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Annals of Mathematics, **135** (1992), 527–549

Fiber polytopes

By LOUIS J. BILLERA and BERND STURMFELS*

Dedicated to Victor Klee on the occasion of his 65^{th} birthday

Introduction

If P and Q are convex polytopes such that $Q = \pi(P)$ is a projection of P, the fibers $\pi^{-1}(x)$, $x \in Q$, are again convex polytopes. In this article we construct the *fiber polytope* $\Sigma(P, Q)$, which, in a well-defined sense, is the *average* over all fibers $\pi^{-1}(x)$. This turns out to be a polytope whose combinatorial structure reflects that of P and Q in an interesting way: the faces of $\Sigma(P, Q)$ correspond to certain polyhedral subdivisions of Q induced by the faces of P.

Our results generalize earlier work on *secondary polytopes* by Gel'fand, Kapranov and Zelevinsky ([11], [12]) and others. See [5] for some discussion of the background in this area. Special cases in which Q has dimension one or two have received much attention in homotopy theory ([1], [4], [19]), theoretical computer science [18] and combinatorics [13].

We briefly summarize some important special cases of fiber polytopes.

(a) If $\dim(P) = \dim(Q) + 1$, then $\Sigma(P, Q)$ is a line segment whose two vertices correspond to the complementary subdivisions of Q induced by the "top" and "bottom" of P.

(b) If Q is one dimensional, then $\Sigma(P, Q)$ is a polytope of dimension $\dim(P) - 1$, which we call the *monotone path polytope* of P *in direction* $L :=$ $\text{aff}(Q)$. Its vertices correspond to certain monotone edge paths on P.

(c) Given any polytope Q with n vertices, there is a canonical map of the regular $(n-1)$-simplex Δ_{n-1} onto Q. The fiber polytope $\Sigma(\Delta_{n-1}, Q)$ equals the *secondary polytope*, as defined in [11]. Recall that the vertices of the secondary polytope correspond to the *regular* triangulations of Q.

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(d) Given any d-zonotope $\mathcal{Z}$ with n zones, there is a canonical map π of the regular n-cube $\mathcal{C}_n$ onto $\mathcal{Z}$. Here the fiber polytope $\Sigma(\mathcal{C}_n, \mathcal{Z})$ equals the $(n-d)$-zonotope generated by all elementary vectors in the kernel of π. Its vertices correspond to coherent *cubical subdivisions* of the zonotope $\mathcal{Z}$ (see Figure 1 in §4).

(e) Given any centrally symmetric d-polytope $\mathcal{S}$ with $2n$ vertices, there is a canonical map of the regular n-cross polytope $\mathcal{T}_n$ onto $\mathcal{S}$. The fiber polytope $\Sigma(\mathcal{T}_n, \mathcal{S})$ is centrally symmetric, and its vertices correspond to *centrally regular* triangulations of $\mathcal{S}$.

This paper is organized as follows: In Section 1 we introduce the Minkowski integral of a polytope bundle, which is the continuous analogue to the Minkowski sum of (finitely many) polytopes. We then define the fiber polytope $\Sigma(P, Q)$ as the Minkowski integral of the projection bundle associated with P and Q. In Section 2 we prove that this integral is a convex polytope whose face lattice is isomorphic to the poset of subdivisions of Q *coherently induced* by P. Note that if $\dim(P) = n$ and $\dim(Q) = d$, then almost all fibers $\pi^{-1}(x)$ have dimension $n - d$, and so does the fiber polytope $\Sigma(P, Q)$. In Section 3 we consider the average $\Sigma(P, \partial Q)$ of all fibers over the boundary of Q. This polytope has dimension at most $n - d$, and its faces correspond to coherently induced polyhedral subdivisions of the $(d-1)$-sphere ∂Q. Section 4 deals with the projection of the n-cube onto a d-zonotope, the case (d) above, while Section 5 deals with centrally symmetric polytopes, the case (e) above, and other examples. In Section 6 we use Gale transforms to describe an embedding of all fibers $\pi^{-1}(x)$ and of $\Sigma(P, Q)$ into $\mathbf{R}^{n-d}$. All results of this paper are also valid for point configurations $\mathcal{A}$ with interior and multiple points. This is important because we generally need to keep track of those interior points of Q that are images of vertices of P under π.

1. Polytope bundles and fiber polytopes

Let Q be a polytope in $\mathbf{R}^d$. A *polytope bundle* over the *base polytope* Q is a set-valued function $\mathcal{B} : Q \to 2^{\mathbf{R}^n}$, $x \mapsto \mathcal{B}_x$, such that $\mathcal{B}_x \subset \mathbf{R}^n$ is a nonempty convex polytope for all $x \in Q$, and such that its *graph* $\bigcup\{\mathcal{B}_x \times x \mid x \in Q\}$ is a bounded Borel subset of $\mathbf{R}^{n+d}$. In particular, a polytope bundle is closed, Borel measurable, and integrably bounded in the sense of Aumann [3]. Polytope bundles are special cases of the *fields of polytopes* introduced in the work of Vershik and Chernjakov [20], where the base space Q can be any manifold with boundary.

We are most interested in the following class of bundles: Let π be a linear map taking the polytope $P \subset \mathbf{R}^n$ onto the polytope $Q \subset \mathbf{R}^d$. The

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corresponding *projection bundle* is the map $\mathcal{B} = \pi^{-1}$, which assigns the fiber $\mathcal{B}_x = \pi^{-1}(x)$ to each $x \in Q$. Note that the graph of the bundle π^{-1} is compact, hence Borel.

A *section* of a polytope bundle $\mathcal{B}$ over Q is a (Borel) measurable function $\gamma : Q \to \mathbf{R}^n$ such that $\gamma(x) \in \mathcal{B}_x$ for all $x \in Q$. The set $\Gamma(\mathcal{B})$ of all sections of $\mathcal{B}$ is a convex subset of the real vector space of all measurable functions $Q \to \mathbf{R}^n$. The *Minkowski integral* $\int \mathcal{B}$ of a polytope bundle $\mathcal{B}$ is the subset of $\mathbf{R}^n$ defined by

$$\int \mathcal{B} := \int_Q \mathcal{B}_x dx := \left\{ \int_Q \gamma(x)\, dx \mid \gamma \in \Gamma(\mathcal{B}) \right\}.$$

PROPOSITION 1.1. *The Minkowski integral $\int \mathcal{B}$ of any polytope bundle $\mathcal{B}$ is a nonempty, compact, convex subset of* $\mathbf{R}^n$.

This result is a special case of Theorems 1 and 4 in [3] when the base polytope Q is the line segment $[0, 1]$. For arbitrary base polytopes Q, one can obtain Proposition 1.1 by integrating over segments in Q.

Let $\mathcal{B}$ be a fixed polytope bundle over Q. Another bundle $\mathcal{F}$ over Q is said to be a *face bundle* of $\mathcal{B}$ if $\mathcal{F}_x$ is a face of $\mathcal{B}_x$ for each $x \in Q$. Under which conditions is $\int \mathcal{F}$ a face of $\int \mathcal{B}$? In order to answer this question we need the concept of coherent face bundles. For a convex set $S \subset \mathbf{R}^n$ and a linear functional ψ on $\mathbf{R}^n$ we denote by S^ψ the face of S on which ψ is minimized. In other words, S^ψ is the extreme face of S in direction ψ. A face bundle $\mathcal{F}$ of $\mathcal{B}$ is said to be *coherent* if there exists a linear functional ψ on $\mathbf{R}^n$ such that $\mathcal{F}_x = (\mathcal{B}_x)^\psi$ for each $x \in Q$. So every $\psi \in (\mathbf{R}^n)^*$ defines a coherent face bundle $\mathcal{B}^\psi$ via $(\mathcal{B}^\psi)_x := (\mathcal{B}_x)^\psi$.

PROPOSITION 1.2. *Given a polytope bundle $\mathcal{B}$ on Q, the faces of the compact convex set $\int_Q \mathcal{B}$ are the integrals of coherent face bundles. More precisely,*

$$\int_Q \mathcal{B}^\psi = \left(\int_Q \mathcal{B} \right)^\psi$$

for every linear functional ψ on $\mathbf{R}^n$.

Proof. It is clear from the definitions of the Minkowksi integral and the bundle $\mathcal{B}^\psi$ that $\int \mathcal{B}^\psi \subseteq \left(\int \mathcal{B}\right)^\psi$. To see the opposite inclusion consider a (measurable) section $\gamma \in \Gamma(\mathcal{B})$ such that $\int \gamma(x)dx \notin \int \mathcal{B}^\psi$. There exist a set $\mathcal{U} \subset Q$ of positive measure and an $\epsilon > 0$ such that $\langle \gamma(x), \psi \rangle > \epsilon + \langle \mathcal{B}_x^\psi, \psi \rangle$ for all $x \in \mathcal{U}$. Now take any measurable section γ_ψ of the bundle $\mathcal{B}^\psi$ (one must exist by von Neumann's measurable choice theorem; cf. [3], [16]). Define $\tilde{\gamma} \in \Gamma(\mathcal{B})$ by $\tilde{\gamma}(x) := \gamma_\psi(x)$ for $x \in \mathcal{U}$, and by $\tilde{\gamma}(x) := \gamma(x)$ otherwise. Then we have $\langle \int \tilde{\gamma}(x)dx,\, \psi \rangle < \langle \int \gamma(x)dx,\, \psi \rangle$ and consequently $\int \gamma(x)dx \notin \left(\int \mathcal{B}\right)^\psi$. $\square$

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In order to compute the face in direction ψ of $\int_Q \mathcal{B}$ we need to integrate the faces in direction ψ of the individual polytopes $\mathcal{B}_x$. The resulting face of $\int_Q \mathcal{B}$ is a vertex if and only if, for almost all $x \in Q$, the face of $\mathcal{B}_x$ is a vertex (cf. [17, Satz 3]). It also follows from Proposition 1.2 that the *support function* $h(\int \mathcal{B},\, \psi) = \min\{\langle y, \psi\rangle \mid y \in \int_Q \mathcal{B}\}$ of the Minkowski integral $\int \mathcal{B}$ equals the integral $\int_Q h(\mathcal{B}_x,\, \psi)dx$ of the individual support functions (see also [9], §19).

