FIBRATIONS AND YONEDA'S LEMMA IN A 2-CATEGORY

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Our purpose is to provide within a 2-category a conceptual proof of a set-free version of the Yoneda lemma using the theory of fibrations. In doing so we carry many definitions of category theory into a 2-category and prove in this more general setting results already familiar for *CAT*.

The La Jolla articles of Lawvere [5] and Gray [2] have strongly influenced this work. Both articles are written in styles which allow easy transfer into a 2-category. However, they also freely use the fact that *CAT* is cartesian closed, a luxury we do not allow ourselves.

The 2-category is required to satisfy an elementary completeness condition amounting to the existence of 2-pullbacks and comma objects. This relates the 2-category closely to a 2-category of category objects in a category. Such considerations appear in §1 and were considered by Gray [3].

Fibrations over B appear in §2 as pseudo algebras for a 2-monad on the 2-category of objects over B. This 2-monad is of a special kind distinguished by Kock [4]. We define lax algebras and lax homomorphisms for general 2-monads and provide alternative descriptions of pseudo algebras and lax homomorphisms for the special 2-monads. We are able then to give an equivalent definition of fibration generalizing the setting for the Chevalley criterion of Gray [2] p 56.

In order to eliminate the need for our 2-category to be cartesian closed in the remainder of our work we are led to introduce an extra variable; we must consider bifibrations from A to B rather than fibrations over B. A particular class of spans from A to B, called covering spans, is introduced in §3. As with their analogue in topology, covering spans are bifibrations. Furthermore, any arrow of spans between covering spans is a homomorphism. In the case of *CAT*, bifibrations correspond to category-valued functors and the last sentence reflects the fact that covering spans correspond to those functors which are discrete-category-valued; that is, set-valued. With this interpretation of covering spans as set-valued functors, we see that Corollary 16 is a generalization of the Yoneda lemma of category theory.

The concept of Kan extension of functors is one of the most fruitful concepts of category theory, and the definition just begs translation into a 2-category. This has already been used to some extent (see [6] and [7]). But the Kan extensions of functors which occur in practice are all pointwise (using the terminology of Dubuc [1]). Using comma objects we define pointwise extensions in a 2-category in §4. Note that, in general, for the 2-category V-Cat, this definition does not agree with Dubuc's; ours is too strong (we hope to remedy this by passing to some related 2-category). For $V=Set$ and $V=\mathbf{2}$, the definitions do agree; for $V=AbGp$ and $V=Cat$, they do not. The closing section gives some applications of the Yoneda lemma and fibration theory to pointwise extensions illustrating their many pleasing properties.

§1. Representable 2-categories.

Let $\mathcal{A}$ denote a category. A *span* from A to B in $\mathcal{A}$ is a diagram (u_0,S,u_1):

$$\begin{tikzcd} & S \arrow[dl, "u_0"'] \arrow[dr, "u_1"] & \\ A & & B \end{tikzcd} .$$

When no confusion is likely, we abbreviate (u_0,S,u_1) to S; then the *reverse span* (u_1,S,u_0) is abbreviated to S^*. Also we identify an arrow $u:A \longrightarrow B$ with the span $(1,A,u)$ from A to B. An arrow of spans $f:(u_0,S,u_1) \longrightarrow (u_0',S',u_1')$ is a commutative diagram

$$\begin{tikzcd} & S \arrow[dl, "u_0"'] \arrow[dr, "u_1"] \arrow[dd, "f"] & \\ A & & B \\ & S' \arrow[ul, "u_0'"] \arrow[ur, "u_1'"'] & \end{tikzcd}$$

Let $SPN(A,B)$ denote the category of spans from A to B and their arrows.

When $\mathcal{A}$ has pullbacks, a span (u_0,S,u_1) from A to B and a span (v_0,T,v_1) from B to C have a *composite span* $(u_0\hat{v}_0,T\circ S,v_1\hat{u}_1)$ from A to C where the following square is a pullback.

$$\begin{tikzcd} T\circ S \arrow[r, "\hat{u}_1"] \arrow[d, "\hat{v}_0"'] & T \arrow[d, "v_0"] \\ S \arrow[r, "u_1"'] & B \end{tikzcd}$$

If $f:S \longrightarrow S'$ is an arrow of spans from A to B and $g:T \longrightarrow T'$ is an arrow of spans from B to C then the arrow $g\circ f:T\circ S \longrightarrow T'\circ S'$ induced on pullbacks is an arrow of spans.

An *opspan* from A to B in $\mathcal{A}$ is a span from A to B in $\mathcal{A}^{op}$; however, arrows of opspans are arrows of diagrams in $\mathcal{A}$.

Suppose $\mathcal{A}$ has pullbacks. A *category object* $\underset{\sim}{A}$ in $\mathcal{A}$ consists of the following data from $\mathcal{A}$:

- an object A_0;
- a span (d_0, A_1, d_1) from A_0 to A_0;
- arrows of spans $i:(1,A_0,1) \longrightarrow (d_0,A_1,d_1)$,

  $c:(d_0\hat{d}_0, A_1 \circ A_1, d_1\hat{d}_1) \longrightarrow (d_0,A_1,d_1)$;

such that the following diagrams commute

$$\begin{tikzcd} A_1 \arrow[r, "1\circ i"] \arrow[dr, "1"'] & A_1\circ A_1 \arrow[d, "c"] & A_1 \arrow[l, "i\circ 1"'] \arrow[dl, "1"] \\ & A_1 & \end{tikzcd} \qquad \begin{tikzcd} A_1\circ A_1\circ A_1 \arrow[r, "1\circ c"] \arrow[d, "c\circ 1"'] & A_1\circ A_1 \arrow[d, "c"] \\ A_1\circ A_1 \arrow[r, "c"'] & A_1 \end{tikzcd} .$$

A *functorial arrow* $\underset{\sim}{f}:\underset{\sim}{A} \longrightarrow \underset{\sim}{B}$ consists of an arrow $f_0:A_0 \longrightarrow B_0$ and an arrow of spans $f_1:(f_0d_0, A_1, f_0d_1) \longrightarrow (d_0,B_1,d_1)$ such that the following commutes

$$\begin{tikzcd} A_0 \arrow[r, "i"] \arrow[d, "f_0"'] & A_1 \arrow[d, "f_1"'] & A_1\circ A_1 \arrow[l, "c"'] \arrow[d, "f_1\circ f_1"] \\ B_0 \arrow[r, "i"'] & B_1 & B_1\circ B_1 \arrow[l, "c"] \end{tikzcd} .$$

If $\underset{\sim}{f}, \underset{\sim}{f}':\underset{\sim}{A} \longrightarrow \underset{\sim}{B}$ are functorial arrows, a *transformation* from $\underset{\sim}{f}$ to $\underset{\sim}{f}'$ is an arrow of spans $n:(f_0,A_0,f_0') \longrightarrow (d_0,B_1,d_1)$ such that the following diagram commutes

$$\begin{tikzcd} A_1 \arrow[r, "(nd_1)\circ f_1"] \arrow[d, "f_1'\circ(nd_0)"'] & B_1\circ B_1 \arrow[d, "c"] \\ B_1\circ B_1 \arrow[r, "c"'] & B_1 \end{tikzcd} .$$

With the natural compositions we obtain a 2-category $CAT(A)$ of category objects in A.

A category object $\underset{\sim}{A}$ in A is determined up to isomorphism by the contravariant category-valued functor on A which assigns to each object X of A the category whose source and target functions are $A(X,d_0)$, $A(X,d_1)$: $A(A,A_1) \longrightarrow A(X,A_0)$ and whose identities and composition are determined by the functions $A(X,i)$, $A(X,c)$. Indeed, we have described the object function of a 2-fully-faithful 2-functor

$$CAT(A) \longrightarrow [A^{op}, CAT] \ .$$

Henceforth we work in a 2-category K. By "span" we shall mean "span in the category K_0".

