# On The Bifibrations Underlying Optimization and Elimination

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## 1 Examples

A prototypical example wherein an adjoint triple

$$f_!, \exists_f \dashv f^*, f^{-1} \dashv f^!, \forall_f$$

arises is that of functions $f : X \to Y$ between sets $X$ and $Y$. The inverse image functor $f^* : \mathscr{P}Y \to \mathscr{P}X$ is defined on a subset $T \subseteq Y$

$$f^*(T) = \{x \in X : f(x) \in T\},$$

and is functorial in the sense that if $T \subseteq T' \subseteq Y$ then $f^*(T) \subseteq f^*(T') \subseteq f^*(T)$. The adjoint functors $\exists_f, \forall_f : \mathscr{P}X \to \mathscr{P}Y$ are defined on $S \subseteq X$ as

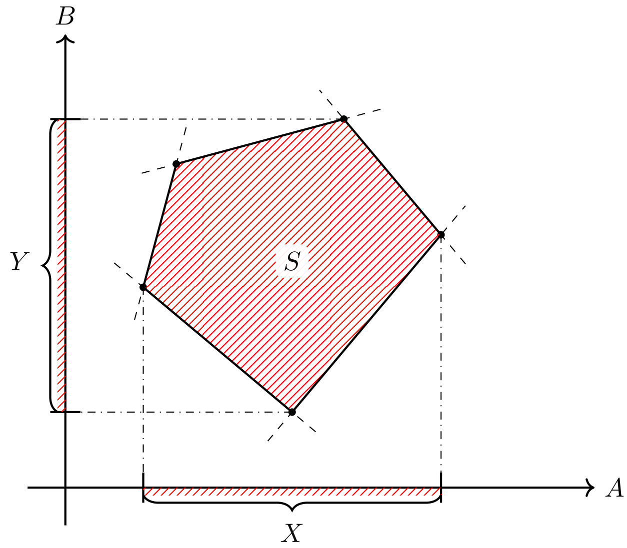
$$\exists_f(S) = \{y \in Y : \exists x \in f^*(y) : x \in S\}$$
$$\forall_f(S) = \{y \in Y : \forall x \in f^*(y) : x \in S\}$$

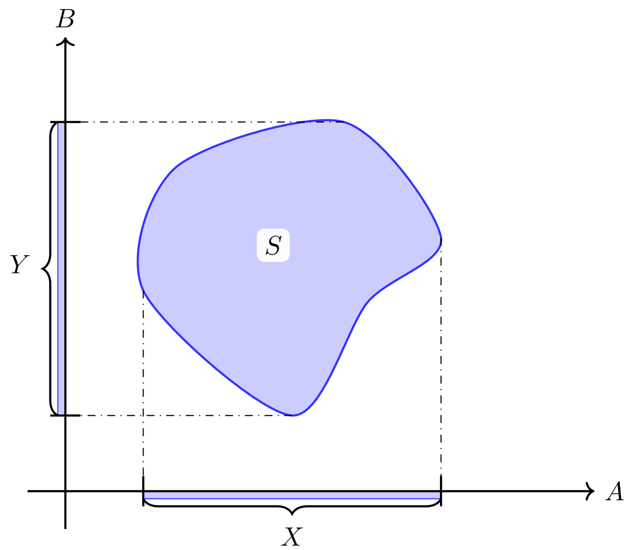
form an adjoint triple in the sense that $\exists_f \dashv f^* \dashv \forall_f$:

$$\exists_f \dashv f^* : \quad \exists_f(S) \subseteq T \iff S \subseteq f^*(T)$$
$$f^* \dashv \forall_f : \quad f^*(T) \subseteq R \iff T \subseteq \forall_f(R)$$

| Context | Fibration | Total $\mathcal{E}$ | Base $\mathcal{E}$ | Fibers | Covariant Functor | Contravariant Functor |
|---|---|---|---|---|---|---|
| Subset Projection | | | | | | |
| Linear Quantifier Elimination | | | | | | |
| Non-linear Quantifier Elimination | | | | | | |
| Real-valued Optimization | | | | | | |
| General Lattice Optimization | | | | | | |
| Convex Projection | | | | | | |
| Convex Optimization | | | | | | |
| Resolution | | | | | | |
| . . . and more | | | | | | |

