# On the fibrations underlying optimization and elimination

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**Abstract**

*As of July 25, 2019:* The theory of fibrations and fibered categories appears to be a natural place to discuss the theory of various optimization and elimination problems, including resolution in logic, linear and non-linear quantifier elimination, polytope projection, lattice optimization over various spaces, etc. These notes aim to investigate that claim and furthermore attempts to determine any and all structural similarities between the various cases.

# Contents



# 1 Potentially relevant examples

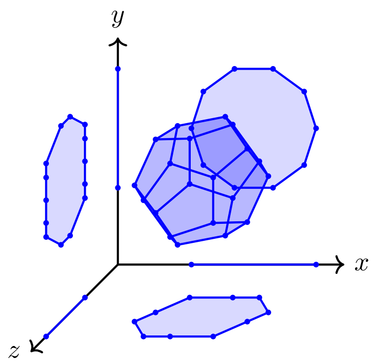
## 1.1 The projections of convex polytopes

For our purposes, a *polytope* is the intersection of a finite number of half-spaces of a vector space. Concretely a polytope $P$ is the solution to a system

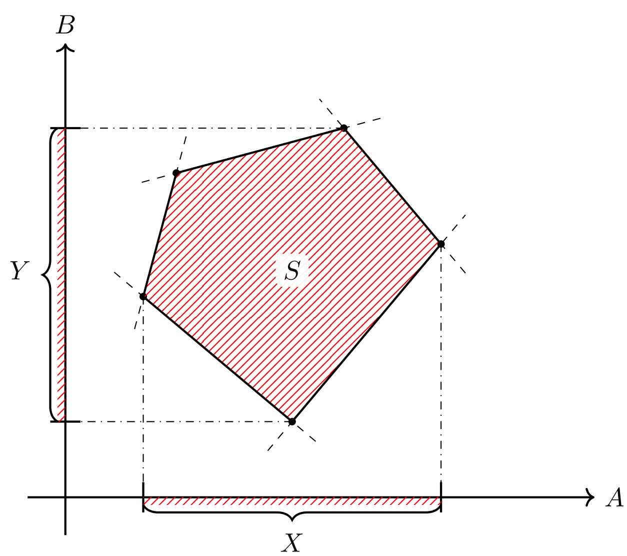
$$P = \{x \in V \mid Ax \succeq b\} \tag{1}$$

where $Ax \succeq b$ is a finite system of $m$ linear inequalities constraining $x = (x_1, x_2, \ldots, x_n)$:

$$
\begin{aligned}
a_{1,1}x_1 + a_{1,2}x_2 + \ldots + a_{1,n}x_n &\geq b_1 \\
a_{2,1}x_1 + a_{2,2}x_2 + \ldots + a_{2,n}x_n &\geq b_2 \\
&\vdots \\
a_{m,1}x_1 + a_{m,2}x_2 + \ldots + a_{m,n}x_n &\geq b_m
\end{aligned}
$$




$$
\begin{tikzcd}
& \mathbf{Poly}_Y \arrow[dd, "\pi_{1,!}"] & & \mathbf{Poly}_{X\otimes Y} \arrow[ll, "\pi_{Y,!}"'] \arrow[dd, "\pi_{X,!}"'] \\
\mathbf{Poly}_{Y\otimes Z} \arrow[ur, "\pi_{Y,!}"'] \arrow[dd, "\pi_{Z,!}"] & & \mathbf{Poly}_{X\otimes Y\otimes Z} \arrow[ll, "\pi_{Y\otimes Z,!}"'] \arrow[ur, "\pi_{X\otimes Y,!}"] \arrow[dd, "\pi_{X\otimes Z,!}"] & \\
& \mathbf{Poly}_1 & & \mathbf{Poly}_X \arrow[ll, "\pi_{1,!}"'] \\
\mathbf{Poly}_Z \arrow[ur, "\pi_{1,!}"] & & \mathbf{Poly}_{X\otimes Z} \arrow[ll, "\pi_{Z,!}"'] \arrow[ur, "\pi_{X,!}"] &
\end{tikzcd}
\tag{2}
$$



| Context | Fibration | Total $\mathcal{E}$ | Base $\mathcal{E}$ | Fibers | Covariant Functor | Contravariant Functor |
|---|---|---|---|---|---|---|
| Subset Projection | | | | | | |
| Linear Quantifier Elimination | | | | | | |
| Non-linear Quantifier Elimination | | | | | | |
| Real-valued Optimization | | | | | | |
| General Lattice Optimization | | | | | | |
| Convex Projection | | | | | | |
| Convex Optimization | | | | | | |
| Resolution | | | | | | |
| . . . and more | | | | | | |