We say that two polytopes P_1 and P_2 are *normally equivalent* if they have the same normal fan $\mathcal{N}(P_1) = \mathcal{N}(P_2)$. (The *normal fan* of a polytope P is the cell complex consisting of all inner normal cones to faces of P.) A polytope bundle $\mathcal{B}$ is said to be *piecewise linear* if there exists a (finite) polyhedral subdivision of the base polytope Q such that, for any two points x and y in the same cell, the polytopes $\mathcal{B}_x$ and $\mathcal{B}_y$ are normally equivalent. Note that this definition of piecewise-linear $\mathcal{B}$ is very general in the following sense: Within each cell corresponding facets of $\mathcal{B}_x$ have to be parallel, but we allow the parallel displacement to be a *nonlinear* function of x. We get the next result from the additivity of the Minkowski integral and Proposition 1.2.

THEOREM 1.3. *Let $\mathcal{B}$ be a piecewise-linear polytope bundle. Then the Minkowski integral $\int \mathcal{B}$ is a convex polytope in $\mathbf{R}^n$. Moreover there exists a finite subset $\{x_1, x_2, \ldots, x_m\} \subset Q$ such that the Minkowski sum $\mathcal{B}_{x_1} + \mathcal{B}_{x_2} + \cdots + \mathcal{B}_{x_m}$ is normally equivalent to $\int \mathcal{B}$.*

We note that Theorem 1.3 remains true if "polyhedral subdivision" is replaced by "finite-measurable subdivision" in the definition of a piecewise-linear bundle.

Given a piecewise-linear polytope bundle $\mathcal{B}$, the set $\{\mathcal{B}^\psi\}$ of coherent face bundles of $\mathcal{B}$ is finite, and we partially order this set by pointwise inclusion.

COROLLARY 1.4. *The poset of coherent face bundles of a piecewise-linear bundle $\mathcal{B}$ is isomorphic via the map $\mathcal{B}^\psi \mapsto \int_Q \mathcal{B}^\psi$ to the face lattice of the polytope $\int_Q \mathcal{B}$.*

We now consider an arbitrary projection $\pi : P \to Q$ of polytopes. The *fiber polytope* of P over Q is defined as the normalized Minkowski integral $\Sigma(P, Q) := 1/\operatorname{vol}(Q) \int \mathcal{B}$ of the corresponding projection bundle $\mathcal{B} := \pi^{-1}$. The projection bundle $\mathcal{B}$ is piecewise linear with respect to the polyhedral subdivision of Q defined as the common refinement of all the images in Q of faces of P. We write $\sigma_1, \sigma_2, \ldots, \sigma_m$ for the maximal cells of this subdivision. If P is a simplex, then the σ_i are precisely the *chambers* of Q studied in [2], while in general the σ_i are convex unions of chambers.

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THEOREM 1.5. *The fiber polytope* $\Sigma(P, Q)$ *is a convex polytope of dimension* $\dim(P) - \dim(Q)$ *equal to the (finite) Minkowski sum*

$$\frac{\mathrm{vol}(\sigma_1)}{\mathrm{vol}(Q)}\pi^{-1}(x_1) + \frac{\mathrm{vol}(\sigma_2)}{\mathrm{vol}(Q)}\pi^{-1}(x_2) + \cdots + \frac{\mathrm{vol}(\sigma_m)}{\mathrm{vol}(Q)}\pi^{-1}(x_m),$$

where the points $x_i \in \sigma_i$ *are the centroids of their respective cells.*

Proof. If $\mathcal{B}$ is the projection bundle, then we have $\int \mathcal{B} = \int_{\sigma_1} \mathcal{B} + \int_{\sigma_2} \mathcal{B} + \cdots + \int_{\sigma_m} \mathcal{B}$. For all $x \in \sigma_i$ the normal fans of the fibers $\mathcal{B}_x$ are the same. Corresponding facets of these $\mathcal{B}_x$ are translates of each other, where the translation vector is an affine function of x. Thus the average location of each facet of $\mathcal{B}_x$ over σ_i is that of the corresponding facet of the centroid fiber $\mathcal{B}_{x_i}$. Applying Proposition 1.2 when ψ is the facet normal, we find that $\int_{\sigma_i} \mathcal{B} = \mathrm{vol}(\sigma_i)\mathcal{B}_{x_i}$. The dimension statement follows, because for all $x \in \mathrm{int}(Q)$ the fibers $\mathcal{B}_x$ have dimension $\dim(P) - \dim(Q)$ and lie parallel to each other. $\square$

We note that if $\pi : Q \to Q$ is the identity, then the fiber polytope $\Sigma(Q, Q)$ equals the centroid $x_Q := 1/\mathrm{vol}(Q) \int_Q x\, dx$, a zero-dimensional convex polytope. If π is the canonical projection of $P \times Q$ onto Q, then $\Sigma(P \times Q, Q) = \pi^{-1}(x_Q)$ equals the *centroid fiber* and thus is a translate of P. In the special case of $Q = \{q\}$ we get $\Sigma(P, \{q\}) = P$. For a general projection $\pi : P \to Q$ we have $\Sigma(P, Q) \subseteq \pi^{-1}(x_Q) \subseteq P$; i.e., the fiber polytope is always contained in the centroid fiber.

2. Coherent subdivisions and vertex coordinates of fiber polytopes

We first give a combinatorial interpretation of the vertices and all other faces of $\Sigma(P, Q)$ in terms of a decomposition theorem for the base polytope Q. We then consider the fiber polytope $\Sigma(\Delta_{n-1}, Q)$ where $\Delta_{n-1} = \mathrm{conv}\{e_1, \ldots, e_n\} \subset \mathbf{R}^n$ is the standard $(n-1)$-simplex mapping onto a d-polytope Q with n vertices. We show that $\Sigma(\Delta_{n-1}, Q)$ is homothetic to the secondary polytope $\Sigma(Q)$ defined in [11], [12]. This result can be interpreted as an integral representation for secondary polytopes. General fiber polytopes will then have coordinates specified in a natural way via projections of secondaries.

We consider a general projection of polytopes. Let $P := \mathrm{conv}(\mathcal{A}_P)$, where $\mathcal{A}_P = \{p_1, p_2, \ldots, p_m\} \subset \mathbf{R}^n$, and let $Q := \mathrm{conv}(\mathcal{A}_Q)$, where $\mathcal{A}_Q = \{q_1, q_2, \ldots, q_m\} \subset \mathbf{R}^d$. (Both $\mathcal{A}_P$ and $\mathcal{A}_Q$ can be multisets.) Suppose that $\pi : \mathbf{R}^n \to \mathbf{R}^d$ is an affine map with $\pi(p_1) = q_1, \ldots, \pi(p_m) = q_m$ and let $\mathcal{B} = \pi^{-1}$ be the corresponding projection bundle.

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Throughout this paper a *polyhedral subdivision of* Q will mean a polyhedral subdivision of $\mathcal{A}_Q$, as defined in [5], [12]; that is, a collection Π of subsets of $\mathcal{A}_Q$ whose convex hulls form a polyhedral complex that covers Q. A polyhedral subdivision Π of Q is said to be *induced by* π *from* P if each cell $\sigma \in \Pi$ is of the form $\pi(F_\sigma)$ for some face F_σ of P. Since both the cells of Π and the faces of P are regarded as labeled, $\sigma = \pi(F_\sigma)$ uniquely specifies the face F_σ. Note also that $\dim \sigma \leq \dim F_\sigma$. An induced subdivision Π is called *tight* if $\dim \sigma = \dim F_\sigma$ for each d-cell $\sigma \in \Pi$. Note that if P is simplicial (and $\dim Q < \dim P$), then all tight induced subdivisions of Q are triangulations. Conversely, if P is a simplex, then all triangulations of Q are induced.

Let $\mathcal{F}$ be a coherent face bundle of $\mathcal{B}$. We define an induced polyhedral subdivision $\Pi(\mathcal{F})$ of Q as follows: Given $x \in Q$, the face $\mathcal{F}_x$ of $\mathcal{B}_x$ is then contained in a unique minimal face $\widetilde{\mathcal{F}_x}$ of P. (In fact, $\mathcal{F}_x = \mathcal{B}_x \cap \widetilde{\mathcal{F}_x}$.) Then $\pi(\widetilde{\mathcal{F}_x})$ is the cell of $\Pi(\mathcal{F})$ that contains x. This defines a subdivision of Q, since, by the coherence of $\mathcal{F}$, $y \in \operatorname{rel\,int} \pi(\widetilde{\mathcal{F}_x})$ if and only if $\widetilde{\mathcal{F}_y} = \widetilde{\mathcal{F}_x}$. Such an induced subdivision of Q will be called *coherent*. (Here rel int denotes the *relative interior* of a set, i.e., its interior relative to its affine span.)

We partially order the set of coherent subdivisions of Q by refinement. Then the map $\mathcal{F} \mapsto \Pi(\mathcal{F})$ defines anti-isomorphism between the poset of coherent face bundles of $\mathcal{B}$ and the poset of coherent subdivisions of Q. Using Corollary 1.4, we find that this implies the following result:

THEOREM 2.1. *The face lattice of the fiber polytope* $\Sigma(P, Q)$ *is isomorphic to the poset of all coherent subdivisions of* Q. *Here the vertices of* $\Sigma(P, Q)$ *correspond to the tight coherent subdivisions of* Q.

We will next study the behavior of fiber polytopes under an affine map $\theta : \mathbf{R}^m \to \mathbf{R}^n$. Every map of polytopes $\theta : P \to Q$ defines an embedding of the face lattice $\mathbf{F}(Q)$ of Q as a sublattice into the face lattice $\mathbf{F}(P)$ of P. This sublattice is called the *shadow boundary* of Q in P. Here each face of Q is associated to the maximal face of P mapping onto it.