### 1.1 Subset Projection

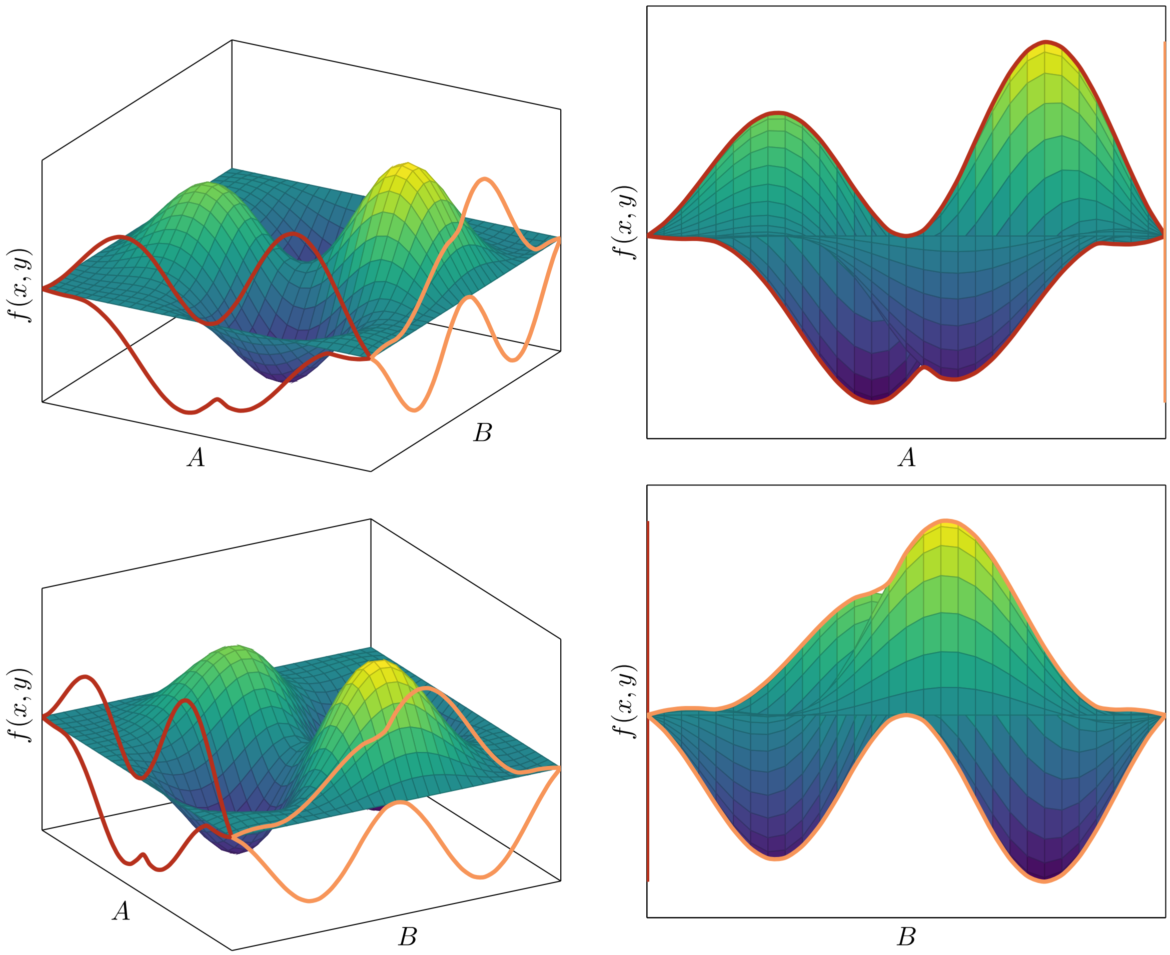



Consider a pair of sets $A$ and $B$ and a subset $S \subseteq A \times B$ of their cartesian product. The projection morphisms associated with $A \times B$ are $p : A \times B \to A$ and $q : A \times B \to B$. The projection of the subset $S$ onto $A$ is then the subset $X \subseteq A$ defined by:

$$X = \{a \in A \mid \exists s \in S, p(s) = a\}$$

$$S \subseteq p^*(X) \iff \exists_p(S) \subseteq X \tag{1}$$


$f(x, y)$
$A$
$B$
$f(x, y)$
$A$
$f(x, y)$
$A$
$B$
$f(x, y)$
$B$

## 2 Categorical Notions

The following unordered list of categorical concepts are anticipated to be utilized:

- adjunctions
- fibered categories
- cleavages
- puesdo functors (and if cleavages are splitting, functors)
- Beck-Chevalley condition
- Frobenius reciprocity (and functors of monoidal categories)

**Definition 2.1.** Let $P : \mathcal{E} \to \mathcal{B}$ be a functor between categories $\mathcal{E}$ and $\mathcal{B}$. An arrow $\phi : \alpha \to \beta$ of $\mathcal{E}$ is *cartesian* with respect to $P$ if for every arrow $\psi : \gamma \to \beta$ sharing a codomain with $\phi$, and for every arrow $g : P(\gamma) \to P(\alpha)$ in $\mathcal{B}$ satisfying $g \circ P(\phi) = P(\psi)$, there exists a unique arrow $\theta : \gamma \to \alpha$ in $\mathcal{E}$ satisfying $\phi \circ \theta = \psi$ and $P(\theta) = g$.

