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MONOIDAL GROTHENDIECK CONSTRUCTION

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Abstract. We lift the standard equivalence between fibrations and indexed categories to an equivalence between monoidal fibrations and monoidal indexed categories, namely weak monoidal pseudofunctors to the 2-category of categories. In doing so, we investigate the relation between this 'global' monoidal structure where the total category is monoidal and the fibration strictly preserves the structure, and a 'fibrewise' one where the fibres are monoidal and the reindexing functors strongly preserve the structure, first hinted by Shulman. In particular, when the domain is cocartesian monoidal, lax monoidal structures on the functor to the 2-category of categories correspond to lifts of the functor to the 2-category of monoidal categories. Finally, we give examples where this correspondence appears, spanning from the fundamental and family fibrations to network models and systems.

Contents

1. Introduction

The Grothendieck construction [Gro61] exhibits one of the most fundamental relations in category theory, namely the equivalence between contravariant pseudofunctors into Cat and fibrations. This equivalence allows us to freely move between the worlds of indexed categories and fibred categories, providing access to tools and results from both. Due to its importance, it is only natural that one would be interested in possible extra structure these objects may have, and how the correspondence extends then. The goal of this paper is to precisely establish the appropriate correspondence in the monoidal setting.

Our main result, Theorem 3.10, accomplishes this goal by lifting the standard equivalence ICat $\simeq$ Fib induced by the Grothendieck construction to an equivalence between the pseudomonoids of each cartesian monoidal 2-category. Using 2-categorical machinery, we obtain a canonical correspondence between *monoidal fibrations* (fibrations which are strict monoidal functors with a cartesian lifting condition on the domain tensor product functor) and *monoidal indexed categories* (weak monoidal pseudofunctors into Cat). The monoidal Grothendieck construction in this sense employs the weak monoidal structure of the pseudofunctor to equip the corresponding total category with a monoidal product, which is strictly preserved by the fibration.

On a different but highly related note, Shulman introduced monoidal fibrations in [Shu08], where he explicitly constructed an equivalence between monoidal fibrations over a cartesian monoidal base and ordinary pseudofunctors into MonCat; the latter were already called *indexed (strong) monoidal categories* in [HM06]. For this result, the existence of finite products is instrumental, making it impossible to extend it to arbitrary monoidal products. Moreover, the involved monoidal fibrations have monoidal fibre categories and strong monoidal reindexing functors between them, which is certainly not always the case for an arbitrary monoidal fibration.

This striking dissimilarity between Shulman's equivalence and the one established here motivated an investigation regarding a 'fibrewise' monoidal structure of a fibration as opposed to a 'global' one. It turns out that from a high level perspective, these structures are encompassed as pseudomonoids in different monoidal 2-categories: fixed-base fibrations $\mathsf{Fib}(\mathcal{X})$ and arbitrary-base fibrations Fib, see (27). Notably, these two versions only meet when the base category has a (co)cartesian monoidal structure, expressed as a bijection between ordinary pseudofunctors into MonCat and weak monoidal pseudofunctors into $(\mathsf{Cat}, \times, \mathbf{1})$. This interesting subtlety concerning the transfer of monoidality from the target category to the very structure of the functor and vice versa could potentially bring new perspective into future variations of the Grothendieck construction. As an example, in [BW19] the authors work towards a 'fibrewise' enriched version of the correspondence between fibrations and indexed categories, hence future work could address the 'global' enriched Grothendieck construction.

Finally, the examples of the monoidal Grothendieck construction are those that render the clarification of this correspondence essential. As is the case for the ordinary Grothendieck construction, applications seem to arise in diverse settings, which extend from categorical and algebraic frameworks, to more applied contexts like network and systems theory. We gather some of them in the last section of the paper, and we are convinced that many more exist and would benefit from such a viewpoint.

Outline of the paper. In Section 2, we review the basic theory of fibrations and indexed categories, as well as that of monoidal 2-categories and pseudomonoids. Section 3 contains the eponymous construction in the form of 2-equivalences between the respective 2-categories of monoidal objects: Sections 3.1 and 3.2 contains elementary descriptions of (braided/symmetric) monoidal variations of fibrations and indexed categories, whereas Section 3.3 details the relevant correspondences. In Section 4, we investigate the relation between the 'global' and 'fibrewise' monoidal Grothendieck construction for cartesian bases. Finally, Section 5 highlights some examples of this construction as it arises in various contexts, and Section 6 presents some of the earlier structures in greater detail.

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2. Preliminaries

We assume familiarity with the basics of 2-category theory, see e.g. [KS74, Lac10]. We denote by $\mathsf{2Cat}$ the paradigmatic example of a 3-category [Gur13] which consists of 2-categories, 2-functors, 2-natural transformations and modifications between them. If we take pseudofunctors $\mathscr{F} : \mathcal{K} \to \mathcal{L}$ between 2-categories, i.e. assignments that preserve the composition and identities up to coherent isomorphism, along with pseudonatural transformations $\mathscr{F} \Rightarrow \mathscr{G}$ between them, i.e. with components for which the usual naturality squares commute only up to coherent isomorphism, we obtain a tricategory denoted by $\mathsf{2Cat}_{ps}$.

2.1. **Fibrations and Indexed Categories.** We recall some basic facts and constructions from the theory of fibrations and indexed categories, as well as the equivalence between them via the Grothendieck construction. A few indicative references for the general theory are [Gra66, Her94, Bor94, Jac99, Joh02].

Consider a functor $P\colon \mathcal{A} \to \mathcal{X}$. A morphism $\phi\colon a \to b$ in $\mathcal{A}$ over a morphism $f = P(\phi)\colon x \to y$ in $\mathcal{X}$ is called **cartesian** if and only if, for all $g\colon x' \to x$ in $\mathcal{X}$ and $\theta\colon a' \to a$ in $\mathcal{A}$ with $P\theta = f \circ g$, there exists a unique arrow $\psi\colon a' \to a$ such that $P\psi = g$ and $\theta = \phi \circ \psi$:

$$\begin{tikzcd}
a' \arrow[rrd, "\theta"] \arrow[rd, dashed, "\exists!\psi"'] \arrow[dd, dotted] & & & \\
& a \arrow[r, "\phi"'] \arrow[dd, dotted] & b \arrow[dd, dotted] & \text{in } \mathcal{A} \\
x' \arrow[rrd, "f\circ g=P\theta"] \arrow[rd, "g"'] & & & \\
& x \arrow[r, "f=P\phi"'] & y & \text{in } \mathcal{X}
\end{tikzcd} \tag{1}$$

For $x \in \mathrm{ob}\mathcal{X}$, the **fibre of P over x** written $\mathcal{A}_x$, is the subcategory of $\mathcal{A}$ which consists of objects a such that $P(a) = x$ and morphisms ϕ with $P(\phi) = 1_x$, called **vertical** morphisms. The functor $P\colon \mathcal{A} \to \mathcal{X}$ is called a **fibration** if and only if, for all $f\colon x \to y$ in $\mathcal{X}$ and $b \in \mathcal{A}_Y$, there is a cartesian morphism ϕ with codomain b above f; it is called a **cartesian lifting** of b along f. The category $\mathcal{X}$ is then called the **base** of the fibration, and $\mathcal{A}$ its **total category**.

Dually, the functor $U\colon \mathcal{C} \to \mathcal{X}$ is an **opfibration** if U^{op} is a fibration, i.e. for every $c \in \mathcal{C}_x$ and $h\colon x \to y$ in $\mathcal{X}$, there is a cocartesian morphism with domain c above h, the **cocartesian lifting** of c along h with the dual universal property:

$$\begin{tikzcd}
& & d' \arrow[dd, dotted] & \\
c \arrow[rru, "\gamma"] \arrow[r, "\beta"'] \arrow[dd, dotted] & d \arrow[ru, dashed, "\exists!\delta"'] \arrow[dd, dotted] & & \text{in } \mathcal{C} \\
& & y' & \\
x \arrow[rru, "k\circ h=U\gamma"] \arrow[r, "h=U\beta"'] & y \arrow[ru, "k"'] & & \text{in } \mathcal{X}
\end{tikzcd}$$

A **bifibration** is a functor which is both a fibration and opfibration.

If $P\colon \mathcal{A} \to \mathcal{X}$ is a fibration, assuming the axiom of choice we may select a cartesian arrow over each $f\colon x \to y$ in $\mathcal{X}$ and $b \in \mathcal{A}_y$, denoted by $\mathrm{Cart}(f, b)\colon f^*(b) \to b$. Such a choice of cartesian liftings is called a **cleavage** for P, which is then called a **cloven fibration**; any fibration is henceforth assumed to be cloven. Dually, if U is an opfibration, for any $c \in \mathcal{C}_x$ and $h\colon x \to y$ in $\mathcal{X}$ we can choose a cocartesian lifting of c along h, $\mathrm{Cocart}(h,c)\colon c \longrightarrow h_!(c)$. The choice of (co)cartesian liftings in an (op)fibration induces a so-called **reindexing functor** between the fibre categories

$$f^*\colon \mathcal{A}_y \to \mathcal{A}_x \qquad \text{and} \qquad h_!\colon \mathcal{C}_x \to \mathcal{C}_y \tag{2}$$

respectively, for each morphism $f\colon x \to y$ and $h\colon x \to y$ in the base category. It can be verified by the (co)cartesian universal property that $1_{\mathcal{A}_x} \cong (1_x)^*$ and that for composable morphism in the base category, $g^* \circ f^* \cong (g \circ f)^*$, as well as $(1_x)_! \cong 1_{\mathcal{C}_x}$ and $(k \circ h)_! \cong k_! \circ h_!$. If these isomorphisms are equalities, we have the notion of a **split** (op)fibration.

A **fibred 1-cell** $(H, F)\colon P \to Q$ between fibrations $P\colon \mathcal{A} \to \mathcal{X}$ and $Q\colon \mathcal{B} \to \mathcal{Y}$ is given by a commutative square of functors and categories

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "H"] \arrow[d, "P"'] & \mathcal{B} \arrow[d, "Q"] \\
\mathcal{X} \arrow[r, "F"'] & \mathcal{Y}
\end{tikzcd} \tag{3}$$

where the top H preserves cartesian liftings, meaning that if ϕ is P-cartesian, then $H\phi$ is Q-cartesian. In particular, when P and Q are fibrations over the same base category, we may consider fibred 1-cells of the form $(H, 1_{\mathcal{X}})$ displayed by

$$\begin{tikzcd}
\mathcal{A} \arrow[rr, "H"] \arrow[dr, "P"'] & & \mathcal{B} \arrow[dl, "Q"] \\
& \mathcal{X} &
\end{tikzcd} \tag{4}$$

and H is then called a **fibred functor**. Dually, we have the notion of an **opfibred 1-cell** and **opfibred functor**. Notice that any such (op)fibred 1-cell induces functors between the fibres, by commutativity of (3):

$$H_x\colon \mathcal{A}_x \longrightarrow \mathcal{B}_{Fx} \tag{5}$$

A **fibred 2-cell** between fibred 1-cells (H, F) and (K, G) is a pair of natural transformations $(\beta\colon H \Rightarrow K, \alpha\colon F \Rightarrow G)$ with β above α, i.e. $Q(\beta_a) = \alpha_{Pa}$ for all $a \in \mathcal{A}$, displayed as

$$\begin{tikzcd}
\mathcal{A} \arrow[r, bend left, "H"{name=U}] \arrow[r, bend right, "K"'{name=D}] \arrow[d, "P"'] & \mathcal{B} \arrow[d, "Q"] \\
\mathcal{X} \arrow[r, bend left, "F"{name=U2}] \arrow[r, bend right, "G"'{name=D2}] & \mathcal{Y}.
\arrow[Rightarrow, from=U, to=D, "\beta"]
\arrow[Rightarrow, from=U2, to=D2, "\alpha"]
\end{tikzcd} \tag{6}$$

A **fibred natural transformation** is of the form $(\beta, 1_{1_{\mathcal{X}}})\colon (H, 1_{\mathcal{X}}) \Rightarrow (K, 1_{\mathcal{X}})$

$$\begin{tikzcd}
\mathcal{A} \arrow[rr, bend left, "H"{name=U}] \arrow[rr, bend right, "K"'{name=D}] \arrow[dr, "P"'] & & \mathcal{B} \arrow[dl, "Q"] \\
& \mathcal{X} &
\arrow[Rightarrow, from=U, to=D, "\beta"]
\end{tikzcd} \tag{7}$$

Dually, we have the notion of an **opfibred 2-cell** and **opfibred natural transformation** between opfibred 1-cells and functors respectively.

We thus obtain a 2-category Fib of fibrations over arbitrary base categories, fibred 1-cells and fibred 2-cells. There is also a 2-category $\mathsf{Fib}(\mathscr{X})$ of fibrations over a fixed base category $\mathscr{X}$, fibred functors and fibred natural transformations. Dually, we have the 2-categories OpFib and $\mathsf{OpFib}(\mathscr{X})$. Moreover, we also have 2-categories $\mathsf{Fib}_{\mathrm{sp}}$ and $\mathsf{OpFib}_{\mathrm{sp}}$ of split (op)fibrations, and (op)fibred 1-cells that preserve the cartesian liftings 'on the nose'.

Remark 2.1. Notice that Fib and OpFib are both sub-2-categories of $\mathsf{Cat}^{\mathbf{2}} = [\mathbf{2}, \mathsf{Cat}]$, the arrow 2-category of Cat. Similarly, $\mathsf{Fib}(\mathscr{X})$ and $\mathsf{OpFib}(\mathscr{X})$ are sub-2-categories of $\mathsf{Cat}/\mathscr{X}$, the slice 2-category of functors into $\mathscr{X}$. Due to that (see also Section 5.1), both these (1-)categories form fibrations themselves. Explicitly, the functor $\mathrm{cod}\colon \mathsf{Fib} \to \mathsf{Cat}$ which maps a fibration to its base is a fibration, with fibres $\mathsf{Fib}(\mathscr{X})$ and cartesian liftings pullbacks along fibrations. In fact, it is a *2-fibration* as explained in [Buc14, 2.3.8].

We now turn to the world of indexed categories. Given an ordinary category $\mathscr{X}$, an **$\mathscr{X}$-indexed category** is a pseudofunctor

$$\mathscr{M}\colon \mathscr{X}^{\mathrm{op}} \to \mathsf{Cat}$$

where $\mathscr{X}$ is viewed as a 2-category with trivial 2-cells; it comes with natural isomorphisms $\delta_{g,f}\colon (\mathscr{M}g) \circ (\mathscr{M}f) \xrightarrow{\sim} \mathscr{M}(g \circ f)$ and $\gamma_x\colon 1_{\mathscr{M}x} \xrightarrow{\sim} \mathscr{M}(1_x)$ for every $x \in \mathscr{X}$ and composable morphisms f and g, satisfying coherence axioms. Dually, an **$\mathscr{X}$-opindexed category** is an $\mathscr{X}^{\mathrm{op}}$-indexed category, i.e. a pseudofunctor $\mathscr{X} \to \mathsf{Cat}$. If an (op)indexed category strictly preserves composition, i.e. is a (2-)functor, then it is called **strict**.

An **indexed 1-cell** $(F, \tau)\colon \mathscr{M} \to \mathscr{N}$ between indexed categories $\mathscr{M}\colon \mathscr{X}^{\mathrm{op}} \to \mathsf{Cat}$ and $\mathscr{N}\colon \mathscr{Y}^{\mathrm{op}} \to \mathsf{Cat}$ consists of an ordinary functor $F\colon \mathscr{X} \to \mathscr{Y}$ along with a pseudonatural transformation $\tau\colon \mathscr{M} \Rightarrow \mathscr{N} \circ F^{\mathrm{op}}$

$$\begin{tikzcd}
\mathscr{X}^{\mathrm{op}} \arrow[d, "F^{\mathrm{op}}"'] \arrow[dr, "\mathscr{M}"] \arrow[dr, phantom, shift right=4ex, "\Downarrow \tau"] & \\
\mathscr{Y}^{\mathrm{op}} \arrow[r, "\mathscr{N}"'] & \mathsf{Cat}
\end{tikzcd} \tag{8}$$

with components functors $\tau_x\colon \mathscr{M}x \to \mathscr{N}Fx$, equipped with coherent natural isomorphisms $\tau_f\colon (\mathscr{N}Ff) \circ \tau_x \xrightarrow{\sim} \tau_y \circ (\mathscr{M}f)$ for any $f\colon x \to y$ in $\mathscr{X}$. For indexed categories with the same base, we may consider indexed 1-cells of the form $(1_{\mathscr{X}}, \tau)$

$$\begin{tikzcd}
\mathscr{X}^{\mathrm{op}} \arrow[r, bend left, "\mathscr{M}"] \arrow[r, bend right, "\mathscr{N}"'] \arrow[r, phantom, "\Downarrow\tau"] & \mathsf{Cat}
\end{tikzcd} \tag{9}$$

which are called **indexed functors**. Dually, we have the notion of an **opindexed 1-cell** and **opindexed functor**.

An **indexed 2-cell** (α, m) between indexed 1-cells (F, τ) and (G, σ), pictured as

$$\begin{tikzcd}
\mathcal{X}^{\mathrm{op}} \arrow[dd, bend right=50, "F^{\mathrm{op}}"'] \arrow[dd, bend left=50, "G^{\mathrm{op}}"] \arrow[dd, phantom, "\overset{\alpha^{\mathrm{op}}}{\Leftarrow}"] \arrow[dr, "\mathscr{M}"] & \\
 & \mathsf{Cat} \arrow[l, phantom, "\tau \overset{m}{\Rrightarrow} \sigma"] \\
\mathcal{Y}^{\mathrm{op}} \arrow[ur, "\mathscr{N}"'] &
\end{tikzcd}$$

consists of an ordinary natural transformation $\alpha\colon F \Rightarrow G$ and a modification m

$$\begin{tikzcd}
\mathcal{X}^{\mathrm{op}} \arrow[rr, bend left, "\mathscr{M}"] \arrow[rr, phantom, "\Downarrow \tau"] \arrow[dr, "F^{\mathrm{op}}"'] & & \mathsf{Cat} & \overset{m}{\Rrightarrow} & \mathcal{X}^{\mathrm{op}} \arrow[rr, bend left, "\mathscr{M}"] \arrow[dr, bend left, "G^{\mathrm{op}}"] \arrow[dr, bend right, "F^{\mathrm{op}}"'] \arrow[dr, phantom, "\Downarrow \alpha^{\mathrm{op}}"] & \Downarrow \sigma & \mathsf{Cat} \\
 & \mathcal{Y}^{\mathrm{op}} \arrow[ur, "\mathscr{N}"'] & & & & \mathcal{Y}^{\mathrm{op}} \arrow[ur, bend right, "\mathscr{N}"'] &
\end{tikzcd} \tag{10}$$

given by a family of natural transformations $m_x\colon \tau_x \Rightarrow \mathscr{N}\alpha_x \circ \sigma_x$. Notice that taking opposites is a 2-functor $(-)^{\mathrm{op}}\colon \mathsf{Cat} \to \mathsf{Cat}^{co}$, on which the above diagrams rely. An **indexed natural transformation** between two indexed functors is an indexed 2-cell of the form $(1_{1_{\mathcal{X}}}, m)$. Dually, we have the notion of an **opindexed 2-cell** and **opindexed natural transformation** between opindexed 1-cells and functors respectively.

Notice that an indexed 2-cell (α, m) is invertible if and only if both α is a natural isomorphism and the modification m is invertible, due to the way vertical composition is formed.

We obtain a 2-category ICat of indexed categories over arbitrary bases, indexed 1-cells and indexed 2-cells. In particular, there is a 2-category $\mathsf{ICat}(\mathcal{X})$ of indexed categories with fixed domain $\mathcal{X}$, indexed functors and indexed natural transformations, which coincides with the functor 2-category $\mathsf{2Cat}_{\mathrm{ps}}(\mathcal{X}^{\mathrm{op}}, \mathsf{Cat})$.

Dually, we have the 2-categories OpICat and $\mathsf{OpICat}(\mathcal{X}) = \mathsf{2Cat}_{\mathrm{ps}}(\mathcal{X}, \mathsf{Cat})$. Notice that due to the absence of opposites in the world of opindexed categories, opindexed 2-cells have a different form than (10), namely

$$\begin{tikzcd}
\mathcal{X} \arrow[rr, bend left, "\mathscr{M}"] \arrow[dr, bend left, "F"] \arrow[dr, bend right, "G"'] \arrow[dr, phantom, "\Downarrow \alpha"] & \Downarrow \tau & \mathsf{Cat} & \overset{m}{\Rrightarrow} & \mathcal{X} \arrow[rr, bend left, "\mathscr{M}"] \arrow[rr, phantom, "\Downarrow \sigma"] \arrow[dr, "G"'] & & \mathsf{Cat} \\
 & \mathcal{Y} \arrow[ur, bend right, "\mathscr{N}"'] & & & & \mathcal{Y} \arrow[ur, "\mathscr{N}"'] &
\end{tikzcd}$$

Moreover, we have 2-categories of strict (op)indexed categories and (op)indexed 1-cells that consist of strict natural transformations τ (8), i.e. $\mathsf{ICat}(\mathcal{X}) = [\mathcal{X}^{\mathrm{op}}, \mathsf{Cat}]$ and $\mathsf{OpICat}_{\mathrm{sp}}(\mathcal{X}) = [\mathcal{X}, \mathsf{Cat}]$ the usual functor 2-categories.

Remark 2.2. Similarly to Remark 2.1, notice that these (1-)categories also form fibrations over Cat, this time essentially using the family fibration also seen in Section 5.3. The functor $\mathsf{ICat} \to \mathsf{Cat}$ sends an indexed category to its domain and an indexed 1-cell to its first component. It is a split fibration, with fibres $\mathsf{ICat}(\mathcal{X})$ and cartesian liftings pre-composition with functors. In fact, it is also a 2-fibration as explained in [Buc14, 2.3.2].