## 1.2 Subset Projection

A prototypical example wherein an adjoint triple

$$f_!, \exists_f \dashv f^*, f^{-1} \dashv f^!, \forall_f$$

arises is that of functions $f : X \to Y$ between sets $X$ and $Y$. The inverse image functor $f^* : \mathscr{P}Y \to \mathscr{P}X$ is defined on a subset $T \subseteq Y$

$$f^*(T) = \{x \in X : f(x) \in T\},$$

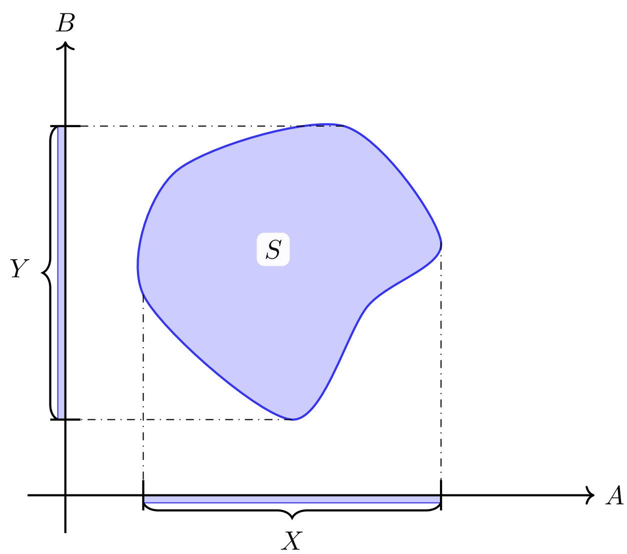
and is functorial in the sense that if $T \subseteq T' \subseteq Y$ then $f^*(T) \subseteq f^*(T') \subseteq f^*(T)$. The adjoint functors $\exists_f, \forall_f : \mathscr{P}X \to \mathscr{P}Y$ are defined on $S \subseteq X$ as

$$\exists_f(S) = \{y \in Y : \exists x \in f^*(y) : x \in S\}$$
$$\forall_f(S) = \{y \in Y : \forall x \in f^*(y) : x \in S\}$$

form an adjoint triple in the sense that $\exists_f \dashv f^* \dashv \forall_f$:

$$\exists_f \dashv f^* : \quad \exists_f(S) \subseteq T \iff S \subseteq f^*(T)$$
$$f^* \dashv \forall_f : \quad f^*(T) \subseteq R \iff T \subseteq \forall_f(R)$$

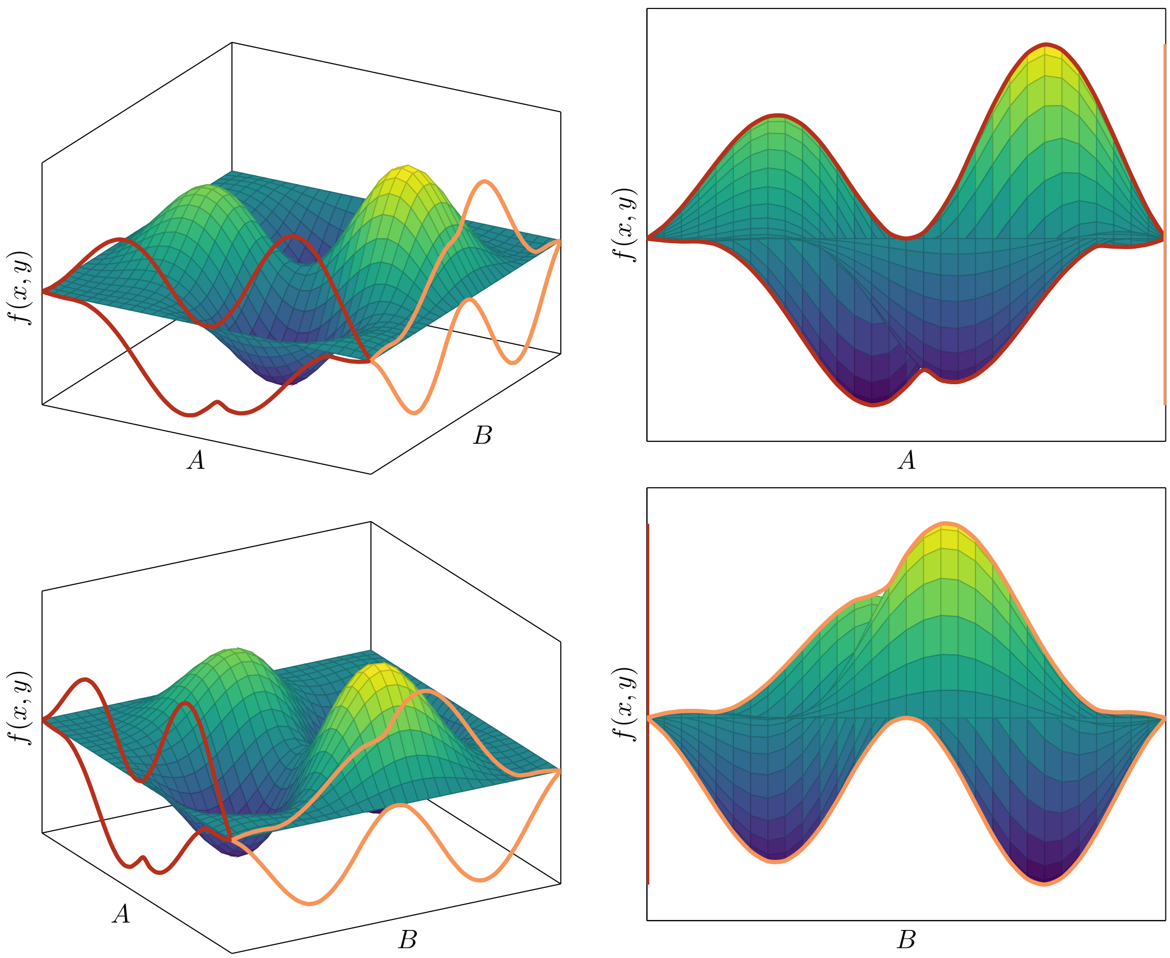
## 1.3 Subset Projection



Consider a pair of sets $A$ and $B$ and a subset $S \subseteq A \times B$ of their cartesian product. The projection morphisms associated with $A \times B$ are $p : A \times B \to A$ and $q : A \times B \to B$. The projection of the subset $S$ onto $A$ is then the subset $X \subseteq A$ defined by:

$$X = \{a \in A \mid \exists s \in S, p(s) = a\}$$

$$S \subseteq p^*(X) \iff \exists_p(S) \subseteq X \tag{3}$$



# 2 Categorical Notions

The following unordered list of categorical concepts are anticipated to be utilized:

- adjunctions
- fibered categories
- cleavages
- puesdo functors (and if cleavages are splitting, functors)
- Beck-Chevalley condition
- Frobenius reciprocity (and functors of monoidal categories)

## 2.1 Cartesian Arrows

**Definition 2.1.** Let $P : \mathcal{E} \to \mathcal{B}$ be a functor between categories $\mathcal{E}$ and $\mathcal{B}$. An arrow $\phi : \alpha \to \beta$ of $\mathcal{E}$ is *cartesian* with respect to $P$ if for every arrow $\psi : \gamma \to \beta$ sharing a codomain with $\phi$, and for every arrow $g : P(\gamma) \to P(\alpha)$ in $\mathcal{B}$ satisfying $g \circ P(\phi) = P(\psi)$, there exists a unique arrow $\theta : \gamma \to \alpha$ in $\mathcal{E}$ satisfying $\phi \circ \theta = \psi$ and $P(\theta) = g$.

$$\begin{tikzcd}
\gamma \arrow[dr, dashed, "\exists ! \theta"'] \arrow[drr, bend left, "\forall \psi"] \arrow[dd, mapsto] & & \\
& \alpha \arrow[r, "\phi"'] \arrow[dd, mapsto] & \beta \arrow[dd, mapsto] \\
P(\gamma) \arrow[dr, "\forall g"'] \arrow[drr, bend left, "P(\psi)"] & & \\
& P(\alpha) \arrow[r, "P(\phi)"'] & P(\beta)
\end{tikzcd} \tag{4}$$

**Corollary 2.0.1.** *A cartesian morphism $\phi : \alpha \to \beta$ in $\mathcal{E}$ with respect to a functor $P : \mathcal{E} \to \mathcal{B}$ establishes an isomorphism of categories [Lur09, Section 2.4.1]*[1]

$$\mathcal{E}/\phi \cong \mathcal{E}/\beta \times_{\mathcal{B}/P(\beta)} \mathcal{B}/P(\phi) \tag{5}$$

*where $\mathcal{E}/\beta \times_{\mathcal{B}/P(\beta)} \mathcal{B}/P(\phi)$ is the pullback of functors.*

$$\begin{tikzcd}
\mathcal{E}/\phi \arrow[dr, dashed, leftrightarrow, "\cong"] \arrow[drr, bend left, "P/\phi"] \arrow[ddr, bend right, "\mathrm{cod}"'] & & \\
& \mathcal{E}/\beta \times_{\mathcal{B}/P(\beta)} \mathcal{B}/P(\phi) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathcal{B}/P(\phi) \arrow[d, "\mathrm{cod}"] \\
& \mathcal{E}/\beta \arrow[r, "P/\beta"] & \mathcal{B}/P(\beta)
\end{tikzcd} \tag{6}$$

The pullback category $\mathcal{E}/\beta \times_{\mathcal{B}/P(\beta)} \mathcal{B}/P(\phi)$ has morphisms associated with diagrams of $\mathcal{B}$ with the

[1] This formulation is also discussed here: `https://ncatlab.org/nlab/show/Cartesian+morphism#CartInOrdCatReformulation`.

following format:

$$
\begin{tikzcd}
 & P(\gamma) \arrow[d, "P(\chi)"] \arrow[ddl, bend right, "f"'] \arrow[ddr, bend left, "P(\omega)"] & \\
 & P(\delta) \arrow[dl, "g"'] \arrow[dr, "P(\psi)"] & \\
P(\alpha) \arrow[rr, "P(\phi)"] & & P(\beta)
\end{tikzcd}
\tag{7}
$$

Evidently, if $\phi : \alpha \to \beta$ is cartesian, then there exists unique morphisms $\zeta : \gamma \to \alpha$ and $\eta : \delta \to \alpha$ such that $P(\zeta) = f$ and $P(\eta) = g$ and the following diagram of $\mathcal{E}$ commutes:

$$
\begin{tikzcd}
 & \gamma \arrow[d, "\chi"] \arrow[ddl, bend right, "\zeta"'] \arrow[ddr, bend left, "\omega"] & \\
 & \delta \arrow[dl, "\eta"'] \arrow[dr, "\psi"] & \\
\alpha \arrow[rr, "\phi"] & & \beta
\end{tikzcd}
\tag{8}
$$

Intuitively, if $\phi$ is cartesian, then in order to determine the category $\mathcal{E}/\phi$ over $\phi$, it is sufficient to specify $\mathcal{E}/\beta \times_{\mathcal{B}/P(\beta)} \mathcal{B}/P(\phi)$.

## 2.2 Fibrations, Fibered Categories, and Cleavages

**Definition 2.2.** A *fibered category over* $\mathcal{B}$ is a category $\mathcal{E}$ associated to the domain of a functor, referred to as the *fibration*, $P : \mathcal{E} \to \mathcal{B}$ with the property that for every morphism $f : a \to b$ of $\mathcal{B}$ and object $\beta$ such that $P(\beta) = b$, there exists a cartesian arrow $\phi : \alpha \to \beta$ with $P(\phi) = f$.

