The Quantum Marginals Problem

March 6, 2020

TC Fraser

Quantum Foundations Group Meeting

Overview

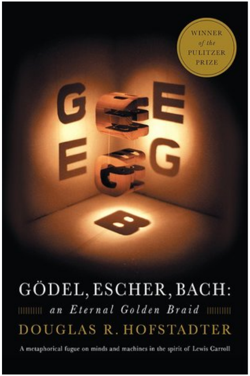
1. Big Picture & Origins of the Problem
2. Definitions and Terminology
3. Various Applications and Known Results
4. Quantum Causality & The Future of Marginal Problems

Big Picture

Big Picture

- When is local information compatible with a global model?
- Already many instances within Quantum Foundations
- Classical marginal problems (application to database integrity, knowledge integration, coalition games)[1]
- Local operational data is contextual when in fails to admit of a global noncontextual ontological model[2]
- Wigner functions arise when trying to view the statistics of local quantum observables as marginals of a covariant global distribution

Example:



[2] Tobias Fritz and Rafael Chaves. "Entropic inequalities and marginal problems". In: *IEEE transactions on information theory* 59.2 (2012), pp. 803–817.

[2] Samson Abramsky and Adam Brandenburger. "The sheaf-theoretic structure of non-locality and contextuality". In: *New Journal of Physics* 13.11 (2011), p. 113036.

Origins of the Quantum Marginal Problem

- Originally appeared with application to quantum chemistry[3]
- Consider a state of N-fermions $|\psi\rangle \in \bigwedge^N \mathcal{H}$
- and any 2-local Hamiltonian

$$H = \sum_{1\leq i\leq N} h_i + \sum_{1\leq i<j\leq N} h_{ij}$$

- Energetics of a state $|\psi\rangle\langle\psi|$ only depend on reduced densities

$$\mathrm{Tr}(H\,|\psi\rangle\langle\psi|) = \sum_{1\leq i<j\leq N} \mathrm{Tr}\left(\tilde{h}_{ij}\rho_{ij}\right)$$

> The advantage of (8.2) is that it does not involve the N-particle wave function but only the 2 matrix $\sigma(12;1'2')$, it is exact, and unlike approximations to the energy in terms of the 1 matrix, it is *linear* in the density matrix and occupation numbers. One could, therefore, hope to obtain the ground-state energy of an N-particle system merely by choosing σ to minimize (8.2). It was while he enjoyed the hospitality of the Summer Research Institute of the Canadian Mathematical Congress in 1951 that this possibility first occurred to the present author. He proceeded to calculate the energy of the ground state of Li and was somewhat surprised to obtain a value about 30% *too low!* This shook his naive and unexamined faith in Ritz and Rayleigh but aroused his interest. In that first attempt σ had been varied over too large a class of functions. The restriction to N-representable σ had not been imposed. If the
>
> — A. J. Coleman, 1963

- The N*-representability problem* aims to characterize the space of 2-body density operators which can be realized as the reduced states of an N-body fermionic pure state

[3] A John Coleman. "Structure of fermion density matrices". In: *Reviews of modern Physics* 35.3 (1963), p. 668.

> *"We make no apology for the consideration of such a special case. The general N-representability problem for one and two body reduced density matrices is so difficult and yet so fundamental to many branches of science that each concrete result is useful in shedding light on the nature of the solution of the more general problem."*
>
> — B. E. Borland, K. Dennis[4]

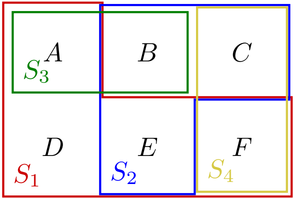
- The N-representability problem is QMA-complete[5].

[4] RE Borland and K Dennis. "The conditions on the one-matrix for three-body fermion wavefunctions with one-rank equal to six". In: *Journal of Physics B: Atomic and Molecular Physics* 5.1 (1972), p. 7.

[5] Yi-Kai Liu, Matthias Christandl, and Frank Verstraete. "Quantum computational complexity of the N-representability problem: QMA complete". In: *Physical review letters* 98.11 (2007), p. 110503.