$$\begin{tikzcd}
\gamma \arrow[drr, bend left, "\forall\psi"] \arrow[dr, dashed, "\exists!\theta"'] \arrow[dd, mapsto] & & \\
& \alpha \arrow[r, "\phi"'] \arrow[dd, mapsto] & \beta \arrow[dd, mapsto] \\
P(\gamma) \arrow[drr, bend left, "P(\psi)"] \arrow[dr, "\forall g"'] & & \\
& P(\alpha) \arrow[r, "P(\phi)"'] & P(\beta)
\end{tikzcd} \tag{2}$$

**Corollary 2.0.1.** *A cartesian morphism $\phi : \alpha \to \beta$ in $\mathcal{E}$ with respect to a functor $P : \mathcal{E} \to \mathcal{B}$ establishes an isomorphism of categories [Lur09, Section 2.4.1]*[1]

$$\mathcal{E}/\phi \cong \mathcal{E}/\beta \times_{\mathcal{B}/P(\beta)} \mathcal{B}/P(\phi) \tag{3}$$

*where $\mathcal{E}/\beta \times_{\mathcal{B}/P(\beta)} \mathcal{B}/P(\phi)$ is the pullback of functors.*

$$\begin{tikzcd}
\mathcal{E}/\phi \arrow[drr, bend left, "P/\phi"] \arrow[dr, dashed, leftrightarrow, "\cong"] \arrow[ddr, bend right, "\mathrm{cod}"'] & & \\
& \mathcal{E}/\beta \times_{\mathcal{B}/P(\beta)} \mathcal{B}/P(\phi) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathcal{B}/P(\phi) \arrow[d, "\mathrm{cod}"] \\
& \mathcal{E}/\beta \arrow[r, "P/\beta"] & \mathcal{B}/P(\beta)
\end{tikzcd} \tag{4}$$

The pullback category $\mathcal{E}/\beta \times_{\mathcal{B}/P(\beta)} \mathcal{B}/P(\phi)$ has morphisms associated with diagrams of $\mathcal{B}$ with the following format:

$$\begin{tikzcd}
& P(\gamma) \arrow[d, "P(\chi)"] \arrow[ddl, bend right, "f"'] \arrow[ddr, bend left, "P(\omega)"] & \\
& P(\delta) \arrow[dl, "g"'] \arrow[dr, "P(\psi)"] & \\
P(\alpha) \arrow[rr, "P(\phi)"] & & P(\beta)
\end{tikzcd} \tag{5}$$

[1]This formulation is also discussed here: https://ncatlab.org/nlab/show/Cartesian+morphism#CartInOrdCatReformulation.

Evidently, if $\phi : \alpha \to \beta$ is cartesian, then there exists unique morphisms $\zeta : \gamma \to \alpha$ and $\eta : \delta \to \alpha$ such that $P(\zeta) = f$ and $P(\eta) = g$ and the following diagram of $\mathcal{E}$ commutes:

$$\begin{tikzcd}
& \gamma \arrow[d, "\chi"] \arrow[ddl, bend right, "\zeta"'] \arrow[ddr, bend left, "\omega"] & \\
& \delta \arrow[dl, "\eta"'] \arrow[dr, "\psi"] & \\
\alpha \arrow[rr, "\phi"] & & \beta
\end{tikzcd} \tag{6}$$

Intuitively, if $\phi$ is cartesian, then in order to determine the category $\mathcal{E}/\phi$ over $\phi$, it is sufficient to specify $\mathcal{E}/\beta \times_{\mathcal{B}/P(\beta)} \mathcal{B}/P(\phi)$.

**Lemma 2.1.** *A fibered category $P : \mathcal{E} \to \mathcal{B}$ is a faithful functor if and only if its fibers are thin.*

*Proof.* Recall that if $P : \mathcal{E} \to \mathcal{B}$ is a faithful functor, then by definition every pair of parallel arrows $\phi, \psi : \alpha \to \beta$ in $\mathcal{E}$ satisfies

$$P(\phi) = P(\psi) : P(\alpha) \to P(\beta) \implies \phi = \psi. \tag{7}$$

$\implies$ : Assuming $P : \mathcal{E} \to \mathcal{B}$ is faithful functor, consider an arbitrary pair of parallel arrows $\phi, \psi : \alpha \to \beta$ in an arbitrary fiber $\mathcal{E}_x$ over $x$; i.e. $P(\phi) = P(\psi) = \mathrm{id}_x$. In such cases, faithfulness of $P$ (Eq. 7) guarantees that $\phi = \psi$ and thus $\mathcal{E}_x$ is a thin category.