**Lemma 2.1.** *A fibration $P : \mathcal{E} \to \mathcal{B}$ is a faithful functor if and only if its fibers are thin.*

*Proof.* Recall that if $P : \mathcal{E} \to \mathcal{B}$ is a faithful functor, then by definition every pair of parallel arrows $\phi, \psi : \alpha \to \beta$ in $\mathcal{E}$ satisfies

$$P(\phi) = P(\psi) : P(\alpha) \to P(\beta) \implies \phi = \psi. \tag{9}$$

$\implies$ : Assuming $P : \mathcal{E} \to \mathcal{B}$ is faithful functor, consider an arbitrary pair of parallel arrows $\phi, \psi : \alpha \to \beta$ in an arbitrary fiber $\mathcal{E}_x$ over $x$; i.e. $P(\phi) = P(\psi) = \mathrm{id}_x$. In such cases, faithfulness of $P$ (Eq. 9) guarantees that $\phi = \psi$ and thus $\mathcal{E}_x$ is a thin category.

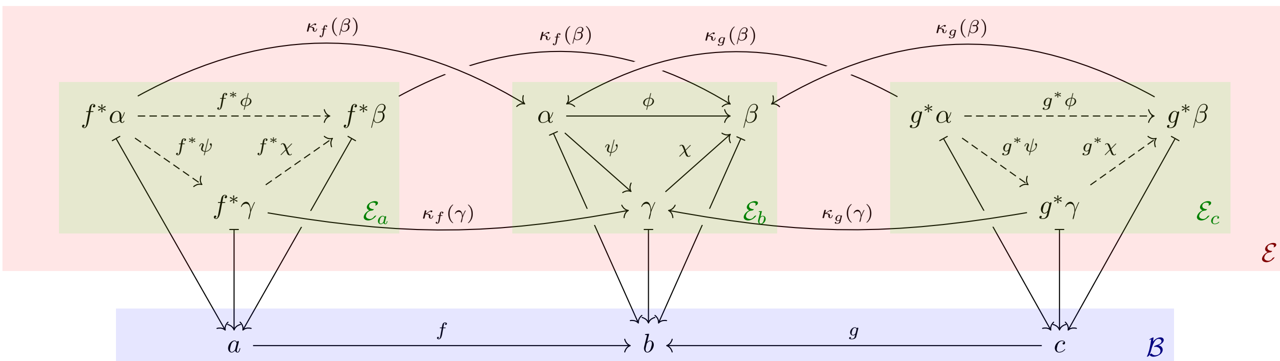
$\impliedby$ : If the fiber $\mathcal{E}_x$ for every object $x$ in $\mathcal{B}$ is a thin category, then clearly $P : \mathcal{E} \to \mathcal{B}$ *must* be faithful when restricted to an individual fiber. The non-trivial case is to consider an arbitrary pair of parallel morphisms $\phi, \psi : \alpha \to \beta$ not belonging to any fibers of $\mathcal{E}$. Denote $a \coloneqq P(\alpha)$ and $b \coloneqq P(\beta)$ and suppose $f \coloneqq P(\phi) = P(\psi) : a \to b$. Then, because $\mathcal{E}$ is a fibered category, there exists a cartesian arrow $\zeta : \gamma \to \beta$, such that $P(\zeta) = f$ (note that $a = P(\alpha) = P(\gamma)$ but $\gamma$ is not necessarily equal to $\alpha$). Since $\zeta$ is a cartesian arrow, there exists a unique arrows $\mu, \nu : \alpha \to \gamma$ completing the top edges of the following diagram:

$$
\begin{tikzcd}
\alpha \arrow[rr, dashed, "\mu"] \arrow[dd, mapsto] \arrow[dr, bend left, "\psi"'] & & \gamma \arrow[dl, "\zeta"'] \arrow[dd, mapsto] & \alpha \arrow[l, dashed, "\nu"'] \arrow[dd, mapsto] \arrow[dll, bend left, "\phi"] \\
 & \beta \arrow[dd, mapsto] & & \\
a \arrow[rr, equal] \arrow[dr, bend right, "f"'] & & a \arrow[r, equal] \arrow[dl, "f"] & a \arrow[dll, bend left, "f"] \\
 & b & &
\end{tikzcd}
.
\tag{10}
$$

However, $P(\nu) = \mathrm{id}_a = P(\mu)$ and therefore $\mu$ and $\nu$ are parallel arrows in the fiber $\mathcal{E}_a$ and therefore $\mu = \nu$ because $\mathcal{E}_a$ is assumed thin. Therefore, $\psi = \zeta \circ \mu = \zeta \circ \nu = \phi$ and thus $P$ is a faithful functor. $\square$

**Definition 2.3.** A *cleavage* for a fibration $P : \mathcal{E} \to \mathcal{B}$ is an assignment to each morphism $f : a \to b$ of $\mathcal{B}$ and object $\beta$ in $\mathcal{E}_b$ (i.e. $P(\beta) = b$), a unique cartesian morphism $\phi$ such that $P(\phi) = f$.