Terminology

The Quantum Marginals Problem: Terminology



$$
\begin{aligned}
\mathcal{T} &= \{A, B, C, D, E, F\} \\
S_1 &= \{A, D, E, F\} \\
S_2 &= \{B, C, E\} \\
S_3 &= \{A, B\} \\
S_4 &= \{C, F\}
\end{aligned}
$$

Definition

A **quantum marginals problem (QMP)** can be identified by a family of subsystems $\{S_1, S_2, \ldots, S_m\} \in 2^{\mathcal{T}}$ of some total system $\mathcal{T}$.
The objective is to characterize the space of families of density operators $\{\rho_{S_1}, \rho_{S_2}, \ldots, \rho_{S_m}\}$ with $\rho_{S_j} \in \mathcal{B}(\mathcal{H}_{S_j})$ for which

$$\text{there exists} \quad \rho_{\mathcal{T}} \in \mathcal{B}(\mathcal{H}_{\mathcal{T}}) \quad \text{such that} \quad \forall j : \operatorname{Tr}_{\mathcal{T} \setminus S_j}(\rho_{\mathcal{T}}) = \rho_{S_j}$$

Variations of the Quantum Marginals Problem

- *pure* QMP : joint density operator $\rho_{\mathcal{T}} = |\psi_{\mathcal{T}}\rangle \langle\psi_{\mathcal{T}}|$ is pure
- *disjoint/univariate* QMP : the subsystems do not overlap $S_i \cap S_j = \delta_{ij} S_i$
- *spectral* QMP : characterize the space of specta $\lambda_j = \mathrm{spec}(\rho_{S_j})$
- *fermionic* QMP : the joint density operator is $\rho_{\mathcal{T}} \in \mathcal{B}(\bigwedge^{|\mathcal{T}|} \mathcal{H})$
- *Gaussian* QMP : the marginal ρ_{S_j} and joint density operators are Gaussian states
- *channel* QMP : replace density operators with quantum channels
- and many more

Results About the Disjoint Spectral QMP

Theorem (Higuchi-Sudbery-Szulc, 2003[6])

For N qubits $\otimes_{i=1}^{N}\mathcal{H}_i$ $(\dim(\mathcal{H}_i)=2)$, all constraints on the margins $\{\rho_i\}_{i=1}^{n}$ of a pure state are given by

$$\lambda_i \leq \sum_{j\neq i}\lambda_i$$

where λ_i is the minimal eigenvalue of ρ_i.

[6] Atsushi Higuchi, Anthony Sudbery, and Jason Szulc. "One-qubit reduced states of a pure many-qubit state: polygon inequalities". In: *Physical review letters* 90.10 (2003), p. 107902.

Theorem (Bravyi, 2003[7])

For 2 qubits $\mathcal{H}_A \otimes \mathcal{H}_B$, the spectral marginal problem for $\{A, B, AB\}$ is solved by

$$\begin{aligned} \min(\lambda_A, \lambda_B) &\geq \lambda_3^{AB} + \lambda_4^{AB}, \\ \lambda_A + \lambda_B &\geq \lambda_2^{AB} + \lambda_3^{AB} + 2\lambda_4^{AB}, \\ |\lambda_A - \lambda_B| &\leq \min(\lambda_1^{AB} - \lambda_3^{AB}, \lambda_2^{AB} - \lambda_4^{AB}), \end{aligned}$$

where λ_A, λ_B are minimal eigenvalues of ρ_A, ρ_B and $\lambda_1^{AB} \geq \lambda_2^{AB} \geq \lambda_3^{AB} \geq \lambda_4^{AB}$ is the spectrum of ρ_{AB}.

[7] Sergey Bravyi. "Requirements for compatibility between local and multipartite quantum states". In: *arXiv preprint quant-ph/0301014* (2003).

Theorem (Higuchi, 2003[8])

For 3 qutrits $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$ ($\dim(\mathcal{H}_i) = 3$), a necessary and sufficient condition on the nine spectral values $0 \leq \lambda_1^{(a)} \leq \lambda_2^{(a)} \leq \lambda_3^{(a)}$ and $\lambda_1^{(a)} + \lambda_2^{(a)} + \lambda_3^{(a)} = 1$ for $a = 1, 2, 3$ to be the reduced spectra of a pure three-qutrit quantum state is given by the following:

$$
\begin{aligned}
\lambda_2^{(a)} + \lambda_1^{(a)} &\leq \lambda_2^{(b)} + \lambda_1^{(b)} + \lambda_2^{(c)} + \lambda_1^{(c)}, \\
\lambda_3^{(a)} + \lambda_1^{(a)} &\leq \lambda_2^{(b)} + \lambda_1^{(b)} + \lambda_3^{(c)} + \lambda_1^{(c)}, \\
\lambda_2^{(a)} + \lambda_3^{(a)} &\leq \lambda_2^{(b)} + \lambda_1^{(b)} + \lambda_2^{(c)} + \lambda_3^{(c)}, \\
2\lambda_2^{(a)} + \lambda_1^{(a)} &\leq 2\lambda_2^{(b)} + \lambda_1^{(b)} + 2\lambda_2^{(c)} + \lambda_1^{(c)}, \\
2\lambda_1^{(a)} + \lambda_2^{(a)} &\leq 2\lambda_2^{(b)} + \lambda_1^{(b)} + 2\lambda_1^{(c)} + \lambda_2^{(c)}, \\
2\lambda_2^{(a)} + \lambda_3^{(a)} &\leq 2\lambda_2^{(b)} + \lambda_1^{(b)} + 2\lambda_2^{(c)} + \lambda_3^{(c)}, \\
2\lambda_2^{(a)} + \lambda_3^{(a)} &\leq 2\lambda_1^{(b)} + \lambda_2^{(b)} + 2\lambda_3^{(c)} + \lambda_2^{(c)},
\end{aligned}
$$

where (abc) are any permutations of (123).

[8] Atsushi Higuchi. "On the one-particle reduced density matrices of a pure three-qutrit quantum state". In: *arXiv preprint quant-ph/0309186* (2003).

- In 2004[9] and 2006[10], Klyachko effectively solves all disjoint spectral QMP
- Utilizes results from *geometric invariant theory*, in particular the Hilbert-Mumford stability criterion to derive necessary and sufficient constraints on spectra[11].
- Essential conclusion: the space of compatible disjoint spectra is always a finite polytope!
- Also applies to the spectra of the Riemann curvature tensor $\mathrm{R} : \wedge^2\mathcal{T} \to \wedge^2\mathcal{T}$ and Ricci tensor $\mathrm{Ric} : \mathcal{T} \to \mathcal{T}$ which are mutually compatible (inequalities are unpublished, but in his slides)
- Establishes connection with representation theory using Berenstein and Sjamaar's work[12]

"For overlapping margins like $\rho_{AB}, \rho_{BC}, \rho_{AC}$, the problem is beyond the scope of the current approach." — A. Klyachko, 2009

[9] Alexander Klyachko. "Quantum marginal problem and representations of the symmetric group". In: *arXiv preprint quant-ph/0409113* (2004).

[10] Alexander A Klyachko. "Quantum marginal problem and N-representability". In: *Journal of Physics: Conference Series*. Vol. 36. 1. IOP Publishing. 2006, p. 72.

[11] Nicolas Ressayre. "Geometric invariant theory and the generalized eigenvalue problem". In: *Inventiones mathematicae* 180.2 (2010), pp. 389–441.

[12] Arkady Berenstein and Reyer Sjamaar. "Coadjoint orbits, moment polytopes, and the Hilbert-Mumford criterion". In: *Journal of the American Mathematical Society* 13.2 (2000), pp. 433–466.

Notable PhD Theses

- Schilling's thesis[13]:
  - Greatly elaborates and expands upon the mathematical formalism behind Klyachko's approach
  - Ephasizes the importance of variational principles (Ky-Fan, Hersch-Zwahlen)
  - Develops the physical inuition behind Klyachko's solution to the fermionic N-representability in terms of *generalized Pauli constraints* and *(quasi-)pinning*.
- Klassen's thesis[14]:
  - Investigates the extent to which various features of the global quantum state can be uniquely determined by its marginals.

[13] Christian Schilling. "The quantum marginal problem". In: *Mathematical Results in Quantum Mechanics: Proceedings of the QMath12 Conference*. World Scientific. 2015, pp. 165–176.

[14] Joel David Klassen. "Existence and Uniqueness in the Quantum Marginal Problem". PhD thesis. 2017.

