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On the Kronecker Product of S_n Characters

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Let χ_ρ, χ_λ, χ_μ be irreducible S_n characters and assume χ_ρ appears in the Kronecker product $\chi_\lambda \otimes \chi_\mu$ with maximal first part ρ_1. Then $\rho_1 = |\lambda \cap \mu| = \sum \min(\lambda_i, \mu_i)$. A similar result holds for the maximal first column. We also give a recursive formula for $\chi_\lambda \otimes \chi_\mu$. As an application, we show that if $n = \lambda_1 + \mu_1 - \rho_1$, then $\langle \chi_\lambda \otimes \chi_\mu, \chi_\rho \rangle_{S_n} = \langle \chi_{(\lambda_2, \lambda_3, \dots)} \hat{\otimes} \chi_{(\mu_2, \mu_3, \dots)}, \chi_{(\rho_2, \rho_3, \dots)} \rangle_{S_{n-\rho_1}}$ where $\hat{\otimes}$ denotes the outer tensor product. These results are applied to study the character $\sum \chi_\lambda \otimes \chi_\lambda$ where λ runs through the partitions with no more then k parts. This character is closely related to the polynomial identities of the algebra of $k \times k$ matrices.

0. INTRODUCTION

Throughout this paper the base field is $\mathbb{C}$. Let S_n denote the symmetric group. The partition $\lambda = (\lambda_1, \lambda_2, \dots)$ of n ($\lambda \vdash n$) corresponds to the irreducible S_n character $\lambda \leftrightarrow \chi_\lambda$. Given λ, $\mu \vdash n$ then $\langle \chi_\mu, \chi_\lambda \rangle_{S_n} = \delta_{\mu\lambda}$ defines an inner product: $\langle \chi, \chi_\lambda \rangle_{S_n}$ is the coefficient of χ_λ in the S_n character χ. Let λ' denote the conjugate partition of λ. Given an S_n character χ, we have

$$\chi = \sum_{\lambda \vdash n} \langle \chi, \chi_\lambda \rangle_{S_n} \cdot \chi_\lambda.$$

Define (the conjugate character)

$$\chi' = \sum_{\lambda \vdash n} \langle \chi, \chi_{\lambda'} \rangle_{S_n} \cdot \chi_\lambda.$$

Define the width of χ as follows: $w(\chi_\lambda) = w(\lambda) = w(\lambda_1, \lambda_2, \dots) = \lambda_1$, and

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$w(\chi) = \max\{w(\lambda) \mid \langle \chi, \chi_\lambda \rangle_{S_n} \neq 0\}$. Define the height $h(\chi) = w(\chi')$. Obviously

$$h(\chi) = \max\{w(\lambda') \mid \langle \chi, \chi_\lambda \rangle_{S_n} \neq 0\}.$$

Let $\chi_\lambda \otimes \chi_\mu$ be the inner (Kronecker) products, and let

$$\chi_\lambda \otimes \chi_\mu = \sum_{\rho \vdash n} c(\lambda, \mu, \rho)\, \chi_\rho \qquad (\text{i.e., } c(\lambda, \mu, \rho) = \langle \chi_\lambda \otimes \chi_\mu, \chi_\rho \rangle_{S_n})$$

denote the decomposition of the inner product as a sum of irreducible characters. The known algorithms for calculating (i.e., decomposing) $\chi_\lambda \otimes \chi_\mu$ [Gar-Rem; Jam-Ker, 2.9] are rather ineffective in the general case. The problem of finding effective ways for calculating, and even estimating $\chi_\lambda \otimes \chi_\mu$, is, probably, the last major problem in the ordinary representation theory of S_n.

The inequality $h(\chi_\lambda \otimes \chi_\mu) \leqslant h(\chi_\lambda) \cdot h(\chi_\mu)$ was proved in [Reg 1], and was crucial for the results there. The main result here is a precise formula which expresses $h(\chi_\lambda \otimes \chi_\mu)$ as the area of the intersection of the young diagrams of λ and of μ', the conjugate of μ. This is

THEOREM (1.6 below). *Write* $\lambda \cap \mu = (\min(\lambda_1, \mu_1), \min(\lambda_2, \mu_2), ...)$, *so* $|\lambda \cap \mu|$ *is the area of the intersection of the diagrams of* λ *and of* μ. *Then*

(a) $w(\chi_\lambda \otimes \chi_\mu) = |\lambda \cap \mu|$

(b) $h(\chi_\lambda \otimes \chi_\mu) = |\lambda \cap \mu'|$.

Note that $|\lambda \cap \mu'| \leqslant \lambda_1' \cdot \mu_1' = h(\lambda) \cdot h(\mu)$. Thus 1.6(b) here replaces the estimate in [Reg 1] (1.7 here) by a precise formula. In 1.8 we extend part of 1.6 to skew characters.

