

# On the Fibrations Underlying Optimization and Elimination

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## Abstract

*As of July 25, 2019:* The theory of fibrations and fibered categories appears to be a natural place to discuss the theory of various optimization and elimination problems, including resolution in logic, linear and non-linear quantifier elimination, polytope projection, lattice optimization over various spaces, etc. These notes aim to investigate that claim and furthermore attempts to determine any and all structural similarities between the various cases.

## Contents

<b>1</b>	<b>Potentially Relevant Examples</b>	<b>2</b>
1.1	The Subobject Fibration . . . . .	2
1.2	The category of convex polyhedra and affine maps . . . . .	3
1.3	Subset Projection . . . . .	4
1.4	Optimization of real-valued functions . . . . .	5
<b>2</b>	<b>Categorical Notions</b>	<b>6</b>
2.1	Cartesian Arrows . . . . .	6
2.2	Fibrations, Fibered Categories, and Cleavages . . . . .	7
2.3	Pseudo-Functors, Splitting Cleavages . . . . .	8
2.4	Nearby Fibrations: Opfibrations and *-fibrations . . . . .	8
2.5	Hom-Functors . . . . .	8
2.6	Adjoint Functors . . . . .	9
2.7	Beck-Chevalley Conditions . . . . .	9
2.8	Slice and Coslice Categories . . . . .	10
2.9	Functors of Monoidal Categories . . . . .	11
2.10	Frobenius Reciprocity . . . . .	11

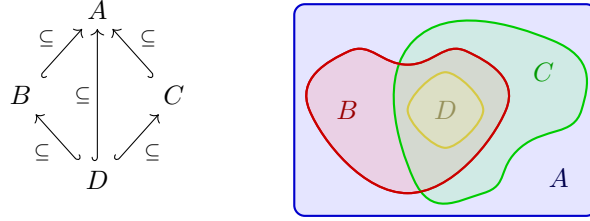
# 1 Potentially Relevant Examples

## 1.1 The Subobject Fibration

Given any category  $\mathcal{B}$ , and an object  $X$  of  $\mathcal{B}$ , a *subobject* of  $X$  is an isomorphism class of monomorphisms  $f : Y \hookrightarrow X$ ; two monomorphisms  $f : Y \hookrightarrow X, g : Z \hookrightarrow X$  with shared codomain  $X$  are isomorphic if there exists an isomorphism  $h : Y \xrightarrow{\sim} Z$  such that  $f = g \circ h$ . The individual monomorphisms of a subobject class can be equipped with a preorder relation  $(f : Y \hookrightarrow X) \leq (g : Z \hookrightarrow X)$  if there exists  $k : Y \rightarrow Z$  such that  $f = g \circ k$ :

$$\begin{array}{ccc} Y & \xrightarrow{k} & Z \\ & \searrow f & \swarrow g \\ & X & \end{array} \quad (1)$$

Note that if such a  $k : Y \rightarrow Z$  exists, then it is unique because if  $g$  is a monomorphism; if  $k' : Y \rightarrow Z$  satisfied  $f = g \circ k'$  as well, then  $g \circ k = g \circ k'$  which implies  $k = k'$ . Moreover, this preorder relation on the individual monomorphisms extends to a poset  $\text{Sub}_{\mathcal{B}}(X)$  between the subobjects. For example, if  $\mathcal{B}$  is the category  $\text{Set}$ , the subobjects of  $X$  are subsets of  $X$  and thus  $\text{Sub}_{\text{Set}}(X) \cong \mathcal{P}(X)$ . Even more concretely, if  $X$  is the set  $\mathbb{R}^2$ , an exemplary diagram of  $\text{Sub}_{\text{Set}}(\mathbb{R}^2)$  is



If the category  $\mathcal{B}$  has pullbacks, the posetal categories  $\text{Sub}_{\mathcal{B}}(X)$  for varying objects  $X$  of  $\mathcal{B}$  can be “stitched together” to form an enveloping category denoted  $\text{Sub}(\mathcal{B})$ . The objects of  $\text{Sub}(\mathcal{B})$  are the objects of  $\text{Sub}_{\mathcal{B}}(X)$  for various  $X$ . The morphisms of  $\text{Sub}(\mathcal{B})$  are defined using the morphisms of  $\mathcal{B}$ . Given an equivalence class of monomorphisms  $B = \{b_i : Z_i \hookrightarrow X\}_{i \in \mathcal{I}} \in \text{Sub}_{\mathcal{B}}(X)$  with shared codomain  $X$ , and morphism  $f : Y \rightarrow X$  (not necessarily a monomorphism) with codomain  $X$ , the pullbacks  $f^*b_i$  of  $b_i$  along  $f$  for each  $i \in \mathcal{I}$  constitute an equivalence class of monomorphisms  $f^*B := \{f^*b_i : Y \times_X Z_i \hookrightarrow Y\}_{i \in \mathcal{I}} \in \text{Sub}_{\mathcal{B}}(Y)$ .<sup>1,2</sup>

$$\begin{array}{ccc} Y \times_X Z_i & \xrightarrow{b_i^* f} & Z_i \\ f^* b_i \downarrow & \lrcorner & \downarrow b_i \\ Y & \xrightarrow{f} & X \end{array} \quad (2)$$

