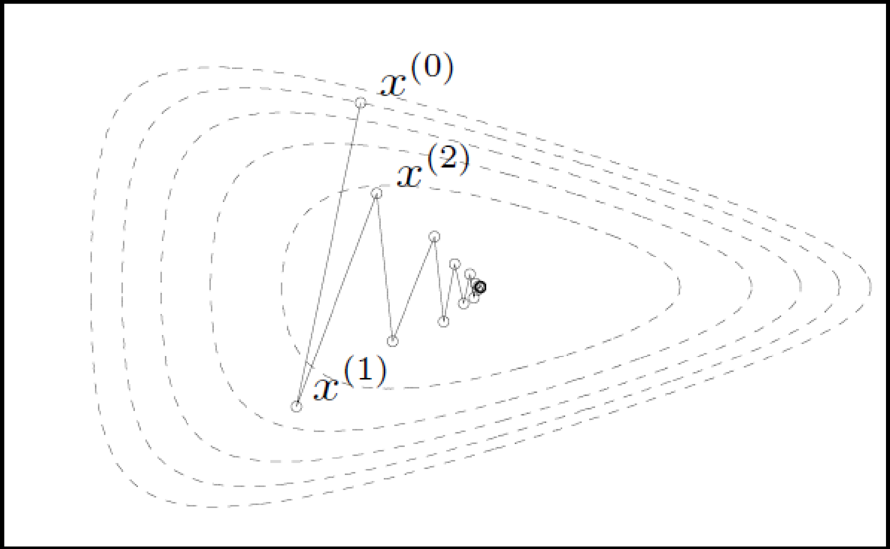
Constrained Optimization II

Using material from Stephen Boyd

Last time…

- We saw how to use local methods to solve unconstrained optimization problems



- What do we do if there are constraints on x?
- Example: certain control inputs are not possible for a robot

Outline

- Defining convex optimization problems
- Linear programming
- Quadratic programming

Defining Convex Optimization Problems with Constraints

Convex Optimization

“With only a bit of exaggeration, we can say that, if you formulate a practical problem as a convex optimization problem, then you have solved the original problem.”

- Stephen Boyd

Definition of a general optimization problem

- The "standard form"

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\ & h_i(x) = 0, \quad i = 1, \ldots, p \end{array}$$

- $x \in \mathbf{R}^n$ is the optimization variable
- $f_0 : \mathbf{R}^n \to \mathbf{R}$ is the objective or cost function
- $f_i : \mathbf{R}^n \to \mathbf{R}$, $i = 1, \ldots, m$, are the inequality constraint functions
- $h_i : \mathbf{R}^n \to \mathbf{R}$ are the equality constraint functions

"inf" is a generalization of "min"

$$p^\star = \inf\{f_0(x) \mid f_i(x) \leq 0, \ i = 1, \ldots, m, \ h_i(x) = 0, \ i = 1, \ldots, p\}$$

- $p^\star = \infty$ if problem is infeasible (no x satisfies the constraints)
- $p^\star = -\infty$ if problem is unbounded below

What if we just want something feasible?

- If any solution will do:

$$
\begin{array}{ll}
\text{minimize} & 0 \\
\text{subject to} & f_i(x) \leq 0, \quad i = 1, \dots, m \\
& h_i(x) = 0, \quad i = 1, \dots, p
\end{array}
$$

- $p^\star = 0$ if constraints are feasible; any feasible x is optimal
- $p^\star = \infty$ if constraints are infeasible

Definition of a convex optimization problem

General Optimization Problem

$$
\begin{array}{ll}
\text{minimize} & f_0(x) \\
\text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\
 & h_i(x) = 0, \quad \cancel{i = 1, \ldots, p}
\end{array}
$$

Convex Optimization Problem

$$
\begin{array}{ll}
\text{minimize} & f_0(x) \\
\text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\
 & Ax = b
\end{array}
$$

