

On the Fibrations Underlying Optimization and Elimination

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Abstract

As of July 25, 2019: The theory of fibrations and fibered categories appears to be a natural place to discuss the theory of various optimization and elimination problems, including resolution in logic, linear and non-linear quantifier elimination, polytope projection, lattice optimization over various spaces, etc. These notes aim to investigate that claim and furthermore attempts to determine any and all structural similarities between the various cases.

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Notation Proposals

- **Aff** the category of affine spaces (?) and affine maps between them
- **Vect** the category of vector spaces and the linear maps between them
- **Poly** the category of polyhedra and the affine maps between them
- **Cone** the category of cones and the linear maps between them

1 Introduction

Below is a provisional list of various notions of “elimination”:

- The **resolution rule** of propositional (and also first order) logics. Two clauses containing a complementary literals (e.g. variable c in one and its negation $\neg c$ in the other) entails a clause with the complementary literals eliminated (see *Ground resolvents and Ground resolution* in [\[Rob+65\]](#)):¹

$$\frac{a_1 \vee a_2 \vee \dots \vee c, \quad b_1 \vee b_2 \vee \dots \vee \neg c}{a_1 \vee a_2 \vee \dots \vee b_1 \vee b_2 \vee \dots}$$

Equivalently,

$$\frac{(\neg a_1 \wedge \neg a_2 \wedge \dots) \rightarrow c, \quad c \rightarrow (b_1 \vee b_2 \vee \dots)}{(\neg a_1 \wedge \neg a_2 \wedge \dots) \rightarrow (b_1 \vee b_2 \vee \dots)}$$

This generalizes to arbitrary conjunctions of literals which may or may not reference c or $\neg c$.

- The incremental step of **Fourier-Motzkin elimination** [\[ziegler2012lectures\]](#) [\[Zie12\]](#) for systems of linear inequalities. Given

$$a_0 + a_1 x_1 + a_2 x_2 + \dots + a_n x_n \geq 0, \quad b_0 + b_1 x_1 + b_2 x_2 + \dots + b_n x_n \geq 0$$

with $a_1 > 0$ and $b_1 < 0$, then

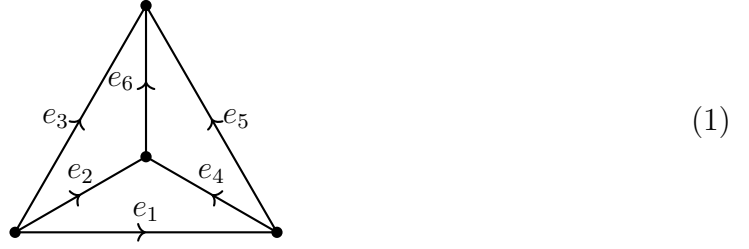
$$\left(\frac{a_0}{a_1} + \frac{a_2}{a_1} x_2 + \dots + \frac{a_n}{a_1} x_n \right) - \left(\frac{b_0}{b_1} + \frac{b_2}{b_1} x_2 + \dots + \frac{b_n}{b_1} x_n \right) \geq 0$$

This generalizes to arbitrary systems of linear inequalities over a set of variables containing x_1 .

- The **elimination axiom of oriented matroids** [\[bjoerner1999oriented\]](#) [\[Bjo+99\]](#), circuit axiom (C3)]. Given two circuits $X_0 = (X_0^+, X_0^-)$, $X_1 = (X_1^+, X_1^-)$ (with $X_0 \neq -X_1$), and an element $e \in X_0^+ \cap X_1^-$ which is positively oriented in one circuit and negatively oriented in the other, then the circuits can be “glued” along e producing a new circuit $X = (X^+, X^-)$

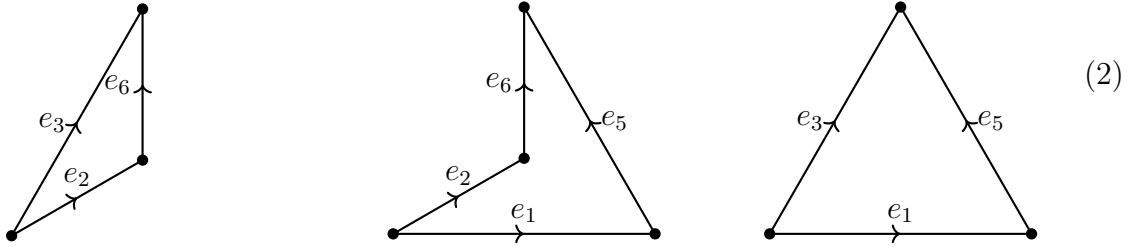
¹Reference [https://en.wikipedia.org/wiki/Resolution_\(logic\)](https://en.wikipedia.org/wiki/Resolution_(logic)).

satisfying $X^+ \subseteq X_0^+ \cup X_1^+ \setminus \{e\}$ and $X^- \subseteq X_0^- \cup X_1^- \setminus \{e\}$ (i.e. it at least eliminates e). For example, the oriented matroid generated by the cycles of the following graph



satisfies the elimination axiom. The following example eliminates e_2 (an indirectly eliminates e_6):

$$X_0 = (\{e_2, e_6\}, \{e_3\}) \quad X_1 = (\{e_1, e_5\}, \{e_2, e_6\}) \quad X = (\{e_1, e_5\}, \{e_3\})$$



TC: Generally, this “elimination” of $n - 1$ surfaces by gluing together n -dimensional surfaces reminds me of the analogous idea in the homology theory of polyhedra; assign to each n -dimensional face the sum of the $n - 1$ faces *incidence* to it (its boundary) as a formal sum in the free abelian group of all $n - 1$ faces modulo 2 (the modulo 2 carries out the unoriented elimination).

2 Category Theory Terminology

The following unordered list of categorical concepts are anticipated to be utilized:

- adjunctions
- fibered categories
- cleavages
- pseudo functors (and if cleavages are splitting, functors)
- Beck-Chevalley condition
- Frobenius reciprocity (and functors of monoidal categories)

Tobias: Cleavages are not really important because for any two different choices of cleavage, the resulting pullback functors are naturally isomorphic. So cleavages are just a technical tool relevant for proving the equivalence between fibred cats and pseudofunctors, but not relevant in practice

TC: The above comment makes sense. Overall there are isomorphisms lurking behind every corner: first, there are natural isomorphisms present when considering the equivalence between pseudo-functors and "cleaved" fibered categories, and second, whenever the cartesian arrows are indeed pullbacks, they are unique up to unique isomorphism and thus entire cleavages are unique up to unique isomorphisms. For a discussion see [vistol12004notes](#) [\[Vis04\]](#) at the end of Section 3.1.3. starting on page 50.

