# On the Fibrations Underlying Optimization and Elimination

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**Abstract**

*As of July 25, 2019:* The theory of fibrations and fibered categories appears to be a natural place to discuss the theory of various optimization and elimination problems, including resolution in logic, linear and non-linear quantifier elimination, polytope projection, lattice optimization over various spaces, etc. These notes aim to investigate that claim and furthermore attempts to determine any and all structural similarities between the various cases.

# Contents

# Notation Proposals

- Aff the category of affine spaces (?) and affine maps between them
- Vect the category of vector spaces and the linear maps between them
- Poly the category of polyhedra and the affine maps between them
- Cone the category of cones and the linear maps between them

# 1 Introduction

Below is a provisional list of various notions of "elimination":

- The **resolution rule** of propositional (and also first order) logics. Two clauses containing a complementary literals (e.g. variable $c$ in one and its negation $\neg c$ in the other) entails a clause with the complementary literals eliminated (see *Ground resolvents and Ground resolution* in robinson1965machine [Rob+65]):[1]

  $$\frac{a_1 \vee a_2 \vee \cdots \vee c, \quad b_1 \vee b_2 \vee \cdots \vee \neg c}{a_1 \vee a_2 \vee \cdots \vee b_1 \vee b_2 \vee \cdots}$$

  Equivalently,

  $$\frac{(\neg a_1 \wedge \neg a_2 \wedge \cdots) \to c, \quad c \to (b_1 \vee b_2 \vee \cdots)}{(\neg a_1 \wedge \neg a_2 \wedge \cdots) \to (b_1 \vee b_2 \vee \cdots)}$$

  This generalizes to arbitrary conjunctions of literals which may or may not reference $c$ or $\neg c$.

- The incremental step of **Fourier-Motzkin elimination** ziegler2012lectures [Zie12] for systems of linear inequalities. Given

  $$a_0 + a_1 x_1 + a_2 x_2 + \cdots + a_n x_n \geq 0, \quad b_0 + b_1 x_1 + b_2 x_2 + \cdots + b_n x_n \geq 0$$

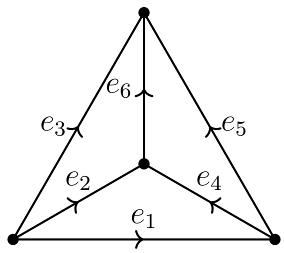
  with $a_1 > 0$ and $b_1 < 0$, then

  $$\left(\frac{a_0}{a_1} + \frac{a_2}{a_1} x_2 + \cdots + \frac{a_n}{a_1} x_n\right) - \left(\frac{b_0}{b_1} + \frac{b_2}{b_1} x_2 + \cdots + \frac{b_n}{b_1} x_n\right) \geq 0$$

  This generalizes to arbitrary systems of linear inequalities over a set of variables containing $x_1$.

[1] Reference https://en.wikipedia.org/wiki/Resolution_(logic).

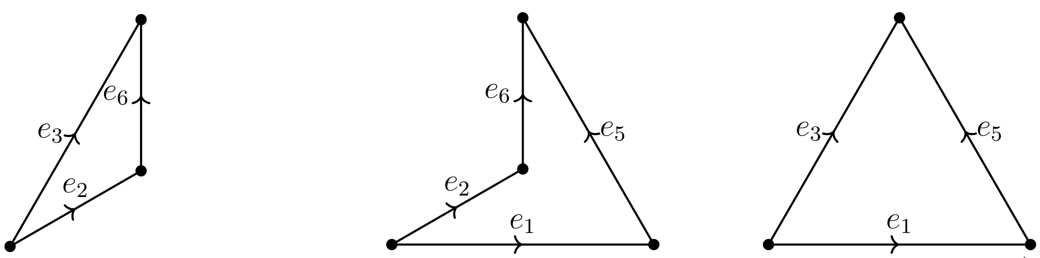
- The **elimination axiom of oriented matroids** bjorner1999oriented [Bjo+99, circuit axiom (C3)]. Given two citcuits $X_0 = (X_0^+, X_0^-), X_1 = (X_1^+, X_1^-)$ (with $X_0 \neq -X_1$), and an element $e \in X_0^+ \cap X_1^-$ which is positively oriented in one circuit and negatively oriented in the other, then the circuits can be "glued" along $e$ producing a new circuit $X = (X^+, X^-)$ satisfying $X^+ \subseteq X_0^+ \cup X_1^+ \setminus \{e\}$ and $X^- \subseteq X_0^- \cup X_1^- \setminus \{e\}$ (i.e. it at least eliminates $e$). For example, the oriented matroid generated by the cycles of the following graph

  (1)

  satisfies the elimination axiom. The following example eliminates $e_2$ (an indirectly eliminates $e_6$):

  $X_0 = (\{e_2, e_6\}, \{e_3\})$ $\qquad$ $X_1 = (\{e_1, e_5\}, \{e_2, e_6\})$ $\qquad$ $X = (\{e_1, e_5\}, \{e_3\})$

  (2)

  **TC:** Generally, this "elimination" of $n-1$ surfaces by gluing together $n$-dimensional surfaces reminds me of the analogous idea in the homology theory of polyhedra; assign to each $n$-dimensional face the sum of the $n-1$ faces *incidence* to it (its boundary) as a formal sum in the free abelian group of all $n-1$ faces modulo 2 (the modulo 2 carries out the unoriented elimination).

## 2 Category Theory Terminology

The following unordered list of categorical concepts are anticipated to be utilized:

- adjunctions
- fibered categories

- cleavages
- pseudo functors (and if cleavages are splitting, functors)
- Beck-Chevalley condition
- Frobenius reciprocity (and functors of monoidal categories)

**Tobias:** Cleavages are not really important because for any two different choices of cleavage, the resulting pullback functors are naturally isomorphic. So cleavages are just a technical tool relevant for proving the equivalence between fibred cats and pseudofunctors, but not relevant in practice

**TC:** The above comment makes sense. Overall there are isomorphisms lurking behind every corner: first, there are natural isomorphisms present when considering the equivalence between pseudo-functors and "cleavaged" fibered categories, and second, whenever the cartesian arrows are indeed pullbacks, they are unique up to unique isomorphism and thus entire cleavages are unique up to unique isomorphisms. For a discussion see vistoli2004notes [Vis04] at the end of Section 3.1.3. starting on page 50.

## 2.1 Cartesian Arrows

**Definition 2.1.** Let $P : \mathcal{E} \to \mathcal{B}$ be a functor between categories $\mathcal{E}$ and $\mathcal{B}$. An arrow $\phi : \alpha \to \beta$ of $\mathcal{E}$ is *cartesian* with respect to $P$ (sometimes $P$-*cartesian*) if for every arrow $\psi : \gamma \to \beta$ sharing a codomain with $\phi$, and for every arrow $g : P(\gamma) \to P(\alpha)$ in $\mathcal{B}$ satisfying $g \circ P(\phi) = P(\psi)$, there exists a unique arrow $\theta : \gamma \to \alpha$ in $\mathcal{E}$ satisfying $\phi \circ \theta = \psi$ and $P(\theta) = g$.

$$\begin{tikzcd}
\gamma \arrow[drr, bend left, "\forall\psi"] \arrow[dr, dashed, "\exists!\theta"'] \arrow[dd, mapsto] & & \\
& \alpha \arrow[r, "\phi"'] \arrow[dd, mapsto] & \beta \arrow[dd, mapsto] \\
P(\gamma) \arrow[dr, "\forall g"'] \arrow[drr, bend left, "P(\psi)"] & & \\
& P(\alpha) \arrow[r, "P(\phi)"'] & P(\beta)
\end{tikzcd} \tag{3}$$

**Corollary 2.0.1.** *A cartesian morphism $\phi : \alpha \to \beta$ in $\mathcal{E}$ with respect to a functor $P : \mathcal{E} \to \mathcal{B}$ establishes an isomorphism of categories* lurie2009higher *[Lur09, Section 2.4.1]*[2]

$$\mathcal{E}/\phi \cong \mathcal{E}/\beta \times_{\mathcal{B}/P(\beta)} \mathcal{B}/P(\phi) \tag{4}$$

[2]This formulation is also discussed here: https://ncatlab.org/nlab/show/Cartesian+morphism#CartInOrdCatReformulation.

