Journal of Pure and Applied Algebra 61 (1989) 243-250
North-Holland

CATEGORIES OF AFFINE SPACES*

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Communicated by P.J. Freyd
Received 9 August 1988
Revised 30 October 1988

We give a characterization of slice categories of additive categories with kernels purely in terms of a property involving finite limits and finite sums which generalizes the notion of a *modular lattice*. In particular, we give a new characterization of additive categories just in terms of finite sums and finite products.

Introduction

In view of the recent remarks of Schanuel (see [1]) that the fundamental construction of the Grassmann algebra is based on the category Aff(k) of *k-affine* spaces and that this last is equivalent to the full subcategory of the *slice category* k-Vect/k determined by surjective maps, it becomes even more sensible to have a simple categorical characterization of the categories of the form $\mathbf{A}/X$, where $\mathbf{A}$ is an additive category with kernels (see [2]). Even if additivity itself is lost for all $X \neq 0$, the slice categories $\mathbf{A}/X$ retain some aspects of additivity, and the problem is to have a precise description of what all slice categories $\mathbf{A}/X$ have in common. Such a problem is reminiscent of a basic one in topos theory, namely, to discover what all the categories $\mathrm{Sh}_J(\mathbf{C})$ of sheaves on a site $(\mathbf{C}, J)$ have in common, and the Giraud Theorem tells us the answer: a category $\mathbf{E}$ is equivalent to a category of sheaves for some site iff

(1) $\mathbf{E}$ is a cocomplete *distributive category*;
(2) equivalence relations in $\mathbf{E}$ are *effective and universal*;
(3) $\mathbf{E}$ has a *set of generators*.

Condition (1) (see Section 1 for a precise definition) is the one which really distinguishes toposes as categories of sheaves of sets from the categories of sheaves of modules, which satisfy conditions (2) and (3) but *not* (1), and which after all were the additive origin of topos theory. Condition (1) is basically the categorical version of a cocomplete *distributive lattice* (also called *locale*), so that a common intuition about a Grothendieck topos is that it is a 'glorified' locale.

* Work partially supported by the Australian Research Grants Scheme.

Keeping in mind the above lattice-theoretical intuition about toposes, the answer for slices of additive categories is surprisingly simple: they are 'glorified' versions (in the same sense as distributive categories are) of *modular lattices*, and we will call them *modular categories* (see again Section 1 for the precise definition). Easily one sees that slices of additive categories with kernels $\mathbf{E}=\mathbf{A}/X$ are modular and that one can recover $\mathbf{A}$ as Pt($\mathbf{E}$), but the theorem is now that a left exact category with finite sums $\mathbf{E}$ is modular iff

(1) Pt($\mathbf{E}$) is additive with kernels, and

(2) the canonical functor $\mathbf{E}\to \mathrm{Pt}(\mathbf{E})/(1\to 1+1)$ is an equivalence (1 denoting a terminal object),

so that *modular categories are precisely the slices of additive categories with kernels*; in particular additive categories are precisely the modular categories in which '$0=1$', that is in which the unique arrow $0\to 1$ from an initial object to a terminal one is invertible. Also, differently from the lattice-theoretical experience, the only category which is both distributive and modular is the trivial one.

The above definitions and theorems are developed in Sections 1 and 2; in Section 3 we discuss a very peculiar property of reflexive graphs in additive categories with kernels which is stable under slicing and which generalizes the above condition (1). This axiom was conjectured by Bill Lawvere as a possible characterization of slices of additive categories in a private conversation when we were both visiting Sydney in February 1988; we still do not know if such a property is in fact equivalent to modularity.

1. Modular categories

The notion of *distributive* category is well known: let $\mathbf{E}$ be a left exact category with finite sums (observe that such a requirement is equivalent to asking that each slice category $\mathbf{E}/U$ have finite sums and finite products); then $\mathbf{E}$ is distributive if:

(1) for each slice category $\mathbf{E}/U$ the canonical arrows

$$(X\times Y)+(X\times Z)\to X\times (Y+Z)$$

and

$$0\to X\times 0$$

(where 0 denotes an initial object) are *invertible*;

(2) the injections $X \xrightarrow{i_X} X+Y \xleftarrow{i_Y} Y$ in a sum are *mono* and *disjoint*.