To every morphism  $f : Y \rightarrow X$  of  $\mathcal{B}$ , and object  $B \in \text{Sub}_{\mathcal{B}}(X)$ , denote  $\kappa_f(B) : f^*B \rightarrow B$  where  $f^*B \in \text{Sub}_{\mathcal{B}}(Y)$ . Finally, a morphism of  $\text{Sub}(\mathcal{B})$  between  $A \in \text{Sub}_{\mathcal{B}}(Y)$  and  $B \in \text{Sub}_{\mathcal{B}}(X)$  is the formal sequence  $A \xrightarrow{\leq} f^*(B) \xrightarrow{\kappa_f(B)} B$ . The projection functor  $P_{\mathcal{B}} : \text{Sub}(\mathcal{B}) \rightarrow \mathcal{B}$  defines the *subobject fibration*<sup>3</sup>.

[TODO: figure out the relationship between a subobject fibration and the codomain fibration via the notion of subterminal objects.]

<sup>1</sup>Evidently, this relies on the fact that pullbacks preserve monomorphisms; i.e.  $b : Z \hookrightarrow X$  is a monomorphism, then the pullback  $f^*b : Y \times_X Z \hookrightarrow Y$  is also.

<sup>2</sup>The isomorphisms connecting the monomorphisms of  $f^*B$  are also given by pullback of the isomorphisms connecting the monomorphisms of  $B$ .

<sup>3</sup>Note that while the fibres  $\text{Sub}_{\mathcal{B}}(X)$  are thin, the total category  $\text{Sub}(\mathcal{B})$  is not necessarily thin.

## 1.2 The category of convex polyhedra and affine maps

One of the primary motivating examples for this project is the theory of *convex polyhedra* and associated maps which implement their projections. Following Boyd and Vandenberghe [BV04], a *polyhedron*<sup>4,5</sup>  $P$  is the intersection of a finite number of halfspaces of some ambient vector space  $V$ . Equivalently, taking  $n = \dim V$ ,  $P$  can be regarded as the solution space of a finite system of  $m$  linear inequalities constraining  $n$  variables  $x = (x_1, x_2, \dots, x_n)$ :

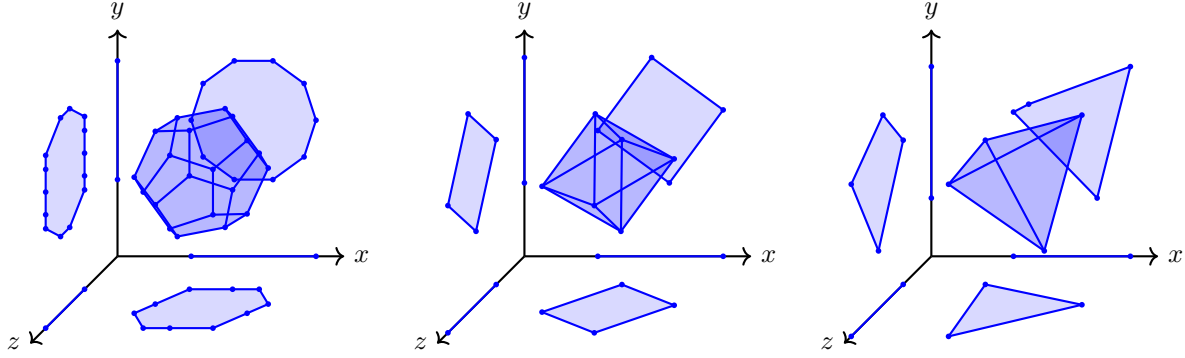
$$\begin{aligned} a_{1,1}x_1 + a_{1,2}x_2 + \dots + a_{1,n}x_n &\geq b_1, \\ a_{2,1}x_1 + a_{2,2}x_2 + \dots + a_{2,n}x_n &\geq b_2, \\ &\vdots, \\ a_{m,1}x_1 + a_{m,2}x_2 + \dots + a_{m,n}x_n &\geq b_m. \end{aligned}$$

Written compactly,

$$P = \{x \in V \mid Ax \succeq b\} \quad (3)$$

where the coefficients  $a_{i,j}$  form a  $m \times n$  matrix  $A$  and the constants  $b_i$  form a vector  $b$ .

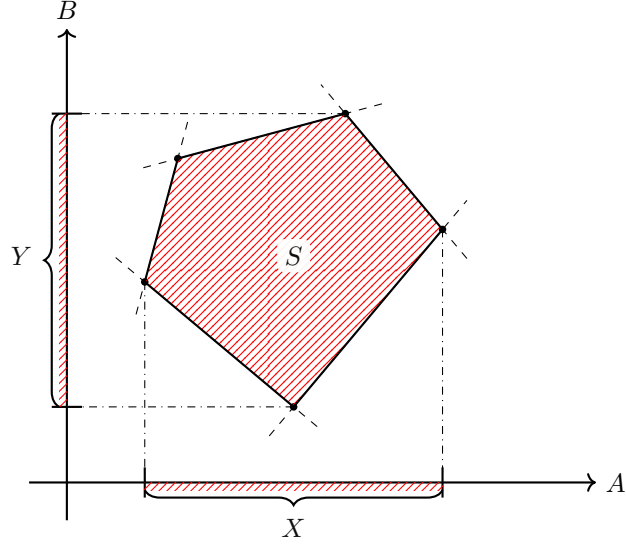
**Definition 1.1.** The category Poly consists of polyhedra as objects and affine maps between them.