2.1 Cartesian Arrows

Definition 2.1. Let $P : \mathcal{E} \rightarrow \mathcal{B}$ be a functor between categories \mathcal{E} and \mathcal{B} . An arrow $\phi : \alpha \rightarrow \beta$ of \mathcal{E} is *cartesian* with respect to P (sometimes *P-cartesian*) if for every arrow $\psi : \gamma \rightarrow \beta$ sharing a codomain with ϕ , and for every arrow $g : P(\gamma) \rightarrow P(\alpha)$ in \mathcal{B} satisfying $g \circ P(\phi) = P(\psi)$, there exists a unique arrow $\theta : \gamma \rightarrow \alpha$ in \mathcal{E} satisfying $\phi \circ \theta = \psi$ and $P(\theta) = g$.

$$\begin{array}{ccccc}
 & & \forall \psi & & \\
 & \gamma & \xrightarrow{\quad} & \beta & \\
 & \downarrow & \searrow \exists! \theta & \xrightarrow{\quad} & \downarrow \\
 & P(\gamma) & \xrightarrow{P(\psi)} & P(\alpha) & \xrightarrow{P(\phi)} & P(\beta) \\
 & & \downarrow & \downarrow & \\
 & & \forall g & &
 \end{array}
 \tag{3}$$

Corollary 2.0.1. A cartesian morphism $\phi : \alpha \rightarrow \beta$ in \mathcal{E} with respect to a functor $P : \mathcal{E} \rightarrow \mathcal{B}$ establishes an isomorphism of categories [\[Lur09, Section 2.4.1\]²](#)

$$\mathcal{E}/\phi \cong \mathcal{E}/\beta \times_{\mathcal{B}/P(\beta)} \mathcal{B}/P(\phi)
 \tag{4}$$

where $\mathcal{E}/\beta \times_{\mathcal{B}/P(\beta)} \mathcal{B}/P(\phi)$ is the pullback of functors.

$$\begin{array}{ccccc}
 \mathcal{E}/\phi & \xrightarrow{P/\phi} & \mathcal{B}/P(\phi) & & \\
 \downarrow \text{cod} & \swarrow \cong & \downarrow \text{cod} & & \\
 \mathcal{E}/\beta \times_{\mathcal{B}/P(\beta)} \mathcal{B}/P(\phi) & \xrightarrow{\quad} & \mathcal{B}/P(\phi) & & \\
 \downarrow \lrcorner & & \downarrow \text{cod} & & \\
 \mathcal{E}/\beta & \xrightarrow{P/\beta} & \mathcal{B}/P(\beta) & &
 \end{array}
 \tag{5}$$

²This formulation is also discussed here: <https://ncatlab.org/nlab/show/Cartesian+morphism#CartInOrdCatReformulation>.

The pullback category $\mathcal{E}/\beta \times_{\mathcal{B}/P(\beta)} \mathcal{B}/P(\phi)$ has morphisms associated with diagrams of \mathcal{B} with the following format:

$$\begin{array}{ccc}
 & P(\gamma) & \\
 f \swarrow & \downarrow P(\chi) & \searrow P(\omega) \\
 & P(\delta) & \\
 g \swarrow & & \searrow P(\psi) \\
 P(\alpha) & \xrightarrow{P(\phi)} & P(\beta)
 \end{array}$$

(6) eq:pullback

Evidently, if $\phi : \alpha \rightarrow \beta$ is cartesian, then there exists unique morphisms $\zeta : \gamma \rightarrow \alpha$ and $\eta : \delta \rightarrow \alpha$ such that $P(\zeta) = f$ and $P(\eta) = g$ and the following diagram of \mathcal{E} commutes:

$$\begin{array}{ccc}
 & \gamma & \\
 \zeta \swarrow & \downarrow \chi & \searrow \omega \\
 & \delta & \\
 \eta \swarrow & & \searrow \psi \\
 \alpha & \xrightarrow{\phi} & \beta
 \end{array}$$

(7) eq:cartesian

Intuitively, if ϕ is cartesian, then in order to determine the category \mathcal{E}/ϕ over ϕ , it is sufficient to specify $\mathcal{E}/\beta \times_{\mathcal{B}/P(\beta)} \mathcal{B}/P(\phi)$.

2.2 Fibrations, Fibered Categories, and Cleavages

Definition 2.2. A *fibered category* over \mathcal{B} is a category \mathcal{E} associated to the domain of a functor, referred to as the *fibration*, $P : \mathcal{E} \rightarrow \mathcal{B}$ with the property that for every morphism $f : a \rightarrow b$ of \mathcal{B} and object β such that $P(\beta) = b$, there exists a cartesian arrow $\phi : \alpha \rightarrow \beta$ with $P(\phi) = f$.

Lemma 2.1. A fibration $P : \mathcal{E} \rightarrow \mathcal{B}$ is a faithful functor if and only if its fibers are thin.

Proof. Recall that if $P : \mathcal{E} \rightarrow \mathcal{B}$ is a faithful functor, then by definition every pair of parallel arrows $\phi, \psi : \alpha \rightarrow \beta$ in \mathcal{E} satisfies

$$P(\phi) = P(\psi) : P(\alpha) \rightarrow P(\beta) \implies \phi = \psi. \quad (8)$$

eq:faithful

\implies : Assuming $P : \mathcal{E} \rightarrow \mathcal{B}$ is faithful functor, consider an arbitrary pair of parallel arrows $\phi, \psi : \alpha \rightarrow \beta$ in an arbitrary fiber \mathcal{E}_x over x ; i.e. $P(\phi) = P(\psi) = \text{id}_x$. In such cases, faithfulness of P (Eq. 8) guarantees that $\phi = \psi$ and thus \mathcal{E}_x is a thin category.

\impliedby : If the fiber \mathcal{E}_x for every object x in \mathcal{B} is a thin category, then clearly $P : \mathcal{E} \rightarrow \mathcal{B}$ must be faithful when restricted to an individual fiber. The non-trivial case is to consider an

arbitrary pair of parallel morphisms $\phi, \psi : \alpha \rightarrow \beta$ not belonging to any fibers of \mathcal{E} . Denote $a := P(\alpha)$ and $b := P(\beta)$ and suppose $f := P(\phi) = P(\psi) : a \rightarrow b$. Then, because \mathcal{E} is a fibered category, there exists a cartesian arrow $\zeta : \gamma \rightarrow \beta$, such that $P(\zeta) = f$ (note that $a = P(\alpha) = P(\gamma)$ but γ is not necessarily equal to α). Since ζ is a cartesian arrow, there exists a unique arrows $\mu, \nu : \alpha \rightarrow \gamma$ completing the top edges of the following diagram:

$$\begin{array}{ccccc}
 \alpha & \xrightarrow{\quad \mu \quad} & \gamma & \xleftarrow{\quad \nu \quad} & \alpha \\
 \downarrow \psi & & \downarrow \zeta & & \downarrow \phi \\
 & & \beta & & \\
 \downarrow & & \downarrow & & \downarrow \\
 a & \xrightarrow{\quad f \quad} & b & \xleftarrow{\quad f \quad} & a
 \end{array}
 \quad (9)$$

However, $P(\nu) = \text{id}_a = P(\mu)$ and therefore μ and ν are parallel arrows in the fiber \mathcal{E}_a and therefore $\mu = \nu$ because \mathcal{E}_a is assumed thin. Therefore, $\psi = \zeta \circ \mu = \zeta \circ \nu = \phi$ and thus P is a faithful functor. \square

Definition 2.3. A *cleavage* for a fibration $P : \mathcal{E} \rightarrow \mathcal{B}$ is an assignment to each morphism $f : a \rightarrow b$ of \mathcal{B} and object β in \mathcal{E}_b (i.e. $P(\beta) = b$), a unique cartesian morphism $\kappa_f(\beta)$ of \mathcal{E} such that $P(\kappa_f(\beta)) = f$.