In Section 2 we establish a recursive formula, of interest in its own, for computing $\langle \chi_\lambda \otimes \chi_\mu, \chi_\rho \rangle_{S_n}$. The recursion in that formula is on n and on $n - \rho_1$. In the special case that $\rho_1 = |\lambda \cap \mu|$ we obtain

THEOREM (2.4 below). *Given partitions* α, β, *let* α/β *denote the skew partition with corresponding skew character* $\chi_{\alpha/\beta}$ [Mac]. *Let* $\lambda, \mu, \rho \vdash n$, $\rho_1 = |\lambda \cap \mu|$. *Then*

$$\langle \chi_\lambda \otimes \chi_\mu, \chi_\rho \rangle_{S_n} = \langle \chi_{\lambda/\lambda \cap \mu} \otimes \chi_{\mu/\lambda \cap \mu}, \chi_{(\rho_2, \rho_3, ...)} \rangle_{S_{n-\rho_1}}.$$

In section 3 we apply these results to decompose $\chi_\lambda \otimes \chi_\mu$ in some special cases. Let $\lambda = (\lambda_1, \lambda_2, ...)$. Then denote $\theta(\lambda) = (\lambda_2, \lambda_3, ...)$ and $d(\lambda) = |\theta(\lambda)| = n - \lambda_1$. Let $\alpha \vdash n$, $\beta \vdash m$. Then $\chi_\alpha \hat{\otimes} \chi_\beta$ denotes the outer product of χ_α and χ_β. The decomposition of $\chi_\alpha \hat{\otimes} \chi_\beta$ is given by the Littlewood–Richardson Rule [Jam-Ker]. Our main result in Section 3 is

THEOREM (3.3 below). *Given λ, μ, ρ partitions of n such that $d(\lambda)+d(\mu)=d(\rho)$ (or equivalently $n=\lambda_1+\mu_1-\rho_1$) then*

$$\langle \chi_\lambda \otimes \chi_\mu, \chi_\rho \rangle_{S_n} = \langle \chi_{\theta(\lambda)} \hat{\otimes} \chi_{\theta(\mu)}, \chi_{\theta(\rho)} \rangle_{S_{d(\rho)}}.$$

In Section 4 we apply the above results to study the character

$$\psi(k, n) = \sum_{\lambda \in H(k, 0; n)} \chi_\lambda \otimes \chi_\lambda$$

and its restrictions

$$\left(\sum_{\lambda \in H(k, 0; n)} \chi_\lambda \otimes \chi_\lambda \right)_{S_m},$$

where $m \leqslant n$ and

$$H(k, 0; n) = \{\lambda \vdash n \mid \lambda_{k+1} = 0\}.$$

When $m=n$ and $m=n-1$, the corresponding characters describe the trace identities of the $k \times k$ matrices, which are closely related to the polynomial identities of such matrices. Write:

$$\left(\sum_{\lambda \in H(k, 0; n)} \chi_\lambda \otimes \chi_\lambda \right)_{S_m} = \sum_{\rho \in H(k^2, 0; m)} a(\rho, k, n)\, \chi_\rho.$$

The calculation of the coefficients $a(\rho, k, n)$ when $m=n$ and $m=n-1$ is a major open problem in the theory of polynomial identities. We show (4.3 below) that if $\rho=(\rho_1, ..., \rho_{k^2})$ is a partition such that $|\rho| \leqslant n$ and $0 \leqslant t \leqslant \rho_{k^2}$ then

$$a((\rho_1, ..., \rho_{k^2}), k, n) = a((\rho_1 - t, ..., \rho_{k^2} - t, 0), k, n - k^2 t),$$

i.e., $k^2 \times t$ rectangles can be removed.

In the special case that $\rho = \phi$ is a $k^2 \times t$ rectangle

$$\phi = \underbrace{(t, t..., t)}_{k^2}$$

(and also for

$$\phi = (t+1, \underbrace{t, ..., t}_{k^2-1}))$$

we have the following intriguing formula (4.5 below):

$$a(\phi, k, n) = \sum_{\lambda \in H(k, 0; n - k^2 t)} (\deg \chi_\lambda)^2.$$

Note that

$$\deg \psi(k,n) = \sum_{\lambda \in H(k,0;n)} (\deg \chi_\lambda)^2.$$

The asymptotics and the generating functions of these sums were studied in [Reg 2, Bec-Reg].

1. The Width and Height of $\chi_\lambda \otimes \chi_\mu$

All groups here are finite. In addition, the following definitions and notations will hold.

Let $F(G)$ be the ring of class functions of G into $\mathbb{C}$, $CH(G)$ the subring generated by the characters of G, $\mathrm{Irr}(G)$ the irreducible characters.

Denote by $\langle\ ,\ \rangle_G : F(G) \times F(G) \to \mathbb{C}$ the natural bilinear form defined by $\langle f_1, f_2 \rangle_G = \delta_{f_1, f_2}$ for all $f_1, f_2 \in \mathrm{Irr}(G)$.

Let H, G be finite groups, $H \subseteq G$. For any $f \in F(H)$, f^G will denote the induced class function and for any $f \in F(G)$, f_H will denote the restricted (to H) class function.

For groups G_1, G_2, $f_i \in F(G_i)$, define $f_1 \times f_2 \in F(G_1 \times G_2)$ by $(f_1 \times f_2)(g_1, g_2) = f_1(g_1) \cdot f_2(g_2)$. We denote the inner (or the Kronecker) product of $f_1, f_2 \in F(G)$ by $f_1 \otimes f_2$. Denote by $f \hat{\otimes} g$ the outer product of $f \in F(S_m)$ and $g \in F(S_n)$ [Mac].

Following [Mac], define the skew character $\chi_{\lambda/\mu}$ of the skew tableau λ/μ as follows: let $n, m \in \mathbb{N} \cup \{0\}$, $\lambda \vdash n$, and $\mu \vdash m$; then $\chi_{\lambda/\mu} = \sum_{\rho \vdash n-m} \langle \chi_\lambda, \chi_\mu \hat{\otimes} \chi_\rho \rangle_{S_n} \chi_\rho$.

Throughout we will assume S_0 is the group of one element, with $\chi_{(0)} = \chi_{(0,0,\ldots)}$ the unique trivial character of S_0. In this case, for any $\lambda \vdash n$, $\chi_\lambda \hat{\otimes} \chi_{(0)} = \chi_\lambda$ (by definition).