$$\begin{array}{ccccc} & & \text{Poly}_Y & \xleftarrow{\pi_{Y,!}} & \text{Poly}_{X \otimes Y} \\ & \nearrow \pi_{Y,!} & \downarrow \pi_{Y \otimes Z,!} & & \nearrow \pi_{X \otimes Y,!} \\ \text{Poly}_{Y \otimes Z} & \xleftarrow{\pi_{Y \otimes Z,!}} & \text{Poly}_{X \otimes Y \otimes Z} & \xleftarrow{\pi_{X,!}} & \text{Poly}_X \\ & \downarrow \pi_{Z,!} & \downarrow \pi_{1,!} & & \downarrow \pi_{X \otimes Z,!} \\ & \text{Poly}_1 & \xleftarrow{\pi_{1,!}} & & \text{Poly}_X \\ & \downarrow \pi_{Z,!} & \downarrow \pi_{1,!} & & \downarrow \pi_{X,!} \\ \text{Poly}_Z & \xleftarrow{\pi_{Z,!}} & \text{Poly}_{X \otimes Z} & & \end{array} \quad (4)$$

<sup>4</sup>The term polytope will be reserved for the context of *bounded polyhedron*. Note that the opposite convention is sometimes used by other authors as pointed out by [BV04].

<sup>5</sup>Alternative and sometimes inequivalent definitions for “polyhedra” do exist; oftentimes, these alternative definitions accommodate more general notions of polyhedra, such as non-convex polyhedra. Understanding the relationship between these various definitions, and the proposal of new ones, is a mathematical endeavour which dates back to antiquity and continues today [Gru03; Lak15].



### 1.3 Subset Projection

A prototypical example wherein an adjoint triple

$$f_!, \exists_f \dashv f^*, f^{-1} \dashv f^!, \forall_f$$

arises is that of functions  $f : X \rightarrow Y$  between sets  $X$  and  $Y$ . The inverse image functor  $f^* : \mathcal{P}Y \rightarrow \mathcal{P}X$  is defined on a subset  $T \subseteq Y$

$$f^*(T) = \{x \in X : f(x) \in T\},$$

and is functorial in the sense that if  $T \subseteq T' \subseteq Y$  then  $f^*(T) \subseteq f^*(T') \subseteq f^*(T)$ . The adjoint functors  $\exists_f, \forall_f : \mathcal{P}X \rightarrow \mathcal{P}Y$  are defined on  $S \subseteq X$  as

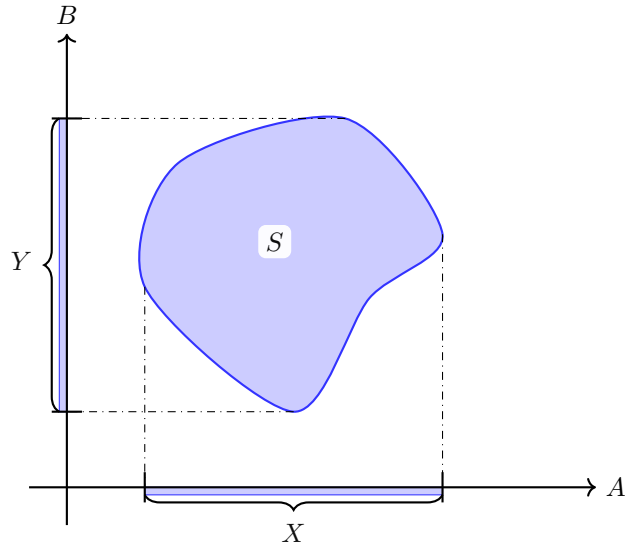
$$\exists_f(S) = \{y \in Y : \exists x \in f^*(y) : x \in S\}$$

$$\forall_f(S) = \{y \in Y : \forall x \in f^*(y) : x \in S\}$$

form an adjoint triple in the sense that  $\exists_f \dashv f^* \dashv \forall_f$ :

$$\exists_f \dashv f^* : \quad \exists_f(S) \subseteq T \iff S \subseteq f^*(T)$$

$$f^* \dashv \forall_f : \quad f^*(T) \subseteq R \iff T \subseteq \forall_f(R)$$