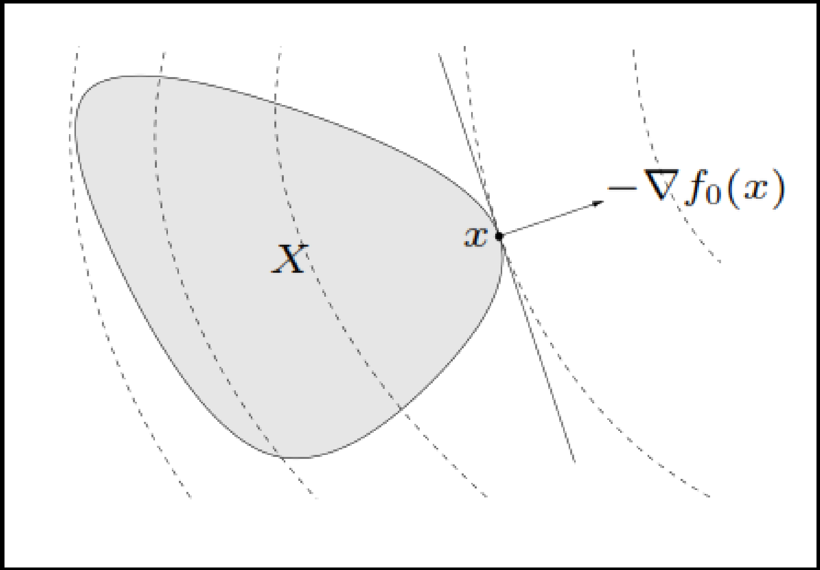
- $f_0, f_1, \ldots, f_m$ are convex

- The feasible set of solutions in a convex optimization problem must be convex
- Any locally-optimal point is globally-optimal!

Optimality for differentiable objective functions in convex optimization

- x is optimal if and only if it is feasible and

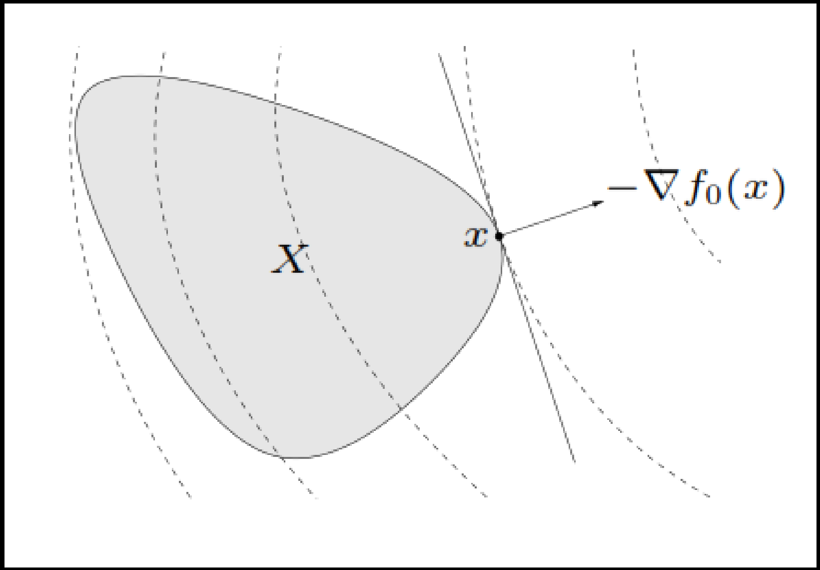
$$\nabla f_0(x)^T(y-x) \geq 0 \quad \text{for all feasible } y$$



- If non-zero, $\nabla f_0(x)$ defines a *supporting hyperplane* to feasible set X at x (all of X is below the hyperplane)

Local methods yield global optimum

- If X is convex and we follow the gradient, we are guaranteed to reach the global optimum