Definition 1.1. Let $\lambda \vdash n$, $\mu \vdash m$, $n \geqslant m$, and $S_m \subseteq S_n$ in the natural embedding. Define $k(\lambda, \mu) = \langle \chi_\lambda, (\chi_\mu)^{S_n} \rangle_{S_n}$. By Frobenius Reciprocity:

$$k(\lambda, \mu) = \langle (\chi_\lambda)_{S_m}, \chi_\mu \rangle_{S_m}.$$

Remarks 1.2. Let $n \geqslant m$, $\lambda \vdash n$, $\mu \vdash m$.

(a) It is well known that $k(\lambda, \mu) > 0$ if and only if $\mu \subseteq \lambda$ [Mac, 5.7, 5.12]. Actually, $k(\lambda, \mu)$ is the degree of $\chi_{\lambda/\mu}$.

In later sections we shall also need:

(b) Let $\alpha \vdash m - m$ and $\langle \chi_\alpha \hat{\otimes} \chi_\mu, \chi_\lambda \rangle_{S_n} \neq 0$. Then $\alpha, \mu \subseteq \lambda$ (by the Littlewood–Richardson Rule).

(c) $(\chi_\lambda)_{S_m \times S_{n-m}} = \sum_{\theta \vdash m} \chi_\theta \times \chi_{\lambda/\theta}$ [Jam-Ker, 2.3.12].

Clearly we can also write:

$$(1.3)\qquad (\chi_\lambda)_{S_m} = \sum_{\substack{\alpha \vdash m \\ \alpha \subseteq \lambda}} k(\lambda, \alpha)\, \chi_\alpha$$

$$(1.4)\qquad (\chi_\mu)^{S_n} = \sum_{\substack{\beta \vdash n \\ \beta \supseteq \mu}} k(\beta, \mu)\, \chi_\beta.$$

PROPOSITION 1.5. *Let $(m) \vdash m$, $\lambda, \mu \vdash n$, and $m \leqslant n$. Then*:

$$\sum_{\substack{\beta \vdash n \\ \beta \supseteq (m)}} k(\beta, (m)) \langle \chi_\beta, \chi_\lambda \otimes \chi_\mu \rangle_{S_n} = \sum_{\substack{\alpha \vdash m \\ \alpha \subseteq \lambda \cap \mu}} k(\lambda, \alpha)\, k(\mu, \alpha).$$

Proof. Let

$$p = \langle (\chi_{(m)})^{S_n}, \chi_\lambda \otimes \chi_\mu \rangle_{S_n}$$

and

$$q = \langle \chi_{(m)}, (\chi_\lambda)_{S_m} \otimes (\chi_\mu)_{S_m} \rangle_{S_m}.$$

By Frobenius Reciprocity (and $(\chi_\lambda \otimes \chi_\mu)_{S_m} = (\chi_\lambda)_{S_m} \otimes (\chi_\mu)_{S_m}$) $p = q$. By (1.3)

$$q = \left\langle \chi_{(m)}, \left(\sum_{\substack{\alpha \vdash m \\ \alpha \subseteq \lambda}} k(\lambda, \alpha)\, \chi_\alpha \right) \otimes \left(\sum_{\substack{\gamma \vdash m \\ \gamma \subseteq \mu}} k(\mu, \gamma)\, \chi_\gamma \right) \right\rangle_{S_m},$$

and by (1.4)

$$p = \left\langle \sum_{\substack{\beta \vdash n \\ \beta \supseteq (m)}} k(\beta, (m))\, \chi_\beta, \chi_\lambda \otimes \chi_\mu \right\rangle_{S_n}.$$

It easily follows from basic properties of character tables that for any $\alpha, \gamma \vdash m$, $\langle \chi_\alpha \otimes \chi_\gamma, \chi_{(m)} \rangle_{S_m} = \delta_{\alpha\gamma}$. Thus

$$q = \sum_{\substack{\alpha \vdash m \\ a \subseteq \lambda \cap \mu}} k(\lambda, \alpha)\, k(\mu, \alpha)$$

and the statement follows. ∎

The main result in this section is (recall $w(\chi)$, $h(\chi)$, and $\lambda \cap \mu$ from Sect. 0):

THEOREM 1.6. *For any partitions λ, μ of n the following holds:*

(a) $w(\chi_\lambda \otimes \chi_\mu) = |\lambda \cap \mu|$

(b) $h(\chi_\lambda \otimes \chi_\mu) = |\lambda \cap \mu'|$.

Proof. (a). We show first that $w(\chi_\lambda \otimes \chi_\mu) \leqslant |\lambda \cap \mu|$. In order to do so, apply 1.5 with $|\lambda \cap \mu| = m - 1$,

$$\sum_{\substack{\beta \vdash n \\ \beta \supseteq (|\lambda \cap \mu| + 1)}} k(\beta, |\lambda \cap \mu| + 1)) \langle \chi_\lambda \otimes \chi_\mu, \chi_\beta \rangle_{S_n}$$

$$= \sum_{\substack{\alpha \vdash |\lambda \cap \mu| + 1 \\ \alpha \subseteq \lambda \cap \mu}} k(\lambda, \alpha)\, k(\mu, \alpha) = 0,$$

since clearly, $\{\alpha \mid \alpha \vdash |\lambda \cap \mu| + 1 \text{ and } \alpha \subseteq \lambda \cap \mu\}$ is empty.

For all $\beta = (\beta_1, \beta_2, ...)$ such that $(|\lambda \cap \mu| + 1) \subseteq \beta$ (i.e., $\beta_1 \geqslant |\lambda \cap \mu| + 1$), $k(\beta, (|\lambda \cap \mu| + 1)) > 0$ (Remark 1.2(a)); hence $\langle \chi_\lambda \otimes \chi_\mu, \chi_\beta \rangle_{S_n} = 0$. By the definition of w, this implies: $w(\chi_\lambda \otimes \chi_\mu) \leqslant |\lambda \cap \mu|$.