A *comma object* for the opspan (r,D,s) from A to B is a span $(d_0, r/s, d_1)$ from A to B together with a 2-cell

$$\begin{tikzcd} r/s \arrow[r, "d_1"] \arrow[d, "d_0"'] \arrow[dr, phantom, "\xrightarrow{\lambda}"] & B \arrow[d, "s"] \\ A \arrow[r, "r"'] & D \end{tikzcd}$$

satisfying the following two conditions

- for any span (u_0, S, u_1) from A to B, composition with λ yields a bijection

$$\begin{tikzcd} & S \arrow[dl, "u_0"'] \arrow[dr, "u_1"] \arrow[dd, "f"] & \\ A & & B \\ & r/s \arrow[ul, "d_0"] \arrow[ur, "d_1"'] & \end{tikzcd} \quad \longleftrightarrow \quad \begin{tikzcd} S \arrow[r, "u_1"] \arrow[d, "u_0"'] \arrow[dr, phantom, "\xrightarrow{\sigma}"] & B \arrow[d, "s"] \\ A \arrow[r, "r"'] & D \end{tikzcd}$$

between arrows of spans f and 2-cells σ;

- given 2-cells ξ, η such that the two composites

$$\begin{tikzcd} & r/s \arrow[rr, "d_0"] \arrow[d, phantom, "\Downarrow\xi"] & & A \arrow[rr, "r"] \arrow[d, phantom, "\Downarrow\lambda"] & & D \\ S \arrow[ur, "f"] \arrow[rr, "f'"'] & {} & r/s \arrow[ur, "d_0"'] \arrow[rr, "d_1"'] & {} & B \arrow[ur, "s"'] & \end{tikzcd} \qquad \begin{tikzcd} S \arrow[rr, "f"] \arrow[dr, "f'"'] & & r/s \arrow[rr, "d_0"] \arrow[dr, "d_1"] & & A \arrow[dr, "r"] & \\ & r/s \arrow[rr, "d_1"'] \arrow[u, phantom, "\Downarrow\eta"] & & B \arrow[rr, "s"'] \arrow[u, phantom, "\Downarrow\lambda"] & & D \end{tikzcd}$$

are equal, then there exists a unique 2-cell $\begin{tikzcd} S \arrow[r, bend left, "f"] \arrow[r, bend right, "f'"'] \arrow[r, phantom, "\Downarrow\phi"] & r/s \end{tikzcd}$ such that

$\xi = d_0\phi$, $\eta = d_1\phi$.

In non-elementary terms, r/s is defined by a 2-natural isomorphism

$$K(S, r/s) \cong K(S,r)/K(S,s),$$

where the expression on the right hand side is the usual comma category of the functors $K(S,r)$, $K(S,s)$.

The comma object of the identity opspan $(1,A,1)$ from A to A is denoted by ΦA. It is defined by a 2-natural isomorphism

$$K(S,\Phi A) \cong K(S,A)^{\mathbf{2}} \quad ,$$

and so is the cotensor in K of the category $\mathbf{2}$ with the object A. When ΦA exists for each object A and when K has 2-pullbacks we say that K is a *representable 2-category* (Gray [3] uses "strongly representable").

Example. If A has pullbacks then $K = CAT(A)$ is a representable 2-category. //

Opcomma objects in K are comma objects in K^{op}. In a 2-category which is both representable and oprepresentable, Φ has a left 2-adjoint Ψ and any limit which exists in K_0 is automatically a 2-limit in K.

Proposition 1. In a representable 2-category each opspan has a comma object.

Proof. The formula is $r/s = s^* \circ \Phi D \circ r$. //

In a representable 2-category, an identity 2-cell $\begin{tikzcd} A \arrow[r, bend left, "1"] \arrow[r, bend right, "1"'] \arrow[r, phantom, "\Downarrow 1"] & A \end{tikzcd}$ corresponds to an arrow $i : A \longrightarrow \Phi A$, and the composite 2-cell

$$\begin{tikzcd} & \Phi A \circ \Phi A \arrow[dl, "\hat{d}_0"'] \arrow[dr, "\hat{d}_1"] & \\ \Phi A \arrow[dr, "d_1"] \arrow[dr, bend right=50, "d_0"'] \arrow[dr, phantom, bend right=20, "\xrightarrow{\lambda}"] & & \Phi A \arrow[dl, "d_0"'] \arrow[dl, bend left=50, "d_1"] \arrow[dl, phantom, bend left=20, "\xrightarrow{\lambda}"] \\ & A & \end{tikzcd}$$

corresponds to an arrow $\Phi A \circ \Phi A \xrightarrow{c} \Phi A$. For each arrow $f : A \longrightarrow B$, the 2-cell $\begin{tikzcd} \Phi A \arrow[r, bend left, "d_0"] \arrow[r, bend right, "d_1"'] \arrow[r, phantom, "\Downarrow\lambda"] & A \arrow[r, "f"] & B \end{tikzcd}$ corresponds to an arrow $\Phi f : \Phi A \longrightarrow \Phi B$.

Proposition 2. In a representable 2-category the following results hold.

(a) For each object A, *the arrows* i, c *enrich* $d_0, d_1 : \Phi A \longrightarrow A$ *to a category object* $\underset{\sim}{A}$ *in* K_0.

(b) For each arrow $f : A \longrightarrow B$, *the pair of arrows* $f, \Phi f$ *constitute a functorial arrow* $\underset{\sim}{f} : \underset{\sim}{A} \longrightarrow \underset{\sim}{B}$.

(c) For each 2-cell $\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "f'"'] \arrow[r, phantom, "\sigma\Downarrow"] & B \end{tikzcd}$, *the corresponding arrow* $\underset{\sim}{\sigma} : A \longrightarrow \Phi B$ *is a transformation from* $\underset{\sim}{f}$ *to* $\underset{\sim}{f}'$.

(d) The assignment

$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "f'"'] \arrow[r, phantom, "\Downarrow\sigma"] & B \end{tikzcd} \longmapsto \begin{tikzcd} \underset{\sim}{A} \arrow[r, bend left, "\underset{\sim}{f}"] \arrow[r, bend right, "\underset{\sim}{f'}"'] \arrow[r, phantom, "\Downarrow\underset{\sim}{\sigma}"] & \underset{\sim}{B} \end{tikzcd}$$

defines a 2-functor from K *to* $CAT(K_0)$.

Proof. (a) For each object X, $|K(X,A)^2| \rightrightarrows |K(X,A)|$ are the source and target functions for the category $K(X,A)$; so $K_0(X,\Phi A) \rightrightarrows K_0(X,A)$ are the source and target functions for a category, functorially in X. So $\Phi A \rightrightarrows A$ carries the structure of a category object in K_0. It is readily checked that this structure agrees with that of the proposition.

(b) For each X, $(K_0(X,f), K_0(X,\Phi f))$ corresponds to the functor $K(X,f)\colon K(X,A) \longrightarrow K(X,B)$.

(c) Similarly, $K_0(X,\underset{\sim}{\sigma})$ corresponds to the natural transformation $K(X,\sigma)\colon K(X,f) \longrightarrow K(X,f')$.

(d) What we have shown is that the composite

$$K \longrightarrow CAT(K_0) \longrightarrow [K_0{}^{op}, CAT]$$

is the Yoneda embedding, a well-known 2-functor. It follows that the first arrow is a 2-functor.//

§2. Lax algebras and fibrations

Suppose D is a 2-monad on a 2-category C and let $i\colon 1 \longrightarrow D$, $c\colon DD \longrightarrow D$ denote the unit and multiplication. A *lax* D-*algebra* consists of an object E, an arrow $c\colon DE \longrightarrow E$ and 2-cells

$$\begin{tikzcd} E \arrow[d, "i_E"'] \arrow[dr, "1"] & \\ DE \arrow[r, "c"'] \arrow[ur, phantom, "\zeta"] & E \end{tikzcd} \qquad \begin{tikzcd} D^2E \arrow[r, "c_E"] \arrow[d, "Dc"'] \arrow[dr, phantom, "\theta"] & DE \arrow[d, "c"] \\ DE \arrow[r, "c"'] & E \end{tikzcd}$$

in the 2-category C such that the composites

$$
\begin{tikzcd}
DE \arrow[rr, "1"] \arrow[ddd, "c"'] \arrow[dr, "i_{DE}"'] & & DE \arrow[ddd, "c"] \\
& D^2E \arrow[ur, "c_E"'] \arrow[d, "Dc"] & \\
& DE \arrow[ddr, "c"] \arrow[r, phantom, "\theta"] & {} \\
E \arrow[ur, "i_E"] \arrow[rr, "1"'] \arrow[r, phantom, "\zeta"] & {} & E
\end{tikzcd}
\overset{(1)}{=}
\begin{tikzcd}
DE \arrow[r, bend left, "c"] \arrow[r, bend right, "c"'] \arrow[r, phantom, "\Downarrow 1"] & E
\end{tikzcd}
\overset{(2)}{=}
\begin{tikzcd}
DE \arrow[drr, "1"] \arrow[dr, "Di_E"'] \arrow[dd, "1"'] & & \\
& D^2E \arrow[r, "c_E"] \arrow[dl, "Dc"] \arrow[l, phantom, "D\zeta"] \arrow[dr, phantom, "\theta"] & DE \arrow[d, "c"] \\
DE \arrow[rr, "c"'] & & E
\end{tikzcd}
$$

$$
(3)\quad
\begin{tikzcd}
D^3E \arrow[rr, "c_{DE}"] \arrow[dr, "Dc_E"] \arrow[dd, "D^2c"'] & & D^2E \arrow[dr, "c_E"] & \\
& D^2E \arrow[rr, "c_E"'] \arrow[dd, "Dc"] & & DE \arrow[dd, "c"] \\
D^2E \arrow[dr, "Dc"'] \arrow[ur, phantom, "D\theta"] & & {} \arrow[d, phantom, "\theta"] & \\
& DE \arrow[rr, "c"'] & {} & E
\end{tikzcd}
=
\begin{tikzcd}
D^3E \arrow[rr, "c_{DE}"] \arrow[dd, "D^2c"'] & & D^2E \arrow[dr] \arrow[dd, "Dc"'] & \\
& & & DE \arrow[dd, "c"] \\
D^2E \arrow[rr, "c_E"] \arrow[dr, "Dc"'] & & DE \arrow[dr, "c"] \arrow[ur, phantom, "\theta"] & \\
& DE \arrow[rr, "c"'] \arrow[ur, phantom, "\theta"] & & E
\end{tikzcd}
$$

are equal as indicated. A *pseudo* D-*algebra* is a lax D-algebra in which ζ, θ are isomorphisms. A *normalized* lax D-algebra has ζ an identity 2-cell. A D-*algebra* is a lax D-algebra with both ζ, θ identities. Of course, for any E in C, DE with $c_E : D^2E \longrightarrow DE$ is the free D-algebra on E.