Given a cleavage for a fibration, the cartesianness of morphisms within a cleavage permits one to establish functors between the fibers of the fibration. This concept is visualized in the following figure:

$$\begin{array}{c}
 \text{Top section (Fiber objects): } \alpha \xrightarrow{\mu} \gamma \xleftarrow{\nu} \alpha \\
 \text{Middle section (Pullback objects): } f^*\alpha \xrightarrow{f^*\phi} f^*\beta, \quad g^*\alpha \xrightarrow{g^*\phi} g^*\beta \\
 \text{Bottom section (Base objects): } a \xrightarrow{f} b \xleftarrow{g} c \\
 \text{Key morphisms: } \kappa_f(\alpha), \kappa_f(\beta), \kappa_g(\alpha), \kappa_g(\beta), \kappa_f(\gamma), \kappa_g(\gamma) \\
 \text{Regions: } \mathcal{E}_a^{\kappa_f(\gamma)}, \mathcal{E}_b, \mathcal{E}_c, \mathcal{E}
 \end{array}
 \quad (10)$$

2.3 Pseudo-Functors, Splitting Cleavages

Pages 47-48 of [\[Vis04\]](#) ^{vistoli2004notes} explicate the notions of pseudo-functors and their equivalence to fibrations with cleavages. Moreover if the cleavage is splitting, the induced pseudo-functor is

in fact a functor.

2.4 Nearby Fibrations: Opfibrations and *-fibrations

Given a functor $P : \mathcal{E} \rightarrow \mathcal{B}$, it can be considered as a fibration in many different ways. For example, if $P^{\text{op}} : \mathcal{E}^{\text{op}} \rightarrow \mathcal{B}^{\text{op}}$ is a fibration, then P is said to be an *opfibration*.

2.5 Hom-Functors

For a locally small category \mathcal{C} , the hom-functor of \mathcal{C} is a functor $\text{Hom}_{\mathcal{C}} : \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathbf{Set}$ constructed in the following manner. Given objects $a, b, c, \dots \in \mathcal{C}_0$ of \mathcal{C} , the hom-functor $\text{Hom}_{\mathcal{C}}$ maps a pair of objects $(a, b) \in (\mathcal{C}^{\text{op}} \times \mathcal{C})_0 = \mathcal{C}_0 \times \mathcal{C}_0 = \mathcal{C}_0^2$ into the set³ of morphisms \mathcal{C}_1 of \mathcal{C} with source a and target b . Therefore, $\text{Hom}_{\mathcal{C}}(a, b)$ is the set of morphisms in \mathcal{C} of type $a \rightarrow b$. Given morphisms $g^{\text{op}} \in \text{Hom}_{\mathcal{C}^{\text{op}}}(a, c)$ and $h \in \text{Hom}_{\mathcal{C}}(b, d)$, the hom-functor $\text{Hom}_{\mathcal{C}}$ constructs a function

$$\text{Hom}_{\mathcal{C}}(g^{\text{op}}, h) : \text{Hom}_{\mathcal{C}}(a, b) \rightarrow \text{Hom}_{\mathcal{C}}(c, d)$$

which takes a morphism $f : a \rightarrow b \in \text{Hom}_{\mathcal{C}}(a, b)$ and produces the morphism $h \circ f \circ g : c \rightarrow d \in \text{Hom}_{\mathcal{C}}(c, d)$. Graphically,

$$\text{Hom}_{\mathcal{C}}(g^{\text{op}}, h) \left(a \xrightarrow{f} b \right) = c \xrightarrow{g} a \xrightarrow{f} b \xrightarrow{h} d$$

2.6 Adjoint Functors

Given two categories \mathcal{C} and \mathcal{D} , a pair of functors $L : \mathcal{C} \rightarrow \mathcal{D}, R : \mathcal{D} \rightarrow \mathcal{C}$ are called an *adjoint pair*, denoted $L \dashv R$ or

$$\begin{array}{ccc} & L & \\ \mathcal{C} & \xrightleftharpoons[\perp]{} & \mathcal{D} \\ & R & \end{array}$$

if there exists a natural isomorphism α between the following pair of hom-functors of type $\mathcal{C}^{\text{op}} \times \mathcal{D} \rightarrow \mathbf{Set}$:

$$\text{Hom}_{\mathcal{D}}(L^{\text{op}}(-), -) \xrightarrow{\alpha} \text{Hom}_{\mathcal{C}}(-, R(-))$$

This relationship can be depicted graphically as 2-cell (and its inverse) in \mathbf{Cat} ,

$$\begin{array}{ccc} \mathcal{C}^{\text{op}} \times \mathcal{D} & \xrightarrow{I_{\mathcal{C}^{\text{op}}} \times R} & \mathcal{C}^{\text{op}} \times \mathcal{C} \\ \downarrow L^{\text{op}} \times I_{\mathcal{D}} & \alpha \swarrow \quad \searrow \alpha^{-1} & \downarrow \text{Hom}_{\mathcal{C}} \\ \mathcal{D}^{\text{op}} \times \mathcal{D} & \xrightarrow{\text{Hom}_{\mathcal{D}}} & \mathbf{Set} \end{array}$$

³The collection of morphisms of type $a \rightarrow b$ forms a set because \mathcal{C} is locally small.

Concretely, the naturality of α means that for every morphism $(f^{\text{op}} : b \rightarrow a, g : c \rightarrow d) \in (\mathcal{C}^{\text{op}} \times \mathcal{D})_1$ the components $\alpha_{(b,c)}$ and $\alpha_{(a,d)}$ of α make the following square commute:

$$\begin{array}{ccc} \text{Hom}_{\mathcal{D}}(L^{\text{op}}(b), c) & \xrightarrow{\text{Hom}_{\mathcal{D}}(L^{\text{op}}(f^{\text{op}}), g)} & \text{Hom}_{\mathcal{D}}(L^{\text{op}}(a), d) \\ \downarrow \alpha_{(b,c)} & & \downarrow \alpha_{(a,d)} \\ \text{Hom}_{\mathcal{C}}(b, R(c)) & \xrightarrow{\text{Hom}_{\mathcal{C}}(f^{\text{op}}, R(g))} & \text{Hom}_{\mathcal{C}}(a, R(d)) \end{array}$$

2.7 Beck-Chevalley Conditions

The Beck-Chevalley Conditions are conditions that may or may not be satisfied by a quadruplet of functors F, H, G, K which form a natural isomorphism $\alpha : KF \Rightarrow HG$ square:

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{F} & \mathcal{B} \\ G \downarrow & \alpha \swarrow & \downarrow K \\ \mathcal{C} & \xrightarrow{H} & \mathcal{D} \end{array}$$

To define the *left* Beck-Chevalley condition, one needs functors $F_L : \mathcal{B} \rightarrow \mathcal{A}$ and $H_L : \mathcal{D} \rightarrow \mathcal{A}$ which are respectively left adjoint functors to F and H ,

$$\mathcal{A} \begin{array}{c} \xleftarrow{F_L} \\ \perp \\ \xrightarrow{F} \end{array} \mathcal{B}, \quad \mathcal{C} \begin{array}{c} \xleftarrow{H_L} \\ \perp \\ \xrightarrow{H} \end{array} \mathcal{D}.$$

Using these left adjoint functors, it becomes possible to construct a natural transformation $\beta : KH_L \Rightarrow GF_L$ from α ⁴. Graphically, β can be identified as the outer cell of the following diagram:

$$\begin{array}{ccc} \begin{array}{ccc} & F_L & \\ \mathcal{A} & \begin{array}{c} \xleftarrow{\quad} \\ \perp \\ \xrightarrow{\quad} \end{array} & \mathcal{B} \\ G \downarrow & \begin{array}{c} F \\ \alpha \swarrow \\ H \end{array} & \downarrow K \\ \mathcal{C} & \begin{array}{c} \xleftarrow{\quad} \\ \top \\ \xrightarrow{\quad} \end{array} & \mathcal{D} \\ & H_L & \end{array} & \text{i.e.} & \begin{array}{ccc} & F_L & \\ \mathcal{A} & \xleftarrow{\quad} & \mathcal{B} \\ G \downarrow & \begin{array}{c} \beta \swarrow \\ \end{array} & \downarrow K \\ \mathcal{C} & \xleftarrow{H_L} & \mathcal{D} \end{array} \end{array}$$

Although the natural transformation α is assumed to be a natural isomorphism, the natural transformation β need not be; if β happens to be a natural isomorphism, then we say that

⁴The natural transformations α and β are known as *mates* or *conjugates*.

the original square satisfies the *left* Beck-Chevalley condition⁵. The *right* Beck-Chevalley condition is defined analogously with functors F_R, H_R which are respectively right adjoints $F \dashv F_R$ and $H \dashv H_R$.

2.8 Slice and Coslice Categories

Given a category \mathcal{C} and an object $c \in \mathcal{C}_0$ of \mathcal{C} , the *slice category* (or *over category*) \mathcal{C}/c is the “stuff in \mathcal{C} that is on top of c ”. Specifically, the objects of \mathcal{C}/c are all the morphisms $f \in \mathcal{C}_1$ from \mathcal{C} whose codomain is $\text{cod}(f) = c$ (alternatively you could write $(\mathcal{C}/c)_0 = \text{Hom}_{\mathcal{C}}(-, c)$). A morphism of \mathcal{C}/c between objects $f : a \rightarrow c, g : b \rightarrow c \in (\mathcal{C}/c)_0$ is a commuting triangle completed by a third morphism $h : a \rightarrow b \in \mathcal{C}_1$:

$$\begin{array}{ccc} a & \xrightarrow{h} & b \\ & \searrow g & \swarrow f \\ & c & \end{array}$$

Composition of morphisms in \mathcal{C}/c is induced by the composition of morphisms in \mathcal{C} :

$$\left(\begin{array}{ccc} y & \xrightarrow{n} & z \\ & \searrow f & \swarrow h \\ & c & \end{array} \right) \circ_{\mathcal{C}/c} \left(\begin{array}{ccc} x & \xrightarrow{m} & y \\ & \searrow g & \swarrow f \\ & c & \end{array} \right) = \begin{array}{ccccc} & x & \xrightarrow{m} & y & \xrightarrow{n} & z \\ & \searrow g & & \swarrow f & & \swarrow h \\ & & c & & & \end{array}$$

The assignment of an overcategory \mathcal{C}/c to each object c can be extended to a *slice functor* $\mathcal{C}/(-) : \mathcal{C} \rightarrow \mathbf{Cat}$ in the following sense. For objects $c \in \mathcal{C}_0$, the slice functor takes c to the slice category \mathcal{C}/c ; for morphisms $f : a \rightarrow b \in \mathcal{C}_1$, the slice functor takes f to the functor $\mathcal{C}/f : \mathcal{C}/a \rightarrow \mathcal{C}/b$ defined graphically; for every morphism of \mathcal{C}/a (commuting triangle in \mathcal{C} over a), construct the morphism of \mathcal{C}/b (commuting triangle in \mathcal{C} over b) as follows:

$$\begin{array}{ccc} l & \xrightarrow{m} & r \\ & \searrow x & \swarrow x' \\ & a & \\ & \downarrow f & \\ & b & \end{array} \quad \begin{array}{c} f \circ_{\mathcal{C}} x \\ \quad \quad \quad \\ f \circ_{\mathcal{C}} x' \end{array}$$

where the inner triangle is a morphism of \mathcal{C}/a and the outer triangle is a morphism of \mathcal{C}/b given by the functor \mathcal{C}/f .

Given a category \mathcal{C} and an object $c \in \mathcal{C}_0$ of \mathcal{C} the *coslice category* (or *under category*) c/\mathcal{C} is the “stuff in \mathcal{C} that is underneath c ”. Specifically, the objects of c/\mathcal{C} are all the

⁵Are the left adjoints F_L, H_L unique? If not, it might be better to say the original square satisfies the left Beck-Chevalley condition with respect to F_L, H_L .

morphisms $f \in \mathcal{C}_1$ from \mathcal{C} whose domain is $\text{dom}(f) = c$ (alternatively you could write $(c/\mathcal{C})_0 = \text{Hom}_{\mathcal{C}}(c, -)$). A morphism of c/\mathcal{C} between objects $f : c \rightarrow a, g : c \rightarrow b \in (c/\mathcal{C})_0$ is a commuting triangle completed by a third morphism $h : a \rightarrow b \in \mathcal{C}_1$:

$$\begin{array}{ccc} & c & \\ g \swarrow & & \searrow f \\ a & \xrightarrow{h} & b \end{array}$$

Everything about coslice categories is defined as expected analogously to that of a slice categories.