$$\begin{tikzcd}
f^*\beta_1 \arrow[r, "\kappa(f;\beta_1)"] \arrow[d, mapsto] & \beta_1 \arrow[d, mapsto] & g^*\beta_1 \arrow[l, "\kappa(g;\beta_1)"'] \arrow[d, mapsto] \\
a \arrow[r, "f"] & b & c \arrow[l, "g"'] \\
f^*\beta_2 \arrow[r, "\kappa(f;\beta_2)"'] \arrow[u, mapsto] & \beta_2 \arrow[u, mapsto] & g^*\beta_2 \arrow[l, "\kappa(g;\beta_2)"] \arrow[u, mapsto]
\end{tikzcd} \tag{11}$$




(12)

# Categorical Definitions

## 2.3 Hom-Functors

For a locally small category $\mathcal{C}$, the hom-functor of $\mathcal{C}$ is a functor $\mathrm{Hom}_{\mathcal{C}} : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Set}$ constructed in the following manner. Given objects $a, b, c, \ldots \in \mathcal{C}_0$ of $\mathcal{C}$, the hom-functor $\mathrm{Hom}_{\mathcal{C}}$ maps a pair of objects $(a, b) \in (\mathcal{C}^{\mathrm{op}} \times \mathcal{C})_0 = \mathcal{C}_0 \times \mathcal{C}_0 = \mathcal{C}_0^2$ into the set[2] of morphisms $\mathcal{C}_1$ of $\mathcal{C}$ with source $a$ and target $b$. Therefore, $\mathrm{Hom}_{\mathcal{C}}(a, b)$ is the set of morphisms in $\mathcal{C}$ of type $a \to b$. Given morphisms $g^{\mathrm{op}} \in \mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(a, c)$ and $h \in \mathrm{Hom}_{\mathcal{C}}(b, d)$, the hom-functor $\mathrm{Hom}_{\mathcal{C}}$ constructs a function

$$\mathrm{Hom}_{\mathcal{C}}(g^{\mathrm{op}}, h) : \mathrm{Hom}_{\mathcal{C}}(a, b) \to \mathrm{Hom}_{\mathcal{C}}(c, d)$$

which takes a morphism $f : a \to b \in \mathrm{Hom}_{\mathcal{C}}(a, b)$ and produces the morphism $h \circ f \circ g : c \to d \in \mathrm{Hom}_{\mathcal{C}}(c, d)$. Graphically,

$$\mathrm{Hom}_{\mathcal{C}}(g^{\mathrm{op}}, h)\left( a \xrightarrow{f} b \right) = c \xrightarrow{g} a \xrightarrow{f} b \xrightarrow{h} d$$

## 2.4 Adjoint Functors

Given two categories $\mathscr{C}$ and $\mathscr{D}$, a pair of functors $L : \mathscr{C} \to \mathscr{D}, R : \mathscr{D} \to \mathscr{C}$ are called an *adjoint pair*, denoted $L \dashv R$ or

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left, "L"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "R"]
\end{tikzcd}$$

[2]The collection of morphisms of type $a \to b$ forms a set because $\mathcal{C}$ is locally small.

if there exists a natural isomorphism $\alpha$ between the following pair of hom-functors of type $\mathscr{C}^{\mathrm{op}} \times \mathscr{D} \to \mathbf{Set}$:

$$\mathrm{Hom}_{\mathscr{D}}(L^{\mathrm{op}}(-), -) \overset{\alpha}{\simeq} \mathrm{Hom}_{\mathscr{C}}(-, R(-))$$

This relationship can be depicted graphically as 2-cell (and its inverse) in **Cat**,

$$\begin{tikzcd}
\mathcal{C}^{\mathrm{op}} \times \mathcal{D} \arrow[r, "I_{\mathcal{C}^{\mathrm{op}}} \times R"] \arrow[d, "L^{\mathrm{op}} \times I_{\mathcal{D}}"'] \arrow[dr, phantom, "\overset{\alpha}{\nearrow}\ \underset{\alpha^{-1}}{\swarrow}"] & \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \arrow[d, "\mathrm{Hom}_{\mathcal{C}}"] \\
\mathcal{D}^{\mathrm{op}} \times \mathcal{D} \arrow[r, "\mathrm{Hom}_{\mathcal{D}}"] & \mathbf{Set}
\end{tikzcd}$$

Concretely, the naturality of $\alpha$ means that for every morphism $(f^{\mathrm{op}} : b \to a, g : c \to d) \in (\mathcal{C}^{\mathrm{op}} \times \mathcal{D})_1$ the components $\alpha_{(b,c)}$ and $\alpha_{(a,d)}$ of $\alpha$ make the following square commute:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{D}}(L^{\mathrm{op}}(b), c) \arrow[r, "\mathrm{Hom}_{\mathcal{D}}(L^{\mathrm{op}}(f^{\mathrm{op}}), g)"] \arrow[d, "\alpha_{(b,c)}"'] & \mathrm{Hom}_{\mathcal{D}}(L^{\mathrm{op}}(a), d) \arrow[d, "\alpha_{(a,d)}"] \\
\mathrm{Hom}_{\mathcal{C}}(b, R(c)) \arrow[r, "\mathrm{Hom}_{\mathcal{C}}(f^{\mathrm{op}}, R(g))"] & \mathrm{Hom}_{\mathcal{C}}(a, R(d))
\end{tikzcd}$$

## 2.5 Beck-Chevalley Conditions

The Beck-Chevalley Conditions are conditions that may or may not be satisfied by a quadruplet of functors $F, H, G, K$ which form a natural isomorphism $\alpha : KF \Rightarrow HG$ square:

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "F"] \arrow[d, "G"'] \arrow[dr, phantom, "\overset{\alpha}{\swarrow}"] & \mathcal{B} \arrow[d, "K"] \\
\mathcal{C} \arrow[r, "H"'] & \mathcal{D}
\end{tikzcd}$$

To define the *left* Beck-Chevalley condition, one needs functors $F_L : \mathcal{B} \to \mathcal{A}$ and $H_L : \mathcal{D} \to \mathcal{A}$ which are respectively left adjoint functors to $F$ and $H$,

$$\begin{tikzcd}
\mathcal{A} \arrow[r, bend right, "F"'] \arrow[r, phantom, "\perp"] & \mathcal{B} \arrow[l, bend right, "F_L"']
\end{tikzcd}, \qquad
\begin{tikzcd}
\mathcal{C} \arrow[r, bend right, "H"'] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend right, "H_L"']
\end{tikzcd}.$$

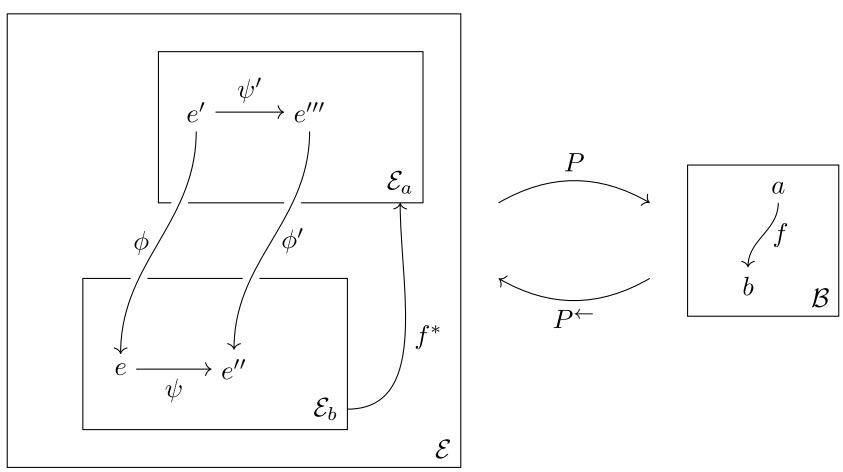
Using these left adjoint functors, it becomes possible to construct a natural transformation $\beta : KH_L \Rightarrow GF_L$ from $\alpha$[3]. Graphically, $\beta$ can be identified as the outer cell of the following diagram:

$$
\begin{tikzcd}
\mathcal{A} \arrow[r, bend right, "F"'] \arrow[r, phantom, "\perp"] \arrow[d, "G"'] & \mathcal{B} \arrow[l, bend right, "F_L"'] \arrow[d, "K"] \arrow[dl, Rightarrow, "\alpha"'] \\
\mathcal{C} \arrow[r, bend left, "H"] \arrow[r, phantom, "\top"] & \mathcal{D} \arrow[l, bend left, "H_L"]
\end{tikzcd}
\quad , \qquad \text{i.e.} \qquad
\begin{tikzcd}
\mathcal{A} \arrow[d, "G"'] & \mathcal{B} \arrow[l, "F_L"'] \arrow[d, "K"] \\
\mathcal{C} & \mathcal{D} \arrow[l, "H_L"] \arrow[ul, Rightarrow, "\beta"']
\end{tikzcd}
\ .
$$

Although the natural transformation $\alpha$ is assumed to be a natural isomorphism, the natural transformation $\beta$ need not be; if $\beta$ happens to be a natural isomorphism, then we say that the original square satisfies the *left* Beck-Chevalley condition[4]. The *right* Beck-Chevalley condition is defined analogously with functors $F_R, H_R$ which are respectively right adjoints $F \dashv F_R$ and $H \dashv H_R$.

## 2.6 The Equivalence of Puesdofunctors and Fibrations

Given a functor $P : \mathcal{E} \to \mathcal{B}$ which is also a Grothendieck fibration equipped with a cleavage (i.e. a choice of cartesian morphism $\phi \in \mathrm{Hom}_{\mathcal{E}}(e', e)$ for each $f \in \mathrm{Hom}_{\mathcal{B}}(a, P(e))$ such that $P(\phi) = f$), it is possible to construct a pseudofunctor (read weak 2-functor between weak 2-categories) $\pi : \mathcal{B}^{\mathrm{op}} \to \mathbf{Cat}$. In particular, each object $b \in \mathcal{B}_0$ is mapped to the *sub-category* $\pi(b) = \mathcal{E}_b$ of $\mathcal{E}$ whose objects are those which map to $b$ under $P$ and whose morphism are those which map to $\mathrm{id}_b$ under $P$; $\mathcal{E}_b$ is the fibre category over $b$ with respect to $P$. For each morphism $f \in \mathrm{Hom}_{\mathcal{B}}(a, b)$ in $\mathcal{B}$, the pseudofunctor $\pi$ maps $f^{\mathrm{op}} : b \to a$ onto a functor $\pi(f^{\mathrm{op}}) = f^* : \mathcal{E}_b \to \mathcal{E}_a$ which is defined accordingly:




Given an object $e \in (\mathcal{E}_b)_0$, the functor $f^*$ finds the unique cartesian morphism $\phi \in \mathrm{Hom}_{\mathcal{E}}(e', e)$ as specified by the cleavage and assigns $f^*(e) = e'$. Next, given a morphism $\psi \in \mathrm{Hom}_{\mathcal{E}_b}(e, e'')$, the functor $f^*$ first finds the unique cartesian morphisms $\phi \in \mathrm{Hom}_{\mathcal{E}}(e', e)$ and $\phi' \in \mathrm{Hom}_{\mathcal{E}}(e''', e'')$. Then, because $g = \mathrm{id}_a$ completes

[3] The natural transformations $\alpha$ and $\beta$ are known as *mates* or *conjugates*.