In the first volume of the *Séminaire de Géométrie Algébrique du Bois Marie* [Gro61], Grothendieck introduced a construction for a fibration $P_{\mathscr{M}} \colon \int\mathscr{M} \to \mathcal{X}$ from a given indexed category $\mathscr{M} \colon \mathcal{X}^{\mathrm{op}} \to \mathsf{Cat}$ as follows. If δ and γ are the structure pseudonatural transformations of the pseudofunctor $\mathscr{M}$, the total category $\int\mathscr{M}$ has

- objects (x, a) with $x \in \mathcal{X}$ and $a \in \mathscr{M}x$;
- morphisms $(f, k) \colon (x, a) \to (y, b)$ with $f \colon x \to y$ a morphism in $\mathcal{X}$, and $k \colon a \to (\mathscr{M}f)(b)$ a morphism in $\mathscr{M}x$;
- composition $(g, \ell) \circ (f, k) \colon (x, a) \to (y, b) \to (z, c)$ is given by $g \circ f \colon a \to b \to c$ in $\mathcal{X}$ and
$$a \xrightarrow{k} (\mathscr{M}f)(b) \xrightarrow{(\mathscr{M}g)(\ell)} (\mathscr{M}g \circ \mathscr{M}f)(c) \xrightarrow{(\delta_{f,g})_c} \mathscr{M}(g \circ f)(c) \quad \text{in } \mathscr{M}x; \tag{11}$$
- unit $1_{(x,a)} \colon (x, a) \to (x, a)$ is given by $1_x \colon x \to x$ in $\mathcal{X}$ and
$$a = 1_{\mathscr{M}x} a \xrightarrow{(\gamma_x)_a} (\mathscr{M}1_x)(a) \quad \text{in } \mathscr{M}x.$$

The fibration $P_{\mathscr{M}} \colon \int\mathscr{M} \to \mathcal{X}$ is given by $(x, a) \mapsto x$ on objects and $(f, k) \mapsto f$ on morphisms, and the cartesian lifting of any (y, b) in $\int\mathscr{M}$ along $f \colon x \to y$ in $\mathcal{X}$ is precisely $(f, 1_{(\mathscr{M}f)b})$. Its fibres are precisely $\mathscr{M}x$ and the reindexing functors between them are $\mathscr{M}f$.

In the other direction, given a (cloven) fibration $P \colon \mathcal{A} \to \mathcal{X}$, we can define an indexed category $\mathscr{M}_P \colon \mathcal{X}^{\mathrm{op}} \to \mathsf{Cat}$ that sends each object x of $\mathcal{X}$ to its fibre category $\mathcal{A}_x$, and each morphism $f \colon x \to y$ to the corresponding reindexing functor $f^* \colon \mathcal{A}_y \to \mathcal{A}_x$ as in (2). The isomorphisms of cartesian liftings $f^* \circ g^* \cong (g \circ f)^*$ and $1_{\mathcal{A}_x} \cong 1_x^*$ render this assignment pseudofunctorial.

Details of the above, as well as the correspondence between 1-cells and 2-cells can be found in the provided references. Briefly, given a pseudonatural transformation $\tau \colon \mathscr{M} \to \mathscr{N} \circ F^{\mathrm{op}}$ (8) with components $\tau_x \colon \mathscr{M}x \to \mathscr{N}Fx$, define a functor $P_\tau \colon \int\mathscr{M} \to \int\mathscr{N}$ mapping $(x \in \mathcal{X}, a \in \mathscr{M}x)$ to the pair $(Fx \in \mathcal{Y}, \tau_x(a) \in \mathscr{N}Fx)$ and accordingly for arrows. This makes the square

$$\begin{tikzcd}
\int\mathscr{M} \arrow[r, "P_\tau"] \arrow[d, "P_{\mathscr{M}}"'] & \int\mathscr{N} \arrow[d, "P_{\mathscr{N}}"] \\
\mathcal{X} \arrow[r, "F"'] & \mathcal{Y}
\end{tikzcd} \tag{12}$$

commute, and moreover P_τ preserves cartesian liftings due to pseudonaturality of τ. Moreover, given an indexed 2-cell $(\alpha, m) \colon (F, \tau) \Rightarrow (G, \sigma)$ as in (10), we can form a fibred 2-cell

$$\begin{tikzcd}
\int\mathscr{M} \arrow[r, bend left, "P_\tau"] \arrow[r, bend right, "P_\sigma"'] \arrow[r, phantom, "\Downarrow P_m"] \arrow[d, "P_{\mathscr{M}}"'] & \int\mathscr{N} \arrow[d, "P_{\mathscr{N}}"] \\
\mathcal{X} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow \alpha"] & \mathcal{Y}
\end{tikzcd} \tag{13}$$

where $\alpha \colon F \Rightarrow G$ is piece of the given structure, whereas P_m is given by components

$$(P_m)_{(x,a)} \colon P_\tau(x, a) = (Fx, \tau_x a) \to P_\sigma(x, a) = (Gx, \sigma_x a) \quad \text{in } \int\mathscr{N}$$

explicitly formed by $\alpha_x\colon Fx\to Gx$ in $\mathcal{Y}$ and $(m_x)_a\colon \tau_x a\to(\mathscr{N}\alpha_x)\sigma_x a$ in $\mathscr{N}Fx$.

The following theorem summarizes these standard results.

Theorem 2.3.

(1) Every fibration $P\colon\mathcal{A}\to\mathcal{X}$ gives rise to a pseudofunctor $\mathscr{M}_P\colon\mathcal{X}^{\mathrm{op}}\to\mathsf{Cat}$.

(2) Every indexed category $\mathscr{M}\colon\mathcal{X}^{\mathrm{op}}\to\mathsf{Cat}$ gives rise to a fibration $P_{\mathscr{M}}\colon\int\mathscr{M}\to\mathcal{X}$.

(3) The above correspondences yield an equivalence of 2-categories

$$\mathsf{ICat}(\mathcal{X})\simeq\mathsf{Fib}(\mathcal{X})$$

so that $\mathscr{M}_{P_{\mathscr{M}}}\cong\mathscr{M}$ and $P_{\mathscr{M}_P}\cong P$.

(4) The above 2-equivalence extends to one between 2-categories of arbitrary-base fibrations and arbitrary-domain indexed categories

$$\mathsf{ICat}\simeq\mathsf{Fib}\tag{14}$$

If we combine the above with Remark 2.1 and Remark 2.2 that point out that the 2-categories Fib and ICat are fibred over Cat with fibres $\mathsf{Fib}(\mathcal{X})$ and $\mathsf{ICat}(\mathcal{X})$ respectively, we obtain the following Cat-fibred equivalence

$$\begin{tikzcd}
\mathsf{ICat} \arrow[rr, "\simeq"] \arrow[dr] & & \mathsf{Fib} \arrow[dl] \\
& \mathsf{Cat} &
\end{tikzcd}\tag{15}$$

There is an analogous story for opindexed categories and opfibrations that results into a 2-equivalences $\mathsf{OpICat}(\mathcal{X})\simeq\mathsf{OpFib}(\mathcal{X})$ and $\mathsf{OpICat}\simeq\mathsf{OpFib}$, as well as for the split versions of (op)indexed and (op)fibred categories.

2.2. Monoidal 2-categories and pseudomonoids.

Below we sketch some basic definitions and constructions relative to monoidal 2-categories, necessary for what follows; relevant references where explicit axioms can be found are [Car95, GPS95, DS97, McC00].

A **monoidal 2-category** $\mathcal{K}$ is a 2-category equipped with a pseudofunctor $\otimes\colon\mathcal{K}\times\mathcal{K}\to\mathcal{K}$ and a unit object $I\colon\mathbf{1}\to\mathcal{K}$ which are associative and unital up to coherent equivalence. A **weak monoidal pseudofunctor** $\mathscr{F}\colon\mathcal{K}\to\mathcal{L}$ between monoidal 2-categories is a pseudofunctor equipped with pseudonatural transformations

$$\begin{tikzcd}
\mathcal{K}\times\mathcal{K} \arrow[r, "\mathscr{F}\times\mathscr{F}"] \arrow[d, "\otimes_{\mathcal{K}}"'] \arrow[dr, phantom, "\overset{\mu}{\Swarrow}"] & \mathcal{L}\times\mathcal{L} \arrow[d, "\otimes_{\mathcal{L}}"] & \mathbf{1} \arrow[d, "I_{\mathcal{K}}"'] \arrow[dr, bend left, "I_{\mathcal{L}}"] \arrow[dr, phantom, "\overset{\mu_0}{\Swarrow}"] & \\
\mathcal{K} \arrow[r, "\mathscr{F}"'] & \mathcal{L} & \mathcal{K} \arrow[r, "\mathscr{F}"'] & \mathcal{L}
\end{tikzcd}\tag{16}$$

with components $\mu_{a,b}\colon\mathscr{F}a\otimes\mathscr{F}b\to\mathscr{F}(a\otimes b)$, $\mu_0\colon I\to\mathscr{F}I$, and invertible modifications

$$\begin{tikzcd}[column sep=small]
& \mathcal{L}\times\mathcal{L}\times\mathcal{L} \arrow[r, "\otimes_{\mathcal{L}}\times 1"] \arrow[d, phantom, "\Downarrow\mu\times 1"] & \mathcal{L}\times\mathcal{L} \arrow[dr, "\otimes_{\mathcal{L}}"] & & & & \mathcal{L}\times\mathcal{L}\times\mathcal{L} \arrow[r, "\otimes_{\mathcal{L}}\times 1"] \arrow[dr, "1\times\otimes_{\mathcal{L}}"'] & \mathcal{L}\times\mathcal{L} \arrow[dr, "\otimes_{\mathcal{L}}"] \arrow[d, phantom, "\cong"] & \\
\mathcal{K}\times\mathcal{K}\times\mathcal{K} \arrow[ur, "\mathscr{F}\times\mathscr{F}\times\mathscr{F}"] \arrow[r, "\otimes_{\mathcal{K}}\times 1"] \arrow[dr, "1\times\otimes_{\mathcal{K}}"'] & \mathcal{K}\times\mathcal{K} \arrow[ur, "\mathscr{F}\times\mathscr{F}"] \arrow[dr, "\otimes_{\mathcal{K}}"] \arrow[d, phantom, "\cong"] & \Downarrow\mu & \mathcal{L} & \overset{\omega}{\Rrightarrow} & \mathcal{K}\times\mathcal{K}\times\mathcal{K} \arrow[ur, "\mathscr{F}\times\mathscr{F}\times\mathscr{F}"] \arrow[dr, "1\times\otimes_{\mathcal{K}}"'] & \Downarrow 1\times\mu & \mathcal{L}\times\mathcal{L} \arrow[r, "\otimes_{\mathcal{L}}"] \arrow[d, phantom, "\Downarrow\mu"] & \mathcal{L} \\
& \mathcal{K}\times\mathcal{K} \arrow[r, "\otimes_{\mathcal{K}}"'] & \mathcal{K} \arrow[ur, "\mathscr{F}"'] & & & & \mathcal{K}\times\mathcal{K} \arrow[ur, "\mathscr{F}\times\mathscr{F}"] \arrow[r, "\otimes_{\mathcal{K}}"'] & \mathcal{K} \arrow[ur, "\mathscr{F}"'] &
\end{tikzcd}\tag{17}$$

$$\begin{tikzcd}
\mathcal{K} \arrow[rr, "\mathscr{F}\times I"] \arrow[rrr, bend left=40, "\mathscr{F}"] \arrow[dr, "1\times I"'] \arrow[drr, bend right=60, "1"'] \arrow[rr, phantom, bend left=20, "\cong"] \arrow[dr, phantom, bend left=30, "1\times\mu_0\ \Downarrow"] & & \mathcal{L}\times\mathcal{L} \arrow[r, "\otimes_{\mathcal{L}}"'] \arrow[d, phantom, "\Downarrow\ \mu"] & \mathcal{L} \\
& \mathcal{K}\times\mathcal{K} \arrow[r, "\otimes_{\mathcal{K}}"'] \arrow[ur, dashed, "\mathscr{F}\times\mathscr{F}"] \arrow[d, phantom, "\cong"] & \mathcal{L} \arrow[ur, "\mathscr{F}"'] & \\
& {} & &
\end{tikzcd}
\overset{\zeta}{\Rrightarrow}
\begin{tikzcd}
\mathcal{K} \arrow[r, bend left, "\mathscr{F}"] \arrow[r, bend right, "\mathscr{F}"'] \arrow[r, phantom, "\Downarrow 1"] & \mathcal{L}
\end{tikzcd}
\overset{\xi}{\Rrightarrow}
\begin{tikzcd}
\mathcal{K} \arrow[rr, "I\times\mathscr{F}"] \arrow[rrr, bend left=40, "\mathscr{F}"] \arrow[dr, "I\times 1"'] \arrow[drr, bend right=60, "1_a"'] \arrow[rr, phantom, bend left=20, "\cong"] \arrow[dr, phantom, bend left=30, "\mu_0\times 1\ \Downarrow"] & & \mathcal{L}\times\mathcal{L} \arrow[r, "\otimes_{\mathcal{L}}"'] \arrow[d, phantom, "\Downarrow\ \mu"] & \mathcal{L} \\
& \mathcal{K}\times\mathcal{K} \arrow[r, "\otimes_{\mathcal{K}}"'] \arrow[ur, dashed, "\mathscr{F}\times\mathscr{F}"] \arrow[d, phantom, "\cong"] & \mathcal{K} \arrow[ur, "\mathscr{F}"'] & \\
& {} & &
\end{tikzcd}$$

subject to coherence conditions. A **monoidal pseudonatural transformation** $\tau\colon \mathscr{F} \Rightarrow \mathscr{G}$ between two weak monoidal pseudofunctors $(\mathscr{F},\mu,\mu_0)$ and $(\mathscr{G},\nu,\nu_0)$ is a pseudonatural transformation equipped with two invertible modifications

$$\begin{tikzcd}
\mathcal{K}\times\mathcal{K} \arrow[r, bend left, "\mathscr{F}\times\mathscr{F}"] \arrow[r, bend right, "\mathscr{G}\times\mathscr{G}"'] \arrow[r, phantom, "\Downarrow\,\tau\times\tau"] \arrow[d, "\otimes"'] & \mathcal{L}\times\mathcal{L} \arrow[d, "\otimes"] \\
\mathcal{K} \arrow[r, "\mathscr{G}"'] \arrow[ur, phantom, "\Downarrow\,\nu"] & \mathcal{L}
\end{tikzcd}
\overset{u}{\Rrightarrow}
\begin{tikzcd}
\mathcal{K}\times\mathcal{K} \arrow[r, "\mathscr{F}\times\mathscr{F}"] \arrow[d, "\otimes"'] & \mathcal{L}\times\mathcal{L} \arrow[d, "\otimes"] \\
\mathcal{K} \arrow[r, bend left, "\mathscr{F}"] \arrow[r, bend right, "\mathscr{G}"'] \arrow[r, phantom, "\Downarrow\,\tau"] \arrow[ur, phantom, "\Downarrow\,\mu"] & \mathcal{L}
\end{tikzcd}
\tag{18}$$

$$\begin{tikzcd}
\mathbf{1} \arrow[r, "I_{\mathcal{L}}"] \arrow[d, "1_{\mathcal{K}}"'] & \mathcal{L} \\
\mathcal{K} \arrow[ur, bend right, "\mathscr{G}"'] \arrow[ur, phantom, "\Downarrow\,\nu_0"] &
\end{tikzcd}
\overset{u_0}{\Rrightarrow}
\begin{tikzcd}
\mathbf{1} \arrow[r, "I_{\mathcal{L}}"] \arrow[d, "I_{\mathcal{K}}"'] \arrow[dr, phantom, bend left=20, "\Downarrow\,\mu_0"] & \mathcal{L} \\
\mathcal{K} \arrow[ur, "\mathscr{F}"] \arrow[ur, bend right=50, "\mathscr{G}"'] \arrow[ur, phantom, bend right=30, "\Downarrow\,\tau"] &
\end{tikzcd}$$

that consist of natural isomorphisms with components

$$u_{a,b}\colon \nu_{a,b}\circ(\tau_a\otimes\tau_b)\xrightarrow{\sim}\tau_{a\otimes b}\circ\mu_{a,b},\quad u_0\colon \nu_0\xrightarrow{\sim}\tau_I\circ\mu_0 \tag{19}$$

satisfying coherence conditions.

The above notions of course generalize those of an ordinary monoidal category, lax monoidal functor and monoidal natural transformation; however notice how standard notation changes *lax* to *weak monoidal* in this setting, potentially to distinguish from the term *lax functor* between 2-categories, which relates to composition rather than monoidal product.

A **monoidal modification** between two monoidal pseudonatural transformations (τ,u,u_0) and (σ,v,v_0) is a modification

$$\begin{tikzcd}
\mathcal{K} \arrow[r, bend left=50, "\mathscr{F}"] \arrow[r, bend right=50, "\mathscr{G}"'] \arrow[r, phantom, "\tau\ \overset{m}{\Rrightarrow}\ \sigma"] & \mathcal{L}
\end{tikzcd}$$

which consists of pseudonatural transformations $m_a\colon \tau_a \Rightarrow \sigma_a$ compatible with the monoidal structures, in the sense that

$$\begin{tikzcd}
 & \mathscr{G}a\otimes\mathscr{G}b \arrow[dr, bend left, "\nu_{a,b}"] & \\
\mathscr{F}a\otimes\mathscr{F}b \arrow[ur, bend left, "\sigma_a\otimes\sigma_b"] \arrow[dr, bend right, "\mu_{a,b}"'] \arrow[r, phantom, "\Downarrow v_{a,b}"] & {} & \mathscr{G}(x\otimes y) \\
 & \mathscr{F}(a\otimes b) \arrow[ur, bend left, "\tau_{a\otimes b}"] \arrow[ur, bend right, "\sigma_{a\otimes b}"'] \arrow[ur, phantom, "\Downarrow m_{a\otimes b}"] &
\end{tikzcd}
=
\begin{tikzcd}
 & \mathscr{G}a\otimes\mathscr{G}b \arrow[dr, bend left, "\nu_{a,b}"] & \\
\mathscr{F}a\otimes\mathscr{F}b \arrow[ur, bend left, "\tau_a\otimes\tau_b"] \arrow[ur, bend right, "\sigma_a\otimes\sigma_b"'] \arrow[ur, phantom, "\Downarrow m_x\otimes m_y"] \arrow[dr, bend right, "\mu_{a,b}"'] \arrow[r, phantom, "\Downarrow u_{a,b}"] & {} & \mathscr{G}(a\otimes b) \\
 & \mathscr{F}(a\otimes b) \arrow[ur, bend right, "\tau_{a\otimes b}"'] &
\end{tikzcd}
\tag{20}$$

$$\begin{tikzcd}
I \arrow[rr, bend left, "\nu_0"] \arrow[dr, bend right, "\mu_0"'] \arrow[r, phantom, "\Downarrow v_0"] & {} & \mathscr{G}(I) \\
 & \mathscr{F}(I) \arrow[ur, bend left, "\tau_I"] \arrow[ur, bend right, "\sigma_I"'] \arrow[ur, phantom, "\Downarrow m_I"] &
\end{tikzcd}
=
\begin{tikzcd}
I \arrow[rr, bend left, "\nu_0"] \arrow[dr, bend right, "\mu_0"'] \arrow[r, phantom, "\Downarrow u_0"] & {} & \mathscr{G}(I) \\
 & \mathscr{F}(I) \arrow[ur, bend right, "\tau_I"'] &
\end{tikzcd}$$

For any monoidal 2-categories $\mathcal{K},\mathcal{L}$ there are 2-categories $\mathsf{Mon2Cat}_{\mathrm{ps}}(\mathcal{K},\mathcal{L})$ denoted by $\mathsf{WMonHom}(\mathcal{K},\mathcal{L})$ in [DS97] for bicategories. If we take weak monoidal 2-functors and monoidal 2-transformations, the corresponding sub-2-category is denoted by $\mathsf{Mon2Cat}(\mathcal{K},\mathcal{L})$.

A **pseudomonoid** (or **monoidale**) in a monoidal 2-category $(\mathcal{K},\otimes,I)$ is an object a equipped with multiplication $m : a\otimes a\to a$, unit $j : I\to a$, and invertible 2-cells

$$\begin{tikzcd}
a\otimes a\otimes a \arrow[r, "1\otimes m"] \arrow[d, "m\otimes 1"'] \arrow[dr, phantom, "\overset{\alpha}{\cong}"] & a\otimes a \arrow[d, "m"] \\
a\otimes a \arrow[r, "m"'] & a
\end{tikzcd}
\qquad
\begin{tikzcd}
a\otimes I \arrow[r, "1\otimes j"] \arrow[dr, "\sim"'] & a\otimes a \arrow[d, "m"] \arrow[dl, phantom, "\overset{\lambda}{\cong}"] \arrow[dr, phantom, "\overset{\rho}{\cong}"] & I\otimes a \arrow[l, "j\otimes 1"'] \arrow[dl, "\sim"] \\
 & a &
\end{tikzcd}
\tag{21}$$

expressing assiociativity and unitality up to isomorphism, that satisfy appropriate coherence conditions. A **lax morphism** between pseudomonoids a,b is a 1-cell $f\colon a\to b$ equipped with 2-cells

$$\begin{tikzcd}
a\otimes a \arrow[r, "f\otimes f"] \arrow[d, "m"'] \arrow[dr, phantom, "\overset{\phi}{\Swarrow}"] & b\otimes b \arrow[d, "m"] \\
a \arrow[r, "f"'] & b
\end{tikzcd}
\qquad
\begin{tikzcd}
I \arrow[d, "j"'] \arrow[dr, bend left, "j"] \arrow[dr, phantom, "\overset{\phi_0}{\Swarrow}"] & \\
a \arrow[r, "f"'] & b
\end{tikzcd}
\tag{22}$$

such that the following conditions hold:

$$\begin{tikzcd}
 & b\otimes b\otimes b \arrow[r, "m\otimes 1"] & b\otimes b\otimes a \arrow[dr, "m"] & \\
a\otimes a\otimes a \arrow[ur, "f\otimes f\otimes f"] \arrow[r, "m\otimes 1"] \arrow[dr, "1\otimes m"'] & a\otimes a \arrow[u, phantom, "\Downarrow \phi\otimes 1_f"] \arrow[ur, "f\otimes f"'] \arrow[dr, "m"] \arrow[d, phantom, "\overset{\alpha}{\cong}"] \arrow[r, phantom, "\Downarrow \phi"] & {} & b \\
 & a\otimes a \arrow[r, "m"'] & a \arrow[ur, "f"'] &
\end{tikzcd}
=
\begin{tikzcd}
 & b\otimes b\otimes b \arrow[r, "m\otimes 1"] \arrow[dr, "1\otimes m"'] & b\otimes b \arrow[dr, "m"] \arrow[d, phantom, "\overset{\alpha}{\cong}"] & \\
a\otimes a\otimes a \arrow[ur, "f\otimes f\otimes f"] \arrow[dr, "1\otimes m"'] \arrow[r, phantom, "\Downarrow 1_f\otimes\phi"] & {} & b\otimes b \arrow[r, "m"] \arrow[d, phantom, "\Downarrow \phi"] & b \\
 & a\otimes a \arrow[ur, "f\otimes f"] \arrow[r, "m"'] & b\otimes b \arrow[ur, "f"'] &
\end{tikzcd}
\tag{23}$$

$$\begin{tikzcd}
a \cong a \otimes I \arrow[rr, "f\otimes j"] \arrow[rrr, bend left=40, "f"] \arrow[dr, "1\otimes j"'] \arrow[dr, phantom, shift left=6, "1_f\otimes\phi_0 \Downarrow"] \arrow[rr, phantom, shift left=4, "1_f\otimes\lambda\cong"] \arrow[drr, bend right=50, "1_a"'] \arrow[drr, phantom, bend right=20, "\lambda\cong"] & & b \otimes b \arrow[r, "m"'] \arrow[d, phantom, shift left=4, "\Downarrow \phi"] & b \\
& a \otimes a \arrow[ur, dashed, "f\otimes f"] \arrow[r, "m"'] & a \arrow[ur, "f"'] &
\end{tikzcd}
=
\begin{tikzcd}
a \arrow[r, dashed, bend left, "f"] \arrow[r, dashed, bend right, "f"'] \arrow[r, phantom, "\Downarrow 1_f"] & b
\end{tikzcd}
=
\begin{tikzcd}
a \cong I \otimes a \arrow[rr, "j\otimes 1"] \arrow[rrr, bend left=40, "f"] \arrow[dr, "j\otimes 1"'] \arrow[dr, phantom, shift left=6, "\phi_0\otimes 1_f \Downarrow"] \arrow[rr, phantom, shift left=4, "\rho\otimes 1_f\cong"] \arrow[drr, bend right=50, "1_a"'] \arrow[drr, phantom, bend right=20, "\rho\cong"] & & b \otimes b \arrow[r, "m"'] \arrow[d, phantom, shift left=4, "\Downarrow \phi"] & b \\
& a \otimes a \arrow[ur, dashed, "f\otimes f"] \arrow[r, "m"'] & a \arrow[ur, "f"'] &
\end{tikzcd}$$

