Weighted-Hypergraph Transversals: A Practical Approach to Non-Contextuality Inequalities

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This is the abstract.

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I. INTRODUCTION

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A. Definitions

To every random variable[1] v there corresponds a prescribed set of **outcomes** $\mathcal{O}_v$ and a set of **events over** v denoted $\Omega(v)$ corresponding to the set of all functions of the form $\omega : \{v\} \to \mathcal{O}_v$. Evidently, $\Omega(v)$ and $\mathcal{O}_v$ are isomorphic structures and their distinction can be confounding. There is rarely any harm in referring synonymously to either as outcomes. Nonetheless, a sheaf-theoretic treatment of contextuality [1] demands the distinction. Specifically for this work, the distinction becomes essential for the exploitation of marginal symmetries in Section III D. As a natural generalization we define the event over a collection of random variables $V = \{v_1, \dots, v_n\}$ in a parallel manner:

$$\Omega(V) \equiv \{\omega : V \to \mathcal{O}_V \mid \forall v \in V, \omega(v) \in \mathcal{O}_v\}$$

Furthermore, the **domain** $\mathcal{D}(\omega)$ of an event ω is the set of random variables it valuates, i.e. if $\omega \in \Omega(V)$ then $\mathcal{D}(\omega) = V$.

For every $V' \subset V$ and $\omega \in \Omega(V)$, the **restriction of** ω **onto** V' (denoted $\omega|_{V'}$) corresponds to the unique event in $\Omega(V')$ that agrees with ω for all valuations of variables in V', i.e. $\forall v' \in V' : \omega|_{V'}(v') = \omega(v')$. Using this notational framework, a probability distribution or simply **distribution** p_V is a probability measure on $\Omega(V)$, assigning to each $\omega \in \Omega(V)$ a real number $\mathsf{p}_V(\omega) \in [0, 1]$ such that $\sum_{\omega \in \Omega(V)} \mathsf{p}_V(\omega) = 1$. The set of all distributions over $\Omega(V)$ is denoted $\mathcal{P}_V$. Moreover, given $\mathsf{p}_V \in \mathcal{P}_V$ and $V' \subset V$, there is an induced distribution $\mathsf{p}_V|_{V'} \in \mathcal{P}_{V'}$ obtained by *marginalizing* p_V:

$$\mathsf{p}_V|_{V'}(\omega') = \sum_{\substack{\omega \in \Omega(V) \\ \omega|_{V'} = \omega'}} \mathsf{p}_V(\omega) \tag{1}$$

Presently, the reader is equipped with sufficient notation and terminology to comprehend the marginal problem.

Definition 1. The Marginal Problem: Given a collection of m distributions $\{\mathsf{p}_{V_1}, \dots, \mathsf{p}_{V_m}\}$, does there exist a distribution $\mathsf{p}_\Lambda \in \mathcal{P}_\Lambda$ with $\Lambda \equiv \bigcup_{i=1}^m V_m$ such that $\forall i : \mathsf{p}_\Lambda|_{V_i} = \mathsf{p}_{V_i}$?

To facilitate further discussion of this problem, several pieces of nomenclature will be introduced. First, the set $\mathcal{V} = \{V_1, \dots, V_m\}$ is called the **marginal scenario** while its elements are called the **marginal contexts**. The collection of distributions $\mathsf{p}_{*\mathcal{V}} \equiv \{\mathsf{p}_{V_1}, \dots, \mathsf{p}_{V_m}\}$[2] is

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[1] Throughout this document, it is assumed that all random variables are discrete and have finite cardinality.