*where $\mathcal{E}/\beta \times_{\mathcal{B}/P(\beta)} \mathcal{B}/P(\phi)$ is the pullback of functors.*

$$\begin{tikzcd}
\mathcal{E}/\phi \arrow[dr, dashed, leftrightarrow, "\cong"] \arrow[drr, bend left, "P/\phi"] \arrow[ddr, bend right, "\mathrm{cod}"'] & & \\
& \mathcal{E}/\beta \times_{\mathcal{B}/P(\beta)} \mathcal{B}/P(\phi) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{B}/P(\phi) \arrow[d, "\mathrm{cod}"] \\
& \mathcal{E}/\beta \arrow[r, "P/\beta"] & \mathcal{B}/P(\beta)
\end{tikzcd} \tag{5}$$

The pullback category $\mathcal{E}/\beta \times_{\mathcal{B}/P(\beta)} \mathcal{B}/P(\phi)$ has morphisms associated with diagrams of $\mathcal{B}$ with the following format:

$$\begin{tikzcd}
& P(\gamma) \arrow[d, "P(\chi)"] \arrow[ddl, bend right, "f"'] \arrow[ddr, bend left, "P(\omega)"] & \\
& P(\delta) \arrow[dl, "g"'] \arrow[dr, "P(\psi)"] & \\
P(\alpha) \arrow[rr, "P(\phi)"] & & P(\beta)
\end{tikzcd} \tag{6}$$ eq:pullback_ca

Evidently, if $\phi : \alpha \to \beta$ is cartesian, then there exists unique morphisms $\zeta : \gamma \to \alpha$ and $\eta : \delta \to \alpha$ such that $P(\zeta) = f$ and $P(\eta) = g$ and the following diagram of $\mathcal{E}$ commutes:

$$\begin{tikzcd}
& \gamma \arrow[d, "\chi"] \arrow[ddl, bend right, "\zeta"'] \arrow[ddr, bend left, "\omega"] & \\
& \delta \arrow[dl, "\eta"'] \arrow[dr, "\psi"] & \\
\alpha \arrow[rr, "\phi"] & & \beta
\end{tikzcd} \tag{7}$$ eq:cartesian_c

Intuitively, if $\phi$ is cartesian, then in order to determine the category $\mathcal{E}/\phi$ over $\phi$, it is sufficient to specify $\mathcal{E}/\beta \times_{\mathcal{B}/P(\beta)} \mathcal{B}/P(\phi)$.

## 2.2 Fibrations, Fibered Categories, and Cleavages

**Definition 2.2.** A *fibered category over* $\mathcal{B}$ is a category $\mathcal{E}$ associated to the domain of a functor, referred to as the *fibration*, $P : \mathcal{E} \to \mathcal{B}$ with the property that for every morphism $f : a \to b$ of $\mathcal{B}$ and object $\beta$ such that $P(\beta) = b$, there exists a cartesian arrow $\phi : \alpha \to \beta$ with $P(\phi) = f$.

**Lemma 2.1.** *A fibration $P : \mathcal{E} \to \mathcal{B}$ is a faithful functor if and only if its fibers are thin.*

*Proof.* Recall that if $P : \mathcal{E} \to \mathcal{B}$ is a faithful functor, then by definition every pair of parallel arrows $\phi, \psi : \alpha \to \beta$ in $\mathcal{E}$ satisfies

$$P(\phi) = P(\psi) : P(\alpha) \to P(\beta) \implies \phi = \psi. \tag{8}$$

eq:faithfulness

$\implies$ : Assuming $P : \mathcal{E} \to \mathcal{B}$ is faithful functor, consider an arbitrary pair of parallel arrows $\phi, \psi : \alpha \to \beta$ in an arbitrary fiber $\mathcal{E}_x$ over $x$; i.e. $P(\phi) = P(\psi) = \mathrm{id}_x$. In such cases, faithfulness of $P$ (Eq. 8) guarantees that $\phi = \psi$ and thus $\mathcal{E}_x$ is a thin category.

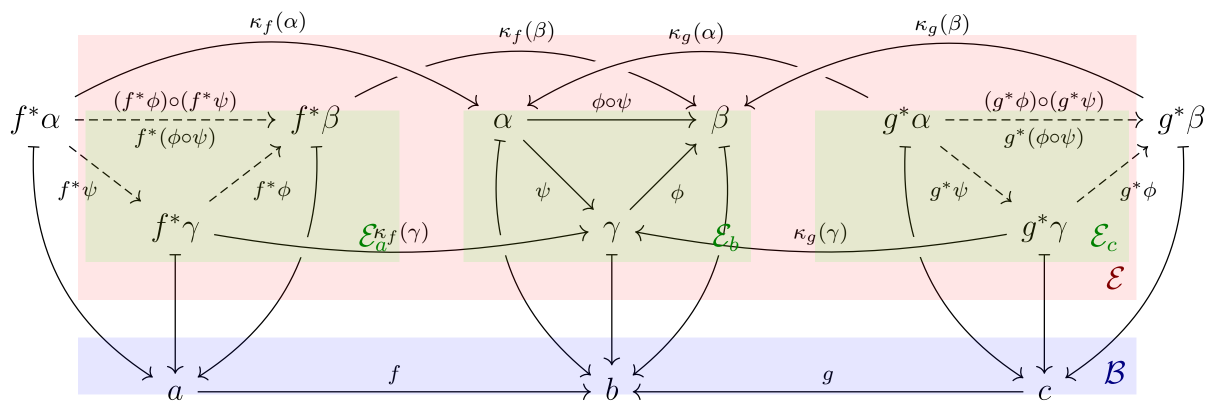
$\impliedby$ : If the fiber $\mathcal{E}_x$ for every object $x$ in $\mathcal{B}$ is a thin category, then clearly $P : \mathcal{E} \to \mathcal{B}$ *must* be faithful when restricted to an individual fiber. The non-trivial case is to consider an arbitrary pair of parallel morphisms $\phi, \psi : \alpha \to \beta$ not belonging to any fibers of $\mathcal{E}$. Denote $a := P(\alpha)$ and $b := P(\beta)$ and suppose $f := P(\phi) = P(\psi) : a \to b$. Then, because $\mathcal{E}$ is a fibered category, there exists a cartesian arrow $\zeta : \gamma \to \beta$, such that $P(\zeta) = f$ (note that $a = P(\alpha) = P(\gamma)$ but $\gamma$ is not necessarily equal to $\alpha$). Since $\zeta$ is a cartesian arrow, there exists a unique arrows $\mu, \nu : \alpha \to \gamma$ completing the top edges of the following diagram:

$$\begin{tikzcd}
\alpha \arrow[rr, dashed, "\mu"] \arrow[dr, bend left, "\psi"'] \arrow[dd, mapsto] & & \gamma \arrow[dl, "\zeta"'] \arrow[dd, mapsto] & \alpha \arrow[l, dashed, "\nu"'] \arrow[dll, bend left, "\phi"] \arrow[dd, mapsto] \\
& \beta \arrow[dd, mapsto] & & \\
a \arrow[rr, equal] \arrow[dr, bend right, "f"'] & & a \arrow[r, equal] \arrow[dl, "f"] & a \arrow[dll, bend left, "f"] \\
& b & &
\end{tikzcd} . \tag{9}$$

However, $P(\nu) = \mathrm{id}_a = P(\mu)$ and therefore $\mu$ and $\nu$ are parallel arrows in the fiber $\mathcal{E}_a$ and therefore $\mu = \nu$ because $\mathcal{E}_a$ is assumed thin. Therefore, $\psi = \zeta \circ \mu = \zeta \circ \nu = \phi$ and thus $P$ is a faithful functor. ■

**Definition 2.3.** A *cleavage* for a fibration $P : \mathcal{E} \to \mathcal{B}$ is an assignment to each morphism $f : a \to b$ of $\mathcal{B}$ and object $\beta$ in $\mathcal{E}_b$ (i.e. $P(\beta) = b$), a unique cartesian morphism $\kappa_f(B)$ of $\mathcal{E}$ such that $P(\kappa_f(B)) = f$.

Given a cleavage for a fibration, the cartesianness of morphisms within a cleavage permits one to establish functors between the fibers of the fibration. This concept is visualized in the following figure:

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## 2.3 Pseudo-Functors, Splitting Cleavages

Pages 47-48 of [Vis04] explicate the notions of pseudo-functors and their equivalence to fibrations with cleavages. Morover if the cleavage is splitting, the induced pseudo-functor is in fact a functor.

## 2.4 Nearby Fibrations: Opfibrations and $*$-fibrations

Given a functor $P : \mathcal{E} \to \mathcal{B}$, it can be considered as a fibration in many different ways. For example, if $P^{\mathrm{op}} : \mathcal{E}^{\mathrm{op}} \to \mathcal{B}^{\mathrm{op}}$ is a fibration, then $P$ is said to be an *opfibration*.