If (f, ϕ, ϕ_0) and (g, ψ, ψ_0) are two lax morphisms between pseudomonoids a and b, a **2-cell** between them $\sigma\colon f \Rightarrow g$ in $\mathcal{K}$ which is compatible with multiplications and units, in the sense that

$$\begin{tikzcd}
& b \otimes b \arrow[dr, "m"] \arrow[d, phantom, "\psi \Downarrow"] & \\
a \otimes a \arrow[ur, bend left, "f\otimes f"] \arrow[ur, bend right, "g\otimes g"'] \arrow[ur, phantom, "\Downarrow \sigma\otimes\sigma"] \arrow[dr, "m"'] & & b \\
& a \arrow[ur, "g"'] &
\end{tikzcd}
=
\begin{tikzcd}
& b \otimes b \arrow[dr, "m"] & \\
a \otimes a \arrow[ur, "f\otimes f"] \arrow[dr, "m"'] \arrow[r, phantom, "\phi \Downarrow"] & & b \\
& a \arrow[ur, bend left, "f"] \arrow[ur, bend right, "g"'] \arrow[ur, phantom, "\Downarrow \sigma"] &
\end{tikzcd}
\tag{24}$$

$$\begin{tikzcd}
I \arrow[rr, bend left, "j"] \arrow[dr, "j"'] \arrow[r, phantom, "\phi_0 \Downarrow"] & & b \\
& a \arrow[ur, bend left, "f"] \arrow[ur, bend right, "g"'] \arrow[ur, phantom, "\Downarrow \sigma"] &
\end{tikzcd}
=
\begin{tikzcd}
I \arrow[rr, bend left, "j"] \arrow[dr, "j"'] \arrow[rr, phantom, "\Downarrow \psi_0"] & & b \\
& a \arrow[ur, "g"'] &
\end{tikzcd}$$

We obtain a 2-category $\mathsf{PsMon}_\ell(\mathcal{K})$ for any monoidal 2-category $\mathcal{K}$, which is sometimes denoted by $\mathsf{Mon}(\mathcal{K})$ [CLS10]. By changing the direction of the 2-cells in (22) and the rest of the axioms appropriately, or asking for them to be invertible, we have 2-categories $\mathsf{PsMon}_{\mathrm{op}\ell}(\mathcal{K})$ and $\mathsf{PsMon}(\mathcal{K})$ of pseudomonoids with **oplax** or **(strong) morphisms** between them.

Example 2.4. The prototypical example is that of the monoidal 2-category $\mathcal{K} = (\mathsf{Cat}, \times, \mathbf{1})$ of categories, functors, and natural transformations with the cartesian product of categories and the unit category with a unique object and arrow. A pseudomonoid in $(\mathsf{Cat}, \times, \mathbf{1})$ is a monoidal category, a lax (resp. oplax, strong) morphism between two of these is precisely a lax (resp. oplax, strong) monoidal functor, and a 2-cell is a monoidal natural transformation. Therefore we obtain the well-known 2-categories MonCat_ℓ, $\mathsf{MonCat}_{\mathrm{op}\ell}$ and MonCat.

Remark 2.5. There is an evident similarity between the structures defined above, e.g (16) and (22), or (18) and (24). This is due to the fact that monoidal 2-categories, weak monoidal pseudofunctors and monoidal pseudonatural transformations are themselves appropriate pseudomonoid-related notions in a higher level; we do not get into such details, as they are not pertinent to the present work.

For our purposes, we are interested in a different observation: any pseudomonoid a in a monoidal 2-category $\mathcal{K}$ can in fact be expressed as a weak monoidal **normal** pseudofunctor $A\colon \mathbf{1} \to \mathcal{K}$ with $A(*) = a$, namely one where $A(1_*)$ is *equal* to 1_a. Moreover, a monoidal pseudonatural transformation $\tau\colon A \Rightarrow B\colon \mathbf{1} \to \mathcal{K}$ bijectively corresponds to a *strong morphism* between the pseudomonoids a and b, and similarly for monoidal modifications and 2-cells. Since every pseudofunctor is equivalent to a normal one, the 2-category of pseudomonoids $\mathsf{PsMon}(\mathcal{K})$ can be equivalently viewed as $\mathsf{Mon2Cat}_{\mathrm{ps}}(\mathbf{1},\mathcal{K})$, the 2-category of weak monoidal pseudofunctors $\mathbf{1}\to\mathcal{K}$, monoidal pseudonatural transformations and monoidal modifications.

As was already shown in [DS97, Prop. 5], any weak monoidal 2-functor $\mathscr{F}\colon\mathcal{K}\to\mathcal{L}$ takes pseudomonoids to pseudomonoids, and in fact [McC00] there is a functor $\mathsf{PsMon}(\mathscr{F})$ that commutes with the respective forgetful functors

$$\begin{tikzcd}
\mathsf{PsMon}(\mathcal{K}) \arrow[r, "\mathsf{PsMon}(\mathscr{F})"] \arrow[d] & \mathsf{PsMon}(\mathcal{L}) \arrow[d] \\
\mathcal{K} \arrow[r, "\mathscr{F}"'] & \mathcal{L}.
\end{tikzcd}$$

Based on the above Remark 2.5, and since every pseudofunctor from $\mathbf{1}$ into a 2-category trivially preserves composition on the nose and every pseudonatural transformation is really 2-natural, we can define a hom-2-functor that clarifies these assignments.

Proposition 2.6. *There is a 2-functor*

$$\mathsf{PsMon}(-)\simeq\mathsf{Mon2Cat}_{(}\mathrm{ps})(\mathbf{1},-)\colon\mathsf{Mon2Cat}\to\mathsf{2Cat} \tag{25}$$

which maps a monoidal 2-category to its 2-category of pseudomonoids, strong morphisms and 2-cells between them.

The theory in [DS97, McC00] extends the above definitions to the case of *braided* and *symmetric* pseudomonoids in *braided* and *symmetric monoidal 2-categories*. Briefly recall that a **braiding** for $(\mathcal{K},\otimes,I)$ is a pseudonatural equivalence with components $\beta_{a,b}\colon a\otimes b\to b\otimes a$ and invertible modifications, whereas a **syllepsis** is an invertible modification

$$a\otimes b\xrightarrow{1}a\otimes b\Rrightarrow a\otimes b\xrightarrow{\beta_{a,b}}b\otimes a\xrightarrow{\beta_{b,a}}a\otimes b$$

which is called **symmetry** if it satisfies extra axioms. With the appropriate notions of *braided* and *symmetric* weak monoidal pseudofunctors and monoidal pseudonatural transformations (and usual monoidal modifications), we have 3-categories $\mathsf{BrMon2Cat}_{\mathrm{ps}}$ and $\mathsf{SymMon2Cat}_{\mathrm{ps}}$. Indicatively, a weak monoidal pseudofunctor comes equipped an invertible modification with components

$$\begin{tikzcd}
\mathscr{F}a\otimes\mathscr{F}b \arrow[r, "\mu_{a,b}"] \arrow[d, "\beta_{\mathscr{F}a,\mathscr{F}b}"'] \arrow[dr, phantom, "\Downarrow v_{a,b}"] & \mathscr{F}b\otimes\mathscr{F}a \arrow[d, "\mathscr{F}(\beta_{a,b})"] \\
\mathscr{F}b\times\mathscr{F}a \arrow[r, "\mu_{b,a}"'] & \mathscr{F}(b\otimes a)
\end{tikzcd} \tag{26}$$

As earlier, there exist 2-categories of braided and symmetric pseudomonoids with strong morphisms between them, expressed as $\mathsf{BrPsMon}(\mathcal{K})=\mathsf{BrMon2Cat}_{(\mathrm{ps})}(\mathbf{1},\mathcal{K})$ and $\mathsf{SymPsMon}(\mathcal{K})=\mathsf{SymMon2Cat}_{(\mathrm{ps})}(\mathbf{1},\mathcal{K})$.

Proposition 2.7. *There are 2-functors*

$$\mathsf{BrPsMon}\colon\mathsf{BrMon2Cat}\to\mathsf{2Cat},\quad\mathsf{SymPsMon}\colon\mathsf{SymMon2Cat}\to\mathsf{2Cat}$$

which map a braided or symmetric monoidal 2-category to its 2-category of braided or symmetric pseudomonoids.

Finally, recall the notion of a monoidal 2-equivalence arising as the equivalence internal to the 2-category $\mathsf{Mon2Cat}$.

Definition 2.8. A **monoidal 2-equivalence** is a 2-equivalence $\mathscr{F}\colon \mathcal{K}\simeq\mathcal{L} : \mathscr{G}$ where both 2-functors are weak monoidal, and the 2-natural isomorphisms $1_{\mathcal{K}}\cong\mathscr{F}\mathscr{G}$, $\mathscr{G}\mathscr{F}\cong 1_{\mathcal{L}}$ are monoidal. Similarly for **braided** and **symmetric** monoidal 2-equivalences.

As is the case for any 2-functor between 2-categories, PsMon as well as $\mathsf{BrPsMon}$ and $\mathsf{SymPsMon}$ map equivalences to equivalences.

Proposition 2.9. *Any monoidal 2-equivalence $\mathcal{K}\simeq\mathcal{L}$ induces a 2-equivalence between the respective 2-categories of pseudomonoids $\mathsf{PsMon}(\mathcal{K})\simeq\mathsf{PsMon}(\mathcal{L})$. Similarly any braided or symmetric monoidal 2-equivalence induces $\mathsf{BrPsMon}(\mathcal{K})\simeq\mathsf{BrPsMon}(\mathcal{L})$ or $\mathsf{SymPsMon}(\mathcal{K})\simeq\mathsf{SymPsMon}(\mathcal{L})$.*

3. The Monoidal Grothendieck Construction

In this section, we give explicit descriptions of the 2-categories of pseudomonoids in the cartesian monoidal 2-categories of fibrations and indexed categories, Fib and ICat, and we exhibit their equivalence induced by the *monoidal* Grothendieck construction. We also consider the fixed-base case, namely pseudomononoids in $\mathsf{Fib}(\mathcal{X})$ and $\mathsf{ICat}(\mathcal{X})$ and the corresponding equivalence that arises. These two cases are in general distinct, and can be summarized in

$$\begin{tikzcd}
 & \mathsf{Fib}\simeq\mathsf{ICat} \arrow[dl, "\mathsf{PsMon}(-)"'] \arrow[dr, "\text{fix } \mathcal{X}"] & \\
\mathsf{MonFib}\simeq\mathsf{MonICat} \arrow[d, "\text{fix } \mathcal{X}"'] & & \mathsf{Fib}(\mathcal{X})\simeq 2\mathsf{Cat}_{\mathrm{ps}}(\mathcal{X}^{\mathrm{op}},\mathsf{Cat}) \arrow[d, "\mathsf{PsMon}(-)"] \\
\mathsf{MonFib}(\mathcal{X})\simeq\mathsf{Mon}2\mathsf{Cat}_{\mathrm{ps}}(\mathcal{X}^{\mathrm{op}},\mathsf{Cat}) & & \mathsf{PsMon}(\mathsf{Fib}(\mathcal{X}))\simeq 2\mathsf{Cat}_{\mathrm{ps}}(\mathcal{X}^{\mathrm{op}},\mathsf{MonCat})
\end{tikzcd} \tag{27}$$

The two feet turn out to coincide only under certain hypotheses, which reveal some interesting subtleties on the potential monoidal structures on fibrations and pseudofunctors, discussed in detail in Section 4.

3.1. Monoidal Fibrations.

The 2-categories Fib and OpFib of (op)fibrations over arbitrary bases, recalled in Section 2.1, have a natural cartesian monoidal structure inherited from $\mathsf{Cat}^{\mathbf{2}}$. For two fibrations P and Q, their (2-)product

$$P\times Q\colon \mathcal{A}\times\mathcal{B}\to\mathcal{X}\times\mathcal{Y} \tag{28}$$

is also an fibration, where a cartesian lifting is a pair of a P-lifting and a Q-lifting; similarly for opfibrations. The monoidal unit is the trivial (op)fibration $1_{\mathbf{1}}\colon\mathbf{1}\to\mathbf{1}$. Since the monoidal structure is cartesian, they are both symmetric monoidal 2-categories.

A pseudomonoid in $(\mathsf{Fib},\times,1_{\mathbf{1}})$ is called a **monoidal fibration**. A detailed argument of how the following proposition captures the required structure can be found in [Vas18], and also that was the original description of this notion in [Shu08].

Proposition 3.1. *A monoidal fibration $P\colon\mathcal{A}\to\mathcal{X}$ is a fibration for which both the total $\mathcal{A}$ and base category $\mathcal{X}$ are monoidal, P is a strict monoidal functor and the the tensor product $\otimes_{\mathcal{A}}$ of $\mathcal{A}$ preserves cartesian liftings.*

Explicitly, the multiplication and unit are fibred 1-cells $m = (\otimes_{\mathcal{A}}, \otimes_{\mathcal{X}})\colon P \times P \to P$ and $j = (I_{\mathcal{A}}, I_{\mathcal{X}})\colon \mathbf{1} \to P$ displayed as

$$\begin{tikzcd}
\mathcal{A}\times\mathcal{A} \arrow[r, "\otimes_{\mathcal{A}}"] \arrow[d, "P\times P"'] & \mathcal{A} \arrow[d, "P"] & \text{and} & \mathbf{1} \arrow[r, "I_{\mathcal{A}}"] \arrow[d, "1_{\mathbf{1}}"'] & \mathcal{A} \arrow[d, "P"] \\
\mathcal{X}\times\mathcal{X} \arrow[r, "\otimes_{\mathcal{X}}"'] & \mathcal{X} & & \mathbf{1} \arrow[r, "I_{\mathcal{X}}"'] & \mathcal{X}
\end{tikzcd} \tag{29}$$

A **monoidal fibred 1-cell** between two monoidal fibrations $P\colon \mathcal{A} \to \mathcal{X}$ and $Q\colon \mathcal{B} \to \mathcal{Y}$ is a (strong) morphism of pseudomonoids between them, as defined in Section 2.2. It amounts to a fibred 1-cell, i.e. a commutative

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "H"] \arrow[d, "P"'] & \mathcal{B} \arrow[d, "Q"] \\
\mathcal{X} \arrow[r, "F"'] & \mathcal{Y}
\end{tikzcd} \tag{30}$$

where H preserves cartesian liftings, along with invertible 2-cells (22) in Fib satisfying axioms (23). By (6), these are fibred 2-cells

$$\begin{tikzcd}
& \mathcal{B}\times\mathcal{B} \arrow[dr, "\otimes_W"] & & & & & \\
\mathcal{A}\times\mathcal{A} \arrow[ur, "H\times H"] \arrow[dr, "\otimes_{\mathcal{A}}"'] \arrow[ddd, "P\times P"'] & \Downarrow \phi & \mathcal{B} \arrow[ddd, "Q"] & & \mathbf{1} \arrow[rr, bend left, "I_{\mathcal{B}}"] \arrow[dr, "I_{\mathcal{A}}"'] \arrow[ddd, equal] & \Downarrow \phi_0 & \mathcal{B} \arrow[ddd, "Q"] \\
& \mathcal{A} \arrow[ur, "H"'] & & & & \mathcal{A} \arrow[ur, "H"'] & \\
& \mathcal{Y}\times\mathcal{Y} \arrow[dr, "\otimes_{\mathcal{Y}}"] & & & & & \\
\mathcal{X}\times\mathcal{X} \arrow[ur, "F\times F"] \arrow[dr, "\otimes_{\mathcal{X}}"'] & \Downarrow \psi & \mathcal{Y} & & \mathbf{1} \arrow[rr, bend left, "I_{\mathcal{Y}}"] \arrow[dr, "I_{\mathcal{X}}"'] & \Downarrow \psi_0 & \mathcal{Y} \\
& \mathcal{X} \arrow[ur, "F"'] & & & & \mathcal{X} \arrow[ur, "F"'] &
\end{tikzcd} \tag{31}$$

where ϕ and ψ are natural isomorphisms with components

$$\phi_{a,b}\colon Ha \otimes Hb \xrightarrow{\sim} H(a \otimes b), \quad \psi_{x,y}\colon Fx \otimes Fy \xrightarrow{\sim} F(x \otimes y)$$

such that ϕ is above ψ, i.e. the following diagram commutes:

$$\begin{tikzcd}
Q(Ha \otimes Hb) \arrow[r, "Q\phi_{a,b}"] \arrow[d, equal, "(29)"'] & QH(a \otimes b) \arrow[d, equal, "(30)"] \\
QHa \otimes QHb \arrow[d, equal, "(30)"'] & FP(a \otimes b) \arrow[d, equal, "(29)"] \\
FPa \otimes FPb \arrow[r, "\psi_{Pa,Pb}"'] & F(Pa \otimes Pb)
\end{tikzcd}$$

Similarly, ϕ_0 and ψ_0 have single components $\phi_0\colon I_{\mathcal{B}} \xrightarrow{\sim} H(I_{\mathcal{A}})$ and $\psi_0\colon I_{\mathcal{Y}} \xrightarrow{\sim} F(I_{\mathcal{X}})$ such that $Q(\phi_0) = \psi_0$. These two conditions in fact say that the identity transformation, a.k.a. commutative square (30) is a monoidal one, as expressed in [Shu08, 12.5]. The relevant axioms dictate that (ϕ, ϕ_0) and (ψ, ψ_0) give H and F the structure of strong monoidal functors, thus we obtain the following characterization.

Proposition 3.2. *A monoidal fibred 1-cell between two monoidal fibrations P and Q is a fibred 1-cell (H, F) where both functors are monoidal, (H, ϕ, ϕ_0) and (F, ψ, ψ_0), such that $Q(\phi_{a,b}) = \psi_{Pa,Pb}$ and $Q\phi_0 = \psi_0$.*

For lax or oplax morphisms of pseudomonoids in Fib, we obtain appropriate notions of monoidal fibred 1-cells, where the top and bottom functors of (30) are lax or oplax monoidal respectively.

Finally, a **monoidal fibred 2-cell** is a 2-cell between lax morphisms (H, F) and (K, G) of pseudomonoids P, Q in Fib. Explicitly, it is a fibred 2-cell as described in Section 2.1

$$\begin{tikzcd}
\mathcal{A} \arrow[r, bend left, "H"{name=U}] \arrow[r, bend right, "K"'{name=D}] \arrow[d, "P"'] & \mathcal{B} \arrow[d, "Q"] \\
\mathcal{X} \arrow[r, bend left, "F"{name=U2}] \arrow[r, bend right, "G"'{name=D2}] & \mathcal{Y}
\arrow[from=U, to=D, Rightarrow, "\beta"]
\arrow[from=U2, to=D2, Rightarrow, "\alpha"]
\end{tikzcd}$$

satisfying the axioms (24); these come down to the fact that both β and α are monoidal natural transformations between the respective lax monoidal functors, $H \Rightarrow K$ and $F \Rightarrow G$.

Proposition 3.3. *A monoidal fibred 2-cell between two monoidal fibred 1-cells is an ordinary fibred 2-cell (α, β) where both natural transformations are monoidal.*

We denote by $\mathsf{PsMon}(\mathsf{Fib}) = \mathsf{MonFib}$ the 2-category of monoidal fibrations, monoi -dal fibred 1-cells and monoidal fibred 2-cells. By changing the notion of morphisms between pseudomonoids to lax or oplax, we obtain 2-categories MonFib_ℓ and $\mathsf{MonFib}_{\mathrm{op}\ell}$. There are also 2-categories $\mathsf{BrMonFib}$ and $\mathsf{SymMonFib}$ of **braided** (resp. **symmetric**) **monoidal fibrations**, **braided** (resp. **symmetric**) **monoidal fibred 1-cells** and monoidal fibred 2-cells, defined to be $\mathsf{BrPsMon}(\mathsf{Fib})$ and $\mathsf{SymPsMon}(\mathsf{Fib})$ respectively; see Proposition 2.7.

Dually, we have appropriate 2-categories of **monoidal opfibrations**, **monoidal opfibred 1-cells** and **monoidal opfibred 2-cells** and their braided and symmetric variations, $\mathsf{MonOpFib}$, $\mathsf{BrMonOpFib}$ and $\mathsf{SymMonOpFib}$. All the structures are constructed dually, where a monoidal opfibration, namely a pseudomonoid in the cartesian monoidal $(\mathsf{OpFib}, \times, 1_{\mathbf{1}})$, is a strict monoidal functor such that the tensor product of the total category preserves cocartesian liftings.

All the above 2-categories have sub-2-categories of monoidal (op)fibrations over a fixed monoidal base $(\mathcal{X}, \otimes, I)$, e.g. $\mathsf{MonFib}(\mathcal{X})$ and $\mathsf{MonOpFib}(\mathcal{X})$. The morphisms are **monoidal (op)fibred functors**, i.e. fibred 1-cells of the form $(H, 1_\mathcal{X})$ with H monoidal, and the 2-cells are **monoidal (op)fibred natural transformations**, i.e. fibred 2-cells of the form $(\beta, 1_{1_\mathcal{X}})$ with β monoidal. These 2-categories constitute the lower left foot of the diagram (27) on the side of fibrations, and correspond to the 'global' monoidal structure part of the story (in a sense to be clarified by Theorem 3.10).