[2] The subscript $*$ preceding ${}_*\mathcal{V}$ is added for clarity; $\mathsf{p}_{*\mathcal{V}}$ is *not* a distribution but a set of distributions over $\mathcal{V}$. The ${}_*\mathcal{V}$ convention is adopted throughout this report.

called the **marginal model** [2][3]. The distribution p_Λ, if it exists, is termed the **joint distribution**. Strictly speaking, as defined by [2], a marginal scenario forms an *abstract simplicial complex*, meaning it satisfies the supplementary requirement that all subsets of contexts are also contexts, i.e. $\forall V \in \mathcal{V} : V' \subset V \implies V' \in \mathcal{V}$. Throughout this work, we exclusively consider (without loss of generality) *maximal* marginal scenarios, restricting our focus to the contexts which are contained in no others. Finally, a marginal model $\mathsf{p}_{*\mathcal{V}}$ is said to be **non-contextual**, and will be denoted $\mathsf{p}_{*\mathcal{V}} \in \mathcal{N} \subseteq \mathcal{P}_{*\mathcal{V}}$ if it admits a joint distribution and **contextual** otherwise ($\mathsf{p}_{*\mathcal{V}} \in \bar{\mathcal{N}}$). Equipped with this additional terminology and notation, the marginal problem now reads: given $\mathsf{p}_{*\mathcal{V}}$, is $\mathsf{p}_{*\mathcal{V}} \in \mathcal{N}$ or not?

B. Linearity

An essential feature of the marginal problem is linearity; the marginalization of p_Λ onto the marginal contexts $\{\mathsf{p}_\Lambda|_V \mid V \in \mathcal{V}\}$ is a linear transformation, requiring only the summations pursuant to Eq. (1). Consequently, it is advantageous to consider the statement of the marginal problem as a matrix multiplication. To this end, for each marginal scenario $\mathcal{V}$ we define a binary matrix $\mathcal{M}$ called the **incidence matrix** which implements this mapping. The columns of $\mathcal{M}$ are indexed by *joint events* $j \in \Omega(\Lambda)$ and the rows are indexed by *marginal events* $m \in \Omega(V)$ for some $V \in \mathcal{V}$. By deliberate abuse of notation, we will denote the set of all marginal events as $\Omega(*\mathcal{V})$ and is defined as the following disjoint union:

$$\Omega(*\mathcal{V}) \equiv \coprod_{V \in \mathcal{V}} \Omega(V)$$

The $|\Omega(*\mathcal{V})| \times |\Omega(\Lambda)|$ incidence matrix $\mathcal{M}$ is then defined element-wise for $m \in \Omega(*\mathcal{V})$ and $j \in \Omega(\Lambda)$:

$$\mathcal{M}_j^m = \begin{cases} 1 & j|_{\mathcal{D}(m)} = m \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

Conceptually, the entries of this matrix are populated with ones whenever the marginal event (row) m is the restriction of some joint event (column) j. For a given marginal scenario $\mathcal{V}$, $\mathcal{M}$ represents the tuple of restriction maps $\mathcal{M} : \Omega(\Lambda) \to \prod_{V \in \mathcal{V}} \Omega(V) :: j \mapsto \{j|_V \mid V \in \mathcal{V}\}$ [1]. Furthermore, note that the component indices of $\mathcal{M}$ in Eq. (2) are deliberately separated. Among other reasons, this is done to allow one to denote the m-th row of $\mathcal{M}$ as $\mathcal{M}^m$ and the j-th column as $\mathcal{M}_j$. For further notational convenience, since $\mathcal{M}$ is a binary matrix, we let $\mathcal{M}^m$ and $\mathcal{M}_j$ analogously correspond their respective *supports*[4], e.g. $m \in \sigma(\mathcal{M}_j)$ if and only if $\mathcal{M}_j^m = 1$. Throughout the remainder of this report, the utility of the incidence matrix $\mathcal{M}$ will be indispensable.

To illustrate this concretely, consider the following example. Let Λ be 3 binary variables $\{a, b, c\}$ and $\mathcal{V}$ be the marginal scenario $\mathcal{V} = \{\{a, b\}, \{b, c\}, \{a, c\}\}$. The incidence matrix for $\mathcal{V}$ becomes:

$$\begin{array}{c} (a,b,c) \mapsto \\ (a{\mapsto}0,b{\mapsto}0) \\ (a{\mapsto}0,b{\mapsto}1) \\ (a{\mapsto}1,b{\mapsto}0) \\ (a{\mapsto}1,b{\mapsto}1) \\ (b{\mapsto}0,c{\mapsto}0) \\ (b{\mapsto}0,c{\mapsto}1) \\ (b{\mapsto}1,c{\mapsto}0) \\ (b{\mapsto}1,c{\mapsto}1) \\ (a{\mapsto}0,c{\mapsto}0) \\ (a{\mapsto}0,c{\mapsto}1) \\ (a{\mapsto}1,c{\mapsto}0) \\ (a{\mapsto}1,c{\mapsto}1) \end{array} \begin{array}{c} \begin{array}{cccccccc} (0,0,0) & (0,0,1) & (0,1,0) & (0,1,1) & (1,0,0) & (1,0,1) & (1,1,0) & (1,1,1) \end{array} \\ \begin{pmatrix} \mathbf{1} & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{1} & \mathbf{1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{1} & \mathbf{1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 \\ 0 & \mathbf{1} & 0 & 0 & 0 & \mathbf{1} & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 & 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & \mathbf{1} \\ \mathbf{1} & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{1} & 0 & \mathbf{1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{1} & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & \mathbf{1} \end{pmatrix} \end{array} \tag{3}$$

In addition, for any joint distribution $\mathsf{p}_\Lambda \in \mathcal{P}_\Lambda$ we associate a joint distribution *vector* p_Λ (identically denoted) indexed by $j \in \Omega(\Lambda)$, i.e. $\mathsf{p}_\Lambda^j \equiv \mathsf{p}_\Lambda(j)$. Analogously, for each marginal model $\mathsf{p}_{*\mathcal{V}} \in \mathcal{P}_{*\mathcal{V}}$ there is an associated marginal distribution *vector* $\mathsf{p}_{*\mathcal{V}}$ indexed by $m \in \Omega(*\mathcal{V})$ such that $\mathsf{p}_{*\mathcal{V}}^m \equiv \mathsf{p}_{\mathcal{D}(m)}(m)$. Using these vectors, the marginal problem becomes the following linear program:

Definition 2. **The Marginal Linear Program (MLP)**:

$$\begin{aligned} \text{minimize:} \quad & \emptyset \cdot \mathsf{p}_\Lambda{}^{5} \\ \text{subject to:} \quad & \mathsf{p}_\Lambda \succeq 0 \\ & \mathcal{M} \cdot \mathsf{p}_\Lambda = \mathsf{p}_{*\mathcal{V}} \end{aligned} \tag{4}$$

As such, $\mathsf{p}_{*\mathcal{V}} \in \mathcal{N}$ if and only if MLP is a *feasible* linear program. Importantly, if MLP is feasible, it will return the joint distribution p_Λ. To every linear program, there exists a dual linear program that characterizes the feasibility of the original [3]. Constructing the dual linear program is a well-defined procedure [4].

Definition 3. **The Dual Marginal Linear Program (DMLP)**:

$$\begin{aligned} \text{minimize:} \quad & \gamma \cdot \mathsf{p}_{*\mathcal{V}} \\ \text{subject to:} \quad & \gamma \cdot \mathcal{M} \succeq 0 \end{aligned}$$

By construction, DMLP completely determines the whether or not $\mathsf{p}_{*\mathcal{V}} \in \mathcal{N}$ or not. If $\mathsf{p}_{*\mathcal{V}} \in \mathcal{N}$, then MLP is feasible and the following holds,

$$\gamma \cdot \mathsf{p}_{*\mathcal{V}} = \gamma \cdot (\mathcal{M} \cdot \mathsf{p}_\Lambda) \geq 0 \tag{5}$$

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[3] In [1], $\mathsf{p}_{*\mathcal{V}}$ is instead called an *empirical model*.

[4] The *support* $\sigma(f)$ of a mapping f is the subset of its domain $\mathcal{D}(f)$ that is not mapped to a zero element: $\sigma(f) = \{x \in \mathcal{D}(f) \mid f(x) \neq 0\}$.