$\impliedby$ : If the fiber $\mathcal{E}_x$ for every object $x$ in $\mathcal{B}$ is a thin category, then clearly $P : \mathcal{E} \to \mathcal{B}$ *must* be faithful when restricted to an individual fiber. The non-trivial case is to consider an arbitrary pair of parallel morphisms $\phi, \psi : \alpha \to \beta$ not belonging to any fibers of $\mathcal{E}$. Denote $a \coloneqq P(\alpha)$ and $b \coloneqq P(\beta)$ and suppose $f \coloneqq P(\phi) = P(\psi) : a \to b$. Then, because $\mathcal{E}$ is a fibered category, there exists a cartesian arrow $\zeta : \gamma \to \beta$, such that $P(\zeta) = f$ (note that $a = P(\alpha) = P(\gamma)$ but $\gamma$ is not necessarily equal to $\alpha$). Since $\zeta$ is a cartesian arrow, there exists a unique arrows $\mu, \nu : \alpha \to \gamma$ completing the top edges of the following diagram:

$$\begin{tikzcd}
\alpha \arrow[rr, dashed, "\mu"] \arrow[dr, bend right, "\psi"'] \arrow[dd, mapsto] & & \gamma \arrow[dl, "\zeta"'] \arrow[dd, mapsto] & \alpha \arrow[l, dashed, "\nu"'] \arrow[dll, bend left, "\phi"] \arrow[dd, mapsto] \\
& \beta \arrow[dd, mapsto] & & \\
a \arrow[rr, equal] \arrow[dr, bend right, "f"'] & & a \arrow[r, equal] \arrow[dl, "f"] & a \arrow[dll, bend left, "f"] \\
& b & &
\end{tikzcd} . \tag{8}$$

However, $P(\nu) = \mathrm{id}_a = P(\mu)$ and therefore $\mu$ and $\nu$ are parallel arrows in the fiber $\mathcal{E}_a$ and therefore $\mu = \nu$ because $\mathcal{E}_a$ is assumed thin. Therefore, $\psi = \zeta \circ \mu = \zeta \circ \nu = \phi$ and thus $P$ is a faithful functor. $\square$

# Categorical Definitions

## 2.1 Hom-Functors

For a locally small category $\mathcal{C}$, the hom-functor of $\mathcal{C}$ is a functor $\mathrm{Hom}_{\mathcal{C}} : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Set}$ constructed in the following manner. Given objects $a, b, c, \ldots \in \mathcal{C}_0$ of $\mathcal{C}$, the hom-functor $\mathrm{Hom}_{\mathcal{C}}$ maps a pair of objects $(a, b) \in (\mathcal{C}^{\mathrm{op}} \times \mathcal{C})_0 = \mathcal{C}_0 \times \mathcal{C}_0 = \mathcal{C}_0^2$ into the set[2] of morphisms $\mathcal{C}_1$ of $\mathcal{C}$ with source $a$ and target $b$. Therefore,

[2]The collection of morphisms of type $a \to b$ forms a set because $\mathcal{C}$ is locally small.

$\mathrm{Hom}_{\mathcal{C}}(a,b)$ is the set of morphisms in $\mathcal{C}$ of type $a \to b$. Given morphisms $g^{\mathrm{op}} \in \mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(a,c)$ and $h \in \mathrm{Hom}_{\mathcal{C}}(b,d)$, the hom-functor $\mathrm{Hom}_{\mathcal{C}}$ constructs a function

$$\mathrm{Hom}_{\mathcal{C}}(g^{\mathrm{op}},h) : \mathrm{Hom}_{\mathcal{C}}(a,b) \to \mathrm{Hom}_{\mathcal{C}}(c,d)$$

which takes a morphism $f : a \to b \in \mathrm{Hom}_{\mathcal{C}}(a,b)$ and produces the morphism $h \circ f \circ g : c \to d \in \mathrm{Hom}_{\mathcal{C}}(c,d)$. Graphically,

$$\mathrm{Hom}_{\mathcal{C}}(g^{\mathrm{op}},h)\left( a \xrightarrow{f} b \right) = \; c \xrightarrow{g} a \xrightarrow{f} b \xrightarrow{h} d$$

## 2.2 Adjoint Functors

Given two categories $\mathscr{C}$ and $\mathscr{D}$, a pair of functors $L : \mathscr{C} \to \mathscr{D}, R : \mathscr{D} \to \mathscr{C}$ are called an *adjoint pair*, denoted $L \dashv R$ or

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "L"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "R"] \end{tikzcd}$$

if there exists a natural isomorphism $\alpha$ between the following pair of hom-functors of type $\mathscr{C}^{\mathrm{op}} \times \mathscr{D} \to \mathbf{Set}$:

$$\mathrm{Hom}_{\mathscr{D}}(L^{\mathrm{op}}(-),-) \overset{\alpha}{\simeq} \mathrm{Hom}_{\mathscr{C}}(-,R(-))$$

This relationship can be depicted graphically as 2-cell (and its inverse) in $\mathbf{Cat}$,

$$\begin{tikzcd} \mathcal{C}^{\mathrm{op}} \times \mathcal{D} \arrow[r, "I_{\mathcal{C}^{\mathrm{op}}} \times R"] \arrow[d, "L^{\mathrm{op}} \times I_{\mathcal{D}}"'] \arrow[dr, phantom, "\overset{\alpha}{\Rightarrow}\ \overset{\alpha^{-1}}{\Leftarrow}"] & \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \arrow[d, "\mathrm{Hom}_{\mathcal{C}}"] \\ \mathcal{D}^{\mathrm{op}} \times \mathcal{D} \arrow[r, "\mathrm{Hom}_{\mathcal{D}}"] & \mathbf{Set} \end{tikzcd}$$