Next we show that $w(\chi_\lambda \otimes \chi_\mu) \geqslant |\lambda \cap \mu|$.

Again apply 1.5, with $m = |\lambda \cap \mu|$:

$$\sum_{\substack{\beta \vdash n \\ \beta \supseteq (|\lambda \cap \mu|)}} k(\beta, (|\lambda \cap \mu|)) \langle \chi_\lambda \otimes \chi_\mu, \chi_\beta \rangle_{S_n}$$

$$= \sum_{\substack{\alpha \vdash |\lambda \cap \mu| \\ \alpha \subseteq \lambda \cap \mu}} k(\lambda, \alpha)\, k(\mu, \alpha) = k(\lambda, \lambda \cap \mu)\, k(\mu, \lambda \cap \mu) > 0.$$

Hence there exists $\beta = (\beta_1, \beta_2, ...) \vdash n$ with $\beta_1 \geqslant |\lambda \cap \mu|$ (i.e., $w(\beta) \geqslant |\lambda \cap \mu|$) such that $\langle \chi_\beta, \chi_\lambda \otimes \chi_\mu \rangle_{S_n} \neq 0$ so:

$$w(\chi_\lambda \otimes \chi_\mu) \geqslant |\lambda \cap \mu|.$$

The proof of (a) is now complete.

(b) Since $\chi_\lambda \otimes \chi_{\mu'} = (\chi_\lambda \otimes \chi_\mu)'$, therefore

$$h(\chi_\lambda \otimes \chi_\mu) = w((\chi_\lambda \otimes \chi_\mu)') = w(\chi_\lambda \otimes \chi_{\mu'}) = |\lambda \cap \mu'| \qquad \text{(by 1.6(a)).} \blacksquare$$

COROLLARY 1.7 [Reg 1]. *Let $\lambda, \mu \vdash n$, $h(\lambda) = k$, and $h(\mu) = l$. Then*

$$h(\chi_\lambda \otimes \chi_\mu) \leqslant kl.$$

Proof. Obviously, $|\lambda \cap \mu'| \leqslant kl$, so 1.7 follows from 1.6(b). $\blacksquare$

Let χ_α be an irreducible character appearing in the skew character $\chi_{\lambda/\eta}$. Then by definition of skew characters and 1.2(b), $\alpha \subseteq \lambda$. Thus, we obviously have the following corollary, which extends part of 1.6 to skew characters:

COROLLARY 1.8. *Let* $m_1, m_2, l \geqslant 0$, $n_1 = m_1 + l$, $n_2 = m_2 + l$, $\lambda \vdash n_1$, $\eta \vdash m_1$, $\mu \vdash n_2$ *and* $\xi \vdash m_2$. *Then*

(a) $w(\chi_{\lambda/\eta} \otimes \chi_{\mu/\xi}) \leqslant |\lambda \cap \mu|$.

(b) $h(\chi_{\lambda/\eta} \otimes \chi_{\mu/\xi}) \leqslant |\lambda \cap \mu'|$.

2. A RECURSIVE FORMULA FOR $\chi_\lambda \otimes \chi_\mu$

Theorems 2.3, 2.4, and 2.4′ below give recursive formulas for the coefficients $c(\lambda, \mu, \rho)$ (see Sect. 0). A major tool here is the well-known Young's Rule [Jam-Ker].

PROPOSITION 2.1. *Let* $m, l \geqslant 0$ and $\theta \vdash l$. Define

$$Y(\theta, m) = \{\eta \mid \eta \vdash m + l, \eta_i \geqslant \theta_i \geqslant \eta_{i+1} \textit{ for all } i \geqslant 1\}.$$

Then

(a) *Young's Rule can be written as follows*: $\chi_{(m)} \hat{\otimes} \chi_\theta = \sum_{\eta \in Y(\theta, m)} \chi_\eta$.

(b) $$\langle \chi_{(m)} \times \chi_\theta, (\chi_\gamma)_{S_m \times S_l} \rangle_{S_m \times S_l} = \begin{cases} 1, & \gamma \in Y(\theta, m) \\ 0, & \textit{otherwise}. \end{cases}$$

(c) *Let* $m \geqslant \theta_1$, *and define* $\rho \vdash m + l$ *as follows*: $\rho_1 = m$, $\rho_i = \theta_{i-1}$ $(i \geqslant 2)$; *then* $\rho \in Y(\theta, m)$. *Also, if* $\eta \in Y(\theta, m)$ *and* $\eta \neq \rho$ *then* $\eta_1 > m$.

Proof. (a) See [Jam-Ker, 2.3.14].

(b) Applying Frobenius Reciprocity, we have

$$\begin{aligned} \langle \chi_{(m)} \times \chi_\theta, (\chi_\gamma)_{S_m \times S_l} \rangle_{S_m \times S_l} &= \langle (\chi_{(m)} \times \chi_\theta)^{S_{m+l}}, \chi_\gamma \rangle_{S_{m+l}} \\ &= \langle \chi_{(m)} \hat{\otimes} \chi_\theta, \chi_\gamma \rangle_{S_{m+l}} \end{aligned}$$

(Young's Rule)

$$= \left\langle \sum_{\eta \in Y(\theta, m)} \chi_\eta, \chi_\gamma \right\rangle_{S_{m+l}} = \begin{cases} 1, & \gamma \in Y(\times, m); \\ 0, & \text{otherwise}. \end{cases}$$

(c) Obviously, $\rho \in Y(\theta, m)$. Let $\eta \in Y(\theta, m)$, $\eta \neq \rho$, and assume $\eta_1 \leqslant m$. Since $\eta \in Y(\theta, m)$, therefore $\eta_i \leqslant \theta_{i-1} = \rho_i$ $(i \geqslant 2)$, by the definition of $Y(\theta, m)$. Thus $\eta_i \leqslant \rho_i$ $(i \geqslant 1)$. But $\eta, \rho \vdash m + l$; hence $\eta = \rho$, contradiction. ∎.

