=1}^n (b_i - a_i)^{\frac{1}{p}}
\end{aligned}$$

which means $u(t)$ is an absolutely continuous function on $[0,1]$. By the Lebesgue fundamental theorem of calculus, we have

$$u(t) = u(0) + \int_0^t u'(x)dx,$$

where $u(0) = F(\chi_\emptyset) = F(0) = 0$. If we let $g(x) = u'(x)$, we have

$$F(\chi_{[0,t)}) = \int_0^t f(x)dx = \int_0^1 \chi_{[0,t)} g(x)dx = F_g(\chi_{[0,t)}).$$

Since F is linear, so

$$F(f) = F_g(f) \text{ for any simple function } f.$$

Let f be any bounded function on $[0, 1]$, there exists uniformly bounded sequence of simple function $\{f_n\}$ such that $f_n \to f$ a.e. in $[0, 1]$ and so $f_n \to f$ in $L_p[0, 1]$. By Lebesgue dominated convergence theorem, we have

$$F(f) = \lim_{n\to\infty} F(f_n) = \lim_{n\to\infty} \int_0^1 f_n(x)g(x)dx = \int_0^1 f(x)g(x)dx = F_g(f).$$

Therefore, the formula above is true for any bounded function on $[0, 1]$.

Then we will show that $g \in L_q[0, 1]$. Consider a family of bounded functions defined by

$$f_n(x) = \left\{ \begin{array}{cc} \mathrm{sign}(g(x))|g(x)|^{q-1} & \text{if } |g(x)| \le n; \\ \\ 0 & \text{if } |g(x)| > n. \end{array} \right\}$$

Then f_n is measurable and bounded, so $F(f_n) = F_g(f_n)$. And $|f_n(x)|^p \to |g(x)|^q$ a.e. x in $[0, 1]$. So we have

$$\begin{aligned} \textstyle\int_0^1 |f_n(x)|^p dx &= \textstyle\int_0^1 |f_n(x)||f_n(x)|^{\frac{1}{q-1}} dx \\ &\le \textstyle\int_0^1 |f_n(x)||g(x)|dx = \int_0^1 f_n(x)g(x)dx = F_g(f_n) \\ &= F(f_n) \le \|F\|\|f_n\|_p = \|F\|(\int_0^1 |f_n(x)|^p dx)^{\frac{1}{p}}, \end{aligned}$$

which means

$$(\int_0^1 |f_n(x)|^p dx)^{\frac{1}{q}} \le \|F\|.$$

Since $|f_n(x)|^p \to |g(x)|^q$ a.e. $x \in [0,1]$, by Fatou's Lemma, we have

$$\begin{aligned} \|g\|_q &= (\int_0^1 |g(x)|^q dx)^{\frac{1}{q}} \\ &= (\int_0^1 \lim_{n\to\infty} |f_n(x)|^p dx)^{\frac{1}{q}} \\ &\le \liminf_{n\to\infty} (\int_0^1 |f_n(x)|^p dx)^{\frac{1}{q}} \\ &\le \|F\| < \infty, \end{aligned}$$

which means $g \in L_q[0,1]$, and so $F_g \in L_p[0,1]^*$.

Finally, we will show that $F = F_g$. Since all measurable and bounded functions are dense in $L_p[0,1]$, and $F, F_g \in L_p[0,1]^*$, $F(f) = F_g(f)$ for any measurable and bounded function f. So $F = F_g$.

(2)When $p = 1$, we can show in the similar way that $L_1[0,1]^* = L_\infty[0,1]$.

2. Let X be a NLS, $x_n, x \in X$ and $x_n \rightharpoonup x$ weakly, then

$$\|x\| \le \liminf_{n\to\infty} \|x_n\|.$$

Proof: Since $x_n \rightharpoonup x$ weakly, then for any $f \in X^*$, we have $f(x_n) \to f(x)$. So

$$|f(x)| = |\lim_{n\to\infty} f(x_n)| = \lim_{n\to\infty} |f(x_n)| \le \liminf_{n\to\infty} \|f\| \|x_n\|,$$

which means

$$\|x\| = \sup_{\|f\|=1} |f(x)| \le \sup_{\|f\|=1} (\liminf_{n\to\infty} \|f\| \|x_n\|) = \liminf_{n\to\infty} \|x_n\|.$$