Quantum State Transformations and the Schubert Calculus

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Abstract

Recent developments in mathematics have provided powerful tools for comparing the eigenvalues of matrices related to each other via a moment map. In this paper we survey some of the more concrete aspects of the approach with a particular focus on applications to quantum information theory. After discussing the connection between Horn's Problem and Nielsen's Theorem, we move on to characterizing the eigenvalues of the partial trace of a matrix.

1 Introduction

This paper presents some applications of recently minted matrix eigenvalue inequalities to problems in quantum information theory. Much of the machinery necessary to understand the results is developed on the fly, with the intention that physicists unfamiliar with the relevant algebraic topology, representation theory and symplectic geometry can use this paper as an introduction to the mathematics. Likewise, while much of the mathematical material will be familiar to experts, we hope that they will enjoy seeing their techniques put to use in the service of a relatively novel application, analysis of the nonlocal structure of quantum states.

One way of understanding quantum information theory is as the identification of the basic resources useful for manipulating information in a quantum-mechanical world, and the pursuit of optimal methods for converting between these resources. (See [3] for a summary of recent work in the field.) In this framework, communication of classical and quantum bits (qubits) are resources, as are the many varieties of entanglement. Our focus here will be on understanding the communication resources required to transform one known, bipartite, pure quantum state into another. In the case where only local operations and classical communication (LOCC) are allowed, we will find that Klyachko's resolution [25] of Horn's Conjecture [21] gives an essentially complete answer, in principle, to the question of which states can be converted into a given other using a fixed finite amount of communication. The bulk of the paper, however, is devoted to the case where qubits are exchanged instead of classical bits, but that problem can again be resolved in principle using a similar collection of techniques.

From a mathematical perspective, the first problem reduces via well-known results to the question of determining the possible eigenvalues of a convex combination of isospectral matrices with defined spectra, a clear special case of Klyachko's result. We also show how using only the much earlier Horn's Theorem, any such convex combination can be replaced by a convex combination with equal weights, an observation with implications for communication. The second problem corresponds to determining the moment polytope for the group U(m) acting on mn by mn Hermitian matrices via conjugation: $(X, H) \mapsto (X \otimes I_n)H(X \otimes I_n)^{-1}$. This is a special case which we examine in detail of a problem considered by Berenstein and Sjamaar [4]. Again we find simplifications not present in the general case.

Guide to the paper: Section 2 introduces Horn's problem and explains its relevance to state transformations. Section 3 then introduces the partial trace problem which is the focus of the rest of the paper, including a discussion of its physical interpretation. Section 4 presents a powerful variational approach to sums of eigenvalues of a Hermitian matrix due to Hersch and Zwahlen that is the source of our inequalities relating the spectra of a matrix and its partial trace, while Section 5 consists entirely of background material about the Schubert calculus. The heart of the paper is contained in Sections 6 and 7 which provide, respectively, the crucial cohomological calculation and an explicit evaluation of the inequalities in low dimension. Also included in section 7 is a discussion of the connection to representation theory and a very brief discussion of the connection to symplectic geometry which is required to complete our argument. (For more information on this aspect of the problem, the reader can do no better than Knutson's excellent review [27].)

2 Horn's Problem and State Transformations

2.1 The problem and its solution

Horn's problem is the following: Given the spectra of $n \times n$ Hermitian matrices X and Y, what are the possible spectra of Z = X + Y? This problem was first seriously attacked by H. Weyl in 1912 [38], but the complete solution has only been achieved recently [2, 25, 26, 28, 29, 35].

Early attempts at Horn's problem involved finding inequalities that the eigenvalues of X, Y, and Z had to satisfy, in order that Z = X + Y. Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be the eigenvalues of X, $\beta = (\beta_1, \ldots, \beta_n)$ be the eigenvalues of Y, and $\gamma = (\gamma_1, \ldots, \gamma_n)$ be the eigenvalues of Z, all written in non-increasing order. One basic constraint that α , β and γ must satisfy is the *trace condition*

$$\sum_{i=1}^{n} \gamma_{i} = \sum_{i=1}^{n} \alpha_{i} + \sum_{i=1}^{n} \beta_{i}. \tag{1}$$

Besides this equality condition, for many years all other known constraints on the eigenvalues could be reduced to linear inequalities among the eigenvalues; in fact, they all had the form

$$\sum_{k \in K} \gamma_k \le \sum_{i \in I} \alpha_i + \sum_{j \in J} \beta_j. \tag{2}$$

where I, J, and K are all subsets of $\{1, \ldots, n\}$ of the same cardinality r. Such inequalities were systematically analyzed by A. Horn in 1962 [21]. He found conditions on triples of index sets (I, J, K) for which he conjectured that inequalities of the form of Inequality (2) would be necessary and sufficient.

Horn defined sets T_r^n of triples (I, J, K), corresponding to the (conjectured) necessary and sufficient inequalities inductively as follows. For each positive integer n and $r \le n$, let

$$U_r^n = \{(I, J, K) | \sum_{i \in I} i + \sum_{j \in J} j = \sum_{k \in K} k + r(r+1)/2 \}.$$
 (3)

Then for r = 1, let $T_1^n = U_1^n$. For r > 1, let

$$\begin{split} T^n_r &=& \{(I,J,K) \in U^n_r| \text{ for all } p < r \text{ and all } (F,G,H) \in T^r_p, \\ &\sum_{f \in F} i_f + \sum_{g \in G} j_g \leq \sum_{h \in H} k_h + p(p+1)/2\}. \end{split}$$

Horn then proposed:

Conjecture 2.1 (Horn) A triple (α, β, γ) can be the eigenvalues of $n \times n$ Hermitian matrices X, Y, and Z, where Z = X + Y, if and only if the trace condition holds, and

$$\sum_{k \in K} \gamma_k \le \sum_{i \in I} \alpha_i + \sum_{j \in J} \beta_j$$

for all $(I, J, K) \in T_r^n$, for all r < n.

He showed that his conjecture was valid for n=3 and n=4 (the case n=2 was already known), and asserted that his proof could be extended for $n\leq 8$. Moreover, he managed to prove that the general form of his inequalities was sufficient:

Theorem 2.2 (Horn) For each positive n and N there exists a finite set L and index sets $\{K_l\} \subset \{1,\ldots,N\}$ and $\{J_{il}\} \subset \{1,\ldots,N\}$, where $l \in L$ and $i \in \{1,\ldots,N\}$, such that the following holds: An $n \times n$ Hermitian matrix A can be written as the sum of N Hermitian $n \times n$ matrices with respective spectra $\lambda^1, \lambda^2, \ldots, \lambda^N$ if and only if

$$\sum_{k \in K_I} (\operatorname{Spec}(A))_k \le \sum_{i=1}^N \sum_{j \in J_{iI}} \lambda_j^i \tag{4}$$

holds for all $l \in L$. (Note that the spectra λ^i are each written in non-increasing order.)

Extending the demonstration of the conjecture to the general case, however, proved elusive. In 1982, B .V. Lidskii [30] announced that he had verified Horn's conjecture, but his proof sketch was very incomplete, and the details have never appeared. The problem was finally definitively solved by Klyachko [25, 26], with important related contributions from Belkale, Knutson, Tao, Totaro and Woodward [2, 28, 29, 35]:

Theorem 2.3 *Horn's conjecture is true.*

A somewhat unexpected complication that arises in that Horn's list of inequalities is redundant for n > 5; as n increases, the number of redundant inequalities grows rapidly. So it is natural to desire a minimal set of inequalities that are necessary and sufficient for (α, β, γ) to be the spectra of Hermitian matrices X, Y, and X + Y. This issue has been resolved as well; Knutson and Tao have developed combinatorial gadgets called "honeycombs" that can be used to determine which of Horn's inequalities is redundant [28, 29].

2.2 An Application to LOCC Protocols

Besides serving as a motivation for the present work, Horn's problem itself yields insights into problems of quantum information theory. We present an application demonstrating that it is sufficient to consider protocols of a special type in performing transformations using LOCC.

Recall that given two probability vectors p and q in \mathbb{R}^N , we say that q is majorized by p, or $q \prec p$, if

$$\sum_{i=1}^{k} q_i^{\downarrow} \le \sum_{i=1}^{k} p_i^{\downarrow} \tag{5}$$

for all $1 \le k < N$, where v^{\downarrow} represents the vector v with entries arranged in non-increasing order. There is a very useful characterization of the majorization relation: $q \prec p$ if and only if q = Ap for some double stochastic matrix A. The doubly stochastic matrices are, in turn, described by

Theorem 2.4 (Birkhoff) The extreme points of the convex set of $n \times n$ doubly stochastic matrices consist of $n \times n$ permutation matrices.

(See [5] for a proof.) Given two Hermitian matrices ρ and σ , we write $\sigma \prec \rho$ if $\operatorname{Spec}(\sigma) \prec \operatorname{Spec}(\rho)$, where $\operatorname{Spec}(X)$ is the vector of eigenvalues of an operator X. (We will adopt the convention that $\operatorname{Spec}(X)$ is always written with components in non-increasing order.) Uhlmann has proved a matrix analog of Birkhoff's Theorem [1, 36]:

Theorem 2.5 (Uhlmann) Let ρ and σ be $n \times n$ density matrices (that is, positive semidefinite with unit trace). Then $\sigma \prec \rho$ if and only if there exists a probability vector p and unitary matrices U_i such that

$$\sigma = \sum_{i} p_i U_i \rho U_i^{\dagger}. \tag{6}$$

Using Theorem 2.3, we can strengthen Uhlmann's Theorem.

Theorem 2.6 Suppose a matrix σ can be written as a convex combination of isospectral Hermitian matrices:

$$\sigma = \sum_{i=1}^{N} p_i U_i \rho U_i^{\dagger},\tag{7}$$

where each U_i is unitary, $p_i \ge 0$ and $\sum_i p_i = 1$. Set $p = (p_1, \dots, p_N)$, and suppose that q is a probability distribution such that $q \prec p$. Then there exist unitary matrices $\{V_i\}_{i=1}^N$ such that

$$\sigma = \sum_{i=1}^{N} q_i V_i \rho V_i^{\dagger}. \tag{8}$$

Proof Let $\mu = \operatorname{Spec}(\sigma)$ and $\lambda = \operatorname{Spec}(\rho)$, so that $p_i \lambda = \operatorname{Spec}(p_i U_i \rho U_i^{\dagger})$. By Theorem 2.3 there is a list of inequalities, each of the form

$$\sum_{k \in K} \mu_k \le \sum_{i=1}^N \sum_{J \in J_i} p_i \lambda_j,\tag{9}$$

that must be satisfied in order for Equation (7) to hold. By the symmetry of interchanging the order of the summands in Equation (7), it must be true for each $\pi \in S_N$ that

$$\sum_{k \in K} \mu_k \le \sum_{i=1}^N \sum_{j \in J_i} p_{\pi(i)} \lambda_j. \tag{10}$$

Now since $q \prec p$, it follows from Theorem 2.4 that there exist coefficients $c_{\pi} \geq 0$, $\sum_{\pi \in S_N} c_{\pi} = 1$, such that for all $i \in \{1, ..., N\}$,

$$q_i = \sum_{\pi \in S_N} c_{\pi} p_{\pi(i)}. \tag{11}$$

Now we take a convex sum of Inequalities (10) over $\pi \in S_N$:

$$\sum_{k \in K} \mu_k = \sum_{\pi \in S_N} c_\pi \sum_{k \in K} \mu_k \tag{12}$$

$$\leq \sum_{\pi \in S_N} c_{\pi} \sum_{i=1}^{N} \sum_{j \in J_i} p_{\pi(i)} \lambda_j \text{ by Inequalities (10)}$$
 (13)

$$= \sum_{i=1}^{N} \sum_{j \in J_i} \lambda_j \sum_{\pi \in S_N} c_{\pi} p_{\pi(i)}$$

$$\tag{14}$$

$$= \sum_{i=1}^{N} \sum_{j \in J_i} q_i \lambda_j. \tag{15}$$

In other words, if an inequality of the form of Inequality (9) holds for values p_i , then it also holds when every p_i is replaced by q_i . Applying Theorem 2.3, we conclude that there must be unitary matrices $\{V_i\}_{i=1}^N$ such that

$$\sigma = \sum_{i=1}^{N} q_i V_i \rho V_i^{\dagger}. \tag{16}$$

In particular, we have

Corollary 2.7 Suppose a matrix σ can be written as a convex combination of unitary conjugations of a fixed Hermitian matrix ρ with N terms:

$$\sigma = \sum_{i=1}^{N} p_i U_i \rho U_i^{\dagger}, \tag{17}$$

where each U_i is unitary, $p_i \ge 0$ and $\sum_i p_i = 1$. Then there exist unitary matrices $\{V_i\}_{i=1}^N$ such that

$$\sigma = \frac{1}{N} \sum_{i=1}^{N} V_i \rho V_i^{\dagger}. \tag{18}$$

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Proof Set $q = (\frac{1}{N}, \dots, \frac{1}{N})$ in Theorem 2.6.

In [34], M. Nielsen described how to transform a (pure) quantum state $|\varphi_{AB}\rangle$, jointly held by two parties, Alice and Bob, into another bipartite quantum state $|\psi_{AB}\rangle$, using only local operations and classical communication; he found that this is possible if and only if

$$\operatorname{Spec}(\varphi_A) \prec \operatorname{Spec}(\psi_A).$$
 (19)

(Note that L. Hardy independently arrived at the same conclusion with the benefit of the benefit of Uhlmann's Theorem [15].) It follows from Uhlmann's Theorem that Condition (19) holds if and only if φ_A can be written as a convex sum

$$\varphi_A = \sum_{i=1}^{N} p_i U_i \psi_A U_i^{\dagger} \tag{20}$$

where each U_i is unitary. Nielsen showed that if $|\varphi_{AB}\rangle$ can be tranformed into $|\psi_{AB}\rangle$ via LOCC, then Equation (20) holds, by presenting a protocol (using $\log_2 N$ bits of classical communication) that exhibits this representation. In the protocol, one party performs a measurement with N possible outcomes, where p_i is the probability of the ith outcome, to her portion of the joint system. The outcome i is communicated to the other party, who then performs a unitary U_i to his portion of the system. Any such protocol carries out the transformation $|\varphi_{AB}\rangle \rightarrow |\psi_{AB}\rangle$.

In a subsequent paper, Harrow and Lo [16] demonstrated that without altering the total number of bits transmitted, any LOCC protocol for transforming known pure quantum states can be transformed into one of the following form:

- 1. Alice performs a generalized measurement (POVM).
- 2. Alice sends the result of the measurement to Bob.
- 3. Bob performs a unitary operation conditioned on the message he receives from Alice.
- 4. Alice and Bob both discard ancillary systems.

With the exception of the discard step, the above protocol is of the same type analyzed by Nielsen. Equation (20) therefore applies and Corollary 2.7 has the following consequence.

Corollary 2.8 Any protocol for transforming known, pure, bipartite quantum states via LOCC may be transformed into an equivalent one in which all communication is from Alice to Bob, the total amount of communication is the same and all measurement outcomes, as well as messages, are equiprobable.

3 The Spectrum of a Partial Trace

We now move on to defining our main problem. Let $A = \mathbb{C}^{d_A}$, $B = \mathbb{C}^{d_B}$, and let ρ_{AB} be an operator on $A \otimes B$. We identify ρ_{AB} with its matrix in the standard basis, which has entries

$$\rho_{AB}^{ij,kl} = \langle i_A | \otimes \langle j_B | \rho_{AB} | k_A \rangle \otimes | l_B \rangle \tag{21}$$

in terms of orthonormal bases $\{|i_A\rangle\}$ and $\{|j_B\rangle\}$ of A and B respectively. Define the *partial* trace $\rho_A = \text{Tr}_B \, \rho_{AB}$ of ρ_{AB} to be the operator

$$\rho_A = \sum_{k} \langle k_B | \rho_{AB} | k_B \rangle \tag{22}$$

on A. The matrix entries of ρ_A are

$$\rho_A^{ij} = \sum_{k} \langle i_A | \otimes \langle k_B | \rho_{AB} | j_A \rangle \otimes | k_B \rangle. \tag{23}$$

Equivalently, given the matrix ρ_{AB} , we can define ρ_A to be the unique matrix such that

$$\operatorname{Tr}(\rho_{AB}X \otimes I_B) = \operatorname{Tr}(\rho_A X)$$
 (24)

for all X on A, where I_B is the identity on B.

