On the Causality of Dynamical Systems

Thomas C. Fraser[1,2]

[1]Perimeter Institute for Theoretical Physics, Waterloo, Ontario, Canada, N2L 2Y5
[2]Dept. of Physics and Astronomy, University of Waterloo, Waterloo, Ontario, Canada, N2L 3G1

November 14, 2019

Disclaimers

1 A Chaotic Origin Story

1.1 Lorenz's Discovery

In "Deterministic nonperiodic flow" (Lorenz 1963), Lorenz (this is Edward Norton Lorenz, not to be confused with the physicist Ludvig Lorenz, or the actor Edward Norton) presented a simplified dynamical model for atmospheric convection, which today is known as the Lorenz system:

$$\begin{aligned} \frac{\mathrm{d}X}{\mathrm{d}t} &= \sigma(Y - X), \\ \frac{\mathrm{d}Y}{\mathrm{d}t} &= X(\rho - Z) - Y, \\ \frac{\mathrm{d}Z}{\mathrm{d}t} &= XY - \beta Z. \end{aligned} \tag{1}$$

The Lorenz system was remarkable because despite its determinism and simplicity, it exhibits flows which are non-periodic, chaotic and yet bounded as in Figure 1. Moreover, Lorenz identifies the non-linearity of the equations as being essential for these features postulates their ubiquity.

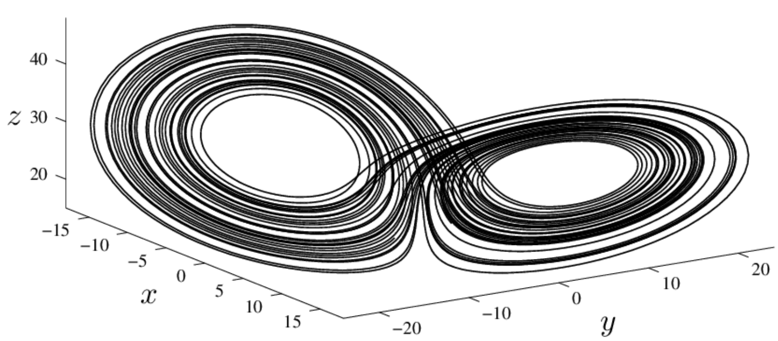

Figure 1: The strange attractor of the Lorenz system for $\sigma = 10, \beta = 8/3, \rho = 28$ with $X(0) = 1.1, Y(0) = -1.9, Z(0) = 13.1$. Figure by P. T. Clemson.

Later in "On the nature of turbulence" (Ruelle and Takens 1971), Ruelle and Takens develop a variety of mathematics to study such quasi-periodic behaviors and refers to them as *strange attractors*. Importantly, they conjectured that universally, strange attractors where the cause of turbulent behvaiors seen in fluid flow. This conjecture recieved further support after "Onset of turbulence in a rotating fluid" (Gollub and Swinney 1975).

1.2 What are attractors anyway?

What exactly is notion of an *attractor*? Intuitively, attractors generalize the more familiar notions of stability for differential equations such as *stable fixed points* and *limit cycles* in the sense that an attractor A is a compact subset such that points starting sufficiently close to A will remain in a given neighborhood of A. For a detailed account of the various notions and formulations of an attractor, see "On the concept of attractor" (Milnor 1985). Nevertheless, the following definition will be adequate.

A dynamical system on a manifold M (for example $\mathbb{R}^n$) is specified by a **flow** (a solution to the system of ODEs)

$$\Phi_{(\cdot)}(\cdot) : \mathbb{R} \times M \to M, \tag{2}$$

where Φ_t is a diffeomorphism of M for all $t \in \mathbb{R}$ such that

$$\forall V \in M : \Phi_0(V) = V, \quad \text{and} \quad \forall s, t \in \mathbb{R} : \Phi_t \circ \Phi_s = \Phi_{s+t}. \tag{3}$$

An **attractor** is any $A \subseteq M$ which is

1. forward invariant
$$\forall a \in A, \forall t \geq 0 : \Phi_t(a) \in A,$$

2. admits an neighborhood $B(A)$, called the **basin of attraction**, such for every open neighborhood N of A, there is a sufficiently large T such that
$$\forall b \in B(A), \forall t \geq T : \Phi_t(b) \in N,$$

3. and finally, there is no other proper subset $A' \subset A$ satisfying the above.

Note that some alternative defintions demand that A has non-zero measure (i.e. to eliminate stable fixed points) or that there exists a orbit of the flow that is dense inside A.

What makes a stange attractor so *strange*? At the time, Ruelle and Takens never articulated a concrete definition for strange attractors, instead is was simply an attractor that was "strange" to them. Today, strange attractors are defined to be attractors whose dimension is not an integer, i.e. a fractal. Of course, the Lorenz attractor is a strange attractor in this sense as well, as proven in "A rigorous ODE solver and Smale's 14th problem" (Tucker 2002).

