Towards a General Theory of Elimination (and Optimization?)

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1 Preliminaries

1.1 Hom-Functors

For a locally small category $\mathscr{C}$, the hom-functor of $\mathscr{C}$ is a functor $\mathrm{Hom}_{\mathscr{C}} : \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathbf{Set}$ constructed in the following manner. Given objects $a, b, c, \ldots \in \mathscr{C}_0$ of $\mathscr{C}$, the hom-functor $\mathrm{Hom}_{\mathscr{C}}$ maps a pair of objects $(a, b) \in (\mathscr{C}^{\mathrm{op}} \times \mathscr{C})_0 = \mathscr{C}_0 \times \mathscr{C}_0 = \mathscr{C}_0^2$ into the set[1] of morphisms $\mathscr{C}_1$ of $\mathscr{C}$ with source a and target b. Therefore, $\mathrm{Hom}_{\mathscr{C}}(a, b)$ is the set of morphisms in $\mathscr{C}$ of type $a \to b$. Given morphisms $g^{\mathrm{op}} \in \mathrm{Hom}_{\mathscr{C}^{\mathrm{op}}}(a, c)$ and $h \in \mathrm{Hom}_{\mathscr{C}}(b, d)$, the hom-functor $\mathrm{Hom}_{\mathscr{C}}$ constructs a function

$$\mathrm{Hom}_{\mathscr{C}}(g^{\mathrm{op}}, h) : \mathrm{Hom}_{\mathscr{C}}(a, b) \to \mathrm{Hom}_{\mathscr{C}}(c, d)$$

which takes a morphism $f : a \to b \in \mathrm{Hom}_{\mathscr{C}}(a, b)$ and produces the morphism $h \circ f \circ g : c \to d \in \mathrm{Hom}_{\mathscr{C}}(c, d)$. Graphically,

$$\mathrm{Hom}_{\mathscr{C}}(g^{\mathrm{op}}, h) \left(a \xrightarrow{f} b \right) = c \xrightarrow{g} a \xrightarrow{f} b \xrightarrow{h} d$$

1.2 Adjoint Functors

Given two categories $\mathscr{C}$ and $\mathscr{D}$, a pair of functors $L : \mathscr{C} \to \mathscr{D}, R : \mathscr{D} \to \mathscr{C}$ are called an *adjoint pair*, denoted $L \dashv R$ or

$$\begin{tikzcd} \mathscr{C} \arrow[r, bend left, "L"] \arrow[r, phantom, "\perp"] & \mathscr{D} \arrow[l, bend left, "R"] \end{tikzcd}$$

if there exists a natural isomorphism α between the following pair of hom-functors of type $\mathscr{C}^{\mathrm{op}} \times \mathscr{D} \to \mathbf{Set}$:

$$\mathrm{Hom}_{\mathscr{D}}(L^{\mathrm{op}}(-), -) \overset{\alpha}{\simeq} \mathrm{Hom}_{\mathscr{C}}(-, R(-))$$

[1] The collection of morphisms of type $a \to b$ forms a set because $\mathscr{C}$ is locally small.

This relationship can be depicted graphically as 2-cell (and its inverse) in **Cat**,

$$\begin{tikzcd}
\mathscr{C}^{\mathrm{op}} \times \mathscr{D} \arrow[r, "I_{\mathscr{C}^{\mathrm{op}}} \times R"] \arrow[d, "L^{\mathrm{op}} \times I_{\mathscr{D}}"'] \arrow[dr, phantom, "\alpha \;\Uparrow\Downarrow\; \alpha^{-1}"] & \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \arrow[d, "\mathrm{Hom}_{\mathscr{C}}"] \\
\mathscr{D}^{\mathrm{op}} \times \mathscr{D} \arrow[r, "\mathrm{Hom}_{\mathscr{D}}"] & \mathbf{Set}
\end{tikzcd}$$