As noticed by S. Schanuel, the above conditions can be equivalently restated by asking that the functors

$$\mathbf{E}/1+1\to \mathbf{E}\times\mathbf{E}$$

(induced by the injections $1\to 1+1$, where 1 denotes a terminal object), and

$$\mathbf{E}/0\to *$$

(where $*$ denotes a terminal category) be *equivalences*. Among the examples of distributive categories are all topos, the category of topological spaces, the dual category of commutative rings; the coproduct completion of any left exact category is also distributive. A large class of *non* distributive categories are the additive ones: an additive category which is distributive is degenerate.

Clearly the notion of a distributive category can be thought as a generalization to categories of a *distributive lattice*: the order reflection of a distributive category is a distributive lattice and the only reason why a distributive lattice $\mathbf{L}$ fails to be a distributive category (unless $\mathbf{L}=*$) is the disjointness of the sums stated in axiom (2). So, following this analogy, we can ask ourselves what could be the notion of a *modular* category; recalling that a lattice $\mathbf{L}$ is modular when for each $x\leq z$ the induced inclusion $x\vee(y\wedge z)\leq(x\vee y)\wedge z$ is in fact an equality for all $y\in\mathbf{L}$, such a generalization should be the following:

Definition. A modular category is a left exact category $\mathbf{E}$ with finite sums such that:

(1) for each slice category $\mathbf{E}/U$ and for each arrow $X\xrightarrow{f}Z\in\mathbf{E}/U$ the canonical arrow

$$\begin{pmatrix}\langle i_X,f\rangle\\ i_Y\times Z\end{pmatrix}:X+(Y\times Z)\to(X+Y)\times Z$$

is invertible for each object $Y\in\mathbf{E}/U$;

(2) for each arrow $f:X\to U$ the commutative square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "i_X"'] & U \arrow[d, "i_1"] \\ X+U \arrow[r, "f+1"'] & U+U \end{tikzcd}$$

is a pullback.

Notice that as in the case of distributive categories the only reason why a modular lattice $\mathbf{L}$ is not a modular category (unless $\mathbf{L}=*$) is condition (2), which in the case of distributive categories is always true. This also shows that axioms (1) and (2) are independent, since there are distributive categories which do not satisfy axiom (1). Finally, notice that the order reflection of a modular category is a modular lattice.

The obvious question is now if there are non trivial examples of modular categories and the answer is yes, *all additive categories with kernels are modular*; more than this, since the notion of a modular category (like the notion of a distributive one) is stable under slicing, *all slices of additive categories with kernels are modular*. The proof of condition (2) is easy, and to prove condition (1) it is enough, by the Yoneda Lemma, to prove that (1) is satisfied by the category of abelian groups: let

$$(X \xrightarrow{h} U) \xrightarrow{f} (Z \xrightarrow{k} U)$$

be a homomorphism of abelian groups over the abelian group U and let $(Y \xrightarrow{g} U)$ be an abelian group over U; then the canonical arrow

$$X + \left(Y \underset{U}{\times} Z \right) \xrightarrow{\phi} (X+Y) \underset{U}{\times} Z$$

turns out to be $\phi(x,y,z) = (x,y,z+f(x))$, which is obviously invertible.

What we will show in the following is that in fact *the slices of additive categories with kernels are all possible examples of modular categories*; and we will prove this fact by a characterization of modular categories reminiscent of the Schanuel characterization of distributive categories.

2. The characterization

If $\mathbf{E}$ is a category with a terminal object 1, denote by $\mathrm{Pt}(\mathbf{E})$ the category whose objects are the points $1 \xrightarrow{p} X$ and whose arrows are the point preserving arrows; if the sums with 1 are representable in $\mathbf{E}$, then $\mathrm{Pt}(\mathbf{E})$ is the category of algebras for the monad on $\mathbf{E}$ defined by $\tilde{X} = X + 1$. Notice that if $\mathbf{A}$ is additive with kernels, then by the usual arguments that can be carried out in additive categories one has $\mathrm{Pt}(\mathbf{A}/U) \simeq \mathbf{A}$, for any object U.