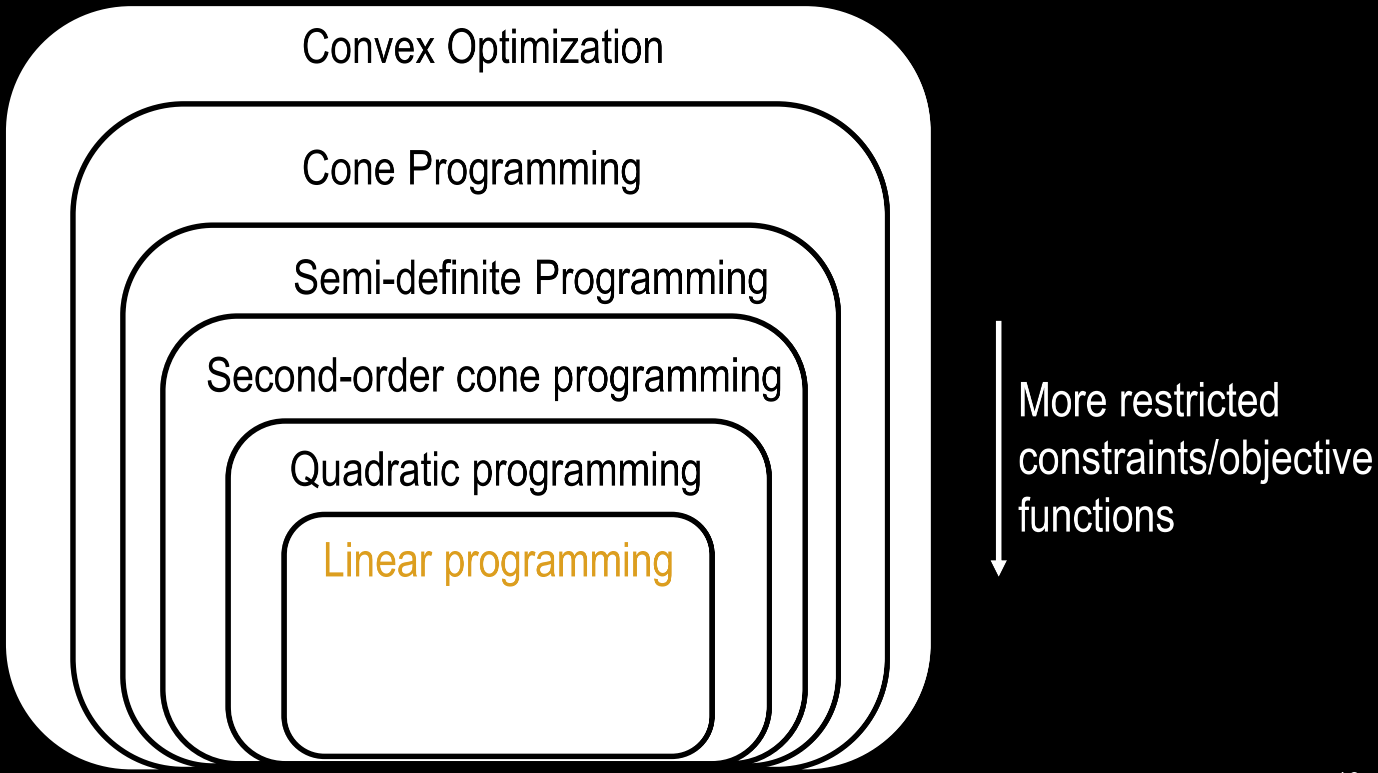
Useful Definitions

- Generalized Inequality $\preceq$ – A generalization of $\leq$
  - Note that generalized inequalities do not necessarily give a linear ordering on elements
  - For our purposes, assume $\preceq$ and $\leq$ are the same
- Infimum (inf) – A generalization of minimum: The greatest lower bound
  - For our purposes, think of this as “min”
- Supremum (sup) – A generalization of maximum: The smallest upper bound
  - For our purposes, think of this as “max”

Linear Programming

Linear Programming

- Most common form of Convex optimization is linear programming
- A “technology” rather than a research field



Linear Programming

- Standard form Linear Program (LP)

<u>General</u> Optimization Problem

$$\begin{array}{lll} \text{minimize} & f_0(x) & \\ \text{subject to} & f_i(x) \le 0, & i = 1, \dots, m \\ & h_i(x) = 0, & i = 1, \dots, p \end{array}$$

<u>Convex</u> Optimization Problem

$$\begin{array}{lll} \text{minimize} & f_0(x) & \\ \text{subject to} & f_i(x) \le 0, & i = 1, \dots, m \\ & Ax = b & \end{array}$$