[5] Note that the primal value of the linear program is of no interest, all that matters is its *feasibility*. Here $\emptyset$ denotes a null vector of all zero entries.

because both $\gamma\cdot\mathcal{M} \succeq 0$ and $\mathsf{p}_\Lambda \succeq 0$. If however, $\gamma\cdot\mathsf{p}_{*\mathcal{V}} < 0$, then Eq. (5) is violated and $\mathsf{p}_{*\mathcal{V}} \notin \mathcal{N}$[6]. In summary, the sign of $d \equiv \min(\gamma \cdot \mathsf{p}_{*\mathcal{V}})$ answers the marginal program; $\mathsf{p}_{*\mathcal{V}} \in \mathcal{N}$ if and only if $d \geq 0$[7].

Corollary 1. *All linear, homogeneous constraints $\gamma \cdot \mathsf{p}_{*\mathcal{V}} \geq 0$ constraining non-contextual marginal models $\mathcal{N} \subseteq \mathcal{P}_{*\mathcal{V}}$ satisfy $\gamma\cdot\mathcal{M} \succeq 0$. Moreover, all vectors γ satisfying $\gamma\cdot\mathcal{M} \succeq 0$ correspond to valid constraints $\gamma\cdot\mathsf{p}_{*\mathcal{V}} \geq 0$ for $\mathcal{N} \subseteq \mathcal{P}_{*\mathcal{V}}$.*

In light of Definitions 2 and 3, when supplied with a particular marginal model $\mathsf{p}_{*\mathcal{V}}$, the marginal problem can be solved computationally by evaluating DMLP to determine the feasibility of MLP. A more difficult variant of the marginal problem is one wherein no particular marginal model is supplied.

Definition 4. The General Marginal Problem (GMP): Given a marginal scenario $\mathcal{V}$ find a set of independent constraints Γ which completely constraint $\mathcal{N} \subseteq \mathcal{P}_{*\mathcal{V}}$; i.e. $\mathsf{p}_{*\mathcal{V}} \in \mathcal{N}$ if and only if it satisfies all constraints in Γ.

The remainder of this paper is concerned with methods for solving (or partially solving) GMP. Specifically, Section II C discusses existing methods for completely solving GMP and outlines some of their disadvantages. Section II D summarizes an existing method for completely solving a possibilistic variant of GMP. Sections II C and II D motivate Section III, wherein a new method for completely solving GMP is presented.

C. Marginal Polytopes

The complete space of marginal models over $\mathcal{V}$ (denoted $\mathcal{P}_{*\mathcal{V}}$) can be partitioned into two spaces: the contextual marginal models ($\bar{\mathcal{N}}$) and the non-contextual marginal models ($\mathcal{N} \equiv \mathcal{P}_{*\mathcal{V}} \setminus \bar{\mathcal{N}}$). Pitowsky [7] demonstrates that $\mathcal{N}$ forms a *convex* polytope commonly referred to as the **marginal polytope** for $\mathcal{V}$. When embedded in $\mathbb{R}^{|\Omega(_*\mathcal{V})|}$, the extremal rays of the marginal polytope correspond to the columns of $\mathcal{M}$ which further correspond to all *deterministic* joint distributions $\mathsf{p}_\Lambda \in \mathcal{P}_\Lambda$[8]. The normalization of p_Λ ($\sum_j \mathsf{p}_\Lambda^j = 1$) defines the convexity of the polytope; each marginal model $\mathsf{p}_{*\mathcal{V}} \in \mathcal{P}_{*\mathcal{V}}$ must be a convex mixture of the deterministic marginal models pursuant to Eq. (4). Consequently, characterizing the contextuality of marginal models is manifestly a problem of polytope description. Notably, the **facets** of a marginal polytope correspond to a finite set of linear inequalities that are complete in the sense that all contextual distributions violate at least one facet inequality [8]. From the perspective of a marginal polytope, convex hull algorithms or linear quantifier elimination can be used to compute a representation of the complete set of facet inequalities and consequently completely solve the GMP. A popular tool for linear quantifier elimination is *Fourier-Motzkin elimination* [9–12]. In this report, we will avoid expounding upon the Fourier-Motzkin procedure and instead recall a few of its notable features and consequences.[9]

Definition 5. [3, Section 12.2] Given a system of linear inequality constraints $\mathcal{S} = \{A \cdot x \leq b\}$ constraining some free variables x, the **Fourier-Motzkin elimination** procedure eliminates some of the variables in x and returns a system of linear inequality constraints $\mathcal{S}' = \{A' \cdot x' \leq b'\}$ over $x' \subset x$ such that any solution x' of $\mathcal{S}'$ will permit at least one compatible solution x of $\mathcal{S}$ (and vice versa).