LEMMA 2.2. *The map* $Q^\nu \mapsto P^\psi$, *where* ψ *spans the orthogonal complement to the hyperplane* $\theta^{-1}(\nu^\perp)$ *in* $\mathbf{R}^m$, *defines a lattice isomorphism*

$$\mathbf{F}(Q) \simeq \left\{ P^\psi \in \mathbf{F}(P) \mid \psi \in \ker(\theta)^\perp \right\}.$$

The following functorial property will prove to be very useful in the sequel:

LEMMA 2.3. *Let* $P \xrightarrow{\theta} Q \xrightarrow{\pi} R$ *be surjective maps of polytopes. Then* $\Sigma(Q, R) = \theta(\Sigma(P, R))$.

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Proof. Using the fact that Minkowski integration is linear, we compute

$$\begin{aligned}\Sigma(Q,R) &= \frac{1}{\operatorname{vol}(R)}\int_R \pi^{-1}(x)dx\\ &= \theta\left(\frac{1}{\operatorname{vol}(R)}\int_R (\pi\circ\theta)^{-1}(x)\right)dx\\ &= \theta\big(\Sigma(P,R)\big).\end{aligned}$$

□

These two lemmas now imply the next result.

THEOREM 2.4. *Let* $P \xrightarrow{\theta} Q \xrightarrow{\pi} R$ *be maps of polytopes,* $\mathcal{B} = (\pi\circ\theta)^{-1}$ *be the composite projection bundle and* Π *be a subdivision of* R. *Then the following are equivalent:*

(1) Π *is coherent with respect to* π;

(2) Π *is coherent with respect to* $\pi\circ\theta$, *and it equals* $\Pi(\mathcal{B}^\psi)$ *for some* $\psi \in \ker(\theta)^\perp \subset \mathbf{R}^m$.

Proof. By Lemma 2.3 we get θ mapping $\Sigma(P,R)$ onto $\Sigma(Q,R)$. Thus by Theorem 2.1 and Lemma 2.2 the set of Q-coherent subdivisions of R is isomorphic to $\mathbf{F}(\Sigma(Q,R)) \simeq \{\Sigma(P,R)^\psi \in \mathbf{F}(\Sigma(P,R)) \mid \psi \in \ker(\theta)^\perp\}$. But by Theorem 2.1 again, this is isomorphic to the set of P-coherent subdivisions of R defined by $\psi \in \ker(\theta)^\perp$. □

We now relate our discussion to the results in [5], [11], [12]. Adapting the notation used in those articles, let $Q = \operatorname{conv}\mathcal{A} \subset \mathbf{R}^d$ be a d-polytope, where $\mathcal{A} = \{a_1,\dots,a_n\}$ is a multiset of points in $\mathbf{R}^d$. Recall that the secondary polytope $\Sigma(\mathcal{A}) = \Sigma(Q)$ has dimension $n-d-1$ and can be embedded into $\mathbf{R}^n$ as follows: Given any $(d+1)$-set $\tau = \{\tau_1,\tau_2,\dots,\tau_{d+1}\} \subset \{1,2,\dots,n\}$, we abbreviate $\mathcal{A}_\tau := \operatorname{conv}\{a_{\tau_1}, a_{\tau_2}\dots, a_{\tau_{d+1}}\}$ and write $\operatorname{vol}(\tau) := \operatorname{vol}(\mathcal{A}_\tau)$. The secondary polytope $\Sigma(Q)$ is the convex hull in $\mathbf{R}^n$ of the vectors $\phi_\Delta := \sum_{\tau\in\Delta}\operatorname{vol}(\tau)(e_{\tau_1}+e_{\tau_2}+\cdots+e_{\tau_{d+1}})$ as Δ ranges over all triangulations of Q. Consider the canonical map $\pi : \Delta_{n-1} \to Q$ and let $\mathcal{B}$ be its projection bundle.

THEOREM 2.5. *The secondary polytope* $\Sigma(Q)$ *equals the scaled fiber polytope* $(d+1)\operatorname{vol}(Q)\Sigma(\Delta_{n-1},Q)$.

Proof. In the following we fix a vertex ϕ_Δ of the secondary polytope $\Sigma(Q) \subset \mathbf{R}^n$ and pick any vector $\psi = (\psi_1,\psi_2,\dots,\psi_n)$ in its open normal cone. Here Δ is the regular triangulation of Q one obtains by assigning the "heights" $\psi_1,\psi_2,\dots,\psi_n$ to the vertices of the base polytope Q. We will see that the face $(\int\mathcal{B})^\psi$ of the fiber polytope is a vertex and then verify that $(d+1)\,(\int\mathcal{B})^\psi = \phi_\Delta$.

Let $\sigma_1,\sigma_2,\dots,\sigma_m$ be the open maximal cells in the polyhedral subdivision of Q obtained by the intersecting of all full-dimensional simplices $\mathcal{A}_\tau$ (the chamber complex of [2]); also let Ω_j denote the collection of all $(d+1)$-sets

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τ with $\sigma_j \subset \mathcal{A}_\tau$. We write $L_\tau^{\tau_i}(x)$ for the unique affine functional on $\mathbf{R}^d$ with $L_\tau^{\tau_i}(a_{\tau_j}) = \delta_{ij}$ (the Kronecker delta). Fix $j \in \{1, 2, \ldots, m\}$ and $x \in \sigma_j$. The fiber

$$\pi^{-1}(x) = \{\lambda \in \Delta_{n-1} \mid \lambda_1 a_1 + \lambda_2 a_2 + \cdots + \lambda_n a_n = x\} \tag{2.1}$$

is the set of convex representations of x with respect to $\mathcal{A}$. The vertices of $\pi^{-1}(x)$ are the convex representations with minimal support; these are the vectors $\sum_{i=1}^{d+1} L_\tau^{\tau_i}(x) e_{\tau_i}$, where $\tau \in \Omega_j$. The *unique* simplex $\tau \in \Omega_j$ that is contained in the triangulation Δ is characterized by the property that $\sum_{i=1}^{d+1} L_\tau^{\tau_i}(x)\psi_{\tau_i}$ is minimal. So the face of $\pi^{-1}(x)$ in direction ψ is the vertex

$$\pi^{-1}(x)^\psi = \sum_{i=1}^{d+1} L_\tau^{\tau_i}(x) e_{\tau_i}. \tag{2.2}$$

Using Proposition 1.2, we find that

$$\left(\int \mathcal{B}\right)^\psi = \int_Q \pi^{-1}(x)^\psi dx = \sum_{j=1}^{m} \int_{\sigma_j} \pi^{-1}(x)^\psi dx \tag{2.3}$$

is the unique extreme vertex in direction ψ. This expression is equal to

$$\begin{aligned} \sum_{\tau\in\Delta}\sum_{j:\tau\in\Omega_j} \int_{\sigma_j} \pi^{-1}(x)^\psi\, dx &= \sum_{\tau\in\Delta} \int_{\operatorname{conv}(\mathcal{A}_\tau)} \left(\sum_{i=1}^{d+1} L_\tau^{\tau_i}(x) e_{\tau_i}\right) dx \\ &= \frac{1}{d+1} \sum_{\tau\in\Delta} \operatorname{vol}(\tau) \sum_{i=1}^{d+1} e_{\tau_i}. \end{aligned} \tag{2.4}$$

This completes the proof of Theorem 2.5. □

We return to the general situation $P = \operatorname{conv}(\mathcal{A}_P)$, $Q = \operatorname{conv}(\mathcal{A}_Q)$ and $\pi : P \to Q$. It follows from Theorems 2.4 and 2.5 that every coherent subdivision Π of Q is regular, in the sense of [5] and [12]. To see this let $\theta : \Delta_{m-1} \to P$ be the canonical projection of the standard $(m-1)$-simplex onto P. If Π is coherent with respect to $\pi : P \to Q$, then, by Theorem 2.4, Π is coherent with respect to $\pi \circ \theta : \Delta_{m-1} \to Q$. The latter condition is equivalent to Π being regular.

In order to compute the fiber polytope $\Sigma(P, Q)$ we use the representation $\Sigma(P, Q) = \theta\big(\Sigma(\Delta_{m-1}, Q)\big)$, given in Lemma 2.3. This implies the following description of fiber polytopes:

COROLLARY 2.6. $\Sigma(P, Q) = \operatorname{conv}\{\Phi_\Delta \mid \Delta$ *a triangulation of* $Q\}$, *where*

$$\Phi_\Delta = \frac{1}{d+1} \sum_{\tau\in\Delta} \frac{\operatorname{vol}(\tau)}{\operatorname{vol}(Q)} \big(p_{\tau_1} + p_{\tau_2} + \cdots + p_{\tau_{d+1}}\big), \tag{2.5}$$

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and where $\mathrm{vol}(\tau)$ *denotes the volume of the d-simplex* $(q_{\tau_1}, q_{\tau_2}, \ldots, q_{\tau_{d+1}})$ *indexed by* τ.

Let Π be a tight induced subdivision of Q. Then any triangulation Δ of Π gives rise to the same vector $\Phi_\Delta \in \mathbf{R}^n$ in equation (2.5). We denote this vector by Φ_Π.

COROLLARY 2.7. *The vertices of the fiber polytope* $\Sigma(P, Q)$ *are the vectors* Φ_Π, *where* Π *ranges over all coherent tight subdivisions of* Q.

We note, finally, that the techniques of this paper have been extended to study the set of triangulations of the $(d-1)$-sphere having a prescribed vertex set and to describe the secondary polyhedra of minors of a configuration $\mathcal{A}$ in terms of the polyhedron $\Sigma(\mathcal{A})$; cf. [6].

3. Boundary fibers

Let $\mathcal{B}$ be a polytope bundle over a d-dimensional polytope Q. Given any i-dimensional polyhedral complex R contained in Q, we can define the Minkowski integral $\int_R \mathcal{B}$ with respect to the i-dimensional Lebesgue measure. In particular, we can form the $(d-1)$-dimensional integral $\int_{\partial Q} \mathcal{B}$ over the boundary of the base polytope Q or, more generally, form the i-dimensional integral $\int_{\partial_i Q} \mathcal{B}$ over its i-skeleton $\partial_i Q$.