- $f_0, f_1, \dots, f_m$ are convex

LP

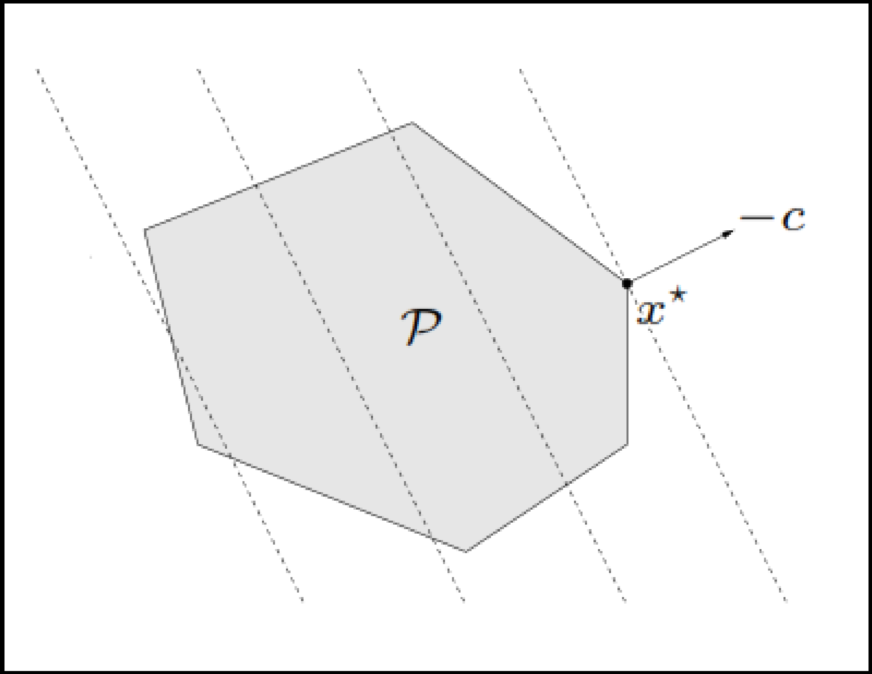
$$\begin{array}{ll} \text{minimize} & c^T x + d \\ \text{subject to} & Gx \preceq h \\ & Ax = b \end{array}$$

- LP is always convex

Linear Programming

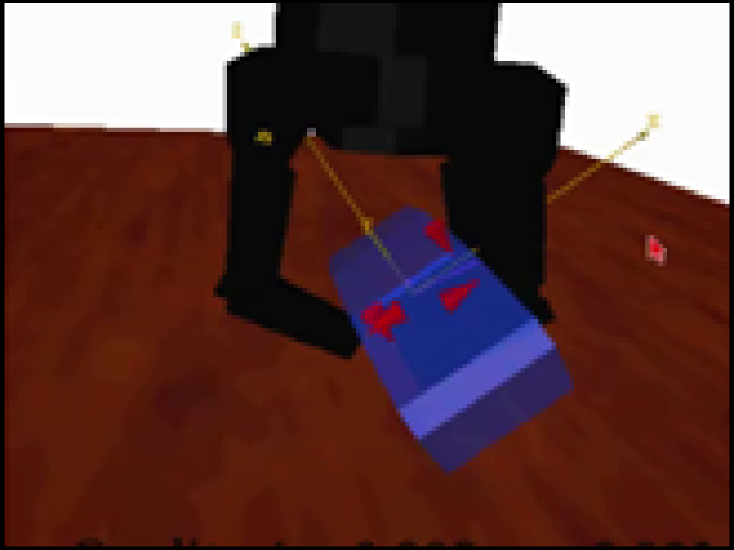
- The feasible set is a polyhedron

$$
\begin{array}{ll}
\text{minimize} & c^T x + d \\
\text{subject to} & Gx \preceq h \\
 & Ax = b
\end{array}
$$



Example: Grasping

- Can use linear programming to check if a grasp immobilizes an object



LP Example Problem

- You want to rent a team of robots to do a certain set of tasks. There are n types of robots and you need to choose how much time to rent each robot $x_1,\ldots,x_n$ while making sure the team can do all the required tasks $t_1,\ldots,t_m$.
  - A type of robot j
    - costs c_j to rent for 1 hour
    - has a_{ij} amount of capability to do t_i in 1 hour
  - The team has to have at least b_i total capability to do t_i
  - Assume every robot can work on all tasks simultaneously
- Problem: Find the least-cost rental times for each robot while ensuring that all the tasks can be completed