We end this section by considering a different monoidal object in the context of (op)fibrations, starting over from the usual 2-categories of (op)fibrations over a fixed base $\mathcal{X}$, (op)fibred functor and (op)fibred natural transformations $\mathsf{Fib}(\mathcal{X})$ and $\mathsf{OpFib}(\mathcal{X})$. Notice that contrary to the earlier devopment, there is no monoidal structure on $\mathcal{X}$. Both these 2-categories are also cartesian monoidal, but in a different manner than Fib and OpFib, due to the cartesian monoidal structure of $\mathsf{Cat}/\mathcal{X}$;

see for example [Jac99, 1.7.4]. Explicitly, for fibrations $P\colon \mathcal{A} \to \mathcal{X}$ and $Q\colon \mathcal{B} \to \mathcal{X}$, their tensor product $P \boxtimes Q$ is given by any of the two equal functors to $\mathcal{X}$ from the following pullback

$$\begin{tikzcd}
\mathcal{A} \times_{\mathcal{X}} \mathcal{B} \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] \arrow[dr, dashed, bend left, "P\boxtimes Q"] & \mathcal{A} \arrow[d, "P"] \\
\mathcal{B} \arrow[r, "Q"'] & \mathcal{X}
\end{tikzcd} \tag{32}$$

since fibrations are closed under pullbacks and of course composition. The monoidal unit is $1_{\mathcal{X}}\colon \mathcal{X} \to \mathcal{X}$.

A pseudomonoid in $(\mathsf{Fib}(\mathcal{X}), \boxtimes, 1_{\mathcal{X}})$ is an ordinary fibration $P\colon \mathcal{A} \to \mathcal{X}$ equipped with two fibred functors $(m, 1_{\mathcal{X}})\colon P \boxtimes P \to P$ and $(j, 1_{\mathcal{X}})\colon 1_{\mathcal{X}} \to P$ displayed as

$$\begin{tikzcd}
\mathcal{A} \times_{\mathcal{X}} \mathcal{A} \arrow[rr, "m"] \arrow[dr, "P\boxtimes P"'] & & \mathcal{A} \arrow[dl, "P"] & \mathcal{X} \arrow[rr, "j"] \arrow[dr, "1_{\mathcal{X}}"'] & & \mathcal{A} \arrow[dl, "P"] \\
& \mathcal{X} & & & \mathcal{X} &
\end{tikzcd}$$

along with invertible fibred 2-cells satisfying the usual axioms. In more detail, the pullback $\mathcal{A} \times_{\mathcal{X}} \mathcal{A}$ consists of pairs of objects of $\mathcal{A}$ which are in the same fibre of P, and $P \boxtimes P$ sends such a pair to their underlying object defining their fibre. The functor m maps any $(a, b) \in \mathcal{A}_x$ to some $m(a, b) := a \otimes_x b \in \mathcal{A}_x$ and the map j sends an object $x \in \mathcal{X}$ to a chosen one, I_x, in its fibre. The invertible 2-cells and the axioms guarantee that these maps define a monoidal structure on each fibre $\mathcal{A}_x$, providing the associativity, left and right unitors. The fact that m and j preserve cartesian liftings translate into a (strong) monoidal structure on the reindexing functors: for any $f\colon x \to y$ and $a, b \in \mathcal{A}_y$, $f^*a \otimes_x f^*b \cong f^*(a \otimes_y b)$ and $I_y \cong f^*(I_x)$.

A (lax) morphism between two such fibrations is a fibred functor (4) such that the induced functors $H_x\colon \mathcal{A}_x \to \mathcal{B}_x$ between the fibres as in (5) are (lax) monoidal, whereas a 2-cell between them is a fibred natural transformation $\beta\colon H \Rightarrow K$ (7) which is monoidal when restricted to the fibers, $\beta_x|_{\mathcal{A}_x}\colon H_x \Rightarrow K_x$. In this way, we obtain the 2-category $\mathsf{PsMon}(\mathsf{Fib}(\mathcal{X}))$ and dually $\mathsf{PsMon}(\mathsf{OpFib}(\mathcal{X}))$. These 2-categories constitute the lower right foot of the diagram (27) on the side of fibrations, and correspond to the 'fibrewise' part of the story.

Remark 3.4. As is evident from the above descriptions, the 2-categories $\mathsf{MonFib}(\mathcal{X})$ and $\mathsf{PsMon}(\mathsf{Fib}(\mathcal{X}))$, as well as their opfibration counterparts, are completely different in general. A monoidal fibration over $\mathcal{X}$ is a strict monoidal functor, whereas a pseudomonoid in fixed-base fibrations is a fibration with monoidal fibres in a coherent way: none of the base or the total category need to be monoidal. Formally, this says that starting with Fib, taking its pseudomonoids and then restricting over some base is *not* the same as first restricting the base and then taking pseudomonoids. This corresponds to the two distinct fibration legs of (27).

3.2. **Monoidal Indexed Categories.** The 2-categories of indexed and opindexed categories ICat and OpICat, recalled in Section 2.1, are both monoidal. Explicitly, given two indexed categories $\mathscr{M}\colon \mathcal{X}^{\mathrm{op}} \to \mathsf{Cat}$ and $\mathscr{N}\colon \mathcal{Y}^{\mathrm{op}} \to \mathsf{Cat}$, their tensor product $\mathscr{M} \otimes \mathscr{N}\colon (\mathcal{X} \times \mathcal{Y})^{\mathrm{op}} \to \mathsf{Cat}$ is the composite

$$(\mathcal{X} \times \mathcal{Y})^{\mathrm{op}} \cong \mathcal{X}^{\mathrm{op}} \times \mathcal{Y}^{\mathrm{op}} \xrightarrow{\mathscr{M} \times \mathscr{N}} \mathsf{Cat} \times \mathsf{Cat} \xrightarrow{\times} \mathsf{Cat} \tag{33}$$

i.e. $(\mathscr{M} \otimes \mathscr{N})(x, y) = \mathscr{M}(x) \times \mathscr{N}(y)$ using the cartesian monoidal structure of Cat. The monoidal unit is the indexed category $\Delta\mathbf{1} \colon \mathbf{1}^{\mathrm{op}} \to \mathsf{Cat}$ that picks out the terminal category $\mathbf{1}$ in Cat, and similarly for opindexed categories. Notice that this monoidal 2-structure, formed pointwise in Cat, is also cartesian.

We call a pseudomonoid in $(\mathsf{ICat}, \otimes, \Delta\mathbf{1})$ a **monoidal indexed category**. Explicitly, it is an indexed category $\mathscr{M} \colon \mathcal{X}^{\mathrm{op}} \to \mathsf{Cat}$ equipped with multiplication and unit indexed 1-cells $(\otimes_{\mathcal{X}}, \mu) \colon \mathscr{M} \otimes \mathscr{M} \to \mathscr{M}$, $(\eta, \mu_0) \colon \Delta\mathbf{1} \to \mathscr{M}$ which by (8) look like

$$\begin{tikzcd}
\mathcal{X}^{\mathrm{op}} \times \mathcal{X}^{\mathrm{op}} \arrow[dd, "\otimes^{\mathrm{op}}"'] \arrow[dr, "\mathscr{M}\otimes\mathscr{M}"] & \\
{} \arrow[r, phantom, "\Downarrow \mu"] & \mathsf{Cat} \\
\mathcal{X}^{\mathrm{op}} \arrow[ur, "\mathscr{M}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{1}^{\mathrm{op}} \arrow[dd, "I^{\mathrm{op}}"'] \arrow[dr, "\Delta\mathbf{1}"] & \\
{} \arrow[r, phantom, "\Downarrow \mu_0"] & \mathsf{Cat} \\
\mathcal{X}^{\mathrm{op}} \arrow[ur, "\mathscr{M}"'] &
\end{tikzcd}$$

These come equipped with invertible indexed 2-cells as in (21); the axioms this data is required to satisfy, on the one hand, render $\mathcal{X}$ a monoidal category with $\otimes \colon \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ its tensor product functor and $I \colon \mathbf{1} \to \mathsf{Cat}$ its unit. On the other hand, the resulting axioms for the components

$$\mu_{x,y} \colon \mathscr{M}x \times \mathscr{M}y \to \mathscr{M}(x \otimes y)$$
$$\mu_0 \colon \mathbf{1} \to \mathscr{M}(I)$$

of the above pseudonatural transformations precisely give $\mathscr{M}$ the structure of a weak monoidal pseudofunctor, recalled in Section 2.2.

Proposition 3.5. *A monoidal indexed category is a weak monoidal pseudofunctor* $(\mathscr{M}, \mu, \mu_0) \colon (\mathcal{X}^{\mathrm{op}}, \otimes^{\mathrm{op}}, I) \to (\mathsf{Cat}, \times, \mathbf{1})$*, where* $(\mathcal{X}, \otimes, I)$ *is an (ordinary) monoidal category.*

We then define a **monoidal indexed 1-cell** to be a (strong) morphism between pseudomonoids in $(\mathsf{ICat}, \otimes, \Delta\mathbf{1})$. In detail, it is an indexed 1-cell $(F, \tau) \colon \mathscr{M} \to \mathscr{N}$

$$\begin{tikzcd}
\mathcal{X}^{\mathrm{op}} \arrow[dd, "F^{\mathrm{op}}"'] \arrow[dr, "\mathscr{M}"] & \\
{} \arrow[r, phantom, "\Downarrow \tau"] & \mathsf{Cat} \\
\mathcal{Y}^{\mathrm{op}} \arrow[ur, "\mathscr{N}"'] &
\end{tikzcd}$$

between two monoidal indexed categories $(\mathscr{M}, \mu, \mu_0) \colon (\mathcal{X}, \otimes, I)^{\mathrm{op}} \to (\mathsf{Cat}, \times, \mathbf{1})$ and $(\mathscr{N}, \nu, \nu_0) \colon (\mathcal{Y}, \otimes, I)^{\mathrm{op}} \to (\mathsf{Cat}, \times, \mathbf{1})$ equipped with two invertible indexed 2-cells (ψ, m) and (ψ_0, m_0) as in (22), which explicitly consist of natural isomorphisms ψ, ψ_0 and invertible modifications

$$\begin{tikzcd}
\mathcal{X}^{\mathrm{op}} \times \mathcal{X}^{\mathrm{op}} \arrow[d, "\otimes^{\mathrm{op}}"'] \arrow[dr, "F^{\mathrm{op}} \times F^{\mathrm{op}}"] \arrow[drrr, bend left, "\mathscr{M}\otimes\mathscr{M}"] & & \Downarrow \tau\times\tau & \\
\mathcal{X}^{\mathrm{op}} \arrow[d, "F^{\mathrm{op}}"'] & \mathcal{Y}^{\mathrm{op}} \times \mathcal{Y}^{\mathrm{op}} \arrow[rr, "\mathscr{N}\otimes\mathscr{N}"] \arrow[dr, "\otimes^{\mathrm{op}}"'] & & \mathsf{Cat} \\
\mathcal{Y}^{\mathrm{op}} \arrow[rr, "\mathrm{id}"'] \arrow[ur, phantom, "\Downarrow \psi"] & & \mathcal{Y}^{\mathrm{op}} \arrow[ur, bend right, "\mathscr{N}"'] \arrow[u, phantom, "\Downarrow \nu"] &
\end{tikzcd}
\overset{m}{\Rrightarrow}
\begin{tikzcd}
\mathcal{X}^{\mathrm{op}} \times \mathcal{X}^{\mathrm{op}} \arrow[d, "\otimes^{\mathrm{op}}"'] \arrow[drr, bend left, "\mathscr{M}\otimes\mathscr{M}"] & \Downarrow \mu & \\
\mathcal{X}^{\mathrm{op}} \arrow[d, "F^{\mathrm{op}}"'] \arrow[rr, "\mathscr{M}"] & & \mathsf{Cat} \\
\mathcal{Y}^{\mathrm{op}} \arrow[urr, bend right, "\mathscr{N}"'] & \Downarrow \tau &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbf{1}^{\mathrm{op}} \arrow[d, "I^{\mathrm{op}}"'] \arrow[ddrr, "I^{\mathrm{op}}" description] \arrow[drrr, bend left, "\Delta\mathbf{1}"] & & \Downarrow \nu_0 & & & \mathbf{1}^{\mathrm{op}} \arrow[d, "I^{\mathrm{op}}"'] \arrow[drr, bend left, "\Delta\mathbf{1}"] & \Downarrow \mu_0 & \\
\mathcal{X}^{\mathrm{op}} \arrow[d, "F^{\mathrm{op}}"'] & \Downarrow \psi_0 & & \mathsf{Cat} & \overset{m_0}{\Rrightarrow} & \mathcal{X}^{\mathrm{op}} \arrow[d, "F^{\mathrm{op}}"'] \arrow[rr, "\mathscr{M}" description] & & \mathsf{Cat} \\
\mathcal{Y}^{\mathrm{op}} \arrow[rr, "\mathrm{id}"'] & & \mathcal{Y}^{\mathrm{op}} \arrow[ur, bend right, "\mathscr{N}"'] & & & \mathcal{Y}^{\mathrm{op}} \arrow[urr, bend right, "\mathscr{N}"'] & \Downarrow \tau &
\end{tikzcd}$$

as dictated by the general form (10) of indexed 2-cells. The natural isomorphisms ψ and ψ_0 have components

$$\psi_{x,z} \colon Fx \otimes Fy \xrightarrow{\sim} F(x \otimes y), \quad \psi_0 \colon I \xrightarrow{\sim} F(I) \quad \text{in } \mathcal{Y}^{\mathrm{op}}$$

whereas the modifications m and m_0 are given by families of invertible natural transformations

$$\begin{tikzcd}
 & \mathscr{N}Fx \times \mathscr{N}Fy \arrow[r, "\overset{\nu_{Fx,Fy}}{\sim}"] \arrow[dr, dashed] & \mathscr{N}(Fx \otimes Fy) \arrow[d, "\mathscr{N}\psi_{x,y}"] \\
\mathscr{M}x \times \mathscr{M}y \arrow[ur, "\tau_x \times \tau_y"] \arrow[dr, "\mu_{x,y}"'] & \Downarrow m_{x,y} & \mathscr{N}F(x \otimes y) \\
 & \mathscr{M}(x \otimes y) \arrow[ur, "\tau_{x \otimes y}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathscr{N}(I) \arrow[dr, "\mathscr{N}\psi_0"] & \\
\mathbf{1} \arrow[ur, "\nu_0"] \arrow[dr, "\mu_0"'] & \Downarrow m_0 & \mathscr{N}(FI) \\
 & \mathscr{M}(I) \arrow[ur, "\tau_I"'] &
\end{tikzcd}$$

The appropriate coherence axioms ensure that the functor $F \colon \mathcal{X} \to \mathcal{Y}$ has a strong monoidal structure (F, ψ, ψ_0), and that the pseudonatural transformation $\tau \colon \mathscr{M} \Rightarrow \mathscr{N} \circ F^{\mathrm{op}}$ is monoidal with $m_{x,y}$, m_0 as in (19). Notice that F^{op} being monoidal makes F monoidal with inverse structure isomorphisms.

Proposition 3.6. *A monoidal indexed 1-cell between two monoidal indexed categories $\mathscr{M}$ and $\mathscr{N}$ is an indexed 1-cell (F, τ), where the functor F is (strong) monoidal and the pseudonatural transformation τ is monoidal.*

Finally, a **monoidal indexed 2-cell** is a 2-cell between morphisms of pseudomonoids in $(\mathsf{ICat}, \otimes, \Delta\mathbf{1})$. Following the definition of Section 2.2, it turns out that an indexed 2-cell $(a, m) \colon (F, \tau) \Rightarrow (G, \sigma) \colon \mathscr{M} \to \mathscr{N}$ as in (10), which consists of a natural transformation $\alpha \colon F \Rightarrow G$ and a modification m with components

$$\begin{tikzcd}
\mathscr{M}x \arrow[rr, bend left, "\tau_x"] \arrow[dr, "\sigma_x"'] & \Downarrow m_x & \mathscr{N}Fx \\
 & \mathscr{N}Gx \arrow[ur, "\mathscr{N}\alpha_x"'] &
\end{tikzcd}$$

is monoidal, exactly when $\alpha \colon F \Rightarrow G$ is compatible with the strong monoidal structures of F and G, and the modification $m \colon \tau \Rrightarrow \mathscr{N}\alpha^{\mathrm{op}} \circ \sigma$ satisfies (20) for the induced monoidal structures on its domain and target pseudonatural transformations.

Proposition 3.7. *A monoidal indexed 2-cell between two monoidal indexed 1-cells (F, τ) and (G, σ) is an indexed 2-cell (α, m) such that α is an ordinary monoidal natural transformation and m is a monoidal modification.*

We write $\mathsf{PsMon}(\mathsf{ICat}) = \mathsf{MonICat}$, the 2-category of monoidal indexed categories, monoidal indexed 1-cells and monoidal indexed 2-cells. Moreover, their braided and symmetric counterparts form $\mathsf{BrMonICat}$ and $\mathsf{SymMonICat}$ respectively, as the 2-categories of braided and symmetric pseudomonoids in $(\mathsf{ICat}, \otimes, \Delta\mathbf{1})$ formally discussed in Section 2.2.

Similarly, we have 2-categories of **(braided** or **symmetric) monoidal opindexed** categories, 1-cells and 2-cells $\mathsf{MonOpICat}$, $\mathsf{BrMonOpICat}$ and $\mathsf{SymMonOpICat}$. Finally, all these 2-categories have sub-2-categories of monoidal (op)indexed categories with a fixed monoidal domain $(\mathcal{X}, \otimes I)$, and specifically

$$\begin{aligned} \mathsf{MonICat}(\mathcal{X}) &= \mathsf{Mon2Cat}_{\mathrm{ps}}(\mathcal{X}^{\mathrm{op}}, \mathsf{Cat}) \\ \mathsf{MonOpICat}(\mathcal{X}) &= \mathsf{Mon2Cat}_{\mathrm{ps}}(\mathcal{X}, \mathsf{Cat}) \end{aligned} \tag{34}$$

the functor 2-categories of weak monoidal pseudofunctors, monoidal pseudonatural transformations and monoidal modifications; these belong to the lower left foot of the diagram (27), now on the side of indexed categories.

Finally, similarly to the previous Section 3.1 on fibrations, we end this section with the study of pseudomonoids in a different but related monoidal 2-category, namely $\mathsf{ICat}(\mathcal{X}) = \mathsf{2Cat}_{\mathrm{ps}}(\mathcal{X}^{\mathrm{op}}, \mathsf{Cat})$ of indexed categories with a fixed domain $\mathcal{X}$. Working in this 2-category, or in $\mathsf{OpICat}(\mathcal{X})$, there is no assumed monoidal structure on $\mathcal{X}$. Their monoidal structure is again cartesian: for two $\mathcal{X}$-indexed categories $\mathscr{M}, \mathscr{N} \colon \mathcal{X}^{\mathrm{op}} \to \mathsf{Cat}$, their product is

$$\mathscr{M} \boxtimes \mathscr{N} \colon \mathcal{X}^{\mathrm{op}} \xrightarrow{\Delta} \mathcal{X}^{\mathrm{op}} \times \mathcal{X}^{\mathrm{op}} \xrightarrow{\mathscr{M} \times \mathscr{N}} \mathsf{Cat} \times \mathsf{Cat} \xrightarrow{\times} \mathsf{Cat} \tag{35}$$

with pointwise components $(\mathscr{M} \boxtimes \mathscr{N})(x) = \mathscr{M}(x) \times \mathscr{N}(x)$ in Cat. The monoidal unit is just $\mathcal{X}^{\mathrm{op}} \xrightarrow{!} \mathbf{1} \xrightarrow{\Delta\mathbf{1}} \mathsf{Cat}$, which we will also call $\Delta\mathbf{1}$.

A pseudomonoid in $(\mathsf{ICat}(\mathcal{X}), \boxtimes, \Delta\mathbf{1})$ is a pseudofunctor $\mathscr{M} \colon \mathcal{X}^{\mathrm{op}} \to \mathsf{Cat}$ equipped with indexed functors (9) $m \colon \mathscr{M} \boxtimes \mathscr{M} \to \mathscr{M}$ and $j \colon \Delta\mathbf{1} \to \mathscr{M}$ i.e. pseudonatural transformations

$$\begin{tikzcd}
 & \mathcal{X}^{\mathrm{op}} \times \mathcal{X}^{\mathrm{op}} \arrow[r, "\mathscr{M} \times \mathscr{M}"] \arrow[d, phantom, "\Downarrow m"] & \mathsf{Cat} \times \mathsf{Cat} \arrow[dr, "\times"] & \\
\mathcal{X}^{\mathrm{op}} \arrow[ur, "\Delta"] \arrow[rrr, bend right, "\mathscr{M}"'] & {} & & \mathsf{Cat}
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathbf{1} \arrow[dr, "\Delta\mathbf{1}"] \arrow[d, phantom, "\Downarrow j"] & \\
\mathcal{X}^{\mathrm{op}} \arrow[ur, "!"] \arrow[rr, bend right, "\mathscr{M}"'] & {} & \mathsf{Cat}
\end{tikzcd}$$

with components $m_x \colon \mathscr{M}x \times \mathscr{M}x \to \mathscr{M}x$ and $j_x \colon \mathbf{1} \to \mathscr{M}x$ for all objects $x \in \mathcal{X}$ and natural isomorphisms

$$\begin{tikzcd}
\mathscr{M}x \times \mathscr{M}x \arrow[r, "\mathscr{M}f \times \mathscr{M}f"] \arrow[d, "m_x"'] \arrow[dr, phantom, "\cong"] & \mathscr{M}y \times \mathscr{M}y \arrow[d, "m_y"] \\
\mathscr{M}x \arrow[r, "\mathscr{M}f"'] & \mathscr{M}y
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{1} \arrow[r, "="] \arrow[d, "j_x"'] \arrow[dr, phantom, "\cong"] & \mathbf{1} \arrow[d, "j_y"] \\
\mathscr{M}x \arrow[r, "\mathscr{M}f"'] & \mathscr{M}y
\end{tikzcd}$$

If we denote $m_x = \otimes_x$ and $j_x = I_x$, the pseudomonoid invertible 2-cells (21) and the axioms these data satisfy make each $\mathscr{M}x$ into a monoidal category $(\mathscr{M}x, \otimes_x, I_x)$, and each $\mathscr{M}f$ into a strong monoidal functor: the above isomorphisms have components $\mathscr{M}f(a) \otimes_y \mathscr{M}f(b) \cong \mathscr{M}f(a \otimes_x b)$ and $I_y \cong \mathscr{M}f(I_x)$ for any $a, b \in \mathscr{M}x$.