[4] Are the left adjoints $F_L, H_L$ unique? If not, it might be better to say the original square satifies the left Beck-Chevalley condition with respect to $F_L, H_L$.

the following diagram

$$\begin{tikzcd} P(e') \arrow[r, "g"] \arrow[dr, bend right, "P(\psi\circ\phi)"'] & P(e''') \arrow[d, "P(\phi')"] \\ & P(e'') \end{tikzcd} \quad = \quad \begin{tikzcd} a \arrow[r, "\mathrm{id}_a"] \arrow[dr, bend right, "\mathrm{id}_b \circ f"'] & a \arrow[d, "f"] \\ & b \end{tikzcd},$$

and because $\phi'$ is cartesian, there must exist a unique $\psi' \in \mathrm{Hom}_{\mathcal{E}_a}(e', e''')$ such that $\psi \circ \phi = \phi' \circ \psi'$. For each $\psi \in \mathrm{Hom}_{\mathcal{E}_b}(e, e'')$, the functor $f^*$ selects this unique morphism $f^*(\psi) = \psi'$. In summary, the pseudofunctor $\pi : \mathcal{B}^{\mathrm{op}} \to \mathbf{Cat}$ induced by $P : \mathcal{E} \to \mathcal{B}$ is defined on objects $b \in \mathcal{B}_0$ as $\pi(b) = \mathcal{E}_b$ and on morphisms $f \in \mathcal{B}_1$ as $\pi(f) = f^*$ and forms a functor [TODO: figure out the 'pseudo' part of the pseudofunctorality.].

## 2.7 Slice and Coslice Categories

Given a category $\mathcal{C}$ and an object $c \in \mathcal{C}_0$ of $\mathcal{C}$, the *slice category* (or *over category*) $\mathcal{C}/c$ is the "stuff in $\mathcal{C}$ that is on top of $c$". Specifically, the objects of $\mathcal{C}/c$ are all the morphisms $f \in \mathcal{C}_1$ from $\mathcal{C}$ whose codomain is $\mathrm{cod}(f) = c$ (alternatively you could write $(\mathcal{C}/c)_0 = \mathrm{Hom}_{\mathcal{C}}(-, c)$). A morphism of $\mathcal{C}/c$ between objects $f : a \to c, g : b \to c \in (\mathcal{C}/c)_0$ is a commuting triangle completed by a third morphism $h : a \to b \in \mathcal{C}_1$:

$$\begin{tikzcd} a \arrow[rr, "h"] \arrow[dr, "g"'] & & b \arrow[dl, "f"] \\ & c & \end{tikzcd}$$

Composition of morphisms in $\mathcal{C}/c$ is induced by the composition of morphisms in $\mathcal{C}$:

$$\left(\begin{tikzcd} y \arrow[rr, "n"] \arrow[dr, "f"'] & & z \arrow[dl, "h"] \\ & c & \end{tikzcd}\right) \circ_{\mathcal{C}/c} \left(\begin{tikzcd} x \arrow[rr, "m"] \arrow[dr, "g"'] & & y \arrow[dl, "f"] \\ & c & \end{tikzcd}\right) = \begin{tikzcd} x \arrow[r, "m"] \arrow[dr, "g"'] & y \arrow[r, "n"] \arrow[d, "f"] & z \arrow[dl, "h"] \\ & c & \end{tikzcd}$$

The assignment of an overcategory $\mathcal{C}/c$ to each object $c$ can be extended to a *slice functor* $\mathcal{C}/(-) : \mathcal{C} \to \mathbf{Cat}$ in the following sense. For objects $c \in \mathcal{C}_0$, the slice functor takes $c$ to the slice category $\mathcal{C}/c$; for morphisms $f : a \to b \in \mathcal{C}_1$, the slice functor takes $f$ to the functor $\mathcal{C}/f : \mathcal{C}/a \to \mathcal{C}/b$ defined graphically; for every morphism of $\mathcal{C}/a$ (commuting triangle in $\mathcal{C}$ over $a$), contruct the morphism of $\mathcal{C}/b$ (commuting triangle in $\mathcal{C}$ over $b$) as follows:

$$\begin{tikzcd} l \arrow[rr, "m"] \arrow[dr, "x"'] \arrow[ddr, bend right=60, "f \circ_{\mathcal{C}} x"'] & & r \arrow[dl, "x'"] \arrow[ddl, bend left=60, "f \circ_{\mathcal{C}} x'"] \\ & a \arrow[d, "f"] & \\ & b & \end{tikzcd}$$

where the inner triangle is a morphism of $\mathcal{C}/a$ and the outer triangle is a morphism of $\mathcal{C}/b$ given by the functor $\mathcal{C}/f$.