The rest of the paper will focus on the following question: What is the relationship between the spectrum of ρ_{AB} and the spectrum of ρ_A ? We generally adopt the point of view that the spectrum of ρ_{AB} is given and we wish to deduce which possible spectra of ρ_A may occur. (Our final results will nonetheless allow us to reason in the other direction as well; given the spectrum of ρ_A , one can deduce the possible spectra of ρ_{AB} .) We let $\mathcal{H}_{AB}(\lambda) = \{\rho_{AB} : \operatorname{Spec}(\rho_{AB}) = \lambda\}$ be the set of Hermitian matrices on $A \otimes B$ with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{d_A d_B}$; then our problem is to fully characterize the set $\mathcal{S}_A(\lambda) = \{\operatorname{Spec}(\rho_A) : \rho_{AB} \in \mathcal{H}_{AB}(\lambda)\}$.

Some of the most useful inequalities in quantum information theory relate the eigenvalues of a density matrix with those of its partial traces. The von Neumann entropy $S(\rho) = -\operatorname{Tr} \rho \log_2 \rho$, for example, satisfies inequalities known as subadditivity:

$$S(\rho_{AB}) \le S(\rho_A) + S(\rho_B) \tag{25}$$

and, more generally, strong subadditivity [31]:

$$S(\rho_{ABC}) + S(\rho_B) \le S(\rho_{AB}) + S(\rho_{BC}). \tag{26}$$

Also, a good deal is known about the possible spectra of reductions when the individual subsystems consist of qubits. Bravyi [7] and, independently Higuchi, Sudbery and Szulc [20], for example, have found necessary and sufficient conditions for a set of qubits states to be the reductions of a larger pure state. In a similar spirit, Linden, Popescu and Wootters [32] have shown that almost all sets of states on subsets of n-1 particles do not correspond to reductions of a single n particle state.

Most directly connected to the present paper, Christandl and Mitchison very recently connected the triples of spectra of given bipartite density matrix and its two reductions to the Kronecker coefficients of the symmetric group [8]. We make a similar connection in Section 7.4, providing a correspondence of the type linking Horn's Problem to the Littlewood-Richardson coefficients. ¹

¹Coincidentally, days after the initial submission of this manuscript, Klyachko also published a very insightful paper on the same topic [24].

3.1 Physical Interpretation

Determining the possible spectra of a partial trace has a number of physical applications. The usual situation is to regard ρ_{AB} as the density matrix of a quantum system AB, a composite of subsystems A and B; ρ_A is then the density matrix of subsystem AB. In this context, we are asking which quantum-mechanical descriptions of a subsystem of a quantum system are compatible with the description of the whole system.

Understanding the relationship between a density operator and its partial trace also allows us to characterize which state transformations are achievable using quantum communication. To illustrate, suppose two parties, Alice and Bob, share a state between them that can be described by a state vector $|\varphi_{ABC}\rangle \in A \otimes B \otimes C$, where Alice holds quantum systems A and C and Bob holds the system B. First, we assume that there will be only one round of quantum communication, from Alice to Bob. Alice's initial description of her subsystem (her *reduced density operator*) is given by $\varphi_{AC} = \operatorname{Tr}_B |\varphi_{ACB}\rangle\langle\varphi_{ACB}|$. If she then sends Bob the system C through a *quantum channel*, her new density operator becomes $\varphi_A = \operatorname{Tr}_{CB} |\varphi_{ACB}\rangle\langle\varphi_{ACB}|$. Thus, understanding how quantum systems change as a result of quantum communication is equivalent to understanding how a density matrix is related to its partial trace. (It is important to note, however, that the model is slightly different than in the LOCC case. Instead of allowing arbitrary local operations, we allow only unitary local operations.)

If many rounds of communication are allowed in a quantum communication protocol, it may seem that the analysis should become more complicated (see Figure 1). Happily, this turns out not to be the case. In fact, the following result [37] shows that it is enough to consider one-round protocols:

Theorem 3.1 Suppose there exists a bipartite quantum communication protocol that transforms the state $|\varphi_{AB}\rangle$ to the state $|\psi_{AB}\rangle$, requiring a total of q qubits of communication. Then there is a oneround protocol that accomplishes the same transformation $|\varphi_{AB}\rangle \to |\psi_{AB}\rangle$, also requiring q qubits of information.

Proof The proof involves showing that at any round of the protocol, any communication from Bob to Alice can be replaced by communication from Alice to Bob; it then follows that all communication can be taken to be in one direction. The effect of Bob sending a qubit to Alice is to transform a state $\sum_i \sqrt{\lambda_i} |i_A\rangle |i_B\rangle$ to a state $\sum_i \sqrt{\lambda_{i'}} |i'_A\rangle |i'_B\rangle$, where the prior and posterior states are written in their Schmidt decompositions. (By an application of the singular value decomposition, any bipartite, quantum state $|\omega\rangle$ is equivalent by local unitary transformations to one of the form $|\omega'\rangle = \sum_i \sqrt{\alpha_i} |i_A\rangle |i_B\rangle$, where $\sum_i \alpha_i = 1$ and $\langle i_A|j_A\rangle = \langle i_B|j_B\rangle = \delta_{ij}$. This is known as the Schmidt decomposition.) But by symmetry, we then see that the swap operator exchanging Alice's and Bob's systems is equivalent to applying some local unitaries $U_A\otimes U_B$ on their joint system. Thus, instead of having Bob send a qubit to Alice, they can apply $U_A\otimes U_B$ and then have Alice send a qubit to Bob (and finally apply some local unitaries $U'_A\otimes U'_B$ to swap Alice and Bob back again) to accomplish the same transformation.

While the problem of comparing the spectrum of a matrix to that of its partial trace has a natural application to density matrices, it may be applied to other settings as well. For example, given the spectrum of an observable for a certain quantum system, one may wish to ask what the spectrum of that observable may be for a subsystem of the given system. In this context ρ_{AB} is the matrix of the observable, rather than a density matrix.

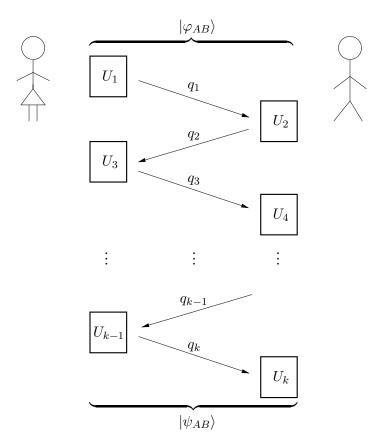


Figure 1 A many-round quantum communication protocol. Two parties, Alice and Bob, initially share a joint system $|\varphi_{AB}\rangle$. Alice applies a local unitary operator U_1 and then sends q_1 quantum bits to Bob, who performs a local unitary U_2 and then sends q_2 quantum bits to Alice, etc.; in the end, they share system $\psi_{AB}\rangle$. By Theorem 3.1, this is equivalent to a protocol in which there is only one round of quantum communication.

4 Variational Principle

We use a variational principle argument to show that inequalities between the eigenvalues of ρ_{AB} and of ρ_A arise whenever certain subsets of the Grassmannian intersect. We also show explicitly that when $d_A=2$, these inequalities are sufficient.

4.1 Some Basic Inequalities

In this section we use a simple argument to derive some inequalities that the spectra of ρ_{AB} and ρ_A must satisfy. Although these inequalities will subsumed by our later results, the proof illustrates the strategy behind the general method. We will make use of the following well-known fact from linear algebra [5]:

Lemma 4.1 (Ky Fan's Maximum Principle) *Let* A *be an* $n \times n$ *Hermitian matrix with spectrum* λ , *where we assume as usual that the components of* λ *are in non-increasing order. Then for all*

$$k \in \{1, \dots, n\},$$

$$\sum_{j=1}^{k} \lambda_j = \max \sum_{j=1}^{k} \langle x_j | A | x_j \rangle$$
(27)

where the maximum is taken over all orthonormal k-tuples of vectors $\{|x_j\rangle\}_{j=1}^k$ in \mathbb{C}^n .

If V is a subspace of the vector space A, then we write $V \leq A$. Thus, $\operatorname{Gr}_k(A) := \{V \leq A : \dim(V) = k\}$ is the Grassmannian of k-dimensional subspaces of the vector space A. We also write $\operatorname{Gr}(k,n)$ for $\operatorname{Gr}_k(\mathbb{C}^n)$. For $V \leq \mathbb{C}^n$ with the standard inner product, let P_V denote the orthogonal projection operator onto the subspace V. Given a vector $v \in \mathbb{C}^d$ and a positive integer n, we define $\Sigma_n(v)$ to be the vector whose components are obtained by summing successive blocks of n components of v:

$$\Sigma_n(v) = (v_1 + \dots + v_n, v_{n+1} + \dots + v_{2n}, \dots, v_{\lfloor d/n \rfloor (n-1)+1} + \dots + v_d).$$
(28)

Recall that we denoted the dimensions of system A and B by d_A and d_B , respectively; and that all vectors of matrix spectra are assumed to be with components in non-increasing order. We will use these conventions throughout.

Lemma 4.2 Let λ be the spectrum of ρ_{AB} , and $\tilde{\lambda}$ be the spectrum of its partial trace ρ_A . Then for every $k \in \{1, \ldots, d_A\}$, the inequality

$$\sum_{i=1}^{k} \tilde{\lambda}_i \le \sum_{i=1}^{d_B k} \lambda_i \tag{29}$$

must hold. We may write the d_A inequalities succinctly as the majorization relation

$$\tilde{\lambda} \prec \Sigma_{d_B}(\lambda).$$
 (30)

Proof

$$\sum_{i=1}^{k} \tilde{\lambda}_{i} = \max_{\{V \in \operatorname{Gr}_{k}(A)\}} \operatorname{Tr}(\rho_{A} P_{V})$$

$$= \max_{\{V \in \operatorname{Gr}_{k}(A)\}} \operatorname{Tr}(\rho_{AB} P_{V \otimes B})$$

$$\leq \max_{\{V \in \operatorname{Gr}_{kd_{B}}(A \otimes B)\}} \operatorname{Tr}(\rho_{AB} P_{V})$$

$$= \sum_{i=1}^{kd_{B}} \lambda_{i}, \tag{31}$$

where the first and last equalities follow from Ky Fan's Maximum Principle, the second equality comes from the definition of partial trace, and the inequality follows because the maximum is being taken over a larger set of projection operators than in the previous expression.

Note the basic idea behind the proof. We expressed the sum of eigenvalues for each matrix in terms of a variational principle on subspaces, and then we looked for an intersection between subspaces in order to relate the variational expressions. This idea will be developed further in the next section.

4.2 General Method

Let A be an $n \times n$ Hermitian matrix with spectrum λ , and let V be a subspace of \mathbb{C}^n . Define the *Rayleigh trace* of A on V to be

$$R_A(V) = \text{Tr}(P_V A). \tag{32}$$

Observe that if B is another Hermitian matrix, then $R_{A+B}(V) = R_A(V) + R_B(V)$. We can restate Ky Fan's Maximum Principle in this notation:

$$\max_{V, \text{ dim } V=r} R_A(V) = \sum_{i=1}^r \lambda_i.$$
(33)

Likewise,

$$\min_{V, \dim V = r} R_A(V) = \sum_{i=n-r+1}^n \lambda_i. \tag{34}$$

Let A_r denote the r-dimensional vector space spanned by eigenvectors corresponding to the r largest eigenvalues of A (if A is degenerate with $\lambda_r = \lambda_{r+1}$, then choose any such A_r .) Now given a binary sequence π of length n and weight r (sometimes written $\pi \in \binom{n}{r}$), the *Schubert cell* in the r-Grassmannian corresponding to π is defined as

$$S_{\pi}(A) = \{ V \le \mathbb{C}^n | \dim(V \cap A_i) / (V \cap A_{i-1}) = \pi(i), 1 \le i \le n \}, \tag{35}$$

where $\pi(i)$ is the *i*th term in the sequence π . Then $\pi(i) = 1$ for r values of i; label these values $i_1 < i_2 < \cdots i_r$. The following variational principle, due to Hersch and Zwahlen [19], provides access to sums of arbitrary combinations of the eigenvalues of A:

Theorem 4.3

$$\min_{V \in S_{\pi}(A)} R_A(V) = \sum_i \pi(i) \lambda_i. \tag{36}$$

Equality occurs when V is the span of eigenvectors corresponding to the eigenvalues $\lambda_{i_1}, \ldots, \lambda_{i_r}$.

Proof Let $V \in S_{\pi}(A)$, and choose orthogonal unit vectors $|u_1\rangle, |u_2\rangle, \dots, |u_r\rangle$ such that $|u_k\rangle \in V \cap A_{i_k}$. Now A_{i_k} is spanned by eigenvectors of A with eigenvalue greater than or equal to λ_{i_k} , so $\langle u_k | A | u_k \rangle \geq \lambda_{i_k}$. It follows that

$$R_A(V) = \sum_{k=1}^r \langle u_k | A | u_k \rangle \ge \sum_{k=1}^r \lambda_{i_k} = \sum_i \pi(i) \lambda_i.$$
 (37)

Now suppose V is the span of eigenvectors corresponding to eigenvalues $\lambda_{i_1}, \lambda_{i_r}$. In this case u_k is an eigenvector of A with eigenvalue λ_{i_k} , so that $R_A(V) = \sum_i \pi(i)\lambda_i$.

Now let $B=\mathbb{C}^{d_B}$. For any $k\leq d_A$, define the map $\phi:\operatorname{Gr}_k(A)\to\operatorname{Gr}_{d_Bk}(A\otimes B)$ by $\phi(V)=V\otimes B$. Let $|y_1\rangle,\ldots,|y_{d_B}\rangle$ be an orthonormal basis of B, and let I_B denote the identity operator on B. For any operator X_A on A, and any $|v\rangle\in\mathbb{C}^{d_A}$, we have that

$$\sum_{i=1}^{d_B} \langle v | \otimes \langle y_i | \left(\frac{1}{d_B} X_A \otimes I_B \right) | v \rangle \otimes | y_i \rangle = \frac{1}{d_B} \sum_{i=1}^{d_B} \langle v | X_A | v \rangle \langle y_i | I_B | y_i \rangle$$

$$= \frac{1}{d_B} \sum_{i=1}^{d_B} \langle v | X_A | v \rangle$$

$$= \langle v | X_A | v \rangle.$$

It follows that $R_{X_A}(V) = R_{\frac{1}{d_B}X_A \otimes I_B}(\phi(V)).$

The following theorem was motivated by an analogous argument, due to Johnson [22] and the pair of Helmke and Rosenthal [18], used in the solution of Horn's problem.

Theorem 4.4 Let X_A be an operator on A and Y_{AB} be an operator on $A \otimes B$ such that $X_A = -\operatorname{Tr}_B(Y_{AB})$. Let $\tilde{\lambda}$ be the spectrum of X_A and λ be the spectrum of Y_{AB} . If $\phi(S_{\pi}(X_A)) \cap S_{\sigma}(Y_{AB}) \neq \emptyset$, then

$$\sum_{i=1}^{d_A} \pi(i)\tilde{\lambda}_i + \sum_{i=1}^{d_A d_B} \sigma(i)\lambda_i \le 0.$$
(38)

Inequality (38) also holds if $\phi(\overline{S_{\pi}(X_A)}) \cap \overline{S_{\sigma}(Y_{AB})} \neq \emptyset$.