Kock [4] has distinguished those 2-monads D with the property that $c \dashv iD$ in the 2-functor 2-category $[C,C]$ with identity counit. Then the identity modification $\begin{tikzcd} D \arrow[r, bend left, "c.Di"] \arrow[r, bend right, "1"'] \arrow[r, phantom, "\Downarrow 1"] & D \end{tikzcd}$ corresponds under the adjunction to a modification $\begin{tikzcd} D \arrow[r, bend left, "Di"] \arrow[r, bend right, "iD"'] \arrow[r, phantom, "\Downarrow \iota"] & D^2 \end{tikzcd}$. Suppose E is a lax D-algebra such that ζ is an isomorphism with inverse $\bar{\zeta}$, and consider the composite

$$
\begin{tikzcd}
DE \arrow[r, bend left, "Di_E"] \arrow[r, bend right, "i_{DE}"'] \arrow[r, phantom, "\Downarrow \iota_E"] & D^2E \arrow[r, bend left, "c.Dc"] \arrow[r, bend right, "cc_E"'] \arrow[r, phantom, "\Downarrow \theta"] & E .
\end{tikzcd}
$$

On the one hand, $\theta\iota_E = (cc_E\iota_E)(\theta.Di_E) = \theta.Di_E = c.D\bar{\zeta}.$

On the other hand, $\theta\iota_E = (\theta i_{DE})(c.Dc.\iota_E) = (\bar{\zeta}c)(c.Dc.\iota_E).$

So we have the equality

$$(4)\qquad \begin{tikzcd} DE \arrow[r, bend left, "Di_E"] \arrow[r, bend right, "i_{DE}"'] \arrow[r, phantom, "\Downarrow\iota_E"] \arrow[d, "c"'] & D^2E \arrow[r, "Dc"] & DE \arrow[r, "c"] \arrow[d, phantom, "\Downarrow\bar{\zeta}"] & E \\ E \arrow[urr, "i_E"] \arrow[urrr, "1"'] & & & \end{tikzcd} \;=\; \begin{tikzcd} DE \arrow[r, bend left, "Dc.Di_E"] \arrow[r, bend right, "1"'] \arrow[r, phantom, "\Downarrow D\bar{\zeta}"] & DE \arrow[r, "c"] & E \end{tikzcd}$$

The next proposition generalizes slightly some of Kock's results; he considers the normalized case.

<u>*Proposition 3*</u>. *Suppose* D *is a 2-monad with the Kock property and suppose the 2-cell*

$$\begin{tikzcd} E \arrow[d, "i_E"'] \arrow[dr, "1"] & \\ DE \arrow[r, "c"'] \arrow[ur, phantom, "\zeta\swarrow"] & E \end{tikzcd}$$

is an isomorphism with inverse $\bar{\zeta}$ *satisfying equality (4). Then:*

(a) $\bar{\zeta}$ *is the counit for an adjunction* $c \dashv i_E$ *with unit given by the composite*

$$\begin{tikzcd} DE \arrow[r, bend left, "Di_E"] \arrow[r, bend right, "i_{DE}"'] \arrow[r, phantom, "\Downarrow\iota_E"] \arrow[d, "c"'] \arrow[rr, bend left=60, "1"] & D^2E \arrow[r, "Dc"'] \arrow[r, phantom, bend left=30, "\Downarrow D\zeta"] & DE \\ E \arrow[urr, "i_E"'] & & \end{tikzcd}$$

(b) the 2-cell $\begin{tikzcd} D^2E \arrow[r, bend left, "c.Dc"] \arrow[r, bend right, "c.c_E"'] \arrow[r, phantom, "\Downarrow\theta"] & E \end{tikzcd}$ *corresponding under adjunction to the identity 2-cell* $\begin{tikzcd} E \arrow[r, bend left, "i_{DE}.i_E"] \arrow[r, bend right, "Di_E.i_E"'] \arrow[r, phantom, "1\Downarrow"] & D^2E \end{tikzcd}$ *is unique with the property that the equality* (1) *holds;*

(c) this 2-cell θ *enriches* E, c, ζ *with the structure of pseudo* D*-algebra.*

Proof. (a) Let π denote the composite 2-cell displayed in *(a)*. Equality (4) gives $\bar{\zeta}c.c\pi = 1$. Since the composite $\iota_E i_E$ is the identity, we also have $i_E\bar{\zeta}.\pi i_E = 1$.

(b) Let τ denote the composite

$$\begin{tikzcd}
DE \arrow[r, "1"] \arrow[dd, "c"'] \arrow[ddr, phantom, "\Downarrow\pi"] & DE \arrow[r, "i_{DE}"] & D^2E \arrow[r, "Dc"] \arrow[d, phantom, "\Downarrow D\bar{\zeta}"] & DE \arrow[r, "c"] \arrow[dd, phantom, "\Downarrow\bar{\zeta}"] & E \\
& & DE \arrow[u, "Di_E"] \arrow[ur, "1"'] & & \\
E \arrow[uur, "i_E"'] \arrow[urr, "i_E"'] \arrow[rrrruu, "1"'] & & & &
\end{tikzcd}$$

Then the 2-cell θ described in *(b)* is the composite

$$\begin{tikzcd}
D^2E \arrow[r, bend left, "Di_{DE}"] \arrow[r, bend right, "i_{D^2E}"'] \arrow[r, phantom, "\Downarrow\iota_{DE}"] \arrow[d, "c_E"'] & D^3E \arrow[r, "Dc_E"] & D^2E \arrow[r, "Dc"] \arrow[d, phantom, "\Downarrow\tau"] & DE \arrow[r, "c"] & E \\
DE \arrow[urr, "i_{DE}"'] \arrow[urrrr, "c"'] & & & &
\end{tikzcd}$$

The 2-naturality of $i : 1 \longrightarrow D$ implies the equality $i_E\bar{\zeta}c = D\bar{\zeta}.Dc.i_{DE}$, from which it easily follows that $\tau = \bar{\zeta}c$. Using this and the equations $c_E i_{DE} = 1$, $\iota_{DE} i_{DE} = 1$, we deduce the equality (1).

To prove uniqueness, suppose θ satisfies (1). The 2-cell corresponding to θ under adjunction is the composite

$$i_{DE}i_E \xrightarrow{(\mathrm{unit})i_{DE}i_E} Di_E.i_Ec.Dc.i_{DE}i_E \xrightarrow{Di_E.i_E\theta i_{DE}i_E} Di_E.i_Ecc_Ei_{DE}i_E$$

$$\xrightarrow{Di_E.i_E(\mathrm{counit})} Di_E.i_E.$$

So (1) implies that this composite is independent of θ. For one such θ the composite is the identity, so the composite is the identity for all such θ. So θ is unique.

(c) Clearly θ is an isomorphism, so it remains to show that θ satisfies (2) and (3). Equality (2) follows from the equations $c_E.Di_E = 1$, $c.Dc.Dc_E.\iota_{DE}.Di_E = c.Dc.Dc_E.Di_{DE}.\iota_E = c.Dc.\iota_E$, $\tau = \bar{\zeta}c$ and (4). By the naturality of "replacing arrows by their right adjoints", equality (3) holds since identity 2-cells appear in the squares of the transformed equality.//

A *lax homomorphism* of lax D-algebras from E to E' consists of an arrow $f:E \longrightarrow E'$ and a 2-cell

$$\begin{tikzcd} DE \arrow[r, "c"] \arrow[d, "Df"'] & E \arrow[d, "f"] \\ DE' \arrow[r, "c"'] \arrow[ur, phantom, "\theta_f"] & E' \end{tikzcd}$$

in C such that the composites

(5)
$$\begin{tikzcd} & DE \arrow[r, "c"] \arrow[dd, "Df"'] & E \arrow[dd, "f"] \\ E \arrow[ur, "i_E"] \arrow[dd, "f"'] & & \\ & DE' \arrow[r, "c"] \arrow[ur, phantom, "\theta_f"] & E' \\ E' \arrow[ur, "i_{E'}"] \arrow[urr, "1"'] & {} \arrow[u, phantom, "\zeta"] & \end{tikzcd} = \begin{tikzcd} & DE \arrow[dr, "c"] & & \\ E \arrow[ur, "i_E"] \arrow[rr, "1"'] & {} \arrow[u, phantom, "\zeta"] & E \arrow[r, "f"] & E' \end{tikzcd}$$