Connections to the Representation Theory of S_k

- Keyl and Werner[15] interested in indirect *measurement* of the spectra of a d-level system ρ
- Their results prove useful for spectral marginal problems
- Find a sequence projections P_k over possible spectra Σ

$$\Sigma = \{s \in \mathbb{R}^d \mid \sum_{j=1}^{d} s_j = 1, \forall j : s_j \geq s_{j+1} \geq 0\}$$

- So that if Δ is the *complement* of a neighborhood around $r = \mathrm{spec}(\rho) \in \Sigma$, the error

$$E_k(\Delta) = \mathrm{Tr}\Big(P_k(\Delta)\rho^{\otimes k}\Big)$$

  vanishes as $k \to \infty$
- Main idea: exploit the symmetry of $\rho^{\otimes k}$

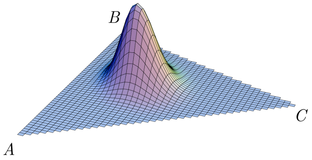


FIG. 1. Probability distribution $\mathrm{tr}(\rho^{\otimes N} P_Y)$ for $d = 3$, $N = 120$ and $r = (0.6, 0.3, 0.1)$. The set Σ is the triangle with corners $A = (1,0,0)$, $B = (1/2,1/2,0)$, $C = (1/3,1/3,1/3)$.

[15] Michael Keyl and Reinhard F Werner. "Estimating the spectrum of a density operator". In: *Asymptotic Theory Of Quantum Statistical Inference: Selected Papers*. World Scientific, 2005, pp. 458–467.

Spectral Estimation Using Representation Theory

- Elegant proof by Christandl and Mitchison[16]
- Spectra of $\rho^{\otimes k}$ is *invariant* under both the action of $SU(d)$ and of S_k
- Utilize Weyl-Schur duality indexed by partitions of $\lambda \in \mathcal{Y}_k$

$$(\mathbb{C}^d)^{\otimes k} = \bigoplus_{\lambda \in \mathcal{Y}_k} \mathcal{U}_\lambda \otimes \mathcal{V}_\lambda$$

  where $\mathcal{U}_\lambda$ and $\mathcal{V}_\lambda$ are irreps of S_k and $SU(d)$ respectively
- Young diagrams $\mathcal{Y}_k$ describe partitions of k

$$\mathcal{Y}_1 = \{\ydiagram{1}\}, \qquad \mathcal{Y}_2 = \{\ydiagram{2}, \ydiagram{1,1}\},$$

$$\mathcal{Y}_3 = \{\ydiagram{3}, \ydiagram{2,1}, \ydiagram{1,1,1}\} \qquad \mathcal{Y}_4 = \{\ydiagram{4}, \ydiagram{3,1}, \ydiagram{2,2}, \ydiagram{2,1,1}, \ydiagram{1,1,1,1}\}, \dots$$

- The eigenvectors of $\rho^{\otimes k}$ which are "sufficiently symmetric" are annihilated by the Young projection P_λ onto the subspace $\mathcal{U}_\lambda \otimes \mathcal{V}_\lambda \subset (\mathbb{C}^d)^{\otimes k}$

[16] Matthias Christandl and Graeme Mitchison. "The spectra of quantum states and the Kronecker coefficients of the symmetric group". In: *Communications in mathematical physics* 261.3 (2006), pp. 789–797.

How to Survive a Young Projector

- Eigen-decomposition of $\rho = \sum_{j=1}^{d} r_j \left|v_j\right\rangle \left\langle v_j\right|$
- Eigenvectors of $\rho^{\otimes k}$ are then of the form

$$\left|V_J\right\rangle \coloneqq \left|V_{(j_1,\ldots,j_k)}\right\rangle \coloneqq \left|v_{j_1}\right\rangle \otimes \left|v_{j_2}\right\rangle \otimes \cdots \otimes \left|v_{j_k}\right\rangle \qquad J \in \{1,\ldots,d\}^k$$

- Example for $1 \leq i, j \leq d$ and $k = 3$ with $\lambda = [1,1,1] \simeq \ydiagram{1,1,1}$:

$$P_{\ydiagram{1,1,1}} \left|V_{(i,i,j)}\right\rangle \propto \left|V_{(i,i,j)}\right\rangle - \left|V_{(i,i,j)}\right\rangle + \left|V_{(i,j,i)}\right\rangle - \left|V_{(i,j,i)}\right\rangle + \left|V_{(j,i,i)}\right\rangle - \left|V_{(j,i,i)}\right\rangle = 0$$

- Similar example for $1 \leq i, j \leq d$ and $k = 4$ with $\lambda = [2,1,1] \simeq \ydiagram{2,1,1}$:

$$P_{\ydiagram{2,1,1}} \left|V_{(i,i,j,j)}\right\rangle \propto \left|V_{(i,i,j,j)}\right\rangle - \left|V_{(i,i,j,j)}\right\rangle + \cdots = 0$$

- Eigenvectors of $\rho^{\otimes k}$ which are "too symmetric" are annihilated!

