Weighted Hypergraph Transversals & Contextuality

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This is the abstract.

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I. INTRODUCTION

A. Applications

II. MARGINAL SATISFIABILITY

A. Definitions

To every random variable[1] v there corresponds a prescribed set of **outcomes** $\mathcal{O}_v$ and a set of **events over** v denoted $\Omega(v)$ corresponding to the set of all functions of the form $\omega : \{v\} \to \mathcal{O}_v$. Evidently, $\Omega(v)$ and $\mathcal{O}_v$ are isomorphic structures and their distinction can be confounding. There is rarely any harm in referring synonymously to either as outcomes. Nonetheless, a sheaf-theoretic treatment of contextuality [1] demands the distinction. Specifically for this work, the distinction becomes essential for the exploitation of marginal symmetries in Section III C. As a natural generalization we define the event over a collection of random variables $V = \{v_1, \dots, v_n\}$ in a parallel manner:

$$\Omega(V) \equiv \{\omega : V \to \mathcal{O}_V \mid \forall v \in V, \omega(v) \in \mathcal{O}_v\}$$

Furthermore, the **domain** $\mathcal{D}(\omega)$ of an event ω is the set of random variables it valuates, i.e. if $\omega \in \Omega(V)$ then $\mathcal{D}(\omega) = V$.

For every $V' \subset V$ and $\omega \in \Omega(V)$, the **restriction of** ω **onto** V' (denoted $\omega|_{V'}$) corresponds to the unique event in $\Omega(V')$ that agrees with ω for all valuations of variables in V', i.e. $\forall v' \in V' : \omega|_{V'}(v') = \omega(v')$. Using this notational framework, a probability distribution or simply **distribution** p_V is a probability measure on $\Omega(V)$, assigning to each $\omega \in \Omega(V)$ a real number $\mathsf{p}_V(\omega) \in [0, 1]$ such that $\sum_{\omega \in \Omega(V)} \mathsf{p}_V(\omega) = 1$. The set of all distributions over $\Omega(V)$ is denoted $\mathcal{P}_V$. Moreover, given $\mathsf{p}_V \in \mathcal{P}_V$ and $V' \subset V$, there is an induced distribution $\mathsf{p}_V|_{V'} \in \mathcal{P}_{V'}$ obtained by *marginalizing* p_V:

$$\mathsf{p}_V|_{V'}(\omega') = \sum_{\substack{\omega \in \Omega(V) \\ \omega|_{V'} = \omega'}} \mathsf{p}_V(\omega)$$

Presently, the reader is equipped with sufficient notation and terminology to comprehend the **marginal satisfiability problem**: given a collection of m distributions $\{\mathsf{p}_{V_1}, \dots, \mathsf{p}_{V_m}\}$, does there exist a distribution $\mathsf{p}_\Lambda \in \mathcal{P}_\Lambda$ where $\Lambda \equiv \bigcup_{i=1}^{m} V_m$ such that $\forall i : \mathsf{p}_\Lambda|_{V_i} = \mathsf{p}_{V_i}$?

To facilitate further discussion of this problem, several pieces of nomenclature will be introduced. First, the set $\mathcal{V} = \{V_1, \dots, V_m\}$ is called the **marginal scenario** while its elements are called the **marginal contexts**. The collection of distributions $\mathsf{p}_{[\mathcal{V}]} \equiv \{\mathsf{p}_{V_1}, \dots, \mathsf{p}_{V_m}\}$[2] is called the **marginal model** [2][3]. The distribution p_Λ, if it exists, is termed the **joint distribution**. Strictly speaking, as defined by [2], a marginal scenario forms an *abstract simplicial complex*, meaning it satisfies the supplementary required that all subsets of contexts are also

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[1] Throughout this document, it is assumed that all random variables are discrete and have finite cardinality.

[2] The subscript $[\mathcal{V}]$ is contained in square brackets for clarity; $\mathsf{p}_{[\mathcal{V}]}$ is *not* a distribution but a set of distributions over $\mathcal{V}$.

[3] In [1], $\mathsf{p}_{[\mathcal{V}]}$ is instead called an *empirical model*.

contexts, i.e. $\forall V \in \mathcal{V} : V' \subset V \implies V' \in \mathcal{V}$. Throughout this section, we exclusively consider (without loss of generality) *maximal* marginal scenarios, restricting our focus to the contexts which are contained in no others. Finally, a marginal model $\mathsf{p}_{[\mathcal{V}]}$ is said to be **contextual**, and will be denoted $\mathsf{p}_{[\mathcal{V}]} \in \mathcal{C}$ if it does not admit a joint a distribution and **non-contextual** otherwise ($\mathsf{p}_{[\mathcal{V}]} \notin \mathcal{C}$). Equipped with additional terminology and notation, the marginal satisfiability problem now reads: given $\mathsf{p}_{[\mathcal{V}]}$, is $\mathsf{p}_{\mathcal{V}} \in \mathcal{C}$ or not?

B. Linearity

C. Polytope Projection

D. Logical Contextuality

III. AN OBSERVATION

A. An Antecedant Hierarchy

B. Irreducibility

C. Marginal Symmetries

D. Curated Inequalities

E. Relaxations

IV. WEIGHTED HYPERGRAPH TRANSERVALS

A. Preliminaries

B. Hypergraph Transversals

C. Adding Weights

V. CONCLUSIONS

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