In the sequel let $\mathcal{B} = \pi^{-1}$ be the projection bundle of a polytope map $\pi : P \to Q$. Here the Minkowski integral

$$\Sigma(P, \partial_i Q) := \frac{1}{\mathrm{vol}(\partial_i Q)} \int_{\partial_i Q} \mathcal{B}$$

will be called the i^{th} *fiber polytope*; and in the case $i = d-1$ we get the *boundary fiber polytope* $\Sigma(P, \partial Q)$. A polytope P_1 is called a *Minkowski summand* of a polytope P if there is a polytope P_2 such that, for some $\lambda > 0$, $P_1 + P_2 = \lambda P$.

PROPOSITION 3.1. *The boundary fiber polytope* $\Sigma(P, \partial Q)$ *is a Minkowski summand of the fiber polytope* $\Sigma(P, Q)$.

Proof. The projection bundle $\mathcal{B}$ is piecewise linear with respect to a decomposition of Q into d-cells $\sigma_1, \sigma_2, \ldots, \sigma_m$. Let F be a facet of Q and suppose that $\sigma_1' := \sigma_1 \cap F, \ldots, \sigma_l' := \sigma_l \cap F$ are the $(d-1)$-cells in the associated subdivision of F. Since $\int_F \mathcal{B} = \sum_{i=1}^{l} \int_{\sigma_i'} \mathcal{B}$ and $\Sigma(P, Q) = \sum_{i=1}^{m} \int_{\sigma_i} \mathcal{B}$, it is enough to show that $\int_{\sigma_i'} \mathcal{B}$ is a Minkowski summand of $\int_{\sigma_i} \mathcal{B}$ for $i = 1, \ldots, l$. But this follows, since, for any $x \in \mathrm{int}\,\sigma_i$ and $y \in \mathrm{rel\,int}\,\sigma_i'$, the normal fan of $\mathcal{B}_x$ is a refinement of that of $\mathcal{B}_y$ (see, e.g., [5], Proposition 1.2.3). $\square$

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Every polyhedral subdivision Π of Q defines a polyhedral subdivision $\partial_i\Pi$ of the i-skeleton $\partial_i Q$. We say that polyhedral subdivision Ω of $\partial_i Q$ is *induced* (resp. *tight*, *coherent*) if there exists an induced (resp. tight, coherent) subdivision Π of Q such that $\Omega = \partial_i\Pi$.

THEOREM 3.2. *The poset of coherent polyhedral subdivisions of the i-skeleton $\partial_i Q$ is isomorphic to the face lattice of the i^{th} fiber polytope $\Sigma(P, \partial_i Q)$.*

Proof. We identify two coherent face bundles of $\mathcal{B}$ if they agree on $\partial_i Q$, thus defining an equivalence relation on the set $\{\mathcal{B}^\psi\}$ of coherent face bundles. Two bundles are equivalent if and only if they induce the same coherent subdivision Ω of $\partial_i Q$. Hence the poset of coherent subdivisions of $\partial_i Q$ is isomorphic to the poset of equivalence classes of coherent face bundles of $\mathcal{B}$ (with the obvious order). We can now apply Corollary 1.4 to the i-faces of Q in order to complete the proof of Theorem 3.2. □

Consider the case where $\mathcal{A} = \{a_1, \ldots, a_n\} \subset \mathbf{R}^d$ and π is the canonical map of the $(n-1)$-simplex Δ_{n-1} onto $Q = \operatorname{conv}(\mathcal{A})$. Writing ∂Q for the boundary of this polytope yields the following corollary from Theorem 3.2:

COROLLARY 3.3. *The vertices of the boundary fiber polytope $\Sigma(\Delta_{n-1}, \partial Q)$ are in one-to-one correspondence with the regular (i.e., coherent) triangulations of the boundary ∂Q.*

The polytope $\Sigma(\Delta_{n-1}, \partial Q)$ has an interesting interpretation in the theory of $\mathcal{A}$-discriminants, which is due to Gel'fand, Kapranov and Zelevinsky. Suppose that $\mathcal{A} \subset \mathbf{Z}^d$ is a set of lattice points. As in [12], let $D_\mathcal{A}$ be the *regular $\mathcal{A}$-determinant* and let $E_\mathcal{A}$ be the *principal $\mathcal{A}$-determinant* which is a multiple of $D_\mathcal{A}$. The Newton polytope $M(E_\mathcal{A})$ of $E_\mathcal{A}$ equals the secondary polytope of $\mathcal{A}$ ([12], Thm. 3A2) and is thus homothetic to the fiber polytope $\Sigma(\Delta_{n-1}, Q)$. After describing the Newton polytope $M(D_\mathcal{A})$ of the regular $\mathcal{A}$-determinant in [12], Thm. 3D2, the authors remark that "$M(E_\mathcal{A})$ is the Minkowski sum of $M(D_\mathcal{A})$ and some other polyhedron." There is good reason to believe that this "missing" polytope $M(D_\mathcal{A}/E_\mathcal{A})$ is homothetic to the boundary fiber polytope $\Sigma(\Delta_{n-1}, \partial Q)$. We illustrate this conjecture for the classical discriminant of the ternary quadric.

Example 3.4. Consider the canonical map π of the 5-simplex $\Delta_5 = \{e_1, \ldots, e_6\} \subset \mathbf{R}^6$ onto the triangle $Q = \operatorname{conv}(\mathcal{A})$, where $\mathcal{A} = \{(2,0,0), (1,1,0), (0,2,0), (0,1,1), (0,0,2), (1,0,1)\}$. The average fiber over the edge $F_{12} := [(2,0,0), (0,2,0)] = \pi(\operatorname{conv}\{e_1, e_2, e_3\})$ equals the segment

$$\frac{1}{\operatorname{vol}(F_{12})} \int_{F_{12}} \pi^{-1}(x)dx = \left[\left(\frac{1}{2}, 0, \frac{1}{2}, 0, 0, 0\right), \left(\frac{1}{4}, \frac{1}{2}, \frac{1}{4}, 0, 0, 0\right)\right].$$

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The average fibers over the other two edges of Q are $\big[\big(0,0,\frac{1}{2},0,\frac{1}{2},0\big),\ \big(0,0,\frac{1}{4},\frac{1}{2},\frac{1}{4},0\big)\big]$ and $\big[\big(\frac{1}{2},0,0,0,\frac{1}{2},0\big),\ \big(\frac{1}{4},0,0,0,\frac{1}{4},\frac{1}{2}\big)\big]$. These three segments are linearly independent in $\mathbf{R}^6$ and, therefore, their Minkowski sum, the scaled boundary fiber polytope $3\Sigma(\Delta_5,\partial Q)$, is a three-dimensional cube. More precisely we find that

$$\begin{aligned}12\Sigma(\Delta_5,\partial Q) = \operatorname{conv}\big\{&(4,0,4,0,4,0),(3,0,4,0,3,2),(4,0,3,2,3,0),\\ &(3,0,3,2,2,2),(3,2,3,0,4,0),(2,2,3,0,3,2),\\ &(3,2,2,2,3,0),(2,2,2,2,2,2)\big\}.\end{aligned}$$

The vertices of this 3-cube correspond to the eight triangulations of ∂Q.

In this example, the set $\mathcal{A}$ consists of the exponent vectors of a ternary quadratic form

$$f(x_1,x_2,x_3) = a_1x_1^2 + 2a_2x_1x_2 + a_3x_2^2 + 2a_4x_2x_3 + a_5x_3^2 + 2a_6x_1x_3,$$

and the regular $\mathcal{A}$-determinant equals the usual discriminant of $f(x_1,x_2,x_3)$:

$$D_{\mathcal{A}} = \det\begin{pmatrix} a_1 & a_2 & a_6\\ a_2 & a_3 & a_4\\ a_6 & a_4 & a_5\end{pmatrix}.$$

Its Newton polytope $M(D_{\mathcal{A}})$ is a bipyramid. The principal $\mathcal{A}$-determinant equals

$$E_{\mathcal{A}} = D_{\mathcal{A}}\det\begin{pmatrix} a_1 & a_2\\ a_2 & a_3\end{pmatrix}\det\begin{pmatrix} a_3 & a_4\\ a_4 & a_5\end{pmatrix}\det\begin{pmatrix} a_5 & a_6\\ a_6 & a_1\end{pmatrix} a_1a_3a_5.$$

Its Newton polytope $M(E_{\mathcal{A}})$ equals the scaled fiber polytope $12\Sigma(\Delta_5,\mathcal{A})$. This 3-polytope is the Minkowski sum of the bipyramid $M(D_{\mathcal{A}})$ and the 3-cube

$$12\Sigma(\Delta_5,\partial Q) - (1,0,1,0,1,0). \qquad \square$$

We end this section by showing that the $(d-1)$-dimensional integral $\int_{\partial Q}\mathcal{B}$ can be expressed as a d-dimensional integral over Q. To this end, for any polytope bundle $\mathcal{B}$ over ∂Q, we define a new bundle $\mathrm{d}\mathcal{B}$ over Q as follows: Let x_Q be the centroid of Q. Given $y \in Q\setminus\{x_Q\}$, let $p_y \in \partial Q$ be the unique boundary point on the halfline emanating from x_Q through y and define $\lambda(y)\in(0,1]$ by $y = \lambda(y)p_y + (1-\lambda(y))x_Q$. Now define $(\mathrm{d}\mathcal{B})_y := (d+1)\lambda(y)\mathcal{B}_{p_y}$. We get the following formal "Stokes Theorem" for polytope bundles:

THEOREM 3.5. *For any polytope bundle $\mathcal{B}$ over ∂Q,*

$$\int_{\partial Q}\mathcal{B} = \int_Q \mathrm{d}\mathcal{B}.$$

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Proof. Let F be any facet of Q and let ψ be a linear functional on $\mathbf{R}^n$. By Proposition 1.2 and the definition of $\mathrm{d}\mathcal{B}$, the face $\left(\int_{\partial Q}\mathcal{B}\right)^{\psi}$ is a vertex if and only $\left(\int_Q \mathrm{d}\mathcal{B}\right)^{\psi}$ is a vertex. For such a ψ compute

$$\int_{\mathrm{conv}(x_Q,F)} (\mathrm{d}\mathcal{B})^{\psi} = \int_0^1 \lambda^d \int_F (d+1)\mathcal{B}_y^{\psi}\, dy\, d\lambda = \int_F \mathcal{B}^{\psi}.$$

Thus $\int_F \mathcal{B} = \int_{\mathrm{conv}(x_Q,F)} \mathrm{d}\mathcal{B}$. Theorem 3.5 follows by the summing of this identity over all facets F of Q. $\square$

4. Fiber zonotopes from projections of the n-cube

In this section we consider the projection $\pi(\mathcal{C}_n)$ of the regular n-cube

$$\mathcal{C}_n := [-e_1, e_1] + [-e_2, e_2] + \cdots + [-e_n, e_n] \tag{4.1}$$

onto a d-dimensional subspace. Here $e_1, e_2, \ldots, e_n$ denote the standard basis vectors of $\mathbf{R}^n$. If π is given by a $(d \times n)$-matrix $\mathbf{A}$ with column vectors $a_1, a_2, \ldots, a_n$, then the image of the n-cube is the d-dimensional *zonotope*

$$\mathcal{Z} := \pi(\mathcal{C}_n) = [-a_1, a_1] + [-a_2, a_2] + \cdots + [-a_n, a_n]. \tag{4.2}$$

We define the *circuits* of $\mathcal{Z}$ to be the elementary vectors in the kernel of $\mathbf{A}$; that is,

$$E_\nu := \sum_{i=1}^{d+1} (-1)^i \det(a_{\nu_1}, \ldots, a_{\nu_{i-1}}, a_{\nu_{i+1}}, \ldots, a_{\nu_{d+1}}) \cdot e_{\nu_i}, \tag{4.3}$$

where ν ranges over all (ordered) $(d+1)$-subsets of $\{1, 2, \ldots, n\}$ (cf. [7]). Here we give an explicit description and combinatorial interpretation of the fiber polytope $\Sigma(\mathcal{C}_n, \mathcal{Z})$.

THEOREM 4.1. *The fiber polytope of a projection of the regular n-cube $\mathcal{C}_n$ onto a d-flat is the $(n-d)$-dimensional zonotope* (*to be called the* fiber zonotope *of* $\mathcal{Z}$)

$$\Sigma(\mathcal{C}_n, \mathcal{Z}) = \frac{1}{\mathrm{vol}\,\mathcal{Z}} \sum_{\nu} [-E_\nu, E_\nu], \tag{4.4}$$

which is generated by all circuits E_ν of $\mathcal{Z} = \pi(\mathcal{C}_n)$.

Our starting point is the observation that the 3^n faces of $\mathcal{C}_n$ have a natural labeling with the sign vectors $\sigma \in \{-, 0, +\}^n$. If σ_-, σ_0, σ_+ denote the index sets of negative, zero and positive entries, then the face of $\mathcal{C}_n$ labeled σ is the $|\sigma_0|$-dimensional cube

$$F_\sigma = \sum_{i \in \sigma^+} e_i - \sum_{j \in \sigma^-} e_j + \sum_{k \in \sigma^0} [-e_k, e_k]. \tag{4.5}$$

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So every subcomplex of the boundary of $\mathcal{C}_n$ is identified with a set of sign vectors. In particular, the faces of $\mathcal{Z}$ are indexed by their preimages in the boundary of $\mathcal{C}_n$ via

$$\mathcal{M} = \{\sigma \in \{-,0,+\}^n : F_\sigma = \pi^{-1}(G),\ G \text{ a face of } \mathcal{Z}\}. \tag{4.6}$$

The set $\mathcal{M}$ is the set of *covectors* of the *oriented matroid* associated with $\mathcal{Z}$; cf. [7]. The *circuits* of the oriented matroid $\mathcal{M}$ are $\operatorname{sign}(E_\nu)$, where ν ranges over all $(d+1)$-subsets of $\{1,\dots,n\}$. Thus by the above identification, the realizable rank-d oriented matroids on n elements are the shadow boundaries of projections of the n-cube into d-flats.

A *lifting* of $\mathcal{Z}$ is a rank $d+1$ oriented matroid $\mathcal{L}$ on $\{1,2,\dots,n,e\}$ such that $\mathcal{M} = \mathcal{L}/e$, the contraction by e. Suppose now that e is a vector of unit length in the kernel of π and let $\psi = (\psi_1,\psi_2,\dots,\psi_n) \in \mathbf{R}^n$. Then the $(d+1)$-dimensional zonotope

$$\mathcal{Z}(\psi) := [-a_1 - \psi_1 e, a_1 + \psi_1 e] + \cdots + [-a_n - \psi_n e, a_n + \psi_n e] + [-e, e] \tag{4.7}$$

defines a one-element lifting $\mathcal{L}(\psi)$. A lifting $\mathcal{L}$ of $\mathcal{Z}$ is *coherent* provided that $\mathcal{L} = \mathcal{L}(\psi)$ for some $\psi \in \mathbf{R}^n$. Note that the coherence of $\mathcal{L}$ depends on the specific zonotope $\mathcal{Z}$, and not only on its combinatorial type $\mathcal{M}$; every coherent lifting is realizable, but, in general, not every realizable lifting is coherent ([7]; §2.2).

A polyhedral subdivision Π of $\mathcal{Z}$ is *zonotopal* if every cell of Π is a zonotope whose edges are translates of the edges of $\mathcal{Z}$. So Π is zonotopal if and only if it is induced from the map $\pi : \mathcal{C}_n \to \mathcal{Z}$. An induced subdivision Π is tight if and only if all cells of Π are affinely equivalent to regular cubes; in this case, Π is called *cubical.* By a theorem of Bohne and Dress [8] (see also [7], §2.2), the assignment

$$\mathcal{L} \mapsto \Pi(\mathcal{L}) := \{\pi(F_\sigma) : (\sigma, -) \text{ is a covector of } \mathcal{L}\} \tag{4.8}$$

is a bijection between the one-element liftings and the zonotopal subdivisions of $\mathcal{Z}$. For the purposes of this paper we need only the following simple variation:

LEMMA 4.2. *A zonotopal subdivision Π of $\mathcal{Z}$ is coherent if and only if $\Pi = \Pi(\mathcal{L})$ for some coherent lifting $\mathcal{L} = \mathcal{L}(\psi)$ of $\mathcal{Z}$.*

Proof. We show that $\Pi(\mathcal{L}(\psi)) = \Pi(\mathcal{B}^\psi)$, where $\mathcal{B} = \pi^{-1}$. Consider the linear map $\theta : \mathbf{R}^n \to \mathbf{R}^{d+1}$, which takes the regular n-cube $\mathcal{C}_n$ onto the $(d+1)$-zonotope

$$\mathcal{Z}'(\psi) := [-a_1 - \psi_1 e, a_1 + \psi_1 e] + \cdots + [-a_n - \psi_n e, a_n + \psi_n e].$$

Let $\phi : \mathbf{R}^{d+1} \to \mathbf{R}^d$ be the coordinate projection, which takes both $\mathcal{Z}'(\psi)$ and $\mathcal{Z}(\psi)$ onto $\mathcal{Z}$. Up to translation by $-e$, the bottom faces of $\mathcal{Z}'(\psi)$ are equal to

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the bottom faces of $\mathcal{Z}(\psi)$. Therefore $\Pi(\mathcal{L}(\psi))$ equals the coherent subdivision $\Pi((\phi^{-1})^e)$. We can now apply Theorem 2.4 to the composition $\pi = \phi\circ\theta$. Since $\psi \in \mathbf{R}^n$ is orthogonal to $\theta^{-1}(e^\perp)$, we have $\Pi((\phi^{-1})^e) = \Pi(\mathcal{B}^\psi)$, as desired. $\square$

Proof of Theorem 4.1. Let π^{-1} be the projection bundle of the projection $\pi : \mathcal{C}_n \to \mathcal{Z}$ and let $\psi \in \mathbf{R}^n$ be any vector such that $\Pi(\mathcal{L})$ is a cubical subdivision of $\mathcal{Z}$, where $\mathcal{L} = \mathcal{L}(\psi)$. We compute

$$\int \pi^{-1}(x)^\psi = \sum_{(\sigma,-)\text{ cocircuit of }\mathcal{L}} \int_{\pi(F_\sigma)} \pi^{-1}(x)^\psi dx.$$

Let $x \in \mathcal{Z}$ be a point that lies in the interior of a maximal face $\pi(F_\sigma)$ of the coherent subdivision $\Pi(\mathcal{L})$. Since $\Pi(\mathcal{L})$ is cubical, the fiber $\pi^{-1}(x)$ has a unique intersection point with the preimage of this face, the d-cube F_σ. (In the notation of Section 2, F_σ is the face $\widetilde{\mathcal{F}_x}$.) This point is the unique extreme vertex of the fiber $\pi^{-1}(x)$ in direction ψ. The resulting assignment

$$x \mapsto \pi^{-1}(x)^\psi = \pi^{-1}(x) \cap F_\sigma \tag{4.9}$$

is an affine function throughout the interior of the d-cube $\pi(F_\sigma)$; so its integral is the barycenter of F_σ times the volume of $\pi(F_\sigma)$. We can see from equation (4.5) that the barycenter of F_σ is the sign vector $\sigma = \sum_{i\in\sigma^+} e_i - \sum_{j\in\sigma^-} e_j$ itself. At the same time, σ is a cocircuit of $\mathcal{L}$ whose complement is a d-element hyperplane of $\mathcal{L}$, say, $\sigma_0 = \{\mu_1, \mu_2, \dots, \mu_d\}$. The volume of the d-cube $\pi(F_\sigma)$ is the absolute value of $\det(a_{\mu_1}, a_{\mu_2}, \dots, a_{\mu_d})$. We have proved

$$\int_{\pi(F_\sigma)} \pi^{-1}(x)^\psi dx = |\det(a_{\mu_1}, a_{\mu_2}, \dots, a_{\mu_d})|\sigma. \tag{4.10}$$

If we sum the expression (4.10) over all maximal cells of $\Pi(\mathcal{L})$, then, by Proposition 1.2, the result is the scaled vertex of $\Sigma(\mathcal{C}_n, \mathcal{Z})$:

$$\begin{aligned}(\operatorname{vol}\mathcal{Z})\Sigma(\mathcal{C}_n, \mathcal{Z})^\psi &= \int_{\mathcal{Z}} \pi^{-1}(x)^\psi dx \\ &= \sum_{(\sigma,-)\text{ cocircuit of }\mathcal{L}} |\det(a_{\mu_1}, a_{\mu_2}, \dots, a_{\mu_d})|\sigma.\end{aligned} \tag{4.11}$$

This sum is over all $\sigma \in \{-,0,+\}^n$ such that $(\sigma,-)$ is a cocircuit of the lifting $\mathcal{L}$. We will now show that the vector in formula (4.11) is also the vertex of the zonotope $\sum_\nu[-E_\nu, E_\nu]$ in direction ψ.