How to Survive a Young Projector, Cont'd

- For $J \in \{1, \ldots, d\}^k$ define the *multiplicity partition* $\mu(J) \in \mathcal{Y}_k$ in the obvious way:

$$\mu((1,1,2,4,4,4)) = [3,2,1] \simeq \ydiagram{3,2,1}$$

Theorem

Let $|V_J\rangle$ be a non-zero eigenvector of $\rho^{\otimes k}$ with multiplicity partition $\mu(J) \in \mathcal{Y}_k$. Then

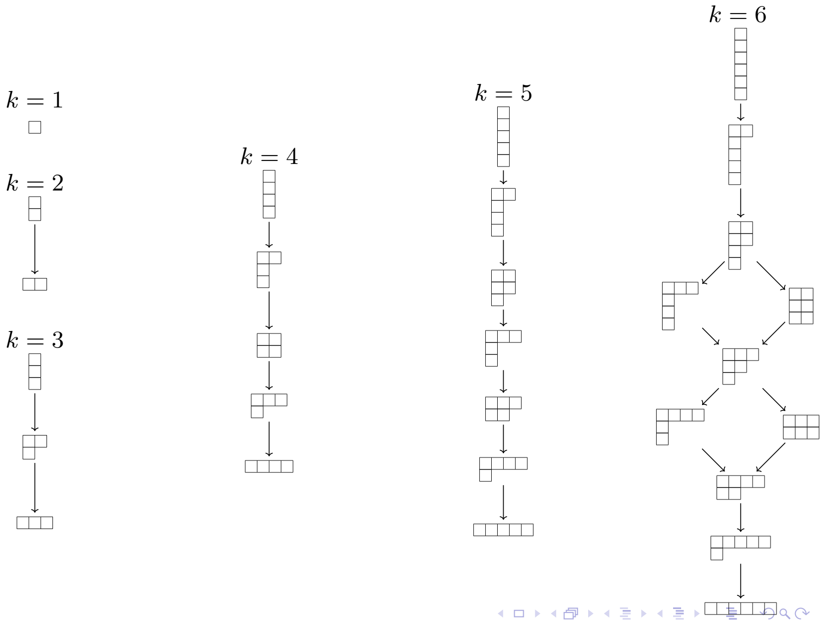
$$P_\lambda |V_J\rangle = 0 \iff \mu(J) \preceq \lambda$$

- Where $\preceq$ is the *majorization preorder* defined on partitions of k as

$$\mu \preceq \lambda \iff \begin{cases} \sum_{i=1}^m \mu_i \leq \sum_{i=1}^m \lambda_i & \forall m \leq k-1, \\ \sum_{i=1}^k \mu_i = \sum_{i=1}^k \lambda_i \end{cases}$$

  where $\mu_1 \geq \mu_2 \geq \cdots \geq 0$ and $\lambda_1 \geq \lambda_2 \geq \cdots \geq 0$.
- As a corollary, eigenvalues of $P_\lambda \rho^{\otimes k} P_\lambda$ are upper bounded by $\prod_i r_i^{\lambda_i}$ (which is largest for $\lambda_i / k \approx r_i$)

Hasse Diagrams for the Majorization Preorder of Partitions



Products of Density Operators and Where to Find Them

Theorem (Christandl and Mitchison 2006)

Let ρ *be a density operator with spectrum* $r = \mathrm{spec}(\rho)$ *and let* P_λ *be the projection onto* $\mathcal{U}_\lambda \otimes \mathcal{V}_\lambda$ *for* $\lambda \in \mathcal{Y}_k$*. Then*

$$\mathrm{Tr}\left(P_\lambda \rho^{\otimes k}\right) \leq (k+1)^{d(d-1)/2} \exp\left(-kD(\bar{\lambda}||r)\right)$$

where $\bar{\lambda}$ *is defined as* $\bar{\lambda}_i = \lambda_i / k$*.*

Definition (KL Divergence)