Proof Let $W \otimes B \in \phi(S_{\pi}(X_A)) \cap S_{\sigma}(Y_{AB})$. Then we have

$$\sum_{i=1}^{d_A} \pi(i)\tilde{\lambda}_i + \sum_{i=1}^{d_A d_B} \sigma(i)\lambda_i$$

$$= \min_{V \in S\pi(X_A)} R_{X_A}(V) + \min_{V' \in S_{\sigma}(Y_{AB})} R_{Y_{AB}}(Y_{AB})$$

$$= \min_{V \in \phi(S\pi(X_A))} R_{\frac{1}{d_B}X_A \otimes I_B}(V) + \min_{V' \in S_{\sigma}(Y_{AB})} R_{Y_{AB}}(Y_{AB})$$

$$\leq R_{\frac{1}{d_B}X_A \otimes I_B}(W \otimes B) + R_{Y_{AB}}(W \otimes B)$$

$$= R_{\frac{1}{d_B}X_A \otimes I_B + Y_{AB}}(W \otimes B)$$

$$= Tr(P_W(X_A + \text{Tr}_B(Y_{AB})))$$

$$= 0. \tag{39}$$

This proves the inequality in the case that $\phi(S_{\pi}(X_A)) \cap S_{\sigma}(Y_{AB}) \neq \emptyset$. If $\phi(\overline{S_{\pi}(X_A)}) \cap \overline{S_{\sigma}(Y_{AB})} \neq \emptyset$, then Theorem 4.3, along with the fact that the Rayleigh trace is continuous, implies that $\min_{V \in \overline{S_{\pi}(A)}} R_A(V) = \sum_i \pi(i)\lambda_i$, and the argument for the case $\phi(S_{\pi}(X_A)) \cap S_{\sigma}(Y_{AB}) \neq \emptyset$ applies equally to this case.

Starting from intersections of Schubert cells, Theorem 4.4 yields inequalities that must be satisfied by the spectra of a matrix and its partial trace. As we will discuss in the next section, the closures of the Schubert cells are generators of the homology of the Grassmannian; thus, we can regard the inequalities as coming from nonzero products in cohomology. Determining which of these products are nonzero and translating these nonzero products into the appropriate inequalities will be the focus of the remainder of the paper.

4.3 Solution for $d_A = 2$

When $d_A=2$, the relationship between the spectrum of ρ_{AB} and that of ${\rm Tr}(\rho_{AB})=\rho_A$ is particularly simple: the only inequalities restricting the spectra are those given by Lemma 4.2. Moreover, in this case it is possible to give a very simple and explicit construction of matrices demonstrating that the inequalities are sufficient. (If we interpret our problem in terms of quantum communication protocols, the $d_A=2$ case corresponds to the situation where Alice sends to Bob her entire quantum system except for one qubit.)

Theorem 4.5 If $d_A = 2$, the inequalities given by Lemma 4.2 are sufficient. That is, given a vector $\lambda \in \mathbb{R}^{2d_B}$ and a vector $\tilde{\lambda} \in \mathbb{R}^2$, each with components in non-increasing order, satisfying $\tilde{\lambda} \prec 0$

 $(\sum_{i=1}^{d_B} \lambda_i, \sum_{i=d_B+1}^{2d_B} \lambda_i)$, there exist matrices ρ_{AB} and ρ_A such that the spectrum of ρ_{AB} is λ , the spectrum of ρ_A is $\tilde{\lambda}$, and $\rho_A = \text{Tr}_B(\rho_{AB})$.

Proof Let $\lambda = (\lambda_{0,0}, \lambda_{0,1}, \dots, \lambda_{0,d_B-1}, \lambda_{1,0}, \lambda_{1,1}, \dots, \lambda_{1,d_B-1})$, let $\{|0_A\rangle, |1_A\rangle\}$ and $\{|0_B\rangle, \dots, |(j-1)_B\rangle$ be orthonormal bases for A and B, respectively, and set

$$\sigma_{AB} = \sum_{i=0}^{1} \sum_{j=0}^{d_B - 1} \lambda_{i,j} |i_A\rangle |j_B\rangle \langle i_A|\langle j_B|$$

$$\tag{40}$$

For $t \in [0, 2\pi]$, let

$$U(t) = \sum_{i=0}^{1} \sum_{j=0}^{d_B - 1} \cos t |i_A\rangle |j_B\rangle \langle i_A|\langle j_B|$$

$$+ \sum_{j=0}^{d_B - 1} \sin t |0_A\rangle |(j-1)_B\rangle \langle 1_A|\langle j_B|$$

$$- \sum_{j=0}^{d_B - 1} \sin t |1_A\rangle |j_B\rangle \langle 0_A|\langle (j-1)_B|,$$
(41)

where the subtraction in the labels of the bra and ket vectors is done modulo d_B . Now U(t) is unitary (in fact, it is real orthogonal) for all t, so the spectrum of $U(t)\sigma_{AB}U(t)^{\dagger}$ is λ . A direct calculation verifies that

$$U(t)\sigma_{AB}U(t)^{\dagger} = \sum_{i=0}^{1} \sum_{j=0}^{d_{B}-1} \lambda_{i,j} \cos^{2}t |i_{A}\rangle |j_{B}\rangle \langle i_{A}|\langle j_{B}|$$

$$+ \sum_{j=0}^{1} (\lambda_{1,j} - \lambda_{0,j-1}) \sin t \cos t |0_{A}\rangle |(j-1)_{B}\rangle \langle 1_{A}|\langle j_{B}|$$

$$+ \sum_{j=0}^{d_{B}-1} (\lambda_{1,j} - \lambda_{0,j-1}) \sin t \cos t |1_{A}\rangle |j_{B}\rangle \langle 0_{A}|\langle (j-1)_{B}|$$

$$+ \sum_{j=0}^{d_{B}-1} \sin^{2}t (\lambda_{0,j-1}|0_{A}\rangle |j_{B}\rangle \langle 0_{A}|\langle j_{B}| + \lambda_{1,j}|1_{A}\rangle |j_{B}\rangle \langle 1_{A}|\langle j_{B}|),$$

$$(42)$$

so that

$$\operatorname{Tr}_{B}(U(t)\sigma_{AB}U(t)^{\dagger}) = \left(\sum_{j=0}^{d_{B}-1} \lambda_{0,j} \cos^{2} t + \sum_{j=0}^{d_{B}-1} \lambda_{1,j} \sin^{2} t\right) |0_{A}\rangle\langle 0_{A}| + \left(\sum_{j=0}^{d_{B}-1} \lambda_{1,j} \cos^{t} + \sum_{j=0}^{d_{B}-1} \lambda_{0,j} \sin^{2} t\right) |1_{A}\rangle\langle 1_{A}|.$$

$$(43)$$

Let $\alpha_1 = \sum_{j=0}^{d_B-1} \lambda_{0,j}$, $\alpha_2 = \sum_{j=0}^{d_B-1} \lambda_{1,j}$. If we let $\rho_{AB}(t) = U(t)\sigma_{AB}U(t)^{\dagger}$, then the spectrum of the partial trace of $\rho_{AB}(t)$ is $(\alpha_1\cos^2 t + \alpha_2\sin^2 t, \alpha_1\sin^2 t + \alpha_2\cos^2 t)$. By choosing the appropriate value of $t \in [0, 2\pi]$, any convex combination of α_1 and α_2 can be achieved for the eigenvalues of $\mathrm{Tr}_B(\rho_{AB}(t))$.

5 Schubert Calculus

This section is intended as a quick introduction to arithmetic in the cohomology ring of the Grassmannian, otherwise known as the Schubert calculus. While we include some proofs in

order to try to help the reader understand the nature of the arguments, our presentation is necessarily incomplete. Full treatments, upon which the following discussion is based, can be found in [33], [12], and [14].

5.1 Symmetric Polynomials

We start with some background on the ring Λ_n of symmetric polynomials in n variables with integer coefficients. A certain class of such polynomials, the Schur polynomials, will be of particular interest, due to its relationship with the cohomology of the Grassmannian. The Schur polynomials (as well as the Grassmannian cohomology classes) are indexed by partitions of integers, so we begin with some terminology relating to partitions.

A partition of an integer n is a finite sequence $\alpha=(\alpha_1,\dots\alpha_l)$ of nonnegative integers, with $n=\sum_i\alpha_i$, arranged in non-increasing order: $\alpha_1\geq\alpha_2\geq\dots\geq\alpha_l\geq0$. These integers α_1,\dots,α_l are called the parts, and the length $\ell(\alpha)$ is the number of nonzero parts. The integer $n=\sum_i\alpha_i$ is the weight of the partition, denoted $|\alpha|$. To any partition α we may associate a Young diagram, whose ith row has length α_i . The conjugate partition α^* is obtained by interchanging rows and columns in the Young diagram of α . For instance, if $\alpha=(5,3,2,2)$, then the Young diagram of α is



so the Young diagram of α^* is



and $\alpha^* = (4, 4, 2, 1, 1)$.

Now let Λ_n be the ring of symmetric polynomials with integer coefficients in n variables. There are a number of computationally useful bases for Λ_n . Perhaps the simplest basis is given by the *monomial symmetric functions*. These are functions obtained by starting with a monomial $x^{\alpha} = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and symmetrizing it, to obtain a polynomial

$$m_{\alpha} = \sum_{\beta \in S_n(\alpha)} x^{\beta}. \tag{46}$$

In this notation, S_n permutes the coefficients of α . Note that the sum is not over all permutations in S_n , but over the image of these permutations; thus, any given monomial appears only once in the sum.

Theorem 5.1 The polynomials m_{α} , where α ranges over partitions with at most n parts, form a basis over \mathbb{Z} for the ring Λ_n .

Proof Given a polynomial $p(x_1, \ldots, x_n) = \sum c_{\alpha} x^{\alpha} \in \Lambda_n$, let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be the maximal n-tuple (with respect to the lexicographic ordering) such that $c_{\alpha} \neq 0$. Because $p(x_1, \ldots, x_n)$ is symmetric, α must be a partition. Now $p(x_1, \ldots, x_n) - c_{\alpha} m_{\alpha}$ is also a symmetric polynomial,

but one whose leading monomial is smaller than x^{α} with respect to the lexicographic ordering. Because $\alpha_i \geq 0$, the lexicographic ordering is a well-ordering, so it follows by induction that $p(x_1, \ldots, x_n)$ can be written as an integer combination of terms m_{α} .

Now suppose $\sum c_{\alpha}m_{\alpha}=0$. Again, let α be the maximal n-tuple with respect to the lexicographic ordering such that $c_{\alpha}\neq 0$. Then the coefficient of x^{α} in the polynomial $\sum c_{\alpha}m_{\alpha}$ is c_{α} , a contradiction.

We will make reference to the following two classes of symmetric polynomials. The *elementary symmetric polynomials* are a subset of the monomial symmetric functions, corresponding to partitions such that all parts are equal to one:

$$e_k = \sum_{1 \le i_1 < \dots < i_k \le n} x_{i_1} \cdots x_{i_k}, \tag{47}$$

for $1 \le k \le n$. The complete symmetric polynomials are

$$h_k = \sum_{1 \le i_i \cdots \le i_k \le n} x_{i_1} \cdots x_{i_k},\tag{48}$$

for $1 \le k \le n$. (If k = 0, then set $e_0 = h_0 = 1$.) We label products of elementary symmetric polynomials, as well as products of complete symmetric polynomials, by partitions α : $e_{\alpha} = e_{\alpha_1} \cdots e_{\alpha_l}$, and $h_{\alpha} = h_{\alpha_1} \cdots h_{\alpha_l}$.

Both the elementary symmetric polynomials and complete symmetric polynomials are important objects in the study of the ring Λ_n . The fundamental theorem of symmetric polynomials states that every symmetric polynomial can be written as a polynomial in the elementary symmetric polynomials [11]; in other words, the polynomials e_{α} , where α ranges through partitions with parts less than or equal to n, form a basis over \mathbb{Z} of the ring Λ_n . We will make use of the following relationship between the polynomials e_k and h_k .

Proposition 5.2 Let $\omega : \Lambda_n \to L_n$ be the ring homomorphism defined by $\omega(e_k) = h_k$. Then ω is an involution.

It follows from the fundamental theorem of elementary symmetric polynomials and Proposition 5.2 that the polynomials h_{α} form a \mathbb{Z} -basis of Λ_n . We now describe another basis for the ring Λ_n : the *Schur polynomials*, which will be a greater focus of our study. In order to do so, we make some observations about the ring of antisymmetric polynomials in n variables. These polynomials have a basis obtained from antisymmetrizing monomials: if γ is an n-tuple of natural numbers, then let

$$a_{\gamma} = \sum_{w \in S_n} \varepsilon(w) x^{w(\gamma)}, \tag{49}$$

where $\varepsilon(w)$ is the sign of the permutation w. Note that if γ has two equal components, then $a_{\gamma}=0$. Thus, we restrict our attention to the case where γ is a strictly decreasing partition. Then γ has the form $\gamma=\alpha+\delta$, where α is a partition and $\delta=(n-1,n-2,\ldots,1,0)$. An argument similar to the proof of Theorem 5.1 shows that the polynomials $a_{\alpha+\delta}$, where α ranges over partitions with at most n parts, form a basis for the ring of antisymmetric polynomials with integer coefficients.

Next, note that every antisymmetric polynomial must be divisible by (x_i-x_j) for all $i\neq j$, and so must be divisible by the Vandermonde determinant $\det(x_i^{n-j})_{1\leq i,j\leq n}=\prod_{1\leq i< j\leq n}(x_i-x_j)$

 x_j). It is not hard to see that multiplying a symmetric polynomial by the Vandermonde determinant produces an antisymmetric polynomial, and that dividing an antisymmetric polynomial by the Vandermonde determinant yields a symmetric polynomial. Thus, multiplication by the Vandermonde determinant gives an isomorphism between symmetric and antisymmetric polynomials. The *Schur polynomials* are obtained by dividing the polynomials a_{γ} by the Vandermonde determinant (which is the same as a_{δ}):

$$s_{\alpha} = \frac{a_{\alpha+\delta}}{a_{\delta}} = \frac{\det(x_i^{\alpha_j+n-j})_{1 \le i,j \le n}}{\det(x_i^{n-j})_{1 \le i,j \le n}}.$$
(50)

By the isomorphism between symmetric and antisymmetric polynomials, we have proven the following theorem.

Theorem 5.3 The Schur polynomials s_{α} , as α ranges over all partitions with at most n parts, form a basis over \mathbb{Z} of the ring Λ_n .

Given a partition α and integer k, let $\alpha \otimes k$ denote the set of partitions obtained by adding k boxes to (the Young diagram of) α , at most one box per column. Let $\alpha \otimes 1^k$ denote the set of partitions obtained by adding k boxes to α , at most one box per row.