## 2.5 Hom-Functors

For a locally small category $\mathcal{C}$, the hom-functor of $\mathcal{C}$ is a functor $\mathrm{Hom}_{\mathcal{C}} : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathsf{Set}$ constructed in the following manner. Given objects $a, b, c, \ldots \in \mathcal{C}_0$ of $\mathcal{C}$, the hom-functor $\mathrm{Hom}_{\mathcal{C}}$ maps a pair of objects $(a, b) \in (\mathcal{C}^{\mathrm{op}} \times \mathcal{C})_0 = \mathcal{C}_0 \times \mathcal{C}_0 = \mathcal{C}_0^2$ into the set[3] of morphisms $\mathcal{C}_1$ of $\mathcal{C}$ with source $a$ and target $b$. Therefore, $\mathrm{Hom}_{\mathcal{C}}(a, b)$ is the set of morphisms in $\mathcal{C}$ of type $a \to b$. Given morphisms $g^{\mathrm{op}} \in \mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(a, c)$ and $h \in \mathrm{Hom}_{\mathcal{C}}(b, d)$, the hom-functor $\mathrm{Hom}_{\mathcal{C}}$ constructs a function

$$\mathrm{Hom}_{\mathcal{C}}(g^{\mathrm{op}}, h) : \mathrm{Hom}_{\mathcal{C}}(a, b) \to \mathrm{Hom}_{\mathcal{C}}(c, d)$$

[3]The collection of morphisms of type $a \to b$ forms a set because $\mathcal{C}$ is locally small.

which takes a morphism $f : a \to b \in \mathrm{Hom}_{\mathcal{C}}(a, b)$ and produces the morphism $h \circ f \circ g : c \to d \in \mathrm{Hom}_{\mathcal{C}}(c, d)$. Graphically,

$$\mathrm{Hom}_{\mathcal{C}}(g^{\mathrm{op}}, h)\left( a \xrightarrow{f} b \right) = c \xrightarrow{g} a \xrightarrow{f} b \xrightarrow{h} d$$

## 2.6 Adjoint Functors

Given two categories $\mathscr{C}$ and $\mathscr{D}$, a pair of functors $L : \mathscr{C} \to \mathscr{D}, R : \mathscr{D} \to \mathscr{C}$ are called an *adjoint pair*, denoted $L \dashv R$ or

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "L"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "R"] \end{tikzcd}$$

if there exists a natural isomorphism $\alpha$ between the following pair of hom-functors of type $\mathscr{C}^{\mathrm{op}} \times \mathscr{D} \to \mathsf{Set}$:

$$\mathrm{Hom}_{\mathscr{D}}(L^{\mathrm{op}}(-), -) \overset{\alpha}{\simeq} \mathrm{Hom}_{\mathscr{C}}(-, R(-))$$

This relationship can be depicted graphically as 2-cell (and its inverse) in $\mathsf{Cat}$,

$$\begin{tikzcd} \mathcal{C}^{\mathrm{op}} \times \mathcal{D} \arrow[r, "I_{\mathcal{C}^{\mathrm{op}}} \times R"] \arrow[d, "L^{\mathrm{op}} \times I_{\mathcal{D}}"'] \arrow[dr, phantom, "\alpha \Uparrow \Downarrow \alpha^{-1}"] & \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \arrow[d, "\mathrm{Hom}_{\mathcal{C}}"] \\ \mathcal{D}^{\mathrm{op}} \times \mathcal{D} \arrow[r, "\mathrm{Hom}_{\mathcal{D}}"] & \mathsf{Set} \end{tikzcd}$$

Concretely, the naturality of $\alpha$ means that for every morphism $(f^{\mathrm{op}} : b \to a, g : c \to d) \in (\mathcal{C}^{\mathrm{op}} \times \mathcal{D})_1$ the components $\alpha_{(b,c)}$ and $\alpha_{(a,d)}$ of $\alpha$ make the following square commute:

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{D}}(L^{\mathrm{op}}(b), c) \arrow[r, "\mathrm{Hom}_{\mathcal{D}}(L^{\mathrm{op}}(f^{\mathrm{op}}), g)"] \arrow[d, "\alpha_{(b,c)}"'] & \mathrm{Hom}_{\mathcal{D}}(L^{\mathrm{op}}(a), d) \arrow[d, "\alpha_{(a,d)}"] \\ \mathrm{Hom}_{\mathcal{C}}(b, R(c)) \arrow[r, "\mathrm{Hom}_{\mathcal{C}}(f^{\mathrm{op}}, R(g))"] & \mathrm{Hom}_{\mathcal{C}}(a, R(d)) \end{tikzcd}$$

## 2.7 Beck-Chevalley Conditions

The Beck-Chevalley Conditions are conditions that may or may not be satisfied by a quadruplet of functors $F, H, G, K$ which form a natural isomorphism $\alpha : KF \Rightarrow HG$ square:

$$
\begin{tikzcd}
\mathcal{A} \arrow[r, "F"] \arrow[d, "G"'] & \mathcal{B} \arrow[d, "K"] \arrow[dl, Rightarrow, "\alpha"'] \\
\mathcal{C} \arrow[r, "H"'] & \mathcal{D}
\end{tikzcd}
$$

To define the *left* Beck-Chevalley condition, one needs functors $F_L : \mathcal{B} \to \mathcal{A}$ and $H_L : \mathcal{D} \to \mathcal{A}$ which are respectively left adjoint functors to $F$ and $H$,

$$
\begin{tikzcd}
\mathcal{A} \arrow[r, bend right, "F"'] \arrow[r, phantom, "\perp"] & \mathcal{B} \arrow[l, bend right, "F_L"']
\end{tikzcd}, \qquad
\begin{tikzcd}
\mathcal{C} \arrow[r, bend right, "H"'] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend right, "H_L"']
\end{tikzcd}.
$$

Using these left adjoint functors, it becomes possible to construct a natural transformation $\beta : KH_L \Rightarrow GF_L$ from $\alpha$[4]. Graphically, $\beta$ can be identified as the outer cell of the following diagram:

$$
\begin{tikzcd}
\mathcal{A} \arrow[r, bend right, "F"'] \arrow[r, phantom, "\perp"] \arrow[d, "G"'] & \mathcal{B} \arrow[l, bend right, "F_L"'] \arrow[d, "K"] \arrow[dl, Rightarrow, "\alpha"'] \\
\mathcal{C} \arrow[r, bend left, "H"] \arrow[r, phantom, "\top"] & \mathcal{D} \arrow[l, bend left, "H_L"]
\end{tikzcd}, \qquad \text{i.e.} \qquad
\begin{tikzcd}
\mathcal{A} \arrow[d, "G"'] & \mathcal{B} \arrow[l, "F_L"'] \arrow[d, "K"] \\
\mathcal{C} & \mathcal{D} \arrow[l, "H_L"] \arrow[ul, Rightarrow, "\beta"']
\end{tikzcd}.
$$

Although the natural transformation $\alpha$ is assumed to be a natural isomorphism, the natural transformation $\beta$ need not be; if $\beta$ happens to be a natural isomorphism, then we say that the original square satisfies the *left* Beck-Chevalley condition[5]. The *right* Beck-Chevalley condition is defined analogously with functors $F_R, H_R$ which are respectively right adjoints $F \dashv F_R$ and $H \dashv H_R$.

[4] The natural transformations $\alpha$ and $\beta$ are known as *mates* or *conjugates*.

[5] Are the left adjoints $F_L, H_L$ unique? If not, it might be better to say the original square satifies the left Beck-Chevalley condition with respect to $F_L, H_L$.

## 2.8 Slice and Coslice Categories

Given a category $\mathcal{C}$ and an object $c \in \mathcal{C}_0$ of $\mathcal{C}$, the *slice category* (or *over category*) $\mathcal{C}/c$ is the "stuff in $\mathcal{C}$ that is on top of $c$". Specifically, the objects of $\mathcal{C}/c$ are all the morphisms $f \in \mathcal{C}_1$ from $\mathcal{C}$ whose codomain is $\mathrm{cod}(f) = c$ (alternatively you could write $(\mathcal{C}/c)_0 = \mathrm{Hom}_{\mathcal{C}}(-, c)$). A morphism of $\mathcal{C}/c$ between objects $f : a \to c, g : b \to c \in (\mathcal{C}/c)_0$ is a commuting triangle completed by a third morphism $h : a \to b \in \mathcal{C}_1$:

$$\begin{tikzcd} a \arrow[rr, "h"] \arrow[dr, "g"'] & & b \arrow[dl, "f"] \\ & c & \end{tikzcd}$$

Composition of morphisms in $\mathcal{C}/c$ is induced by the composition of morphisms in $\mathcal{C}$:

$$\left( \begin{tikzcd} y \arrow[rr, "n"] \arrow[dr, "f"'] & & z \arrow[dl, "h"] \\ & c & \end{tikzcd} \right) \circ_{\mathcal{C}/c} \left( \begin{tikzcd} x \arrow[rr, "m"] \arrow[dr, "g"'] & & y \arrow[dl, "f"] \\ & c & \end{tikzcd} \right) = \begin{tikzcd} x \arrow[r, "m"] \arrow[dr, "g"'] & y \arrow[r, "n"] \arrow[d, "f"] & z \arrow[dl, "h"] \\ & c & \end{tikzcd}$$

The assignment of an overcategory $\mathcal{C}/c$ to each object $c$ can be extended to a *slice functor* $\mathcal{C}/(-) : \mathcal{C} \to \mathbf{Cat}$ in the following sense. For objects $c \in \mathcal{C}_0$, the slice functor takes $c$ to the slice category $\mathcal{C}/c$; for morphisms $f : a \to b \in \mathcal{C}_1$, the slice functor takes $f$ to the functor $\mathcal{C}/f : \mathcal{C}/a \to \mathcal{C}/b$ defined graphically; for every morphism of $\mathcal{C}/a$ (commuting triangle in $\mathcal{C}$ over $a$), contruct the morphism of $\mathcal{C}/b$ (commuting triangle in $\mathcal{C}$ over $b$) as follows:

$$\begin{tikzcd} l \arrow[rr, "m"] \arrow[dr, "x"'] \arrow[ddr, bend right=60, "f \circ_{\mathcal{C}} x"'] & & r \arrow[dl, "x'"] \arrow[ddl, bend left=60, "f \circ_{\mathcal{C}} x'"] \\ & a \arrow[d, "f"] & \\ & b & \end{tikzcd}$$

where the inner triangle is a morphism of $\mathcal{C}/a$ and the outer triangle is a morphism of $\mathcal{C}/b$ given by the functor $\mathcal{C}/f$.