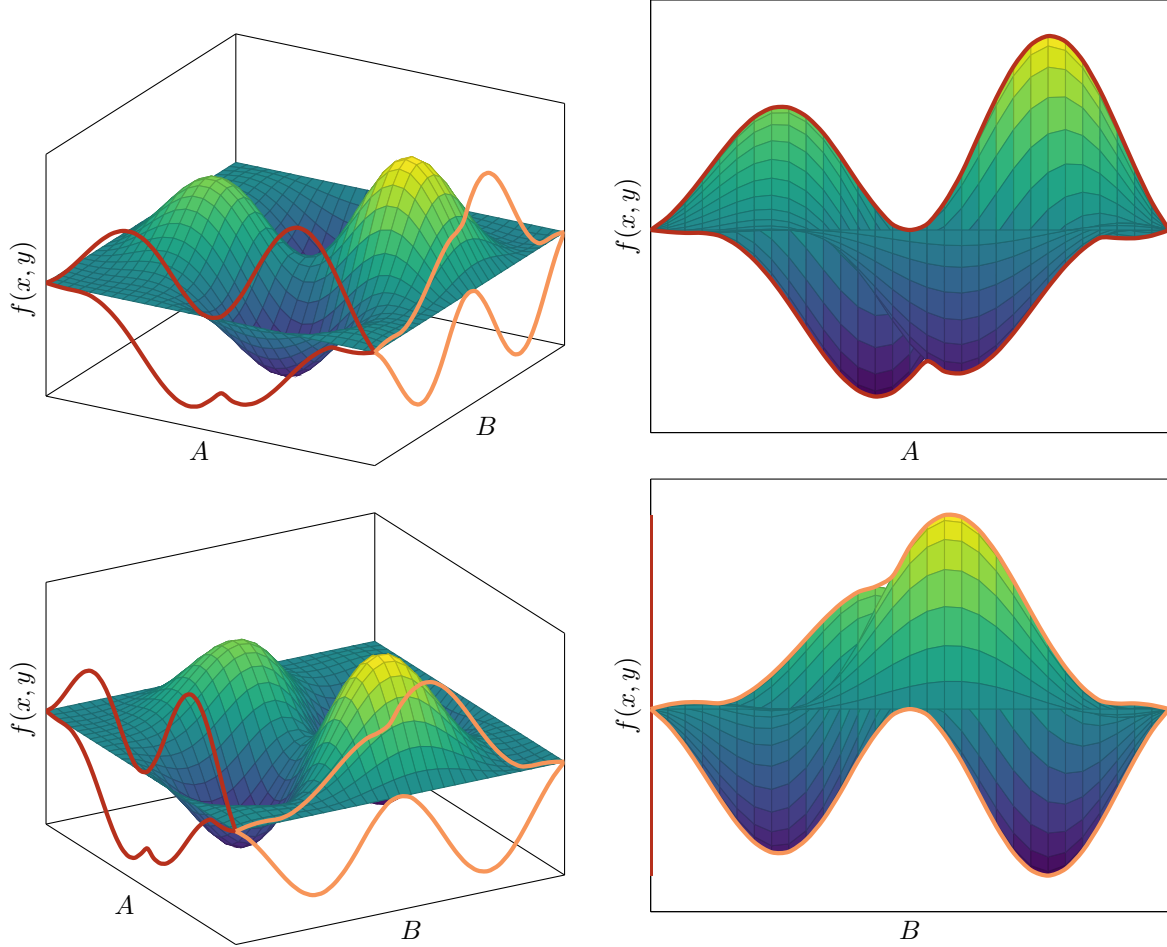


Consider a pair of sets  $A$  and  $B$  and a subset  $S \subseteq A \times B$  of their cartesian product. The projection morphisms associated with  $A \times B$  are  $p : A \times B \rightarrow A$  and  $q : A \times B \rightarrow B$ . The projection of the subset  $S$  onto  $A$  is then the subset  $X \subseteq A$  defined by:

$$X = \{a \in A \mid \exists s \in S, p(s) = a\}$$

$$S \subseteq p^*(X) \iff \exists_p(S) \subseteq X \quad (5)$$

#### 1.4 Optimization of real-valued functions



## 2 Categorical Notions

The following unordered list of categorical concepts are anticipated to be utilized:

- adjunctions
- fibered categories
- cleavages
- pseudo functors (and if cleavages are splitting, functors)
- Beck-Chevalley condition
- Frobenius reciprocity (and functors of monoidal categories)

**T:** Cleavages are not really important because for any two different choices of cleavage, the resulting pullback functors are naturally isomorphic. So cleavages are just a technical tool relevant for proving the equivalence between fibered cats and pseudofunctors, but not relevant in practice

### 2.1 Cartesian Arrows

**Definition 2.1.** Let  $P : \mathcal{E} \rightarrow \mathcal{B}$  be a functor between categories  $\mathcal{E}$  and  $\mathcal{B}$ . An arrow  $\phi : \alpha \rightarrow \beta$  of  $\mathcal{E}$  is *cartesian* with respect to  $P$  (sometimes *P-cartesian*) if for every arrow  $\psi : \gamma \rightarrow \beta$  sharing a codomain with  $\phi$ , and for every arrow  $g : P(\gamma) \rightarrow P(\alpha)$  in  $\mathcal{B}$  satisfying  $g \circ P(\phi) = P(\psi)$ , there exists a unique arrow  $\theta : \gamma \rightarrow \alpha$  in  $\mathcal{E}$  satisfying  $\phi \circ \theta = \psi$  and  $P(\theta) = g$ .

$$(6)$$

**Corollary 2.0.1.** A cartesian morphism  $\phi : \alpha \rightarrow \beta$  in  $\mathcal{E}$  with respect to a functor  $P : \mathcal{E} \rightarrow \mathcal{B}$  establishes an isomorphism of categories <sup>Lurie2009higher</sup>[Lur09, Section 2.4.1]<sup>6</sup>

$$\mathcal{E}/\phi \cong \mathcal{E}/\beta \times_{\mathcal{B}/P(\beta)} \mathcal{B}/P(\phi) \quad (7)$$

where  $\mathcal{E}/\beta \times_{\mathcal{B}/P(\beta)} \mathcal{B}/P(\phi)$  is the pullback of functors.

$$(8)$$

<sup>6</sup>This formulation is also discussed here: <https://ncatlab.org/nlab/show/Cartesian+morphism#CartInOrdCatReformulation>.