(6)
$$\begin{tikzcd} D^2E \arrow[rr, "c_E"] \arrow[dr, "Dc"] \arrow[dd, "D^2f"'] & & DE \arrow[dr, "c"] & \\ & DE \arrow[rr, "c"'] \arrow[dd, "Df"] \arrow[ur, phantom, "\theta"] & & E \arrow[dd, "f"] \\ D^2E' \arrow[dr, "Dc"'] \arrow[ur, phantom, "D\theta_f"] & & & \\ & DE' \arrow[rr, "c"'] & {} \arrow[uur, phantom, "\theta_f"] & E' \end{tikzcd} = \begin{tikzcd} D^2E \arrow[r, "c_E"] \arrow[d, "D^2f"'] & DE \arrow[d, "Df"] \arrow[dr, "c"] & \\ D^2E' \arrow[r, "c_{E'}"'] \arrow[dr, "Dc"'] & DE' \arrow[dr, "c"] \arrow[r, phantom, "\theta_f"] & E \arrow[d, "f"] \\ & DE' \arrow[r, "c"'] \arrow[u, phantom, "\theta"] & E' \end{tikzcd}$$

are equal as indicated. A lax homomorphism f is called a *pseudo homomorphism* when θ_f is an isomorphism, and is called a *homomorphism* when θ_f is an identity.

Proposition 4. *Suppose* D *is a 2-monad with the Kock property and suppose* $f:E \longrightarrow E'$ *is an arrow between pseudo* D*-algebras. Then the 2-cell* $\theta_f:c.Df \longrightarrow fc$ *which corresponds under adjunction to the identity 2-cell* $1:Df.i_E \longrightarrow i_{E'}.f$ *is unique with the property that equality* (5) *holds. Furthermore, this* θ_f *enriches* f *with the structure of lax homomorphism.*

Proof. Suppose θ_f is as explained in the proposition. Equality (5) holds since both the 2-cells $ci_{E'}f \longrightarrow f$ correspond to the identity 2-cell $i_{E'}f \xrightarrow{1} i_{E'}f$ under adjunction (recall that $\bar{\zeta}$ is the counit for $c \dashv i_{E'}$). On the other hand, suppose θ_f satisfies (5). Then θ_f corresponds under adjunction to the composite

$$Df.i_E \xrightarrow{\pi.Df.i_E} i_{E'}c.Df.i_E \xrightarrow{i_{E'}\theta_f i_E} i_{E'}fci_E \xrightarrow{i_{E'}f\bar{\zeta}} i_{E'}f \quad ,$$

which is independent of θ_f by (5); so θ_f is unique. Finally, θ_f satisfies (6) since both the 2-cells $c.Dc.D^2f \longrightarrow fcc_E$ correspond under adjunction to the identity 2-cell $D^2f.i_{DE'}i_E = i_{DE'}.i_{E'}f \xrightarrow{1} Di_{E'}.i_{E'}f$ (recall that $\theta:c.Dc \longrightarrow cc_E$ corresponds to $1:i_{DE}i_E \longrightarrow Di_E.i_E$). //

For convenience we henceforth work in a representable 2-category *K*.

Proposition 5. *Suppose* $f:A \longrightarrow B$ *is an arrow with a right adjoint* u, *counit* ε *and unit* η. *For any arrow* $g:C \longrightarrow B$, *the arrow* $v:C \longrightarrow f/g$ *corresponding to the 2-cell* εg *is a right adjoint for* $d_1:f/g \longrightarrow C$ *with counit the identity and unit* $\beta:1 \longrightarrow vd_1$ *defined by the equations*

$$d_0\beta = u\lambda.\eta d_0 \quad , \quad d_1\beta = 1.$$

Proof. Using $\varepsilon f.f\eta = 1$, we see that the two composite 2-cells

$$\begin{tikzcd} & A \arrow[rr, "f"] \arrow[rrrr, bend left, "1"] \arrow[rrrr, phantom, bend left=20, "\Downarrow\eta"] && B \arrow[rr, "u"] \arrow[dl, phantom, "\Downarrow\lambda"] && A \arrow[rr, "f"] && B \arrow[dl, phantom, "\Downarrow\lambda"] \\ f/g \arrow[ur, "d_0"] \arrow[rr, "d_1"'] \arrow[rrrr, bend right, "1"'] && C \arrow[ur, "g"'] \arrow[rr, "v"] && f/g \arrow[ur, "d_0"] \arrow[rr, "d_1"'] && C \arrow[ur, "g"'] \end{tikzcd} \qquad \begin{tikzcd} f/g \arrow[rr, "1"] \arrow[dr, "1"'] && f/g \arrow[rr, "d_0"] \arrow[dr, "d_1"] && A \arrow[dr, "f"] \arrow[dl, phantom, "\Downarrow\lambda"] \\ & f/g \arrow[rr, "d_1"'] && C \arrow[rr, "g"'] && B \end{tikzcd}$$

are equal; so there exists a unique 2-cell β as asserted. Using $u\varepsilon.\eta u = 1$, we also see that $\beta v = 1$. So β is a unit for $d_1 \dashv v$ with identity counit.//

Corollary 6. For any arrow $p:E \longrightarrow B$, *the arrow* $d_0:p/B \longrightarrow E$ *has a left adjoint* i_p *with unit the identity. Explicitly,* i_p *is the unique arrow whose composite with* λ *is the identity 2-cell* $\begin{tikzcd} E \arrow[r, bend left, "p"{name=U}] \arrow[r, bend right, "p"'{name=D}] & B \arrow[from=U, to=D, Rightarrow, "1"] \end{tikzcd}$.

Proof. Since $1:B \longrightarrow B$ has a left adjoint, a dual of the proposition yields the result.//

Corollary 7. An arrow $f:A \longrightarrow B$ *has a right adjoint if and only if the arrow* $d_1:f/B \longrightarrow B$ *has a right adjoint. In this case there is a right adjoint for* d_1 *with counit the identity.*

Proof. If $d_1 \dashv v$ we can compose with $i_f \dashv d_0$ of Corollary 6 to obtain $f = d_1 i_f \dashv d_0 v$. The converse and the last sentence follow directly from Proposition 5.//

Corollary 4 applied to $p = 1_B$ gives i as left adjoint for $d_0:\Phi B \longrightarrow B$. The unit of this adjunction is the identity and the counit $\begin{tikzcd} \Phi B \arrow[r, bend left, "id_0"{name=U}] \arrow[r, bend right, "1"'{name=D}] & \Phi B \arrow[from=U, to=D, Rightarrow, "\iota_0"] \end{tikzcd}$ is the 2-cell defined by the equations $d_0\iota_0 = 1$, $d_1\iota_0 = \lambda$. Dually, $d_1:\Phi B \longrightarrow B$ has i as right adjoint with counit the identity and unit $\begin{tikzcd} \Phi B \arrow[r, bend left, "1"{name=U}] \arrow[r, bend right, "id_1"'{name=D}] & \Phi B \arrow[from=U, to=D, Rightarrow, "\iota_1"] \end{tikzcd}$ defined by $d_0\iota_1 = \lambda$, $d_1\iota_1 = 1$. Using the 2-pullback property of the square

$$\begin{tikzcd} \Phi B \circ \Phi B \arrow[r, "\hat{d}_1"] \arrow[d, "\hat{d}_0"'] & \Phi B \arrow[d, "d_0"] \\ \Phi B \arrow[r, "d_1"'] & B \end{tikzcd},$$

we see that $d_1\iota_0 = 1 = d_0\iota_1$ imply the existence of a unique 2-cell
$\begin{tikzcd} \Phi B \arrow[r, bend left, "1\circ i"] \arrow[r, bend right, "i\circ 1"'] \arrow[r, phantom, "\Downarrow\iota"] & \Phi B\circ\Phi B \end{tikzcd}$
such that $\hat{d}_0\iota = \iota_0$, $\hat{d}_1\iota = \iota_1$.

Proposition 8. *(a) The composite 2-cell*

$$\begin{tikzcd} \Phi B \arrow[r, bend left, "1\circ i"] \arrow[r, bend right, "i\circ 1"'] \arrow[r, phantom, "\Downarrow\iota"] & \Phi B\circ\Phi B \arrow[r, "c"] & \Phi B \end{tikzcd}$$

is the identity 2-cell $\begin{tikzcd} \Phi B \arrow[r, bend left, "1"] \arrow[r, bend right, "1"'] \arrow[r, phantom, "\Downarrow 1"] & \Phi B \end{tikzcd}$.

(b) The composite 2-cell

$$\begin{tikzcd} B \arrow[r, "i"] & \Phi B \arrow[r, bend left, "1\circ i"] \arrow[r, bend right, "i\circ 1"'] \arrow[r, phantom, "\Downarrow\iota"] & \Phi B\circ\Phi B \end{tikzcd}$$

is the identity 2-cell $\begin{tikzcd} B \arrow[r, bend left, "i\circ i"] \arrow[r, bend right, "i\circ i"'] \arrow[r, phantom, "\Downarrow 1"] & \Phi B\circ\Phi B \end{tikzcd}$.