Given a category $\mathcal{C}$ and an object $c \in \mathcal{C}_0$ of $\mathcal{C}$ the *coslice category* (or *under category*) $c/\mathcal{C}$ is the "stuff in $\mathcal{C}$ that is underneath $c$". Specifically, the objects of $c/\mathcal{C}$ are all the morphisms $f \in \mathcal{C}_1$ from $\mathcal{C}$ whose domain is $\mathrm{dom}(f) = c$ (alternatively you could write $(c/\mathcal{C})_0 = \mathrm{Hom}_{\mathcal{C}}(c, -)$). A morphism of $c/\mathcal{C}$ between objects $f : c \to a, g : c \to b \in (c/\mathcal{C})_0$ is a commuting triangle completed by a third morphism $h : a \to b \in \mathcal{C}_1$:

$$
\begin{tikzcd}
& c \arrow[dl, "g"'] \arrow[dr, "f"] & \\
a \arrow[rr, "h"'] & & b
\end{tikzcd}
$$

Everything about coslice categories is defined as expected analogously to that of a slice categories.

**TODO:** determine how the details of the Grothendieck construction transform the slice (pseudo-)functor $\mathcal{C}/(-) : \mathcal{C} \to \mathbf{Cat}$ into the codomain fibration

### 2.9 Functors of Monoidal Categories

[TODO]

### 2.10 Frobenius Reciprocity

[TODO]

## 3 Case Studies of Interest

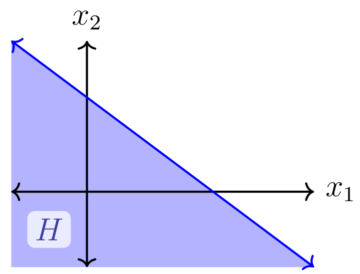
### 3.1 Polyhedra and Affine Maps

One of the primary motiviating examples for this project is the theory of *(finite) convex polyhedra* and the affine maps between them. Following Boyd and Vandenberghe [BV04], a *polyhedron*[6,7] $P$ is the intersection of a finite number of *halfspaces* of some ambient vector space $V \cong \mathbb{R}^n$. A *halfspace* $H \subseteq \mathbb{R}^n$ is a subset of a vector space (of dimension $n$) which is the solution set of a linear inequality constraint over canonical coordinates $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$:

$$H = \{x \in V \mid a^{\intercal} x = \sum_{i=1}^{n} a_i x_i \geq b\} \tag{11}$$

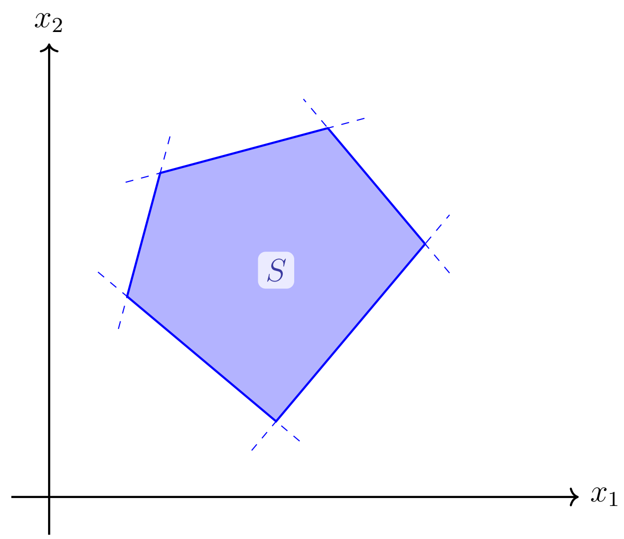
[6] The term polytope will be reserved for the context of *bounded polyhedron*. Note that the opposite convention is sometimes used by other authors as pointed out by [BV04].

[7] Alternative and sometimes inequivalent definitions for "polyhedra" do exist; oftentimes, these alternative definitions accomodate more general notions of polyhedra, such as non-convex polyhedra. Understanding the relationship between these various definitions, and the proposal of new ones, is a mathematical endeavour which dates back to antiquity and continues today [Grü03; Lak15].




As previously mentioned, a polyhedra is the intersection of finitely many halfspaces and therefore corresponds to

$$P = \{x \in V \mid \bigwedge_{j=1}^{k} (a_j^\intercal x \geq b_j)\} \tag{12}$$




$$\begin{tikzcd} f^*(\mathbb{R}^k_+) \arrow[r, "f'"] \arrow[d, hook, "j"] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{R}^k_+ \arrow[d, hook, "i"] \\ \mathbb{R}^n \arrow[r, "f"] & \mathbb{R}^k \end{tikzcd} \tag{13}$$

eq:pullback_of

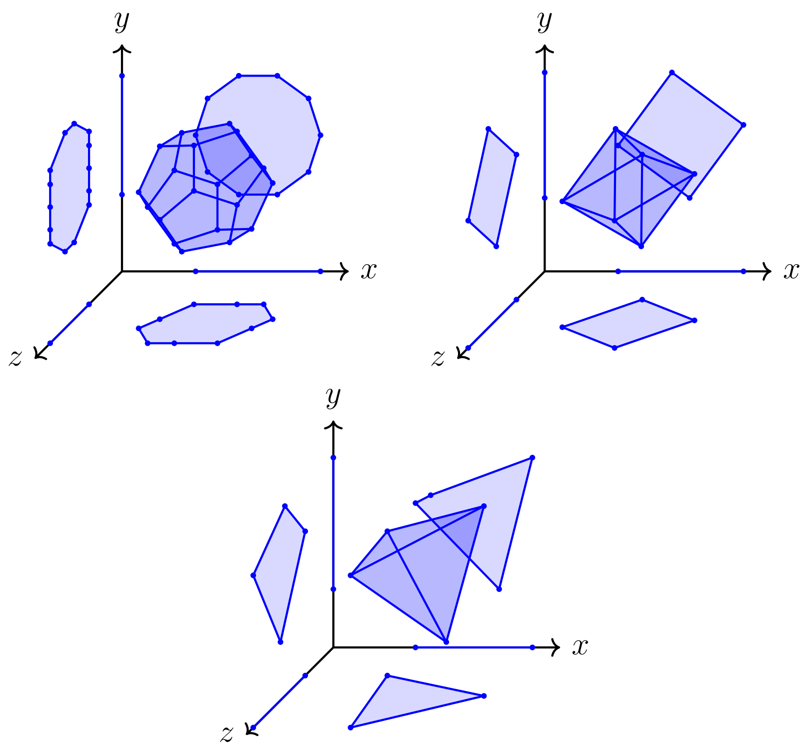
eq:pullback_of_positive_orthant

In Equation 13, the morphism $j$ is simply the inclusion of the polyhedra $f^*(\mathbb{R}^k_+)$ into its ambient vector space $\mathbb{R}^n$ and the morphism $f'$ is provided by the restriction of $f$ onto $f^*(\mathbb{R}^k_+)$.