$$\exists x' : A' \cdot x' \leq b' \iff \exists x : A \cdot x \leq b \tag{6}$$

In particular, the following system of linear inequalities defines the marginal polytope for $\mathcal{V}$:

$$\begin{aligned}
\forall m \in \Omega(_*\mathcal{V}) : \quad & \mathsf{p}_{*\mathcal{V}}^m - \sum\nolimits_j \mathcal{M}_j^m \mathsf{p}_\Lambda^j \geq 0 \\
\forall m \in \Omega(_*\mathcal{V}) : \quad & -\mathsf{p}_{*\mathcal{V}}^m + \sum\nolimits_j \mathcal{M}_j^m \mathsf{p}_\Lambda^j \geq 0 \\
\forall j \in \Omega(\Lambda) : \quad & \mathsf{p}_\Lambda^j \geq 0 \\
& \sum\nolimits_j \mathsf{p}_\Lambda^j \geq 1 \\
& -\sum\nolimits_j \mathsf{p}_\Lambda^j \geq -1
\end{aligned} \tag{7}$$

Using the Fourier-Motzkin elimination procedure, it is possible to eliminate the variables p_Λ^j relating to joint events and obtain a system of linear inequalities constraining only marginal events $\mathsf{p}_{*\mathcal{V}}^m$ which completely characterizes the set of non-contextual marginal models $\mathcal{N} \subseteq \mathcal{P}_{*\mathcal{V}}$.

Lemma 1. [10] *There exists a finite set of integral vectors Γ such that for all $\mathsf{p}_{*\mathcal{V}} \in \mathcal{P}_{*\mathcal{V}}$:*

$$\mathsf{p}_{*\mathcal{V}} \in \mathcal{N} \iff \forall \gamma \in \Gamma : \gamma \cdot \mathsf{p}_{*\mathcal{V}} \geq 0 \tag{8}$$

Proof. The finiteness and existence of Γ is a fundamental property of polytopes [9, 11–14]. The fact that each vector $\gamma \in \Gamma$ need only be integer-valued follows from

[6] These observations are collectively a consequence of Farkas's Lemma [5].

[7] In particular, if $d \geq 0$, then $d = 0$ due to the existence of the trivial solution $\gamma = \emptyset$. This observation is an instance of the *Complementary Slackness Property* [6]. Alternatively, if $d < 0$, then it is unbounded $d = -\infty$ due to the *Unbounded Property* [6].

[8] A deterministic distribution p_Λ is a distribution in which a singular event $j \in \Omega(\Lambda)$ occurs with certainty, i.e. $\mathsf{p}_\Lambda^j = 1$ and $\forall j' \neq j : \mathsf{p}_\Lambda^{j'} = 0$.

[9] Applying the Fourier-Motzkin procedure to completely solve GMP is discussed in more detail in Fritz and Chaves [2].

[10] This is a stronger variant of [11, Proposition 7].

the integer-valued coefficients that constrain Eq. (7). Finally, the homogeneity of the constraints in Eq. (8) follows from the assumption that each $\mathsf{p}_{*\mathcal{V}} \in \mathcal{P}_{*\mathcal{V}}$ a priori satisfies normalization constraints context-wise; i.e. $\forall V \in \mathcal{V} : \sum_{m\in\Omega(V)} \mathsf{p}_V^m = 1$[11]. □