TODO: determine how the details of the Grothendieck construction transform the slice (pseudo-)functor $\mathcal{C}/(-) : \mathcal{C} \rightarrow \mathbf{Cat}$ into the codomain fibration

2.9 Functors of Monoidal Categories

[TODO]

2.10 Frobenius Reciprocity

[TODO]

3 Case Studies of Interest

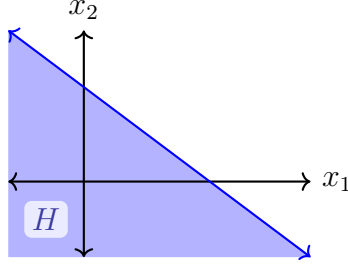
3.1 Polyhedra and Affine Maps

One of the primary motivating examples for this project is the theory of *(finite) convex polyhedra* and the affine maps between them. Following Boyd and Vandenberghe [BV04], a *polyhedron*^{6,7} P is the intersection of a finite number of *halfspaces* of some ambient vector space $V \cong \mathbb{R}^n$. A *halfspace* $H \subseteq \mathbb{R}^n$ is a subset of a vector space (of dimension n) which is the solution set of a linear inequality constraint over canonical coordinates $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$:

$$H = \{x \in V \mid a^\top x = \sum_{i=1}^n a_i x_i \geq b\} \quad (11)$$

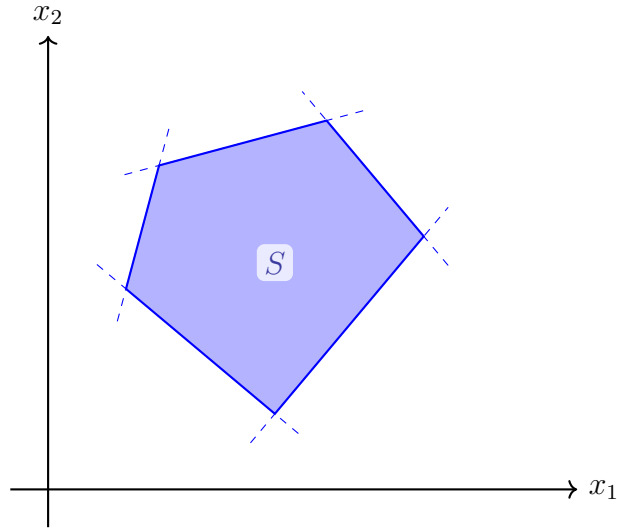
⁶The term polytope will be reserved for the context of *bounded polyhedron*. Note that the opposite convention is sometimes used by other authors as pointed out by [BV04].

⁷Alternative and sometimes inequivalent definitions for “polyhedra” do exist; oftentimes, these alternative definitions accomodate more general notions of polyhedra, such as non-convex polyhedra. Understanding the relationship between these various definitions, and the proposal of new ones, is a mathematical endeavour which dates back to antiquity and continues today [Grü03; Lak15].



As previously mentioned, a polyhedra is the intersection of finitely many halfspaces and therefore corresponds to

$$P = \{x \in V \mid \bigwedge_{j=1}^k (a_j^T x \geq b_j)\} \quad (12)$$



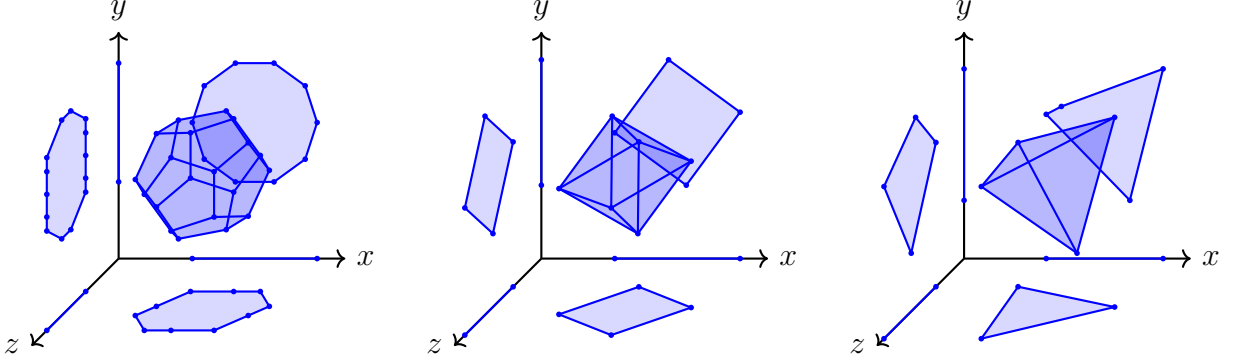
$$\begin{array}{ccc} f^*(\mathbb{R}_+^k) & \xrightarrow{f'} & \mathbb{R}_+^k \\ \downarrow j & \lrcorner & \downarrow i \\ \mathbb{R}^n & \xrightarrow{f} & \mathbb{R}^k \end{array} \quad (13) \quad \boxed{\text{eq:pullback}}$$

In Equation [13](#), the morphism j is simply the inclusion of the polyhedra $f^*(\mathbb{R}_+^k)$ into its ambient vector space \mathbb{R}^n and the morphism f' is provided by the restriction of f onto $f^*(\mathbb{R}_+^k)$.

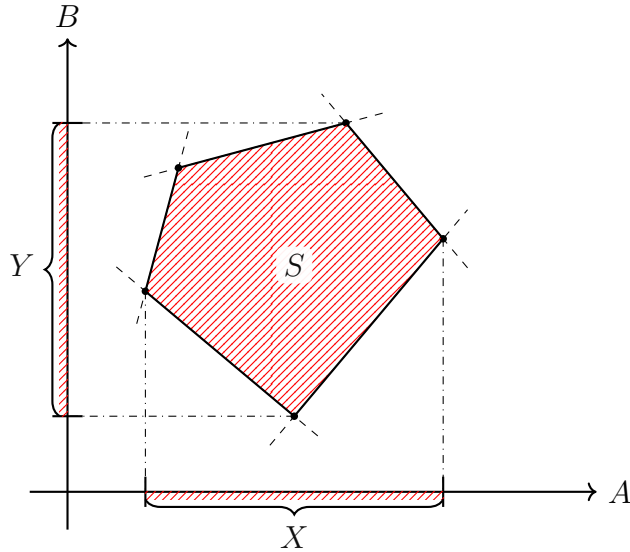
3.2 The Beck-Chevalley Condition for the Polyhedral Fibration

Consider a commuting square of affine maps between vector spaces in the base category of affine maps

Definition 3.1. The category **Poly** consists of polyhedra as objects and affine maps between them.



$$\begin{array}{ccccc}
 & \text{Poly}_Y & \xleftarrow{\pi_{Y,!}} & \text{Poly}_{X \otimes Y} & \\
 \nearrow \pi_{Y,!} & \downarrow \pi_{Y \otimes Z,!} & & \nearrow \pi_{X \otimes Y,!} & \\
 \text{Poly}_{Y \otimes Z} & \xleftarrow{\pi_{1,!}} & \text{Poly}_{X \otimes Y \otimes Z} & \xrightarrow{\pi_{X,!}} & \text{Poly}_X \\
 \downarrow \pi_{Z,!} & \downarrow \pi_{1,!} & \downarrow \pi_{X \otimes Z,!} & & \downarrow \pi_{X,!} \\
 & \text{Poly}_1 & \xleftarrow{\pi_{1,!}} & & \text{Poly}_X \\
 \downarrow \pi_{Z,!} & \nearrow \pi_{1,!} & & \nearrow \pi_{X,!} & \\
 \text{Poly}_Z & \xleftarrow{\pi_{Z,!}} & \text{Poly}_{X \otimes Z} & &
 \end{array} \tag{14}$$