The pullback category  $\mathcal{E}/\beta \times_{\mathcal{B}/P(\beta)} \mathcal{B}/P(\phi)$  has morphisms associated with diagrams of  $\mathcal{B}$  with the following format:

$$\begin{array}{ccc}
 & P(\gamma) & \\
 f \swarrow & \downarrow P(\chi) & \searrow P(\omega) \\
 & P(\delta) & \\
 g \swarrow & & \searrow P(\psi) \\
 P(\alpha) & \xrightarrow{P(\phi)} & P(\beta)
 \end{array} \tag{9}$$

eq:pullback\_

Evidently, if  $\phi : \alpha \rightarrow \beta$  is cartesian, then there exists unique morphisms  $\zeta : \gamma \rightarrow \alpha$  and  $\eta : \delta \rightarrow \alpha$  such that  $P(\zeta) = f$  and  $P(\eta) = g$  and the following diagram of  $\mathcal{E}$  commutes:

$$\begin{array}{ccc}
 & \gamma & \\
 \zeta \swarrow & \downarrow \chi & \searrow \omega \\
 & \delta & \\
 \eta \swarrow & & \searrow \psi \\
 \alpha & \xrightarrow{\phi} & \beta
 \end{array} \tag{10}$$

eq:cartesian

Intuitively, if  $\phi$  is cartesian, then in order to determine the category  $\mathcal{E}/\phi$  over  $\phi$ , it is sufficient to specify  $\mathcal{E}/\beta \times_{\mathcal{B}/P(\beta)} \mathcal{B}/P(\phi)$ .

## 2.2 Fibrations, Fibered Categories, and Cleavages

**Definition 2.2.** A *fibered category over  $\mathcal{B}$*  is a category  $\mathcal{E}$  associated to the domain of a functor, referred to as the *fibration*,  $P : \mathcal{E} \rightarrow \mathcal{B}$  with the property that for every morphism  $f : a \rightarrow b$  of  $\mathcal{B}$  and object  $\beta$  such that  $P(\beta) = b$ , there exists a cartesian arrow  $\phi : \alpha \rightarrow \beta$  with  $P(\phi) = f$ .

**Lemma 2.1.** A fibration  $P : \mathcal{E} \rightarrow \mathcal{B}$  is a faithful functor if and only if its fibers are thin.

*Proof.* Recall that if  $P : \mathcal{E} \rightarrow \mathcal{B}$  is a faithful functor, then by definition every pair of parallel arrows  $\phi, \psi : \alpha \rightarrow \beta$  in  $\mathcal{E}$  satisfies

$$P(\phi) = P(\psi) : P(\alpha) \rightarrow P(\beta) \implies \phi = \psi. \tag{11}$$

eq:faithfuln

$\implies$  : Assuming  $P : \mathcal{E} \rightarrow \mathcal{B}$  is faithful functor, consider an arbitrary pair of parallel arrows  $\phi, \psi : \alpha \rightarrow \beta$  in an arbitrary fiber  $\mathcal{E}_x$  over  $x$ ; i.e.  $P(\phi) = P(\psi) = \text{id}_x$ . In such cases, faithfulness of  $P$  (Eq. [11](#)) guarantees that  $\phi = \psi$  and thus  $\mathcal{E}_x$  is a thin category.

$\impliedby$  : If the fiber  $\mathcal{E}_x$  for every object  $x$  in  $\mathcal{B}$  is a thin category, then clearly  $P : \mathcal{E} \rightarrow \mathcal{B}$  must be faithful when restricted to an individual fiber. The non-trivial case is to consider an arbitrary pair of parallel morphisms  $\phi, \psi : \alpha \rightarrow \beta$  not belonging to any fibers of  $\mathcal{E}$ . Denote  $a := P(\alpha)$  and  $b := P(\beta)$  and suppose  $f := P(\phi) = P(\psi) : a \rightarrow b$ . Then, because  $\mathcal{E}$  is a fibered category, there exists a cartesian arrow  $\zeta : \gamma \rightarrow \beta$ , such that  $P(\zeta) = f$  (note that  $a = P(\alpha) = P(\gamma)$  but  $\gamma$  is not necessarily equal to  $\alpha$ ). Since  $\zeta$  is a cartesian

arrow, there exists a unique arrows  $\mu, \nu : \alpha \rightarrow \gamma$  completing the top edges of the following diagram:

$$\begin{array}{ccccc}
 \alpha & \xrightarrow{\mu} & \gamma & \xleftarrow{\nu} & \alpha \\
 \downarrow \psi & & \downarrow \zeta & & \downarrow \phi \\
 \beta & & \beta & & \beta \\
 \downarrow & & \downarrow & & \downarrow \\
 a & \xrightarrow{=} & a & \xrightarrow{=} & a \\
 \downarrow f & & \downarrow f & & \downarrow f \\
 b & & b & & b
 \end{array}
 \quad (12)$$