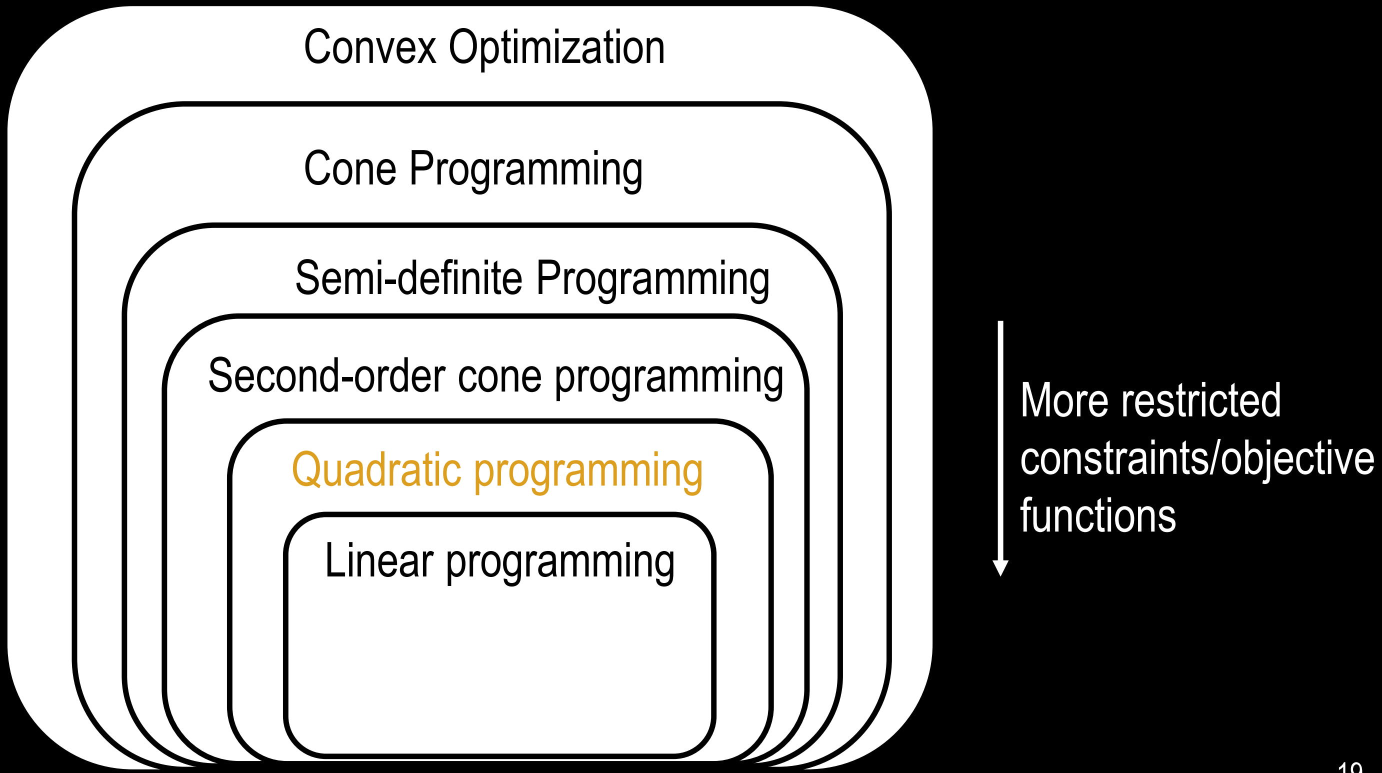
$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \succeq b, \quad x \succeq 0 \end{array}$$

- Where a_{ij} are the elements of the A matrix

Quadratic Programming

Quadratic Programming

- Common form of Convex optimization used in control and robotics
- Many solvers available



Quadratic Programming

- Standard form Quadratic Program (QP)

General Optimization Problem

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \le 0, \quad i = 1, \dots, m \\ & h_i(x) = 0, \quad i = 1, \dots, p \end{array}$$

Convex Optimization Problem

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \le 0, \quad i = 1, \dots, m \\ & Ax = b \end{array}$$

- $f_0, f_1, \dots, f_m$ are convex

QP

$$\begin{array}{ll} \text{minimize} & (1/2)x^T Px + q^T x + r \\ \text{subject to} & Gx \preceq h \\ & Ax = b \end{array}$$

