Quantum Physics α

§Postulates

1. States describe the system
2. States evolve
3. States yield Probabilistic measurements
4. Systems can be composite states

Notation

U: Unitary operator
M_m: Measurement operator with outcome m
$p(m)$: Probability to measure outcome m
p_i: Ensemble probability to be in the ith state

§§Vector States

state: $|\psi\rangle$
evolution: $|\psi\rangle \Rightarrow U\,|\psi\rangle$
measurement: $p(m) = \langle\psi|M_m^\dagger M_m|\psi\rangle$ and
$|\psi_m\rangle = M_m\,|\psi\rangle\,/p(m)$
composition: $|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle \otimes \cdots \otimes |\psi_n\rangle$

§§Ensemble States

state: $\rho = \sum_i p_i\,|\psi_i\rangle\langle\psi_i|$
evolution: $\rho \Rightarrow \sum_i p_i U\,|\psi_i\rangle\langle\psi_i|\,U^\dagger$
measurement: $p(m) = \sum_i p_i\,\mathrm{tr}(M_m^\dagger M_m\,|\psi_i\rangle\langle\psi_i|)$ and
$\rho_m = \sum_i p_i M_m\,|\psi_i\rangle\langle\psi_i|\,M_m^\dagger/p(m)$
composition: $\rho = \sum_i p_i\rho_i = \sum_i p_i \sum_j p_{ij}\,|\psi_{ij}\rangle\langle\psi_{ij}|$

§§Density Operator

state: $\rho = \sum_i p_i\rho_i$
evolution: $\rho \Rightarrow U\rho U^\dagger$
measurement: $p(m) = \mathrm{tr}(M_m^\dagger M_m\rho)$ and
$\rho_m = M_m\rho M_m^\dagger/p(m)$
composition: $\rho = \rho_1 \otimes \rho_2 \otimes \cdots \otimes \rho_n$

§Types of States

Quantum states are either pure or mixed. A state can be a superposition of other states, which is patently *not* a mixed state since the coefficients of the states are

§§Commutators

$[\hat{a}, \hat{a}^\dagger] = 1$
$[\hat{a}, (\hat{a}^\dagger)^n] = n(\hat{a}^\dagger)^{n-1}$
$[\hat{a}^n, \hat{a}^\dagger] = n(\hat{a})^{n-1}$
$[\hat{a}, \exp(\beta\hat{a}^\dagger)] = \beta(\exp(\beta\hat{a}^\dagger)$

§§Pure or Mixed

§§§Pure State (1 or more systems)

A pure state can describe a single or multiple quantum systems.

$|\psi_{\text{pure}}\rangle$: known exactly

$\left|\psi^{(\text{notEntangled})}_{\text{pure}}\right\rangle = |\psi_1\rangle \otimes |\psi_2\rangle \otimes \cdots \otimes |\psi_n\rangle$: all constituent systems can be factored

$\rho_{\text{pure}} \equiv |\psi_{\text{pure}}\rangle\langle\psi_{\text{pure}}|$: density operator is outer product

$\text{tr}(\rho^2_{\text{pure}}) = 1$: unity trace of squared density operator

§§§Mixed State (1 or more systems)

A mixed state is a combination of other states where the coefficients are the probability to be in the ith state p_i, representing our ignorance of the actual state it is in. Note that mixtures are represented by the density matrix ρ rather than the states themselves.

$\rho_{\text{mixed}} = \sum_i p_i |\psi_i\rangle\langle\psi_i|$: *statistical* combination, not exactly known

$\text{tr}(\rho^2_{\text{mixed}}) < 1$: less than unity trace of squared density operator

§§Superposition State

A superposition state is a combination of other states where the coefficients are complex *probability* ***amplitudes*** π_i to be in the ith state.

$p_i = |\pi_i|^2 = \pi_i^* \pi_i$: probability to be in ith state is the squared absolute value of the probability amplitude

$|\psi\rangle = \sum_i \pi_i |\psi\rangle$:

§§Entangled State (2 or more systems)

$|\psi_{\text{entangled}}\rangle$: *joint* system is exactly known

$|\psi_{\text{entangled}}\rangle \neq |\psi_1\rangle \otimes |\psi_2\rangle \otimes \cdots \otimes |\psi_n\rangle$: constituent systems *cannot* be factored

$\text{tr}(\rho^2_{\text{entangled}}) = 1$: unity trace of squared density operator for *joint* system

$\rho_{\text{entangled}} \equiv |\psi_{\text{entangled}}\rangle\langle\psi_{\text{entangled}}|$: density operator is outer product

$\rho^A_{\text{entangled}} = \text{tr}_B(\rho^{AB}_{\text{entangled}})$: susbsystem A obtained via *partial trace* over subsystem B