The Kullback-Liebler divergence $D(p||q)$ between two normalized probability distributions p and q is

$$D(p||q) = \sum_i p_i(\log p_i - \log q_i).$$

- Apply this result to the spectral QMP $\{r_{AB}, r_A, r_B\}$
- Rough conclusion: for each k, there must be three partitions λ, μ, ν of k with

$$\bar{\lambda} \approx r_{AB}, \quad \bar{\mu} \approx r_A, \quad \bar{\nu} \approx r_B$$

such that $(P_\mu \otimes P_\nu)P_\lambda \neq 0$

Definition

The *Kronecker coefficients* dictate the irreps of S_k which are sub-representations of products of irreps of S_k:

$$\mathcal{U}_\mu \otimes \mathcal{U}_\nu = \bigoplus_{\lambda \in \mathcal{Y}_k} g_{\lambda\mu\nu} U_\lambda$$

Theorem (Christandl-Mitchison, 2006)

For every density operator ρ_{AB}, there is a sequence $(\lambda_j, \mu_j, \nu_j)$ of partitions with $j \in \mathbb{N}_+$ of $k_j = |\lambda_j| = |\mu_j| = |\nu_j|$ such that

$$g_{\lambda_j \mu_j \nu_j} \neq 0 \quad \textit{i.e.} \quad \mathcal{U}_{\lambda_j} \subset \mathcal{U}_{\mu_j} \otimes \mathcal{U}_{\nu_j}$$

$$\lim_{j\to\infty} \bar{\lambda}_j = \mathrm{spec}(\rho_{AB}), \qquad \lim_{j\to\infty} \bar{\mu}_j = \mathrm{spec}(\rho_A), \qquad \lim_{j\to\infty} \bar{\nu}_j = \mathrm{spec}(\rho_B).$$

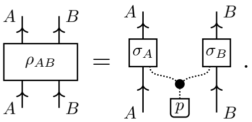
"Calculation of the Kronecker coefficients is a tricky problem, arguably considered as '... the last major problem in ordinary representation theory of S_k' – Yoav Dvir."

— A. Klyachko

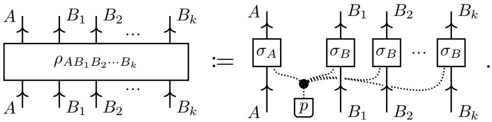
Results about Overlapping QMP

Bipartite Entanglement & Symmetric Extensions

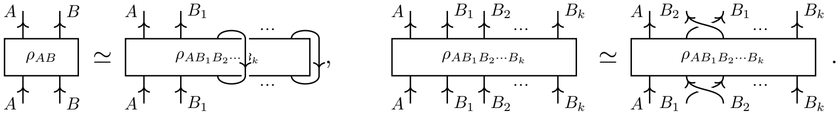
- The separability of a bipartite state $\rho_{AB} = \sum_j p_j \sigma_A^{(j)} \otimes \sigma_B^{(j)}$ can encoded as quantum marginals problem



- Assuming separability, there must exist a new state $\rho_{AB_1B_2\cdots B_k}$ called the *k-symmetric extension* of ρ_{AB} defined by



- The k-symmetric extension has ρ_{AB_i} marginals equal to ρ_{AB} for all i:



A Complete Hierarchy of Separability Tests

Theorem (Doherty-Parrilo-Spedalieri, 2004)

A bipartite mixed state ρ on $\mathcal{H}_A \otimes \mathcal{H}_B$ is separable if and only if ρ has a symmetric extension to k copies for any and all k.

Theorem (Chen-Ji-Kribs-Lütkenhaus-Zeng, 2014)

A bipartite qubit mixed state ρ on $\mathcal{H}_A \otimes \mathcal{H}_B$ admits of a $k = 2$ symmetric extension if and only if

$$\mathrm{Tr}\left(\rho_B^2\right) \geq \mathrm{Tr}\left(\rho_{AB}^2\right) - 4\sqrt{\det(\rho_{AB})}.$$

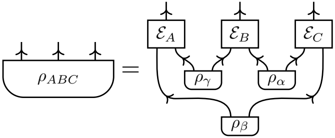
- Serves as first analytic complete solution to an overlapping quantum marginals problem (as far as I know).
- Can be written entirely in terms of spectra:

$$\sum_j (\lambda_j^B)^2 \geq \sum_i (\lambda_i^{AB})^2 - 4\left(\prod_i \lambda_i^{AB}\right)^{1/2}.$$

[16] Andrew C Doherty, Pablo A Parrilo, and Federico M Spedalieri. "Complete family of separability criteria". In: *Physical Review A* 69.2 (2004), p. 022308.