Theorem 5.4 (Pieri formulas) With the above notation,

$$s_{\alpha}e_k = \sum_{\beta \in \alpha \otimes 1^k} s_{\beta},\tag{51}$$

and

$$s_{\alpha}h_k = \sum_{\beta \in \alpha \otimes k} s_{\beta}. \tag{52}$$

The Pieri formulas, for example, can be used to write the Schur polynomials in terms of the complete symmetric polynomials:

Theorem 5.5 (Jacobi-Trudi formula) *Let* α *be a partition with at most* n *parts. Then*

$$s_{\alpha} = \det(h_{\alpha_i - i + j})_{1 \le i, j \le n}. \tag{53}$$

Proof Let l be the length of α . Because $h_0 = 1$, $\det(h_{\alpha_i - i + j})_{1 \le i, j \le n} = \det(h_{\alpha_i - i + j})_{1 \le i, j \le l}$. Expand $\det(h_{\alpha_i - i + j})_{1 < i, j < l}$ along the last column, using induction on l:

$$\det(h_{\alpha_i - i + j})_{1 \le i, j \le l} = \sum_{i=1}^{l} (-1)^{l-i} s_{\lambda_1, \dots, \lambda_{i-1}, \lambda_{i+1} - 1, \dots, \lambda_l - 1} \times h_{\lambda_i + l - i}.$$
(54)

Now it follows from Theorem 5.4 that the *i*th term of the above sum may be written as

$$\sum_{\beta \in J_i} s_{\beta} + \sum_{\beta \in J_{i+1}} s_{\beta},\tag{55}$$

where J_i is the set of partitions β having the same weight as α , satisfying the conditions $\alpha_j \leq \beta_j \leq \alpha_{j-1}$ for j < i, and $\alpha_{j+1} - 1 \leq \beta_j \leq \alpha_j - 1$ for $j \geq i$. Therefore, the right hand sum of Equation 54 telescopes to give us the desired formula.

5.2 Grassmannians as Varieties

Let E be an n-dimensional complex vector space. The Grassmannian Gr(k,n) can be realized as the homogeneous space $U(n)/(U(k)\times U(n-k))$ since the larger group acts transitively on subspaces of \mathbb{C}^n while the smaller one is the stabilizer of a fixed subspace. Gr(k,n) is, in fact, a complex manifold of dimension k(n-k).

If V is a k-dimensional subspace of E, then $\wedge^k V$ is a line in $\wedge^k E$, giving us a map

$$\phi: \operatorname{Gr}_k(E) \to \mathbb{P}(\wedge^k E), \tag{56}$$

where we have introduced the notation $\mathbb{P}(V)$ for the projectivization of the vector space V. Let $A=(a_{ij})$ be a $k\times n$ matrix representing V, so that V is the span of the rows of A. Then a set of homogeneous coordinates in $\phi(V)$ is given by the determinants of the $k\times k$ minors of this matrix: if I is a subset of $\{1,\ldots,n\}$ of cardinality k, then define the coordinate

$$x_I = \det A_I, \tag{57}$$

where A_I denotes the Ith $k \times k$ minor of A. These coordinates are known as $Pl\ddot{u}cker$ coordinates, and the map ϕ is called the $Pl\ddot{u}cker$ embedding. It can be shown [33] that the $Pl\ddot{u}cker$ embedding is indeed an embedding of the Grassmannian $Gr_k(E)$ into the projective space $\mathbb{P}(\wedge^k E)$, and that the homogeneous coordinates are the solutions of a set of (quadratic) polynomial equations, giving $Gr_k(E)$ the structure of a projective algebraic variety.

5.3 Schubert Varieties

Define a (complete) flag F_{\bullet} on E to be a nested sequence

$$F_{\bullet}: 0 = F_0 \subset F_1 \subset F_2 \subset \ldots \subset F_n = E \tag{58}$$

with $\dim(F_i) = i$. For any such flag, we obtain a cell decomposition of $\operatorname{Gr}_k(E)$, as follows. Let α be a partition contained in a $k \times (n-k)$ rectangle (this means that α has length at most k and that all parts are less than or equal to n-k). To each such α we associate the *Schubert cell*

$$\Omega_{\alpha} = \{ V \in \operatorname{Gr}_k(E) | \dim(V \cap F_j) = i \text{ if } n - k + i - \alpha_i \le j \le n - k + i - \alpha_{i+1} \}.$$
 (59)

and the Schubert variety

$$X_{\alpha} = \{ V \in \operatorname{Gr}_{k}(E) | \dim(V \cap F_{n-k+i-\alpha_{i}}) \ge i \}.$$

$$(60)$$

This definition of Schubert cell differs from the one given in the previous section, but the two definitions refer to the same object, as we now show. Given any binary string π of length n and weight k, associate to it a partition α_{π} as follows. Let a_i be the number of zeroes that appear in π before the ith one. Then let $\alpha_{\pi}=(a_k,a_{k-1},\ldots,a_1)$. For instance if $\pi=010011$, then $\alpha_{\pi}=(3,3,1)$. It is not hard to see that this gives a one-to-one correspondence between binary strings of length n and weight k, and partitions contained in a $k\times(n-k)$ rectangle, and that $S_{\pi}=\Omega_{\alpha_{\pi}}$.

When we wish to emphasize the flag, we write $\Omega_{\alpha}(F_{\bullet})$ and $X_{\alpha}(F_{\bullet})$ for Ω_{α} and X_{α} , respectively. Schubert varieties corresponding to partitions with only one nonzero part are called *special Schubert varieties*

$$X_l = \{ V \in Gr_k(E) | V \cap F_{n-k+1-l} \neq 0 \}.$$
(61)

We now show that Schubert varieties are indeed algebraic varieties. Note that $\dim(V \cap F_i) \geq j$ if and only if the rank of the map

$$V \hookrightarrow \mathbb{C}^n \twoheadrightarrow \mathbb{C}^n / F_i \tag{62}$$

is less than or equal to k-j. This means that, in local coordinates, all minors of order k-j+1 of the matrix of this map must have vanishing determinant, a requirement governed by polynomial equations. The Schubert varieties are therefore algebraic subvarieties of $Gr_k(E)$.

In what follows, let f_1, \ldots, f_n be a basis respecting the flag F_{\bullet} of E; in other words, these vectors are such that $F_i = \langle f_1, \ldots f_i \rangle$ for all i. Let α be a partition contained in a $k \times (n-k)$ rectangle. In terms of the basis $\langle f_1, \ldots, f_n \rangle$, any $V \in \Omega_{\alpha}$ can be expressed in terms of a unique basis, consisting of the rows of a $k \times (n-k)$ matrix with the following properties: the ith row contains a 1 in the $(n-k+i-\alpha_i)$ th position, and zeros in all subsequent positions; and all other entries in the $(n-k+i-\alpha_i)$ th column are zero. For instance, if n=7, k=3, and $\alpha=(3,2,1)$, such matrices are of the form

$$\begin{pmatrix} * & 1 & 0 & 0 & 0 & 0 & 0 \\ * & 0 & * & 1 & 0 & 0 & 0 \\ * & 0 & * & 0 & * & 1 & 0 \end{pmatrix}, \tag{63}$$

where the stars denote arbitrary entries. Clearly any such matrix corresponds to a $V \in \Omega_{\alpha}$, so we have a homeomorphism of Ω_{α} with $\mathbb{C}^{k(n-k)-|\alpha|}$. In general, V can be written (not uniquely) as the span of the rows of any $k \times (n-k)$ matrix with a nonzero entry in the $(n-k+i-\alpha_i)$ th position of the ith row, and zeros afterwards. Using our example n=7, k=3, and $\alpha=(3,2,1)$, such matrices can be written as

$$\begin{pmatrix} * & * & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & 0 & 0 & 0 \\ * & * & * & * & * & * & 0 \end{pmatrix}, \tag{64}$$

where the last star in each row represents any nonzero term, and all other stars represent arbitrary terms. From this representation, we see that if $\alpha \subset \beta$ (this means that the Young diagram of α is contained in the diagram of β), then $\Omega_{\beta} \subset \overline{\Omega_{\alpha}}$.

The following theorem tells how to determine the incidence of Schubert varieties.

Theorem 5.6 *For all partitions* $\alpha \subset k \times (n-k)$ *,*

(a)
$$X_{\alpha} = \overline{\Omega_{\alpha}} = \coprod_{\beta \supset \alpha} \Omega_{\beta}$$
, and

(b)
$$X_{\beta} \subset X_{\alpha}$$
 if and only if $\alpha \subset \beta$.

The Schubert cells Ω_{α} , as a result, form a cellular decomposition of the Grassmannian. Therefore, the fundamental classes of their closures are a basis of the integral cohomology of $\operatorname{Gr}_k(E)$. (Because all cells are of even real dimension, the integral cohomology is torsion-free.) For any Schubert variety X_{α} , let $\sigma_{\alpha} = [X_{\alpha}]$ denote its class in cohomology, called a *Schubert class*. The results of this section then imply the following theorem.

Theorem 5.7 The integral cohomology of the Grassmannian $Gr_k(E)$ has a basis given by the Schubert classes σ_{α} , where α ranges over all partitions contained in a $k \times (n-k)$ rectangle:

$$H^*(Gr_k(E)) = \bigoplus_{\alpha \subset k \times (n-k)} \mathbb{Z}\sigma_{\alpha}.$$
 (65)

The Schubert class σ_{α} is an element of $H^{2|\alpha|}(Gr_k(E))$.

5.4 Intersections of Schubert Varieties

Let us now determine when two Schubert varieties must intersect. Given a flag F_{\bullet} , let \tilde{F}_{\bullet} be the opposite flag to F_{\bullet} . That is, if $\{f_1,\ldots,f_n\}$ is a basis for E such that $F_k=\langle f_1,\ldots,f_k\rangle$, then $\tilde{F}_k=\langle f_{n-k+1},\ldots,f_n\rangle$. For any partition α with at most k rows and n-k columns, let $\Omega_a=\Omega_\alpha(F_{\bullet})$ and let $\tilde{\Omega}_\alpha=\Omega_\alpha(\tilde{F}_{\bullet})$. Because GL(E) acts transitively on the flags, Ω_α and $\tilde{\Omega}_\alpha$ have the same fundamental class, denoted σ_α .

We have seen that any element of Ω_{α} can be written as the span of the rows of a unique $k \times (n-k)$ matrix of the form

where the ith row has a 1 in the $(n-k+i-\alpha_i)$ th position. Similarly, each element of $\tilde{\Omega}_{\beta}$ can be written in terms of a basis whose elements are the rows of a unique $k \times (n-k)$ matrix of the form

where the *i* row has a 1 in position $\beta_{n-k-i+1} + i$.

If $\Omega_{\alpha} \cap \tilde{\Omega}_{\beta} \neq \emptyset$, then there must be a k-plane W such that each of the two above matrices determines a basis for W. Now, the first row of the first matrix cannot be a linear combination of rows of the second unless $\beta_{n-k}+1 \leq n+1-\alpha_1 \Longrightarrow \alpha_1+\beta_{n-k} \leq n$. In general, in order for the i row of the first matrix to be a linear combination of rows of the second matrix, but not a linear combination of the first i-1 rows of the second matrix, we must have that $\alpha_i+\beta_{n-k-i+1}\leq n$.

For any partition α contained in an $k \times (n-k)$ rectangle, define $\hat{\alpha}$ to be the complementary partition of α in the rectangle: that is, $\hat{\alpha}_i = n - \alpha_{n-k-i+1}$. (If the Young diagram of $\hat{\alpha}$ is turned upside down, it fits perfectly with the diagram of α to form a $k \times (n-k)$ rectangle.) The argument of the previous paragraph shows that $\Omega_{\alpha} \cap \tilde{\Omega}_{\beta} = \emptyset$ unless $\beta \subset \hat{\alpha}$. We now have

Theorem 5.8 Suppose α and β are two partitions with at most k rows and n-k columns, and that $|\alpha| + |\beta| = k(n-k)$. Then the cup product in cohomology of the fundamental classes corresponding to α and β is zero unless $\beta = \hat{\alpha}$, in which case it is one; that is,

$$\sigma_{\alpha} \cup \sigma_{\beta} = \delta_{\beta,\hat{\alpha}}. \tag{68}$$

The classes σ_{α} *and* $\sigma_{\hat{\alpha}}$ *are therefore said to be* dual.

Proof We have seen that $\Omega_{\alpha} \cap \tilde{\Omega}_{\beta} = \emptyset$ unless $\alpha_i + \beta_{n-k+i-1} \leq n$ for all i. Since $|\alpha| + |\beta| = k(n-k)$, we must have equality hold in all these inequalities in order for them to be

simultaneously satisfied, and so $\Omega_{\alpha} \cap \tilde{\Omega}_{\beta} = \emptyset$ unless $\beta = \hat{\alpha}$. It follows that if $\beta \neq \hat{\alpha}$, then the intersection of Schubert varieties $X_{\alpha} \cap \tilde{X}_{\beta} = \emptyset$, so $\sigma_{\alpha} \cup \sigma_{b} = 0$. On the other hand, if $\beta = \hat{\alpha}$, then $X_{\alpha} \cap \tilde{X}_{\beta} = \Omega_{\alpha} \cap \tilde{\Omega}_{\beta}$. The above parametrizations of Ω_{α} and $\tilde{\Omega}_{\beta}$ in terms of matrices show that Ω_{α} intersects $\tilde{\Omega}_{\beta}$ in exactly one point, determined by the basis vectors corresponding to the positions of the 1's in both of these matrices. Now the stars in the matrices correspond to local coordinates of Ω_{α} and Ω_{β} ; taking all the stars together yields coordinates for a neighborhood of the intersection in the Grassmannian. The intersection is obtained at the point where all coordinates are equal to zero, so it follows that the intersection of Ω_{α} and Ω_{β} is transverse at that point. Therefore, $\sigma_{\alpha} \cup \sigma_{\hat{\alpha}} = 1$.

This observation is the starting point for determining the multiplication rule for Schubert varieties and illustrates the convenience of dealing with intersections between varieties associated with flags opposite to each other. For an integer l between 1 and n-k, let σ_l denote the Schubert class corresponding to the special Schubert variety X_l . Then the Pieri rule holds for Schubert classes:

Theorem 5.9 (Pieri rule for Schubert classes) *Let* a *be a partition contained in an* $k \times (n - k)$ *rectangle, and let* l *be an integer between* 1 *and* n - k. Then

$$\sigma_{\alpha} \cup \sigma_{l} = \sum_{\nu \subset k \times (n-k), \nu \in \lambda \otimes k} \sigma_{\nu}. \tag{69}$$

The proof even of this theorem consists only of linear algebra, but is too lengthy to include here. Because the Schubert classes in cohomology satisfy the Pieri rule, we have the following result.

Corollary 5.10 The map $\Lambda_k \longrightarrow H^*(Gr_k(E))$, which sends the Schur function s_α to the Schubert class σ_α if α is a partition contained in a $k \times (n-k)$ rectangle, and sends s_α to zero otherwise, is a surjective ring homomorphism.

6 Computing ϕ^*

Using Theorem 4.4 we can obtain inequalities relating an operator ρ_{AB} and its partial trace ρ_A whenever there is a non-empty intersection of the Schubert variety $X_\beta(F)$ with $\phi(X_\alpha(F'))$, where F and F' are the flags determined by eigenbases of ρ_{AB} and ρ_A , respectively. The condition that there must be a nonzero intersection corresponds cohomologically to there being nonzero product of the Schubert classes, $\sigma_\alpha \cup \phi^*(\sigma_\beta) \neq 0$, where $\phi^*: H^*(\mathrm{Gr}_{d_Bk}(A \otimes B)) \longrightarrow H^*(\mathrm{Gr}_k(A))$ is the map on cohomology induced by ϕ . In order to compute when this product is nonzero, we wish to know the behavior of ϕ^* , which is easier to determine using another presentation for the ring $H^*(\mathrm{Gr}(k,n))$, in terms of Chern classes of vector bundles. In this section we develop this presentation, show how it corresponds to the previous description of $H^*(\mathrm{Gr}(k,n))$ in terms of fundamental classes of Schubert varieties, and use it to describe how ϕ^* acts on $H^*(\mathrm{Gr}_{d_Bk}(A \otimes B))$.