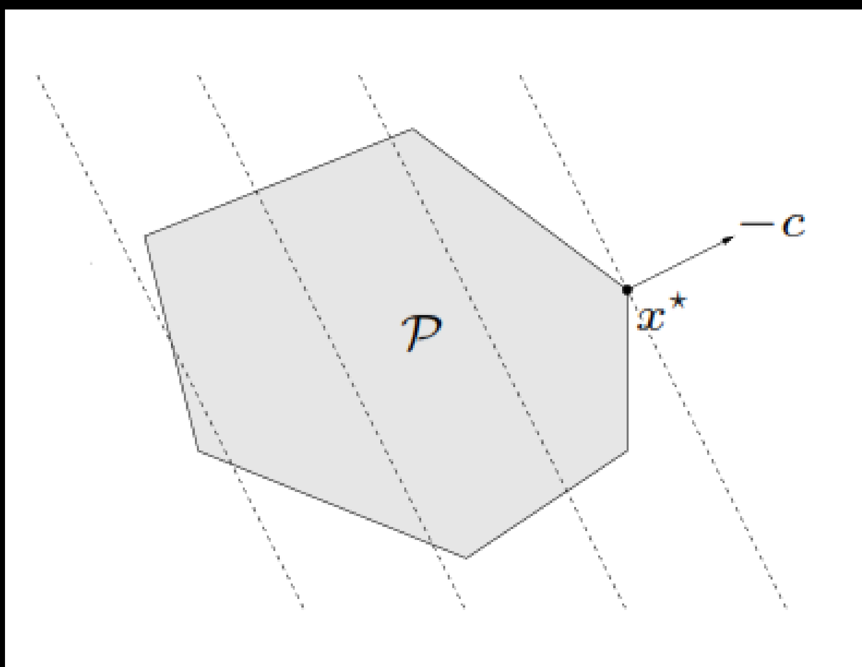
LP

$$\begin{array}{ll} \text{minimize} & c^T x + d \\ \text{subject to} & Gx \preceq h \\ & Ax = b \end{array}$$

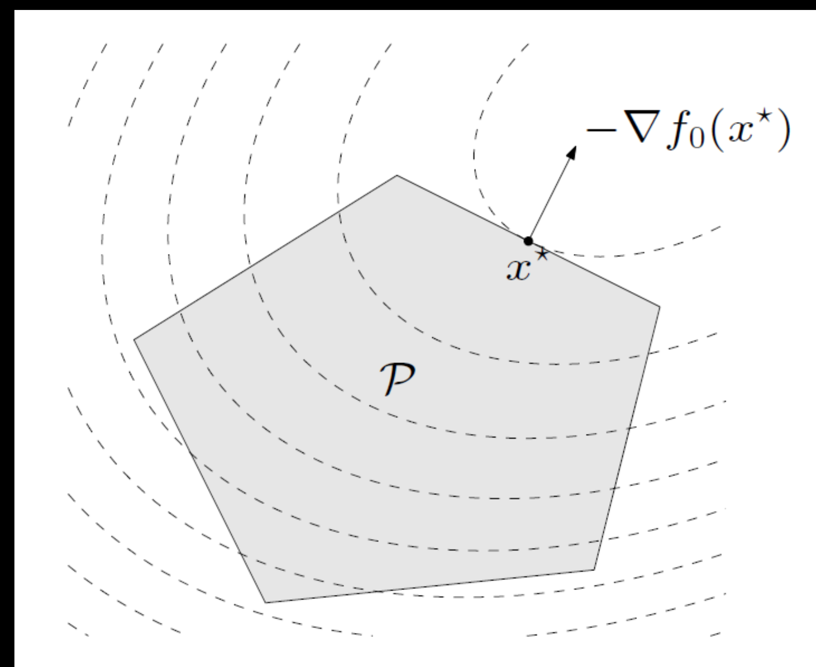
- P must be a symmetric positive semi-definite matrix
  - $z^T P z \ge 0$ for any z
- Constraints are same as LP
- Objective function is quadratic

Quadratic Programming

- The feasible set is a polyhedron (same as LP)
- Objective function is more expressive than LP



LP



QP

Example: Optimal Control with a QP

- Assume we have a robot with linear dynamics:

$$x_{t+1} = Ax_t + Bu_t$$

where x is the state and u is the control input

- We also have some constraints on each dimension of u

$$u_{i,min} \leq u_i \leq u_{i,max}$$

- We start at state x_0 and want to reach a goal x_{goal}
- Problem: Find a control input u^* that gets the robot as close to the goal as possible.

Example: Optimal Control with a QP

- Let's write this as a QP:

$$\underset{u}{minimize} \quad \left\| x_{goal} - (Ax_0 + Bu) \right\|^2$$

This will make the math simpler/faster without changing u^*

$$Subject\ to \quad u_{i,min} \le u_i \le u_{i,max} \quad \text{for } i = 1 \ldots |u|$$

Can convert to standard form

$$\begin{array}{ll} \text{minimize} & (1/2)x^T P x + q^T x + r \\ \text{subject to} & Gx \preceq h \\ & Ax = b \end{array}$$

Summary

- Saw how to define convex optimization problems <u>with constraints</u>

- Linear programming is a popular and powerful convex optimization problem class

- Quadratic programming can be used for many control problems in robotics

Homework

- Homework 5