[16] Jianxin Chen et al. "Symmetric extension of two-qubit states". In: *Physical Review A* 90.3 (2014), p. 032318.

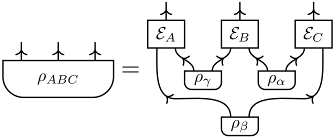
- The ability to test causal hypotheses is intrinsically important to science
- For quantum theory, the quantum inflation technique due to Wolfe et al.[17] transforms quantum causal hypotheses into QMPs
- Quick example with *triangle network*:



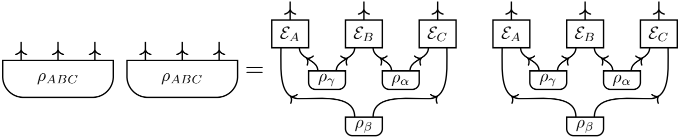
- The tripartite quantum state ρ_{ABC} can be prepared using the triangle network only if there exists a 6-partite quantum state $\Gamma_{A_1 B_1 C_1 A_2 B_2 C_2}$ admitting certain marginals

[17] Elie Wolfe et al. "Quantum inflation: a general approach to quantum causal compatibility". In: *arXiv preprint arXiv:1909.10519* (2019).

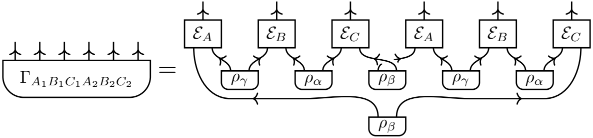
- If ρ_{ABC} is triangle network compatible,

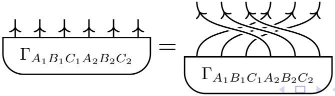


- then prepare ρ_{ABC} twice,

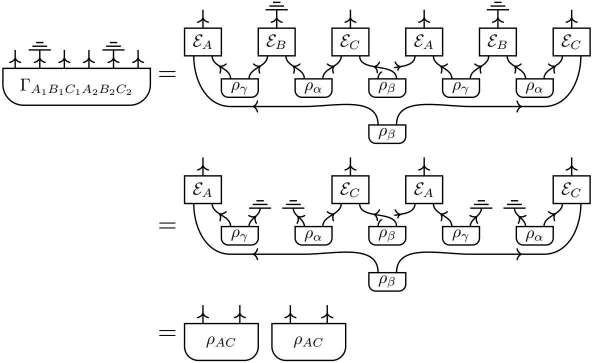


- but instead rewire the sources,



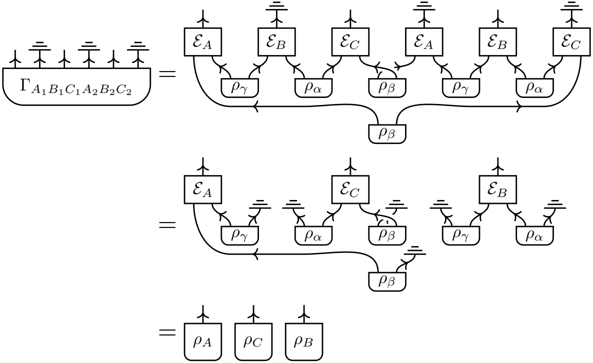


- Some marginals of $\Gamma_{A_1 B_1 C_1 A_2 B_2 C_2}$ are constrained by ρ_{ABC},



$$
\begin{aligned}
\mathrm{Tr}_{B_1 B_2}(\Gamma_{A_1 B_1 C_1 A_2 B_2 C_2}) &= \rho_{AC} \otimes \rho_{AC} \\
\mathrm{Tr}_{A_1 A_2}(\Gamma_{A_1 B_1 C_1 A_2 B_2 C_2}) &= \rho_{BC} \otimes \rho_{BC} \\
\mathrm{Tr}_{C_1 C_2}(\Gamma_{A_1 B_1 C_1 A_2 B_2 C_2}) &= \rho_{AB} \otimes \rho_{AB}
\end{aligned}
$$

A QMP for Triangle Network Compatibility, Cont'd



$$\mathrm{Tr}_{B_1A_2C_2}(\Gamma_{A_1B_1C_1A_2B_2C_2}) = \rho_A \otimes \rho_B \otimes \rho_C$$
$$\mathrm{Tr}_{A_1C_1B_2}(\Gamma_{A_1B_1C_1A_2B_2C_2}) = \rho_B \otimes \rho_A \otimes \rho_C$$

References

Abramsky, Samson and Adam Brandenburger. “The sheaf-theoretic structure of non-locality and contextuality”. In: *New Journal of Physics* 13.11 (2011), p. 113036.