Recall the coefficients $c(\lambda, \mu, \rho)$ (see Sect. 0). We shall need the following technical

PROPOSITION 2.2. *Let* $m, l \geqslant 0$, $\lambda, \mu \vdash m+l$, $\theta \vdash l$. *Then*

(a) $\langle (\chi_\lambda \otimes \chi_\mu)_{S_m \times S_l}, \chi(m) \times \chi_\theta \rangle_{S_m \times S_l} = \sum_{\rho \in Y(\theta, m)} c(\lambda, \mu, \rho)$, *and also*

(b) $\langle (\chi_\lambda \otimes \chi_\mu)_{S_m \times S_l}, \chi_{(m)} \times \chi_\theta \rangle_{S_m \times S_l} = \sum_{\alpha \vdash m} \langle \chi_{\lambda/\alpha} \otimes \chi_{\mu/\alpha}, \chi_\theta \rangle_{S_l}$.

Proof. (a)

$$\langle (\chi_\lambda \otimes \chi_\mu)_{S_m \times S_l}, \chi_{(m)} \times \chi_\theta \rangle_{S_m \times S_l}$$

$$= \left\langle \left(\sum_{\rho \vdash m+l} c(\lambda, \mu, \rho)\, \chi_\rho \right)_{S_m \times S_l}, \chi_{(m)} \times \chi_\theta \right\rangle_{S_m \times S_l} \qquad \text{(by linearity)}$$

$$= \sum_{\rho \vdash m+l} c(\lambda, \mu, \rho) \langle (\chi_\rho)_{S_m \times S_l}, \chi_{(m)} \times \chi_\theta \rangle_{S_m \times S_l} \qquad \text{(by 2.1(b))}$$

$$= \sum_{\rho \in Y(\theta, m)} c(\lambda, \mu, \rho).$$

(b) The following is well known: Let H, G be finite groups, $H \subseteq G$, and $f_1, f_2 \in F(G)$. Then $(f_1 \otimes f_2)_H = f_{1_H} \otimes f_{2_H}$. Therefore

$$\langle (\chi_\lambda \otimes \chi_\mu)_{S_m \times S_l}, \chi_{(m)} \times \chi_\theta \rangle_{S_m \times S_l}$$

$$= \langle (\chi_\lambda)_{S_m \times S_l} \otimes (\chi_\mu)_{S_m \times S_l}, \chi_{(m)} \times \chi_\theta \rangle_{S_m \times S_l} \qquad \text{(by 1.2(c))}$$

$$= \left\langle \left(\sum_{\alpha \vdash m} \chi_\alpha \times \chi_{\lambda/\alpha} \right) \otimes \left(\sum_{\beta \vdash m} \chi_\beta \times \chi_{\mu/\beta} \right), \chi_{(m)} \times \chi_\theta \right\rangle_{S_m \times S_l}$$

(by linearity)

$$= \sum_{\alpha, \beta \vdash m} \langle (\chi_\alpha \times \chi_{\lambda/\alpha}) \otimes (\chi_\beta \times \chi_{\mu/\beta}), \chi_{(m)} \times \chi_\theta \rangle_{S_m \times S_l} = D.$$

For any finite groups H, G, if $f_1, f_2 \in F(H)$, $g_1, g_2 \in F(G)$ then $(f_1 \times g_1) \otimes (f_2 \times g_2) = (f_1 \otimes f_2) \times (g_1 \otimes g_2)$. Also, $\langle f_1 \times g_1, f_2 \times g_2 \rangle_{H \times G} = \langle f_1, f_2 \rangle_H \cdot \langle g_1, g_2 \rangle_G$. Thus

$$D = \sum_{\alpha, \beta \vdash m} \langle (\chi_\alpha \otimes \chi_\beta) \times (\chi_{\lambda/\alpha} \otimes \chi_{\mu/\beta}), \chi_{(m)} \times \chi_\theta \rangle_{S_m \times S_l}$$

$$= \sum_{\alpha, \beta \vdash m} \langle \chi_\alpha \otimes \chi_\beta, \chi_{(m)} \rangle_{S_m} \langle \chi_{\lambda/\alpha} \otimes \chi_{\mu/\beta}, \chi_\theta \rangle_{S_l}.$$

Note that $\chi_{(m)}$ is the identity character of S_m; hence $\langle \chi_\alpha \otimes \chi_\beta, \chi_{(m)} \rangle = \delta_{\alpha, \beta}$. Thus

$$D = \langle (\chi_\lambda \otimes \chi_\mu)_{S_m \times S_l}, \chi_{(m)} \otimes \chi_\theta \rangle_{S_m \times S_l}$$

$$= \sum_{\alpha \vdash m} \langle \chi_{\lambda/\alpha} \otimes \chi_{\mu/\alpha}, \chi_\theta \rangle_{S_l}. \quad \blacksquare$$

We can now give a recursion formula for the coefficients $c(\lambda, \mu, \rho)$.