$\text{tr}_A([\rho^A_{\text{entangled}}]^2) < 1$: less than unity trace of squared density operator for *subsystem*

- Subsystem is in a *mixed* state whereas the full system is a *pure* state — fingerprint of entanglement

§§Eigenstates

$\mathcal{O}\left|\psi^{(\text{eigen})}\right\rangle = \lambda\left|\psi^{(\text{eigen})}\right\rangle$: an eigenstate $\left|\psi^{(\text{eigen})}\right\rangle$ *of an operator* $\mathcal{O}$ remains unchanged after action by the operator except for a scale factor called the eigenvalue λ

$c(\lambda) = \det|\mathcal{O} - \lambda I|$: the characteristic function $c(\lambda)$ is *independent of the representation* of the operator $\mathcal{O}$ (NOTE: I is the identity matrix.)

$0 = \det|\mathcal{O} - \lambda I|$: solutions where the characteristic equation is zero yield the eigenvalues λ_i

$\mathcal{O}^{\text{diag}} = \sum_i \lambda_i \left|\psi_i^{(\text{eigen})}\right\rangle\left\langle\psi_i^{(\text{eigen})}\right|$: a diagonal representation of the operator $\mathcal{O}^{\text{diag}}$, if it exists, presents the eigenvalues λ_i along the diagonal that correspond to eigenstates $\left|\psi_i^{(\text{eigen})}\right\rangle$ which form a complete, orthonormal set for $\mathcal{O}^{\text{diag}}$

$$\mathcal{O}^{\text{diag}} = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$

§Two-state System

$\mathcal{O}$	λ	$\left\vert\psi^{(\text{eigen})}\right\rangle$
$\sigma_x \equiv X$	1	$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$
$\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}$	-1	$\frac{1}{\sqrt{2}}\begin{pmatrix}-1\\1\end{pmatrix}$
$\sigma_y \equiv Y$	1	$\frac{1}{\sqrt{2}}\begin{pmatrix}-1\\-i\end{pmatrix}$
$\begin{pmatrix}0 & -i\\i & 0\end{pmatrix}$	-1	$\frac{1}{\sqrt{2}}\begin{pmatrix}-1\\i\end{pmatrix}$
$\sigma_z \equiv Z$	1	$\begin{pmatrix}1\\0\end{pmatrix}$
$\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$	-1	$\begin{pmatrix}0\\1\end{pmatrix}$

$$|\psi\rangle = \cos(\theta/2)|0\rangle + e^{i\phi}\sin(\theta/2)|1\rangle$$
$$(x, y, z) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$

$\mathcal{O}$	λ	$\left\vert\psi^{(\text{eigen})}\right\rangle$	(θ, ϕ)
$\sigma_x \equiv X$	1	$\frac{1}{\sqrt{2}}(\vert 0\rangle + \vert 1\rangle)$	$(\frac{\pi}{2}, 0)$
$\vert 0\rangle\langle 1\vert + \vert 1\rangle\langle 0\vert$	-1	$\frac{1}{\sqrt{2}}(\vert 0\rangle - \vert 1\rangle)$	$(\frac{\pi}{2}, -\pi)$
$\sigma_y \equiv Y$	1	$\frac{1}{\sqrt{2}}(\vert 0\rangle + i\vert 1\rangle)$	$(\frac{\pi}{2}, \frac{\pi}{2})$
$-i(\vert 0\rangle\langle 1\vert - \vert 1\rangle\langle 0\vert)$	-1	$\frac{1}{\sqrt{2}}(\vert 0\rangle - i\vert 1\rangle)$	$(\frac{\pi}{2}, -\frac{\pi}{2})$
$\sigma_z \equiv Z$	1	$\vert 0\rangle$	$(0, 0)$
$\vert 0\rangle\langle 0\vert - \vert 1\rangle\langle 1\vert$	-1	$\vert 1\rangle$	$(\pi, 0)$

§§Commutators for Two-State System

§Number State

$|n\rangle$: *number* state with exactly n quanta (AKA Fock)

$\hat{a}|n\rangle = \sqrt{n}|n-1\rangle$: *annihilation* operator $\hat{a}$ removes a quanta (AKA destruction, lowering)

$\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$: *creation* operator $\hat{a}^\dagger$ introduces a quanta (AKA raising)

$\hat{n}|n\rangle \equiv \hat{a}^\dagger\hat{a}|n\rangle = n|n\rangle$: *number* operator $\hat{n} \equiv \hat{a}^\dagger\hat{a}$ counts quanta

$|n\rangle = (n!)^{-1/2}(\hat{a}^\dagger)^n|0\rangle$: generate number state $|n\rangle$ from a vacuum state $|0\rangle$