D. Logical Contextuality

Let $a \in \Omega(_*\mathcal{V})$ be *any* marginal event and $C = \{c_1, \ldots, c_n\} \subseteq \Omega(_*\mathcal{V})$ be a subset of marginal events such that the following logical implication holds for *all* marginal models $\mathsf{p}_{*\mathcal{V}} \in \mathcal{P}_{*\mathcal{V}}$:

$$a \implies c_1 \vee \cdots \vee c_n = \bigvee_{c\in C} c \tag{9}$$

Which can be dictated: *whenever the event a occurs, at least one event in C occurs.* In accordance with the logical form of Eq. (9), a will be referred to as the **antecedent** and C as the **consequent set**. To clarify, a marginal model $\mathsf{p}_{*\mathcal{V}} \in \mathcal{P}_{*\mathcal{V}}$ satisfies Eq. (9) if there always at least one $c \in C$ that is *possible* ($\mathsf{p}_{*\mathcal{V}}^c > 0$) whenever a is possible. A marginal model violates Eq. (9) whenever *none* of events in c are possible while a remains possible. Marginal models that violate logical statements such as Eq. (9) are known as **Hardy Paradoxes** [10, 15, 16]. Motivated by a greater sense of robustness compared to possibilistic constraints, the concept of witnessing quantum contextuality on a logical level has be analyzed thoroughly for decades [11, 17].

All logical implications of the form of Eq. (9) can be derived by first selecting am antecedent marginal event a, then constructing a consequent set C such that Eq. (9) holds. This is accomplished by making use of Lemma 2.

Lemma 2. *Let $m \in \Omega(_*\mathcal{V})$ be a marginal event. Then for all non-contextual marginal models $\mathsf{p}_{*\mathcal{V}} \in \mathcal{N}$,*

$$m \iff \bigvee_{j\in\sigma(\mathcal{M}^m)} j$$

Essentially, if a joint distribution does exist ($\mathsf{p}_{*\mathcal{V}} \in \mathcal{N}$), then event m represents partial knowledge of the entire system of variables Λ; whenever m occurs, exactly one joint event j has occurred in reality and m must be a restriction of j. Applying Lemma 2 to the antecedent $a \in \Omega(_*\mathcal{V})$ and consequent set $C \subseteq \Omega(_*\mathcal{V})$,

$$\begin{aligned} a &\iff \bigvee_{j\in\sigma(\mathcal{M}^a)} j \\ \bigvee_{c\in C} c &\iff \bigvee_{c\in C}\bigvee_{j\in\sigma(\mathcal{M}^c)} j \end{aligned} \tag{10}$$

Therefore, if a subset C of $\Omega(_*\mathcal{V})$ (preferably excluding a) can be found such that,

$$\sigma(\mathcal{M}^a) \subseteq \bigcup_{c\in C} \sigma(\mathcal{M}^c) \tag{11}$$

then Eq. (9) follows from Eqs. (10,11).

It is possible to show that for each logical constraint of the form Eq. (9), there exists a corresponding probabilistic constraint that is tighter. Corollary 2[12] generalizes Lemma 2.

Corollary 2. *Let $m \in \Omega(_*\mathcal{V})$ be a marginal event. Then for all non-contextual marginal models $\mathsf{p}_{*\mathcal{V}} \in \mathcal{N}$,*

$$\mathsf{p}_{*\mathcal{V}}^m = \sum_{j\in\sigma(\mathcal{M}^m)} \mathsf{p}_\Lambda^j$$

III. AN OBSERVATION

A. An Antecedent Hierarchy

B. The Antecedent Hypergraph

Given an antecedent multi-set γ where $\gamma \preceq 0$, we identify the **inhibiting set** of joint events $\mathcal{I}(\gamma) \subseteq \Omega(\Lambda)$ preventing $\gamma \cdot \mathcal{M}$ from being positive semi-definite:

$$\mathcal{I}(\gamma) \equiv \left\{ j \in \Omega(\Lambda) \mid (\gamma\cdot\mathcal{M})_j < 0 \right\}$$

The inhibiting set $\mathcal{I}(\gamma)$ of γ completely characterizes the **antecedent hypergraph** $\mathcal{H}(\gamma)$ whose edges $\mathcal{E}_j$ are indexed by the inhibiting events $j \in \mathcal{I}(\gamma)$. Each edge $\mathcal{E}_j \subseteq \Omega(_*\mathcal{V})$ corresponds to the set of the marginal events $m \in \Omega(_*\mathcal{V})$ which are restrictions of j. Specifically,

$$\begin{aligned} \mathcal{H}(\gamma) &\equiv \{\mathcal{E}_j \mid j \in \mathcal{I}(\gamma)\} \\ \mathcal{E}_j &\equiv \left\{ m \in \Omega(_*\mathcal{V}) \mid m = j|_{\mathcal{D}(m)}, \gamma_m = 0 \right\} \end{aligned}$$

