A FORCING ARGUMENT?

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Here is the complete argument which I tried to summarize on MathOverflow.

Definition 0.1. Let $\mathcal{C}$ be a triangulated category with zero object 0 and $\Delta \subseteq (\operatorname{Ob}\mathcal{C})^3$ the set of objects in a triple. Let $1 \in \mathcal{C}$ be a distinguished object. Assume that $\mathcal{C}$ is *monogenic* in the sense that the smallest triangulated subcategory of $\mathcal{C}$ containing 1 is $\mathcal{C}$ itself.[1]

A function $L : \operatorname{Ob}\mathcal{C} \to \mathbb{R}_{\geq 0}$ is said to be a *norm* if $L(0) = 0$, $L(1) \leq 1$, and for every triangle $X \to Y \to Z$ we have $L(Y) \leq L(X) + L(Z)$.

A norm is said to be *gapped* if there are no $X \in \mathcal{C}$ with $0 < L(X) < 1$, and moreover $L(X \oplus Y) = L(X) + L(Y)$ and $L(X) = L(X[1])$ for all $X, Y \in \mathcal{C}$.[2]

I will assume for simplicity that $\mathcal{C}$ has (countably many objects and) countably many morphisms, though this doesn't seem to be essential.

Theorem 0.2. *Let $\mathcal{C}$ be a triangulated category and $1 \in \mathcal{C}$ a distinguished object. Let $L : \operatorname{Ob}\mathcal{C} \to \mathbb{R}_{\geq 0}$ be a gapped norm. Then L is integer-valued.*

Proof. (Note that if $L(1) = 0$, then $L = 0$ by induction on number of cells and the result is trivial. Otherwise by gappedness we have $L(1) = 1$, so we will assume this is the case.)

Step 0: Let p be a prime.

Step 1: Let $L_0(X) = \lceil L(X) \rceil$ be the smallest integer $\geq L(X)$. Then $L_0 : \mathcal{C} \to [0, \infty]$ is a norm, and satisfies all the conditions to be gapped except the condition $L(X \oplus Y) = L(X) + L(Y)$. Then we have $L(X) = \lim_{k\to\infty} \frac{1}{p^k} L_0(p^k X)$. Moreover, $\frac{1}{p^{k+l}} L_0(p^{k+l} X) \leq \frac{1}{p^k} L_0(p^k X)$, so this limit is decreasing in k.

Enumerate the triangles of $\mathcal{C}$ as $E_1, E_2, \dots$ where E_i is $A_i \to B_i \to C_i$. We now define a sequence of functions $L_0, L_1, \dots : \mathcal{C} \to [0, \infty]$ satisfying the following conditions:

(1) $L_0 = \lceil L \rceil$ as above;
(2) For each $X \in \mathcal{C}$ and $n \in \mathbb{N}$ we have $L_n(X) \in \mathbb{Z}[1/p]$;
(3) For each $X \in \mathcal{C}$ and $n \in \mathbb{N}$, we have $L(X) \leq L_n(X) \leq \frac{1}{p^n} L_0(p^n X)$;
(4) For each $n \in \mathbb{N}$ and $1 \leq i \leq n$ we have $L_n(B_i) \leq L_n(A_i) + L_n(C_i)$;
(5) For each $i \leq n \in \mathbb{N}$ with $L(A_i), L(C_i) < L(B_i)$, we have $L(A_i) + L(C_i) - L(B_i) \leq L_n(A_i) + L_n(C_i) - L_n(B_i)$.

(Note that by construction, if $L(X) \in \mathbb{Z}$, then $L_n(X) = L(X)$ will be satisfied for all n.)

To wit, we define L_n as follows. For $X \in \mathcal{C}$ not of the form $B_i[k]$ for $i \leq n$, define $L_n(X) = \frac{1}{p^n} \lceil L(p^n X) \rceil$. Let $\lambda_1 < \cdots < \lambda_k \in [0, \infty]$ be those numbers λ such that there exists $i \leq n$ with $L(B_i) = \lambda$. Now for $j \leq k$, assume inductively that we have defined $L_n(B_i)$ whenever $L(B_i) < \lambda_j$. Let $B \in \{B_1, \dots, B_n\}$ be such that $L(B) = \lambda_j$; we wish now to define $L_n(B)$. Let $\mathcal{I}$ be the collection

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[1]This means that objects of $\mathcal{C}$ can always be built of by a finite sequence of "cell attachments". More precisely, for every $X \in \mathcal{C}$ there is a finite chain of maps $0 = X_0 \to X_1 \to \cdots \to X_n = X$ such that the map $X_i \to X_{i+1}$ fits in a triangle of the form $X_i to X_{i+1} \to 1[k]$ for some $k \in \mathbb{Z}$. The least such $n \in \mathbb{N}$ is the *number of cells* in X.