6.1 Vector Bundles

Recall that if M is a manifold, then a d-dimensional complex vector bundle is a map $p: E \to M$ such that the fiber $E_p \equiv p^{-1}(b)$ is an d-dimensional complex vector space for each $b \in M$, and

the following local triviality condition is satisfied: there is an open cover $\{U_{\alpha}\}$ of M, together with homeomorphisms

$$h_{\alpha}: p^{-1}(U_{\alpha}) \to U_{\alpha} \times \mathbb{C}^d$$
 (70)

that are vector space isomorphisms on each fiber. Often the total space E is referred to as the vector bundle, with the rest of the bundle structure implicit. If d = 1, then E is also referred to as a *line bundle*.

We will use several standard constructions of bundles:

- (1) For any manifold M, and any d, there is the *trivial* or *product* bundle $E = M \times \mathbb{C}^d$, where p is the projection onto the first factor.
- (2) If E and E' are bundles, then their direct sum $E \oplus E'$, their tensor product $E \otimes E'$, and the dual E^* are all defined in a natural way [6].
- (3) Let M and N be manifolds and $p: E \to M$ a vector bundle over M. Then if $f: N \to M$ is a (continuous) map, it induces a vector bundle $f^*(E)$ on N, given by the following subset of $N \times E$:

$$\{(n,e): f(n) = p(e)\}.$$
 (71)

This bundle $f^*(E)$, called the *pullback of* E *by* f, is the unique maximal subset of $N \times E$ that makes the following diagram commute:

$$f^*(E) \longrightarrow E$$

$$\downarrow \qquad \qquad \downarrow^p$$

$$N \longrightarrow M.$$

(4) Let V be a d-dimensional complex vector space and let $\mathbb{P}(V)$ be its projectivization, that is, $\mathbb{P}(V) = \operatorname{Gr}_1(V)$ is the set of one-dimensional subspaces of V. Let \hat{V} be the product bundle $\mathbb{P}(V) \times V$. Then the *universal subbundle* S is the subbundle of V given by

$$S = \{(\ell, v) \in \mathbb{P}(V) \times V | v \in \ell\},\tag{72}$$

also called the *tautological line bundle*; and the *universal quotient bundle* Q is defined by the exact sequence

$$0 \to S \to \hat{V} \to Q \to 0. \tag{73}$$

This is known as the *tautological exact sequence* over $\mathbb{P}(V)$. The dual S^* is called the *hyperplane bundle*.

We will also use the following fact [17].

Proposition 6.1 Let $0 \to A \to B \to C \to 0$ be an exact sequence of vector bundles. Then B is isomorphic as a bundle to $A \oplus C$.

Instead of requiring the fiber of each point of a manifold M to be a vector space in our definition, we may have it be any topological space F, thus obtaining a *fiber bundle* with fiber F [17]. The main example of this will be the *projective bundle* $\mathbb{P}(E) \to B$ associated to any d-dimensional vector bundle $E \to B$. The fiber at each point of $\mathbb{P}(E)$ is isomorphic to the

complex projective space \mathbb{P}^{d-1} , and the local trivializations of $\mathbb{P}(E)$ are induced by those of E [6]. If we let p denote the projection from $\mathbb{P}(E)$ to M, then we may pull back E by p to obtain a bundle $p^*(E)$ over $\mathbb{P}(E)$, whose fiber at any point ℓ_p is E_p . As in example (4) above, this pullback bundle has a universal subbundle $S = \{(\ell_p, v) \in p^*(E) | v \in \ell_p\}$ and a universal quotient bundle Q defined by exactness of the sequence $0 \to S \to p^*(E) \to Q \to 0$.

6.2 Chern Classes

Chern classes are integral cohomology classes naturally associated to complex vector bundles. We will need the following fact. Let \mathbb{P}^d be the d-dimensional complex projective space. Since PGL_{d+1} is a connected group acting transitively on the hyperplanes of \mathbb{P}^d , the fundamental class in cohomology associated to a hyperplane H does not depend on the chosen hyperplane. Let h denote this class, which we call the *hyperplane class*.

Chern classes are defined axiomatically as follows [17]:

Theorem 6.2 There are unique functions $c_1, c_2, ...$ on complex vector bundles $E \to N$ over compact differentiable varieties, with $c_i(E) \in H^{2i}(M)$, that depend only on the isomorphism type of E and satisfy the following properties:

- (a) (functoriality) For any continuous map $f: N \to M$, $c_i(f^*(E)) = f^*(c_i(E))$.
- (b) (Whitney sum formula) Writing $c = 1 + c_1 + c_2 + ...$, we have $c(E_1 \oplus E_2) = c(E_1) \cup c(E_2)$.
- (c) If $i > \dim E$, then $c_i(E) = 0$.
- (d) (normalization) For the tautological line bundle S on \mathbb{P}^d , $c_1(S) = -h$, the negative of the hyperplane class.

These classes $c_i(E)$ are called Chern classes of the vector bundle E, and $c(E) = \sum_k c_k(E)$ is called the total Chern class of E (setting $c_0(E) = 1$).

We note that the Whitney sum formula may be written as

$$c_k(E \oplus F) = \sum_{i+j=k} c_i(E) \cup c_j(F). \tag{74}$$

It can be shown [17] that the axiomatic properties of Chern classes imply that if L_1 and L_2 are line bundles, then $c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2)$. From this fact, it readily follows that $c_1(L) = 0$ if L is a trivial line bundle, and hence that $c_k(E)$ is zero for any trivial bundle E, by the Whitney formula.

We now specialize to the problem at hand. Let T be the tautological bundle of dimension k over Gr(k,n), for which the fiber over a subspace V is V itself. Let Q be the quotient bundle over Gr(k,n) whose fiber over a vector space V is \mathbb{C}^n/V . Then the properties of Chern classes imply the following result [33].

Theorem 6.3 The 1th Chern class of the quotient bundle, $c_l(Q)$, is equal to the class of the special Schubert variety σ_l .

Proof Fix a complete flag F_{\bullet} for the n-dimensional complex vector space. Let $\alpha(1,l)$ be the partition corresponding to the complement of a $1 \times l$ rectangle in the $k \times (n-k)$ rectangle. We must show that for any partition $\alpha \subset k \times (n-k)$ of weight k(n-k)-l, $c_l(Q) \cup \sigma_\alpha = 1$ if $\alpha = \alpha(1,l)$, and $c_l(Q) \cup \sigma_\alpha = 0$ otherwise.

Suppose that α has weight k(n-k)-l but $\alpha \neq \alpha(1,l)$. Then $\alpha_k \geq n-k-l+1$, so any $V \in X_\alpha$ satisfies $\dim(V \cap F_{k+l-1}) \geq k$. This means that $V \subset F_{k+l-1}$, so that X_α is contained in the smaller Grassmannian $G = \operatorname{Gr}(k,k+l-1)$ of k-dimensional subspaces of F_{k+l-1} . Let $j: G \hookrightarrow G(k,n)$ be the inclusion map. Using the projection formula from topology [33], we have that

$$c_l(Q) \cup \sigma_\alpha = j_*(j^*(c_l(Q)) \cup [X_\alpha]), \tag{75}$$

where j_* is the Gysin homomorphism on cohomology arising from Poincaré duality. But by the exact sequence of bundles over G,

$$0 \to F_{k+l-1}/V \to \mathbb{C}^n/V \to \mathbb{C}^n/F_{k+l-1} \to 0, \tag{76}$$

the restriction Q_G of the quotient bundle to G can be written $Q_G = F_{k+l-1}/V \oplus \mathbb{C}^n/F_{k+l-1}$, where the latter bundle in the direct sum is trivial. It follows from the Whitney formula that $c_l(Q_G) = 0$, so since $c_l(Q_G) = j^*(c_l(Q))$, we must have that $c_l(Q) \cup \sigma_\alpha = 0$ by Equation 75.

Now suppose that $\alpha = \alpha(1, l)$. In this case

$$X_{\alpha} = \{ V \in \operatorname{Gr}(n,k) | F_{k-1} \subset V \subset F_{k-l} \}, \tag{77}$$

which is isomorphic to the l-dimensional projective space $\mathbb{P} = \mathbb{P}(F_{k+l}/F_{k-1})$. Let i denote the natural isomorphism from X_{α} to \mathbb{P} . On \mathbb{P} we have the exact sequence

$$0 \to V/F_{k-1} \to F_{k+l}/F_{k-1} \to Q_{\mathbb{P}} \to 0.$$
 (78)

Here V/F_{k-1} is the tautological line bundle, $Q_{\mathbb{P}}$ is the quotient bundle, and F_{k+l}/F_{k-1} is a trivial bundle. It follows that the total Chern class of $Q_{\mathbb{P}}$ is $c(Q_{\mathbb{P}}) = (1-h)^{-1}$ (where h is the class of the hyperplane). Now the projection formula tells us that

$$c_l(Q) \cup \sigma_{\alpha} = i_*(i^*(c_l(Q)) \cup [X_{\alpha}])$$

$$= i_*(i^*(c_l(Q)))$$

$$= i_*(c_l(Q_{\mathbb{P}}))$$

$$= 1.$$

6.3 The Splitting Principle

We have seen that the Chern classes of the quotient bundle Q correspond to special Schubert classes. Since all Schubert classes can be obtained as products of these special Schubert classes, characterizing the effect of ϕ^* on the Chern classes of Q will be sufficient to determine the effect of ϕ^* on $H^*(Gr(k,n))$. To do this, we will need the *splitting principle*, a fundamental observation from the theory of Chern classes. In what follows, let E be any vector bundle over a manifold M, whose dimension we denote by m. We shall have in mind the case where M = Gr(k,n) and E is the quotient bundle Q defined above (so that m = n - k).

Starting with the bundle E over M, let $\mathbb{P}(E)$ be the projectivization of E, and let f_1 be the induced map from $\mathbb{P}(E)$ to M. Let $f_1^*(E)$ be the pullback bundle:

$$f_1^*(E) \longrightarrow E$$

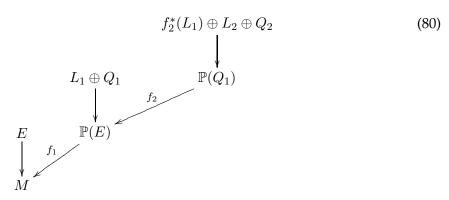
$$\downarrow \qquad \qquad \downarrow$$

$$\mathbb{P}(E) \xrightarrow{f_1} M.$$

Let L_1 be the tautological line bundle of the pullback $f^*(E)$. Then we have an exact sequence

$$0 \to L_1 \to f_1^*(E) \to Q_1 \to 0,$$
 (79)

where E is an (m-1)-dimensional bundle over M, so $f_1^*(E)$ is isomorphic to $L_1 \oplus Q_1$. Similarly, let $\mathbb{P}(Q_1)$ be the projectivization of Q_1 , with f_2 as the map from $\mathbb{P}(Q_1)$ to $\mathbb{P}(Q)$. If L_2 is the tautological line bundle of $\mathbb{P}(Q_1)$, then L_2 gives rise to a quotient Q_2 such that $f_2^*(Q_1)$ is isomorphic to $L_2 \oplus Q_2$. We can thus pull back E to a direct sum of Q_2 and two line bundles:



Continuing in this way, we obtain bundles Q_3,\ldots,Q_{m-1} , and projectivizations $\mathbb{P}(Q_2),\ldots,\mathbb{P}(Q_{m-2})$, such that the pullback of E by the map from $\mathbb{P}(Q_{m-2})$ to M is a direct sum of line bundles. If $f=f_1\circ f_2\circ\ldots f_{m-2}$ is the map from $\mathbb{P}(Q_{m-2})$ to M, then it can be shown that the induced map on cohomology $f^*:H^*(M)\to H^*(\mathbb{P}(Q_{m-2}))$ is injective [6]. We summarize these facts in the following theorem, known as the splitting principle:.

Theorem 6.4 (The Splitting Principle) For any vector bundle E on a manifold M, there exists a manifold N and a continuous $f: N \to M$ such that $f^*(M) \to f^*(N)$ is injective, and pullback bundle $f^*(E)$ is a direct sum of line bundles.

We now illustrate the splitting principle by using it to derive a result that will be useful to us. Let E be a vector bundle, and let $f: N \to M$ be the map given by Theorem 6.1, so that the pullback $f^*(E)$ splits as the direct sum of line bundles L_1, \ldots, L_n . Let $x_i = c_1(L_i)$. Then the Whitney sum formula $c_k(E_1 \oplus E_2) = \sum_{i+j=k} c_i(E_1) \cup c_j(E_2)$ implies that

$$c_k(f^*(E)) = c_k(x_1, \dots, x_n)$$
 (81)

is the kth elementary symmetric polynomial in the first Chern classes of $f^*(E)$. By the functoriality of the Chern classes, it follows that $f^*(c_k(E))$ is the kth elementary symmetric polynomial in $c_1(L_1), \ldots, c_1(L_n)$.

Let us revisit the construction of the split manifold of a vector bundle E. $\mathbb{P}(E)$ consists of pairs (x,ℓ) , where $x\in M$ and ℓ is a line in E_x . Proposition 6.1 allows us to consider all the bundles $Q_1,\ldots Q_{n-1}$ as subbundles of E. Now $\mathbb{P}(Q_1)$ consists of triples (x,ℓ_1,ℓ_2) where ℓ_2 is a line in the linear complement of ℓ_1 in E_p . In general, a point of $\mathbb{P}(Q_j)$ over (x,ℓ_1,\ldots,ℓ_j) in $\mathbb{P}(Q_{j-1})$ is a (j+2)-tuple $(x,\ell_1,\ldots,\ell_j,\ell_{j+1})$ where ℓ_{j+1} is a line in the complement of ℓ_1,\ldots,ℓ_j . We conclude that the split manifold $\mathbb{P}(Q_{m-2})$ is in fact the *flag bundle*:

$$FI(E) = \{ (x, \ell_1 \subset \langle \ell_1, \ell_2 \rangle \subset \langle \ell_1, \ell_2, \ell_3 \rangle \subset \dots \subset E_x) | x \in M \}.$$
 (82)

6.4 Representations and Line Bundles

We have seen that the splitting principle allows us to regard the Chern classes of a vector bundle E as (symmetric) polynomials in the first Chern classes of the line bundles of a flag bundle associated to E. Given an m-dimensional vector space V, the space FI(V) of all complete flags on V can be identified with GL(V)/T, where T is now the group of upper triangular matrices. This follows because GL(V) is transitive on the flags and T, the stabilizer of the standard flag $0 \subset \langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset \cdots \subset \langle e_1, \ldots, e_m \rangle = V$, is isomorphic to the stabilizer of any given flag. We can associate to any one-dimensional representation $\chi: T \to \mathbb{C}^*$ a line bundle over the flag manifold FI(V) as follows:

$$L(\chi) = \operatorname{GL}(V) \times \mathbb{C}/((gt, z) \sim (g, \chi(t)z))$$
(83)

for $g \in GL(V)$, $t \in T$, and $z \in \mathbb{C}$. The projection of $L(\chi)$ onto Fl(V) is just $(g,z) \stackrel{\pi}{\to} (gT)$. Under the action of GL(V) given by h(gt,z) = (hgt,z), the following diagram commutes:

$$\begin{array}{ccc} L(\chi) & \xrightarrow{GL(V)} & L(\chi) \\ \pi \Big\downarrow & & & \Big\downarrow \pi \\ & \mathrm{Fl}(V) & \xrightarrow{GL(V)} & \mathrm{Fl}(V), \end{array}$$

since $(hgt, z) = (hg, \chi(t)z)$. The line bundle $L(\chi)$ is thus *equivariant* with respect to the bundle projection.