Todo (TC Fraser): Incorporate weights

$$\omega(\gamma) = \left\{ \omega_j = -(\gamma\cdot\mathcal{M})_j \mid j \in \mathcal{I}(\gamma) \right\}$$

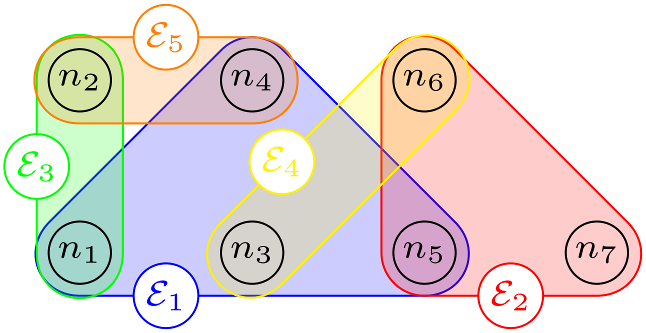
Todo (TC Fraser): Define set of right-minimal inequalities

$$I(\gamma) = \left\{ (\gamma + \tilde{\gamma}) \cdot \mathsf{p}_{*\mathcal{V}} \geq 0 \mid \tilde{\gamma} \in \mathrm{Tr}_{\omega(\gamma)}[\mathcal{H}(\gamma)] \right\}$$

---

[11] Specifically, any inhomogeneous constraint $\gamma \cdot \mathsf{p}_{*\mathcal{V}} \geq \alpha$ can be *homogenized* by replacing α with $\sum_{m\in\Omega(V)} \alpha \mathsf{p}_V^m$.

[12] Corollary 2 is simply the m-th row of Eq. (4).



$$
\begin{array}{c}
 \\ n_1 \\ n_2 \\ n_3 \\ n_4 \\ n_5 \\ n_6 \\ n_7
\end{array}
\begin{array}{c}
\begin{array}{ccccc} \mathcal{E}_1 & \mathcal{E}_2 & \mathcal{E}_3 & \mathcal{E}_4 & \mathcal{E}_5 \end{array} \\
\begin{pmatrix}
\mathbf{1} & 0 & \mathbf{1} & 0 & 0 \\
0 & 0 & \mathbf{1} & 0 & \mathbf{1} \\
\mathbf{1} & 0 & 0 & \mathbf{1} & 0 \\
\mathbf{1} & 0 & 0 & 0 & \mathbf{1} \\
\mathbf{1} & \mathbf{1} & 0 & 0 & 0 \\
0 & \mathbf{1} & 0 & \mathbf{1} & 0 \\
0 & \mathbf{1} & 0 & 0 & 0
\end{pmatrix}
\end{array}
$$

FIG. 1. Dual-representations of a hypergraph $\mathcal{H} = \{\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3, \mathcal{E}_4, \mathcal{E}_5\}$.

C. Irreducibility

D. Marginal Symmetries

E. Curated Inequalities

F. Targeted Searches

G. Relaxations

IV. EDGE-WEIGHTED HYPERGRAPH TRANSVERSALS

A. Preliminaries

B. Hypergraph Transversals

The space of all positive integer multi-sets over the universe of elements U is denoted $\mathbb{Z}^U_{\geq 0}$.

Definition 6. A multi-set $t \in \mathbb{Z}^{\mathcal{N}}_{\geq 0}$ is a **ω-super-transversal of $\mathcal{H}$** if $t \cdot \mathcal{H} \succeq \omega$. A ω-transversal t of $\mathcal{H}$ is **minimal** if there does not exist another ω-transversal t' of $\mathcal{H}$ such that $t' \preceq t$. The set of all minimal ω-transversals of $\mathcal{H}$ will be denoted $\mathrm{Tr}_{\succeq \omega}[\mathcal{H}]$.