Such a structure, namely a pseudofunctor $\mathscr{M} \colon \mathcal{X}^{\mathrm{op}} \to \mathsf{MonCat}$ into the 2-category of monoidal categories, (strong) monoidal functors and monoidal natural transformations, was directly defined as an **indexed strong monoidal category** in [HM06].

We will avoid this notation in order to not create confusion with *monoidal indexed categories* earlier.

A strong morphism of pseudomonoids in $(\mathsf{ICat}(\mathcal{X}), \boxtimes, \Delta\mathbf{1})$ ends up being a pseudonatural trasformation $\tau\colon \mathscr{M} \Rightarrow \mathscr{N}\colon \mathcal{X}^{\mathrm{op}} \to \mathsf{Cat}$ (indexed functor) whose components $\tau_x\colon \mathscr{M}x \to \mathscr{N}x$ are strong monoidal functors, whereas a 2-cell between strong morphisms of pseudomonoids is an ordinary modification

$$\begin{tikzcd}[column sep=huge] \mathcal{X}^{\mathrm{op}} \arrow[r, bend left=50, "\mathscr{M}", ""{name=U, below}] \arrow[r, bend right=50, "\mathscr{N}"', ""{name=D}] & \mathsf{Cat} \arrow[Rightarrow, from=U, to=D, bend right=40, "\tau"'] \arrow[Rightarrow, from=U, to=D, bend left=40, "\sigma"] \arrow[from=U, to=D, phantom, "\overset{m}{\Rrightarrow}"] \end{tikzcd}$$

whose components $m_x\colon \tau_x \Rightarrow \sigma_x$ are monoidal natural transformations.

We thus obtain the 2-categories $\mathsf{PsMon}(\mathsf{ICat}(\mathcal{X}))$ as well as $\mathsf{PsMon}(\mathsf{OpICat}(\mathcal{X}))$; from the above descriptions, it is clear that

$$\begin{aligned} \mathsf{PsMon}(\mathsf{ICat}(\mathcal{X})) &= \mathsf{2Cat}_{\mathrm{ps}}(\mathcal{X}^{\mathrm{op}}, \mathsf{MonCat}) \\ \mathsf{PsMon}(\mathsf{OpICat}(\mathcal{X})) &= \mathsf{2Cat}_{\mathrm{ps}}(\mathcal{X}, \mathsf{MonCat}) \end{aligned} \tag{36}$$

which will also be rediscovered by Proposition 4.5. These 2-categories correspond to the right foot of (27) on the side of indexed categories.

Remark 3.8. Similarly to what was noted in Remark 3.4, it is evident that $\mathsf{MonICat}(\mathcal{X})$ and $\mathsf{PsMon}(\mathsf{ICat}(\mathcal{X}))$ are in principle completely different. A monoidal indexed category with base $\mathcal{X}$ is a weak monoidal pseudofunctor into Cat (and $\mathcal{X}$ is required to be monoidal already), whereas a a pseudomonoid in $\mathcal{X}$-indexed categories is a pseudofunctor from an ordinary category $\mathcal{X}$ into MonCat. Formally, this says that starting with ICat, taking its pseudomonoids, and then restricting them over some domain is *not* the same as first choosing a domain and then taking pseudomonoids. This is also highlighted by the indexed category legs of (27).

3.3. The equivalence $\mathsf{MonFib} \cong \mathsf{MonICat}$.

In Section 2.1, we recalled the standard equivalence between fibrations and indexed categories via the Grothendieck construction. We will now lift this correspondence to their monoidal versions studied in Section 3.1 and 3.2, using general results about pseudomonoids in arbitrary monoidal 2-categories described in Section 2.2.

Since both Fib and ICat are cartesian monoidal 2-categories, via (28) and (33) respectively, our first task is to ensure that they are *monoidally* equivalent.

Lemma 3.9. *The 2-equivalence* $\mathsf{Fib} \simeq \mathsf{ICat}$ *between the cartesian monoidal 2-categories of fibrations and indexed categories is (symmetric) monoidal.*

Proof. Since they form an equivalence, both 2-functors from Theorem 2.3 preserve limits, therefore are monoidal 2-functors. Moreover, it can be verified that the natural isomorphisms with components $\mathscr{F} \cong \mathscr{F}_{P_{\mathscr{F}}}$ and $P \cong P_{\mathscr{F}_P}$ are monoidal with respect to the cartesian structure, due to universal properties of products. □

As a result, and since $\mathsf{MonFib} = \mathsf{PsMon}(\mathsf{Fib})$ and $\mathsf{MonICat} = \mathsf{PsMon}(\mathsf{ICat})$, we obtain the following equivalence as a special case of Proposition 2.9; also for $\mathsf{OpFib} \simeq \mathsf{OpICat}$.

Theorem 3.10. *There are 2-equivalences*

$$\begin{aligned}\mathsf{MonFib} &\simeq \mathsf{MonICat}\\ \mathsf{BrMonFib} &\simeq \mathsf{BrMonICat}\\ \mathsf{SymMonFib} &\simeq \mathsf{SymMonICat}\end{aligned}$$

between the 2-categories of monoidal fibrations and monoidal indexed categories, as well as their braided and symmetric versions.

Dually, there is a 2-equivalence $\mathsf{MonOpFib} \simeq \mathsf{MonOpICat}$ *between the 2-categories of monoidal opfibrations and monoidal opindexed categories, as well as their braided and symmetric versions.*

Corollary 3.11. *The above 2-equivalences restrict to the sub-2-categories of fixed bases/domains, which by (34) are*

$$\begin{aligned}\mathsf{MonFib}(\mathscr{X}) &\simeq \mathsf{Mon2Cat}_{\mathrm{ps}}(\mathscr{X}^{\mathrm{op}}, \mathsf{Cat})\\ \mathsf{MonOpFib}(\mathscr{X}) &\simeq \mathsf{Mon2Cat}_{\mathrm{ps}}(\mathscr{X}^{\mathrm{op}}, \mathsf{Cat})\end{aligned}$$

These results are summarized by the equivalences on the left foot of (27), and correspond to the *global* monoidal structure of fibrations and indexed categories. Even though they were directly derived via abstract reasoning, for exposition purposes we briefly describe this equivalence on the level of objects; some relevant details can also be found in [BFMP17, §6]. Independently and much earlier, in his thesis [Shu09] Shulman explores such a fixed-base equivalence on the level of double categories (of monoidal fibrations and monoidal pseudofunctors over the same base).

Suppose that $(\mathscr{M}, \mu, \mu_0) \colon (\mathscr{X}^{\mathrm{op}}, \otimes, I) \to (\mathsf{Cat}, \times, \mathbf{1})$ is a monoidal indexed category, i.e. a weak monoidal pseudofunctor with structure maps (16) with components $\mu_{x,y} \colon \mathscr{M}x \times \mathscr{M}y \to \mathscr{M}(x \otimes y)$ and $\mu_0 \colon \mathbf{1} \to \mathscr{M}I$. The Grothendieck category $\int\mathscr{M}$ obtains a monoidal structure in the following way: its tensor product $\otimes_\mu \colon \int\mathscr{M} \times \int\mathscr{M} \to \int\mathscr{M}$ is defined on objects by

$$(x, a) \otimes_\mu (y, b) = (x \otimes y, \mu_{x,y}(a, b)) \tag{37}$$

where $a \in \mathscr{M}x$ and $b \in \mathscr{M}y$, and $I_\mu = (I, \mu_0(*))$ is the unit object. Clearly, the induced fibration $\int\mathscr{M} \to \mathscr{X}$ which maps each pair to the underlying $\mathscr{X}$-object strictly preserves the monoidal structure. Moreover, pseudonaturality of μ implies that $\otimes_\mu$ preserves cartesian liftings, so of course all clauses of Proposition 3.1 are satisfied. For a more detailed exposition of the structure, as well as the braided and symmetric version, we refer the reader to the Appendix, Section 6.1.

The equivalences of Theorem 3.10 and Corollary 3.11 restrict in a straightforward way to the split context; notice that a weak monoidal 2-functor from a 1-category reduces to a lax monoidal functor in the ordinary sense.

Theorem 3.12. *There are 2-equivalences*

$$\begin{aligned}\mathsf{MonFib}_{sp} &\simeq \mathsf{MonICat}_{sp}\\ \mathsf{MonOpFib}_{sp} &\simeq \mathsf{MonOpICat}_{sp}\end{aligned}$$

between monoidal split (op)fibrations and strict indexed (op)categories. Moreover, fixing the base or the domain respectively, we obtain

$$\begin{aligned}\mathsf{MonFib}_{sp}(\mathscr{X}) &\simeq \mathsf{MonCat}_{\ell}(\mathscr{X}^{\mathrm{op}}, \mathsf{Cat})\\ \mathsf{MonOpFib}_{sp}(\mathscr{X}) &\simeq \mathsf{MonCat}_{\ell}(\mathscr{X}, \mathsf{Cat})\end{aligned}$$

Remark 3.13. Based on an observation made by Mike Shulman in private correspondence with the authors, this monoidal version of the Grothendieck construction may in fact be further generalized to the context of double categories. More specifically, there is evidence of a correspondence between discrete fibrations of double categories, and lax double functors into the double category $\mathbb{S}\mathsf{pan}$ of sets and spans. If such a result was also true for arbitrary fibrations of double categories, Theorem 3.10 would be a special case for double categories with one object and one vertical arrow, namely monoidal categories.

We close this section in a similar manner to Sections 3.1 and 3.2, namely by working in the cartesian monoidal 2-categories $(\mathsf{Fib}(\mathcal{X}),\boxtimes,1_{\mathcal{X}})$ and $(\mathsf{ICat}(\mathcal{X}),\boxtimes,\Delta\mathbf{1})$ of fibrations and indexed categories with fixed bases and domains, to begin with. As already noted in Remarks 3.4 and 3.8, the categories of pseudomonoids in the fixed base 2-categories are of a very different flavor compared to monoidal fibrations and monoidal indexed categories. Since $\mathsf{Fib}(\mathcal{X})\simeq\mathsf{ICat}(\mathcal{X})$ is also a monoidal 2-equivalence, Proposition 2.9 applies once more and the following is true, see (36).

Theorem 3.14. *There are 2-equivalences between (split) fibrations with monoidal fibres and strong monoidal reindexing functors, and (pseudo)functors into* MonCat

$$\begin{aligned}\mathsf{PsMon}(\mathsf{Fib}(\mathcal{X}))&\simeq\mathsf{2Cat}_{\mathrm{ps}}(\mathcal{X}^{\mathrm{op}},\mathsf{MonCat})\\ \mathsf{PsMon}(\mathsf{OpFib}(\mathcal{X}))&\simeq\mathsf{2Cat}_{\mathrm{ps}}(\mathcal{X}^{\mathrm{op}},\mathsf{MonCat})\\ \mathsf{PsMon}(\mathsf{Fib}_{\mathsf{sp}}(\mathcal{X}))&\simeq[\mathcal{X}^{\mathrm{op}},\mathsf{MonCat}]\\ \mathsf{PsMon}(\mathsf{OpFib}_{\mathsf{sp}}(\mathcal{X}))&\simeq[\mathcal{X}^{\mathrm{op}},\mathsf{MonCat}]\end{aligned}$$

These equivalences establish the right leg of (27), and correspond to the *fibrewise* monoidal structure on fibrations and indexed categories. In more detail, a pseudofunctor $\mathscr{M}\colon\mathcal{X}^{\mathrm{op}}\to\mathsf{MonCat}$ maps every object x to a monoidal category $\mathscr{M}x$ and every morphism $f\colon x\to y$ to a strong monoidal functor $\mathscr{M}f\colon\mathscr{M}y\to\mathscr{M}x$; under the usual Grothendieck construction (Section 2.1), these are precisely the fibre categories and the reindexing functors between them for the induced fibration, as described at the end of Section 3.1. Notice how, in particular, $\mathcal{X}$ is *not* a monoidal category, as was the case in Corollary 3.11.

Remark 3.15. A very similar, relaxed version of the fibrewise monoidal correspondence seems to connect the concepts of an *indexed monoidal category*, defined in [HM06] as a pseudofunctor $\mathscr{M}\colon\mathcal{X}^{\mathrm{op}}\to\mathsf{MonCat}_{\ell}$, and that of of a *lax monoidal fibration*, defined in [Zaw11]. Notice that these terms are misleading with respect to ours: an indexed monoidal category is *not* a monoidal indexed category, and also a lax monoidal fibration is *not* a functor with a lax monoidal stucture.

Briefly, there is a full sub-2-category $\mathsf{Fib}_{\mathrm{op}\ell}(\mathcal{X})\subseteq\mathsf{Cat}/\mathcal{X}$ of fibrations, namely fibred 1-cells (3) which are not required to have a cartesian functor on top. As discussed in [Shu08, Prop.3.6], this is 2-equivalent to $\mathsf{2Cat}_{ps,opl}(\mathcal{X}^{\mathrm{op}},\mathsf{Cat})$, the 2-category of pseudofunctors, oplax natural transformations and modifications. Describing pseudomonoids therein appears to give rise to a fibration with monoidal fibres and *lax* monoidal reindexing functors between them, or equivalently a pseudofunctor into MonCat_{ℓ}. We omit the details so as to not digress from our main development.

In summary, Theorems 3.10 and 3.14 establish the two monoidal variants of the Grothendieck construction, depicted in (27). In the following section, we compare them in a special case where they actually coincide.

4. (Co)cartesian case: fibrewise and global monoidal structures

In the previous section, we obtain two different equivalences between fixed-base fibrations and fixed-domain indexed categories of monoidal flavor. Corollary 3.11 establishes a correspondence between weak monoidal pseudofunctors $\mathscr{M}\colon \mathcal{X}^{\mathrm{op}} \to \mathsf{Cat}$ and monoidal fibrations $\int\mathscr{M} \to \mathcal{X}$, where the induced monoidal structure on the fibration is *global*: both total and base categories are monoidal, and the fibration strictly preserves the structure. On the other hand, Theorem 3.14 establishes a correspondence between ordinary pseudofunctors $\mathscr{M}\colon \mathcal{X}^{\mathrm{op}} \to \mathsf{MonCat} \hookrightarrow \mathsf{Cat}$ and ordinary fibrations $\int\mathscr{M} \to \mathcal{X}$ equipped with a *fibrewise* monoidal structure: none of the base or total categories are monoidal, but each fibre is, and the reindexing functors strongly preserve the structure.

Clearly, neither of these two cases implies the other in general. The global monoidal structure as defined in (37) sends two objects in arbitrary fibres to a new object lying in the fibre of the tensor of their underlying objects in the base. Therefore multiplying within the same fibre $(\int\mathscr{M})_x$ gives an object in the fibre $(\int\mathscr{M})_{x\otimes x}$. On the other hand, having a fibrewise tensor products does not, of course, give a way of multiplying objects in different fibres of the total category.

The above two different equivalences are formally expressed as

$$\mathsf{MonFib}(\mathcal{X}) \simeq \mathsf{Mon2Cat}_{\mathrm{ps}}(\mathcal{X}^{\mathrm{op}}, \mathsf{Cat}) \tag{38}$$

$$\mathsf{PsMon}(\mathsf{Fib}(\mathcal{X})) \simeq \mathsf{2Cat}_{\mathrm{ps}}(\mathcal{X}^{\mathrm{op}}, \mathsf{MonCat}) \tag{39}$$

where $\mathcal{X}$ is a monoidal category in the first isomorphism, and is an ordinary category in the second; these correspond to the two different legs of (27).

In [Shu08], Shulman introduces monoidal fibrations (Proposition 3.1) as a building block for fibrant double categories. Due to the nature of the examples, the results restrict to the case where the base of the monoidal fibration $P\colon \mathcal{A} \to \mathcal{X}$ is equipped with specifically a cartesian or cocartesian monoidal structure; the main idea is that these fibrations form a "parameterized family of monoidal categories". Formally, a central result therein lifts the Grothendieck construction to the monoidal setting, by showing an equivalence between monoidal fibrations over a fixed (co)cartesian base and ordinary pseudofunctors into MonCat.

Theorem 4.1. [Shu08, Thm. 12.7] *If $\mathcal{X}$ is cartesian monoidal,*

$$\mathsf{MonFib}(\mathcal{X}) \simeq \mathsf{2Cat}_{\mathrm{ps}}(\mathcal{X}^{\mathrm{op}}, \mathsf{MonCat}) \tag{40}$$

Dually, if $\mathcal{X}$ is cocartesian monoidal, $\mathsf{MonOpFib}(\mathcal{X}) \simeq \mathsf{2Cat}_{\mathrm{ps}}(\mathcal{X}, \mathsf{MonCat})$.

Evidently, this result provides an equivalence between the left part of (38) and the right part of (39), namely the two separate feet of (27). Bringing all these structures together, we obtain the following.

Theorem 4.2. *If $\mathcal{X}$ is a cartesian monoidal category,*

$$\begin{tikzcd}
\mathsf{MonFib}(\mathcal{X}) \arrow[r, "\simeq"] \arrow[d, "\wr"'] & \mathsf{Mon2Cat}_{\mathrm{ps}}(\mathcal{X}^{\mathrm{op}}, \mathsf{Cat}) \arrow[d, "\wr"] \\
\mathsf{PsMon}(\mathsf{Fib}(\mathcal{X})) \arrow[r, "\simeq"] & \mathsf{2Cat}_{\mathrm{ps}}(\mathcal{X}^{\mathrm{op}}, \mathsf{MonCat})
\end{tikzcd}$$

Dually, if $\mathcal{X}$ is a cocartesian monoidal category,

$$\begin{tikzcd}
\mathsf{MonOpFib}(\mathcal{X}) \arrow[r, "\simeq"] \arrow[d, "\wr"'] & \mathsf{Mon2Cat}_{\mathrm{ps}}(\mathcal{X}, \mathsf{Cat}) \arrow[d, "\wr"] \\
\mathsf{PsMon}(\mathsf{OpFib}(\mathcal{X})) \arrow[r, "\simeq"] & \mathsf{2Cat}_{\mathrm{ps}}(\mathcal{X}, \mathsf{MonCat})
\end{tikzcd}$$

The original proof of Theorem 4.1 is an explicit, piece-by-piece construction of an equivalence, and employs the reindexing functors Δ^* and π^* induced by the diagonal and projections in order to move between the appropriate fibres and build the required structures. The global monoidal structure is therein called "external" and the fibrewise is called "internal".

Here we present a different, high-level argument that does not focus on the fibrations side. The equivalence between weak monoidal pseudofunctors $\mathcal{X}^{\mathrm{op}} \to \mathsf{Cat}$ and ordinary pseudofunctors $\mathcal{X}^{\mathrm{op}} \to \mathsf{MonCat}$, which essentially provides a way of transferring the monoidal structure from the target category to the functor itself and vice versa, brings a new perspective on the behavior of such objects.

Lemma 4.3. *For any two monoidal 2-categories $\mathcal{K}$ and $\mathcal{L}$, the following are true.*

(1) For an arbitrary 2-category $\mathcal{A}$,

$$\mathsf{2Cat}_{\mathrm{ps}}(\mathcal{A}, \mathsf{Mon2Cat}_{\mathrm{ps}}(\mathcal{K}, \mathcal{L})) \simeq \mathsf{Mon2Cat}_{\mathrm{ps}}(\mathcal{K}, \mathsf{2Cat}_{\mathrm{ps}}(\mathcal{A}, \mathcal{L})) \tag{41}$$

(2) For a cocartesian 2-category $\mathcal{A}$,

$$\mathsf{2Cat}_{\mathrm{ps}}(\mathcal{A}, \mathsf{Mon2Cat}_{\mathrm{ps}}(\mathcal{K}, \mathcal{L})) \simeq \mathsf{Mon2Cat}_{\mathrm{ps}}(\mathcal{A} \times \mathcal{K}, \mathcal{L}) \tag{42}$$

Proof. First of all, recall [Str80, 1.34] that there are equivalences

$$\mathsf{2Cat}_{\mathrm{ps}}(\mathcal{A}, \mathsf{2Cat}_{\mathrm{ps}}(\mathcal{K}, \mathcal{L})) \simeq \mathsf{2Cat}_{\mathrm{ps}}(\mathcal{A} \times \mathcal{K}, \mathcal{L}) \simeq \mathsf{2Cat}_{\mathrm{ps}}(\mathcal{K}, \mathsf{2Cat}_{\mathrm{ps}}(\mathcal{A}, \mathcal{L}))$$

which underlie (41) and (42) for the respective pseudofunctors; so the only part needed is the correspondence between the respective monoidal structures. Notice that $\mathcal{A} \times \mathcal{K}$ is a monoidal 2-category since both $\mathcal{A}$ and $\mathcal{K}$ are, and also $\mathsf{2Cat}_{\mathrm{ps}}(\mathcal{A}, \mathcal{L})$ is monoidal since $\mathcal{L}$ is: define $\otimes_{[]}$ and $I_{[]}$ by $(\mathscr{F} \otimes_{[]} \mathscr{G})(a) = \mathscr{F}a \otimes_{\mathcal{L}} \mathscr{G}a$ (similarly to (35)) and $I_{[]} \colon \mathcal{A} \xrightarrow{!} \mathbf{1} \xrightarrow{I_{\mathcal{L}}} \mathcal{L}$.

(1) Take a pseudofunctor $\mathscr{F} \colon \mathcal{A} \to \mathsf{Mon2Cat}_{\mathrm{ps}}(\mathcal{K}, \mathcal{L})$. For every $a \in \mathcal{A}$, its image pseudofunctor $\mathscr{F}a$ is weak monoidal, i.e. comes equipped with morphisms in $\mathcal{L}$

$$\phi^a_{x,y} \colon (\mathscr{F}a)(x) \otimes_{\mathcal{L}} (\mathscr{F}a)(y) \to (\mathscr{F}a)(x \otimes_{\mathcal{K}} y), \quad \phi^a_0 \colon I_{\mathcal{L}} \to (\mathscr{F}a)I_{\mathcal{K}} \tag{43}$$

for every $x, y \in \mathcal{K}$, satisfying coherence axioms.

Now define the pseudofunctor $\bar{\mathscr{F}} \colon \mathcal{K} \to \mathsf{2Cat}_{\mathrm{ps}}(\mathcal{A}, \mathcal{L})$, with $(\bar{\mathscr{F}}x)(a) := (\mathscr{F}a)(x)$. It has a weak monoidal structure, given by pseudonatural transformations

$$\bar{\mathscr{F}}x \otimes_{[]} \bar{\mathscr{F}}y \Rightarrow \bar{\mathscr{F}}(x \otimes_{\mathcal{K}} y), \quad I_{[]} \Rightarrow \bar{\mathscr{F}}(I_{\mathcal{K}})$$

whose components evaluated on some $a \in \mathcal{A}$ are defined to be (43). Pseudonaturality and weak monoidal axioms follow, and in a similar way we can establish the opposite direction and verify the equivalence.