However,  $P(\nu) = \text{id}_a = P(\mu)$  and therefore  $\mu$  and  $\nu$  are parallel arrows in the fiber  $\mathcal{E}_a$  and therefore  $\mu = \nu$  because  $\mathcal{E}_a$  is assumed thin. Therefore,  $\psi = \zeta \circ \mu = \zeta \circ \nu = \phi$  and thus  $P$  is a faithful functor.  $\square$

**Definition 2.3.** A *cleavage* for a fibration  $P : \mathcal{E} \rightarrow \mathcal{B}$  is an assignment to each morphism  $f : a \rightarrow b$  of  $\mathcal{B}$  and object  $\beta$  in  $\mathcal{E}_b$  (i.e.  $P(\beta) = b$ ), a unique cartesian morphism  $\kappa_f(\beta)$  of  $\mathcal{E}$  such that  $P(\kappa_f(\beta)) = f$ .

Given a cleavage for a fibration, the cartesianness of morphisms within a cleavage permits one to establish functors between the fibers of the fibration. This concept is visualized in the following figure:

(13)

## 2.3 Pseudo-Functors, Splitting Cleavages

Pages 47-48 of [\[Vis04\]](#) <sup>vistoli2004notes</sup> explicate the notions of pseudo-functors and their equivalence to fibrations with cleavages. Moreover if the cleavage is splitting, the induced pseudo-functor is in fact a functor.

## 2.4 Nearby Fibrations: Opfibrations and \*-fibrations

Given a functor  $P : \mathcal{E} \rightarrow \mathcal{B}$ , it can be considered as a fibration in many different ways. For example, if  $P^{\text{op}} : \mathcal{E}^{\text{op}} \rightarrow \mathcal{B}^{\text{op}}$  is a fibration, then  $P$  is said to be an *opfibration*.

# Categorical Definitions

## 2.5 Hom-Functors

For a locally small category  $\mathcal{C}$ , the hom-functor of  $\mathcal{C}$  is a functor  $\text{Hom}_{\mathcal{C}} : \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathbf{Set}$  constructed in the following manner. Given objects  $a, b, c, \dots \in \mathcal{C}_0$  of  $\mathcal{C}$ , the hom-functor  $\text{Hom}_{\mathcal{C}}$  maps a pair of objects  $(a, b) \in (\mathcal{C}^{\text{op}} \times \mathcal{C})_0 = \mathcal{C}_0 \times \mathcal{C}_0 = \mathcal{C}_0^2$  into the set<sup>7</sup> of morphisms  $\mathcal{C}_1$  of  $\mathcal{C}$  with source  $a$  and target  $b$ . Therefore,

<sup>7</sup>The collection of morphisms of type  $a \rightarrow b$  forms a set because  $\mathcal{C}$  is locally small.



$\text{Hom}_{\mathcal{C}}(a, b)$  is the set of morphisms in  $\mathcal{C}$  of type  $a \rightarrow b$ . Given morphisms  $g^{\text{op}} \in \text{Hom}_{\mathcal{C}^{\text{op}}}(a, c)$  and  $h \in \text{Hom}_{\mathcal{C}}(b, d)$ , the hom-functor  $\text{Hom}_{\mathcal{C}}$  constructs a function

$$\text{Hom}_{\mathcal{C}}(g^{\text{op}}, h) : \text{Hom}_{\mathcal{C}}(a, b) \rightarrow \text{Hom}_{\mathcal{C}}(c, d)$$

which takes a morphism  $f : a \rightarrow b \in \text{Hom}_{\mathcal{C}}(a, b)$  and produces the morphism  $h \circ f \circ g : c \rightarrow d \in \text{Hom}_{\mathcal{C}}(c, d)$ . Graphically,

$$\text{Hom}_{\mathcal{C}}(g^{\text{op}}, h) \left( a \xrightarrow{f} b \right) = c \xrightarrow{g} a \xrightarrow{f} b \xrightarrow{h} d$$

## 2.6 Adjoint Functors

Given two categories  $\mathcal{C}$  and  $\mathcal{D}$ , a pair of functors  $L : \mathcal{C} \rightarrow \mathcal{D}, R : \mathcal{D} \rightarrow \mathcal{C}$  are called an *adjoint pair*, denoted  $L \dashv R$  or

$$\begin{array}{ccc} & L & \\ \mathcal{C} & \xrightleftharpoons{\perp} & \mathcal{D} \\ & R & \end{array}$$

if there exists a natural isomorphism  $\alpha$  between the following pair of hom-functors of type  $\mathcal{C}^{\text{op}} \times \mathcal{D} \rightarrow \text{Set}$ :

$$\text{Hom}_{\mathcal{D}}(L^{\text{op}}(-), -) \xrightarrow{\alpha} \text{Hom}_{\mathcal{C}}(-, R(-))$$