Concretely, the naturality of α means that for every morphism $(f^{\mathrm{op}} : b \to a, g : c \to d) \in (\mathscr{C}^{\mathrm{op}} \times \mathscr{D})_1$ the components $\alpha_{(b,c)}$ and $\alpha_{(a,d)}$ of α make the following square commute:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathscr{D}}(L^{\mathrm{op}}(b), c) \arrow[r, "\mathrm{Hom}_{\mathscr{D}}(L^{\mathrm{op}}(f^{\mathrm{op}}), g)"] \arrow[d, "\alpha_{(b,c)}"'] & \mathrm{Hom}_{\mathscr{D}}(L^{\mathrm{op}}(a), d) \arrow[d, "\alpha_{(a,d)}"] \\
\mathrm{Hom}_{\mathscr{C}}(b, R(c)) \arrow[r, "\mathrm{Hom}_{\mathscr{C}}(f^{\mathrm{op}}, R(g))"] & \mathrm{Hom}_{\mathscr{C}}(a, R(d))
\end{tikzcd}$$

1.3 Beck-Chevalley Conditions

The Beck-Chevalley Conditions are conditions that may or may not be satisfied by a quadruplet of functors F, H, G, K which form a natural isomorphism $\alpha : KF \Rightarrow HG$ square:

$$\begin{tikzcd}
\mathscr{A} \arrow[r, "F"] \arrow[d, "G"'] \arrow[dr, phantom, "\swarrow\alpha"] & \mathscr{B} \arrow[d, "K"] \\
\mathscr{C} \arrow[r, "H"'] & \mathscr{D}
\end{tikzcd}$$

To define the *left* Beck-Chevalley condition, one needs functors $F_L : \mathscr{B} \to \mathscr{A}$ and $H_L : \mathscr{D} \to \mathscr{A}$ which are respectively left adjoint functors to F and H,

$$\begin{tikzcd}
\mathscr{A} \arrow[r, bend right, "F"'] \arrow[r, phantom, "\perp"] & \mathscr{B} \arrow[l, bend right, "F_L"']
\end{tikzcd}, \qquad
\begin{tikzcd}
\mathscr{C} \arrow[r, bend right, "H"'] \arrow[r, phantom, "\perp"] & \mathscr{D} \arrow[l, bend right, "H_L"']
\end{tikzcd} .$$

Using these left adjoint functors, it becomes possible to construct a natural transformation $\beta : KH_L \Rightarrow GF_L$ from α[2]. Graphically, β can be identified as the outer cell of the following

[2]The natural transformations α and β are known as *mates* or *conjugates*.

diagram:

$$
\begin{tikzcd}
\mathscr{A} \arrow[r, bend right, "F"'] \arrow[r, phantom, "\perp"] \arrow[d, "G"'] & \mathscr{B} \arrow[l, bend right, "F_L"'] \arrow[d, "K"] \arrow[dl, Rightarrow, "\alpha"'] \\
\mathscr{C} \arrow[r, bend left, "H"] \arrow[r, phantom, "\top"] & \mathscr{D} \arrow[l, bend left, "H_L"]
\end{tikzcd}
,\qquad \text{i.e.} \qquad
\begin{tikzcd}
\mathscr{A} \arrow[d, "G"'] & \mathscr{B} \arrow[l, "F_L"'] \arrow[d, "K"] \\
\mathscr{C} & \mathscr{D} \arrow[l, "H_L"] \arrow[ul, Rightarrow, "\beta"']
\end{tikzcd}
.
$$

Although the natural transformation α is assumed to be a natural isomorphism, the natural transformation β need not be; if β happens to be a natural isomorphism, then we say that the original square satisfies the *left* Beck-Chevalley condition[3]. The *right* Beck-Chevalley condition is defined analogously with functors F_R, H_R which are respectively right adjoints $F \dashv F_R$ and $H \dashv H_L$.

1.4 Cartesian Morphism

A morhpism $\phi : e' \to e$ in $\mathscr{E}$ is *cartesian* with respect to a functor $P : \mathscr{E} \to \mathscr{B}$ if for every $\psi : e'' \to e$ in $\mathscr{E}$ and for every $s : P(e'') \to P(e)$ such that $P(\phi) \circ_{\mathscr{B}} g = P(\psi)$ (i.e. such that the second diagram commutes), there exists a unique morphism $\sigma : e'' \to e'$ in $\mathscr{E}$ such that $\phi \circ_{\mathscr{E}} \sigma = \psi$ (i.e. such that the first diagram commutes):[4]

$$
\begin{tikzcd}
e'' \arrow[r, dashed, "\exists!\sigma"] \arrow[dr, bend right, "\forall\psi"'] & e' \arrow[d, "\phi"] \\
& e
\end{tikzcd}
\qquad \overset{P}{\mapsto} \qquad
\begin{tikzcd}
P(e'') \arrow[r, "\forall g"] \arrow[dr, bend right, "P(\psi)"'] & P(e') \arrow[d, "P(\phi)"] \\
& P(e)
\end{tikzcd}
$$