Conversely, suppose L is an equivariant line bundle over $\mathrm{Fl}(V)$. Then T acts on the fiber over eT, so this fiber is a one-dimensional representation χ of T. Indeed, the the line bundle $L(\chi)$ corresponding to this representation is isomorphic to L. Let $y \in L$ lie in the fiber over eT. The isomorphism is given by the map $r:L(\chi)\to L$ given by $r(g,z)=z(g\cdot y)$. At first glance, the map isn't obviously well-defined, but note that $r(gt,z)=z(gt\cdot y)=z(g\chi(t)y)=r(g,\chi(t)z)$. Since G acts transitively on the fibers of L, and multiplication by z is a surjective map on any given fiber, r is surjective. For injectivity, suppose that $z_1(g_1\cdot y)=z_2(g_2\cdot y)$. If $z_1\neq 0$, then $y=z_1^{-1}z_2g_1^{-1}g_2\cdot y$, so $g_1^{-1}g_2\in T$. This means that $g_2=g_1t$ for some $t\in T$ and $z_1^{-1}z_2\chi(t)=1$, so $z_2=z_1\chi(t)^{-1}$. Thus, as elements of $L(\chi)$, $(g_2,z_2)=(g_1t,\chi(t)^{-1}z_1)=(g_1,z_1)$, so r is indeed injective. It is likewise easy to see that r commutes with the bundle projection and is linear on fibers. The correspondence between line bundles and one-dimensional representations of T is therefore a bijection.

Actually, we can identify the characters χ with line bundles on a flag manifold Fl(V) more explicitly. Consider the *tautological filtration* [12]

$$0 = U_0 \subset U_1 \subset U_2 \subset \cdots \subset U_m = \operatorname{Fl}(V) \times V \tag{84}$$

of vector bundles over $\mathrm{Fl}(V)$, where $\mathrm{Fl}(V) \times V$ is the product bundle, and U_k is the k-dimensional bundle over $\mathrm{Fl}(V)$ whose fiber over a flag $V_1 \subset \cdots \subset V_m$ is V_k . It follows from the splitting principle that the cohomology ring $H^*(\mathrm{Fl}(V))$ is generated by the first Chern classes of the line bundles $L_i = U_i/U_{i-1}$, setting $x_i = c_1(L_1)$. The identity matrix fixes the standard flag $\{e_1,\ldots,e_m\}$. Therefore, over eT, the fiber of L_i is V_i/V_{i-1} , where $V_i = \langle e_1,\ldots,e_i\rangle$. If $v = \sum_{k=1}^i \alpha_i e_i \in V_i$ and $t \in T$, then $t \cdot v = w + t_{ii}e_i$, where $w \in V_{i-1}$ and t_{ii} is the ith diagonal entry of t. We have shown the following:

Theorem 6.5 If L_i is the line bundle over a flag manifold defined as above, then the character χ associated to L_i is the map taking t to t_{ii} .

Let us adapt this machinery to the problem at hand. Recall that we have two complex vector spaces A and B of dimensions d_A and d_B , respectively, together with a map $\phi: \operatorname{Gr}_k(A) \to \operatorname{Gr}_{kd_B}(A \otimes B)$ given by $\phi(V) = V \otimes B$. We wish to compute the action of the induced map $\phi^*: H^*(\operatorname{Gr}_{kd_B}(A \otimes B)) \to H^*(\operatorname{Gr}_k(A))$.

Let Q_A and Q_{AB} be the quotient bundles of Theorem 6.3 over the Grassmannians $\operatorname{Gr}_k(A)$ and $\operatorname{Gr}_{kd_B}(A\otimes B)$ respectively. The Chern classes of these bundles are the classes of the special Schubert varieties in the cohomology rings. By the splitting principle, the associated flag bundles $\operatorname{Fl}(A)$ and $\operatorname{Fl}(A\otimes B)$ have pullbacks which split as a direct sum of line bundles L_i of the respective tautological filtrations. The cohomology of the Grassmannians embeds in the cohomology of these pullbacks, so we may determine ϕ^* by its action on the Chern classes of the pullback bundle of $\operatorname{Fl}(A\otimes B)$.

It follows from the definition of pullback bundles that the bundle $\phi^*(L(\chi_i))$ is the set of triples $(gT_A,\phi(g),z)\in \operatorname{GL}(A)/T_A\times\operatorname{GL}(A\otimes B)\times\mathbb{C}$ with the identification $(gT_A,\phi(g\cdot t),z)\sim (gT_A,\phi(g),\chi(\phi(t))z)$. This means that $\phi^*(L(\chi_i))=L(\phi^*(\chi_i))$. The pullback of the map induced by ϕ on the characters of the group T_{AB} is readily computed: for a matrix $X\in T_A$, and the character χ_i taking a matrix to its ith diagonal entry, we have $\phi^*(\chi_i)(X)=\chi_i(\phi(X))=\chi_i(X\otimes I)=\chi_{\lceil i/d_B\rceil}(X)$. So $\phi^*(\chi_i)=\chi_{\lceil i/d_B\rceil}$. Now we can calculate the action of ϕ^* on the Chern classes:

$$\phi^{*}(x_{i}) = \phi^{*}(c_{1}(L(\chi_{i})))
= c_{1}(\phi^{*}(L(\chi_{i})))
= c_{1}(L(\phi^{*}(\chi_{i})))
= c_{1}(L(\chi_{\lceil i/d_{B} \rceil})).$$
(85)

Since that short calculation was the reason for developing so much machinery, we give the conclusion the status of a theorem:

Theorem 6.6 $\phi^*(x_i) = c_1(L(\chi_{\lceil i/d_R \rceil})).$

7 Determining the Inequalities

In this section we use our knowledge of how ϕ^* behaves to explicitly derive inequalities relating the spectra of ρ_{AB} and of ρ_A and work out some examples in low dimensions. We also restate how to obtain the inequalities in the language of representation theory. Next, we discuss recent progress in symplectic geometry that shows that the inequalities derived using the

method described here are sufficient. Finally, we prove that if $d_B \ge \frac{1}{2}d_A^2$, then the inequalities simplify greatly.

7.1 Putting It All Together

Let $\rho_A = \text{Tr}_B \, \rho_{AB}$, and let λ , μ , and $\tilde{\lambda}$ denote the spectra of ρ_{AB} , $-\rho_{AB}$, and ρ_A , respectively. Theorem 4.4 can be interpreted cohomologically as saying that if

$$\phi^*(\sigma_\pi) \cup \tilde{\sigma}_\nu \neq 0, \tag{86}$$

where $\sigma_{\pi} \in H^*(Gr(kd_B, d_Ad_B))$ and $\tilde{\sigma}_{\nu} \in H^*(Gr(k, d_a))$ are Schubert classes, then the spectra μ and $\tilde{\lambda}$ must satisfy the inequalities

$$\sum \nu(i)\tilde{\lambda}_i + \sum \pi(i)\mu_i \le 0. \tag{87}$$

Now $\phi^*(\sigma_\pi)$ is an integer combination of Schubert classes,

$$\phi^*(\sigma_{\pi}) = \sum_i n_i \tilde{\sigma}_{\pi_i}. \tag{88}$$

For each of these classes, $\tilde{\sigma}_{\pi_i} \cup \tilde{\sigma}_{\nu} \neq 0$ iff ν contains the complement of π_i in the $k \times (n-k)$ rectangle. But if we consider the case where ν is in fact the complement of π_i , then we see that the Inequalities (87) are the strongest in this case; for any other $\nu' \supset \nu$, the inequalities determined by ν' are implied by the inequalities determined by ν . So it is sufficient to consider complements of each Schubert class $\tilde{\sigma}_{\pi_i}$ contained in $\phi^*(\sigma_\pi)$, in order to obtain the inequalities relating $-\rho_{AB}$ and ρ_A . Now if μ is the spectrum of $-\rho_{AB}$, then the spectrum λ of ρ_{AB} is given by $\lambda_i = -\mu_{d_A-i+1}$ (since the ordering of the eigenvalues is reversed). Given binary strings $\pi, \hat{\pi} \in \binom{d_A d_B}{k}$ satisfying $\hat{\pi}(i) = \pi(d_A d_B - i + 1)$, so that $\hat{\pi}$ is simply the string π in reverse, the Schubert cell $S_{\hat{\pi}}$ corresponds to the complementary partition to that of S_{π} . This means that we obtain inequalities

$$\sum \nu(i)\tilde{\lambda}_i \le \sum \pi(i)\lambda_i \tag{89}$$

whenever $\phi^*(\sigma_{\hat{\pi}})$ contains $\sigma_{\hat{\nu}}$ (where $\hat{\nu}$ is the complementary partition to ν) as a summand. It then follows that Inequalities (89) are obtained whenever $\phi^*(\sigma_{\pi})$ contains σ_{ν} as a summand.

Theorem 6.3 says that the lth Chern class $c_l(Q)$ of the universal quotient bundle Q over the Grassmannian Gr(k,n) is equal to the special Schubert class $\sigma_l \in H^*(Gr(k,n))$. And the splitting principle allows us to conclude that

$$c_l(Q) = e_l(x_1, \dots, x_{n-k}),$$
 (90)

where $x_i = c_1(L_i)$ is the first Chern class of the *i*th split component of $f^*(Q)$, and e_l is the *l*th elementary symmetric polynomial. Because the special Schubert classes σ_l generate the cohomology ring, we therefore have a surjective ring homomorphism

$$\tilde{\psi}: \Lambda_{n-k} \to H^*(\operatorname{Gr}(k,n))$$

$$e_l(x_1, \dots, x_{n-k}) \mapsto \sigma_l.$$

We may compose the map $\tilde{\psi}$ with the involution $\omega: \Lambda_{n-k} \to \Lambda_{n-k}$, $\omega(e_k) = h_k$, to obtain a map

$$\psi: \Lambda_{n-k} \to H^*(Gr(k,n))$$

 $h_l(x_1, \dots x_{n-k}) \mapsto \sigma_l.$

Now, by the Pieri rule, it follows that for any partition λ , $\psi(s_{\lambda}(x_1,\ldots,x_{n-k}))=\sigma_{\lambda}$. Thus, we may determine how ϕ^* acts on $H^*(Gr(kd_B,d_Ad_B))$ by determining how the map $x_i\mapsto x_{\lceil i/d_B\rceil}$ acts on Schur functions.

7.2 Some Observations

In this section we make some observations about the map ϕ^* that will simplify our computations to some degree. First, we note that ϕ^* is particularly easy to calculate on the Newton power sums $p_j = \sum_i x_i^j$:

$$\phi^*(p_j(x_1, \dots, x_{(d_A - k)d_B})) = \phi^*\left(\sum_{i=1}^{(d_A - k)d_B} x_i^j\right)$$
(91)

$$= \sum_{i=1}^{(d_A - k)d_B} x^j_{\lceil i/d_B \rceil} \tag{92}$$

$$=\sum_{i=1}^{d_A-k} d_B x_i^j \tag{93}$$

$$= d_B p_j(x_1, \dots, x_{d_A - k}). (94)$$

We further note that the total degree of a polynomial in the Chern classes $x_1, \ldots x_{n-k}$ is equal to the weight of the corresponding partition, and ϕ^* maps every monomial in $x_1, \ldots, x_{(d_A-k)d_B}$ to a monomial in x_1, \ldots, x_{d_A-k} of the same total degree, so that $\phi^*(\sigma_\pi)$ is a sum of Schubert classes of the same weight as π .

Applying this observation to the empty partition $\alpha = (0)$, which corresponds to the binary string $\underbrace{11\dots1}_k\underbrace{00\dots0}_{n-k}$ in $\operatorname{Gr}(k,n)$, we obtain the inequalities

$$\sum_{i=1}^{k} \tilde{\lambda}_i \le \sum_{i=1}^{d_B k} \lambda_i \tag{95}$$

for every $k \in \{1, ..., d_A\}$. These are the same inequalities previously derived in Lemma 4.2 using only Ky Fan's Maximum Principle. We will call Inequalities (95) *basic inequalities*. As we shall see, many of the inequalities that arise from considering the intersections of Schubert classes will not contain additional information; rather, they will be consequences of the basic inequalities. We call such inequalities *redundant inequalities*.

Finally, we argue that it is sufficient to consider inequalities derived from ϕ^* acting on $H^*(Gr(kd_B, d_Ad_B))$, where $k \leq \frac{d_A}{2}$. To see this, suppose there is an inequality of the form

$$\sum_{i=1}^{d_A} \nu(i)\tilde{\lambda}_i \le \sum_{i=1}^{d_A d_B} \pi(i)\lambda_i,\tag{96}$$

α	s_{lpha}	$\pi_{\alpha} \in H^*(\operatorname{Gr}(2,6))$	$\pi_{\alpha} \in H^*(\operatorname{Gr}(1,3))$
	p_1	101000	010
	$\frac{1}{2}(p_1^2+p_2)$	100100	001
LE	$\frac{1}{2}(p_1^2-p_2)$	011000	_

Figure 2 Partitions, their Schur polynomials and binary strings

where the weight of ν is greater than $\frac{d_A}{2}$. We may apply this inequality to the matrices $-\rho_{AB}$ and $-\rho_A$ and use the trace condition to conclude that

$$\sum_{i=1}^{d_A} \nu'(i)\tilde{\lambda}_i \le \sum_{i=1}^{d_A d_B} \pi'(i)\lambda_i,\tag{97}$$

where $\nu'(i)=1-\nu(i)$ for all i, and similarly for π' . If the weight of ν is greater than $\frac{d_A}{2}$, then the weight of ν' is less than $\frac{d_A}{2}$. Thus, the desired inequality is a consequence of an inequality involving fewer than $\frac{d_A}{2}$ eigenvalues. (This argument is not valid unless we know that our method generates all possible valid inequalities. This is indeed the case, but we postpone the discussion for Section 7.5.)

7.3 Examples

We now work out the inequalities for some examples. The case $d_A=2$ was already solved in Section 4.3, where it was shown that the basic inequalities were the only constraints on the eigenvalues of ρ_A and ρ_{AB} . Thus, the simplest remaining case is $d_A=3$, $d_B=2$, which we will now illustrate. We use h_l to refer to the lth complete symmetric function, and p_l to refer to the lth Newton power sum symmetric function. We identify Schur functions with their images as Schubert classes, denoting either by a (Young diagram of a) partition.

As we have argued, we may restrict attention to inequalities involving at most $\frac{d_A}{2}$ eigenvalues; in the case $d_A=3$, this means that it suffices to consider maps $\phi^*:H^*(\mathrm{Gr}(2,6))\to H^*(\mathrm{Gr}(1,3))$. The Schubert classes of $H^*(\mathrm{Gr}(1,3))$ correspond to partitions that fit inside a 1×2 rectangle, of which there are only two (excluding the empty partition, for which we obtain the basic inequalities): \square and \square . Because ϕ^* preserves the weight of a partition, we need only consider partitions of weight one and two in $H^*(\mathrm{Gr}(2,6))$: namely, \square , and \square . Figure 2 lists the Schur polynomials and binary strings associated to each of these partitions (the polynomials are readily computed using the Jacobi-Trudi formula).

Using this information, we can calculate ϕ^* on each of the Schubert classes \Box , \Box , and $\exists \in H^*(Gr(2,6))$:

- (1) $\phi^*(\square) = \phi^*(p_1) = 2p_1 = 2\square$. This yields the inequality $\tilde{\lambda}_2 \leq \lambda_1 + \lambda_3$.
- (2) $\phi^*(\square) = \phi^*(\frac{1}{2}(p_1^2 + p_2)) = 2p_1^2 + p_2 = 3\square + \square$. For the \square term on the right side, we get the inequality $\tilde{\lambda}_3 \leq \lambda_1 + \lambda_4$. The \square term does not yield an inequality because $\square = 0$ in $H^*(\operatorname{Gr}(1,3))$.

(3)
$$\phi^*(\Box) = \phi^*(\frac{1}{2}(p_1^2 - p_2)) = 2p_1^2 - p_2 = 3\Box + \Box$$
. As before, the \Box term does not yield an inequality. The \Box term yields the inequality $\tilde{\lambda}_3 \leq \lambda_2 + \lambda_3$.