[2]In a more leisurely treatment, I would separate out these conditions into several different adjectives, but for our purposes I lump them together to minimize the terminological load.

of $1 \leq i \leq n$ such that $B_i = B$. For each $i \in \mathcal{I}$, we will derive an inequality that $L_n(B)$ must satisfy, and then we will take $L_n(B)$ to be the largest real number satisfying these inequalities, along with the inequalities $L_n(B) \leq \frac{1}{p^n} L(p^n B)$ and $L_n(B) \leq L_{n-1}(B)$ and $L_n(X[k]) = L_n(X)$. There are a few cases:

- If $L(A_i) = 0$, so that $L(B_i) = L(C_i)$, then we impose no further inequality, and similarly if $L(C_i) = 0$.
- Otherwise, if $L(B_i) \leq L(A_i)$ or $L(B_i) \leq L(C_i)$ and $L(A_i), L(C_i) > 0$, then $L(B_i) < L(A_i) + L(C_i)$. So pick m large enough so that $\frac{1}{p^m} L_0(p^m B_i) \leq L(A_i) + L(C_i)$, and set $L_n(B_i) \leq \frac{1}{p^m} L_0(p^m B_i)$.

Otherwise, we have $L(A_i), L(C_i) < L(B_i)$, so that by induction $L_n(A_i), L_n(C_i)$ are already defined.

- If $L_n(A_i) > L(A_i)$ or $L_n(C_i) > L(C_i)$ or $L(B_i) < L(A_i) + L(C_i)$, then we may find $m \geq n$ large enough so that $\frac{1}{p^m} L_0(p^m B_i) \leq L_n(A_i) + L_n(C_i)$ and moreover that $L(A_i) + L(C_i) - L(B_i) \leq L_n(A_i) + L_n(C_i) - \frac{1}{p^m} L(p^m B_i)$. Choose such an m, and set $L_n(B_i) \leq \frac{1}{p^m} L_0(p^m B_i)$.
- Otherwise, we have $L_n(A_i) = L(A_i)$, $L_n(C_i) = L(C_i)$, and $L(B_i) = L(A_i) + L(C_i) = L_n(A_i) + L_n(C_i)$. In this case, set $L_n(B_i) = L(B_i)$. In this case we have $L(A_i) + L(C_i) - L(B_i) = 0 = L_n(A_i) + L_n(C_i) - L_n(B_i)$.

Step 2: Now let $L_\infty : \mathcal{C} \to \mathcal{R}$ be any nonprincipal ultraproduct of the L_n, valued in an ultrapower $\mathcal{R}$ of the field $\mathbb{R}$. Embedding $\mathbb{R}$ into the ultrapower $\mathcal{R}$, we have $L \leq L_\infty$ because $L \leq L_n$ for all n by (3). By (4) L_∞ is a "$\mathcal{R}$-valued norm". Moreover, by (5) we have $L(A) + L(C) - L(B) \leq L_\infty(A) + L_\infty(C) - L_\infty(B)$ for all sequences with $L(A), L(C) < L(B)$.

Step 3: Now we induct on the number of cells in $0 \neq X \in \mathcal{C}$ to show that $L(X) = L_\infty(X)$. When $n = 1$ this is given by normalization and shift-invariance. Let $1[k] \to Y \to X$ be a cofiber sequence where Y has fewer cells. There are two cases:

- If $L(X) \leq 1$, then by hypothesis we either have $L(X) = 0$, so that $L_\infty(X) = 0$ too, or else we have $L(X) = 1$, so that $L_n(X) = 1$ for all n by construction, and then $L_\infty(X) = 1$ too.
- If $L(X) > 1 = L(1)$, then the fiber sequence $Y \to X \to 1[k+1]$ satisfies $L(Y) + L(1) - L(X) \leq L_\infty(Y) + L_\infty(1) - L_\infty(X)$ by (5). By induction we have $L_\infty(Y) = L(Y)$ and $L_\infty(1) = L(1)$. Hence we have $-L(X) \leq -L_\infty(X)$. The reverse inequality we have already guaranteed. Hence $L_\infty(X) = L(X)$ as desired.

Thus $L_\infty = L$, for any prime p and any ultrafilter.

Step 4: Since this is true for any ultrafilter, for each $X \in \mathcal{C}$ the sequence $(L_n(X))_{n \in \mathbb{N}}$ must be eventually constant at $L(X)$, so that $L(X) \in \mathbb{Z}[1/p]$.

Step 5: Since this is true for any p, we have $L(X) \in \mathbb{Z}$. □