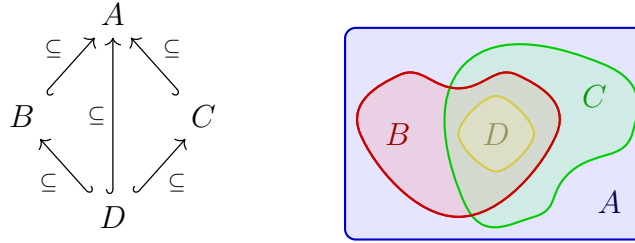
3.3 The Subobject Fibration

Given any category \mathcal{B} , and an object X of \mathcal{B} , a *subobject* of X is an isomorphism class of monomorphisms $f : Y \hookrightarrow X$; two monomorphisms $f : Y \hookrightarrow X, g : Z \hookrightarrow X$ with shared codomain X are isomorphic if there exists an isomorphism $h : Y \xrightarrow{\sim} Z$ such that $f = g \circ h$. The individual monomorphisms of a subobject class can be equipped with a preorder relation

$(f : Y \hookrightarrow Z) \leq (g : Z \hookrightarrow X)$ if there exists $k : Y \rightarrow Z$ such that $f = g \circ k$:

$$\begin{array}{ccc} Y & \xrightarrow{k} & Z \\ & \searrow f & \swarrow g \\ & X & \end{array} \quad (15)$$

Note that if such a $k : Y \rightarrow Z$ exists, then it is unique because g is a monomorphism; if $k' : Y \rightarrow Z$ satisfied $f = g \circ k'$ as well, then $g \circ k = g \circ k'$ which implies $k = k'$. Moreover, this preorder relation on the individual monomorphisms extends to a poset $\mathbf{Sub}_{\mathcal{B}}(X)$ between the subobjects. For example, if \mathcal{B} is the category \mathbf{Set} , the subobjects of X are subsets of X and thus $\mathbf{Sub}_{\mathbf{Set}}(X) \cong \mathbf{P}(X)$. Even more concretely, if X is the set \mathbb{R}^2 , an exemplary diagram of $\mathbf{Sub}_{\mathbf{Set}}(\mathbb{R}^2)$ is



If the category \mathcal{B} has pullbacks, the posetal categories $\mathbf{Sub}_{\mathcal{B}}(X)$ for varying objects X of \mathcal{B} can be “stitched together” to form an enveloping category denoted $\mathbf{Sub}(\mathcal{B})$. The objects of $\mathbf{Sub}(\mathcal{B})$ are the objects of $\mathbf{Sub}_{\mathcal{B}}(X)$ for various X . The morphisms of $\mathbf{Sub}(\mathcal{B})$ are defined using the morphisms of \mathcal{B} . Given an equivalence class of monomorphisms $B = \{b_i : Z_i \hookrightarrow X\}_{i \in \mathcal{I}} \in \mathbf{Sub}_{\mathcal{B}}(X)$ with shared codomain X , and morphism $f : Y \rightarrow X$ (not necessarily a monomorphism) with codomain X , the pullbacks f^*b_i of b_i along f for each $i \in \mathcal{I}$ constitute an equivalence class of monomorphisms $f^*B := \{f^*b_i : Y \times_X Z_i \hookrightarrow Y\}_{i \in \mathcal{I}} \in \mathbf{Sub}_{\mathcal{B}}(Y)$:^{8,9}

$$\begin{array}{ccc} Y \times_X Z_i & \xrightarrow{b_i^* f} & Z_i \\ f^* b_i \downarrow & \lrcorner & \downarrow b_i \\ Y & \xrightarrow{f} & X \end{array} \quad (16)$$

To every morphism $f : Y \rightarrow X$ of \mathcal{B} , and object $B \in \mathbf{Sub}_{\mathcal{B}}(X)$, denote $\kappa_f(B) : f^*B \rightarrow B$ where $f^*B \in \mathbf{Sub}_{\mathcal{B}}(Y)$. Finally, a morphism of $\mathbf{Sub}(\mathcal{B})$ between $A \in \mathbf{Sub}_{\mathcal{B}}(Y)$ and $B \in \mathbf{Sub}_{\mathcal{B}}(X)$ is the formal sequence $A \xrightarrow{\leq} f^*(B) \xrightarrow{\kappa_f(B)} B$. The projection functor $P_{\mathcal{B}} : \mathbf{Sub}(\mathcal{B}) \rightarrow \mathcal{B}$ defines the *subobject fibration*¹⁰.

TODO: figure out the relationship between a subobject fibration and the codomain fibration via the notion of **subterminal objects**.

⁸Evidently, this relies on the fact that pullbacks preserve monomorphisms; i.e. $b : Z \hookrightarrow X$ is a monomorphism, then the pullback $f^*b : Y \times_X Z \hookrightarrow Y$ is also.

⁹The isomorphisms connecting the monomorphisms of f^*B are also given by pullback of the isomorphisms connecting the monomorphisms of B .

¹⁰Note that while the fibres $\mathbf{Sub}_{\mathcal{B}}(X)$ are thin, the total category $\mathbf{Sub}(\mathcal{B})$ is not necessarily thin.

3.4 The Beck-Chevalley Condition for Any Subobject Fibration

Given any category \mathcal{B} with all pullbacks, the functor $P_{\mathcal{B}} : \mathbf{Sub}_{\mathcal{B}} \rightarrow \mathcal{B}$ which sends subobjects $[\psi]$ to their shared codomains, constitutes a bifibration. In particular, given a morphism $f : a \rightarrow b$ of \mathcal{B} , there is an induced adjoint pair of functors between the subobject fibers:

$$\begin{array}{ccc} & \xrightarrow{f_!} & \\ \mathbf{Sub}_{\mathcal{B}}(a) & \perp & \mathbf{Sub}_{\mathcal{B}}(b) \\ & \xleftarrow{f^*} & \end{array} \quad (17)$$

Specifically, left adjoint functor $f_!$ acting on a subobject $[\psi] : \mathbf{Sub}_{\mathcal{B}}(a)$ (where $[\psi]$ is the equivalence class of monomorphisms into a containing $\psi : a' \hookrightarrow a$) is given by post-composition $f_!([\psi]) = [f\psi] \in \mathbf{Sub}_{\mathcal{B}}(b)$ in \mathcal{B} .