This relationship can be depicted graphically as 2-cell (and its inverse) in  $\text{Cat}$ ,

$$\begin{array}{ccc} \mathcal{C}^{\text{op}} \times \mathcal{D} & \xrightarrow{I_{\mathcal{C}^{\text{op}}} \times R} & \mathcal{C}^{\text{op}} \times \mathcal{C} \\ \downarrow L^{\text{op}} \times I_{\mathcal{D}} & \alpha \swarrow \quad \searrow \alpha^{-1} & \downarrow \text{Hom}_{\mathcal{C}} \\ \mathcal{D}^{\text{op}} \times \mathcal{D} & \xrightarrow{\text{Hom}_{\mathcal{D}}} & \text{Set} \end{array}$$

Concretely, the naturality of  $\alpha$  means that for every morphism  $(f^{\text{op}} : b \rightarrow a, g : c \rightarrow d) \in (\mathcal{C}^{\text{op}} \times \mathcal{D})_1$  the components  $\alpha_{(b, c)}$  and  $\alpha_{(a, d)}$  of  $\alpha$  make the following square commute:

$$\begin{array}{ccc} \text{Hom}_{\mathcal{D}}(L^{\text{op}}(b), c) & \xrightarrow{\text{Hom}_{\mathcal{D}}(L^{\text{op}}(f^{\text{op}}), g)} & \text{Hom}_{\mathcal{D}}(L^{\text{op}}(a), d) \\ \downarrow \alpha_{(b, c)} & & \downarrow \alpha_{(a, d)} \\ \text{Hom}_{\mathcal{C}}(b, R(c)) & \xrightarrow{\text{Hom}_{\mathcal{C}}(f^{\text{op}}, R(g))} & \text{Hom}_{\mathcal{C}}(a, R(d)) \end{array}$$

## 2.7 Beck-Chevalley Conditions

The Beck-Chevalley Conditions are conditions that may or may not be satisfied by a quadruplet of functors  $F, H, G, K$  which form a natural isomorphism  $\alpha : KF \Rightarrow HG$  square:

$$\begin{array}{ccc}
A & \xrightarrow{F} & B \\
G \downarrow & \alpha \swarrow & \downarrow K \\
C & \xrightarrow{H} & D
\end{array}$$

To define the *left* Beck-Chevalley condition, one needs functors  $F_L : \mathcal{B} \rightarrow \mathcal{A}$  and  $H_L : \mathcal{D} \rightarrow \mathcal{A}$  which are respectively left adjoint functors to  $F$  and  $H$ ,

$$\begin{array}{ccc}
& F_L & \\
A & \xleftarrow{\quad} & B \\
& \perp & \\
& F &
\end{array}, \quad
\begin{array}{ccc}
& H_L & \\
C & \xleftarrow{\quad} & D \\
& \perp & \\
& H &
\end{array}.$$

Using these left adjoint functors, it becomes possible to construct a natural transformation  $\beta : KH_L \Rightarrow GF_L$  from  $\alpha$ <sup>8</sup>. Graphically,  $\beta$  can be identified as the outer cell of the following diagram:

$$\begin{array}{ccc}
\begin{array}{ccc}
& F_L & \\
A & \xleftarrow{\quad} & B \\
& \perp & \\
& F & \\
G \downarrow & \alpha \swarrow & \downarrow K \\
C & \xrightarrow{H} & D \\
& \top & \\
& H_L &
\end{array} & \text{i.e.} & \begin{array}{ccc}
& F_L & \\
A & \xleftarrow{\quad} & B \\
& \beta \swarrow & \downarrow K \\
C & \xleftarrow{H_L} & D
\end{array}
\end{array}$$

Although the natural transformation  $\alpha$  is assumed to be a natural isomorphism, the natural transformation  $\beta$  need not be; if  $\beta$  happens to be a natural isomorphism, then we say that the original square satisfies the *left* Beck-Chevalley condition<sup>9</sup>. The *right* Beck-Chevalley condition is defined analogously with functors  $F_R, H_R$  which are respectively right adjoints  $F \dashv F_R$  and  $H \dashv H_R$ .

## 2.8 Slice and Coslice Categories

Given a category  $\mathcal{C}$  and an object  $c \in \mathcal{C}_0$  of  $\mathcal{C}$ , the *slice category* (or *over category*)  $\mathcal{C}/c$  is the “stuff in  $\mathcal{C}$  that is on top of  $c$ ”. Specifically, the objects of  $\mathcal{C}/c$  are all the morphisms  $f \in \mathcal{C}_1$  from  $\mathcal{C}$  whose codomain is  $\text{cod}(f) = c$  (alternatively you could write  $(\mathcal{C}/c)_0 = \text{Hom}_{\mathcal{C}}(-, c)$ ). A morphism of  $\mathcal{C}/c$  between objects  $f : a \rightarrow c, g : b \rightarrow c \in (\mathcal{C}/c)_0$  is a commuting triangle completed by a third morphism  $h : a \rightarrow b \in \mathcal{C}_1$ :

$$\begin{array}{ccc}
a & \xrightarrow{h} & b \\
g \searrow & & \swarrow f \\
& c &
\end{array}$$

<sup>8</sup>The natural transformations  $\alpha$  and  $\beta$  are known as *mates* or *conjugates*.