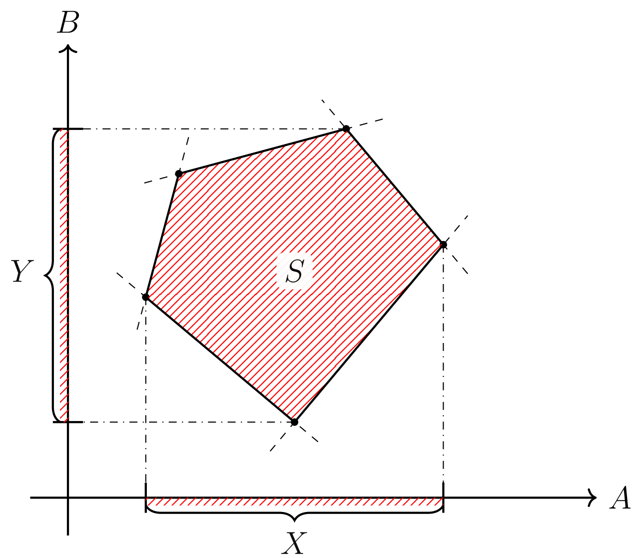
## 3.2 The Beck-Chevalley Condition for the Polyhedral Fibration

Consider a commuting square of affine maps between vector spaces in the base category of affine maps

**Definition 3.1.** The category $\mathsf{Poly}$ consists of polyhedra as objects and affine maps between them.




$$
\begin{tikzcd}
& \mathsf{Poly}_Y \arrow[dd, "\pi_{1,!}"] & & \mathsf{Poly}_{X\otimes Y} \arrow[ll, "\pi_{Y,!}"'] \arrow[dd, "\pi_{X,!}"'] \\
\mathsf{Poly}_{Y\otimes Z} \arrow[ur, "\pi_{Y,!}"'] \arrow[dd, "\pi_{Z,!}"] & & \mathsf{Poly}_{X\otimes Y\otimes Z} \arrow[ll, "\pi_{Y\otimes Z,!}"'] \arrow[ur, "\pi_{X\otimes Y,!}"] \arrow[dd, "\pi_{X\otimes Z,!}"] & \\
& \mathsf{Poly}_1 & & \mathsf{Poly}_X \arrow[ll, "\pi_{1,!}"'] \\
\mathsf{Poly}_Z \arrow[ur, "\pi_{1,!}"] & & \mathsf{Poly}_{X\otimes Z} \arrow[ll, "\pi_{Z,!}"'] \arrow[ur, "\pi_{X,!}"] &
\end{tikzcd}
\tag{14}
$$




## 3.3 The Codomain Fibration

arrow_category

**Definition 3.2.** For any category $\mathcal{C}$, its **arrow category** $\mathsf{Arr}(\mathcal{C})$ has as objects the morphisms $f : f_0 \to f_1$ of $\mathcal{C}$ and has as morphisms $\alpha : f \to g$ the commuting squares of $\mathcal{C}$, i.e.

$$\begin{tikzcd} f_0 \arrow[r, "f"] \arrow[d, "\alpha_0"'] & f_1 \arrow[d, "\alpha_1"] \\ g_0 \arrow[r, "g"'] & g_1 \end{tikzcd} \tag{15}$$

The arrow category can be equivalently defined as a functor category $[\mathsf{I}, \mathcal{C}] \simeq \mathsf{Arr}(\mathcal{C}) = \mathcal{C}^{\mathsf{I}}$ where $\mathsf{I}$ is the *interval category*

$$\begin{tikzcd} 0 \arrow[loop left, "\mathrm{id}_0"] \arrow[r, "i"] & 1 \arrow[loop right, "\mathrm{id}_1"] \end{tikzcd} \tag{16}$$

consisting of two objects and a non-identity morphism $i : 0 \to 1$ between them.

**Definition 3.3.** The **codomain functor** $\mathrm{cod} : \mathsf{Arr}(\mathcal{C}) \to \mathcal{C}$ takes a morphism of $\mathsf{Arr}(\mathcal{C})$ (commuting square of $\mathcal{C}$) $\alpha : f \to g$ to its codomain $\mathrm{cod}(\alpha) = g$,

$$\mathrm{cod}\left(\begin{tikzcd} f_0 \arrow[r, "f"] \arrow[d, "\alpha_0"'] & f_1 \arrow[d, "\alpha_1"] \\ g_0 \arrow[r, "g"'] & g_1 \end{tikzcd}\right) = \begin{tikzcd} g_0 \arrow[r, "g"'] & g_1 \end{tikzcd} \tag{17}$$

The fibers of the codomain functor $\mathrm{cod} : \mathsf{Arr}(\mathcal{C}) \to \mathcal{C}$ are therefore isomorphism to the slice categories; given an object $a$ in $\mathcal{C}$, the fiber over $a$ is the slice category $\mathsf{Arr}_a(\mathcal{C}) \simeq \mathcal{C}/a$ whose morphisms are commuting triangles over $a$ in $\mathcal{C}$, i.e.

$$\begin{tikzcd} c \arrow[rr, "t"] \arrow[dr, "g"'] & & d \arrow[dl, "h"] \\ & a & \end{tikzcd} . \tag{18}$$

Automatically, observe that the codomain functor $\mathrm{cod} : \mathsf{Arr}(\mathcal{C}) \to \mathcal{C}$ constitutes an opfibration; for each morphism $f : a \to b$, the associated fiber-convariant functor $f_! : \mathcal{C}/a \simeq \mathsf{Arr}_a(\mathcal{C}) \to \mathsf{Arr}_b(\mathcal{C}) \simeq \mathcal{C}/b$ is specified by post-composition with $f$:

$$f_! \left( \begin{tikzcd} c \arrow[rr, "t"] \arrow[dr, "g"'] & & d \arrow[dl, "h"] \\ & a & \end{tikzcd} \right) = \begin{tikzcd} c \arrow[rr, "t"] \arrow[dr, "fg"'] & & d \arrow[dl, "fh"] \\ & b & \end{tikzcd} \tag{19}$$

$$\begin{tikzcd} c \arrow[r, "t"] \arrow[d, "g"'] \arrow[dr, "fg" near start] & d \arrow[d, "fh"] \arrow[dl, "h" near end] \\ a \arrow[r, "f"'] & b \end{tikzcd} . \tag{20}$$

Under the right conditions, a codomain functor is also a *fibration* and thus an *bifibration.*

**Proposition 3.1.** *If a category $\mathcal{B}$ has pullbacks, the codomain functor* $\mathrm{cod} : \mathsf{Arr}(\mathcal{B}) \to \mathcal{B}$ *is a fibration called the* ***codomain fibration***.

For each morphism $f : a \to b$ in the base $\mathcal{B}$, the associated fiber-contravariant functor $f^* : \mathcal{B}/a \to \mathcal{B}/b$ is specified by pullback along $f$:

$$f^* \left( \begin{tikzcd} c \arrow[rr, "t"] \arrow[dr, "g"'] & & d \arrow[dl, "h"] \\ & b & \end{tikzcd} \right) = \begin{tikzcd} c' \arrow[rr, "t'"] \arrow[dr, "f^*g"'] & & d' \arrow[dl, "f^*h"] \\ & b & \end{tikzcd} \tag{21}$$

$$\begin{tikzcd} c' \arrow[rr, "g^*f"] \arrow[dd, "f^*g"'] \arrow[dr, dashed, "t'"] \arrow[dr, phantom, "\lrcorner", very near start] & & c \arrow[dr, "t"] \arrow[dd, "g" near end] & \\ & d' \arrow[rr, "h^*f" near start] \arrow[dl, "f^*h"] & & d \arrow[dl, "h"] \\ a \arrow[rr, "f"'] & & b & \end{tikzcd} . \tag{22}$$

Note that the morphism $t' : c' \to d'$ completing the resulting commuting triangle is unique by the universality of $d'$ as the pullback of $a \xrightarrow{f} b \xleftarrow{h} d$.

Given a category $\mathcal{B}$ with pullbacks, and a morphism $f : a \to b$ of $\mathcal{B}$, the codomain bifibration $\mathrm{cod} : \mathsf{Arr}(\mathcal{B}) \to \mathcal{B}$ induces a adjoint pair of functors between the fibers

$$\begin{tikzcd} \mathcal{B}/a \arrow[r, bend left, "f_!"] \arrow[r, phantom, "\perp"] & \mathcal{B}/b \arrow[l, bend left, "f^*"] \end{tikzcd} \tag{23}$$

such that $f_! : \mathcal{B}/a \to \mathcal{B}/b$ is given by post-composition and $f^* : \mathcal{B}/b \to \mathcal{B}/a$ is given by pullback.

**Lemma 3.2.** *Given a category $\mathcal{B}$ with pullbacks, the codomain bifibration* cod : $\mathsf{Arr}(\mathcal{B}) \to \mathcal{B}$ *satisfies the Beck-Chevalley condition at every pullback square in $\mathcal{B}$.*

*Proof.* Consider an arbitrary pullback square in base category $\mathcal{B}$,

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[d, "h"'] \arrow[dr, phantom, "\lrcorner", very near start] & b \arrow[d, "g"] \\ c \arrow[r, "k"'] & d \end{tikzcd} . \tag{24}$$

eq:generic_pul

Automatically, there exists a natural isomorphism $\alpha : f^*g^* \to h^*k^*$:

$$\begin{tikzcd} \mathcal{B}/a & \mathcal{B}/b \arrow[l, "f^*"'] \arrow[dl, Rightarrow, "\simeq"', "\alpha"] \\ \mathcal{B}/c \arrow[u, "h^*"] & \mathcal{B}/d \arrow[l, "k^*"] \arrow[u, "g^*"'] \end{tikzcd} \tag{25}$$

eq:right_adjo

The individual components $\alpha_\psi$ for each object $\psi : p \to d$ of $\mathcal{B}/d$ is specified by the following diagram:

$$\begin{tikzcd} a' \arrow[rr, "h^*k^*\psi"'] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & & a \arrow[dl, "h"'] \arrow[dr, "f"] & & a'' \arrow[ll, "f^*g^*\psi"] \arrow[d] \arrow[llll, dashed, bend right, "\simeq"', "\alpha_\psi"] \\ c' \arrow[r, "k^*\psi"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & c \arrow[dr, "k"] & & b \arrow[dl, "g"'] & b' \arrow[l, "g^*\psi"'] \arrow[d] \\ p \arrow[rr, "\psi"] \arrow[rrrr, equal, bend right] & & d & & p \arrow[ll, "\psi"'] \end{tikzcd} \tag{26}$$

eq:right_adjo

Using the composition of pullback squares, it is clear that *both* $h^*k^*\psi$ and $f^*g^*\psi$ are projection morphisms (onto $a$) in the pullback square of $a \xrightarrow{gf=kh} d \xleftarrow{\psi} p$ and therefore, they are unique up to a unique isomorphism; the component $\alpha_\psi$ is precisely that isomorphism. Notice that this argument *does not* make use of the fact that the original commuting square in Equation eq:generic_pullback 24 was a pullback square.