Theorem. *Let* $\mathbf{E}$ *be a left exact category with finite sums; then* $\mathbf{E}$ *is modular if and only if:*

(1) $\mathrm{Pt}(\mathbf{E})$ *is additive with kernels, and*

(2) *the canonical functor*

$$\mathbf{E} \to \mathrm{Pt}(\mathbf{E})/(1 \to 1+1)$$

($1 \to 1+1$ *denoting an injection*) *is an equivalence.*

Proof. Clearly if conditions (1) and (2) are satisfied, then $\mathbf{E}$ is modular, being equivalent to an affine category.

To prove the converse, let us first show that every free tilde algebra, which turns out to be simply a point of the form $i_1 : 1 \to X+1$, has a unique abelian group structure for which i_1 is the zero element and moreover that every arrow $X \to Y+1$ extends to a group homomorphism $X+1 \to Y+1$; then additivity of $\mathrm{Pt}(\mathbf{E})$ will follow by showing that the Kleisli category of the tilde monad is equivalent to the Eilenberg–Moore one. Now observe that the arrow

$$\phi_X : (X+X+1) \to (X+1) \times (X+1)$$

obtained by applying modularity to the arrow $X \xrightarrow{t_X} 1 \xrightarrow{i_1} X+1$ is invertible (t_X denoting the unique arrow from X to a terminal object); so we can define an addition

$$(X+1)\times(X+1)\to(X+1)$$

on $(X+1)$ as

$$(X+1)\times(X+1) \xrightarrow{\phi_X^{-1}} (X+X+1) \xrightarrow{\delta_X+1} (X+1)$$

where δ_X denotes the codiagonal map. The proof that such an operation is associative and commutative is tedious but straightforward (just work out the canonical isomorphism $(X+X+X+1)\to(X+1)\times(X+1)\times(X+1)$ and apply associativity and commutativity of δ_X), as is the proof that the injection $i_1: 1\to X+1$ is the zero element. Now the opposite map υ_X for the addition can be verified to be

$$\upsilon_X=(X+1) \xrightarrow{(X+1)\times i_1} (X+1)\times(X+1) \xrightarrow{\theta_X^{-1}} X+(X+1) \xrightarrow{X+t_{(X+1)}} (X+1),$$

where θ_X denotes the invertible arrow induced from modularity applied to the arrow $i_X: X\to X+1$ and the object 1.

Easily one has that every arrow $f:(X+1)\to(Y+1)$ of pointed objects is in fact a group homomorphism with respect to the above group structures. Finally, the uniqueness of the group structure on the objects $X+1$ is also easily proved using the condition of i_1 being the zero element and the definition of the canonical isomorphism ϕ_X; the basic computation consists in showing that the composite $\phi_X^{-1}[(X+1)\times i_1]$ is $X+i_1$.

If $1 \xrightarrow{p} X$ is a pointed object, then modularity implies that the canonical arrow

$$1+(0\times X)\to(1+0)\times X=X$$

is invertible, so that every pointed object is isomorphic to a free one; hence the Kleisli category of the tilde monad in equivalent to the Eilenberg–Moore one, and Pt(**E**) is an additive category with kernels.

If $i_2: 1\to(1+1)$ is the second injection, the correspondence sending an object X of **E** to the object $1 \xrightarrow{i_1} (X+1) \xrightarrow{t_X+1} (1+1)$ of Pt(**E**)/$(1\to 1+1)$ defines a functor

$$\mathbf{E}\to \mathrm{Pt}(\mathbf{E})/(1\to 1+1)$$

which is full and faithful: faithfulness follows since from axiom (2) with $U=1$ we get that injections $i_Y: Y\to Y+1$ are mono, being inverse images of the point $1\to 1+1$ which is always a mono; fullness means that if $\phi:(X+1)\to(Y+1)$ is an arrow such that $\phi i_1=i_1$ and $(t_Y+1)\phi=t_X+1$, then there exists an arrow $f: X\to Y$ such that $\phi=f+1$; but from $(t_Y+1)\phi=t_X+1$ and axiom (2) with $U=1$, we get that in the pullback

$$\begin{tikzcd} P \arrow[r, "f"] \arrow[d, "j"'] & Y \arrow[d, "i_Y"] \\ X+1 \arrow[r, "\phi"'] & Y+1 \end{tikzcd}$$

the left-hand vertical arrow can be assumed to be the injection $i_X : X \to X+1$ (since from axiom (2) the square

$$\begin{tikzcd} Y \arrow[r, "t_Y"] \arrow[d, "i_Y"'] & 1 \arrow[d, "i_2"] \\ Y+1 \arrow[r, "t_Y+1"'] & 1+1 \end{tikzcd}$$

is a pullback), so that the condition $\phi i_1 = i_1$ ensures that $\phi = f+1$.