$$\mathrm{Tr}_{\succeq \omega}[\mathcal{H}] \equiv \min\left\{t \in \mathbb{Z}^{\mathcal{N}}_{\geq 0} \mid t \cdot \mathcal{H} \succeq \omega\right\}$$

Dual to ω-super-transversals are **ω-sub-transversals**;

$$\mathrm{Tr}_{\preceq \omega}[\mathcal{H}] \equiv \{t \mid t \cdot \mathcal{H} \preceq \omega, \nexists t' \succeq t : t' \cdot \mathcal{H} \preceq \omega\}$$

$$
\begin{aligned}
t \cdot \mathcal{H} \preceq \omega \iff & -t \cdot \mathcal{H} \succeq -\omega \\
\iff & k\mathbf{1}_{\mathcal{N}} \cdot \mathcal{H} - t \cdot \mathcal{H} \succeq k\mathbf{1}_{\mathcal{N}} \cdot \mathcal{H} - \omega \qquad k \equiv \max(\omega) \\
\implies & k\mathbf{1}_{\mathcal{N}} \cdot \mathcal{H} - t \cdot \mathcal{H} \succeq k\mathbf{1}_{\mathcal{N}} \cdot \mathcal{H} - \omega \qquad \mathbf{1}_{\mathcal{N}} \cdot \mathcal{H} \succeq \mathbf{1}_{\mathcal{N}} \cdot \mathcal{H} \\
\implies & (k\mathbf{1}_{\mathcal{N}} - t) \cdot \mathcal{H} \succeq k\mathbf{1}_{\mathcal{N}} \cdot \mathcal{H} - \omega
\end{aligned}
$$

$$
\begin{aligned}
\tilde{t} \in \mathrm{Tr}_{\succeq (k\mathbf{1}_{\mathcal{N}} \cdot \mathcal{H} - \omega)}[\mathcal{H}] \implies & \tilde{t} \cdot \mathcal{H} \succeq k\mathbf{1}_{\mathcal{N}} \cdot \mathcal{H} - \omega \\
\implies & k\mathbf{1}_{\mathcal{N}} \cdot \mathcal{H} - \tilde{t} \cdot \mathcal{H} \preceq \omega \\
\implies & k\mathbf{1}_{\mathcal{N}} \cdot \mathcal{H} - (k\mathbf{1}_{\mathcal{N}} - t) \cdot \mathcal{H} \preceq \omega \\
\implies & k(\mathbf{1}_{\mathcal{N}} \cdot \mathcal{H} - \mathbf{1}_{\mathcal{N}} \cdot \mathcal{H}) + t \cdot \mathcal{H} \preceq \omega \\
\implies & t \cdot \mathcal{H} \preceq \omega
\end{aligned}
$$

If $\tilde{t}$ is minimal, is $t = k\mathbf{1}_{\mathcal{N}} - \tilde{t}$? Suppose $t' \succeq t$ was also a transversal,

$$t' \cdot \mathcal{H} \preceq \omega$$

$$\left(k\mathbf{1}_{\mathcal{N}} - \tilde{t}'\right) \cdot \mathcal{H} \preceq \omega$$

$$k\mathbf{1}_{\mathcal{N}} \cdot \mathcal{H} - \tilde{t}' \cdot \mathcal{H} \preceq \omega$$

C. Adding Weights

D. Unsolved

Consider γ_1, γ_2 and $\gamma_1 + \gamma_2$.

$$I(\gamma_a) = \left\{(\gamma_a + \tilde{\gamma}_a) \cdot \mathsf{p}_{*\mathcal{V}} \geq 0 \mid \tilde{\gamma}_a \in \mathrm{Tr}_{\omega(\gamma_a)}[\mathcal{H}(\gamma_a)]\right\} \quad a = 1, 2$$

$$I(\gamma_1 + \gamma_2) = \left\{(\gamma_1 + \gamma_2 + \tilde{\gamma}_{12}) \cdot \mathsf{p}_{*\mathcal{V}} \geq 0 \mid \tilde{\gamma}_{12} \in \mathrm{Tr}_{\omega(\gamma_1)+\omega(\gamma_2)}[\mathcal{H}(\gamma_1 +$$

V. CONCLUSIONS

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