Let χ denote the chirotope of the lifted oriented matroid $\mathcal{L}$ and consider any ordered $(d+1)$-subset ν of $\{1,2,\dots,n\}$. If the circuit E_ν is nonzero, then ν is a basis of $\mathcal{L}$. Its orientation $\chi(\nu) \in \{-1,+1\}$ equals the sign of the determinant

$$\det(a_{\nu_1} + \psi_{\nu_1}e, a_{\nu_2} + \psi_{\nu_2}e, \dots, a_{\nu_{d+1}} + \psi_{\nu_{d+1}}e) = \langle E_\nu, \psi\rangle. \tag{4.12}$$

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This proves that $\sum_\nu \chi(\nu)E_\nu$ is the vertex of the zonotope $\sum_\nu[-E_\nu, E_\nu]$ in direction ψ. We rewrite the vertex $\sum_\nu \chi(\nu)E_\nu$ as follows:

$$\begin{aligned}
&\sum_{1\le\nu_1<\cdots<\nu_{d+1}\le n} \chi(\nu_1,\ldots,\nu_{d+1}) \\
&\qquad\cdot \sum_{i=1}^{d+1}(-1)^i \det(a_{\nu_1},\ldots,a_{\nu_{i-1}},a_{\nu_{i+1}},\ldots,a_{\nu_{d+1}})\cdot e_{\nu_i} \\
&\qquad= \sum_{1\le\mu_1<\cdots<\mu_d\le n} \det(a_{\mu_1},\ldots,a_{\mu_d}) \sum_{j=1}^{n}\chi(\mu_1,\ldots,\mu_d,j)\cdot e_j \\
&\qquad= \sum_{1\le\mu_1<\cdots<\mu_d\le n} |\det(a_{\mu_1},\ldots,a_{\mu_d})| \\
&\qquad\cdot \sum_{j=1}^{n}\chi(\mu_1,\ldots,\mu_d,e)\chi(\mu_1,\ldots,\mu_d,j)\cdot e_j.
\end{aligned} \tag{4.13}$$

Here the sign of $\det(a_{\mu_1},\ldots,a_{\mu_d})$ equals $\chi(\mu_1,\ldots,\mu_d,e)$, because $\chi(\cdots,e)$ is the chirotope of the contraction $\mathcal{M} = \mathcal{L}/e$. Using further the relationship between the sign of the determinant and cocircuits, we see that either the sign vector

$$\sigma = \sum_{j=1}^{n}\chi(\mu_1,\ldots,\mu_d,e)\chi(\mu_1,\ldots,\mu_d,j)\cdot e_j \tag{4.14}$$

is zero or $(\sigma, -)$ is a cocircuit of $\mathcal{L}$. Conversely every such cocircuit of $\mathcal{L}$ can be written uniquely as equation (4.14) for some ordered d-set $\mu \subset \{1,2,\ldots,n\}$. Therefore the expression in (4.13) equals the scaled vertex of $\Sigma(\mathcal{C}_n,\mathcal{Z})^\psi$ defined in equation (4.11). This completes the proof of Theorem 4.1. $\square$

From the results in Section 2 we conclude the following:

COROLLARY 4.2. *The face lattice of the fiber zonotope* $\Sigma(\mathcal{C}_n,\mathcal{Z}) = 1/\operatorname{vol}\mathcal{Z}\sum_\nu[-E_\nu,E_\nu]$ *is isomorphic to the poset of coherent zonotopal subdivisions of* $\mathcal{Z}$. *Here the vertices correspond to the coherent cubical subdivisions of* $\mathcal{Z}$.

Example 4.3. *The fiber zonotope of a centrally symmetric octagon.* Let $\pi : \mathcal{C}_4 \to \mathcal{Z}$ be the projection of the regular 4-cube onto a centrally symmetric octagon $\mathcal{Z}$ in the plane. The oriented matroid $\mathcal{M}$ of the zonotope $\mathcal{Z}$ is uniform of rank 2 on four elements, which means it has four distinct signed circuits $E_{123}, E_{124}, E_{134}, E_{234}$. These form a vector configuration in general position in the two-dimensional space $\ker(\pi)$ so that the fiber zonotope

$$\Sigma(\mathcal{C}_4,\mathcal{Z}) = [-E_{123},E_{123}] + [-E_{124},E_{124}] + [-E_{134},E_{134}] + [-E_{234},E_{234}]$$

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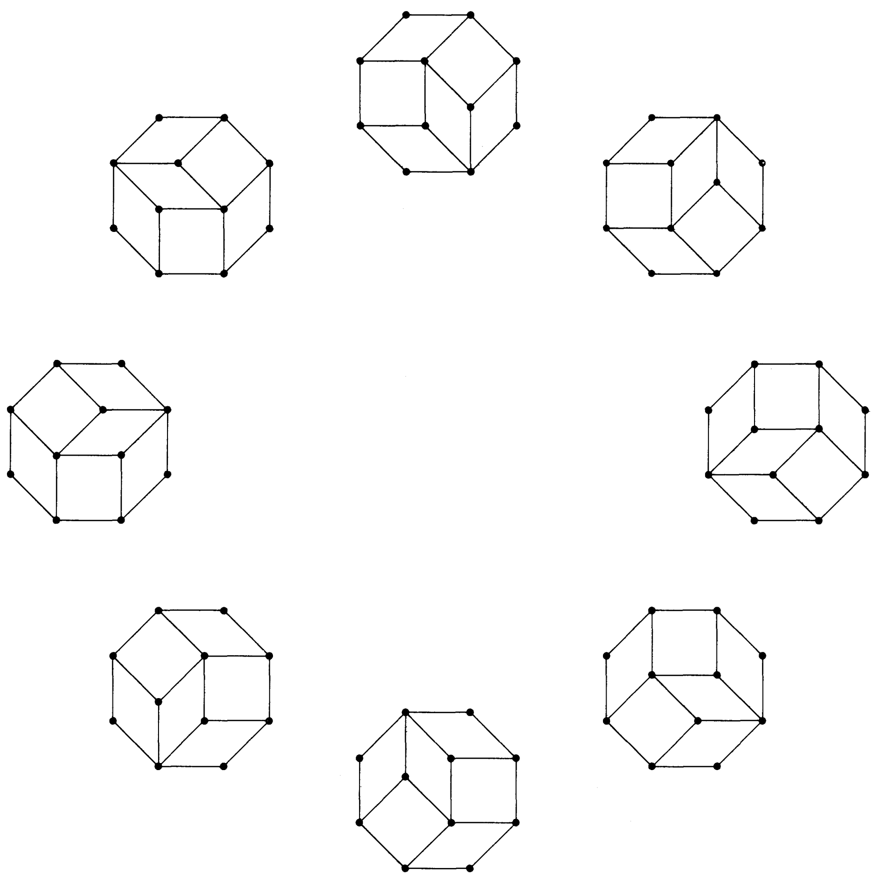

FIGURE 1. The fiber zonotope of the 4-cube with respect to an octagon

is an octagon again. The centrally symmetric octagon $\mathcal{Z}$ has eight cubical subdivisions, which are all coherent. Their correspondence with the vertices of $\Sigma(\mathcal{C}_4, \mathcal{Z})$ is illustrated in Figure 1 above. The eight edges of the fiber octagon $\Sigma(\mathcal{C}_4, \mathcal{Z})$ correspond to the zonotopal subdivisions whose coherent face bundle involves precisely one 3-cube in the boundary of the 4-cube $\mathcal{C}_4$.

5. Centrally symmetric polytopes and other examples

We will now discuss, in reverse order, the special cases (a), (b) and (e) mentioned in the Introduction. Consider any centrally symmetric d-polytope $\mathcal{S} = \mathrm{conv}\{\pm a_1, \ldots, \pm a_n\}$ in $\mathbf{R}^d$, $n > d$. Then $\mathcal{S}$ is the image of the standard *n-cross polytope* $\mathcal{T}_n = \mathrm{conv}\{\pm e_1, \ldots, \pm e_n\}$ under the linear map $\pi : \mathbf{R}^n \to \mathbf{R}^d$, $e_i \mapsto a_i$. What is the combinatorial interpretation of the fiber polytope $\Sigma(\mathcal{T}_n, \mathcal{S})$ in this centrally symmetric situation?

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Recall that a polyhedral subdivision Π of $\mathcal{S}$ is regular if there exists a function $g : \mathcal{S} \to \mathbf{R}$ that is piecewise linear and strictly convex with respect to Π. A subdivision Π is said to be *centrally regular* if there exists a g that additionally satisfies $g(-a_i) = -g(a_i)$ for $i = 1, \ldots, n$. We partially order the set of centrally regular subdivisions of $\mathcal{S}$ by refinement. (Note that a more general notion of central regularity is needed in the case in which the points $\pm a_i$ are not in convex position.)