Finally the functor can be proved to be essentially surjective on objects: if

$$1 \xrightarrow{p} X \xrightarrow{f} 1+1 = 1 \xrightarrow{i_2} 1+1$$

is an object of $\mathrm{Pt}(\mathbf{E})/(1 \to 1+1)$, then the modularity of the slice category $\mathbf{E}/1+1$, applied to the arrow

$$(1 \xrightarrow{i_2} 1+1) \xrightarrow{p} (X \xrightarrow{f} 1+1)$$

and to the object $(1 \xrightarrow{i_1} 1+1)$ given by the first injection, guarantees that this object is isomorphic to the image of the object U of $\mathbf{E}$ defined by the pullback of i_1 along f. □

Notice that in the proof we just used axiom (2) with $U=1$, so that such a not stable form of axiom (2) is in fact equivalent to the stable one. Notice also that axiom (2) alone can not imply property (2) of the theorem, as it is shown by the example of the category of sets. Finally, from the above proof emerges the following characterization of additive categories:

Corollary. *A category* $\mathbf{A}$ *is additive iff*

(1) $\mathbf{A}$ *has finite sums and products and for each arrow* $f: X \to Z$ *of* $\mathbf{A}$ *the canonical arrow*

$$\begin{pmatrix} \langle i_X, f\rangle \\ i_Y \times Z \end{pmatrix} : X+(Y\times Z) \to (X+Y)\times Z$$

is invertible, and

(2) $\mathbf{A}$ *is pointed, i.e. the unique arrow* $0 \to 1$ *is invertible.* □

3. Groupoids and reflexive graphs

If $\mathbf{A}$ is the category of abelian groups and if

$$\begin{tikzcd} C_1 \arrow[r, shift left=2, "d_0"] \arrow[r, shift right=2, "d_1"'] & C_0 \arrow[l, "i"'] \end{tikzcd}, \qquad d_0 i = 1 = d_1 i$$

is an oriented *reflexive graph* in $\mathbf{A}$, then addition provides a way to define a homomorphism (*composition*)

$$C_2 \to C_1$$

on the abelian group $C_2 = \{\langle f, g\rangle \in C_1 \times C_1 \mid d_1(f) = d_0(g)\}$ of composable arrows by

$$gf = f + g - i(d_1(f)).$$

Easily one can show that $d_0(gf) = d_0(f)$ and $d_1(gf) = d_1(g)$ and that composition is *associative* and has *identities* provided by i. In other words, the given reflexive graph extends to a *category* object in $\mathbf{A}$, which in fact is a *groupoid* object, the inverse of an arrow f being the arrow

$$f^{-1} = i(d_0(f)) + i(d_1(f)) - f.$$

Notice that every graph morphism is automatically functorial with respect to the above composition. Notice also that the above composition is *unique* over the given reflexive graph, since the condition of being a homomorphism means the functoriality of the sum:

$$(kf) + (gh) = (k+g)(f+h);$$

so, denoting identities simply with 1, we get:

$$f + g = 1f + g1 = (1+g)(f+1) = (g+1)(f+1) = gf + 1.$$

Clearly the above argument carries over to any *additive* category $\mathbf{A}$ *with kernels*, so that we have just proved that any such a category satisfies the following axiom:

Axiom A. *The forgetful functor from the category of groupoids in* $\mathbf{A}$ *to the category of reflexive graphs in* $\mathbf{A}$ *is an isomorphism.*

Since the above axiom makes sense in any left exact category, we can ask if there are examples of left exact categories not necessarily additive for which the axiom holds. A quite obvious class of examples is provided by the left exact categories of the form $\mathbf{E} = \mathbf{A}/X$, where $\mathbf{A}$ is an additive category with kernels, since it is not hard to see that such an axiom is stable under slicing; so all modular categories satisfy axiom A, hence modularity implies axiom A and the question arises of deciding whether in a left exact category **E** with finite sums it is in fact *equivalent* to modularity.

References

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[2] S. Mac Lane, Categories for the Working Mathematician, (Springer, Berlin, 1971).