Concretely, the naturality of $\alpha$ means that for every morphism $(f^{\mathrm{op}} : b \to a, g : c \to d) \in (\mathcal{C}^{\mathrm{op}} \times \mathcal{D})_1$ the components $\alpha_{(b,c)}$ and $\alpha_{(a,d)}$ of $\alpha$ make the following square commute:

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{D}}(L^{\mathrm{op}}(b),c) \arrow[r, "\mathrm{Hom}_{\mathcal{D}}(L^{\mathrm{op}}(f^{\mathrm{op}}),g)"] \arrow[d, "\alpha_{(b,c)}"'] & \mathrm{Hom}_{\mathcal{D}}(L^{\mathrm{op}}(a),d) \arrow[d, "\alpha_{(a,d)}"] \\ \mathrm{Hom}_{\mathcal{C}}(b,R(c)) \arrow[r, "\mathrm{Hom}_{\mathcal{C}}(f^{\mathrm{op}},R(g))"] & \mathrm{Hom}_{\mathcal{C}}(a,R(d)) \end{tikzcd}$$

## 2.3 Beck-Chevalley Conditions

The Beck-Chevalley Conditions are conditions that may or may not be satisfied by a quadruplet of functors $F, H, G, K$ which form a natural isomorphism $\alpha : KF \Rightarrow HG$ square:

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "F"] \arrow[d, "G"'] & \mathcal{B} \arrow[d, "K"] \arrow[dl, Rightarrow, "\alpha"'] \\
\mathcal{C} \arrow[r, "H"'] & \mathcal{D}
\end{tikzcd}$$

To define the *left* Beck-Chevalley condition, one needs functors $F_L : \mathcal{B} \to \mathcal{A}$ and $H_L : \mathcal{D} \to \mathcal{A}$ which are respectively left adjoint functors to $F$ and $H$,

$$\begin{tikzcd}
\mathcal{A} \arrow[r, bend right, "F"'] \arrow[r, phantom, "\perp"] & \mathcal{B} \arrow[l, bend right, "F_L"']
\end{tikzcd}, \qquad
\begin{tikzcd}
\mathcal{C} \arrow[r, bend right, "H"'] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend right, "H_L"']
\end{tikzcd}.$$

Using these left adjoint functors, it becomes possible to construct a natural transformation $\beta : KH_L \Rightarrow GF_L$ from $\alpha$[3]. Graphically, $\beta$ can be identified as the outer cell of the following diagram:

$$\begin{tikzcd}
\mathcal{A} \arrow[r, bend right, "F"'] \arrow[r, phantom, "\perp"] \arrow[d, "G"'] & \mathcal{B} \arrow[l, bend right, "F_L"'] \arrow[d, "K"] \arrow[dl, Rightarrow, "\alpha"'] \\
\mathcal{C} \arrow[r, bend left, "H"] \arrow[r, phantom, "\top"] & \mathcal{D} \arrow[l, bend left, "H_L"]
\end{tikzcd}, \qquad \text{i.e.} \qquad
\begin{tikzcd}
\mathcal{A} \arrow[d, "G"'] & \mathcal{B} \arrow[l, "F_L"'] \arrow[d, "K"] \\
\mathcal{C} & \mathcal{D} \arrow[l, "H_L"] \arrow[ul, Rightarrow, "\beta"']
\end{tikzcd}.$$

Although the natural transformation $\alpha$ is assumed to be a natural isomorphism, the natural transformation $\beta$ need not be; if $\beta$ happens to be a natural isomorphism, then we say that the original square satisfies the *left* Beck-Chevalley condition[4]. The *right* Beck-Chevalley condition is defined analogously with functors $F_R, H_R$ which are respectively right adjoints $F \dashv F_R$ and $H \dashv H_L$.

## 2.4 Cartesian Morphism

A morhpism $\phi : e' \to e$ in $\mathcal{E}$ is *cartesian* with respect to a functor $P : \mathcal{E} \to \mathcal{B}$ if for every $\psi : e'' \to e$ in $\mathcal{E}$ and for every $s : P(e'') \to P(e)$ such that $P(\phi) \circ_{\mathcal{B}} g = P(\psi)$ (i.e. such that the second diagram commutes), there exists a unique morphism $\sigma : e'' \to e'$ in $\mathcal{E}$ such that $\phi \circ_{\mathcal{E}} \sigma = \psi$ (i.e. such that the first diagram commutes):[5]

$$\begin{tikzcd}
e'' \arrow[r, dashed, "\exists!\sigma"] \arrow[dr, bend right, "\forall\psi"'] & e' \arrow[d, "\phi"] \\
& e
\end{tikzcd}
\quad \overset{P}{\mapsto} \quad
\begin{tikzcd}
P(e'') \arrow[r, "\forall g"] \arrow[dr, bend right, "P(\psi)"'] & P(e') \arrow[d, "P(\phi)"] \\
& P(e)
\end{tikzcd}$$

[3] The natural transformations $\alpha$ and $\beta$ are known as *mates* or *conjugates*.

