Edge-Weighted Hypergraph Transversals & Contextuality

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This is the abstract.

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A. Definitions

To every random variable[1] v there corresponds a prescribed set of **outcomes** $\mathcal{O}_v$ and a set of **events over** v denoted $\Omega(v)$ corresponding to the set of all functions of the form $\omega : \{v\} \to \mathcal{O}_v$. Evidently, $\Omega(v)$ and $\mathcal{O}_v$ are isomorphic structures and their distinction can be confounding. There is rarely any harm in referring synonymously to either as outcomes. Nonetheless, a sheaf-theoretic treatment of contextuality [1] demands the distinction. Specifically for this work, the distinction becomes essential for the exploitation of marginal symmetries in Section III D. As a natural generalization we define the event over a collection of random variables $V = \{v_1, \ldots, v_n\}$ in a parallel manner:

$$\Omega(V) \equiv \{\omega : V \to \mathcal{O}_V \mid \forall v \in V, \omega(v) \in \mathcal{O}_v\}$$

Furthermore, the **domain** $\mathcal{D}(\omega)$ of an event ω is the set of random variables it valuates, i.e. if $\omega \in \Omega(V)$ then $\mathcal{D}(\omega) = V$.

For every $V' \subset V$ and $\omega \in \Omega(V)$, the **restriction of** ω **onto** V' (denoted $\omega|_{V'}$) corresponds to the unique event in $\Omega(V')$ that agrees with ω for all valuations of variables in V', i.e. $\forall v' \in V' : \omega|_{V'}(v') = \omega(v')$. Using this notational framework, a probability distribution or simply **distribution** p_V is a probability measure on $\Omega(V)$, assigning to each $\omega \in \Omega(V)$ a real number $\mathsf{p}_V(\omega) \in [0, 1]$ such that $\sum_{\omega\in\Omega(V)} \mathsf{p}_V(\omega) = 1$. The set of all distributions over $\Omega(V)$ is denoted $\mathcal{P}_V$. Moreover, given $\mathsf{p}_V \in \mathcal{P}_V$ and $V' \subset V$, there is an induced distribution $\mathsf{p}_V|_{V'} \in \mathcal{P}_{V'}$ obtained by *marginalizing* p_V:

$$\mathsf{p}_V|_{V'}(\omega') = \sum_{\substack{\omega\in\Omega(V)\\ \omega|_{V'}=\omega'}} \mathsf{p}_V(\omega) \tag{1}$$

Presently, the reader is equipped with sufficient notation and terminology to comprehend the **marginal (satisfiability) problem**: given a collection of m distributions $\{\mathsf{p}_{V_1}, \ldots, \mathsf{p}_{V_m}\}$, does there exist a distribution $\mathsf{p}_\Lambda \in \mathcal{P}_\Lambda$ where $\Lambda \equiv \bigcup_{i=1}^m V_m$ such that $\forall i : \mathsf{p}_\Lambda|_{V_i} = \mathsf{p}_{V_i}$?

To facilitate further discussion of this problem, several pieces of nomenclature will be introduced. First, the set $\mathcal{V} = \{V_1, \ldots, V_m\}$ is called the **marginal scenario** while its elements are called the **marginal contexts**. The collection of distributions $\mathsf{p}_{*\mathcal{V}} \equiv \{\mathsf{p}_{V_1}, \ldots, \mathsf{p}_{V_m}\}$[2] is called the **marginal model** [2][3]. The distribution p_Λ,

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[1] Throughout this document, it is assumed that all random variables are discrete and have finite cardinality.

[2] The subscript $*$ preceding $_*\mathcal{V}$ is added for clarity; $\mathsf{p}_{*\mathcal{V}}$ is *not* a distribution but a set of distributions over $\mathcal{V}$. The $_*\mathcal{V}$ convention is adopted throughout this report.

[3] In [1], $\mathsf{p}_{*\mathcal{V}}$ is instead called an *empirical model.*

if it exists, is termed the **joint distribution**. Strictly speaking, as defined by [2], a marginal scenario forms an *abstract simplicial complex*, meaning it satisfies the supplementary required that all subsets of contexts are also contexts, i.e. $\forall V \in \mathcal{V} : V' \subset V \implies V' \in \mathcal{V}$. Throughout this work, we exclusively consider (without loss of generality) *maximal* marginal scenarios, restricting our focus to the contexts which are contained in no others. Finally, a marginal model $\mathsf{p}_{*\mathcal{V}}$ is said to be **contextual**, and will be denoted $\mathsf{p}_{*\mathcal{V}} \in \mathcal{C} \subseteq \mathcal{P}_{*\mathcal{V}}$ if it does not admit a joint distribution and **non-contextual** otherwise ($\mathsf{p}_{*\mathcal{V}} \notin \mathcal{C}$). Equipped with additional terminology and notation, the marginal problem now reads: given $\mathsf{p}_{*\mathcal{V}}$, is $\mathsf{p}_{*\mathcal{V}} \in \mathcal{C}$ or not?