(c) The arrow $c:\Phi B\circ\Phi B \longrightarrow \Phi B$ *is left adjoint to* $i\circ 1$ *with counit the identity and with unit given by the composite*

$$\begin{tikzcd} \Phi B\circ\Phi B \arrow[r, bend left, "1\circ i\circ 1"] \arrow[r, bend right, "i\circ 1\circ 1"'] \arrow[r, phantom, "\iota\circ 1\Downarrow"] \arrow[rr, bend left=50, "1"] \arrow[dr, "c"'] & \Phi B\circ\Phi B\circ\Phi B \arrow[r, "1\circ c"] & \Phi B\circ\Phi B \\ & \Phi B \arrow[ur, "i\circ 1"'] & \end{tikzcd}$$

Proof. (a) This follows from the calculations

$$d_0c\iota = d_0\hat{d}_0\iota = d_0\iota_0 = 1, \quad d_1c\iota = d_1\hat{d}_1\iota = d_1\iota_1 = 1.$$

(b) This follows from the calculations

$$\hat{d}_0\iota i = \iota_0 i = 1_i \ , \quad \hat{d}_1\iota i = \iota_1 i = 1_i .$$

(c) Using (a), we have

$$c(1\circ c)(\iota\circ 1) = c(c\circ 1)(\iota\circ 1) = c(c\iota\circ 1) = 1_c ;$$

and using (b), we have

$$(1\circ c)(\iota\circ 1)(i\circ 1) = (1\circ c)(\iota i\circ 1) = (1\circ c)1_{i\circ i\circ 1} = 1_{i\circ c(i\circ 1)} = 1_i .$$

So $c \dashv i\circ 1$ with counit and unit as stated.//

Let K_B denote the comma object of the opspan

$$\begin{tikzcd} K \arrow[dr, "1"'] & & 1 \arrow[dl, "B"] \\ & K & \end{tikzcd}$$

in 2-CAT. So K_B is the 2-category whose objects are pairs (E,p) where $p:E \longrightarrow B$ is an arrow in K, whose arrows $f:(E,p) \longrightarrow (E',p')$ are arrows $f:E \longrightarrow E'$ in K such that $p'f = p$, and whose 2-cells $\begin{tikzcd}(E,p) \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\sigma"] & (E',p')\end{tikzcd}$ are 2-cells $\begin{tikzcd}E \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\sigma"] & E'\end{tikzcd}$ in K such that $p'\sigma = 1_p$. We often write E for (E,p).

Let $L:K_B \longrightarrow K_B$ denote the 2-functor given by:

$$\begin{tikzcd}(E,p) \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\sigma"] & (E',p')\end{tikzcd} \longmapsto \begin{tikzcd}(p/B,d_1) \arrow[r, bend left, "f/B"] \arrow[r, bend right, "g/B"'] \arrow[r, phantom, "\Downarrow\sigma/B"] & (p'/B,d_1)\end{tikzcd};$$

or, in other words, $L(E,p) = (\Phi B \circ p, d_1\hat{p})$, $Lf = 1 \circ f$, $L\sigma = 1 \circ \sigma$. Let $i:1 \longrightarrow L$, $c:L^2 \longrightarrow L$ denote the 2-natural transformations with (E,p)-components

$$E \xrightarrow{i \circ 1} \Phi B \circ p \ , \quad \Phi B \circ \Phi B \circ E \xrightarrow{c \circ 1} \Phi B \circ E \ .$$

The diagrams which say that $\underset{\sim}{B}$ (see Proposition 2(a)) is a category object precisely say that L is a 2-monad on K_B with unit i and multiplication c. Moreover, Proposition 8 shows that L has the Kock property so that Propositions 3 and 4 apply.

An arrow $p:E \longrightarrow B$ is called a *0-fibration over* B when (E,p) supports the structure of pseudo L-algebra. The 0-fibration is called *split* when (E,p) supports the structure of an L-algebra.

Proposition 9. (Chevalley criterion). The arrow $p:E \longrightarrow B$ *is a 0-fibration over* B *if and only if the arrow* $\bar{p}:\Phi E \longrightarrow p/B$ *corresponding to the 2-cell*

$$\begin{tikzcd} \Phi E \arrow[r, "pd_1"] \arrow[d, "d_0"'] \arrow[dr, phantom, "p\lambda"] & B \arrow[d, "1"] \\ E \arrow[r, "p"'] & B \end{tikzcd}$$

has a left adjoint with unit an isomorphism.

Proof. Suppose (E,p) is a pseudo L-algebra. The counit of Corollary 6 is readily seen to be

$$p/B = \Phi B\circ E \begin{tikzcd} {} \arrow[r, bend left, "(id_0)\circ 1"] \arrow[r, bend right, "1"'] \arrow[r, phantom, "\Downarrow\imath_0\circ 1"] & {} \end{tikzcd} \Phi B\circ E = p/B \ ;$$

this 2-cell corresponds to an arrow $k\colon p/B \longrightarrow \Phi(p/B)$. Let ℓ be the composite $p/B \xrightarrow{k} \Phi(p/B) \xrightarrow{\Phi c} \Phi E$. One readily verifies that $\bar{p}\ell = L(c\imath_E)$. Let $p/B \begin{tikzcd} {} \arrow[r, bend left, "1"] \arrow[r, bend right, "\bar{p}\ell"'] \arrow[r, phantom, "\Downarrow\eta"] & {} \end{tikzcd} p/B$ denote the 2-cell $L\zeta$; it is an isomorphism. Let $\Phi E \begin{tikzcd} {} \arrow[r, bend left, "\ell\bar{p}"] \arrow[r, bend right, "1"'] \arrow[r, phantom, "\Downarrow\varepsilon"] & {} \end{tikzcd} \Phi E$ denote the unique 2-cell satisfying $d_0\varepsilon = \bar{\zeta}d_0$, $d_1\varepsilon = (\bar{\zeta}d_1)(c\bar{p}\imath_1)$. (Note that $\bar{p}i = i_E$). By applying d_0, d_1 to $\bar{p}\varepsilon.\eta\bar{p}$ it is readily seen that $\bar{p}\varepsilon.\eta\bar{p} = 1$. Also $d_0(\varepsilon\ell.\ell\eta) = 1$ is immediate. To complete the proof that $\ell \dashv \bar{p}$ with counit ε and unit η, we must show that $d_1(\varepsilon\ell.\ell\eta) = 1$. But $d_1(\varepsilon\ell.\ell\eta) = (\bar{\zeta}d_1\ell)(c\bar{p}\imath_1\ell)(cL\zeta)$.

From the calculations

$d_0\bar{p}\imath_1\ell = d_0\imath_1\ell = \lambda\ell = \lambda.\Phi c.k = c\lambda k = c(\imath_0\circ 1) = c(\hat{d}_0\circ 1)(\imath\circ 1) = d_0(1\circ c)(\imath\circ 1)$,

$d_1\bar{p}\imath_1\ell = \Phi p.d_1\imath_1\ell = 1_{\Phi(pc)k} = 1_{d_1k} = 1_{d_1} = d_1(\imath_1\circ 1) = d_1(\hat{d}_1\circ 1)(\imath\circ 1) = d_1(1\circ c)(\imath\circ 1)$,

we deduce that $\bar{p}\imath_1\ell = (1\circ c)(\imath\circ 1) = Lc.\imath_E$. So, by condition (4), we have $d_1(\varepsilon\ell.\ell\eta) = (\bar{\zeta}c)(c.Lc.\imath_E)(cL\zeta) = 1$.

Conversely, suppose $\ell \dashv \bar{p}$ with counit ε and isomorphism unit η. Since $\ell \dashv \bar{p}$ with counit ε and $d_1 \dashv i$ with counit 1, we have $d_1\ell \dashv \bar{p}i = i_E$ with counit $d_1\varepsilon i\colon d_1\ell pi \longrightarrow d_1 i = 1$. So put c equal to the composite $p/B \xrightarrow{\ell} \Phi E \xrightarrow{d_1} E$ and $\bar{\zeta} = d_1\varepsilon i$. It is readily checked that the composite

$$\begin{tikzcd}
& & p/B \arrow[dr, "\ell"] \arrow[d, phantom, "\Downarrow\varepsilon"] & \\
E \arrow[r, "i"] & \Phi E \arrow[ur, "\bar{p}"] \arrow[rr, bend right, "1"'] & {} & \Phi E
\end{tikzcd}$$

is an isomorphism with inverse the composite

$$\begin{tikzcd}
E \arrow[r, "i"] \arrow[rrrrr, bend left, "1"] & \Phi E \arrow[r, "\bar{p}"] & p/B \arrow[rr, "1"] \arrow[dr, "\ell"'] & {} \arrow[d, phantom, "\Downarrow\eta"] & p/B \arrow[r, "d_0"] & E \arrow[r, "i"] \arrow[d, phantom, "\Downarrow\iota_0"] & \Phi E \\
& & & \Phi E \arrow[ur, "\bar{p}"'] \arrow[urr, bend right, "d_0"] \arrow[urrr, bend right, "1"'] & & {} &
\end{tikzcd}$$.