[4] Are the left adjoints $F_L, H_L$ unique? If not, it might be better to say the original square satifies the left Beck-Chevalley condition with respect to $F_L, H_L$.

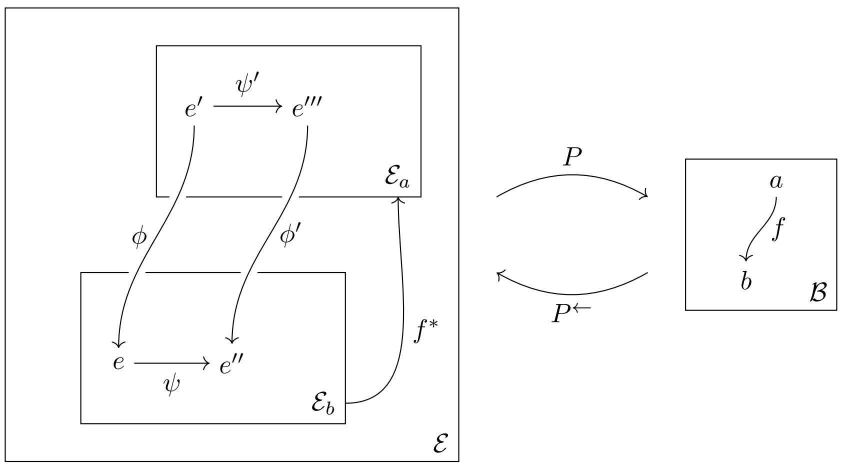
[5] The definition and treatment of Cartesian morphisms found in the *Reformulations* section of `https://ncatlab.org/nlab/show/Cartesian+morphism#CartInOrdCatReformulation` is probably better suited here.

## 2.5 Grothendieck Fibrations

A functor $P : \mathcal{E} \to \mathcal{B}$ is a *Grothendieck fibration* if it satisfies the following "lifting" property that for every morphism $f : b \to P(e)$ of $\mathcal{B}$ (i.e. if the codomain of $f$ is contained in the image of $P$), there exists a *cartesian* morphism $\phi : e' \to e$ of $\mathcal{E}$ in the fibered category $\mathcal{E}_{P(e)}$ (i.e. $P(\phi) = f$).

## 2.6 The Equivalence of Puesdofunctors and Fibrations

Given a functor $P : \mathcal{E} \to \mathcal{B}$ which is also a Grothendieck fibration equipped with a clevage (i.e. a choice of cartesian morphism $\phi \in \mathrm{Hom}_{\mathcal{E}}(e', e)$ for each $f \in \mathrm{Hom}_{\mathcal{B}}(a, P(e))$ such that $P(\phi) = f$), it is possible to construct a pseudofunctor (read weak 2-functor between weak 2-categories) $\pi : \mathcal{B}^{\mathrm{op}} \to \mathbf{Cat}$. In particular, for each object $b \in \mathcal{B}_0$ is mapped to the *sub-category* $\pi(b) = \mathcal{E}_b$ of $\mathcal{E}$ whose objects are those which map to $b$ under $P$ and whose morphism are those which map to $\mathrm{id}_b$ under $P$; $\mathcal{E}_b$ is the fibre category over $b$ with respect to $P$. For each morphism $f \in \mathrm{Hom}_{\mathcal{B}}(a, b)$ in $\mathcal{B}$, the pseudofunctor $\pi$ maps $f^{\mathrm{op}} : b \to a$ onto a functor $\pi(f^{\mathrm{op}}) = f^* : \mathcal{E}_b \to \mathcal{E}_a$ which is defined accordingly:




Given an object $e \in (\mathcal{E}_b)_0$, the functor $f^*$ finds the unique cartesian morphism $\phi \in \mathrm{Hom}_{\mathcal{E}}(e', e)$ as specified by the cleavage and assigns $f^*(e) = e'$. Next, given a morphism $\psi \in \mathrm{Hom}_{\mathcal{E}_b}(e, e'')$, the functor $f^*$ first finds the unique cartesian morphisms $\phi \in \mathrm{Hom}_{\mathcal{E}}(e', e)$ and $\phi' \in \mathrm{Hom}_{\mathcal{E}}(e''', e'')$. Then, because $g = \mathrm{id}_a$ completes the following diagram

