BMO AND FAIRNESS ESTIMATES FOR POISSON

Fix an integer N; our goal is to determine an N depending on the dimension d and the associated A_2 weight ν so that if $\varphi \in \mathrm{BMO}_\nu$, then we have

$$\int_{\mathbb{R}^d} \frac{|\varphi(x)|}{1+|x|^N}\, dx \lesssim \|\varphi\|_{\mathrm{BMO}_\nu}.$$

Rather than applying the doubling property of $\nu \in A_2$, we will use the fairness property: since $\nu \in A_\infty$, there exists a $\delta > 0$ for which we have

$$\frac{\nu(A)}{\nu(B)} \lesssim \left(\frac{|A|}{|B|}\right)^\delta$$

for sets $A \subseteq B$.

Decompose $\mathbb{R}^d$ into a sequence of shells $\mathcal{A}_j = B(0,2^j) \setminus B(0,2^{j-1})$ at scale 2^j with $A_0 = B(0,1)$. Let φ_j denote the average of φ over the ball $B_j = B(0,2^j)$. Without loss of generality, we take the weighted BMO norm of φ to be 1; in this case, we have

$$\begin{aligned}
\int_{\mathbb{R}^d} \frac{|\varphi|}{1+|x|^N}\, dx &= \sum_{j=0}^{\infty} \int_{A_j} \frac{|\varphi|}{1+|x|^N}\, dx \\
&\le \sum_{j=0}^{\infty} \int_{A_j} \frac{|\varphi - \varphi_{B_j}| + |\varphi_{B_j}|}{1+|x|^N}\, dx \\
&\lesssim \sum_{j=0}^{\infty} \int_{B_j} \frac{|\varphi - \varphi_{B_j}| + |\varphi_{B_j}|}{2^{jN}}\, dx \\
&= \sum_{j=0}^{\infty} \frac{1}{2^{jN}} \left(\int_{B_j} |\varphi - \varphi_{B_j}|\, dx + \int_{B_j} |\varphi_{B_j}|\, dx \right) \\
&\le \sum_{j=0}^{\infty} \frac{1}{2^{jN}} \left(\nu(B_j) \cdot \frac{1}{\nu(B_j)} \int_{B_j} |\varphi - \varphi_{B_j}|\, dx + |B_j| \cdot |\varphi_{B_j}| \right) \\
&\le \sum_{j=0}^{\infty} \frac{1}{2^{jN}} \left(\nu(B_j) \cdot \|\varphi\|_{\mathrm{BMO}_\nu} + |B_j| \cdot |\varphi_{B_j}| \right) \\
&= \sum_{j=0}^{\infty} \frac{\nu(B_j) + |B_j| \cdot |\varphi_{B_j}|}{2^{jN}}.
\end{aligned}$$

In the case that $\nu \equiv 1$ induces Lebesgue measure, we would then have the estimates $\nu(B_j) = |B_j| \approx 2^{jd}$ and $|\varphi_{B_j}| \lesssim j$; this sub-exponential growth in the second term is then sufficient to give an estimate when $N = d+1$.

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