So we have three inequalities, $\tilde{\lambda}_2 \leq \lambda_1 + \lambda_3$, $\tilde{\lambda}_3 \leq \lambda_1 + \lambda_4$, and $\tilde{\lambda}_3 \leq \lambda_2 + \lambda_3$. Let us check these inequalities for redundancy. From the basic inequalities, we have that $\tilde{\lambda}_2 \leq \frac{1}{2}(\tilde{\lambda}_1 + \tilde{\lambda}_2) \leq \frac{1}{2}(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4) \leq \lambda_1 + \lambda_3$, so the first inequality is redundant. And $\tilde{\lambda}_3 \leq \frac{1}{3}(\tilde{\lambda}_1 + \tilde{\lambda}_2 + \tilde{\lambda}_3) \leq \frac{1}{3}(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6) \leq \lambda_1 + \lambda_4$, so the second inequality is also redundant. However, the inequality $\tilde{\lambda}_3 \leq \lambda_2 + \lambda_3$ is not redundant. (For example, $\lambda = (1,0,0,0,0,0)$ and $\tilde{\lambda} = (\frac{1}{3},\frac{1}{3},\frac{1}{3})$ satisfy the basic inequalities, but $\tilde{\lambda}_3 \leq \lambda_2 + \lambda_3$ does not.)

So $\tilde{\lambda}_3 \leq \lambda_2 + \lambda_3$ is the only new inequality we get involving one eigenvalue of ρ_A . By duality, we also have the inequality $\tilde{\lambda}_2 + \tilde{\lambda}_3 \leq \lambda_1 + \lambda_2 + \lambda_3 + \lambda_6$, or $\tilde{\lambda}_1 \geq \lambda_4 + \lambda_5$. Thus, our complete list of eigenvalue constraints on ρ_{AB} and ρ_A is

$$\tilde{\lambda}_1 \leq \lambda_1 + \lambda_2, \tag{98}$$

$$\tilde{\lambda}_3 \geq \lambda_5 + \lambda_6,$$
 (99)

$$\tilde{\lambda}_3 \leq \lambda_2 + \lambda_3,$$
 (100)

$$\tilde{\lambda}_1 \geq \lambda_4 + \lambda_5,$$
 (101)

together with the trace condition $(\tilde{\lambda}_1 + \tilde{\lambda}_2 + \tilde{\lambda}_3) = (\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6)$. Now we consider the case $d_A = 3$, $d_B = 3$. We have that

$$\phi^*(\square) = 3\square, \tag{102}$$

$$\phi^*(\square) = 6\square + 3\square, \tag{103}$$

$$\phi^*(\Box) = 6\Box + 3\Box\Box, \tag{104}$$

yielding inequalities

$$\tilde{\lambda}_2 \leq \lambda_1 + \lambda_2 + \lambda_4, \tag{105}$$

$$\tilde{\lambda}_3 \leq \lambda_1 + \lambda_2 + \lambda_5, \tag{106}$$

$$\tilde{\lambda}_3 \leq \lambda_1 + \lambda_3 + \lambda_4. \tag{107}$$

It is not hard to check that all of these inequalities are redundant. Thus, our only inequalities for the case $d_A = 3$, $d_b = 3$ are the basic inequalities

$$\tilde{\lambda}_1 \leq \lambda_1 + \lambda_2 + \lambda_3, \tag{108}$$

$$\tilde{\lambda}_3 \geq \lambda_7 + \lambda_8 + \lambda_9. \tag{109}$$

7.4 Representation Theory Perspective

Given a Schur polynomial s_{λ} , we have seen how to determine $\phi^*(s_{\lambda})$ as follows: write s_{λ} in terms of Newton power sums, evaluate ϕ^* on each of the power sums, and then express the results in terms of Schur polynomials. While this algorithm is fairly straightforward, the relationship between s_{λ} and the terms appearing in $\phi^*(s_{\lambda})$ is less clear. In this section, we

see that we can interpret this relationship from the standpoint of group representation theory. Asking which Schur polynomials appear in $\phi^*(s_\lambda)$ is equivalent to asking which irreducible representations appear in a certain tensor product of representations of the symmetric group.

While we are concerned with the action of ϕ^* on Schur polynomials acting on a fixed number of variables, we will simplify our discussion by working in the ring of symmetric functions. Define a *symmetric function* to be a set of symmetric polynomials $p(x_1, \ldots, x_l)$, one for each positive integer l, such that

$$p(x_1, \dots, x_l, 0, \dots, 0) = p(x_1, \dots, x_1).$$
 (110)

Recall that the Newton power sum symmetric functions are defined as follows. For a nonnegative integer s (which we may also think of as a partition of one part of size s), $p_s(X_1, \ldots, X_k) = X_1^s + \cdots + X_k^s$. For a partition $\lambda = (\lambda_1, \ldots, \lambda_l)$ of length l, define

$$p_{\lambda}(X_1, \dots, X_k) = \prod_{i=1}^{l} p_{\lambda_i}(X_1, \dots, X_k).$$
 (111)

As we have seen, $\phi^*(p_s) = d_B p_s$, so that $\phi^*(p_\lambda) = d_B^{l(\lambda)} p_\lambda$, where $l(\lambda)$ is the length of the partition λ .

We use the following basic facts about the representation theory of the symmetric group [12, 13]. The irreducible representations of the symmetric group S_n on n letters can be put in one-to-one correspondence with the partitions of n, in a standard way. (And the partitions of n also correspond naturally to the conjugacy classes of S_n .) Furthermore, the Newton power sum symmetric functions p_μ and the Schur polynomials s_λ are related as follows. For any partition μ of n, define

$$z(\mu) = \prod_{r} r^{m_r}(m_r!),$$
(112)

where m_r is the number of times r occurs in μ . Now for any partition μ of n_r

$$p_{\mu} = \sum_{\lambda} \chi_{\mu}^{\lambda} s_{\lambda}; \tag{113}$$

and for any partition λ of n,

$$s_{\lambda} = \sum_{\mu} \frac{1}{z(\mu)} \chi_{\mu}^{\lambda} p_{\mu},\tag{114}$$

where χ^{λ}_{μ} is the character of the representation labelled by λ evaluated on a permutation in the conjugacy class labelled by μ .

Let us now return to the fact that $\phi^*(p_\lambda) = d_B^{l(\lambda)} p_\lambda$. This means that ϕ^* is a class function on S_{d_B} (where $d_B = |\lambda|$), so we wish to find a representation ρ of S_n such that the character χ^ρ of ρ is equal to ϕ^* . Consider the representation ρ of S_n on $B^{\otimes n}$ that acts by permuting the tensor factors: if $\{e_i\}_{i=1}^{d_B}$ is an orthogonal basis for B, then for $w \in S_n$,

$$\rho(w)(e_{i_1} \otimes \cdots \otimes e_{i_n}) = e_{i_{w(1)}} \otimes \cdots \otimes e_{i_{w(n)}}.$$
(115)

We claim that the character $\chi^{\rho} = \phi^*$, or in other words, for any $w \in S_n$, the character of ρ evaluated at w is $d_B^{l(w)}$, where l(w) is the number of cycles in w. To see this, recall that by

definition, $\chi^{\rho}(w) = \text{Tr}(\rho(w))$. So $\chi^{\rho}(w)$ is the number of elements of the basis $\{e_{i_1} \otimes \cdots \otimes e_{i_n}\}$ fixed by the map ρ ; in other words,

$$\chi^{\rho}(w) = |\{(i_1, \dots, i_n) = (i_{w(1)}, \dots, i_{w(n)})\}|. \tag{116}$$

Now, if (i_1,\ldots,i_n) is fixed by w, then for any r_1 and r_2 in the same cycle of w, we must have $i_{r_1}=i_{r_2}$. Conversely, if (i_1,\ldots,i_{d_B}) satisfies the property that $i_{r_1}=i_{r_2}$ for any r_1 and r_2 in the same cycle of w, then (i_1,\ldots,i_n) is fixed by w. We conclude that the number of elements in the set $\{(i_1,\ldots,i_n)=(i_{w(1)},\ldots,i_{w(n)})\}$ is equal to the number of ways to assign a basis element to each cycle of w, which is $d_B^{l(w)}$.

Let V_{λ} be the irreducible representation of S_n labelled by λ . Let

$$V_{\lambda} \otimes \rho = \bigoplus_{\pi} (V_{\pi})^{\otimes m_{\pi}} \tag{117}$$

be a decomposition of $V_{\lambda} \otimes \rho$ into irreducible representations (each irrep V_{π} occurs with multiplicity m_{π}). Then we have that

$$\chi^{\lambda}(\mu)\chi^{\rho}(\mu) = \sum_{\pi} m_{\pi}\chi^{\pi}(m), \tag{118}$$

a result we will use in the next calculation, the evaluation of $\phi^*(s_\lambda)$:

$$\phi^*(s_{\lambda}) = \sum_{\mu} \frac{1}{z(\mu)} \chi^{\lambda}(\mu) \phi^*(p_{\mu})$$
(119)

$$= \sum_{\mu} \frac{1}{z(\mu)} \chi^{\lambda}(\mu) d_B^{l(\mu)} p_{\mu} \tag{120}$$

$$= \sum_{\mu} \frac{1}{z(\mu)} \chi^{\lambda}(\mu) \chi^{\rho}(\mu) p_{\mu} \tag{121}$$

$$= \sum_{\mu} \frac{1}{z(\mu)} \sum_{\pi} m_{\pi} \chi^{\pi}(\mu)(p_{\mu})$$
 (122)

$$= \sum_{\pi} \sum_{\mu} \frac{1}{z(\mu)} \chi^{\pi}(\mu)(p_{\mu}) \tag{123}$$

$$= \sum_{\pi} m_{\pi} s_{\pi}. \tag{124}$$

So the Schur polynomials s_{π} appearing in $\phi^*(s_{\lambda})$ are precisely those corresponding to the representations V_{π} appearing in $V_{\lambda} \otimes \rho$.

7.5 Sufficiency

We have described an approach using a variational principle to determine inequalities relating a matrix ρ_{AB} to its partial trace ρ_A , along with some observations for simplifying the list of inequalities. While our method has the advantage of relative straightforwardness and simplicity, our techniques do not (to our knowledge) allow us to demonstrate that the inequalities obtained are in fact sufficient: that is, if λ and $\tilde{\lambda}$ satisfy the inequalities, then there exists matrices ρ_{AB} and $\rho_A = \operatorname{Tr}_B \rho_{AB}$ such that λ is the spectrum of ρ_{AB} and $\tilde{\lambda}$ is the spectrum of ρ_A . It turns out that the inequalities obtained from our variational principle approach

are indeed sufficient. This follows from recent work in symplectic geometry [4], of which we became aware after deriving the inequalities through our methods. In this section, we will state the main result from [4] and show that it yields inequalities equivalent to the ones we have obtained.

We can express our problem in the language of symplectic geometry. (See [4, 9, 27] for definitions and further discussion.) Consider the Lie group $U(A \otimes B)$ of unitary matrices acting on the space $A \otimes B$. For any vector $\lambda = (\lambda_1, \ldots, \lambda_{d_A d_B})$ with terms arranged in nonincreasing order, the set $\mathcal{O}_{\lambda}^{AB}$ of Hermitian matrices on $A \otimes B$ with spectrum λ is a coadjoint orbit of $K = U(A \otimes B)$. Now consider the action of the Lie group $\tilde{K} = U(A)$ of unitary matrices on A, by conjugation on the symplectic manifold $\mathcal{O}_{\lambda}^{AB}$: for $U \in U(A)$,

$$U: \rho_{AB} \mapsto (U \otimes I_B)\rho_{AB}(U^{\dagger} \otimes I_B). \tag{125}$$

It is not hard to verify that this is a Hamiltonian group action whose moment map is Tr_B , the partial trace with respect to B. So our problem, then, is to describe the image of the symplectic manifold $\mathcal{O}_{\lambda}^{AB}$ under the moment map Tr_B .

This formulation is useful because considerable work has been done in the study of the image of moment maps. For instance, the following result is due to Kirwan [23, 27]:

Theorem 7.1 Let M be a compact connected Hamiltonian K-manifold, with moment map Φ . Then the intersection of the image of Φ with the positive Weyl chamber \mathfrak{t}_+^* is a convex polytope.

In our case, the positive Weyl chamber of U(A) consists of diagonal matrices whose diagonal entries are in nonincreasing order (every matrix in the image of Φ has the same spectrum as one such matrix). Kirwan's theorem thus allows us to conclude that the set of all ordered spectra of matrices obtainable by taking the partial traces of matrices with a fixed spectrum must be a region bounded by a finite set of inequalities.

Interestingly, Horn's problem can also be viewed in this framework. Recall that Horn's problem asks for the possible spectra of X+Y, given the spectra of $n\times n$ matrices X and Y. Suppose that λ is the spectrum of X and μ is the spectrum of Y. Now we consider the action of the group U(n) of $n\times n$ unitary matrices on the symplectic manifold $\mathcal{O}_{\lambda}\times\mathcal{O}_{\mu}$ by diagonal conjugation:

$$U: (X,Y) \mapsto (UXU^{\dagger}, UYU^{\dagger}). \tag{126}$$

This is a Hamiltonian group action whose moment map takes two Hermitian matrices to their sum. Thus, Horn's problem can be viewed as the problem of determining the image of this moment map.

The following theorem of Berenstein and Sjamaar [4] generalizes Klyachko's solution to Horn's problem. Before we state it, some new notation is required. Let K be a compact connected Lie group, and let \tilde{K} be a closed connected subgoup. Let f be the inclusion map of \tilde{K} into K, $f_*: \tilde{\mathfrak{t}} \to \mathfrak{k}$ be the embedding of Lie algebras induced by f, and $f^*: \mathfrak{t}^* \to \mathfrak{t}^{\tilde{*}}$ be the dual projection. Choose maximal tori T of K and \tilde{T} of \tilde{K} , and Weyl chambers $\mathfrak{t}_+^* \subset \mathfrak{t}^*$ and $\tilde{\mathfrak{t}}_+^* \subset \tilde{\mathfrak{t}}_+^*$, where \mathfrak{t} and $\tilde{\mathfrak{t}}$ are the Lie algebras of T and \tilde{T} , respectively. For $\alpha \in \mathfrak{t}_+^*$, let $\Delta(\mathcal{O}_{\alpha}) = f^*(\mathcal{O}_{\alpha}) \cap \tilde{\mathfrak{t}}_+^*$. Let \mathcal{C} be the cone spanned by the simple roots of \mathfrak{t}^* . Let W and W be the Weyl groups of K and W respectively. Let W be the embedding of the flag variety W and W into the flag variety W which is induced by the map W. We now state the main result from [4]:

Theorem 7.2 Let $(\tilde{\alpha}, \alpha) \in \tilde{\mathfrak{t}}_{+}^{*} \times \mathfrak{t}_{+}^{*}$. Then $\tilde{\alpha} \in \Delta(\mathcal{O}_{\alpha})$ if and only if

$$\tilde{w}^{-1}\tilde{\alpha} \in f^*(w^{-1}\alpha - v\mathcal{C}) \tag{127}$$

for all triples $(\tilde{w}, w, v) \in \tilde{W} \otimes W \otimes W_{rel}$ such that $\phi^*(v\sigma_{wv})(\tilde{c}_{\tilde{w}}) \neq 0$.

(Here $W_{\rm rel}$ is the *relative Weyl set*, defined in [4]. We shall not be concerned with the details of its description; it is equal to $\{1\}$ for our case.) For any $w \in W$, $f^*(w^{-1}\lambda - \mathcal{C})$ is a polyhedral cone in $\tilde{\mathfrak{t}}^*$, so Equation (127) represents a finite number of inequalities. The theorem gives us inequalities whenever the condition $\phi^*(v\sigma_{wv})(\tilde{c}_{\tilde{w}}) \neq 0$ is satisfied, where σ_{wv} is the element of the cohomology of the flag variety labelled by Weyl group element wv, and $\tilde{c}_{\tilde{w}}$ is the element of the homology of the flag variety labelled by \tilde{w} . This is equivalent to the condition that $\tilde{\sigma}_{\tilde{w}}$ appears in $\phi^*(\sigma_w)$, remembering that v=1 for us.