(2) If $\mathcal{A}$ is a cocartesian monoidal 2-category, a weak monoidal pseudofunctor $\mathscr{F} \colon \mathcal{A} \to \mathsf{Mon2Cat}_{\mathrm{ps}}(\mathcal{K}, \mathcal{L})$ induces a pseudofunctor $\tilde{\mathscr{F}} \colon \mathcal{A} \times \mathcal{K} \to \mathcal{L}$ by $\tilde{\mathscr{F}}(a, x) := (\mathscr{F}a)(x)$. Its weak monoidal structure is given by the composite

$$\begin{tikzcd}
\tilde{\mathscr{F}}(a,x) \otimes_{\mathcal{L}} \tilde{\mathscr{F}}(b,y) \arrow[rr, dashed, "\psi_{(a,x),(b,y)}"] \arrow[d, equal] & & \tilde{\mathscr{F}}(a+b, x \otimes_{\mathcal{K}} y) \arrow[d, equal] \\
(\mathscr{F}a)(x) \otimes_{\mathcal{L}} (\mathscr{F}b)(y) \arrow[dr, "(\mathscr{F}\iota_a)_x \otimes (\mathscr{F}\iota_b)_y"'] & & (\mathscr{F}(a+b))(x \otimes_{\mathcal{K}} y) \\
& (\mathscr{F}(a+b))(x) \otimes_{\mathcal{L}} (\mathscr{F}(a+b))(y) \arrow[ur, "\phi^{a+b}_{x,y}"'] &
\end{tikzcd}$$

where $a \xrightarrow{\iota_a} a + b \xleftarrow{\iota_b} b$ are the inclusions, and $\psi_0 \colon I_{\mathcal{L}} \xrightarrow{\phi^0_0} \tilde{\mathscr{F}}(0, I_{\mathcal{K}})$; the respective axioms follow.

In the opposite direction, starting with some pseudofunctor $\mathscr{G} \colon \mathcal{A} \times \mathcal{K} \to \mathcal{L}$ equipped with a weak monoidal structure $\psi_{(a,x),(b,y)}$ and ψ_0, we can build $\hat{\mathscr{G}} \colon \mathcal{A} \to \mathsf{Mon2Cat}_{\mathrm{ps}}(\mathcal{K}, \mathcal{L})$ for which every $\hat{\mathscr{G}}a$ is a weak monoidal pseudofunctor, via

$$\begin{tikzcd}
(\hat{\mathscr{G}}a)(x) \otimes_{\mathcal{L}} (\hat{\mathscr{G}}b)(y) \arrow[rr, dashed, "\phi^a_{(x,y)}"] \arrow[d, equal] & & (\hat{\mathscr{G}}a)(x \otimes_{\mathcal{K}} y) \arrow[d, equal] \\
\mathscr{G}(a,x) \otimes_{\mathcal{L}} \mathscr{G}(a,y) \arrow[r, "\psi_{(a,x),(a,y)}"'] & \mathscr{G}(a+a, x \otimes_{\mathcal{K}} y) \arrow[r, "G(\nabla,1)"'] & \mathscr{G}(a, x \otimes_{\mathcal{K}} y)
\end{tikzcd}$$

$$\phi^a_0 \colon I_{\mathcal{L}} \xrightarrow{\psi_0} G(0, I_{\mathcal{K}}) \xrightarrow{G(!,1)} G(a, I_{\mathcal{K}})$$

The equivalence follows, using the universal properties of coproducts and initial object. ☐

In a very similar way, for strict indexed categories we can prove isomorphisms involving Cat and MonCat_ℓ as shown below. These exhibit the existence of certains cotensors and tensors; for these weighted (co)limit notions, see [Kel05, 3.7].

Corollary 4.4. *For any two monoidal categories $\mathcal{X}$, $\mathcal{Y}$, we have isomorphisms*

$$\mathsf{MonCat}_\ell(\mathcal{A} \times \mathcal{X}, \mathcal{Y}) \cong [\mathcal{A}, \mathsf{MonCat}_\ell(\mathcal{X}, \mathcal{Y})] \cong \mathsf{MonCat}_\ell(\mathcal{X}, [\mathcal{A}, \mathcal{Y}])$$

where the right one holds of any ordinary category $\mathcal{A}$, whereas the left one holds only for a cocartesian monoidal $\mathcal{A}$. Therefore MonCat_ℓ is a cotensored 2-category with $\mathcal{A} \pitchfork \mathcal{Y} = [\mathcal{A}, \mathcal{Y}]$ and tensored only by cocartesian categories with $\mathcal{A} \otimes \mathcal{X} = \mathcal{A} \times \mathcal{X}$.

Using the above general results about (weak monoidal) pseudofunctors, we can deduce Theorem 4.2 in an alternate way via the functor 2-categories side.

Proof of Theorem 4.2. The top and bottom right 2-categories of the first square are equivalent as follows, where $\mathcal{X}^{\mathrm{op}}$ is cocartesian.

$$\begin{aligned}
\mathsf{2Cat}_{\mathrm{ps}}(\mathcal{X}^{\mathrm{op}}, \mathsf{MonCat}) &\simeq \mathsf{2Cat}_{\mathrm{ps}}(\mathcal{X}^{\mathrm{op}}, \mathsf{PsMon}(\mathsf{Cat})) \\
&\simeq \mathsf{2Cat}_{\mathrm{ps}}(\mathcal{X}^{\mathrm{op}}, \mathsf{Mon2Cat}_{\mathrm{ps}}(\mathbf{1}, \mathsf{Cat})) && (25) \\
&\simeq \mathsf{Mon2Cat}_{\mathrm{ps}}(\mathcal{X}^{\mathrm{op}} \times \mathbf{1}, \mathsf{Cat}) && (42) \\
&\simeq \mathsf{Mon2Cat}_{\mathrm{ps}}(\mathcal{X}^{\mathrm{op}}, \mathsf{Cat})
\end{aligned}$$

□

The decisive step in the above proof is the much broader Lemma 4.3; for a grounded explanation of the correspondence of the relevant structures, see Section 6.2. In simpler words for the strict case, a lax monoidal structure of a functor $F\colon (\mathcal{A},+,0)\to(\mathsf{Cat},\times,\mathbf{1})$ gives an ordinary $F\colon\mathcal{A}\to\mathsf{MonCat}$, and vice versa: in a sense, 'monoidality' can move between the functor and its target.

As another corollary of Lemma 4.3, we can formally deduce that pseudomonoids in $(\mathsf{ICat}(\mathcal{X}),\boxtimes,\Delta\mathbf{1})$ are functors into MonCat, as described at the end of Section 3.2.

Proposition 4.5. *For any* $\mathcal{X}$, $\mathsf{PsMon}(\mathsf{ICat}(\mathcal{X}))\simeq\mathsf{2Cat}_{\mathrm{ps}}(\mathcal{X}^{\mathrm{op}},\mathsf{MonCat})$.

Proof. There are equivalences

$$\begin{aligned}
\mathsf{PsMon}(\mathsf{ICat}(\mathcal{X})) &= \mathsf{PsMon}(\mathsf{2Cat}_{\mathrm{ps}}(\mathcal{X}^{\mathrm{op}},\mathsf{Cat})) & \\
&\simeq \mathsf{Mon2Cat}_{\mathrm{ps}}(\mathbf{1},\mathsf{2Cat}_{\mathrm{ps}}(\mathcal{X}^{\mathrm{op}},\mathsf{Cat})) & (41)\\
&\simeq \mathsf{2Cat}_{\mathrm{ps}}(\mathcal{X}^{\mathrm{op}},\mathsf{Mon2Cat}_{\mathrm{ps}}(\mathbf{1},\mathsf{Cat})) & (25)\\
&\simeq \mathsf{2Cat}_{\mathrm{ps}}(\mathcal{X}^{\mathrm{op}},\mathsf{PsMon}(\mathsf{Cat})) & \\
&\simeq \mathsf{2Cat}_{\mathrm{ps}}(\mathcal{X}^{\mathrm{op}},\mathsf{MonCat}) &
\end{aligned}$$

□

As a first and meaningful example of Theorem 4.2, recall by Remarks 2.1 and 2.2 that the categories Fib and ICat are themselves fibred over Cat, with fibres $\mathsf{Fib}(\mathcal{X})$ and $\mathsf{ICat}(\mathcal{X})$ respectively. The base category in both cases is the cartesian monoidal category $(\mathsf{Cat},\times,1)$, therefore Theorem 4.2 applies. The following proposition shows that the monoidal structures of Fib, ICat and $\mathsf{Fib}(\mathcal{X})$, $\mathsf{ICat}(\mathcal{X})$, instrumental for the study of global and fibrewise monoidal structures, follow the very same abstract pattern.

Proposition 4.6. *The fibrations* $\mathsf{Fib}\to\mathsf{Cat}$ *and* $\mathsf{ICat}\to\mathsf{Cat}$ *are monoidal, and moreover their fibres* $\mathsf{Fib}(\mathcal{X})$ *and* $\mathsf{ICat}(\mathcal{X})$ *are monoidal and the reindexing functors are strong monoidal.*

Proof. The pseudofunctors inducing $\mathsf{Fib}\to\mathsf{Cat}$ and $\mathsf{ICat}\to\mathsf{Cat}$ are

$$\begin{tikzcd}
\mathsf{Cat}^{\mathrm{op}} \arrow[r] & \mathsf{Cat} \\
\mathcal{X} \arrow[r, mapsto] \arrow[d, "F"'] & \mathsf{Fib}(\mathcal{X}) \\
\mathcal{Y} \arrow[r, mapsto] & \mathsf{Fib}(\mathcal{Y}) \arrow[u, "F^*"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathsf{Cat}^{\mathrm{op}} \arrow[r] & \mathsf{Cat} \\
\mathcal{X} \arrow[r, mapsto] \arrow[d, "F"'] & \mathsf{ICat}(\mathcal{X}) \\
\mathcal{Y} \arrow[r, mapsto] & \mathsf{ICat}(\mathcal{Y}) \arrow[u, "-\circ F^{\mathrm{op}}"']
\end{tikzcd}$$

where F^* takes pullbacks along F and $-\circ F^{\mathrm{op}}$ precomposes with the opposite of F. These are both weak monoidal, with the respective structures essentially being (28) and (33) giving the global monoidal structure on the fibrations.

Since the base of both monoidal fibrations is cartesian, the global monoidal structure is equivalent to a fibrewise monoidal structure, as per the theme of this whole section. The induced monoidal structure on each $\mathsf{Fib}(\mathcal{X})$ is given by (32) and on each $\mathsf{ICat}(\mathcal{X})$ by (35), and F^*, $-\circ F^{\mathrm{op}}$ are strong monoidal functors accordingly. □

The above essentially lifts the global and fibrewise monoidal structure development one level up, exhibiting fibrations and indexed categories as examples of the monoidal Grothendieck construction themselves.

Concluding this investigation on monoidal structures of fibrations and indexed categories, we consider the (co)cartesian monoidal (op)fibration case, i.e. a fibration $P\colon (\mathcal{A}, \times, 1) \to (\mathcal{X}, \times, 1)$ as in Proposition 3.1 where P preserves products (or coproducts for opfibrations) on the nose. As remarked in [Shu08, 12.9], the equivalence (40) restricts to one between pseudofunctors which land to cartesian monoidal categories, and monoidal fibrations where the total category is cartesian monoidal. With the appropriate 1-cells and 2-cells that preserve the structure, we can write the respective equivalences as

$$\begin{gathered} \mathsf{2Cat}_{\mathrm{ps}}(\mathcal{X}^{\mathrm{op}}, \mathsf{cMonCat}) \simeq \mathsf{cMonFib}(\mathcal{X}) \text{ for cartesian } \mathcal{X} \\ \mathsf{2Cat}_{\mathrm{ps}}(\mathcal{X}, \mathsf{cocMonCat}) \simeq \mathsf{cocMonOpFib}(\mathcal{X}) \text{ for cocartesian } \mathcal{X} \end{gathered} \tag{44}$$

where the prefixes c and coc correspond to the respective (co)cartesian structures. Explicitly, in order for the total category to specifically be endowed with (co)cartesian monoidal structure, it is required not only that the base category is but also the fibres are and the reindexing functors preserve finite (co)products.

Remark 4.7. This special case of the monoidal Grothendieck construction that connects the existence of (co)products and initial/terminal object in the fibres and in the total category, is reminiscent (and also an example of) the general theory of *fibred limits*. Explicitly, [Her93, 3.3.6] deduces that if the base of a fibration $P\colon \mathcal{A} \to \mathcal{X}$ has $\mathcal{J}$-limits for any small category $\mathcal{J}$, then the fibres have and the reindexing functors preserve $\mathcal{J}$-limits if and only if $\mathcal{A}$ has $\mathcal{J}$-limits and P strictly preserves them, and dually for opfibrations and colimits.

Hence for finite (co)products in (op)fibrations, (44) re-discovers that result using the monoidal Grothendieck correspondence. The fact that this global and fibrewise interplay carries over to more general (co)limits in the theory of fibrations is only indicative of a broader sense in which the Grothendieck construction may transfer varied pieces structure from indexed categories to fibrations. We leave such considerations for future work.

Moreover, since the squares of Theorem 4.2 reduce to their (co)cartesian variants, we would like to identify the conditions that the corresponding weak monoidal pseudofunctor into Cat needs to satisfy in order to give rise to a (co)cartesian monoidal (op)fibration. Recall that by a folklore result presented in [HV12], any symmetric monoidal category equipped with suitably well-behaved diagonals and augmentations must in fact be cartesian monoidal. We employ its dual version to tackle the opfibration case: if, in a symmetric monoidal category $\mathcal{X}$, there exist monoidal natural transformations with components

$$\nabla_x\colon x \otimes x \to x, \quad u_x\colon I \to x$$

satisfying the commutativity of

$$\begin{tikzcd}
I \otimes x \arrow[r, "u_x \otimes 1"] \arrow[dr, "\sim", "\ell_x"'] & x \otimes x \arrow[d, "\nabla_x"] & & x \otimes I \arrow[r, "1 \otimes u_x"] \arrow[dr, "\sim", "r_x"'] & x \otimes x \arrow[d, "\nabla_x"] \\
& x & & & x
\end{tikzcd} \tag{45}$$

then $\mathcal{X}$ is cartesian monoidal. In fact, it is the case that a symmetric monoidal category is cocartesian if and only if $\mathsf{Mon}(\mathcal{X}) \cong \mathcal{X}$.

Suppose $(\mathscr{M},\mu,\mu_0)\colon\mathcal{X}\to\mathsf{Cat}$ is a (symmetric) weak monoidal pseudofunctor, such that the corresponding Grothendieck category $(\int\mathscr{M},\otimes_\mu,I_\mu)$ described in Section 3.3 is cocartesian monoidal. This means there are monoidal natural transformations with components

$$\nabla_{(x,a)}\colon(x,a)\otimes_\mu(x,a)\to(x,a)\quad\text{and}\quad u_{(x,a)}\colon(I,\mu_0(*))\to(x,a)$$

making the diagrams (45) commute. Explicitly, by (37), $\nabla_{(x,a)}$ consists of morphisms $f_x\colon x\otimes x\to x$ in $\mathcal{X}$ and $\kappa_a\colon(\mathscr{M}f_x)(\mu_{x,x}(a,a))\to a$ in $\mathscr{M}x$, whereas $u_{(x,a)}$ consists of $i_x\colon I\to x$ in $\mathcal{X}$ and $\lambda_a\colon(\mathscr{M}i_x)\mu_0\to a$ in $\mathscr{M}x$.

The conditions (45) say that the composites

$$(I,\mu_0)\otimes_\mu(x,a)\xrightarrow{u_{(x,a)}\otimes_\mu 1_{(x,a)}}(x,a)\otimes_\mu(x,a)\xrightarrow{\nabla_{(x,a)}}(x,a)$$
$$(x,a)\otimes_\mu(I,\mu_0)\xrightarrow{1_{(x,a)}\otimes_\mu u_{(x,a)}}(x,a)\otimes_\mu(x,a)\xrightarrow{\nabla_{(x,a)}}(x,a)$$

are equal to the left and right unitor on x, where all respective structures are detailed in Section 6.1. Using the composition inside $\int\mathscr{M}$ analogously to (11), these conditions translate, on the one hand, to the base being cocartesian monoidal $(\mathcal{X},+,0)$ with $f_x=\nabla_x$ and $i_x=u_x$. On the other hand, κ_a and λ_a form natural transformations

$$\begin{tikzcd}
& \mathscr{M}x\times\mathscr{M}x \arrow[r, "\mu_{x,x}"] & \mathscr{M}(x+x) \arrow[dr, "\mathscr{M}(\nabla_x)"] & \\
\mathscr{M}x \arrow[ur, "\Delta"] \arrow[rrr, bend right, "1"'] & \arrow[r, phantom, "\Downarrow\kappa^x"] & & \mathscr{M}x
\end{tikzcd}\tag{46}$$

$$\begin{tikzcd}
& \mathbf{1} \arrow[r, "\mu_0"] & \mathscr{M}(0) \arrow[dr, "\mathscr{M}(u_x)"] & \\
\mathscr{M}x \arrow[ur, "!"] \arrow[rrr, bend right, "1"'] & \arrow[r, phantom, "\Downarrow\lambda^x"] & & \mathscr{M}x
\end{tikzcd}$$

satisfying the commutativity of

$$\begin{tikzcd}
\mathscr{M}(\nabla_x\circ(u_x+1))(\mu_{0,x}(\mu_0(*),a)) \arrow[r, "\delta", "\sim"'] \arrow[ddd, "\mathrm{id}"'] & (\mathscr{M}(\nabla_x)\circ\mathscr{M}(u_x+1))((\mu_{0,x}(\mu_0(*),a)) \arrow[d, "\mathscr{M}(\nabla_x)(\mu_{u_x,1})", "\wr"'] \\
& \mathscr{M}(\nabla_x)(\mu_{x,x}(\mathscr{M}(u_x)(\mu_0(*),a))) \arrow[d, "\mathscr{M}(\nabla_x)(\mu_{x,x}(\lambda^x_a,\gamma))"] \\
& \mathscr{M}(\nabla_x)(\mu_{x,x}(a,a)) \arrow[d, "\kappa^x_a"] \\
\mathscr{M}(\ell_x)(\mu_{0,x}(\mu_0(*),a)) \arrow[r, "\xi", "\sim"'] & a
\end{tikzcd}$$

and a similar one with μ_0 on second arguments. The above greatly simplifies if $\mathscr{M}$ is just a lax monoidal functor: the first condition becomes $1_a\cong\kappa^x_a\circ\mathscr{M}(\nabla_x)(\mu_{x,x}(\lambda^x_a,1))$, and the second one $1_a\cong\kappa^x_a\circ\mathscr{M}(\nabla_x)(\mu_{x,x}(1_a,\lambda^x_a))$.

Corollary 4.8. *A weak monoidal pseudofunctor $\mathscr{M}\colon(\mathcal{X},+,0)\to(\mathsf{Cat},\times,\mathbf{1})$ equipped with natural transformations κ and λ as in (46) corresponds to an ordinary pseudofunctor $\mathscr{M}\colon\mathcal{X}\to\mathsf{cocMonCat}$, or equivalently (44) to a cocartesian monoidal opfibration.*

5. Examples

In this section, we explore certain settings where the equivalence between monoidal fibrations and monoidal indexed categories naturally arises. Instead of going into details that would result in a much longer text, we mostly sketch the appropriate example cases up to the point of exhibition of the monoidal Grothendieck correspondence, providing indications of further work and references for the interested reader.

5.1. **Fundamental (bi)fibration.** For any category $\mathcal{X}$, the *codomain* or *fundamental* opfibration is the usual functor from its arrow category

$$\mathrm{cod}\colon \mathcal{X}^2 \longrightarrow \mathcal{X}$$

mapping every morphism to its codomain and every commutative square to its right-hand side leg. It uniquely corresponds to the (strict) opindexed category, i.e. mere functor

$$\begin{tikzcd} \mathcal{X} \arrow[r] & \mathsf{Cat} \\ x \arrow[r, mapsto] \arrow[d, "f"'] & \mathcal{X}/x \arrow[d, "f_!"] \\ y \arrow[r, mapsto] & \mathcal{X}/y \end{tikzcd} \tag{47}$$

that maps an object to the slice category over it and a morphism to the post-composition functor $f_! = f \circ -$ induced by it.

If the category has a monoidal structure $(\mathcal{X}, \otimes, I)$, this functor naturally becomes lax monoidal with structure maps

$$\mathcal{X}/x \times \mathcal{X}/y \xrightarrow{\otimes} \mathcal{X}/(x \otimes y) \tag{48}$$
$$\mathbf{1} \xrightarrow{1_I} \mathcal{X}/I$$

which also render $(\mathcal{X}^{\mathbf{2}}, \otimes, 1_I)$ monoidal and the split opfibration cod strict monoidal, in a straightforward way or even as a result of (37). However notice that in general, the slice categories $\mathcal{X}/x$ do not inherit the monoidal structure: there is no way to restrict the global monoidal structure to a fibrewise one.

In accordance with Theorem 4.2, there is an induced monoidal structure on the categories $\mathcal{X}/x$, and a strong monoidal structure on all $f_!$, only when the monoidal structure on $\mathcal{X}$ is given by binary coproducts and an initial object (i.e. cocartesian). In that case, for each $k\colon a \to x$ and $\ell\colon b \to x$ in the same fibre $\mathcal{X}/x$, their tensor product in $\mathcal{X}/x$ is given by

$$a + b \xrightarrow{k+\ell} x + x \xrightarrow{\nabla_x} x$$

as a simple example of (60). In fact, this is precisely the coproduct of two objects in $\mathcal{X}/x$, and $0 \xrightarrow{!} x$ the initial object, due to the way colimits in the slice categories are constructed. Therefore this falls under the cocartesian-fibres special case (44), bijectively corresponding to the cocartesian structure on $\mathcal{X}^{\mathbf{2}}$ inherited from $\mathcal{X}$.

Now suppose an ordinary category $\mathcal{X}$ has pullbacks. This endows the codomain functor also with a fibration structure, corresponding to the indexed category

$$\begin{tikzcd}
\mathcal{X}^{\mathrm{op}} \arrow[r] & \mathsf{Cat} \\
x \arrow[r, mapsto] \arrow[d, "f"'] & \mathcal{X}/x \\
y \arrow[r, mapsto] & \mathcal{X}/y \arrow[u, "f^*"']
\end{tikzcd}$$

with the same mapping on objects as (47) but by taking pullbacks rather than post-composing along morphisms, a pseudofunctorial assignment. This gives $\mathrm{cod}\colon \mathcal{X}^2 \to \mathcal{X}$ a bifibration structure, also by that classic fact that $f_! \dashv f^*$.