So $\bar{\zeta}$ is an isomorphism. The equality (4) for Proposition 3 follows easily. So E is a pseudo L-algebra.//

Compare the above proposition with Gray [2] p.56; so we have related the definition of 0-fibration here with the definition of opfibration in [2] when $K = \mathcal{C}at$. Notice that the unit of the adjunction $\ell \dashv \bar{p}$ for Gray is not just an isomorphism but an identity. It is worth pointing out the reason for this since we will need the observation in the next paper. A 0-fibration will be called *normal* when there is a normalized pseudo L-algebra structure on it. In *Cat* every 0-fibration is normal, but in other 2-categories this need not be the case. In the proof of the Chevalley criterion, if ζ is an identity then so is η. So, *for a normal 0-fibration,* $\bar{p}:\Phi E \longrightarrow p/B$ *has a left adjoint with unit an identity.*

For any arrow $g:B' \longrightarrow B$, "pulling back along g" is a 2-functor $g^*:K_B \longrightarrow K_{B'}$;

for each E in K_B, the diagram

$$\begin{tikzcd}
g^*E \arrow[r, "\hat{g}"] \arrow[d, "\hat{p}"'] & E \arrow[d, "p"] \\
B' \arrow[r, "g"'] & B
\end{tikzcd}$$

is a pullback. The composite 2-cell

$$\begin{tikzcd}
\hat{p}/B' \arrow[r, "d_1"] \arrow[d, "d_0"'] & B' \arrow[d, "1"] \\
g^*E \arrow[r, "\hat{p}"'] \arrow[d, "\hat{g}"'] \arrow[ur, phantom, "\lambda"] & B' \arrow[d, "g"] \\
E \arrow[r, "\hat{p}"'] & B
\end{tikzcd}$$

induces an arrow of spans $\check{g}_E$:

$$\begin{tikzcd}
 & \hat{p}/B' \arrow[dl, "d_0"'] \arrow[d, "\check{g}_E"] \arrow[ddr, "d_1"] & \\
g^*E \arrow[d, "\hat{g}"'] & p/g \arrow[dl, "d_0"] \arrow[dr, "d_1"'] & \\
E & & B'
\end{tikzcd}$$

for each E in K_B. One readily checks that $\check{g}_E$, $E \in K_B$ are the components of a 2-natural transformation $\check{g}$:

$$\begin{tikzcd}
K_B \arrow[r, "g^*"] \arrow[d, "L"'] & K_{B'} \arrow[d, "L"] \arrow[dl, phantom, "\check{g}"] \\
K_B \arrow[r, "g^*"'] & K_{B'}
\end{tikzcd} .$$

Indeed, in the language of Street [6], the pair $(g^*,\check{g})$ is a monad functor from

(K_B, L) to $(K_{B'}, L)$ in the 2-category 2-CAT.

<u>*Proposition 10.*</u> *Suppose* $g: B' \longrightarrow B$ *is an arrow in* K. *For each lax* L-*algebra* E, *the arrow*

$$Lg^*E \xrightarrow{\check{g}_E} g^*LE \xrightarrow{g^*(c)} g^*E$$

enriches g^*E *with the structure of lax* L-*algebra. For each lax homomorphism* $f: E \longrightarrow E'$ *of lax* L-*algebras, the 2-cell*

$$\begin{tikzcd}
Lg^*E \arrow[r, "\check{g}_E"] \arrow[d, "Lg^*E"'] & g^*LE \arrow[r, "g^*(c)"] \arrow[d, "g^*(Lf)"'] \arrow[dr, phantom, "\overset{g^*(\theta_f)}{\nearrow}"] & g^*E \arrow[d, "g^*(f)"] \\
Lg^*E \arrow[r, "\check{g}_{E'}"'] & g^*LE' \arrow[r, "g^*(c)"'] & g^*E'
\end{tikzcd}$$

enriches $g^*(f): g^*E \longrightarrow g^*E'$ *with the structure of lax homomorphism. If* E *is a pseudo* L-*algebra or an* L-*algebra then so is* g^*E. *If* f *is a pseudo homomorphism or a homomorphism then so is* $g^*(f)$.//

<u>*Corollary 11.*</u> *The pullback of a (split) 0-fibration along any arrow is a (split) 0-fibration.*//

Let $R: K_A \longrightarrow K_A$ denote the 2-functor given by:

$$\begin{tikzcd}
(E,q) \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\sigma"] & (E',q')
\end{tikzcd} \longmapsto \begin{tikzcd}
(A/q, d_0) \arrow[r, bend left, "A/f"] \arrow[r, bend right, "A/g"'] \arrow[r, phantom, "\Downarrow A/\sigma"] & (A/q', d_0)
\end{tikzcd}.$$

There is a 2-monad structure on R and the theory develops as for L; just replace K by K^{co}. An arrow $q: E \longrightarrow A$ is called a *1-fibration over* A when (E,q) supports the structure of pseudo R-algebra.

Note that the category $SPN(A,B)$ of spans from A to B becomes a 2-category by taking as 2-cells the 2-cells σ of K as in the diagram

$$\begin{tikzcd}
 & E \arrow[dl, "q"'] \arrow[dr, "p"] \arrow[dd, bend right, "f"'] \arrow[dd, bend left, "g"] \arrow[dd, phantom, "\overset{\sigma}{\Rightarrow}"] & \\
A & & B \\
 & E' \arrow[ul, "q'"] \arrow[ur, "p'"'] &
\end{tikzcd}$$

where $q'\sigma = 1_q$, $p'\sigma = 1_p$. Let $M:SPN(A,B) \longrightarrow SPN(A,B)$ denote the 2-functor given by:

$$\begin{tikzcd} E \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\sigma\Downarrow"] & E' \end{tikzcd} \longmapsto \begin{tikzcd} \Phi B\circ E\circ\Phi A \arrow[r, bend left, "1\circ f\circ 1"] \arrow[r, bend right, "1\circ g\circ 1"'] \arrow[r, phantom, "\Downarrow 1\circ\sigma\circ 1"] & \Phi B\circ E'\circ\Phi A. \end{tikzcd}$$

This 2-functor supports the structure of 2-monad too; the unit $i:1 \longrightarrow M$ and multiplication $c:MM \longrightarrow M$ have as components

$$E \xrightarrow{i\circ 1\circ i} \Phi B\circ E\circ\Phi A \quad\text{and}\quad \Phi B\circ\Phi B\circ E\circ\Phi A\circ\Phi A \xrightarrow{c\circ 1\circ c} \Phi B\circ E\circ\Phi A\ .$$

A span (q,E,p) for A to B is called a *bifibration from* A *to* B when it supports the structure of pseudo M-algebra. A *split bifibration* is an M-algebra.

Results on L-algebras and R-algebras can be transferred to M-algebras via the following result. The corresponding statement for lax algebras is left to the reader.

Proposition 12. *Suppose* E *is a span from* A *to* B. *The* M-*algebra structures* $c:\Phi B\circ E\circ\Phi A \longrightarrow E$ *are in bijective correspondence with pairs of arrows of spans* $c_L:\Phi B\circ E \longrightarrow E$, $c_R:E\circ\Phi A \longrightarrow E$ *such that* c_L, c_R *are* L-*algebra*, R-*algebra structures on* E *related by the condition that*

$$\begin{tikzcd} ME \arrow[r, "1\circ c_R"] \arrow[d, "c_L\circ 1"'] & LE \arrow[d, "c_L"] \\ RE \arrow[r, "c_R"'] & E \end{tikzcd}$$

commutes; the bijection is determined by

$$c_L = (\Phi B\circ E \xrightarrow{1\circ 1\circ i} \Phi B\circ E\circ\Phi A \xrightarrow{c} E)$$

$$c_R = (E\circ\Phi A \xrightarrow{i\circ 1\circ 1} \Phi B\circ E\circ\Phi A \xrightarrow{c} E)$$

$$c = c_L(1\circ c_R) = c_R(c_L\circ 1).$$

Furthermore, an arrow of spans is a homomorphism of M-*algebras if and only if it is a homomorphism of both the corresponding* L-*algebras and the corresponding* R-*algebras.* //

Combining this with Corollary 11 and the dual for 1-fibrations we have:

Corollary 13. *For any arrows* $f:A' \longrightarrow A$, $g:B' \longrightarrow B$, *each (split) bifibration* E *from* A *to* B *induces a (split) bifibration* $g^* \circ E \circ f$ *from* A' *to* B'.//

There is a more general composition of bifibrations which we will not need. If E is a bifibration from A to B and F a bifibration from B to C then the bifibration $F \otimes E$ from A to C can be defined by the usual "tensor product of bimodules" coequalizer, provided this coequalizer exists and is preserved by certain pullbacks.