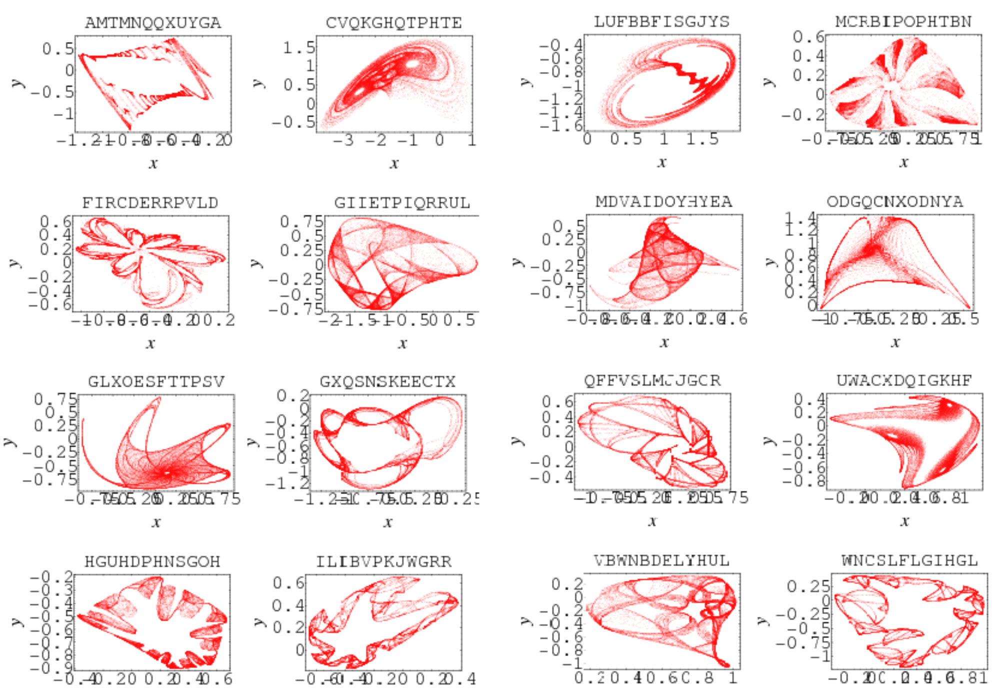

Figure 2: Some strange attractors. Figures from E. W. Weisstein.

1.3 Reconstructing Attractors

In "Geometry from a time series" (Packard et al. 1980), Packard et al. were interested in problem of *dynamical reconstruction*: from experimental observations of a turbulent fluid flow, how does one show the existence of a low-dimensional chaotic dynamical system which explains the observed behaviors? Additionally, how does one determine the dimensionality of the underlying attractor? They demonstrate their ideas using another non-linear dynamical system studied by (Rössler 1976):

$$
\begin{aligned}
\frac{\mathrm{d}X}{\mathrm{d}t} &= -(Y+Z), \\
\frac{\mathrm{d}Y}{\mathrm{d}t} &= X + 0.2Y, \\
\frac{\mathrm{d}Z}{\mathrm{d}t} &= 0.4 + XZ - 5.7Z.
\end{aligned} \tag{4}
$$

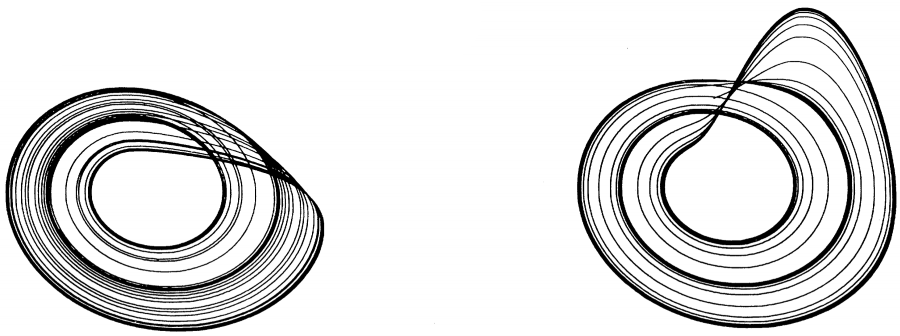

FIG. 1. (x,y) projection of Rossler (Ref. 7).

FIG. 2. $(x,\dot{x})$ reconstruction from the time series.

Figure 3: The Rössler system flow projected onto the X, Y plane and the induced flow of $(X, \dot{X}, \ddot{X})$ projected onto the $X, \dot{X}$ plane. Figure from (Packard et al. 1980).

2 Attractors and Takens' Theorem

3 Selected References

3.1 Some Recent Developments in a Concept of Causality

Granger

3.2 Detecting Causality in Complex Ecosystems

Sugihara et al.

3.3 Distinguishing Time-delayed Causal Interactions using Convergent Cross Mapping

Ye et al.

3.4 Exact Inference of Causal Relations in Dynamical Systems

Benkő et al.

References

Allen, John-Mark A et al. (2017). "Quantum common causes and quantum causal models". In: *Physical Review X* 7.3, p. 031021.