In this case, if $\mathcal{X}$ has a general monoidal structure there is no natural weak monoidal structure of that pseudofunctor as before: there is no reason for the pullback of a tensor to be isomorphic to the tensor of two pullbacks, so (pseudo)naturality of (48) fails. However, if $\mathcal{X}$ is cartesian monoidal (hence has all finite limits), the components

$$\mathcal{X}/x \times \mathcal{X}/y \xrightarrow{\times} \mathcal{X}/(x \times y)$$
$$\mathbf{1} \xrightarrow{\Delta_!} \mathcal{X}/1$$

are pseudonatural since pullbacks commute with products. Moreover, this bijectively corresponds to monoidal fibres and strong monoidal reindexing functors, in fact also cartesian ones: for morphisms $k\colon a \to x$ and $\ell\colon b \to x$ in $\mathcal{X}/x$, their induced product is given by

$$\begin{tikzcd}
\bullet \arrow[r] \arrow[d, "\delta^*(k\times\ell)"'] \arrow[dr, phantom, "\lrcorner", very near start] & a \times b \arrow[d, "k\times\ell"] \\
x \arrow[r, "\delta"] & x \times x
\end{tikzcd}$$

and $1_x\colon x \to x$ is the unit of each slice $\mathcal{X}/x$, this indexed monoidal category also described in [HM06, 3.3(1)]. The monoidal fibration structure on $\mathrm{cod}\colon (\mathcal{X}^2, \times, 1_1) \to (X, \times, 1)$ is the evident one, so it again falls in the special case (44) now for cartesian fibres, by construction of products in slice categories.

As a final remark, analogous constructions hold for the domain functor which is again a bifibration: its fibration structure comes from pre-composing along morphisms, whereas its opfibration structure comes from taking pushouts along morphisms. In fact, Remark 2.1 as well as Proposition 4.6 can be thought as special cases of these more general settings, for the categories of fibrations themselves.

5.2. **Modeling graphs and networks.** Denote by $\mathsf{Grph} = \mathsf{Set}^{\bullet\rightrightarrows\bullet}$ the usual category of (directed, multi) graphs, and consider the functor

$$V\colon \mathsf{Grph} \to \mathsf{Set} \tag{49}$$

which sends a graph to its set of vertices. It is well-known that this functor is a split opfibration, which can also be obtained as the Grothendieck construction on the strict opindexed category (i.e. functor) $\mathsf{Grph}_{(-)}\colon \mathsf{Set} \to \mathsf{Cat}$ described as follows. A set X is mapped to the category Grph_X of graphs with vertex set X

and homomorphisms which fix the vertices, namely the slice category $\mathsf{Set}/X\times X$ of objects $E \to X \times X$ and morphisms functions $k\colon E \to E'$ such that

$$\begin{tikzcd} E \arrow[rr, "k"] \arrow[dr, "{(s,t)}"'] & & E' \arrow[dl, "{(s',t')}"] \\ & X \times X & \end{tikzcd} \quad \text{or equivalently} \quad \begin{tikzcd} E \arrow[rr, "k"] \arrow[dr, shift left, "t"] \arrow[dr, shift right, "s"'] & & E' \arrow[dl, shift right, "s'"'] \arrow[dl, shift left, "t'"] \\ & X & \end{tikzcd}$$

Moreover, any function $f\colon X \to Y$ gives rise to the post-composition functor

$$\mathsf{Grph}_X = \mathsf{Set}/X \times X \xrightarrow{(f\times f)\circ -} \mathsf{Set}/Y \times Y = \mathsf{Grph}_Y$$

that maps an X-graph $(s,t)\colon E \rightrightarrows X$ to the Y-graph $(f \circ s, f \circ t)\colon E \rightrightarrows Y$. Clearly, this functor $\mathsf{Grph}_{(-)} = \mathsf{Set}/(-\times-)$ is a special case of the codomain functor (47) described earlier. As explained via (48), considering Set with is cocartesian monoidal structure induces a symmetric lax monoidal structure on the functor, namely

$$(\mathsf{Grph}_{(-)}, \sqcup, 1_0)\colon (\mathsf{Set}, +, 0) \to (\mathsf{Cat}, \times, \mathbf{1}). \tag{50}$$

Explicitly, its structure maps are

$$\sqcup_{X,Y}\colon \mathsf{Grph}_X \times \mathsf{Grph}_Y \to \mathsf{Grph}_{X+Y}$$
$$1_0\colon \mathbf{1} \to \mathsf{Grph}_0$$

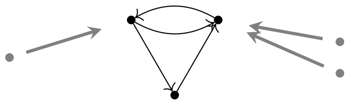
where $\sqcup_{X,Y}(\, E \overset{s}{\underset{t}{\rightrightarrows}} X \,,\, F \overset{s'}{\underset{t'}{\rightrightarrows}} Y \,) = \, E + F \overset{s+s'}{\underset{t+t'}{\rightrightarrows}} X + Y$ and $\mathbf{1} \xrightarrow{!} \mathsf{Grph}_0 \cong \mathbf{1}$. Composing (50) with the forgetful functor to Set, we can discard the morphisms in each of the fibres Grph_X and just consider the (large) set of graphs on X via the symmetric lax monoidal functor

$$(\mathsf{Grph}_{(-)}, \sqcup, 1_0)\colon (\mathsf{Set}, +, 0) \to (\mathsf{Set}, \times, 1).$$

Restricting its domain to the monoidal subcategory FinSet of finite sets, we obtain the motivating example in Fong's so-called *theory of decorated cospans*. In more detail, in [Fon15] a category $F\mathsf{Cospan}$ is constructed from a given lax symmetric monoidal functor

$$(F, \phi, \phi_0)\colon (\mathsf{FinSet}, +, 0) \to (\mathsf{Set}, \times, 1)$$

with the broad goal of modelling open networks. The objects of such a category are finite sets, and a morphism between two finite sets X and Y is a cospan of finite sets $X \xrightarrow{i} N \xleftarrow{o} Y$ equipping the 'network' N with a certain notion of input and output, along with a **decoration** of the apex, namely an object $s \in F(N)$. For example, a $\mathsf{Grph}_{(-)}$-decorated cospan with one input and two output designated nodes looks like



where the middle graph is the chosen decoration of the three-element set. In fact, the apex of any F-decorated cospan can be viewed as an object of the (discrete) Grothendieck category $\int F$ on that functor, since it is a finite set that always comes together with an element of the set of all its possible decorations.

We call such a functor F a **decorator**, and can be viewed as an object in the category $\mathsf{SymMonCat}_\ell((\mathsf{FinSet}, +, 0), (\mathsf{Set}, \times, 1))$, with monoidal natural transformations as morphisms. We call this the **category of decorators**; notice that the construction of a (symmetric monoidal) category $F\mathsf{Cospan}$ of F-decorated cospans from F induces a functor $\mathsf{SymMonCat}_\ell((\mathsf{FinSet},+,0),(\mathsf{Set},\times,1))\to\mathsf{SymMonCat}$. More details about this can be found in [BCV19], where a correspondence between decorated and *structured* cospans is established, partially due to the monoidal Grothendieck construction.

Back to the symmetric lax monoidal functor $\mathsf{Grph}_{(-)}$ of (50), since its domain Set is taken with its cocartesian monoidal structure, Theorem 4.2 ensures that its monoidal structure bijectively corresponds to a monoidal structure on the fibres Grph_X which is strongly preserved by the reindexing functors $-\circ f\colon \mathsf{Grph}_X\to\mathsf{Grph}_Y$. In fact, it can be verified that the fibres are cocartesian themselves, falling under the equivalence (44): for any two graphs $(s,t)\colon E\rightrightarrows X$, $(s',t')\colon E'\rightrightarrows X$ over the set of vertices X, (60) gives

$$E+E' \overset{s+s'}{\underset{t+t'}{\rightrightarrows}} X+X \xrightarrow{\nabla_X} X \tag{51}$$

explicitly constructed by

$$\begin{tikzcd}
E \arrow[dr] \arrow[ddr, bend right=50, "s"'] \arrow[ddr, bend right=25, "t"] & & E' \arrow[dl] \arrow[ddl, bend left=25, "s'"'] \arrow[ddl, bend left=50, "t'"] \\
& E+E' \arrow[d, shift right, "\exists! s''"'] \arrow[d, shift left, "\exists! t''"] & \\
& X &
\end{tikzcd}$$

as is the case of colimits in slice categories. The resulting graph is the *overlay* of the two given graphs, identifying corresponding vertices. This is the same as computing the pushout over the obvious inclusions of the graph with vertex set X and no edges into each of the given graphs. The initial object of its fibre Grph_X is $(!,!)\colon\emptyset\rightrightarrows X$.

Therefore the lax symmetric monoidal $(\mathsf{Grph}_{(-)},\sqcup,1_0)$ bijectively corresponds to a mere functor

$$\mathsf{Grph}_{(-)}\colon\mathsf{Set}\to\mathsf{SymMonCat}.$$

Should we want to view the symmetric monoidal categories Grph_X as commutative monoids of X-graphs with overlay (51) as the binary operation on the set of objects, we can formally take isomorphism classes of objects and then forget the morphisms in each Grph_X. Putting all this data together, for CMon the category of commutative monoids, there is an induced symmetric lax monoidal functor

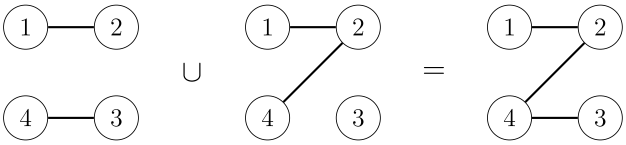
$$(\mathsf{Grph}_{(-)},\sqcup,1_0)\colon(\mathsf{Set},+,0)\to(\mathsf{CMon},\times,\mathbf{1}).$$

If we furthermore restrict its domain to the symmetric groupoid of finite sets and bijections S, we obtain the so-called *network model* for graphs. In more detail, in [BFMP17] an operad $\mathcal{O}_F$ is constructed from a given lax symmetric monoidal functor

$$(F,\phi,\phi_0)\colon(\mathsf{S},+,0)\to(\mathsf{Mon},\times,\mathbf{1})$$

and such a functor is called a **network model**. The monoids $F(\mathbf{n})$ are called the **constituent monoids** of F, and $\mathcal{O}_F$ is the *underlying* operad of the induced monoidal category $(\textstyle\int F,\otimes_\phi,I_\phi)$ described in Section 6.1. The category of network models is denoted $\mathsf{NetMod}=\mathsf{SymMonCat}_\ell(\mathsf{S},+,0),(\mathsf{Mon},\times,\mathbf{1}))$ and the mapping on $F\mapsto\mathcal{O}_F$ defines a functor $\mathsf{NetMod}\to\mathsf{Opd}$ into the category of operads.

The intuition behind this work is that a large complex network can be built from smaller ones by gluing them together in ways written as combinations of a few basic operations, expressed via monoid multiplications and monoidal functors. As an example, let X be a finite set and $F(X)$ be the set of simple graphs with vertex set X; the monoid operation, similarly to (51), says that two simple graphs with the same vertex set can be overlaid by identifying the corresponding vertices and simplifying the edge sets:



The above descriptions exhibit the relation between the theory of decorated cospans and network models, via the machinery of the monoidal Grothendieck correspondence. Mimicking the above development for $\mathsf{Grph}_{(-)}$ (50), we can start with any decorator $(F,\phi,\phi_0)\colon(\mathsf{FinSet},+,0)\to(\mathsf{Set},\times,\mathbf{1})$ and for each finite set X define a monoid structure on each set $F(X)$ as follows. If $\nabla_X\colon X+X\to X$ is the *folding map* given by universal property of coproduct, the induced multiplication and unit on $F(X)$ are the composites

$$F(X)\times F(X)\xrightarrow{\phi_{X,X}}F(X+X)\xrightarrow{F(\nabla_X)}F(X)$$

$$\mathbf{1}\xrightarrow{\phi_0}F(\emptyset)\xrightarrow{F(!)}F(X)$$

which coincide with (51) for $F=\mathsf{Grph}_{(-)}$.

Theorem 5.1. *There is a faithful embedding of the category of decorators into the category of network models*

$$\mathsf{SymMonCat}_\ell((\mathsf{FinSet},+,0),(\mathsf{Set},\times,1))\to\mathsf{SymMonCat}_\ell(\mathsf{S},+,0),(\mathsf{Mon},\times,\mathbf{1}))$$

Proof. This is a special case of Theorem 3.14, where we are considering discrete symmetric monoidal strict FinSet-opindexed categories, and extending a monoidal structure to the fibres as a special case of (60). □

This theorem says that from any decorator we can construct a network model. Although this functor is faithful embedding, it is not an equivalence: the network models constructed are always commutative, i.e. the constituent monoids are always commutative. Network models with noncommutative constituent monoids exist, and arise in applications [Moe18].

Concluding this section, it is important to note that the lax symmetric monoidal functor $\mathsf{Grph}_{(-)}$ as in (50) gives rise to a symmetric monoidal opfibration structure on the initial vertex functor (49), namely $V\colon(\mathsf{Grph},+,0)\to(\mathsf{Set},+,0)$. This corresponds to the well-known fact that the forgetful V (strictly) preserves all coproducts and the initial object, falling under the cocartesian monoidal fibration case of (44) and Corollary 4.8.

5.3. **Family Fibration: Zunino and Turaev categories.** Recall that for any category $\mathcal{C}$, the standard *family fibration* is induced by the (strict) functor

$$[-,\mathcal{C}]\colon\mathsf{Set}^{\mathrm{op}}\to\mathsf{Cat}\tag{52}$$

which maps every discrete category X to the functor category $[X,\mathcal{C}]$ and every function $f\colon X\to Y$ to the functor $f^*=[f,1]$, i.e. pre-composition with f. The total category of the induced fibration $\mathsf{Fam}(\mathcal{C}) \to \mathcal{C}$ has as objects pairs $(X, M \colon X \to \mathcal{C})$ essentially given by a family of X-indexed objects in $\mathcal{C}$, written $\{M_x\}_{x\in X}$, whereas the morphisms are

$$\begin{tikzcd} X \arrow[dd, "f"'] \arrow[dr, "M"] & \\ {} \arrow[r, phantom, "\Downarrow \alpha"] & \mathcal{C} \\ Y \arrow[ur, "N"'] & \end{tikzcd}$$

namely a function $f \colon X \to Y$ together with families of morphisms $\alpha_x \colon M_x \to N_{fx}$ in $\mathcal{C}$. Notice the similarity of this description with (8), which for the strict indexed categories case looks like a non-discrete version of the family fibration, for $\mathcal{C} = \mathsf{Cat}$; see also Remark 2.2. Moreover, it is a folklore fact that $\mathsf{Fam}(\mathcal{C})$ is the free coproduct cocompletion on the category $\mathcal{C}$.

On the other hand, we could consider the opfibration induced by the very same functor (52), denoted by $\mathsf{Maf}(\mathcal{C}) \to \mathsf{Set}^{\mathrm{op}}$. The objects of $\mathsf{Maf}(\mathcal{C})$ are the same as $\mathsf{Fam}(\mathcal{C})$, but morphisms $\{M_x\}_{x\in X} \to \{N_y\}_{y\in Y}$ between them are functions $g \colon Y \to X$ (i.e. $X \to Y$ in $\mathsf{Set}^{\mathrm{op}}$) together with families of arrows $\beta_y \colon M_{gy} \to N_y$ in $\mathcal{C}$. Notice that these are now indexed over the set Y rather than X like before, and in fact $\mathsf{Maf}(\mathcal{X}) = \mathsf{Fam}(\mathcal{X}^{\mathrm{op}})^{\mathrm{op}}$.

In the case that the category is monoidal $(\mathcal{C}, \otimes, I)$, the functor $[-, \mathcal{C}]$ has a canonical lax monoidal structure. Explicitly, by taking its domain $\mathsf{Set}^{\mathrm{op}}$ to be cocartesian by the usual cartesian monoidal structure $(\mathsf{Set}, \times, 1)$, the structure maps are

$$\phi_{X,Y} \colon [X, \mathcal{C}] \times [Y, \mathcal{C}] \to [X \times Y, \mathcal{C}]$$

$$\phi_0 \colon \mathbf{1} \xrightarrow{I_{\mathcal{C}}} [\mathbf{1}, \mathcal{C}] \cong \mathcal{C}$$

where $\phi_{X,Y}$ corresponds, under the tensor-hom adjunction in Cat, to

$$[X, \mathcal{C}] \times [Y, \mathcal{C}] \times X \times Y \xrightarrow{\sim} [X, \mathcal{C}] \times X \times [Y, \mathcal{C}] \times Y \xrightarrow{\mathrm{ev}_X \times \mathrm{ev}_Y} \mathcal{C} \times \mathcal{C} \xrightarrow{\otimes} \mathcal{C}.$$

By Theorem 3.12, this lax monoidal functor makes the corresponding split fibration $\mathsf{Fam}(\mathcal{X}) \to \mathsf{Set}$ monoidal as well, via $\{M_x\} \otimes \{N_y\} \coloneqq \{M_x \otimes N_y\}_{X\times Y}$. On the other hand, we could use the dual part of the same theorem, and instead consider the induced monoidal opfibration $\mathsf{Maf}(\mathcal{X}) \to \mathsf{Set}^{\mathrm{op}}$ corresponding to the same strict monoidal indexed category $([-, \mathcal{C}], \phi, \phi_0)$.

Moreover, since Set is cartesian, Theorem 4.2 also applies in both cases, giving a monoidal structure to the fibres as well: for $M \colon X \to \mathcal{C}$ and $N \colon X \to \mathcal{C}$, their fibrewise tensor product and unit are given by

$$X \xrightarrow{\Delta} X \times X \xrightarrow{M\times N} \mathcal{C} \times \mathcal{C} \xrightarrow{\otimes} \mathcal{C}$$

$$X \xrightarrow{!} 1 \xrightarrow{I} \mathcal{C}$$

which are precisely constructed as in (60). Once again, notice the direct similary with (35), the fibrewise monoidal structure on $\mathsf{ICat}(\mathcal{X})$; see also Proposition 4.6.

As an interesting example, consider $\mathcal{C} = \mathsf{Mod}_R$ for a commutative ring R, with its usual tensor product $\otimes_R$. In [CDL06], the authors introduce a category $\mathcal{T}$ of *Turaev* R-modules, as well as a category $\mathcal{Z}$ of *Zunino* R-modules, which serve as symmetric monoidal categories where group-(co)algebras and Hopf group-(co)algebras, [Tur00], live as (co)monoids and Hopf monoids respectively.

In more detail, the objects of both $\mathcal{T}$ and $\mathcal{Z}$ are defined to be pairs (X, M) where X is a set and $\{M_x\}_{x\in X}$ is an X-indexed family of R-modules, and their morphisms are respectively

$$(\mathcal{T})\begin{cases} s\colon M_{g(y)} \to N_y \text{ in } \mathsf{Mod}_R \\ g\colon Y \to X \text{ in } \mathsf{Set} \end{cases} \qquad (\mathcal{Z})\begin{cases} t\colon M_x \to N_{f(x)} \text{ in } \mathsf{Mod}_R \\ f\colon X \to Y \text{ in } \mathsf{Set} \end{cases}$$

There is a symmetric pointwise monoidal structure, $\{M_x \otimes_R N_y\}_{X\times Y}$, and there are strict monoidal forgetful functors $\mathcal{T} \to \mathsf{Set}^{\mathrm{op}}$, $\mathcal{Z} \to \mathsf{Set}$. It is therein shown that comonoids in $\mathcal{T}$ are *monoid-coalgebras* and monoids in $\mathcal{Z}$ are *monoid-algebras*, i.e. families of R-modules indexed over a monoid, together with respective families of linear maps

$$\begin{array}{llll} (\mathcal{T}) & C_{g*h} \to C_g \otimes C_h & \qquad (\mathcal{Z}) & A_g \otimes A_h \to A_{g*h} \\ & C_e \to R & & R \to A_e \end{array}$$

satisfying appropriate axioms. Based on the above, it is clear that $\mathcal{T} = \mathsf{Maf}(\mathsf{Mod}_R)$ and $\mathcal{Z} = \mathsf{Fam}(\mathsf{Mod}_R)$, which clarifies the origin of these categories and can be used to further generalize the notions of Hopf group-(co)monoids in other monoidal categories.

5.4. **Global categories of modules and comodules.** For any monoidal category $(\mathcal{V}, \otimes, I)$, there exist *global* categories of modules and comodules, denoted by Mod and Comod [Vas14, 6.2]. Their objects are all (co)modules over (co)monoids in $\mathcal{V}$, whereas for example a morphism between an A-module M and a B-module N is given by a monoid map $f\colon A \to B$ together with a morphism $k\colon M \to N$ in $\mathcal{V}$ satisfying the commutativity of

$$\begin{tikzcd} A \otimes M \arrow[rr, "\mu"] \arrow[d, "1\otimes k"'] & & M \arrow[d, "k"] \\ A \otimes N \arrow[r, "f\otimes 1"'] & B \otimes N \arrow[r, "\mu"'] & N \end{tikzcd}$$

where μ denotes the respective action. Both these categories arise as the total categories induced by the (split) Grothendieck construction on the functors

$$\begin{tikzcd} \mathsf{Mon}(\mathcal{V})^{\mathrm{op}} \arrow[r] & \mathsf{Cat} & \mathsf{Comon}(\mathcal{V}) \arrow[r] & \mathsf{Cat} \\ A \arrow[r, mapsto, dotted] \arrow[d, "f"'] & \mathsf{Mod}_{\mathcal{V}}(A) & C \arrow[r, mapsto, dotted] \arrow[d, "g"'] & \mathsf{Comod}_{\mathcal{V}}(C) \arrow[d, "g_!"] \\ B \arrow[r, mapsto, dotted] & \mathsf{Mod}_{\mathcal{V}}(B) \arrow[u, "f^*"'] & D \arrow[r, mapsto, dotted] & \mathsf{Comod}_{\mathcal{V}}(D) \end{tikzcd} \tag{53}$$

where f^* and $g_!$ are (co)restriction of scalars: if M is a B-module, $f^*(M)$ is an A-module via the action

$$A \otimes N \xrightarrow{f\otimes 1} B \otimes N \xrightarrow{\mu} N.$$

The induced split fibration and opfibration, $\mathsf{Mod} \to \mathsf{Mon}(\mathcal{V})$ and $\mathsf{Comod} \to \mathsf{Comon}(\mathcal{V})$, map a (co)module to its respective (co)monoid.