THEOREM 2.3. *Let $\lambda, \mu, \rho \vdash m+l=n$, such that $m=\rho_1$. Define $\theta \vdash l$ as follows: $\theta_i=\rho_{i+1}$, $i \geqslant 1$. Then*

$$c(\lambda, \mu, \rho)=\sum_{\substack{\alpha \vdash m \\ \alpha \subseteq \lambda \cap \mu}} \langle \chi_{\lambda/\alpha} \otimes \chi_{\mu/\alpha}, \chi_\theta \rangle_{S_l} - \sum_{\substack{\eta \in Y(\theta, m) \\ \eta \neq \rho \\ \eta_1 \leqslant |\lambda \cap \mu|}} c(\lambda, \mu, \eta).$$

Note that this is indeed a recursive formula: the skew characters can be decomposed into irreducible ones of S_l [Gar-Rem, Mac]; thus the problem (in the first summand) is reduced to smaller symmetric groups. Also, in the second summand, $\eta_1>\rho_1$ (by 2.1.c); hence $n-\eta_1<n-\rho_1$ and the recursion is on the invariant $I(\lambda, \mu, \rho)=n-\rho_1$ (note that when $I(\lambda, \mu, \rho)=0$, $\rho=(n)$, and $c(\lambda, \mu,(n))=\delta_{\lambda\mu}$).

Proof. Denote $L=\sum_{\eta \in Y(\theta, m)} c(\lambda, \mu, \eta)$ and $R=\sum_{\alpha \vdash m} \langle \chi_{\lambda/\alpha} \otimes \chi_{\mu/\alpha}, \chi_\theta \rangle_{S_l}$. By 1.6,

$$L=\sum_{\substack{\eta \in Y(\theta, m) \\ \eta_1 \leqslant |\lambda \cap \mu|}} c(\lambda, \mu, \eta)$$

and it easily follows from 1.2(a) that

$$R=\sum_{\substack{\alpha \vdash m \\ \alpha \subseteq \lambda \cap \mu}} \langle \chi_{\lambda/\alpha} \otimes \chi_{\mu/\alpha}, \chi_\theta \rangle_{S_l}.$$

By 2.2 $L=R$; thus

$$\sum_{\substack{\eta \in Y(\theta, m) \\ \eta_1 \leqslant |\lambda \cap \mu|}} c(\lambda, \mu, \eta)=\sum_{\substack{\alpha \vdash m \\ \alpha \subseteq \lambda \cap \mu}} \langle \chi_{\lambda/\alpha} \otimes \chi_{\mu/\alpha}, \chi_\theta \rangle_{S_l}.$$

By the definition of $Y(\theta, m)$ it follows that $\rho \in Y(\theta, m)$, and the formula follows. ∎

In the case $\rho_1=|\lambda \cap \mu|$ the formula is particularly efficient:

THEOREM 2.4. *Let $\lambda, \mu, \rho \vdash m+l$, $\rho_1=|\lambda \cap \mu|=m$, and define $\theta=\theta(\rho) \vdash l$ by $\theta_i=\rho_{i+1}$, $i \geqslant 1$. Then*

$$c(\lambda, \mu, \rho)=\langle \chi_{\lambda/\lambda \cap \mu} \otimes \chi_{\mu/\lambda \cap \mu}, \chi_\theta \rangle_{S_l}.$$

Proof. If $\alpha \vdash |\lambda \cap \mu|$ and $\alpha \subseteq \lambda \cap \mu$ then $\alpha=\lambda \cap \mu$. For any $\eta \in Y(\theta, m)$ with $\eta \neq \rho$, $\eta_1>\rho_1=|\lambda \cap \mu|$ (by 2.1(c)) and therefore $c(\lambda, \mu, \eta)=0$ (by 1.6(a)). The proof now follows from 2.3. ∎

There is a conjugate to 2.4:

THEOREM 2.4′. *Let* $\lambda, \mu, \rho \vdash m+l$, $h(\rho)=|\lambda\cap\mu'|=m$, *and define* $\zeta=\zeta(\rho)\vdash n$ *by* $\zeta_i=\rho_i-1$, $i\geqslant 1$, *and* $\rho_i\geqslant 1$. *Then*

$$c(\lambda,\mu,\rho)=\langle\chi_{\lambda/\lambda\cap\mu'}\otimes\chi_{\mu/\lambda'\cap\mu},\chi_\zeta\rangle_{S_l}.$$

Proof. We make use of the following well-known properties of conjugation:

Let χ, ψ be S_n characters, then $\langle\chi,\psi\rangle_{S_n}=\langle\chi',\psi'\rangle_{S_n}$ and $\chi\otimes\psi'=(\chi\otimes\psi)'$. Thus $c(\lambda,\mu,\rho)=c(\lambda,\mu',\rho')$. Finally let $\alpha\vdash m\leqslant n$. Then $(\chi_{\lambda/\alpha})'=\chi_{\lambda'/\alpha'}$. Hence

$$\begin{aligned}
c(\lambda,\mu,\rho)&=c(\lambda,\mu',\rho') \qquad \text{(by 2.4)}\\
&=\langle\chi_{\lambda/\lambda\cap\mu'}\otimes\chi_{\mu'/\lambda\cap\mu'},\chi_{\zeta'}\rangle_{S_{n-m}}\\
&=\langle(\chi_{\lambda/\lambda\cap\mu'}\otimes\chi_{\mu'/\lambda\cap\mu'})',\chi_\zeta\rangle_{S_{n-m}}\\
&=\langle\chi_{\lambda/\lambda\cap\mu'}\otimes(\chi_{\mu'/\lambda\cap\mu'})',\chi_\zeta\rangle_{S_{n-m}}\\
&=\langle\chi_{\lambda/\lambda\cap\mu'}\otimes\chi_{\mu/\lambda'\cap\mu},\chi_\zeta\rangle_{S_{n-m}}. \quad ∎
\end{aligned}$$