B. Linearity

An essential feature of the marginal problem is linearity; the marginalization of p_Λ onto the marginal contexts $\{\mathsf{p}_\Lambda|_V \mid V \in \mathcal{V}\}$ is a linear transformation, requiring only the summations pursuant to Eq. (1). Consequently, it is advantageous to consider the statement of the marginal problem as a matrix multiplication. To this end, for each marginal scenario $\mathcal{V}$ we define a bitwise matrix $\mathcal{M}$ called the **incidence matrix** which implements this mapping. The columns of $\mathcal{M}$ are indexed by *joint events* $j \in \Omega(\Lambda)$ and the rows are indexed by *marginal events* $m \in \Omega(V)$ for some $V \in \mathcal{V}$. By deliberate abuse of notation, we will denote the set of all marginal events as $\Omega(_*\mathcal{V})$ and is defined as the following disjoint union:

$$\Omega(_*\mathcal{V}) \equiv \coprod_{V \in \mathcal{V}} \Omega(V)$$

The $|\Omega(_*\mathcal{V})| \times |\Omega(\Lambda)|$ incidence matrix $\mathcal{M}$ is then defined element-wise for $m \in \Omega(_*\mathcal{V})$ and $j \in \Omega(\Lambda)$:

$$\mathcal{M}_j^m = \begin{cases} 1 & j|_{\mathcal{D}(m)} = m \\ 0 & \text{otherwise} \end{cases}$$

Conceptually, the entries of this matrix are populated with ones whenever the marginal event (row) m is the restriction of some joint event (column) j. For a given marginal scenario $\mathcal{V}$, $\mathcal{M}$ represents the tuple of restriction maps $\mathcal{M} : \Omega(\Lambda) \to \prod_{V \in \mathcal{V}} \Omega(V) :: j \mapsto \{j|_V \mid V \in \mathcal{V}\}$ [1].

To illustrate this concretely, consider the following example. Let Λ be 3 binary variables $\{a, b, c\}$ and $\mathcal{V}$ be the marginal scenario $\mathcal{V} = \{\{a, b\}, \{b, c\}, \{a, c\}\}$. The incidence matrix for $\mathcal{V}$ becomes:

$$\begin{array}{rc}
(a,b,c) \mapsto & \begin{array}{cccccccc} (0,0,0) & (0,0,1) & (0,1,0) & (0,1,1) & (1,0,0) & (1,0,1) & (1,1,0) & (1,1,1) \end{array} \\
\begin{array}{r}
(a{\mapsto}0,b{\mapsto}0) \\ (a{\mapsto}0,b{\mapsto}1) \\ (a{\mapsto}1,b{\mapsto}0) \\ (a{\mapsto}1,b{\mapsto}1) \\ (b{\mapsto}0,c{\mapsto}0) \\ (b{\mapsto}0,c{\mapsto}1) \\ (b{\mapsto}1,c{\mapsto}0) \\ (b{\mapsto}1,c{\mapsto}1) \\ (a{\mapsto}0,c{\mapsto}0) \\ (a{\mapsto}0,c{\mapsto}1) \\ (a{\mapsto}1,c{\mapsto}0) \\ (a{\mapsto}1,c{\mapsto}1)
\end{array} &
\left(\begin{array}{cccccccc}
\mathbf{1} & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \mathbf{1} & \mathbf{1} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \mathbf{1} & \mathbf{1} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & \mathbf{1} \\
\mathbf{1} & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 \\
0 & \mathbf{1} & 0 & 0 & 0 & \mathbf{1} & 0 & 0 \\
0 & 0 & \mathbf{1} & 0 & 0 & 0 & \mathbf{1} & 0 \\
0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & \mathbf{1} \\
\mathbf{1} & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 \\
0 & \mathbf{1} & 0 & \mathbf{1} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \mathbf{1} & 0 & \mathbf{1} & 0 \\
0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & \mathbf{1}
\end{array}\right)
\end{array} \tag{2}$$

In addition, for any joint distribution $\mathsf{p}_\Lambda \in \mathcal{P}_\Lambda$ we associate a joint distribution *vector* p_Λ (identically denoted) indexed by $j \in \Omega(\Lambda)$, i.e. $\mathsf{p}_\Lambda^j \equiv \mathsf{p}_\Lambda(j)$. Analogously, for each marginal model $\mathsf{p}_{*\mathcal{V}} \in \mathcal{P}_{*\mathcal{V}}$ there is an associated marginal distribution *vector* $\mathsf{p}_{*\mathcal{V}}$ indexed by $m \in \Omega(_*\mathcal{V})$ such that $\mathsf{p}_{*\mathcal{V}}^m \equiv \mathsf{p}_{\mathcal{D}(m)}(m)$. Using these vectors, the marginal problem becomes the following linear program: given a marginal distribution vector $\mathsf{p}_{*\mathcal{V}}$, does there exist a joint distribution vector $\mathsf{p}_\Lambda \succeq 0$ such that Eq. (3) holds?