To prove the Beck-Chevalley condition, one must demonstrate that the natural transformation $\beta = \varepsilon_h \alpha \eta_g : h_! f^* \to k^* g_!$ is also a *natural isomorphism.* Remembering that the left adjoints can be computed by post-composition, the component of $\beta$ for an object $\psi : p \to b$ of $\mathcal{B}/b$, denoted $\beta_\psi$, and its inverse can be determined by the following diagram:

$$
\begin{tikzcd}
c' \arrow[r, "k^* g_! \psi"] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & c \arrow[d, "k"] & a \arrow[l, "h"'] \arrow[d, "f"] \arrow[dl, phantom, very near start, "\llcorner"] & a' \arrow[l, "f^* \psi"'] \arrow[d] \arrow[ll, bend right, "h_! f^* \psi"'] \arrow[lll, dashed, bend right, "\simeq", "\beta_\psi"'] \arrow[dl, phantom, very near start, "\llcorner"] \\
p \arrow[r, "g_! \psi"] \arrow[rrr, equal, bend right] & d & b \arrow[l, "g"'] & p \arrow[l, "\psi"']
\end{tikzcd}
\tag{27}
$$

Similarly, $k^* g_! \psi$ and $h_! f^* \psi$ are *both* projection morphisms (onto $c$) in the pullback square of $p \xrightarrow{g\psi} d \xleftarrow{k} c$ and therefore, they are unique up to a unique isomorphism $\beta_\psi$. The key difference in this case is that this argument *does* rely on $a$ being the pullback object in the original pullback square in the base $\mathcal{B}$ (Equation 24). eq:generic_pullback ∎

## 3.4 The Subobject Fibration

Given a category $\mathcal{B}$, and any object $a$ of $\mathcal{B}$, there is a preorder relation $\preceq_a$ among monomorphisms with shared codomain $a$; given $f : b \hookrightarrow a, g : c \hookrightarrow a$,

$$f \preceq_a g \iff \exists k : b \to c, f = gk. \tag{28}$$

$$
\begin{tikzcd}
b \arrow[rr, "k"] \arrow[dr, hook, "f"'] & & c \arrow[dl, hook', "g"] \\
& a &
\end{tikzcd}
\tag{29}
$$

Note that if such a $k : b \to c$ exists, then it is unique because it $g$ is a monomorpism; if $k' : b \to c$ satisfied $f = gk'$ as well, then $gk = gk'$ which implies $k = k'$. The natural equivalence relation between monomorphisms into $a$ induced by this preorder will be denoted $\simeq_a$,

$$f \simeq_a g \iff (f \preceq_a g) \wedge (g \preceq_a f). \tag{30}$$

$$
\begin{tikzcd}
b \arrow[rr, bend left=10, "k"] \arrow[dr, hook, "f"'] & & c \arrow[ll, bend left=10, "k'"] \arrow[dl, hook', "g"] \\
& a &
\end{tikzcd}
\tag{31}
$$

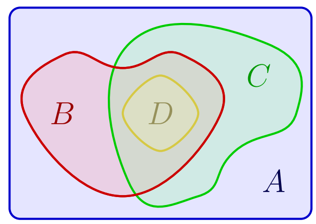
**Definition 3.4.** Given a category $\mathcal{B}$, a **subobject** of $a$ is an equivalence class of monomorphisms with codomain $a$ up to isomorphism.

The unique subobject class containing $f : b \hookrightarrow a$ will be denoted with square brackets $[f] = \{g : c \hookrightarrow a \mid f \sim_a g\}$.

**TC:** What happens when $[f]$ is a proper class and not a set? Is there a set-builder notation for proper classes?

The preorder relation $\preceq_a$ on the individual monomorphisms extends to a poset between the subobjects denoted $\mathsf{Sub}_a(\mathsf{Set})$ between the subobjects. For example, if $\mathcal{B}$ is the category $\mathsf{Set}$, then the subobjects of any set $X$ are isomorphic to its subsets (i.e. a subset $A \subseteq X$ is isomorphic to the class $[i_A] \simeq A$ where $i_A : A \to X$ is the standard inclusion morphism), and thus $\mathsf{Sub}_X(\mathsf{Set}) \cong \mathsf{P}(X)$ is the powerset poset ordered by inclusion. Even more concretely, if $X$ is the set $\mathbb{R}^2$, an exemplary diagram of $\mathsf{Sub}_{\mathbb{R}^2}(\mathsf{Set})$ is

$$\begin{tikzcd}
 & {[i_A]} & \\
{[i_B]} \arrow[ur, "\subseteq"] & & {[i_C]} \arrow[ul, "\subseteq"'] \\
 & {[i_D]} \arrow[ul, "\subseteq"] \arrow[ur, "\subseteq"'] \arrow[uu, hook, "\subseteq"] &
\end{tikzcd}$$




lem:mono_stable **Lemma 3.3.** *Monomorphisms are stable under pullback; given a monomorphism $m : a \hookrightarrow b$, and morphism $f : c \to b$, the pullback of $m$ along $f$, denoted $f^*m : f^*a \hookrightarrow c$ is also a monomorphism.*

$$\begin{tikzcd}
f^*a \arrow[r, "m^*f"] \arrow[d, hook, "f^*m"'] \arrow[dr, phantom, "\lrcorner", very near start] & a \arrow[d, hook, "m"] \\
c \arrow[r, "k"] & b
\end{tikzcd} \tag{32}$$

If the category $\mathcal{B}$ has pullbacks, the posetal categories $\mathsf{Sub}_a(\mathcal{B})$ for varying objects $a$ of $\mathcal{B}$ can be "stitched together" to form an enveloping category denoted $\mathsf{Sub}(\mathcal{B})$. The objects of $\mathsf{Sub}(\mathcal{B})$ are the objects of $\mathsf{Sub}_a(\mathcal{B})$ for all objects $a$ of $\mathcal{B}$. The morphisms of $\mathsf{Sub}(\mathcal{B})$ are a little more complicated; given subobjects $[f] \in \mathsf{Sub}_a(\mathcal{B})$, $[g] \in \mathsf{Sub}_b(\mathcal{B})$ with possibly different codomains, a morphism $k : [g] \to [f]$ of $\mathsf{Sub}(\mathcal{B})$ is a morphism $k : b \to a$ of $\mathcal{B}$ such that the subobject $k^*[f] := \{k^*\tilde{f} \mid \tilde{f} \in [f]\} \in \mathsf{Sub}_{\mathcal{B}}(b)$[8] is above $[g]$ with respect to the poset $\mathsf{Sub}_b(\mathcal{B})$. As a diagram of $\mathcal{B}$, $k : [g] \to [f]$ provided that for all $\tilde{g} \in [g]$, $\tilde{f} \in [f]$ there exists a morphism $h$ such that:

$$\begin{tikzcd}
\cdot \arrow[r, "h"] \arrow[d, hook, "\tilde{g}"'] & \cdot \arrow[r, "\tilde{f}^*k"] \arrow[d, hook, "k^*\tilde{f}"'] \arrow[dr, phantom, "\lrcorner", very near start] & c \arrow[d, hook, "\tilde{f}"] \\
b \arrow[r, equal] & b \arrow[r, "k"] & a
\end{tikzcd} \tag{33}$$

The fact that diagrams of the above sort compose to form a formal category $\mathsf{Sub}(\mathcal{B})$ relies on the fact that the morphisms of $\mathsf{Sub}_a(\mathcal{B})$ which order the subobjects of $a$ are analogously transported by pullback along $k$ to become morphisms of $\mathsf{Sub}_b(\mathcal{B})$; in fact $k^* : \mathsf{Sub}_a(\mathcal{B}) \to \mathsf{Sub}_b(\mathcal{B})$ forms a functor between the individual subobject categories.

[8]The fact that $k^*[f]$ constitutes a subobject relies on Lemma lem:mono_stable 3.3.

**Proposition 3.4.** *For any category $\mathcal{B}$ with pullbacks, the functor $S : \mathsf{Sub}(\mathcal{B}) \to \mathcal{B}$ which sends $[f] \in \mathsf{Sub}_a(\mathcal{B})$ to the object $a$ defines the* ***subobject fibration****.*

Note that while the fibers $\mathsf{Sub}_a(\mathcal{B})$ are posets and thus thin, the total category $\mathsf{Sub}(\mathcal{B})$ is not necessarily thin.