Tobias: That works for the codomain fibration, but not here, because $f\psi$ is generally not a subobject. Rather, $f_!([\psi])$ should be the *image* of the morphism $f\psi$, in the sense of [image factorization](#) (please check whether this works). Thus one needs to assume that image factorizations exist

The right adjoint functor f^* acting on a subobject $[\phi] \in \mathbf{Sub}_{\mathcal{B}}(b)$ is given by pullback of $a \xrightarrow{f} b \xleftarrow{\phi} b'$ in \mathcal{B} :

$$\begin{array}{ccc} f^*b' & \longrightarrow & b' \\ f^*\phi \downarrow & \lrcorner & \downarrow \phi \\ a & \xrightarrow{f} & b \end{array} \quad (18)$$

Generally speaking, it is important to determine how diagrams in the base category \mathcal{B} behave when lifted to the total category $\mathbf{Sub}_{\mathcal{B}}$ under the associated pseudo-functors. For example, given a pullback square in base category \mathcal{B} ,

$$\begin{array}{ccc} a & \xrightarrow{f} & b \\ h \downarrow & \lrcorner & \downarrow g \\ c & \xrightarrow{k} & d \end{array} \quad (19)$$

There exists a natural isomorphism α between the functors f^*g^* and h^*k^* :

$$\begin{array}{ccc} \mathbf{Sub}_{\mathcal{B}}(a) & \xleftarrow{f^*} & \mathbf{Sub}_{\mathcal{B}}(b) \\ h^* \uparrow & \xleftarrow[\alpha]{\sim} & \uparrow g^* \\ \mathbf{Sub}_{\mathcal{B}}(c) & \xleftarrow{k^*} & \mathbf{Sub}_{\mathcal{B}}(d) \end{array} \quad (20)$$

The existence of components $\alpha_{[\psi]}$ for the subobject class containing the monomorphism

$\psi : p \hookrightarrow d$ can be determined by examining the following diagram:

$$\begin{array}{ccccc}
 & & & f^*g^*\psi & f^*g^*p \\
 & & & \curvearrowright & \downarrow \lrcorner \\
 & & a & \xrightarrow{f} & b \xleftarrow{g^*\psi} g^*p \\
 & & \downarrow h \lrcorner & & \downarrow g \lrcorner \\
 & & c & \xrightarrow{k} & d \\
 & & \uparrow k^*\psi \lrcorner & & \swarrow \psi \\
 h^*k^*p & \longrightarrow & k^*p & \longrightarrow & p
 \end{array}
 \quad (21)$$

Using the composition of pullback squares, it is clear that *both* f^*g^*p and h^*k^*p are pullbacks of $a \xrightarrow{gf=kh} d \xleftarrow{\psi} p$ and therefore, they are unique up to a unique isomorphism which will be denoted β . It is important to notice that this does not depend on a being a pullback itself.

$$\begin{array}{ccccc}
 & & \beta & & \\
 & & \curvearrowright & & \\
 & & \beta^{-1} & & \\
 h^*k^*p & & & & f^*g^*p \\
 \downarrow h^*k^*\psi \lrcorner & & & & \downarrow \lrcorner \\
 a & \xrightarrow{gf=kh} & d & \xleftarrow{\psi} & p
 \end{array}
 \quad (22)$$

These isomorphisms actually establish that $h^*k^*\psi \simeq f^*g^*\psi$ and therefore they belong to the same equivalence class $[h^*k^*\psi] = [f^*g^*\psi]$ for each ψ . Therefore, it becomes clear that the natural isomorphism is indeed the identity natural transformation $\text{id}_{f^*g^*} = \text{id}_{h^*k^*} = \alpha : f^*g^* \rightarrow h^*k^*$.¹¹

Alternatively, one can consider whether or not there exists natural transformations (or natural isomorphisms) of the form $\alpha : h_!f^* \rightarrow k^*g_!$. Remembering that the left adjoints $f_!$ are specified by post-composition $f_![\psi] \rightarrow [f\psi]$, this question can be answered by considering the following diagram:

$$\begin{array}{ccccccc}
 & & & hf^*\psi & & & \\
 & & & \curvearrowright & & & \\
 k^*gp & \xrightarrow{k^*g\psi} & c & \xleftarrow{h} & a & \xleftarrow{f^*\psi} & f^*p \\
 \downarrow \lrcorner & & \downarrow k \lrcorner & & \downarrow f \lrcorner & & \downarrow \lrcorner \\
 p & \xrightarrow{g\psi} & d & \xleftarrow{g} & b & \xleftarrow{\psi} & p
 \end{array}
 \quad (23)$$

Similarly, k^*gp and f^*p are *both* pullbacks of $p \xrightarrow{g\psi} d \xleftarrow{k} c$ and therefore, we obtain an analogous statement: since $hf^*\psi \simeq k^*g\psi$, we have $[hf^*\psi] = [k^*g\psi]$ and therefore $h_!f^* = k^*g_!$.

¹¹Ultimately, the reason for $\alpha = \text{id}$ is due to considering $\text{Sub}_{\mathcal{B}}(a)$, the *poset* of subobjects of a , instead of $\text{Mono}_{\mathcal{B}}(a)$ the *preorder* of monomorphisms into a .

Tobias: Nice! Even though the pushforward functors are missing the image factorizations, I think that this is the correct argument in the case of the codomain fibration. For the subobject fibration, we might need some additional stability property, presumably the one stated in the alternative definition of [regular categories](#)

The key difference in this case is that this argument *does* rely on a being the pullback of a pullback square.

In summary, the subobject fibration *satisfies* the Beck-Chevalley condition. Moreover, since none of the above argument relies on the fact that the objects of total category are represented by *monomorphisms* in the base, just that they correspond to *morphisms* in the base, it is clear that any codomain bifibration also satisfies the Beck-Chevalley condition.

3.5 The Category of Convex Cones and Linear Maps

Given any \mathbb{R} -vector space V , a (*closed*) *cone* $C \subseteq V$ is a subset of V such that for any elements $c_1, c_2 \in C$ and for any positive coefficients $\gamma_1, \gamma_2 \geq 0$, $\gamma_1 c_1 + \gamma_2 c_2 \in C$. A polyhedral cone $C \subseteq V$ is one which admits a half-space representation in terms a finite number of linear constraints:

$$C = \{x \in V \mid \bigwedge_{i=1}^K (a_i \cdot x \geq 0)\} \quad (24)$$

Alternatively, a **Cone** can be expressed in terms of the pullback of the positive orthant

$$\mathbb{R}_+^n := \{v \in \mathbb{R}^n \mid \forall i \in [n] : v_i \geq 0\} \quad (25)$$

by a linear transformation $f : V \rightarrow \mathbb{R}^n$ into \mathbb{R}^n .

$$\begin{array}{ccc} f^*(\mathbb{R}_+^n) & \longrightarrow & \mathbb{R}_+^n \\ \downarrow \lrcorner & & \downarrow i_+ \\ V & \xrightarrow{f} & \mathbb{R}^n \end{array} \quad (26)$$

$$f^*(\mathbb{R}_+^n) \cong \{v \in V \mid f(v) \in \mathbb{R}_+^n\} \quad (27)$$

Given a cone $f^*(\mathbb{R}_+^n) \subseteq V$ associated with a finite set of n linear expressions $f : V \rightarrow \mathbb{R}^n$, and a linear transformation $g : V \rightarrow W$,

3.6 Subset Projection

A prototypical example wherein an adjoint triple

$$f_!, \exists_f \dashv f^*, f^{-1} \dashv f^!, \forall_f$$

arises is that of functions $f : X \rightarrow Y$ between sets X and Y . The inverse image functor $f^* : \mathcal{P}Y \rightarrow \mathcal{P}X$ is defined on a subset $T \subseteq Y$