3. APPLICATIONS: EVALUATING $\langle\chi_\lambda\otimes\chi_\mu,\chi_\rho\rangle_{S_n}$ WHEN $d(\lambda)+d(\mu)=d(\rho)$

Let $\alpha=(\alpha_1,\alpha_2,...)$ be a partition, and define $d(\alpha)=\sum_{i\geqslant 2}\alpha_i=|\alpha|-\alpha_1$. In this section we consider triples of partitions $\lambda,\mu,\rho\vdash n$ for which

$$d(\lambda)+d(\mu)=d(\rho).$$

For such partitions we reduce the calculation of $c(\lambda,\mu,\rho)$ to that of outer products (given by the Littlewood–Richardson rule). This is 3.3 below.

Given $\alpha\vdash n$ we denote $\theta(\alpha)=(\alpha_2,\alpha_3,...)$: $\theta(\alpha)\vdash d(\alpha)$.

The proof of 3.3 requires the following two lemmas.

LEMMA 3.1. *Let* $\lambda,\mu,\rho\vdash n$ *be any partitions. Then*

(a) $n-|\lambda_1-\rho_1|\geqslant|\lambda\cap\rho|$.

(b) *If in addition* $d(\lambda)+d(\mu)=d(\rho)$ *then*

(b.1) $\mu_1=n-\lambda_1+\rho_1$.

(b.2) $\lambda_1\geqslant\rho_1$.

(b.3) $\mu_1=|\lambda\cap\rho|$ *if and only if* $\theta(\lambda)\subseteq\theta(\rho)$.

Proof. (a) Without loss of generality we may assume that $\lambda_1 \geqslant \rho_1$. Now $n = \lambda_1 + \lambda_2 + \ldots$; hence

$$\begin{aligned} n - |\lambda_1 - \rho_1| &= \rho_1 + \lambda_2 + \lambda_3 + \cdots \\ &\geqslant \min(\rho_1, \lambda_1) + \min(\rho_2, \lambda_2) + \cdots = |\lambda \cap \rho|. \end{aligned}$$

(b.1) By assumption $d(\lambda) + d(\mu) = d(\rho)$, so $(n - \lambda_1) + (n - \mu_1) = (n - \rho_1)$; hence $\mu_1 = n - \lambda_1 + \rho_1$.

(b.2) Since $\mu_1 = n - \lambda_1 + \rho_1$ and $n \geqslant \mu_1$, therefore $\lambda_1 \geqslant \rho_1$.

(b.3) Assume first that $\mu_1 = |\lambda \cap \rho|$, so (b.1) $|\lambda \cap \rho| = n - \lambda_1 + \rho_1 = \rho_1 + \sum_{i \geqslant 2} \lambda_i$. Also $|\lambda \cap \rho| = \sum_{i \geqslant 1} \min(\lambda_i, \rho_i) = \rho_1 + \sum_{i \geqslant 2} \min(\lambda_i, \rho_i)$ (by (b.2)). Thus $\sum_{i \geqslant 2} \lambda_i = \sum_{i \geqslant 2} \min(\lambda_i, \rho_i)$: hence $\theta(\lambda) \subseteq \theta(\rho)$.

Next, assume $\theta(\lambda) \subseteq \theta(\rho)$, or equivalently, $\lambda_i \leqslant \rho_i$ $(i \geqslant 2)$. By (b.2) $\lambda_1 \geqslant \rho_1$, so $|\lambda \cap \rho| = \sum_{i \geqslant 1} \min(\lambda_i, \rho_i) = \rho_1 + \sum_{i \geqslant 2} \lambda_i = \rho_1 + n - \lambda_1 = \mu_1$ (b.1). ∎

LEMMA 3.2. *As in* 3.1(b), *let* $\lambda, \mu, \rho \vdash n$ *with* $d(\lambda) + d(\mu) = d(\rho)$, *and assume in addition that* $\langle \chi_\lambda \otimes \chi_\mu, \chi_\rho \rangle_{S_n} \neq 0$. *Then*

$$\mu_1 = |\lambda \cap \rho|.$$

Proof. Since $\langle \chi_\lambda \otimes \chi_\rho, \chi_\mu \rangle_{S_n} = \langle \chi_\lambda \otimes \chi_\mu, \chi_\rho \rangle_{S_n} \neq 0$, by 1.6(a)

$$\begin{aligned} \mu_1 &\leqslant |\lambda \cap \rho| \\ &\leqslant n - |\lambda_1 - \rho_1| && \text{(by 3.1(a))} \\ &= n - \lambda_1 + \rho_1 && \text{(by 3.1(b.2))} \\ &= \mu_1 && \text{(by 3.1(b.1))}. \end{aligned}$$

Thus $\mu_1 = |\lambda \cap \rho|$. ∎

THEOREM 3.3. *Let* $\lambda, \mu, \rho \vdash n$ and $d(\lambda) + d(\mu) = d(\rho)$. *Then*

$$\langle \chi_\lambda \otimes \chi_\mu, \chi_\rho \rangle_{S_n} = \langle \chi_{\theta(\lambda)} \hat{\otimes} \chi_{\theta(\mu)}, \chi_{\theta(\rho)} \rangle_{S_{d(\rho)}}.$$

Proof.