<sup>9</sup>Are the left adjoints  $F_L, H_L$  unique? If not, it might be better to say the original square satisfies the left Beck-Chevalley condition with respect to  $F_L, H_L$ .

Composition of morphisms in  $\mathcal{C}/c$  is induced by the composition of morphisms in  $\mathcal{C}$ :

$$\left( \begin{array}{ccc} y & \xrightarrow{n} & z \\ & \searrow f & \nearrow h \\ & c & \end{array} \right) \circ_{\mathcal{C}/c} \left( \begin{array}{ccc} x & \xrightarrow{m} & y \\ & \searrow g & \nearrow f \\ & c & \end{array} \right) = \begin{array}{ccccc} & x & \xrightarrow{m} & y & \xrightarrow{n} & z \\ & & \searrow g & & \nearrow f & \nearrow h \\ & & c & & & \end{array}$$

The assignment of an overcategory  $\mathcal{C}/c$  to each object  $c$  can be extended to a *slice functor*  $\mathcal{C}/(-) : \mathcal{C} \rightarrow \mathbf{Cat}$  in the following sense. For objects  $c \in \mathcal{C}_0$ , the slice functor takes  $c$  to the slice category  $\mathcal{C}/c$ ; for morphisms  $f : a \rightarrow b \in \mathcal{C}_1$ , the slice functor takes  $f$  to the functor  $\mathcal{C}/f : \mathcal{C}/a \rightarrow \mathcal{C}/b$  defined graphically; for every morphism of  $\mathcal{C}/a$  (commuting triangle in  $\mathcal{C}$  over  $a$ ), construct the morphism of  $\mathcal{C}/b$  (commuting triangle in  $\mathcal{C}$  over  $b$ ) as follows:

$$\begin{array}{ccc} l & \xrightarrow{m} & r \\ & \searrow x & \nearrow x' \\ & a & \\ & \downarrow f & \\ & b & \end{array} \quad \begin{array}{c} f \circ_{\mathcal{C}} x \\ \quad \quad \quad f \circ_{\mathcal{C}} x' \end{array}$$

where the inner triangle is a morphism of  $\mathcal{C}/a$  and the outer triangle is a morphism of  $\mathcal{C}/b$  given by the functor  $\mathcal{C}/f$ .

Given a category  $\mathcal{C}$  and an object  $c \in \mathcal{C}_0$  of  $\mathcal{C}$  the *coslice category* (or *under category*)  $c/\mathcal{C}$  is the “stuff in  $\mathcal{C}$  that is underneath  $c$ ”. Specifically, the objects of  $c/\mathcal{C}$  are all the morphisms  $f \in \mathcal{C}_1$  from  $\mathcal{C}$  whose domain is  $\text{dom}(f) = c$  (alternatively you could write  $(c/\mathcal{C})_0 = \text{Hom}_{\mathcal{C}}(c, -)$ ). A morphism of  $c/\mathcal{C}$  between objects  $f : c \rightarrow a, g : c \rightarrow b \in (c/\mathcal{C})_0$  is a commuting triangle completed by a third morphism  $h : a \rightarrow b \in \mathcal{C}_1$ :

$$\begin{array}{ccc} & c & \\ g \swarrow & & \searrow f \\ a & \xrightarrow{h} & b \end{array}$$

Everything about coslice categories is defined as expected analogously to that of a slice categories. [TODO: determine how the details of the Grothendieck construction transform the slice (pseudo-)functor  $\mathcal{C}/(-) : \mathcal{C} \rightarrow \mathbf{Cat}$  into the codomain fibration.]

## 2.9 Functors of Monoidal Categories

[TODO]

## 2.10 Frobenius Reciprocity

[TODO]

## Comments on selected references

This section is temporary and reserved for recording comments toward various references.

- [Vistoli 2004 notes](#) [Vis04]
- [Street 1974 and 1974 fibrations](#) [Str74]

- [Koudenburg2018](#) [Koudenburg2018categorical](#)  
Koudenburg [Kou18]
- [brown2009algebraic](#) [brown2009algebraic](#)  
Brown and Sivera [BS09]
- [lurie2009higher](#) [lurie2009higher](#)  
Lurie [Lur09]
- [shulman2008framed](#) [shulman2008framed](#)  
Shulman [Shu08]
- [boyd2004convex](#) [boyd2004convex](#)  
Boyd and Vandenberghe [BV04]
- [bogart2013hom](#) [bogart2013hom](#)  
Bogart, Contois, and Gubeladze [BCG13]
- [gubeladze2016affine](#) [gubeladze2016affine](#)  
Gubeladze [Gub16]
- [fausk2003isomorphisms](#) [fausk2003isomorphisms](#)  
Fausk, Hu, and May [FHM03]
- [hofstra2011dialectica](#) [hofstra2011dialectica](#)  
Hofstra [Hof11]
- [ponto2012duality](#) [ponto2012duality](#)  
Ponto and Shulman [PS12]
- Spectrahedron are interesting semi-algebraic sets. (<https://www.youtube.com/watch?v=AevFRN5sx0U>).

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