THEOREM 5.1. *The fiber polytope $\Sigma(\mathcal{T}_n, \mathcal{S})$ is a centrally symmetric $(n-d)$-polytope whose face lattice is isomorphic to the poset of centrally regular subdivisions of $\mathcal{S}$. In particular, the vertices of $\Sigma(\mathcal{T}_n, \mathcal{S})$ are in one-to-one correspondence with the centrally regular triangulations.*

Proof. Consider the canonical map

$$\theta : \mathbf{R}^{2n} \to \mathbf{R}^n, e_1 \mapsto e_1, \ldots, e_n \mapsto e_n, e_{n+1} \mapsto -e_1, \ldots, e_{2n} \mapsto -e_n,$$

which takes the standard $(2n-1)$-simplex Δ_{2n-1} onto the standard n-cross polytope.

Let Π be any subdivision of $\mathcal{S}$ that is coherent with respect to π. Then Π is also coherent with respect to $\pi \circ \theta$, and the coherent face bundle of Π in $\mathbf{R}^{2n}$ is defined by some linear functional $\psi \in \ker(\theta)^\perp$. But $\psi \in \ker(\theta)^\perp$ is equivalent to $\psi = (\psi_1, \ldots, \psi_n, -\psi_1, \ldots, -\psi_n)$ for some $\psi_1, \ldots, \psi_n \in \mathbf{R}$. The piecewise-linear convex function g_ψ over Π, defined by the "height vector" ψ, satisfies $g_\psi(-a_i) = -\psi_i = -g_\psi(a_i)$; thus Π is centrally regular. This argument can easily be reversed: If Π is centrally regular, then Π is coherent with respect to π. Using Theorem 2.1 shows that the poset of centrally regular subdivisions of $\mathcal{S}$ is isomorphic to the face lattice of $\Sigma(\mathcal{T}_n, \mathcal{S})$.

The cross polytope $\mathcal{T}_n$ is a simplicial polytope of dimension $n > d$; so the tight coherent subdivisions of $\mathcal{S}$ (with respect to π) are precisely the coherent triangulations of $\mathcal{S}$. This proves that the vertices of $\Sigma(\mathcal{T}_n, \mathcal{S})$ are in one-to-one correspondence with the centrally regular triangulations.

In order to see that the fiber polytope $\Sigma(\mathcal{T}_n, \mathcal{S})$ is itself centrally symmetric we observe that, for any triangulation Δ of $\mathcal{S}$, there is a triangulation $-\Delta$ obtained by central reflection. The vector Φ_Δ, defined as in equation (2.5), satisfies $-\Phi_\Delta = \Phi_{(-\Delta)}$. □

Example 5.2. Consider the two centrally symmetric octagons $\mathcal{S} = \{\pm a_1, \pm a_2, \pm a_3, \pm a_4\}$ and $\mathcal{S}' = \{\pm a_1', \pm a_2', \pm a_3', \pm a_4'\}$ in $\mathbf{R}^2$, where $a_1 = (0, \sqrt{8})$, $a_2 = (2, 2)$, $a_3 = (\sqrt{8}, 0)$, $a_4 = (2, -2)$, and where $a_1' = (0, 3)$, $a_2' = (2, 2)$, $a_3' = (3, 0)$, $a_4' = (2, -2)$. The octagon $\mathcal{S}$ is *regular* (i.e., has full rotational symmetry), and its fiber polytope $\Sigma(\mathcal{T}_4, \mathcal{S})$ is an octagon in $\mathbf{R}^4$, which is also regular. The second octagon $\mathcal{S}'$ is *not regular*, and its fiber polytope is a centrally symmetric

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16-gon in $\mathbf{R}^4$. Eight of the 16 centrally regular triangulations of $\mathcal{S}'$ are also centrally regular for $\mathcal{S}$. The other eight vertices of $\Sigma(\mathcal{T}_4, \mathcal{S}')$ become points in the relative interiors of the eight edges of $\Sigma(\mathcal{T}_4, \mathcal{S})$. The behavior of this example contrasts with that of Example 4.3, where the fiber zonotope is an octagon for all centrally symmetric embeddings of $\mathcal{Z}$.

Moving on to special case (b), we consider a projection π of a d-polytope $P \subset \mathbf{R}^d$ onto a line segment $Q \subset \mathbf{R}^1$. The map $\pi : \mathbf{R}^d \to \mathbf{R}$ is defined by a vector $\gamma \in \mathbf{R}^d$ via $\pi(x) = \langle \gamma, x \rangle$. We assume that the vertices $p_1, \ldots, p_n$ of P are sorted according to the linear functional π; that is, $q_1 = \langle \gamma, p_1 \rangle \leq q_2 = \langle \gamma, p_2 \rangle \leq \cdots \leq q_n = \langle \gamma, p_n \rangle$.

Now let $\psi \in \mathbf{R}^d$ be any vector that induces a tight coherent subdivision $\Pi = \Pi((\pi^{-1})^{\psi})$ of $Q = [q_1, q_n]$. The maximal cells of Π are $\{[q_{i_{j-1}}, q_{i_j}],\ j = 1, \ldots, k\}$ for some increasing sequence $q_1 = q_{i_0} < q_{i_1} < q_{i_2} < \cdots < q_{i_{k-1}} < q_{i_k} = q_n$. The corresponding points $p_1 = p_{i_0}, p_{i_1}, p_{i_2}, \ldots, p_{i_{k-1}}, p_{i_k} = p_n$ are the vertices of P, extreme in the directions $\psi + \alpha\gamma$, as the real parameter α ranges from $-\infty$ to $+\infty$. Such a monotone edge path on P is called a *parametric simplex path* in linear programing.

In summary, the tight coherent face bundles of the map $\pi : P \to Q$ are the parametric, γ-monotone paths between the minimum vertex p_1 and the maximum vertex p_n of P. In view of this interpretation of its vertices, we write $\Sigma_\gamma(P) := \Sigma(P, Q)$ for the fiber polytope, and we call it the *monotone path polytope of P in direction γ*. Using Corollary 2.6 gives us the following coordinate representation for the $(d-1)$-polytope $\Sigma_\gamma(P)$:

THEOREM 5.3. *The monotone path polytope $\Sigma_\gamma(P)$ is the convex hull of the points*

$$\sum_{j=1}^{k} \frac{\langle \gamma, p_{i_j} - p_{i_{j-1}} \rangle}{2\langle \gamma, p_n - p_1 \rangle} \left(p_{i_{j-1}} + p_{i_j}\right),$$

defined by increasing index sequences $1 = i_0 < i_1 < \cdots < i_k = n$. Its vertices are in one-to-one correspondence with the parametric, γ-monotone edge paths on P.

As a special case, we note that the monotone path polytope $\Sigma_\gamma(\Delta_{n-1})$ of the $(n-1)$-simplex, with respect to a general direction γ, is combinatorially equivalent to the $(n-2)$-cube $\mathcal{C}_{n-2}$. Indeed $\Sigma_\gamma(\Delta_{n-1})$ is the secondary polytope of n distinct points on a line; Adams [1] has shown that the poset of all subdivisions of a line segment having n points is a cell complex having the combinatorial structure of an $(n-2)$-cube.

Example 5.4. We compute the monotone path polytope $\Sigma_\gamma(\mathcal{C}_n)$ of the n-cube with respect to $\gamma = (1, 1, \ldots, 1)$. The elementary vectors in the kernel of $\pi : \mathbf{R}^n \to \mathbf{R}$, $x \mapsto \langle x, \gamma \rangle$ are $e_i - e_j$, $1 \leq i < j \leq n$. Theorem 4.1 implies that

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$\Sigma_\gamma(\mathcal{C}_n)$ is (homothetic to) $\sum_{1\le i<j\le n}[e_i - e_j, e_j - e_i]$. Hence $\Sigma_\gamma(\mathcal{C}_n)$ is (homothetic to) the *permutahedron*, defined as the convex hull of all permutations of the vector $(1, 2, \ldots, n)$. Here the vertices of $\Sigma_\gamma(\mathcal{C}_n)$ correspond to monotone paths between $(-1, -1, \ldots, -1)$ and $(1, 1, \ldots, 1)$ in $\mathcal{C}_n$; these are naturally labeled by elements of the symmetric group on n letters. The topology and combinatorics of such paths in the cube have been noted by Baues [4]. In this case, we note that all induced subdivisions are coherent.

Continuing the iterated formation of monotone path polytopes, one considers the monotone edge paths in the permutahedron between $(1, 2, \ldots, n)$ and $(n, n-1, \ldots, 1)$ and finds that the poset of all induced subdivisions is not topologically a sphere. This is a reflection of the fact that not all induced subdivisions are coherent. Indeed a conjecture of Baues ([4]; Conj. 6.4) would follow if one could show that including the (spherical) lattice of coherent subdivisions into the poset of all induced subdivisions gives a homotopy equivalence in this case.[1] It would be interesting to determine, for general P and Q, the topological relationship between the poset of all induced subdivisions of Q and the sublattice of coherent subdivisions. In the case of zonotopes, this would relate to the problem of determining the topological type of the extension lattice of an oriented matroid.