**Definition 3.5.** Given a category $\mathcal{B}$, and a morphism $f : a \to b$, the **image factorization** of $f$, if it exists, is a pair of morphisms $e : a \to c$ and $\operatorname{im}(f) : c \hookrightarrow b$ such that $f = me$ and is *universal* in the following sense:

$$\begin{tikzcd} a \arrow[rr, "f"] \arrow[dr, "e"'] \arrow[ddr, bend right, "e'"'] & & b \\ & c \arrow[ur, hook, "\operatorname{im}(f)"'] \arrow[d, dashed, "\exists!"] & \\ & c' \arrow[uur, hook, bend right, "m"'] & \end{tikzcd} . \tag{34}$$

The monomorphism $\operatorname{im}(f)$ is called the **image** of $f$.

**TC:** It seems to me that the common notation is to let $\operatorname{im}(f)$ denote the *object*, not the associated monomorphism. I would like to improve the notation somehow.

**TODO:** Clean up the discussion here, pullbacks are limits and images are some sort of colimits right?

Given any category $\mathcal{B}$ that has pullbacks and also admits image factorizations, the subobject fibration $S : \mathsf{Sub}(\mathcal{B}) \to \mathcal{B}$ also constitutes an opfibration, and thus a bifibration. Given a morphism $f : a \to b$ of $\mathcal{B}$, the left adjoint functor $f_!$ acting on a subobject $[\psi] : \mathsf{Sub}_a(\mathcal{B})$ satisfies

$$f_!([\psi]) = [\operatorname{im}(f\psi)] \in \mathsf{Sub}_b(\mathcal{B}).$$

To summarize, given a morphism $f : a \to b$ of $\mathcal{B}$, there is an induced adjoint pair of functors between the subobject fibers:

$$\begin{tikzcd} \mathsf{Sub}_a(\mathcal{B}) \arrow[r, bend left, "f_!"] \arrow[r, phantom, "\perp"] & \mathsf{Sub}_b(\mathcal{B}) \arrow[l, bend left, "f^*"] \end{tikzcd} \tag{35}$$

As previously discussed, the right adjoint functor $f^*$ acting on a subobject $[\psi] \in \mathsf{Sub}_b(\mathcal{B})$ is given by pullback of $a \xrightarrow{f} b \xhookleftarrow{\psi} b'$ in $\mathcal{B}$, i.e. $f^*([\psi]) = [f^*\psi]$:

$$\begin{tikzcd} f^*b' \arrow[r] \arrow[d, hook, "f^*\psi"'] \arrow[dr, phantom, very near start, "\lrcorner"] & b' \arrow[d, hook, "\psi"] \\ a \arrow[r, "f"'] & b \end{tikzcd} \tag{36}$$

**Lemma 3.5.** *Given a category $\mathcal{B}$ with pullbacks and pullback-stable image factorizations (this includes regular categories), the subobject bifibration $S : \mathsf{Sub}(\mathcal{B}) \to \mathcal{B}$ satisfies the Beck-Chevalley condition at every pullback square in $\mathcal{B}$.*

*Proof.* Again, consider an arbitrary pullback square in base category $\mathcal{B}$,

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[d, "h"'] \arrow[dr, phantom, "\lrcorner", very near start] & b \arrow[d, "g"] \\ c \arrow[r, "k"'] & d \end{tikzcd} . \tag{37}$$

Automatically, there exists a natural isomorphism $\alpha : f^*g^* \to h^*k^*$:

$$\begin{tikzcd} \mathsf{Sub}_a(\mathcal{B}) & \mathsf{Sub}_b(\mathcal{B}) \arrow[l, "f^*"'] \arrow[dl, Rightarrow, "\simeq"', "\alpha"] \\ \mathsf{Sub}_c(\mathcal{B}) \arrow[u, "h^*"] & \mathsf{Sub}_d(\mathcal{B}) \arrow[l, "k^*"] \arrow[u, "g^*"'] \end{tikzcd} \tag{38}$$

By Lemma lem:mono_stable 3.3, the argument is directly analogous to the diagram of Equation eq:right_adjoints_at 26. The key difference is that while

$$\begin{tikzcd} b \arrow[rr, dashed, "\simeq", "\alpha_\psi"'] \arrow[dr, "f^*g^*\psi"'] & & c \arrow[dl, "h^*k^*\psi"] \\ & a & \end{tikzcd} \tag{39}$$

and thus $h^*k^*\psi \simeq f^*g^*\psi$ as before, the subobjects are precisely equal, *not isomorphic*:

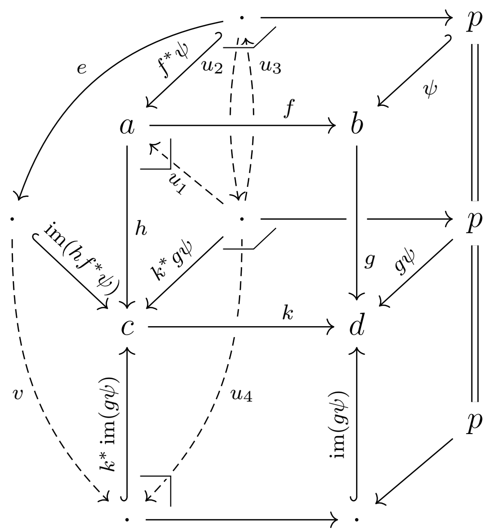
$$[h^*k^*\psi] = [f^*g^*\psi].$$

Therfore, the components of the natural isomorphism $\alpha$ are identities $\alpha_{[\psi]} = \mathrm{id}_{[\psi]}$.

To prove the Beck-Chevalley condition, one must demonstrate that the natural transformation $\beta = \varepsilon_h \alpha \eta_g : h_! f^* \to k^* g_!$ is also a *natural isomorphism*. This will be accomplished by showing that for all monomorphisms $\psi : p \to b$ into $b$, $h_! f^* \psi \simeq k^* g_! \psi$ as monomorphisms into $c$; explicity

$$\mathrm{im}(hf^*\psi) \simeq k^* \,\mathrm{im}(g\psi). \tag{40}$$

Without relying the stability of image factorizations through pullback, the following diagram demonstrates the forward direction: $\operatorname{im}(hf^*\psi) \preceq k^* \operatorname{im}(g\psi)$.




(41)

There are four unique morphisms, $u_1, u_2, u_3$ and $u_4$, which follow from the universality of various pullback constructions. In order,

1. $u_1$ follows from the pullback of $c \xrightarrow{k} d \xleftarrow{g} b$,
2. $u_2$ by pullback of $c \xrightarrow{k} d \xleftarrow{g\psi} p$,
3. $u_3$ by pullback of $a \xrightarrow{f} b \xhookleftarrow{\psi} p$ (this relies on the presence of $u_1$),
4. and finally, $u_4$ by pullback of $c \xrightarrow{k} d \xhookleftarrow{\operatorname{im}(g\psi)} \cdot$.

An additional unique morphism, $v$, is a result of an image factorization:

$$\begin{tikzcd}
\cdot \arrow[rr, "hf^*\psi"] \arrow[dr] \arrow[ddr, bend right, "u_4u_2"'] & & c \\
& \cdot \arrow[ur, hook, "\operatorname{im}(hf^*\psi)"] \arrow[d, dashed, "\exists! v"] & \\
& \cdot \arrow[uur, hook, bend right, "k^*\operatorname{im}(g\psi)"'] &
\end{tikzcd} . \tag{42}$$

This final morphism, $v$, shows that $\operatorname{im}(hf^*\psi) \preceq k^* \operatorname{im}(g\psi)$. To show $\operatorname{im}(hf^*\psi) \succeq k^* \operatorname{im}(g\psi)$, one needs an additional condition, namely that image factorizations are stable under pullback: e.g. $\operatorname{im}(k^*g\psi) = k^* \operatorname{im}(g\psi)$:

$$\begin{tikzcd}
\cdot \arrow[rr, "k^*g\psi"] \arrow[dr, "u_4"] \arrow[ddr, bend right, "eu_3"'] & & c \\
& \cdot \arrow[ur, hook, "\operatorname{im}(k^*g\psi)", "k^*\operatorname{im}(g\psi)"'] \arrow[d, dashed, "\exists! w"] & \\
& \cdot \arrow[uur, hook, bend right, "\operatorname{im}(hf^*\psi)"'] &
\end{tikzcd} . \tag{43}$$

Therefore $\mathrm{im}(hf^*\psi) \simeq k^* \mathrm{im}(g\psi)$, and thus the Beck-Chevalley condition holds with components $\beta_{[\psi]} = \mathrm{id}_{[\psi]}$. ■