Aloy, Albert, Matteo Fadel, and Jordi Tura. “The quantum marginal problem for symmetric states: applications to variational optimization, nonlocality and self-testing”. In: *arXiv preprint arXiv:2001.04440* (2020).

Berenstein, Arkady and Reyer Sjamaar. “Coadjoint orbits, moment polytopes, and the Hilbert-Mumford criterion”. In: *Journal of the American Mathematical Society* 13.2 (2000), pp. 433–466.

Borland, RE and K Dennis. “The conditions on the one-matrix for three-body fermion wavefunctions with one-rank equal to six”. In: *Journal of Physics B: Atomic and Molecular Physics* 5.1 (1972), p. 7.

Bravyi, Sergey. “Requirements for compatibility between local and multipartite quantum states”. In: *arXiv preprint quant-ph/0301014* (2003).

Chen, Jianxin et al. “Symmetric extension of two-qubit states”. In: *Physical Review A* 90.3 (2014), p. 032318.

Christandl, Matthias and Graeme Mitchison. “The spectra of quantum states and the Kronecker coefficients of the symmetric group”. In: *Communications in mathematical physics* 261.3 (2006), pp. 789–797.

Coleman, A John. “Structure of fermion density matrices”. In: *Reviews of modern Physics* 35.3 (1963), p. 668.

Daftuar, Sumit and Patrick Hayden. “Quantum state transformations and the Schubert calculus”. In: *Annals of Physics* 315.1 (2005), pp. 80–122.

Doherty, Andrew C, Pablo A Parrilo, and Federico M Spedalieri. “Complete family of separability criteria”. In: *Physical Review A* 69.2 (2004), p. 022308.

Dvir, Yoav. “On the Kronecker product of Sn characters”. In: *Journal of Algebra* 154.1 (1993), pp. 125–140.

Eisert, Jens et al. “Gaussian quantum marginal problem”. In: *Communications in mathematical physics* 280.1 (2008), pp. 263–280.

Fritz, Tobias and Rafael Chaves. “Entropic inequalities and marginal problems”. In: *IEEE transactions on information theory* 59.2 (2012), pp. 803–817.

Higuchi, Atsushi. “On the one-particle reduced density matrices of a pure three-qutrit quantum state”. In: *arXiv preprint quant-ph/0309186* (2003).

Higuchi, Atsushi, Anthony Sudbery, and Jason Szulc. “One-qubit reduced states of a pure many-qubit state: polygon inequalities”. In: *Physical review letters* 90.10 (2003), p. 107902.

Keyl, Michael and Reinhard F Werner. “Estimating the spectrum of a density operator”. In: *Asymptotic Theory Of Quantum Statistical Inference: Selected Papers*. World Scientific, 2005, pp. 458–467.

Klassen, Joel David. “Existence and Uniqueness in the Quantum Marginal Problem”. PhD thesis. 2017.

Klyachko, Alexander. “Quantum marginal problem and representations of the symmetric group”. In: *arXiv preprint quant-ph/0409113* (2004).

Klyachko, Alexander A. “Quantum marginal problem and N-representability”. In: *Journal of Physics: Conference Series*. Vol. 36. 1. IOP Publishing. 2006, p. 72.

Liu, Yi-Kai, Matthias Christandl, and Frank Verstraete. “Quantum computational complexity of the N-representability problem: QMA complete”. In: *Physical review letters* 98.11 (2007), p. 110503.

Ressayre, Nicolas. “Geometric invariant theory and the generalized eigenvalue problem”. In: *Inventiones mathematicae* 180.2 (2010), pp. 389–441.

Schilling, Christian. “The quantum marginal problem”. In: *Mathematical Results in Quantum Mechanics: Proceedings of the QMath12 Conference*. World Scientific. 2015, pp. 165–176.

Walter, Michael et al. “Entanglement polytopes: multiparticle entanglement from single-particle information”. In: *Science* 340.6137 (2013), pp. 1205–1208.

Wolfe, Elie et al. “Quantum inflation: a general approach to quantum causal compatibility”. In: *arXiv preprint arXiv:1909.10519* (2019).