We review some facts about the cohomology of flag varieties of a complex vector space V [12]. Fix a flag F_{\bullet} of V. The cohomology classes σ_w , known as *Schubert classes*, are indexed by elements of S_n , where $n = \dim V$. For $w \in S_n$, σ_w corresponds to the class of the *Schubert variety* X_w , which is the closure of the *Schubert cell*

$$\Omega_w = \{ E_{\bullet} \in \text{Fl}(V) | \dim(E_p \cap F_q) = \# \{ i \le p : w(i) \le q \} \text{ for } 1 \le p, q \le m \}.$$
 (128)

Let us specialize to the case of our problem of finding the spectrum of a partial trace. For this case $f^*(\mathcal{C}) = \tilde{\mathcal{C}}$. If $\tilde{\sigma}_{\tilde{w}}$ appears in $\phi^*(\sigma_w)$, Equation (127) tells us that

$$f^*(w^{-1}\alpha) - \tilde{w}^{-1}\tilde{\alpha} \in \tilde{\mathcal{C}}$$
(129)

for elements of the dual space $\alpha \in \mathfrak{t}_+^*, \tilde{\alpha} \in \tilde{\mathfrak{t}}_+^*$. These functionals α , $\tilde{\alpha}$ act on the spectra λ , $\tilde{\lambda}$; we have

$$(w^{-1}\alpha)(\lambda) = \alpha(w^{-1}(\lambda)) = \alpha(\lambda_{w(1)}, \lambda_{w(2)}, \dots, \lambda_{w(n)}).$$
(130)

Identifying \mathfrak{t} and $\tilde{\mathfrak{t}}$ with their dual spaces, we have the conditions that

$$f^*(\lambda_{w(1)}, \lambda_{w(2)}, \dots, \lambda_{w(d_A d_B)}) - (\lambda_{\tilde{w}(1)}, \lambda_{\tilde{w}(2)}, \dots, \lambda_{\tilde{w}(d_A)}) \in \tilde{\mathcal{C}}$$

$$(131)$$

whenever $\tilde{\sigma}_{\tilde{w}}$ appears in $\phi^*(\sigma_w)$. But the root cone \mathcal{C} is generated by the simple roots $\lambda_1 - \lambda_2, \lambda_2 - \lambda_3, \ldots, \lambda_{d_A-1} - \lambda_{d_A}$ where $\lambda_i \geq \lambda_{i+1}$; in order words, \mathcal{C} is generated by the set of μ such that

$$\sum_{i=1}^{k} \mu_i \ge 0, \text{ for } k < d_A, \tag{132}$$

and

$$\sum_{i=1}^{d_A} \mu_i = 0. (133)$$

So our conditions are that

$$(0,0,\ldots,0) \prec f^*(\lambda_{w(1)},\lambda_{w(2)},\ldots,\lambda_{w(d_Ad_B)}) - (\lambda_{\tilde{w}(1)},\lambda_{\tilde{w}(2)},\ldots,\lambda_{\tilde{w}(d_A)}), \tag{134}$$

or

$$(0,0,...,0) \prec (\lambda_{w(1)} + ... + \lambda_{w(d_B)}, \lambda_{w(d_B+1)} + ... + \lambda_{w(2d_B)}, ..., \lambda_{(w((d_A-1)d_B+1)} + ... + \lambda_{w(d_Ad_B)}) - (\lambda_{\tilde{w}(1)}, \lambda_{\tilde{w}(2)}, ..., \lambda_{\tilde{w}(d_A)}).$$

This is turn yields $(d_A - 1)$ inequalities:

$$\sum_{i=1}^{d_B} \lambda_{w(i)} \leq \tilde{\lambda}_{\tilde{w}(1)}, \tag{135}$$

$$\sum_{i=1}^{2d_B} \lambda_{w(i)} \leq \tilde{\lambda}_{\tilde{w}(1)} + \lambda_{\tilde{w}(2)}, \tag{136}$$

$$\vdots (137)$$

$$\sum_{i=1}^{(d_A-1)d_B} \leq \sum_{i=1}^{d_A-1} \tilde{\lambda}_{\tilde{w}(i)}.$$
(137)

These inequalities arise from intersections of Schubert cells of the flag varieties but any such inequality can be obtained as a consequence of an intersection of Grassmannian Schubert varieties. Choose a flag variety F_{\bullet} of $A \otimes B$ corresponding to the eigenspaces of ρ_{AB} arranged in nonincreasing order of eigenvalues, and a flag variety \tilde{F}_{\bullet} of A corresponding to the eigenspaces of ρ_A arranged in nonincreasing order of eigenvalues. Now define π_w to be the binary string of length $d_A d_B$ such that

$$\pi_w(i) = 1$$
 if $w^{-1}(i) \le kd_B$, $\pi_w(i) = 0$ otherwise.

Similarly, define $\tilde{\pi}_{\tilde{w}}$ to be the binary string of length d_A which takes on the value 1 only at those positions i such that $\tilde{w}^{-1}(i) \leq k$.

Now consider any inequality of the form

$$\sum_{i=1}^{kd_B} \lambda_{w(i)} \le \sum_{i=1}^k \tilde{\lambda}_{\tilde{w}(i)},\tag{139}$$

for some permuations w and \tilde{w} , arising from the intersection of $\Omega_w(F_{\bullet})$ and $\phi(\Omega_{\tilde{w}}(F_{\bullet}))$. Suppose $E_{\bullet} \in \Omega_w(F_{\bullet})$ and $\tilde{E}_{\bullet} \in \Omega_{\tilde{w}}(\tilde{F}_{\bullet})$, such that $\phi(\tilde{E}_{\bullet}) = E_{\bullet}$. Therefore, the subspaces E_{nk} and \tilde{E}_k satisfy $\phi(\tilde{E}_k) = E_{nk}$. Note that $E_{nk} \in \Omega_{\pi_w}(F_{\bullet})$, and $\tilde{E}_k \in \Omega_{\pi_{\tilde{w}}}(\tilde{F}_{\bullet})$, where $\Omega_{\pi_w}(F_{\bullet})$ and $\Omega_{\pi_{\tilde{n}}}(F_{\bullet})$ are Grassmannian Schubert cells. Therefore, we have a nonempty intersection $\Omega_{\pi_w}(F_{\bullet}) \cap \phi(\Omega_{\pi_{\widetilde{w}}}(F_{\bullet})) \neq \emptyset$, which by Theorem 4.4 yields the same inequality

$$\sum_{i=1}^{kd_B} \lambda_{w(i)} \le \sum_{i=1}^k \tilde{\lambda}_{\tilde{w}(i)}. \tag{140}$$

Thus, considering only Grassmannian intersections is enough to derive any inequality of Theorem 7.2 applied to our problem. So the inequalities derived by the approach we have described are indeed sufficient.

The large message limit

Having determined how to find the inequalities relating ρ_{AB} and ρ_{A} , we can seek methods of simplifying the list of inequalities. It turns out that the inequalities governing the relationship

between the spectra of ρ_{AB} and of ρ_A are particularly simple when d_B is large compared to d_A . In this section we will show that if $d_B \geq \frac{1}{2} d_A^2$, then the basic inequalities are sufficient. (All other inequalities are redundant.) Physically, thinking in terms of a quantum communication protocol where Alice sends $\log_2 d_B$ qubits to Bob, such a result is plausible because a large amount of communication gives Alice a great deal of freedom in manipulating her portion of the system, so we should not expect there to be much restriction in the states she might end up with.

Suppose that $d_B \geq \frac{1}{2} d_A^2$, and consider an arbitrary inequality resulting from the nonzero cup product $\tilde{\sigma}_{\nu} \cup \phi^*(\sigma_{\pi}) \neq 0$. (As discussed in Section 7.1, we may assume that $\tilde{\sigma}_{\nu}$ is a summand in the expansion of $\phi^*(\sigma_{\pi}) \neq 0$ as a sum of Schubert classes.) Such an inequality is of the form

$$\sum_{i \in I} \tilde{\lambda}_i \le \sum_{j \in J} \lambda_j \tag{141}$$

where if |I|=k, then $|J|=d_Bk$. As in Section 7.2, we may assume that $k\leq \frac{d_A}{2}$. Consider the partitions π and ν in the equation $\tilde{\sigma}_{\nu}\cup\phi^*(\sigma_{\pi})\neq 0$ to be binary strings. Let u be the (0,1) vector of length d_Ad_B , whose ith component is equal to 1 if and only if $\pi(i)=1$. Similarly, let \tilde{u} be the (0,1) vector of length d_A , whose ith component is equal to 1 if and only if $\nu(i)=1$. Then Inequality (141) can be rewritten as

$$\tilde{\lambda} \cdot \tilde{u} < \lambda \cdot u. \tag{142}$$

We now prove some easy facts about this situation, ending with our desired result.

Observation 7.3 The Young diagram corresponding to π can't have more than $(\frac{d_A}{2})^2$ boxes.

This follows because the Young diagram corresponding to ν must fit in a $k \times (d_A - k)$ rectangle, and so cannot have more than $(\frac{d_A}{2})^2$ boxes; and π must have the same number of boxes in its Young diagram as ν .

Observation 7.4 *If* $u \prec u'$, then $\lambda \cdot u \leq \lambda \cdot u'$.

This follows easily from the fact that λ has its terms arranged in nonincreasing order.

Observation 7.5 If $j > d_B k + (\frac{d_A}{2})^2$, then $j \notin J$ in Inequality (141) (in other words, λ_j is not one of the terms in the right hand sum).

Proof If $j \in J$, then the Young diagram corresponding to π would have more than $(\frac{d_A}{2})^2$ boxes in its jth row.

Observation 7.6 The first zero of u can't appear before the $(d_Bk - \lfloor (\frac{d_A}{2})^2 \rfloor)$ th component. In other words, if $j \leq d_Bk - (\frac{d_A}{2})^2$, then $j \in J$ in Inequality (141).

Proof Otherwise, the Young diagram corresponding to π would have more than $(\frac{d_A}{2})^2$ rows.

Lemma 7.7

$$(\underbrace{1,\ldots,1}_{d_Bk-\lfloor(\frac{d_A}{2})^2\rfloor},\underbrace{0,\ldots,0}_{\lfloor(\frac{d_A}{2})^2\rfloor},\underbrace{1,\ldots,1}_{\lfloor(\frac{d_A}{2})^2\rfloor},0,\ldots,0) \prec u.$$

$$(143)$$

Consequently, since $d_B \geq \frac{d_A^2}{2}$

$$\left(\underbrace{1,\ldots,1}_{d_Bk-\lfloor\frac{d_B}{2}\rfloor},\underbrace{0,\ldots,0}_{\lfloor\frac{d_B}{2}\rfloor},\underbrace{1,\ldots,1}_{\lfloor\frac{d_B}{2}\rfloor},0,\ldots,0\right) \prec u. \tag{144}$$

Proof This follows from Observations 7.5 and 7.6.

Theorem 7.8 If $d_B \ge \frac{1}{2}d_A^2$, then Inequality (142) is redundant. In other words, the basic inequalities are sufficient to characterize the relationship between the spectrum of ρ_{AB} and the spectrum of ρ_A .

Proof It is sufficient to assume that $\tilde{u} \prec (\underbrace{1,\ldots,1}_{k-1},0,1,0,\ldots,0)$ (the only possible \tilde{u} that does not satisfy this condition is $\tilde{u} = (\underbrace{1,\ldots,1}_{k},0,\ldots,0)$, which gives rise to the basic inequalities).

Then we have

$$\begin{split} \tilde{\lambda} \cdot \tilde{u} & \leq \lambda \cdot \underbrace{(1, \dots, 1, 0, 1, 0, \dots, 0)}_{k-1} \\ & = \sum_{i=1}^{k-1} \tilde{\lambda}_i + \tilde{\lambda}_{k+1} \\ & \leq \frac{1}{2} \left[\sum_{i=1}^{k-1} \tilde{\lambda}_i + \tilde{\lambda}_k + \tilde{\lambda}_{k+1} + \sum_{i=1}^{k-1} \tilde{\lambda}_i \right] \\ & = \underbrace{(1, \dots, 1, \frac{1}{2}, \frac{1}{2}, 0, \dots, 0) \cdot \tilde{\lambda}}_{k-1} \\ & = \frac{1}{2} \underbrace{(1, \dots, 1, 0, \dots, 0) \cdot \tilde{\lambda} + \frac{1}{2} \underbrace{(1, \dots, 1, 0, \dots, 0) \cdot \tilde{\lambda}}_{k+1}}_{k+1} \\ & \leq \frac{1}{2} \underbrace{(1, \dots, 1, 0, \dots, 0) \cdot \lambda + \frac{1}{2} \underbrace{(1, \dots, 1, 0, \dots, 0) \cdot \lambda}_{d_B(k-1)}}_{d_B(k-1)} \\ & = \underbrace{(1, \dots, 1, \frac{1}{2}, \dots, \frac{1}{2}, 0, \dots, 0) \cdot \lambda}_{2d_B} \\ & \leq \underbrace{(1, \dots, 1, 0, \dots, 0, 1, \dots, 1, 0, \dots, 0) \cdot \lambda}_{d_B(k-1)} \\ & \leq \underbrace{(1, \dots, 1, 0, \dots, 0, 1, \dots, 1, 0, \dots, 0) \cdot \lambda}_{d_B(k-1)} \\ & \leq \underbrace{(1, \dots, 1, 0, \dots, 0, 1, \dots, 1, 0, \dots, 0) \cdot \lambda}_{d_B(k-1)} \\ \end{split}$$

But the right hand side of Inequality (142) must be greater than equal to

$$(\underbrace{1,\ldots,1}_{d_Bk-\lfloor\frac{d_B}{2}\rfloor},\underbrace{0,\ldots,0}_{\lfloor\frac{d_B}{2}\rfloor},\underbrace{1,\ldots,1}_{\lfloor\frac{d_B}{2}\rfloor},0,\ldots,0)\cdot\lambda,$$

by Lemma 7.7 and Observation 7.4. Thus, we have shown that Inequality (142) must hold, assuming only the basic inequalities; so this inequality must be redundant, for an arbitrary inequality arising from $\tilde{\sigma}_{\nu} \cup \phi^*(\sigma_{\pi}) \neq 0$.

We conjecture a stronger result, which we have verified for $d_A=2,3$, and 4. (The cases $d_A=2$ and $d_A=3$ have been shown explicitly in this paper.)

Conjecture 7.9 If $d_B \ge d_A$, then the basic inequalities are sufficient to characterize the relationship between the spectrum of ρ_{AB} and the spectrum of ρ_A .

8 Conclusion

We have seen that the question of whether a particular quantum state transformation can be accomplished with a given finite amount of communication, classical or quantum, can be answered by testing a set of inequalities determined by a cohomological condition. In the classical communication case, the question essentially reduces to Horn's Problem. We found, however, that there is a simplification in the sense that all matrices can be assumed to be isospectral or, in communication language, all messages equiprobable. The case of state transformations using quantum communication and only unitary local operations was found to be amenable to a similar analysis but the cohomological condition was different. Nonetheless, in the limit that the amount of communication is large relative to the size of the state kept behind, a significant simplification occurred, reducing the complicated set of inequalities to a type of majorization. The techniques presented here, in particular the theorem of Berenstein and Sjamaar, are applicable to wide range of problems in linear algebra. It is our hope that they will find further applications in quantum information theory.

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