$$\begin{tikzcd} P(e') \arrow[r, "g"] \arrow[dr, bend right, "P(\psi \circ \phi)"'] & P(e''') \arrow[d, "P(\phi')"] \\ & P(e'') \end{tikzcd} \quad = \quad \begin{tikzcd} a \arrow[r, "\mathrm{id}_a"] \arrow[dr, bend right, "\mathrm{id}_b \circ f"'] & a \arrow[d, "f"] \\ & b \end{tikzcd},$$

and because $\phi'$ is cartesian, there must exist a unique $\psi' \in \mathrm{Hom}_{\mathcal{E}_a}(e', e''')$ such that $\psi \circ \phi = \phi' \circ \psi'$. For each $\psi \in \mathrm{Hom}_{\mathcal{E}_b}(e, e'')$, the functor $f^*$ selects this unique morphism $f^*(\psi) = \psi'$. In summary, the pseudofunctor $\pi : \mathcal{B}^{\mathrm{op}} \to \mathbf{Cat}$ induced by $P : \mathcal{E} \to \mathcal{B}$ is defined on objects $b \in \mathcal{B}_0$ as $\pi(b) = \mathcal{E}_b$ and on morphisms $f \in \mathcal{B}_1$ as $\pi(f) = f^*$ and forms a functor [TODO: figure out the 'pseudo' part of the pseudofunctorality.].

## 2.7 Slice and Coslice Categories

Given a category $\mathcal{C}$ and an object $c \in \mathcal{C}_0$ of $\mathcal{C}$ the *slice category* (or *over category*) $\mathcal{C}/c$ is the "stuff in $\mathcal{C}$ that is on top of $c$". Specifically, the objects of $\mathcal{C}/c$ are all the morphisms $f \in \mathcal{C}_1$ from $\mathcal{C}$ whose codomain is $\mathrm{cod}(f) = c$ (alternatively you could write $(\mathcal{C}/c)_0 = \mathrm{Hom}_{\mathcal{C}}(-, c)$). A morphism of $\mathcal{C}/c$ between objects $f : a \to c, g : b \to c \in (\mathcal{C}/c)_0$ is a commuting triangle completed by a third morphism $h : a \to b \in \mathcal{C}_1$:

$$\begin{tikzcd} a \arrow[rr, "h"] \arrow[dr, "g"'] & & b \arrow[dl, "f"] \\ & c & \end{tikzcd}$$

Composition of morphisms in $\mathcal{C}/c$ is induced by the composition of morphisms in $\mathcal{C}$:

$$\left(\begin{tikzcd} y \arrow[rr, "n"] \arrow[dr, "f"'] & & z \arrow[dl, "h"] \\ & c & \end{tikzcd}\right) \circ_{\mathcal{C}/c} \left(\begin{tikzcd} x \arrow[rr, "m"] \arrow[dr, "g"'] & & y \arrow[dl, "f"] \\ & c & \end{tikzcd}\right) = \begin{tikzcd} x \arrow[r, "m"] \arrow[dr, "g"'] & y \arrow[r, "n"] \arrow[d, "f"] & z \arrow[dl, "h"] \\ & c & \end{tikzcd}$$

The assignment of an overcategory $\mathcal{C}/c$ to each object $c$ can be extended to a *slice functor* $\mathcal{C}/(-) : \mathcal{C} \to \mathbf{Cat}$ in the following sense. For objects $c \in \mathcal{C}_0$, the slice functor takes $c$ to the slice category $\mathcal{C}/c$; for morphisms $f : a \to b \in \mathcal{C}_1$, the slice functor takes $f$ to the functor $\mathcal{C}/f : \mathcal{C}/a \to \mathcal{C}/b$ defined graphically; for every morphism of $\mathcal{C}/a$ (commuting triangle in $\mathcal{C}$ over $a$), contruct the morphism of $\mathcal{C}/b$ (commuting triangle in $\mathcal{C}$ over $b$) as follows:

$$\begin{tikzcd} l \arrow[rr, "m"] \arrow[dr, "x"'] \arrow[ddr, bend right=60, "f \circ_{\mathcal{C}} x"'] & & r \arrow[dl, "x'"] \arrow[ddl, bend left=60, "f \circ_{\mathcal{C}} x'"] \\ & a \arrow[d, "f"] & \\ & b & \end{tikzcd}$$

where the inner triangle is a morphism of $\mathcal{C}/a$ and the outer triangle is a morphism of $\mathcal{C}/b$ given by the functor $\mathcal{C}/f$.