§3. Yoneda's Lemma within a 2-category.

Again we work in a representable 2-category K.

A *covering span* is defined to be a span which is the comma object of some opspan.

Theorem 14. *Any covering span is a split bifibration. Any arrow of spans between covering spans is a homomorphism.*

Proof. Any comma object r/s is a composite $s^* \circ \Phi D \circ r$. But ΦD is the value of M at the identity span of D; so ΦD is a free split bifibration. So r/s is a split bifibration by Corollary 13.

Suppose $f:r/s \longrightarrow u/v$ is an arrow of spans from A to B. We must prove that f commutes with the M-algebra structures on r/s and u/v. By Proposition 12, it suffices to show that f commutes with the L-algebra and R-algebra structures separately. By duality, it suffices to show that f commutes with just the L-algebra structures.

The L-algebra structure $c:\Phi B \circ (r/s) \longrightarrow r/s$ comes from that of ΦD via the commutative square

$$\begin{tikzcd} \Phi B \circ (r/s) \arrow[r, "\Phi s \circ \underset{\sim}{\lambda}"] \arrow[d, "c"'] & \Phi D \circ \Phi D \arrow[d, "c"] \\ r/s \arrow[r, "\underset{\sim}{\lambda}"'] & \Phi D \end{tikzcd} .$$

Equivalently, note that $\Phi B \circ (r/s)$ is the comma object of the opspan (r, D, sd_0) from A to ΦB (composed with $d_1 : \Phi B \longrightarrow B$) since we have the pullback

$$\begin{tikzcd}
r/sd_0 \arrow[r, "\hat{d}_1"] \arrow[d, "\hat{d}_0"'] & \Phi B \arrow[d, "d_0"] \\
r/s \arrow[r, "d_1"'] & B
\end{tikzcd}$$

and $c:r/sd_0 \longrightarrow r/s$ corresponds to the composite 2-cell

$$\begin{tikzcd}
& r/sd_0 \arrow[dl, "d_0"'] \arrow[dr, "\hat{d}_1"] & \\
A \arrow[dr, "r"'] \arrow[rr, phantom, "\xrightarrow{\lambda}"] & & \Phi B \arrow[d, bend right, "d_0"'] \arrow[d, bend left, "d_1"] \arrow[d, phantom, "\xrightarrow{\lambda}"] \\
& D & B \arrow[l, "s"]
\end{tikzcd}$$

.

The main trick of the proof is to introduce the 2-cell $\begin{tikzcd} r/sd_0 \arrow[r, bend left, "\hat{d}_0"] \arrow[r, bend right, "c"'] \arrow[r, phantom, "\Downarrow\alpha"] & r/s \end{tikzcd}$ defined by $d_0\alpha = 1_{d_0}$, $d_1\alpha = \lambda\hat{d}_1$; of course, we also have such an α for u/v. The arrow $L(f)$ is defined by the commutative diagram

$$\begin{tikzcd}
r/s \arrow[d, "f"'] & r/sd_0 \arrow[l, "\hat{d}_0"'] \arrow[r, "\hat{d}_1"] \arrow[d, "L(f)"'] & \Phi B \arrow[d, "1"] \\
u/v & u/vd_0 \arrow[l, "\hat{d}_0"] \arrow[r, "\hat{d}_1"'] & \Phi B
\end{tikzcd}$$

.

The calculations

$$d_0\alpha L(f) = 1_{d_0L(f)} = 1_{d_0} = d_0\alpha = d_0f\alpha$$
$$d_1\alpha L(f) = \lambda\hat{d}_1L(f) = \lambda\hat{d}_1 = d_1\alpha = d_1f\alpha$$

show that the following composites are equal

$$\begin{tikzcd}
r/sd_0 \arrow[r, bend left, "\hat{d}_0"] \arrow[r, bend right, "c"'] \arrow[r, phantom, "\Downarrow\alpha"] & r/s \arrow[r, "f"] & u/v
\end{tikzcd} = \begin{tikzcd}
r/sd_0 \arrow[r, "L(f)"] & u/vd_0 \arrow[r, bend left, "\hat{d}_0"] \arrow[r, bend right, "c"'] \arrow[r, phantom, "\Downarrow\alpha"] & u/v
\end{tikzcd}.$$

So $c.L(f) = fc$, which proves that f is a homomorphism.//

Let $COV(A,B)$ denote the full subcategory $SPN(A,B)$ whose objects are the covering spans. Let $SPL(A,B)$ denote the category of algebras for the monad M on $SPN(A,B)$; it is the category of split bifibrations from A to B and their homomorphisms (up to equivalence).

<u>*Corollary 15*</u>. *The inclusion functor* $COV(A,B) \longrightarrow SPN(A,B)$ *factors through the underlying functor* $SPL(A,B) \longrightarrow SPN(A,B)$.//

<u>*Corollary 16 (Yoneda lemma)*</u>. *Suppose* $f: A \longrightarrow B$ *is an arrow and* E *is a covering span from* A *to* B. *Composition with the arrow of spans* $i_f : f \longrightarrow f/B$ *yields a bijection between arrows of spans from* f/B *to* E *and arrows of spans from* f *to* E.

<u>*Proof*</u>. Note that f/B is the free M-algebra on the span f from A to B. This gives a bijection between arrows of spans $f \longrightarrow E$ and homomorphisms $f/B \longrightarrow E$. But by Theorem 14, any arrow of spans $f/B \longrightarrow E$ is a homomorphism.//

<u>*Remark*</u>. Take $K = CAT$ and $A = \mathbb{1}$ in Corollary 16. Covering spans E from $\mathbb{1}$ to B functorially correspond to functors e from B into some category of sets. An arrow $f: \mathbb{1} \longrightarrow B$ is just an object b of B. The bijection of the corollary becomes the usual bijection between natural transformations $B(b,-) \longrightarrow e$ and elements of eb obtained by evaluating at 1_b.

The following special case of Corollary 16 appears in Gray [3].

<u>*Corollary 17*</u>. *The functor* $K(A,B)^{op} \longrightarrow SPN(A,B)$ *given by*

$$\begin{tikzcd} A \arrow[r, bend left, "g"] \arrow[r, bend right, "f"'] \arrow[r, phantom, "\Downarrow\sigma"] & B \end{tikzcd} \qquad \longmapsto \qquad f/B \xrightarrow{\sigma/B} g/B$$

is fully faithful.

<u>*Proof*</u>. The definition of comma objects gives the bijection

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "1"'] \arrow[dr, phantom, "\sigma"] & B \arrow[d, "1"] \\ A \arrow[r, "g"'] & B \end{tikzcd} \qquad \longleftrightarrow \qquad f \xrightarrow{h} g/B$$

between 2-cells σ and arrows of spans h. The Yoneda lemma provides the bijection between such h and arrows of spans $f/A \longrightarrow g/B$.//

<u>§4. Pointwise extensions.</u>

Recall the definition of left extension in a 2-category (see [6]).

Proposition 18. *There is a bijection between 2-cells*

$$\begin{tikzcd} j/B \arrow[r, "d_1"] \arrow[d, "d_0"'] & B \arrow[d, "k"] \\ A \arrow[r, "f"'] \arrow[ur, phantom, "\zeta"] & X \end{tikzcd} \quad \longleftrightarrow \quad \begin{tikzcd} A \arrow[rr, "j"] \arrow[dr, "f"'] \arrow[rr, phantom, shift right=3ex, "\kappa"] & & B \arrow[dl, "k"] \\ & X & \end{tikzcd}$$

obtained by composition with i_j. *The 2-cell* κ *exhibits* k *as a left extension of* f *along* j *if and only if the corresponding* ζ *exhibits* k *as a left extension of* fd_0 *along* d_1.

Proof. By definition of comma objects there is a bijection between 2-cells ζ and arrows of spans $j/B \longrightarrow f/k$, and a bijection between 2-cells κ and arrows of spans $j \longrightarrow f/k$. The first sentence of the proposition now follows by the Yoneda lemma. For any arrow $\ell : B \longrightarrow X$, there are bijections

$$f \longrightarrow \ell j \quad \longleftrightarrow \quad j \longrightarrow f/\ell \quad \longleftrightarrow \quad j/B \longrightarrow f/\ell$$
$$\longleftrightarrow \quad j/B \longrightarrow \Phi X \quad \longleftrightarrow \quad fd_0 \longrightarrow \ell d_1$$

where the first, third and fourth are from the definition of comma object and the second is by Yoneda.//

The 2-cell

$$\begin{tikzcd} A \arrow[rr, "j"] \arrow[dr, "f"'] \arrow[rr, phantom, shift right=3ex, "\kappa"] & & B \arrow[dl, "k"] \\ & X & \end{tikzcd}$$

is said to exhibit k as a *pointwise left extension* of f along j when, for each arrow $g : C \longrightarrow B$, the composite 2-cell

$$\begin{tikzcd}
j/g \arrow[r, "d_1"] \arrow[d, "d_0"'] & C \arrow[d, "g"] \\
A \arrow[r, "j"] \arrow[dr, "f"'] \arrow[ur, phantom, "\lambda"] & B \arrow[d, "k"] \\
& X \arrow[u, phantom, "\kappa"]
\end{tikzcd}$$

exhibits kg as a left extension of fd_0 along d_1.