$$\mathsf{p}_{*\mathcal{V}} = \mathcal{M} \cdot \mathsf{p}_\Lambda \iff \mathsf{p}_{*\mathcal{V}}^m = \sum_{j \in \Omega(\Lambda)} \mathcal{M}_j^m \mathsf{p}_\Lambda^j \tag{3}$$

C. Marginal Polytopes

D. Logical Contextuality

Let $a \in \Omega(_*\mathcal{V})$ be *any* marginal event and $C = \{c_1, \ldots, c_n\} \subseteq \Omega(_*\mathcal{V})$ be a subset of marginal events such that the following logical implication holds for *all* marginal models $\mathsf{p}_{*\mathcal{V}} \in \mathcal{P}_{*\mathcal{V}}$:

$$a \implies c_1 \vee \cdots \vee c_n = \bigvee_{c \in C} c \tag{4}$$

Which can be dictated: *whenever the event a occurs, at least one event in C occurs.* In accordance with the logical form of Eq. (4), a will be referred to as the **antecedent** and C as the **consequent set**. To clarify, a marginal model $\mathsf{p}_{*\mathcal{V}} \in \mathcal{P}_{*\mathcal{V}}$ satisfies Eq. (4) if there always at least one $c \in C$ that is *possible* ($\mathsf{p}_{*\mathcal{V}}^c > 0$) whenever a is possible. A marginal model violates Eq. (4) whenever *none* of events in c are possible while a remains possible. Marginal models that violate logical statements such as Eq. (4) are known as **Hardy Paradoxes** [3–5]. Motivated by a greater sense of robustness compared to possibilistic constraints, the concept of witnessing quantum contextuality on a logical level has be analyzed thoroughly for decades [6, 7].

III. AN OBSERVATION

A. An Antecedent Hierarchy

B. The Antecedent Hypergraph

Given an antecedent multi-set γ where $\gamma \preceq 0$, we identify the **inhibiting set** of joint events $\mathcal{I}(\gamma) \subseteq \Omega(\Lambda)$ preventing $\gamma \cdot \mathcal{M}$ from being positive semi-definite:

$$\mathcal{I}(\gamma) \equiv \left\{ j \in \Omega(\Lambda) \mid \sum_{m \in \Omega({}_*\mathcal{V})} \gamma^m \mathcal{M}^j_m < 0 \right\}$$

The inhibiting set $\mathcal{I}(\gamma)$ of γ completely characterizes the **antecedent hypergraph** $\mathcal{H}(\gamma)$ whose edges $\mathcal{E}_j$ are indexed by the inhibiting events $j \in \mathcal{I}(\gamma)$. Each edge $\mathcal{E}_j \subseteq \Omega({}_*\mathcal{V})$ corresponds to the set of the marginal events $m \in \Omega({}_*\mathcal{V})$ which are restrictions of j. Specifically,

$$\mathcal{H}(\gamma) \equiv \{\mathcal{E}_j \mid j \in \mathcal{I}(\gamma)\}$$
$$\mathcal{E}_j \equiv \left\{ m \in \Omega({}_*\mathcal{V}) \mid m = j|_{\mathcal{D}(m)}, \gamma^m = 0 \right\}$$

C. Irreducibility

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E. Curated Inequalities

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IV. EDGE-WEIGHTED HYPERGRAPH TRANSVERSALS

A. Preliminaries

B. Hypergraph Transversals

C. Adding Weights

V. CONCLUSIONS

ACKNOWLEDGMENTS

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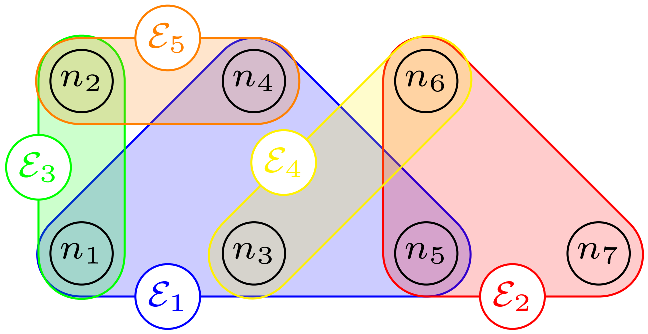

$$\begin{array}{c} \\ n_1 \\ n_2 \\ n_3 \\ n_4 \\ n_5 \\ n_6 \\ n_7 \end{array} \begin{array}{c} \begin{array}{ccccc} \mathcal{E}_1 & \mathcal{E}_2 & \mathcal{E}_3 & \mathcal{E}_4 & \mathcal{E}_5 \end{array} \\ \begin{pmatrix} \mathbf{1} & 0 & \mathbf{1} & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 & \mathbf{1} \\ \mathbf{1} & 0 & 0 & \mathbf{1} & 0 \\ \mathbf{1} & 0 & 0 & 0 & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & 0 & 0 & 0 \\ 0 & \mathbf{1} & 0 & \mathbf{1} & 0 \\ 0 & \mathbf{1} & 0 & 0 & 0 \end{pmatrix} \end{array}$$

FIG. 1. Dual-representations of a hypergraph $\mathcal{H} = \{\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3, \mathcal{E}_4, \mathcal{E}_5\}$.