1.5 Grothendieck Fibrations

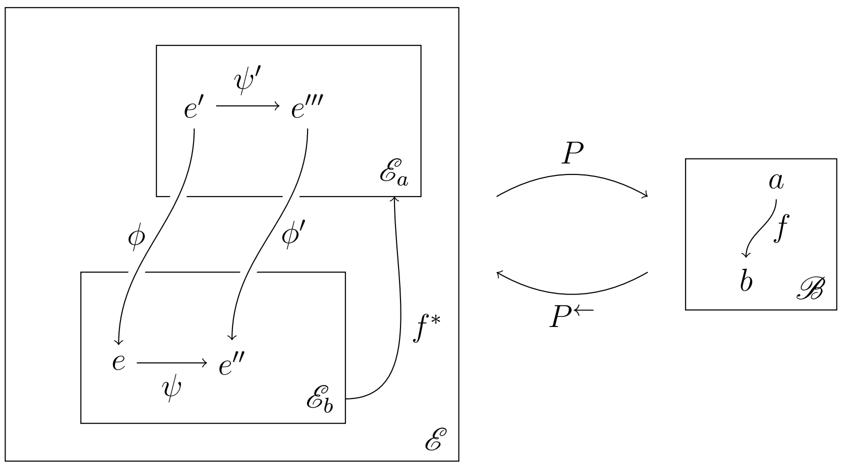
A functor $P : \mathscr{E} \to \mathscr{B}$ is a *Grothendieck fibration* if it satisfies the following “lifting” property that for every morphism $f : b \to P(e)$ of $\mathscr{B}$ (i.e. if the codomain of f is contained in the image of P), there exists a *cartesian* morphism $\phi : e' \to e$ of $\mathscr{E}$ in the fibered category $\mathscr{E}_{P(e)}$ (i.e. $P(\phi) = f$).

[3] Are the left adjoints F_L, H_L unique? If not, it might be better to say the original square satifies the left Beck-Chevalley condition with respect to F_L, H_L.

[4] The definition and treatment of Cartesian morphisms found in the *Reformulations* section of `https://ncatlab.org/nlab/show/Cartesian+morphism#CartInOrdCatReformulation` is probably better suited here.

1.6 The Equivalence of Puesdofunctors and Fibrations

Given a functor $P : \mathscr{E} \to \mathscr{B}$ which is also a Grothendieck fibration equipped with a clevage (i.e. a choice of cartesian morphism $\phi \in \mathrm{Hom}_{\mathscr{E}}(e', e)$ for each $f \in \mathrm{Hom}_{\mathscr{B}}(a, P(e))$ such that $P(\phi) = f$), it is possible to construct a pseudofunctor (read weak 2-functor between weak 2-categories) $\pi : \mathscr{B}^{\mathrm{op}} \to \mathbf{Cat}$. In particular, for each object $b \in \mathscr{B}_0$ is mapped to the *sub-category* $\pi(b) = \mathscr{E}_b$ of $\mathscr{E}$ whose objects are those which map to b under P and whose morphism are those which map to id_b under P; $\mathscr{E}_b$ is the fibre category over b with respect to P. For each morphism $f \in \mathrm{Hom}_{\mathscr{B}}(a, b)$ in $\mathscr{B}$, the pseudofunctor π maps $f^{\mathrm{op}} : b \to a$ onto a functor $\pi(f^{\mathrm{op}}) = f^* : \mathscr{E}_b \to \mathscr{E}_a$ which is defined accordingly:



Given an object $e \in (\mathscr{E}_b)_0$, the functor f^* finds the unique cartesian morphism $\phi \in \mathrm{Hom}_{\mathscr{E}}(e', e)$ as specified by the cleavage and assigns $f^*(e) = e'$. Next, given a morphism $\psi \in \mathrm{Hom}_{\mathscr{E}_b}(e, e'')$, the functor f^* first finds the unique cartesian morphisms $\phi \in \mathrm{Hom}_{\mathscr{E}}(e', e)$ and $\phi' \in \mathrm{Hom}_{\mathscr{E}}(e''', e'')$. Then, because $g = \mathrm{id}_a$ completes the following diagram