Given a category $\mathcal{C}$ and an object $c \in \mathcal{C}_0$ of $\mathcal{C}$ the *coslice category* (or *under category*) $c/\mathcal{C}$ is the "stuff in $\mathcal{C}$ that is underneath $c$". Specifically, the objects of $c/\mathcal{C}$ are all the morphisms $f \in \mathcal{C}_1$ from $\mathcal{C}$ whose domain is $\mathrm{dom}(f) = c$ (alternatively you could write $(c/\mathcal{C})_0 = \mathrm{Hom}_{\mathcal{C}}(c, -)$). A morphism of $c/\mathcal{C}$ between objects $f : c \to a, g : c \to b \in (c/\mathcal{C})_0$ is a commuting triangle completed by a third morphism $h : a \to b \in \mathcal{C}_1$:

$$\begin{tikzcd} & c \arrow[dl, "g"'] \arrow[dr, "f"] & \\ a \arrow[rr, "h"'] & & b \end{tikzcd}$$

Everything about coslice categories is defined as expected analogously to that of a slice categories. [TODO: determine how the details of the Grothendieck construction transform the slice (pseudo-)functor $\mathcal{C}/(-) : \mathcal{C} \to \mathbf{Cat}$ into the codomain fibration.]

## 2.8 The Pullback and Pushforward Functors

Given a category $\mathcal{C}$ and a morphism $f : a \to b \in \mathcal{C}_1$, the image of $f$ under the slice functor $\mathcal{C}/(-)$ produces a functor $\mathcal{C}/f : \mathcal{C}/a \to \mathcal{C}/b$ between slice categories of $\mathcal{C}$ in the "same direction" as $f$ TODO: confirm that $\mathcal{C}/f$ is the pushforward functor $f_!$ of $f \in \mathcal{C}_1$.

If the given category $\mathcal{C}$ admits pullbacks, in becomes possible to define, for a morphism $f : a \to b$ a pullback functor $f^* : \mathcal{C}/b \to \mathcal{C}/a$. Given a morphism in $\mathcal{C}/b$ (commuting triangle in $\mathcal{C}$ with base at $b$),

$$\begin{tikzcd} c \arrow[rr, "k"] \arrow[dr, "g"'] & & d \arrow[dl, "h"] \\ & b & \end{tikzcd}$$

the pullback functor $f^* : \mathcal{C}/b \to \mathcal{C}/a$ associated with $f$ takes the objects $g : c \to b, h : d \to b$ of $\mathcal{C}/b$ (morphisms in $\mathcal{C}$) completes the pullback squares associated with $f$

$$\begin{tikzcd} & c' \arrow[dl, "f^*g := g_f"'] \arrow[dr, "f_g"] & \\ a \arrow[dr, "f"'] & & c \arrow[dl, "g"] \\ & b & \end{tikzcd} \qquad \begin{tikzcd} & d' \arrow[dl, "f_h"'] \arrow[dr, "f^*h := h_f"] & \\ d \arrow[dr, "h"'] & & a \arrow[dl, "f"] \\ & b & \end{tikzcd}$$

where a subscript notation $g_f$ means "the pullback of $g$ along $f$". Defining the action of $f^* : \mathcal{C}/b \to \mathcal{C}/a$ on objects to be $f^*g = g_f$ and $f^*h = h_f$, the action on morphisms in $\mathcal{C}/b$ is defined by composing the pullback squares with the commuting triangle morphism:

$$\begin{tikzcd} c \arrow[rr, "k"] \arrow[dr, "g"'] & & d \arrow[dl, "h"] \\ & b & \\ & a \arrow[u, "f"'] & \\ c' \arrow[uuu, bend left, "f_g"] \arrow[ur, "f^*g"] \arrow[rr, dashed, "\exists! f^*k"'] & & d' \arrow[uuu, bend right, "f_h"'] \arrow[ul, "f^*h"'] \end{tikzcd}$$

The commuting triangle in $\mathcal{C}/a$ appearing at the bottom is completed by a unique morphism [TODO: why does this morphism need to be unique and exist?] denoted to be $f^*k$ ($\neq k_f$ obviously). The functoriality of $f^*$ has a simple proof found here `https://proofwiki.org/wiki/Pullback_Functor_is_Functor`.

### 2.9 Functors of Monoidal Categories

[TODO]

### 2.10 Frobenius Reciprocity

[TODO]

## Comments on selected references

This section is temporary and reserved for recording comments toward various references.

- Vistoli [Vis04]
- Street [Str74]
- Koudenburg [Kou18]
- Brown and Sivera [BS09]
- Lurie [Lur09]
- Shulman [Shu08]

## References


[BS09] Ronald Brown and Rafael Sivera. "Algebraic colimit calculations in homotopy theory using fibred and cofibred categories". In: *Theory and Applications of Categories* 22.8 (2009), pp. 222–251.

[Kou18] Seerp Roald Koudenburg. "A categorical approach to the maximum theorem". In: *Journal of Pure and Applied Algebra* 222.8 (2018), pp. 2099–2142.

[Lur09] Jacob Lurie. *Higher Topos Theory (AM-170)*. Vol. 189. Princeton University Press, 2009.

[Shu08] Michael Shulman. "Framed bicategories and monoidal fibrations". In: *Theory and applications of categories* 20.18 (2008), pp. 650–738.

[Str74] Ross Street. "Fibrations and Yoneda's lemma in a 2-category". In: *Category seminar*. Springer. 1974, pp. 104–133.

[Vis04] Angelo Vistoli. "Notes on Grothendieck topologies, fibered categories and descent theory". In: *arXiv preprint math/0412512* (2004).