## 3.5 The Category of Convex Cones and Linear Maps

Given any $\mathbb{R}$-vector space $V$, a *(closed) cone* $C \subseteq V$ is a subset of $V$ such that for any elements $c_1, c_2 \in C$ and for any positive coefficients $\gamma_1, \gamma_2 \geq 0$, $\gamma_1 c_1 + \gamma_2 c_2 \in C$. A polyhedral cone $C \subseteq V$ is one which admits a half-space representation in terms a finite number of linear constraints:

$$C = \{x \in V \mid \bigwedge_{i=1}^{K} (a_i \cdot x \geq 0)\} \tag{44}$$

Alternatively, a $\mathsf{Cone}$ can be expressed in terms of the pullback of the positive orthant

$$\mathbb{R}^n_+ \coloneqq \{v \in \mathbb{R}^n \mid \forall i \in [n] : v_i \geq 0\} \tag{45}$$

by a linear transformation $f : V \to \mathbb{R}^n$ into $\mathbb{R}^n$.

$$\begin{tikzcd}
f^*(\mathbb{R}^n_+) \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{R}^n_+ \arrow[d, hook, "i_+"] \\
V \arrow[r, "f"] & \mathbb{R}^n
\end{tikzcd} \tag{46}$$

$$f^*(\mathbb{R}^n_+) \cong \{v \in V \mid f(v) \in \mathbb{R}^n_+\} \tag{47}$$

Given a cone $f^*(\mathbb{R}^n_+) \subseteq V$ associated with a finite set of $n$ linear expressions $f : V \to \mathbb{R}^n$, and a linear transformation $g : V \to W$,

## 3.6 Subset Projection

A prototypical example wherein an adjoint triple

$$f_!, \exists_f \dashv f^*, f^{-1} \dashv f^!, \forall_f$$

arises is that of functions $f : X \to Y$ between sets $X$ and $Y$. The inverse image functor $f^* : \mathscr{P}Y \to \mathscr{P}X$ is defined on a subset $T \subseteq Y$

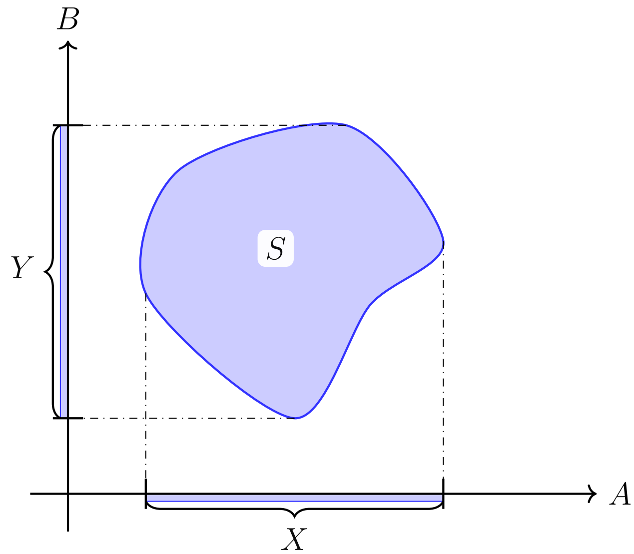
$$f^*(T) = \{x \in X : f(x) \in T\},$$

and is functorial in the sense that if $T \subseteq T' \subseteq Y$ then $f^*(T) \subseteq f^*(T') \subseteq f^*(T)$. The adjoint functors $\exists_f, \forall_f : \mathscr{P}X \to \mathscr{P}Y$ are defined on $S \subseteq X$ as

$$\exists_f(S) = \{y \in Y : \exists x \in f^*(y) : x \in S\}$$
$$\forall_f(S) = \{y \in Y : \forall x \in f^*(y) : x \in S\}$$

form an adjoint triple in the sense that $\exists_f \dashv f^* \dashv \forall_f$:

$$\begin{aligned} \exists_f \dashv f^* : \quad & \exists_f(S) \subseteq T \iff S \subseteq f^*(T) \\ f^* \dashv \forall_f : \quad & f^*(T) \subseteq R \iff T \subseteq \forall_f(R) \end{aligned}$$

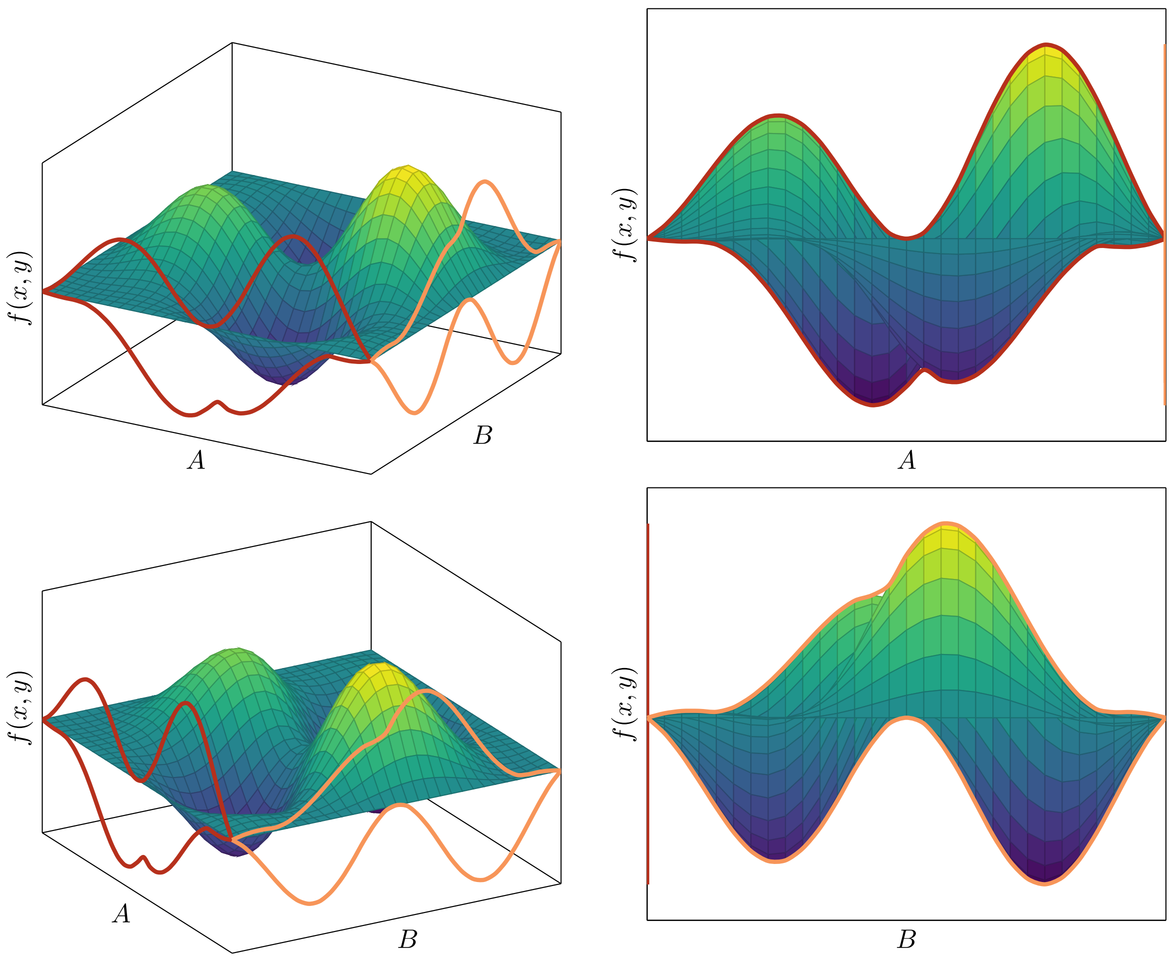



Consider a pair of sets $A$ and $B$ and a subset $S \subseteq A \times B$ of their cartesian product. The projection morphisms associated with $A \times B$ are $p : A \times B \to A$ and $q : A \times B \to B$. The projection of the subset $S$ onto $A$ is then the subset $X \subseteq A$ defined by:

$$X = \{a \in A \mid \exists s \in S, p(s) = a\}$$

$$S \subseteq p^*(X) \iff \exists_p(S) \subseteq X \tag{48}$$

### 3.7 Optimization of real-valued functions




# Potentially Annotated Bibliography

This section is temporary and reserved for recording comments toward various references.


- vistoli2004notes vistoli2004notes
  Vistoli [Vis04]
- street1974fibrations street1974fibrations
  Street [Str74]
- koudenburg2018categorical koudenburg2018categorical
  Koudenburg [Kou18]
- brown2009algebraic brown2009algebraic
  Brown and Sivera [BS09]
- lurie2009higher lurie2009higher
  Lurie [Lur09]

- shulman2008framed shulman2008framed
  Shulman [Shu08]
- boyd2004convex boyd2004convex
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- bogart2013hom bogart2013hom
  Bogart, Contois, and Gubeladze [BCG13]
- gubeladze2016affine gubeladze2016affine
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- fausk2003isomorphisms fausk2003isomorphisms
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- hofstra2011dialectica hofstra2011dialectica
  Hofstra [Hof11]
- ponto2012duality ponto2012duality
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- ziegler2012lectures ziegler2012lectures
  Ziegler [Zie12]
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