$$\begin{tikzcd} P(e') \arrow[r, "g"] \arrow[dr, bend right, "P(\psi \circ \phi)"'] & P(e''') \arrow[d, "P(\phi')"] \\ & P(e'') \end{tikzcd} \quad = \quad \begin{tikzcd} a \arrow[r, "\mathrm{id}_a"] \arrow[dr, bend right, "\mathrm{id}_b \circ f"'] & a \arrow[d, "f"] \\ & b \end{tikzcd},$$

and because ϕ' is cartesian, there must exist a unique $\psi' \in \mathrm{Hom}_{\mathscr{E}_a}(e', e''')$ such that $\psi \circ \phi = \phi' \circ \psi'$. For each $\psi \in \mathrm{Hom}_{\mathscr{E}_b}(e, e'')$, the functor f^* selects this unique morphism $f^*(\psi) = \psi'$. In summary, the pseudofunctor $\pi : \mathscr{B}^{\mathrm{op}} \to \mathbf{Cat}$ induced by $P : \mathscr{E} \to \mathscr{B}$ is defined on objects $b \in \mathscr{B}_0$ as $\pi(b) = \mathscr{E}_b$ and on morphisms $f \in \mathscr{B}_1$ as $\pi(f) = f^*$ and forms a functor [TODO: figure out the 'pseudo' part of the pseudofunctorality.].

1.7 Slice and Coslice Categories

Given a category $\mathscr{C}$ and an object $c \in \mathscr{C}_0$ of $\mathscr{C}$ the *slice category* (or *over category*) $\mathscr{C}/c$ is the "stuff in $\mathscr{C}$ that is on top of c". Specifically, the objects of $\mathscr{C}/c$ are all the morphisms $f \in \mathscr{C}_1$ from $\mathscr{C}$ whose codomain is $\mathrm{cod}(f) = c$ (alternatively you could write

$(\mathscr{C}/c)_0 = \mathrm{Hom}_{\mathscr{C}}(-, c))$. A morphism of $\mathscr{C}/c$ between objects $f : a \to c, g : b \to c \in (\mathscr{C}/c)_0$ is a commuting triangle completed by a third morphism $h : a \to b \in \mathscr{C}_1$:

$$\begin{tikzcd}
a \arrow[rr, "h"] \arrow[dr, "g"'] & & b \arrow[dl, "f"] \\
& c &
\end{tikzcd}$$

Composition of morphisms in $\mathscr{C}/c$ is induced by the composition of morphisms in $\mathscr{C}$:

$$\left(\begin{tikzcd}
y \arrow[rr, "n"] \arrow[dr, "f"'] & & z \arrow[dl, "h"] \\
& c &
\end{tikzcd}\right) \circ_{\mathscr{C}/c} \left(\begin{tikzcd}
x \arrow[rr, "m"] \arrow[dr, "g"'] & & y \arrow[dl, "f"] \\
& c &
\end{tikzcd}\right) = \begin{tikzcd}
x \arrow[r, "m"] \arrow[dr, "g"'] & y \arrow[r, "n"] \arrow[d, "f"] & z \arrow[dl, "h"] \\
& c &
\end{tikzcd}$$

The assignment of an overcategory $\mathscr{C}/c$ to each object c can be extended to a *slice functor* $\mathscr{C}/(-) : \mathscr{C} \to \mathbf{Cat}$ in the following sense. For objects $c \in \mathscr{C}_0$, the slice functor takes c to the slice category $\mathscr{C}/c$; for morphisms $f : a \to b \in \mathscr{C}_1$, the slice functor takes f to the functor $\mathscr{C}/f : \mathscr{C}/a \to \mathscr{C}/b$ defined graphically; for every morphism of $\mathscr{C}/a$ (commuting triangle in $\mathscr{C}$ over a), contruct the morphism of $\mathscr{C}/b$ (commuting triangle in $\mathscr{C}$ over b) as follows:

$$\begin{tikzcd}
l \arrow[rr, "m"] \arrow[dr, "x"] \arrow[ddr, bend right=60, "f \circ_{\mathscr{C}} x"'] & & r \arrow[dl, "x'"'] \arrow[ddl, bend left=60, "f \circ_{\mathscr{C}} x'"] \\
& a \arrow[d, "f"] & \\
& b &
\end{tikzcd}$$

where the inner triangle is a morphism of $\mathscr{C}/a$ and the outer triangle is a morphism of $\mathscr{C}/b$ given by the functor $\mathscr{C}/f$.