Given a category $\mathcal{C}$ and an object $c \in \mathcal{C}_0$ of $\mathcal{C}$ the *coslice category* (or *under category*) $c/\mathcal{C}$ is the "stuff in $\mathcal{C}$ that is underneath $c$". Specifically, the objects of $c/\mathcal{C}$ are all the morphisms $f \in \mathcal{C}_1$ from $\mathcal{C}$ whose domain is $\mathrm{dom}(f) = c$ (alternatively you could write $(c/\mathcal{C})_0 = \mathrm{Hom}_{\mathcal{C}}(c, -)$). A morphism of $c/\mathcal{C}$ between objects $f : c \to a, g : c \to b \in (c/\mathcal{C})_0$ is a commuting triangle completed by a third morphism $h : a \to b \in \mathcal{C}_1$:

$$\begin{tikzcd} & c \arrow[dl, "g"'] \arrow[dr, "f"] & \\ a \arrow[rr, "h"'] & & b \end{tikzcd}$$

Everything about coslice categories is defined as expected analogously to that of a slice categories. [TODO: determine how the details of the Grothendieck construction transform the slice (pseudo-)functor $\mathcal{C}/(-) : \mathcal{C} \to \mathbf{Cat}$ into the codomain fibration.]

## 2.8 The Pullback and Pushforward Functors

Given a category $\mathcal{C}$ and a morphism $f : a \to b \in \mathcal{C}_1$, the image of $f$ under the slice functor $\mathcal{C}/(-)$ produces a functor $\mathcal{C}/f : \mathcal{C}/a \to \mathcal{C}/b$ between slice categories of $\mathcal{C}$ in the "same direction" as $f$ TODO: confirm that $\mathcal{C}/f$ is the pushforward functor $f_!$ of $f \in \mathcal{C}_1$.

If the given category $\mathcal{C}$ admits pullbacks, in becomes possible to define, for a morphism $f : a \to b$ a pullback functor $f^* : \mathcal{C}/b \to \mathcal{C}/a$. Given a morphism in $\mathcal{C}/b$ (commuting triangle in $\mathcal{C}$ with base at $b$),

$$\begin{tikzcd} c \arrow[rr, "k"] \arrow[dr, "g"'] & & d \arrow[dl, "h"] \\ & b & \end{tikzcd}$$

the pullback functor $f^* : \mathcal{C}/b \to \mathcal{C}/a$ associated with $f$ takes the objects $g : c \to b, h : d \to b$ of $\mathcal{C}/b$ (morphisms in $\mathcal{C}$) completes the pullback squares associated with $f$

$$\begin{tikzcd} & c' \arrow[dl, "f^*g := g_f"'] \arrow[dr, "f_g"] & \\ a \arrow[dr, "f"'] & & c \arrow[dl, "g"] \\ & b & \end{tikzcd} \qquad \begin{tikzcd} & d' \arrow[dl, "f_h"'] \arrow[dr, "f^*h := h_f"] & \\ d \arrow[dr, "h"'] & & a \arrow[dl, "f"] \\ & b & \end{tikzcd}$$

where a subscript notation $g_f$ means "the pullback of $g$ along $f$". Defining the action of $f^* : \mathcal{C}/b \to \mathcal{C}/a$ on objects to be $f^*g = g_f$ and $f^*h = h_f$, the action on morphisms in $\mathcal{C}/b$ is defined by composing the pullback squares with the commuting triangle morphism:

$$\begin{tikzcd} c \arrow[rr, "k"] \arrow[dr, "g"'] & & d \arrow[dl, "h"] \\ & b & \\ & a \arrow[u, "f"'] & \\ c' \arrow[uuu, bend left, "f_g"] \arrow[ur, "f^*g"] \arrow[rr, dashed, "\exists! f^*k"'] & & d' \arrow[uuu, bend right, "f_h"'] \arrow[ul, "f^*h"'] \end{tikzcd}$$

The commuting triangle in $\mathcal{C}/a$ appearing at the bottom is completed by a unique morphism [TODO: why does this morphism need to be unique and exist?] denoted to be $f^*k$ ($\neq k_f$ obviously). The functoriality of $f^*$ has a simple proof found here `https://proofwiki.org/wiki/Pullback_Functor_is_Functor`.

### 2.9 Functors of Monoidal Categories

[TODO]

### 2.10 Frobenius Reciprocity

[TODO]

## Comments on selected references

This section is temporary and reserved for recording comments toward various references.

- Vistoli [Vis04]
- Street [Str74]
- Koudenburg [Kou18]
- Brown and Sivera [BS09]
- Lurie [Lur09]

# References


[BS09] Ronald Brown and Rafael Sivera. "Algebraic colimit calculations in homotopy theory using fibred and cofibred categories". In: *Theory and Applications of Categories* 22.8 (2009), pp. 222–251.

[Kou18] Seerp Roald Koudenburg. "A categorical approach to the maximum theorem". In: *Journal of Pure and Applied Algebra* 222.8 (2018), pp. 2099–2142.

[Lur09] Jacob Lurie. *Higher Topos Theory (AM-170)*. Vol. 189. Princeton University Press, 2009.

[Str74] Ross Street. "Fibrations and Yoneda's lemma in a 2-category". In: *Category seminar*. Springer. 1974, pp. 104–133.

[Vis04] Angelo Vistoli. "Notes on Grothendieck topologies, fibered categories and descent theory". In: *arXiv preprint math/0412512* (2004).