Taking $g = 1_B$ in this definition we obtain the following corollary to the last proposition.

Corollary 19. *A pointwise left extension is a left extension.*//

Remark. When K = CAT, the pointwise left extensions are precisely those given by the formula

$$kb = \varinjlim \left(j/b \xrightarrow{d_0} A \xrightarrow{f} X\right).$$

To see this, take $C = \mathbb{1}$, $g = b$ and note that left extension along $j/b \longrightarrow \mathbb{1}$ is direct limit.

Recall that left extensions along an arrow with a right adjoint always exist and are obtained by composing with the right adjoint. The following result is a direct corollary of Proposition 5.

Proposition 20. *Any left extension along an arrow which has a right adjoint is pointwise.*//

Proposition 21. *An arrow* $f:B \longrightarrow A$ *is a left adjoint to* $u:A \longrightarrow B$ *if and only if there is an isomorphism* $f/A \cong B/u$ *of spans from* A *to* B.

Proof. Using the Yoneda lemma, we have bijections

$$1 \xrightarrow{\eta} uf \longleftrightarrow f \longrightarrow B/u \longleftrightarrow f/A \xrightarrow{m} B/u$$
$$fu \xrightarrow{\varepsilon} 1 \longleftrightarrow u^* \longrightarrow f/A \longleftrightarrow B/u \xrightarrow{n} f/A \quad .$$

It is readily checked that η, ε are a unit and counit for an adjunction $f \dashv u$ if and only if the corresponding m,n are mutually inverse isomorphisms.//

An arrow $j:A \longrightarrow B$ is said to be *fully faithful* when, given any 2-cell $C \underset{jv}{\overset{ju}{\Downarrow\tau}} B$, there exists a unique 2-cell $C \underset{v}{\overset{u}{\Downarrow\sigma}} A$ such that τ is the composite $C \underset{v}{\overset{u}{\Downarrow\sigma}} A \xrightarrow{j} B$. It is readily seen that j is fully faithful if and only if the arrow of spans $\Phi A \longrightarrow j/j$ corresponding to $j\lambda$ is an isomorphism.

Proposition 22. *If* $j : A \longrightarrow B$ *is fully faithful and if the 2-cell*

$$\begin{tikzcd} A \arrow[rr, "j"] \arrow[dr, "f"'] & \arrow[d, phantom, "\overset{\sigma}{\Rightarrow}"] & B \arrow[dl, "k"] \\ & X & \end{tikzcd}$$

exhibits k *as a pointwise left extension of* f *along* j, *then* σ *is an isomorphism.*

Proof. Since k is a pointwise left extension and $\Phi A \longrightarrow j/j$ is an isomorphism, the composite 2-cell

$$\begin{tikzcd} \Phi A \arrow[d] \arrow[dr, "d_1"] \arrow[dd, bend right=60, "d_0"'] & \\ j/j \arrow[r, "d_1"] \arrow[d, "d_0"] & A \arrow[d, "j"] \\ A \arrow[r, "j"] \arrow[ur, phantom, "\lambda"] \arrow[dr, "f"'] & B \arrow[d, "k"] \\ & X \end{tikzcd}$$

exhibits kj as a left extension of fd_0 along d_1. By Proposition 18, the corresponding 2-cell

$$\begin{tikzcd} A \arrow[rr, "1"] \arrow[dr, "f"'] & \arrow[d, phantom, "\overset{\sigma}{\Rightarrow}"] & A \arrow[dl, "kj"] \\ & X & \end{tikzcd}$$

exhibits kj as a left extension of f along 1_A. But also the identity 2-cell exhibits f as a left extension of f along 1_A. So σ is an isomorphism.//

For a 0-fibration $p : E \longrightarrow B$ and arrow $b : G \longrightarrow B$, we denote by E_b the pullback of b along p.

$$\begin{tikzcd} E_b \arrow[r, "\hat{p}"] \arrow[d, "\hat{b}"'] & G \arrow[d, "b"] \\ E \arrow[r, "p"'] & B \end{tikzcd}$$

Proposition 23. *Suppose in the diagram*

$$\begin{tikzcd} E \arrow[rr, "p"] \arrow[dr, "f"'] & \arrow[d, phantom, "\overset{\sigma}{\Rightarrow}"] & B \arrow[dl, "k"] \\ & X & \end{tikzcd}$$

that p *is a normal* 0-*fibration. The* 2-*cell* σ *exhibits* k *as a pointwise left extension of* f *along* p *if and only if, for each arrow* $b:G \longrightarrow B$, *the* 2-*cell* $\sigma\hat{b}$ *exhibits* kb *as a left extension of* $f\hat{b}$ *along* $\hat{p}$.

Proof. The following square is readily seen to be a pullback.

$$\begin{tikzcd} E/\hat{b} \arrow[r, "p'"] \arrow[d] & p/b \arrow[d] \\ \Phi E \arrow[r, "\bar{p}"'] & p/B \end{tikzcd}$$

Since p is a normal 0-fibration, $\bar{p}$ has a left adjoint with unit an identity (Chevalley criterion). This property is preserved by pullback: so p' has a left adjoint ℓ' with unit an identity. The arrow $d_1:E/\hat{b} \longrightarrow E_b$ has a right adjoint $i_{\hat{b}}:E_b \longrightarrow E/\hat{b}$ (dual of Corollary 6). So the composite $d_1\ell':p/b \longrightarrow E_b$ has a right adjoint $p'i_{\hat{b}}$. Let η denote the unit of this adjunction. One readily checks the equations

$$\hat{b} = d_0p'i_b \text{ , } \hat{p}d_1\ell' = d_1 \quad \text{and} \quad pd_0\eta = \lambda \text{ .}$$

$$\begin{tikzcd} p/b \arrow[rr, bend left, "d_1\ell'"] \arrow[drr, "d_0"'] & \arrow[dr, phantom, "\overset{d_0\eta}{\Rightarrow}"] & E_b \arrow[ll, bend left, "p'i_{\hat{b}}"] \arrow[rr, "\hat{p}"] \arrow[d, "\hat{b}"] & & G \arrow[d, "b"] \\ & & E \arrow[rr, "p"] \arrow[dr, "f"'] & \arrow[d, phantom, "\overset{\sigma}{\Rightarrow}"] & B \arrow[dl, "k"] \\ & & & X & \end{tikzcd}$$

So $fd_0\eta$ exhibits $f\hat{b}$ as a left extension of fd_0 along $d_1\ell'$. It follows that the composite 2-cell

$$\begin{tikzcd} p/b \arrow[r, "d_1"] \arrow[d, "d_0"'] & G \arrow[d, "b"] \\ E \arrow[r, "p"] \arrow[ur, phantom, "\nearrow\lambda"] \arrow[dr, "f"'] \arrow[r, phantom, bend right, "\xrightarrow{\sigma}"'] & B \arrow[d, "k"] \\ & X \end{tikzcd}$$

exhibits kb as a left extension of fd_0 along d_1 if and only if $\sigma\hat{b}$ exhibits kb as a left extension of $f\hat{b}$ along $\hat{p}$.//

Proposition 24. Suppose in the diagram

$$\begin{tikzcd} j/g \arrow[r, "d_1"] \arrow[d, "d_0"'] & C \arrow[d, "g"] \\ A \arrow[r, "j"] \arrow[ur, phantom, "\nearrow\lambda"] \arrow[dr, "f"'] \arrow[r, phantom, bend right, "\xrightarrow{\kappa}"'] & B \arrow[d, "k"] \\ & X \end{tikzcd}$$

that κ *exhibits* k *as a pointwise left extension of* f *along* j. *Then the composite* 2-*cell exhibits* kg *as a pointwise left extension of* fd_0 *along* d_1.

Proof. Take $b:G \longrightarrow C$. The following square is a pullback.

$$\begin{tikzcd} j/gb \arrow[r, "d_1"] \arrow[d] & G \arrow[d] \\ j/g \arrow[r, "d_1"'] & C \end{tikzcd} .$$

If this is mounted on the top of the diagram of the proposition we obtain the diagram

$$\begin{tikzcd} j/gb \arrow[r, "d_1"] \arrow[d] & G \arrow[d, "gb"] \\ A \arrow[r, "j"] \arrow[ur, phantom, "\nearrow\lambda"] \arrow[dr, "f"'] \arrow[r, phantom, bend right, "\xrightarrow{\kappa}"'] & B \arrow[d, "k"] \\ & X \end{tikzcd} ;$$

and this composite 2-cell does exhibit kgb as a left extension of fd_0 along d_1 (from the pointwise property of κ). By Proposition 12 and Theorem 14 we have that $d_1:j/g \longrightarrow C$ is a normal 0-fibrations (indeed, split). So Proposition 23 applies with $p = d_1:j/g \longrightarrow C$ to yield the result.//

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