We finally consider a map of polytopes $\pi : P \to Q$, where $\dim(P) - \dim(Q) = 1$. Supposing that Q is a d-polytope with vertices $q_1, q_2, \ldots, q_n \in \mathbf{R}^d$, we may assume that $\pi : \mathbf{R}^{d+1} \to \mathbf{R}^d$ is the projection onto the first d coordinates. Then the vertices of P are $p_1 = (q_1, \psi_1), p_2 = (q_2, \psi_2), \ldots p_n = (q_n, \psi_n)$. The fiber polytope $\Sigma(P, Q)$ is a line segment whose two vertices correspond to the coherent face bundles defined by the vectors $-e_{d+1}$ and e_{d+1} in $\mathbf{R}^{d+1}$. The two resulting subdivisions of Q are the projections of the top and bottom of the boundary of P onto Q. Equivalently, these are the two regular subdivisions of Q defined by the vectors $-\psi$ and ψ, where $\psi = (\psi_1, \psi_2, \ldots, \psi_n) \in \mathbf{R}^n$, with respect to the canonical map $\Delta_{n-1} \to Q$. Thus for any projection bundle $\mathcal{B}$ over Q, the coherent subdivisions $\Pi(\mathcal{B}^\psi)$ and $\Pi(\mathcal{B}^{-\psi})$ can always be found as the top and bottom of some P projecting onto Q, as above.

6. Gale bundles

The fiber polytope $\Sigma(P, Q)$ of an n-polytope P and a d-polytope Q is an $(n-d)$-polytope contained in n-space; for many purposes it is useful to have an explicit embedding of the fibers $\pi^{-1}(x)$ and their average $\Sigma(P, Q)$ into a

[1] The authors and M.M. Kapranov [21] have since shown this for *any* P whenever $\dim Q = 1$, thereby proving the conjecture of Baues.

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space of dimension $n-d$. In this section we intend to give such an embedding. Our main emphasis will lie on the cases where P is a simplex, cube or cross polytope.

To begin with we consider the projection $\pi : \Delta_{n-1} \to Q$ of the regular $(n-1)$-simplex onto a d-polytope $Q \subset \mathbf{R}^d$ with n vertices. Let $\mathbf{A}$ be the $(d+1) \times n$-matrix whose columns are the homogeneous coordinates for Q. Pick any $n \times (n-d-1)$-matrix $\mathbf{B}$ such that

$$0 \longrightarrow \mathbf{R}^{n-d-1} \xrightarrow{\mathbf{B}} \mathbf{R}^n \xrightarrow{\mathbf{A}} \mathbf{R}^{d+1} \longrightarrow 0 \tag{6.1}$$

is an exact sequence of $\mathbf{R}$-linear maps. Then the configuration of row vectors $b_1, b_2, \ldots, b_n$ of $\mathbf{B}$ is a *Gale transform* of Q. For any interior point $\lambda \in \Delta_{n-1}$ we define the polytope

$$\begin{aligned} \mathcal{G}_\lambda := & \left[\operatorname{conv}\left\{-\frac{b_1}{\lambda_1}, -\frac{b_2}{\lambda_2}, \ldots, -\frac{b_n}{\lambda_n}\right\}\right]^* \\ = & \{x \in \mathbf{R}^{n-d-1} \mid \mathbf{B}x + \lambda \geq 0\}, \end{aligned} \tag{6.2}$$

where * denotes the polar polytope with respect to the standard inner product on $\mathbf{R}^{n-d-1}$. This definition extends naturally to boundary points of Δ_{n-1}, in which case some λ_i are zero and the origin lies in the boundary of $\mathcal{G}_\lambda$. The polytope bundle $\mathcal{G}$ on Δ_{n-1}, $\lambda \mapsto \mathcal{G}_\lambda \subset \mathbf{R}^{n-d-1}$, will be called the *Gale bundle.* The proof of the following lemma is straightforward from the preceding definitions.

LEMMA 6.1. *The image of the polytope $\mathcal{G}_\lambda$ under the monomorphism $\mathbf{B}$ equals the translated fiber $\pi^{-1}(\pi(\lambda)) - \lambda$.*

THEOREM 6.2. *The fiber polytope $\Sigma(\Delta_{n-1}, Q)$ is normally equivalent to $\mathbf{B}(\int_{\Delta_{n-1}} \mathcal{G})$, the image of the Minkowski integral of the Gale bundle. In fact, $\Sigma(\Delta_{n-1}, Q)$ is a translate of the polytope*

$$\frac{1}{\operatorname{vol} Q}\mathbf{B}\left(\int_{\Delta_{n-1}} \frac{1}{\operatorname{vol}\bigl(\pi^{-1}(\pi(\lambda))\bigr)} \mathcal{G}_\lambda \, d\lambda\right).$$

Proof. We abbreviate $V(\lambda) := 1/\operatorname{vol}\bigl(\pi^{-1}(\pi(\lambda))\bigr)$. Consider the polytope bundle $\mathcal{H}$ on Δ_{n-1} defined by $\mathcal{H}_\lambda := \pi^{-1}(\pi(\lambda)) \subset \mathbf{R}^n$. Application of Fubini's theorem to the extreme sections of $\mathcal{H}$ implies that

$$\int_{\Delta_{n-1}} V(\lambda)\mathcal{H}_\lambda \, d\lambda = \int_Q \pi^{-1}(x)\, dx. \tag{6.3}$$

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Using Lemma 6.1 and the linearity of Minkowski integrals, we find that

$$
\begin{aligned}
\mathbf{B}\Big(\int_{\Delta_{n-1}} V(\lambda)\mathcal{G}_\lambda \, d\lambda\Big) &= \int_{\Delta_{n-1}} V(\lambda)\mathbf{B}(\mathcal{G}_\lambda)\, d\lambda \\
&= \int_{\Delta_{n-1}} V(\lambda)(\mathcal{H}_\lambda - \lambda)\, d\lambda \\
&= \int_Q \pi^{-1}(x)\, dx - x_\Delta,
\end{aligned}
\tag{6.4}
$$

where $x_\Delta = \int_{\Delta_{n-1}} V(\lambda)\lambda\, d\lambda$. The first statement follows, since the polytopes $\int \mathcal{G}$ and $\int V(\lambda)\mathcal{G}_\lambda\, d\lambda$ are normally equivalent, and since normal equivalence is preserved under monomorphisms. □

Note that this result generalizes to any projection of polytopes, $\pi : \mathbf{R}^n \to \mathbf{R}^d, P \to Q$. Using an exact sequence as above, we identify the kernel of π with $\mathbf{R}^{n-d}$ and consider the complementary orthogonal projection $\theta : \mathbf{R}^n \to \mathbf{R}^{n-d}$. Here the *Gale bundle* is the polytope bundle

$$\mathcal{G} : p \mapsto \theta\big(\pi^{-1}(\pi(p)) - p\big) \tag{6.5}$$

over P. As in Theorem 6.2, the Minkowski integral $\int_P \mathcal{G}$ of the Gale bundle is normally equivalent to the fiber polytope $\Sigma(P, Q)$.

Our definition of Gale bundle is consistent with the definition of generalized Gale transforms given by Filliman [10]. As is shown in [10, §3], these transforms specialize to McMullen's *zonal diagrams* [14] when P is a regular cube and to *centrally symmetric diagrams* [15] when P is a regular cross polytope.

We compute the Gale bundle $\mathcal{G}$ in the case where $\mathbf{A}$ is the projection of the regular n-cube $\mathcal{C}_n$ onto a d-zonotope $\mathcal{Z}$, as in Section 4. Let $p = (p_1, \ldots, p_n) \in \mathcal{C}_n$; that is, $|p_i| \le 1$ for $i = 1, \ldots, n$. Then

$$\pi^{-1}(\pi(p)) - p = \{q \in \mathbf{R}^n \mid \mathbf{A}q = 0 \text{ and } |p_i + q_i| \le 1 \text{ for } i = 1, \ldots, n\}. \tag{6.6}$$

Since $\mathbf{A}q = 0$ is equivalent to $q = \mathbf{B}x$ for some $x \in \mathbf{R}^{n-d}$, we have

$$\mathcal{G}_p = \{x \in \mathbf{R}^{n-d} \mid -1 - p_i \le \langle x, b_i\rangle \le 1 - p_i \text{ for } i = 1, \ldots, n\}. \tag{6.7}$$

Thus the Gale bundle of the zonotope $\mathcal{Z} = \sum_{i=1}^n [-a_i, a_i] \subset \mathbf{R}^d$ consists of all polytopes in $\mathbf{R}^{n-d}$ that are obtained as intersections of n "parallel strips" perpendicular to the vectors in the Gale transform $\{b_1, \ldots, b_n\} \subset \mathbf{R}^{n-d}$.

Finally we describe the Gale bundle of the map $\pi : \mathcal{T}_n \to \mathcal{S}$ of the n-cross polytope onto a centrally symmetric d-polytope, as in Section 5. This map is represented by a $d \times n$-matrix $\mathbf{A}$, and we can choose a Gale transform $\mathbf{B}$ as in the sequence (6.1). The resulting vector configuration $\{\pm b_1, \ldots, \pm b_n\} \subset$

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$\mathbf{R}^{n-d}$ is the *centrally symmetric diagram* of $\mathcal{S}$, as introduced by McMullen and Shephard [15]. Here the Gale bundle $\mathcal{G} : p \mapsto \mathcal{G}_p \subset \mathbf{R}^{n-d}$ is given by

$$\mathcal{G}_p = \left\{ x \in \mathbf{R}^{n-d} : \sum_{i=1}^{n} |\langle x, b_i \rangle + p_i| \leq 1 \right\},$$

for $p \in \mathcal{T}_n$. Using the 1-norm, we can rewrite this expression as $\mathcal{G}_p = \{x : \|\mathbf{B}x + p\|_1 \leq 1\}$. Similarly, using the ∞-norm shows that the expression (6.7) for the Gale bundle of a d-zonotope equals $\mathcal{G}_p = \{x : \|\mathbf{B}x + p\|_\infty \leq 1\}$, $p \in \mathcal{C}_n$. It may be of interest to generalize this construction and to study the polytope bundle $\mathcal{G}_p = \{x \in \mathbf{R}^{n-d} : \|\mathbf{B}x + p\| \leq 1\}$ over the unit ball of an arbitrary piecewise-linear norm $\|\cdot\|$ on $\mathbf{R}^n$. By Theorem 5.1, integrating the bundle $\mathcal{G}_p$ yields a centrally symmetric $(n-d)$-polytope in $\mathbf{R}^{n-d}$ and hence a piecewise-linear norm on $\mathbf{R}^{n-d}$.

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