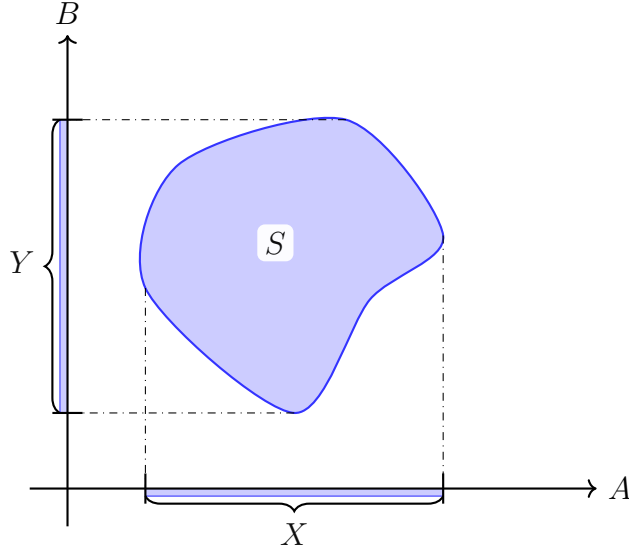
$$f^*(T) = \{x \in X : f(x) \in T\},$$

and is functorial in the sense that if $T \subseteq T' \subseteq Y$ then $f^*(T) \subseteq f^*(T') \subseteq f^*(T)$. The adjoint functors $\exists_f, \forall_f : \mathcal{P}X \rightarrow \mathcal{P}Y$ are defined on $S \subseteq X$ as

$$\begin{aligned}\exists_f(S) &= \{y \in Y : \exists x \in f^*(y) : x \in S\} \\ \forall_f(S) &= \{y \in Y : \forall x \in f^*(y) : x \in S\}\end{aligned}$$

form an adjoint triple in the sense that $\exists_f \dashv f^* \dashv \forall_f$:

$$\begin{aligned}\exists_f \dashv f^* : \quad \exists_f(S) \subseteq T &\iff S \subseteq f^*(T) \\ f^* \dashv \forall_f : \quad f^*(T) \subseteq R &\iff T \subseteq \forall_f(R)\end{aligned}$$

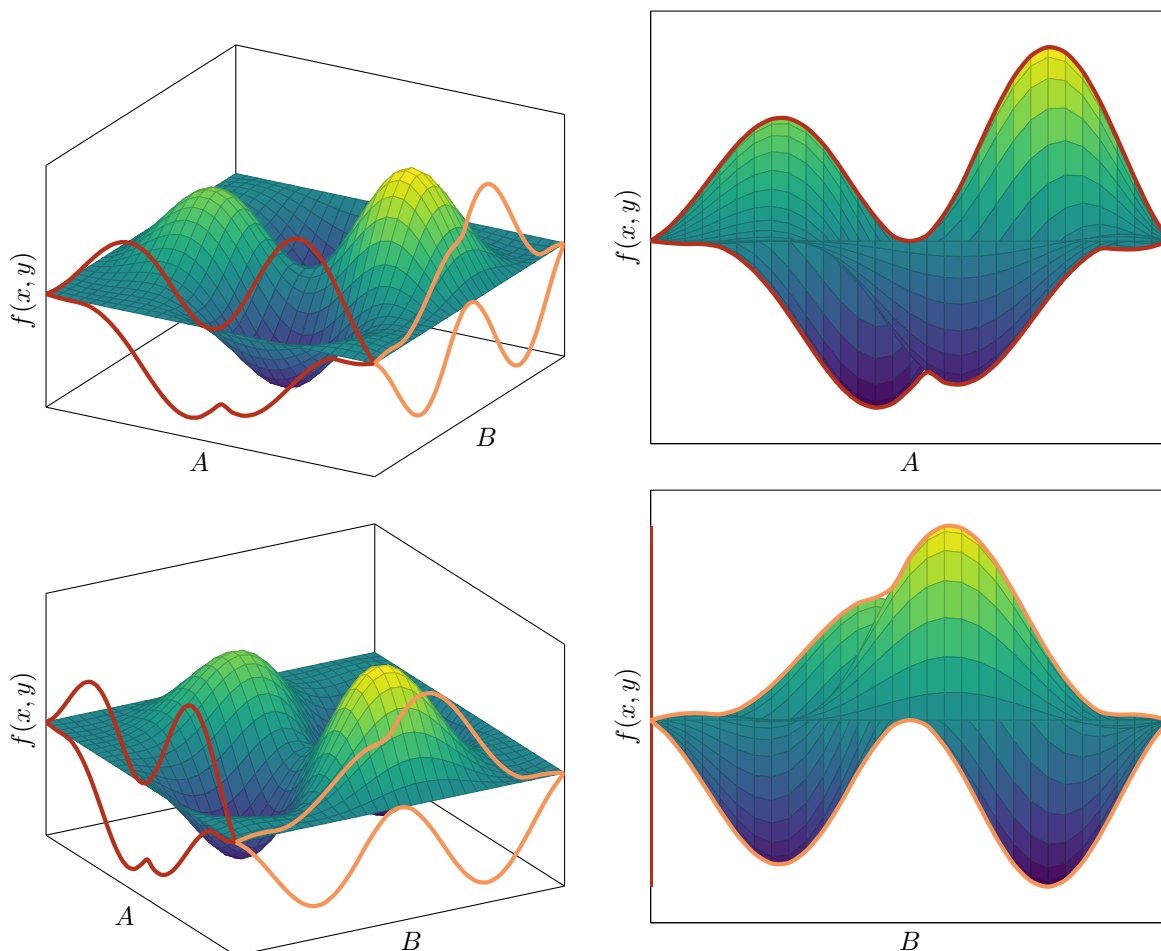


Consider a pair of sets A and B and a subset $S \subseteq A \times B$ of their cartesian product. The projection morphisms associated with $A \times B$ are $p : A \times B \rightarrow A$ and $q : A \times B \rightarrow B$. The projection of the subset S onto A is then the subset $X \subseteq A$ defined by:

$$X = \{a \in A \mid \exists s \in S, p(s) = a\}$$

$$S \subseteq p^*(X) \iff \exists_p(S) \subseteq X \tag{28}$$

3.7 Optimization of real-valued functions



Potentially Annotated Bibliography

This section is temporary and reserved for recording comments toward various references.

- [vistoli2004notes](#) [Vis04]
- [street1974fib](#) [Str74]
- [koudenburg2018categorical](#) [Kou18]
- [brown2009algebraic](#) [BS09]
- [lurie2009higher](#) [Lur09]
- [shulman2008framed](#) [Shu08]
- [boyd2004convex](#) [BV04]
- [bogart2013hom](#) [BCG13]

- ~~gubeladze2016affine~~ ~~Gubeladze2016affine~~ Gubeladze [Gub16]
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- Spectrahedron are interesting semi-algebraic sets. (<https://www.youtube.com/watch?v=AevFRN5sxOU>).

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