Barnett, Lionel and Anil K Seth (2014). "The MVGC multivariate Granger causality toolbox: a new approach to Granger-causal inference". In: *Journal of neuroscience methods* 223, pp. 50–68.

Benkő, Zsigmond et al. (2018). "Exact Inference of Causal Relations in Dynamical Systems". In: *arXiv preprint arXiv:1808.10806*.

Blom, Tineke, Stephan Bongers, and Joris M Mooij (2019). "Beyond Structural Causal Models: Causal Constraints Models". In: *Proceedings of the Conference on Uncertainty in Artificial Intelligence*.

Bollt, Erik M, Jie Sun, and Jakob Runge (2018). "Introduction to Focus Issue: Causation inference and information flow in dynamical systems: Theory and applications". In: *Chaos: An Interdisciplinary Journal of Nonlinear Science* 28.7, p. 075201.

Breuer, Heinz-Peter (2004). "Genuine quantum trajectories for non-Markovian processes". In: *Physical Review A* 70.1, p. 012106.

Friston, Karl J et al. (2014). "Granger causality revisited". In: *NeuroImage* 101, pp. 796–808.

Gollub, Jerry P and Harry L Swinney (1975). "Onset of turbulence in a rotating fluid". In: *Physical Review Letters* 35.14, p. 927.

Granger, Clive WJ (1988). "Some recent development in a concept of causality". In: *Journal of econometrics* 39.1-2, pp. 199–211.

Harnack, Daniel et al. (2017). "Topological causality in dynamical systems". In: *Physical review letters* 119.9, p. 098301.

Lorenz, Edward N (1963). "Deterministic nonperiodic flow". In: *Journal of the atmospheric sciences* 20.2, pp. 130–141.

Lusch, Bethany, Pedro D Maia, and J Nathan Kutz (2016). "Inferring connectivity in networked dynamical systems: Challenges using Granger causality". In: *Physical Review E* 94.3, p. 032220.

Milnor, John (1985). "On the concept of attractor". In: *The theory of chaotic attractors*. Springer, pp. 243–264.

Mogensen, Søren Wengel, Daniel Malinsky, and Niels Richard Hansen (2018). "Causal Learning for Partially Observed Stochastic Dynamical Systems." In: *UAI*, pp. 350–360.

Packard, Norman H et al. (1980). "Geometry from a time series". In: *Physical review letters* 45.9, p. 712.

Pearl, Judea (2000). *Causality: models, reasoning and inference*. Vol. 29. Springer.

Pearl, Judea et al. (2009). "Causal inference in statistics: An overview". In: *Statistics surveys* 3, pp. 96–146.

Penny, Will D et al. (2004). "Comparing dynamic causal models". In: *Neuroimage* 22.3, pp. 1157–1172.

Poole, David and Mark Crowley (2013). "Cyclic causal models with discrete variables: Markov chain equilibrium semantics and sample ordering". In: *Twenty-Third International Joint Conference on Artificial Intelligence*.

Rössler, Otto E (1976). "An equation for continuous chaos". In: *Physics Letters A* 57.5, pp. 397–398.

Rubenstein, Paul K et al. (2016). "From deterministic ODEs to dynamic structural causal models". In: *arXiv preprint arXiv:1608.08028*.

Ruelle, David and Floris Takens (1971). "On the nature of turbulence". In: *Les rencontres physiciens-mathématiciens de Strasbourg-RCP25* 12, pp. 1–44.

Spekkens, Robert W (2015). "The paradigm of kinematics and dynamics must yield to causal structure". In: *Questioning the Foundations of Physics*. Springer, pp. 5–16.

Stephan, Klaas Enno et al. (2010). "Ten simple rules for dynamic causal modeling". In: *Neuroimage* 49.4, pp. 3099–3109.

Sugihara, George et al. (2012). "Detecting causality in complex ecosystems". In: *science* 338.6106, pp. 496–500.

Szepesvári, Csaba, Jean-Yves Audibert, et al. (2007). "Manifold-adaptive dimension estimation". In: *Proceedings of the 24th international conference on Machine learning*. ACM, pp. 265–272.

Takens, Floris (1981). "Detecting strange attractors in turbulence". In: *Dynamical systems and turbulence, Warwick 1980*. Springer, pp. 366–381.

Tucker, Warwick (2002). "A rigorous ODE solver and Smale's 14th problem". In: *Foundations of Computational Mathematics* 2.1, pp. 53–117.

Wagner, Andreas (1999). "Causality in complex systems". In: *Biology and Philosophy* 14.1, pp. 83–101.

White, Halbert, Karim Chalak, and Xun Lu (2011). "Linking granger causality and the pearl causal model with settable systems". In: *NIPS Mini-Symposium on Causality in Time Series*, pp. 1–29.

Wood, Christopher J and Robert W Spekkens (2015). "The lesson of causal discovery algorithms for quantum correlations: Causal explanations of Bell-inequality violations require fine-tuning". In: *New Journal of Physics* 17.3, p. 033002.

Ye, Hao et al. (2015). "Distinguishing time-delayed causal interactions using convergent cross mapping". In: *Scientific reports* 5, p. 14750.