Recall that when $(\mathcal{V}, \otimes, I, \sigma)$ is braided monoidal, its categories of monoids and comonoids inherit the monoidal structure: if A and B are monoids, then $A \otimes B$ has also a monoid structure via

$$A \otimes B \otimes A \otimes B \xrightarrow{1 \otimes \sigma \otimes 1} A \otimes A \otimes B \otimes B \xrightarrow{m \otimes m} A \otimes B$$

$$I \cong I \otimes I \xrightarrow{j \otimes j} A \otimes B$$

where m and j give the respective monoid structures. In that case, the induced split fibration and opfibration are both monoidal. This can be deduced by directly checking the conditions of Proposition 3.1, or using Theorem 3.12 since both functors (53) are lax monoidal. For example, for any $A, B \in \mathsf{Mon}(\mathcal{V})$ there are natural maps

$$\phi_{A,B} \colon \mathsf{Mod}_{\mathcal{V}}(A) \times \mathsf{Mod}_{\mathcal{V}}(B) \to \mathsf{Mod}_{\mathcal{V}}(A \otimes B)$$
$$\phi_0 \colon \mathbf{1} \to \mathsf{Mod}_{\mathcal{V}}(I)$$

with $\phi_{A,B}(M, N) = M \otimes N$, with the $A \otimes B$-module structure being

$$A \otimes B \otimes M \otimes N \xrightarrow{1 \otimes \sigma \otimes 1} A \otimes M \otimes B \otimes N \xrightarrow{\mu \otimes \mu} M \otimes N$$

and $\phi_0(*) = I$.

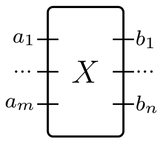
In fact, the monoidal opfibration $\mathsf{Comod} \to \mathsf{Comon}(\mathcal{V})$ serves as the monoidal base of an *enriched fibration* structure on $\mathsf{Mod} \to \mathsf{Mon}(\mathcal{V})$, as explained in [Vas18]. Moreover, analogous monoidal structures are induced on the (op)fibrations of monads and comonads in any fibrant monoidal double category, see [Vas19, Prop. 3.18].

Notice that in general, the monoidal bases $\mathsf{Mon}(\mathcal{V})$ and $\mathsf{Comon}(\mathcal{V})$ are not (co)cartesian, since they match with the monoidal structure of $(\mathcal{V}, \otimes, I, \sigma)$. Therefore this case does not fall under Theorem 4.2, hence the fibre categories are not monoidal. For example in $(\mathsf{Vect}_k, \otimes_k, k)$, the k-tensor product of two A-modules for a k-algebra A is not an A-module as well.

5.5. **Systems as monoidal indexed categories.** In [SSV19] as well as in earlier works e.g. [VSL15], the authors investigate a categorical framework for modeling systems of systems using algebras for a monoidal category. In more detail, systems in a broad sense are perceived as weak monoidal pseudofunctors

$$\mathcal{W}_{\mathcal{C}} \to \mathsf{Cat}$$

where $\mathcal{W}_{\mathcal{C}}$ is the monoidal category of *$\mathcal{C}$-labeled boxes* and *wiring diagrams* with types in a finite product category $\mathcal{C}$. Briefly, the objects in $\mathcal{W}_{\mathcal{C}}$ are pairs $X = (X^{\text{in}}, X^{\text{out}})$ of finite sets equipped with functions to ob$\mathcal{C}$, thought of as boxes

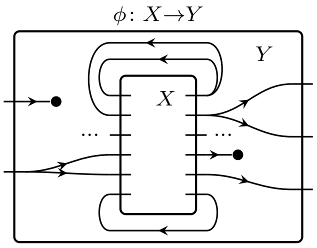


(54)

where $X^{\text{in}} = \{a_1, \dots, a_m\}$ are the input ports, $X^{\text{out}} = \{b_1, \dots, b_n\}$ the output ones and all wires are associated to a $\mathcal{C}$-object expressing the type of information that can go through them. A morphism $\phi \colon X \to Y$ in this category consists of a pair

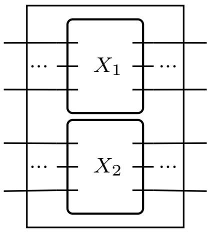
$$\begin{cases} \phi^{\text{in}} \colon X^{\text{in}} \to X^{\text{out}} + Y^{\text{in}} \\ \phi^{\text{out}} \colon Y^{\text{out}} \to X^{\text{out}} \end{cases} \tag{55}$$

which roughly expresses which port is 'fed information' by which. Graphically, we can picture it as

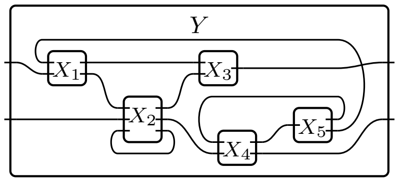


(56)

Composition of morphisms can be thought of a zoomed-in picture of three boxes, and the monoidal structure amounts to parallel placement of boxes as in



The systems-as-algebras formalism uses weak monoidal pseudofunctors from this category $\mathcal{W}_\mathcal{C}$ to Cat that essentially receive a general picture such as



(which really takes place in its underlying operad) and assign systems of a certain kind, sometimes called *inhabitants*, to all inner boxes; the weak monoidal and pseudofunctorial structure of this assignment formally produce an inhabitant of the outer box, as a system of the same kind.

Examples of such systems are discrete dynamical systems (Moore machines in the finite case), continuous dynamical systems (using differential equations) but also more general systems with deterministic or total conditions on their inputs and outputs; details can be found in the provided references. Since all these systems are weak monoidal pseudofunctors from the non-cocartesian monoidal category of wiring diagrams to Cat, i.e. monoidal indexed category, the monoidal Grothendieck construction Theorem 3.10 induces a corresponding monoidal fibration in each system case, and this global structure does not reduce to a fibrewise one.

For example, the algebra for discrete dynamical systems [SSV19, §2.3]

$$\mathrm{DDS}\colon \mathcal{W}_{\mathsf{Set}} \to \mathsf{Cat} \tag{57}$$

assigns to each box $X = (X^{\mathrm{in}}, X^{\mathrm{out}})$ the category of all discrete dynamical systems with fixed input and output sets being $\prod_{x\in X^{\mathrm{in}}} x$ and $\prod_{y\in X^{\mathrm{out}}} y$ respectively. There exist morphisms between systems of the same input and output set, but not between those with different ones. To each morphism, i.e. wiring diagram as in (56), DDS produces a functor that maps an inner discrete dynamical system to a new outer one, with changed input and output sets accordingly. (Pseudo)functoriality of this assignment allows the coherent zoom-in and zoom-out on dynamical systems built out of smaller dynamical systems, and monoidality allows the creation of new dynamical systems on parallel boxes.

Being a monoidal indexed category, (57) gives rise to a monoidal opfibration over $\mathcal{W}_{\mathsf{Set}}$. Its total category $\int\mathrm{DDS}$ has objects all dynamical systems with arbitrary input and output sets, morphisms that can now go between systems of different inputs/outputs, and also a natural tensor product inherited from that in $\mathcal{W}_{\mathsf{Set}}$ and the laxator of $\int\mathrm{DDS}$. In a sense, this category has all the required flexibility for the direct communication (via morphisms in the total category) between any discrete dynamical system, or any composite of systems or parallel placement of them, whereas the wiring diagram algebra (57) focuses on the machinery of building new discrete dynamical systems systems from old.

This classic change of point of view also transfers over to maps of algebras, i.e. indexed monoidal 1-cells. As an example, see [SSV19, §5.1], discrete dynamical systems can naturally be viewed as general *total* and *deterministic* machines denoted by $\mathrm{Mch}^{\mathrm{td}}$, via a monoidal pseudonatural transformation

$$\begin{tikzcd}
\mathcal{W}_{\mathsf{Set}} \arrow[dd] \arrow[dr, "\mathrm{DDS}"] & \\
\Downarrow & \mathsf{Cat} \\
\mathcal{W}_{\widetilde{\mathrm{Int}}_N} \arrow[ur, "\mathrm{Mch}^{\mathrm{td}}"'] &
\end{tikzcd}$$

which also changes the type of input and output wires from sets to *discrete interval sheaves* $\widetilde{\mathrm{Int}}_N$. This gives rise to a monoidal opfibred 1-cell

$$\begin{tikzcd}
\int\mathrm{DDS} \arrow[r] \arrow[d] & \int\mathrm{Mch}^{\mathrm{td}} \arrow[d] \\
\mathcal{W}_{\mathsf{Set}} \arrow[r] & \mathcal{W}_{\widetilde{\mathrm{Int}}_N}
\end{tikzcd}$$

which provides a direct functorial translation between the one sort of system to the other in a way compatible with the monoidal structure.

As a final note, this method of modeling certain objects as algebras for a monoidal category (a.k.a. strict or general monoidal indexed categories) carries over to further contexts than systems and the wiring diagram category. Examples include hypergraph categories as algebras on cospans [FS18] and traced monoidal categories as algebras on cobordisms [SSR17]. In all these cases, the monoidal Grothendieck construction gives a potentially fruitful change of perspective that should be further investigated.

6. APPENDIX: SUMMARY OF STRUCTURES

The bulk of the main body of this paper is dedicated to proving various monoidal variations of the equivalence between fibrations and indexed categories, using general results in monoidal 2-category theory in a high-level approach. In this appendix, we detail the descriptions of the (braided/symmetric) monoidal structures on the total category of the Grothendieck construction, assuming the appropriate data is present. We also provide a hands-on correspondence that underlies the proof of Theorem 4.2 regarding the transfer of monoidal structure from a functor to its target and vice versa. We hope this section can serve as a quick and clear reference on some fundamental constructions of this work.

6.1. **Monoidal structures.** As sketched under Corollary 3.11, let $(\mathcal{X}, \otimes, I)$ be a monoidal category, and

$$(\mathscr{M}, \mu, \mu_0)\colon (\mathcal{X}^{\mathrm{op}}, \otimes^{\mathrm{op}}, I) \to (\mathsf{Cat}, \times, \mathbf{1})$$

a monoidal indexed category, a.k.a. weak monoidal pseudofunctor. Recall that μ is pseudonatural transformation consisting of functors $\mu_{x,y}\colon \mathscr{M}x \times \mathscr{M}y \to \mathscr{M}(x \otimes y)$ for any objects x and y of $\mathcal{X}$, and natural isomorphisms

$$\begin{tikzcd}
\mathscr{M}z \times \mathscr{M}w \arrow[r, "\mathscr{M}f \times \mathscr{M}g"] \arrow[d, "\mu_{z,w}"'] \arrow[dr, phantom, "\Downarrow \mu_{f,g}"] & \mathscr{M}x \times \mathscr{M}y \arrow[d, "\mu_{x,y}"] \\
\mathscr{M}(z \otimes w) \arrow[r, "\mathscr{M}(f \otimes g)"'] & \mathscr{M}(x \otimes y)
\end{tikzcd}$$

for any arrows $f\colon x \to z$ and $g\colon y \to w$ in $\mathcal{X}$. Also the unique component of μ_0 is the functor $\mu_0\colon \mathbf{1} \to \mathscr{M}(I)$.

The induced tensor product functor on the total category, denoted as $\otimes_\mu\colon \int\mathscr{M} \times \int\mathscr{M} \to \int\mathscr{M}$, is given on objects by

$$(x, a) \otimes_\mu (y, b) = (x \otimes y, \mu_{x,y}(a, b)) \tag{58}$$

On morphisms $(f\colon x \to z, k\colon a \to (\mathscr{M}f)c)$ and $(g\colon y \to w, \ell\colon b \to (\mathscr{M}g)d)$, we get

$$(f, k) \otimes_\mu (g, \ell) = (x \otimes y \xrightarrow{f \otimes g} z \otimes w, \mu_{f,g}(\mu_{x,y}(k, \ell)))$$

where the latter is the composite morphism

$$\mu_{x,y}(a, b) \xrightarrow{\mu_{x,y}(k,\ell)} \mu_{x,y}\left((\mathscr{M}f)(c), (\mathscr{M}g)(d)\right) \xrightarrow{\sim} \mathscr{M}(f \otimes g)(\mu_{z.w}(c, d)) \text{ in } \mathscr{M}(x \otimes y).$$

The monoidal unit is $I_\mu = (I, \mu_0)$.

If $\mathscr{M}$ is a *strict* monoidal indexed category, i.e. a lax monoidal functor as in Theorem 3.12, the formula for objects remains the same but on morphisms it simplifies:

$$(f, k) \otimes_\mu (g, \ell) = (f \otimes g, \mu_{x,y}(k, \ell)) \tag{59}$$

If $a_{x,y,z}\colon (x \otimes y) \otimes z \to x \otimes (y \otimes z)$ denotes the associator in $\mathcal{X}$, the associator for $(\int\mathscr{M}, \otimes_\mu, I_\mu)$ is given by

$$\alpha_{(x,b),(y,c),(z,d)} = (\alpha_{x,y,z}, \omega_{x,y,z}(b, c, d))$$

where ω is the invertible modification (17). In the strict case ω is the identity, namely it reduces to one of the commutative diagrams a lax monoidal functor is required to satisfy; therefore the associator becomes $a_{(x,b),(y,c),(z,d)} = (\alpha_{x,y,z}, \mathrm{id})$.

If $l_x\colon I \otimes x \to x$ and $r_x\colon x \otimes I \to x$ are the left and right unitors in $\mathcal{X}$, the unitors in $\int\mathscr{M}$ are defined as

$$\lambda_x = (l_x, \xi_x^{-1}(a))\colon (I, \mu_0) \otimes_\mu (x, a) \to (x, a)$$
$$\rho_x = (r_x, \zeta_x(a))\colon (x, a) \otimes_\mu (I, \mu_0) \to (x, a)$$

where ζ and ξ are invertible modifications as in (17). In the strict case, these reduce to $\lambda_x = (l_x, \mathrm{id})$ and $\rho_x = (r_x, \mathrm{id})$ respectively.

We now turn to the correspondence between 1-cells of Theorem 3.10: given a monoidal indexed 1-cell

$$\begin{tikzcd}
(\mathcal{X},\otimes,I)^{\mathrm{op}} \arrow[dd, "(F{,}\psi{,}\psi_0)^{\mathrm{op}}"'] \arrow[drr, "(\mathscr{M}{,}\mu{,}\mu_0)"] & & \\
& \Downarrow\tau & (\mathsf{Cat},\times,\mathbf{1}) \\
(\mathcal{Y},\otimes,I)^{\mathrm{op}} \arrow[urr, "(\mathscr{N}{,}\nu{,}\nu_0)"'] & &
\end{tikzcd}$$

where $\mathscr{M}$ and $\mathscr{N}$ are weak monoidal pseudofunctors and F is a monoidal functor, as in Proposition 3.6, we first of all obtain an ordinary fibred 1-cell $(P_\tau, F)\colon P_{\mathscr{M}} \to P_{\mathscr{N}}$ as explained above (12)

$$\begin{tikzcd}
\int\mathscr{M} \arrow[r, "P_\tau"] \arrow[d, "P_{\mathscr{M}}"'] & \int\mathscr{N} \arrow[d, "P_{\mathscr{N}}"] \\
\mathcal{X} \arrow[r, "F"'] & \mathcal{Y}
\end{tikzcd}$$

with $P_\tau(x,a) = (Fx, \tau_x(a))$. The functor F is already monoidal, and P_τ obtains a monoidal structure too: for example, there are isomorphisms

$$P_\tau(x,a) \otimes_\nu P_\tau(y,b) \xrightarrow{\sim} P_\tau((x,a)\otimes_\mu (y,b)) \quad \text{in } \int\mathscr{N}$$

between the objects

$$\begin{aligned}
P_\tau(x,a) \otimes_\nu P_\tau(y,b) = (Fx, \tau_x(a)) \otimes_\nu (Fy, \tau_y(b) = (Fx \otimes Fy, \nu_{Fx,Fy}(\tau_x(a), \tau_y(b)) \\
P_\tau((x,a)\otimes_\mu (y,b)) = P_\tau(x\otimes y, \mu_{x,y}(a,b)) = (F(x\otimes y), \tau_{x\otimes y}(\mu_{x,y}(a,b)))
\end{aligned}$$

given by $\psi_{x,y}\colon Fx \otimes Fy \xrightarrow{\sim} F(x\otimes y)$ and by

$$\nu_{Fx,Fy}(\tau_x(a), \tau_y(b)) \cong \mathscr{N}(\psi_{x,y})(\tau_{x\otimes y}(\mu_{x,y}(a,b)))$$

essentially given by the monoidal pseudonatural isomorphism (19) for $\tau\colon \mathscr{M} \Rightarrow \mathscr{N}F^{\mathrm{op}}$. As a result, (P_τ, F) is indeed a monoidal fibred 1-cell as in Proposition 3.2.

Finally, it can be verified that starting with a monoidal indexed 2-cell as in Proposition 3.7, the induced fibred 2-cell (13) is monoidal, i.e. P_m satisfies the conditions of a monoidal natural transformation.

Regarding the induced braided and symmetric monoidal structures, suppose that $(\mathcal{X},\otimes,I)$ is a braided monoidal category, with braiding b with components

$$\beta_{x,y}\colon x\otimes y \xrightarrow{\sim} y\otimes x;$$

then $\mathcal{X}^{\mathrm{op}}$ is braided monoidal with the inverse braiding, namely $(\mathcal{X}^{\mathrm{op}}, \otimes^{\mathrm{op}}, I, \beta^{-1})$. Now if $(\mathscr{M},\mu,\mu_0)\colon \mathcal{X}^{\mathrm{op}} \to \mathsf{Cat}$ is a *braided* weak monoidal pseudofunctor, i.e. a braided monoidal indexed category, by Theorem 3.10 we have an induced braided monoidal structure on $(\int\mathscr{M}, \otimes_\mu, I_\mu)$, namely

$$B_{(x,a),(y,b)}\colon (x,a)\otimes_\mu (y,b) = (x\otimes y, \mu_{x,y}(a,b)) \to (y,b)\otimes_\mu (x,a) = (y\otimes x, \mu_{y,x}(b,a))$$

are given by $\beta_{x,y}\colon x\otimes y \cong y\otimes x$ in $\mathcal{X}$ and $(v_{x,y})_{(a,b)}\colon \mu_{x,y}(a,b) \cong \mathscr{M}(\beta^{-1}_{x,y})(\mu_{y,x}(b,a))$. where v is as in (26).

In the strict case, the components of v are identities, reducing to the condition a lax monoidal functor needs to satisfy in order to be braided.

If $\mathscr{M}$ is a symmetric weak monoidal pseudofunctor, it can be verified that $B_{(y,b),(x,a)} \circ B_{(x,a),(y,b)} = 1_{(x,a)\otimes_\mu(y,b)}$ therefore $\textstyle\int\mathscr{M}$ is also symmetric monoidal, as is the monoidal fibration $P_\mathscr{M}\colon \textstyle\int\mathscr{M} \to \mathcal{X}$.

6.2. **Monoidal Indexed Categories as ordinary pseudofunctors.** Here we detail the correspondence between monoidal opindexed categories and a pseudofunctors into MonCat when the domain is a cocartesian monoidal category, as established by Theorem 4.2; the one for indexed categories is of course similar. We denote by $\nabla_x\colon x + x \to x$ the induced natural components due to the universal property of coproduct, and $iota_x\colon x \to x + y$ the inclusion into a coproduct.

Start with a weak monoidal pseudofunctor $\mathscr{M}\colon (\mathcal{X}, +, 0) \to (\mathsf{Cat}, \times, \mathbf{1})$ equipped with $\mu_{x,y}\colon \mathscr{M}(x) \times \mathscr{M}(y) \to \mathscr{M}(x + y)$ and $\mu_0\colon \mathbf{1} \to \mathscr{M}(0)$, which gives the global monoidal structure (37) of the corresponding opfibration. There exists an induced monoidal structure on each fibre $\mathscr{M}(x)$ as follows:

$$\otimes_x\colon \mathscr{M}(x) \times \mathscr{M}(x) \xrightarrow{\mu_{x,x}} \mathscr{M}(x + x) \xrightarrow{\mathscr{M}(\nabla)} \mathscr{M}(x) \tag{60}$$

$$I_x\colon \mathbf{1} \xrightarrow{\mu_0} \mathscr{M}(0) \xrightarrow{\mathscr{M}(!)} \mathscr{M}(x)$$

Moreover, each $\mathscr{M}f\colon \mathscr{M}x \to \mathscr{M}y$ is a strong monoidal functor, with $\phi_{a,b}\colon (\mathscr{M}f)(a) \otimes_y (\mathscr{M}f)(b) \xrightarrow{\sim} \mathscr{M}f(a \otimes_x b)$ and $\phi_0\colon I_y \xrightarrow{\sim} (\mathscr{M}f)I_x$ essentially given by the following isomorphisms

$$\begin{tikzcd}
\mathscr{M}x \times \mathscr{M}x \arrow[r, "\mathscr{M}f \times \mathscr{M}f"] \arrow[d, "\mu_{x,x}"'] \arrow[dr, phantom, "\overset{\mu^{f,f}}{\cong}"] & \mathscr{M}y \times \mathscr{M}y \arrow[d, "\mu_{y,y}"] & & \mathbf{1} \arrow[r, "\mu_0"] \arrow[d, "\mu_0"'] & \mathscr{M}(0) \arrow[dd, "\mathscr{M}(!)"] \\
\mathscr{M}(x + x) \arrow[r, "\mathscr{M}(f+f)" description] \arrow[d, "\mathscr{M}(\nabla_x)"'] \arrow[dr, phantom, "\cong"] & \mathscr{M}(y + y) \arrow[d, "\mathscr{M}(\nabla_y)"] & & \mathscr{M}(0) \arrow[d, "\mathscr{M}(!)"'] \arrow[r, phantom, "\cong"] & \\
\mathscr{M}x \arrow[r, "\mathscr{M}f"'] & \mathscr{M}y & & \mathscr{M}x \arrow[r, "\mathscr{M}f"'] & \mathscr{M}y
\end{tikzcd}$$

since ∇ and ! are natural and $\mathscr{M}$ is a pseudofunctor.

In the opposite direction, take an ordinary pseudofunctor $\mathscr{M}: \mathcal{X} \to \mathsf{MonCat}$ into the 2-category of monoidal categories, strong monoidal functors and monoidal natural transformations, with $\otimes_x\colon \mathscr{M}(x) \times \mathscr{M}(x) \to \mathscr{M}(x)$ and I_x the fibrewise monoidal structures in every $\mathscr{M}x$. We can use those to endow $\mathscr{M}$ with a weak monoidal structure via

$$\mu_{x,y}\colon \mathscr{M}(x) \times \mathscr{M}(y) \xrightarrow{\mathscr{M}(\iota_x) \times \mathscr{M}(\iota_y)} \mathscr{M}(x + y) \times \mathscr{M}(x + y) \xrightarrow{\otimes_{x+y}} \mathscr{M}(x + y) \tag{61}$$

$$\mu_0\colon \mathbf{1} \xrightarrow{I_0} \mathscr{M}(0)$$

The fact that all $\mathscr{M}f$ are strong monoidal imply that the above components form pseudonatural transformations, and all appropriate conditions are satisfied.

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