Given a category $\mathscr{C}$ and an object $c \in \mathscr{C}_0$ of $\mathscr{C}$ the *coslice category* (or *under category*) $c/\mathscr{C}$ is the “stuff in $\mathscr{C}$ that is underneath c”. Specifically, the objects of $c/\mathscr{C}$ are all the morphisms $f \in \mathscr{C}_1$ from $\mathscr{C}$ whose domain is $\mathrm{dom}(f) = c$ (alternatively you could write $(c/\mathscr{C})_0 = \mathrm{Hom}_{\mathscr{C}}(c, -))$. A morphism of $c/\mathscr{C}$ between objects $f : c \to a, g : c \to b \in (c/\mathscr{C})_0$ is a commuting triangle completed by a third morphism $h : a \to b \in \mathscr{C}_1$:

$$\begin{tikzcd}
& c \arrow[dl, "g"'] \arrow[dr, "f"] & \\
a \arrow[rr, "h"'] & & b
\end{tikzcd}$$

Everything about coslice categories is defined as expected analogously to that of a slice categories. [TODO: determine how the details of the Grothendieck construction transform the slice (pseudo-)functor $\mathscr{C}/(-) : \mathscr{C} \to \mathbf{Cat}$ into the codomain fibration.]

1.8 The Pullback and Pushforward Functors

Given a category $\mathscr{C}$ and a morphism $f : a \to b \in \mathscr{C}_1$, the image of f under the slice functor $\mathscr{C}/(-)$ produces a functor $\mathscr{C}/f : \mathscr{C}/a \to \mathscr{C}/b$ between slice categories of $\mathscr{C}$ in the "same direction" as f TODO: confirm that $\mathscr{C}/f$ is the pushforward functor $f_!$ of $f \in \mathscr{C}_1$.

If the given category $\mathscr{C}$ admits pullbacks, in becomes possible to define, for a morphism $f : a \to b$ a pullback functor $f^* : \mathscr{C}/b \to \mathscr{C}/a$. Given a morphism in $\mathscr{C}/b$ (commuting triangle in $\mathscr{C}$ with base at b),

$$
\begin{tikzcd}
c \arrow[rr, "k"] \arrow[dr, "g"'] & & d \arrow[dl, "h"] \\
& b &
\end{tikzcd}
$$

the pullback functor $f^* : \mathscr{C}/b \to \mathscr{C}/a$ associated with f takes the objects $g : c \to b, h : d \to b$ of $\mathscr{C}/b$ (morphisms in $\mathscr{C}$) completes the pullback squares associated with f

$$
\begin{tikzcd}
& c' \arrow[dl, "f^*g := g_f"'] \arrow[dr, "f_g"] & \\
a \arrow[dr, "f"'] & & c \arrow[dl, "g"] \\
& b &
\end{tikzcd}
\qquad
\begin{tikzcd}
& d' \arrow[dl, "f_h"'] \arrow[dr, "f^*h := h_f"] & \\
d \arrow[dr, "h"'] & & a \arrow[dl, "f"] \\
& b &
\end{tikzcd}
$$

where a subscript notation g_f means "the pullback of g along f". Defining the action of $f^* : \mathscr{C}/b \to \mathscr{C}/a$ on objects to be $f^*g = g_f$ and $f^*h = h_f$, the action on morphisms in $\mathscr{C}/b$ is defined by composing the pullback squares with the commuting triangle morphism:

$$
\begin{tikzcd}
c \arrow[rr, "k"] \arrow[dr, "g"'] & & d \arrow[dl, "h"] \\
& b & \\
& a \arrow[u, "f"'] & \\
c' \arrow[uuu, bend left, "f_g"] \arrow[ur, "f^*g"] \arrow[rr, dashed, "\exists! f^*k"'] & & d' \arrow[uuu, bend right, "f_h"'] \arrow[ul, "f^*h"']
\end{tikzcd}
$$

The commuting triangle in $\mathscr{C}/a$ appearing at the bottom is completed by a unique morphism [TODO: why does this morphism need to be unique and exist?] denoted to be f^*k ($\neq k_f$ obviously). The functoriality of f^* has a simple proof found here https://proofwiki.org/wiki/Pullback_Functor_is_Functor.

1.9 Functors of Monoidal Categories

[TODO]

1.10 Frobenius Reciprocity

[TODO]