Case 1. $\mu_1 = |\lambda \cap \rho|$.
Denote $\eta = \lambda \cap \rho$ and apply 2.4:

$$\begin{aligned} \langle \chi_\lambda \otimes \chi_\mu, \chi_\rho \rangle_{S_n} &= \langle \chi_\lambda \otimes \chi_\rho, \chi_\mu \rangle_{S_n} \\ &= \langle \chi_{\lambda/\eta} \otimes \chi_{\rho/\eta}, \chi_{\theta(\mu)} \rangle_{S_{n-\mu_1}}. \end{aligned}$$

Now $\theta(\lambda) \subseteq \theta(\rho)$ (3.1(b.3)), so $\lambda_i \leqslant \rho_i$ and $\eta_i = \min(\lambda_i, \rho_i) = \lambda_i$ $(i \geqslant 2)$; hence $\theta(\eta) = \theta(\lambda)$. Also, by 3.1(b.2) $\lambda_1 \geqslant \rho_1$ and therefore $\eta_1 = \rho_1$. Thus $\rho/\eta = \theta(\rho)/\theta(\lambda)$ so $\chi_{\rho/\eta} = \chi_{\theta(\rho)/\theta(\lambda)}$. Also, $\lambda/\eta = (\lambda_1 - \rho_1)$ so $\chi_{\lambda/\eta} = \chi_{(\lambda_1 - \rho_1)}$. It follows that

$$
\begin{aligned}
\langle \chi_{\lambda/\eta} \otimes \chi_{\rho/\eta}, \chi_{\theta(\mu)} \rangle_{S_{n-\mu_1}} &= \langle \chi_{(\lambda_1 - \rho_1)} \otimes \chi_{\theta(\rho)/\theta(\lambda)}, \chi_{\theta(\mu)} \rangle_{S_{n-\mu_1}} \\
&= \langle \chi_{\theta(\rho)/\theta(\lambda)}, \chi_{\theta(\mu)} \rangle_{S_{n-\mu_1}} \\
&\qquad (\chi_{(\lambda_1 - \rho_1)} \text{ is the identity character}) \\
&= \langle \chi_{\theta(\lambda)} \hat{\otimes}\, \chi_{\theta(\mu)}, \chi_{\theta(\rho)} \rangle_{S_{n-\rho_1}} \\
&\qquad (\text{by definition of skew characters; see Sect. 1}).
\end{aligned}
$$

Case 2. $\mu_1 \neq |\lambda \cap \rho|$.

We shall show that

$$\langle \chi_\lambda \otimes \chi_\mu, \chi_\rho \rangle_{S_n} = \langle \chi_{\theta(\lambda)} \hat{\otimes}\, \chi_{\theta(\mu)}, \chi_{\theta(\rho)} \rangle_{S_{d(\rho)}} = 0.$$

By 3.2

$$\langle \chi_\lambda \otimes \chi_\mu, \chi_\rho \rangle_{S_n} = 0.$$

By 3.1(b.3)

$$\theta(\lambda) \subsetneq \theta(\rho);$$

hence, by 1.2(b),

$$\langle \chi_{\theta(\lambda)} \hat{\otimes}\, \chi_{\theta(\mu)}, \chi_{\theta(\rho)} \rangle_{S_{d(\rho)}} = 0. \quad \blacksquare$$

EXAMPLE.

$$
\begin{aligned}
&\langle \chi_{(7,3,2)} \otimes \chi_{(9,2,1)}, \chi_{(4,4,2,1^2)} \rangle_{S_{12}} \qquad (\text{by Theorem 3.3}) \\
&\quad = \langle \chi_{(3,2)} \hat{\otimes}\, \chi_{(2,1)}, \chi_{(4,2,1^2)} \rangle_{S_8} \\
&\quad = 2 \qquad (\text{by [Jam-Ker, p. 439]}).
\end{aligned}
$$

4. APPLICATIONS: REMOVING HIGHEST RECTANGLES IN SOME CHARACTERS

DEFINITION 4.1. (a) Let $H(k, l; n) = \{\lambda \vdash n \mid \text{for all } i > k,\ \lambda_i \leqslant l\}$.

(b) Let $m \leqslant n$. Then denote

$$\psi(k, l; n, m) = \left(\sum_{\lambda \in H(k,l;n)} \chi_\lambda \otimes \chi_\lambda \right)_{S_m}.$$

In this paper we only study the case $l=0$. When $l=0$ and $m=n$, $n-1$, these are the cocharacters describing the various "trace identities" of the $k\times k$ matrices. These are very close to the cocharacters of the polynomial identities (P.I.'s) of the $k\times k$ matrices. For that reason, the analysis of these characters is of special interest in the theory of P.I. algebras. When $k=2$, $\psi(2,0;n,n-1)$ was calculated by Procesi (see [Reg 3]). No further results are known for $k\geqslant 3$. We calculate below (4.5) a rather intriguing formula for the multiplicities of highest rectangles, for any k, n, and m.

By 1.7 we can write:

DEFINITION 4.2. Let $n\geqslant m$. The equation

$$\psi(k,0;n,m)=\Big(\sum_{\lambda\in H(k,0;n)}\chi_\lambda\otimes\chi_\lambda\Big)_{S_m}=\sum_{\rho\in H(k^2,0;m)}a(\rho,k,n)\,\chi_\rho$$

defines the coefficients $a(\rho,k,n)$. Thus $a(\rho,k,n)=\langle\psi(k,0;n,|\rho|),\chi_\rho\rangle_{S_{|\rho|}}$. We shall prove

PROPOSITION 4.3. *Let $0\leqslant m\leqslant n$ and $\rho\vdash m$. Let $t\geqslant 0$ and assume ρ contains a $k^2\times t$ rectangle: $\rho=(\hat\rho+t,...,\hat\rho_{k^2}+t)$, $\hat\rho_{k^2}\geqslant 0$. Finally let $a(\rho,k,n)$, $a(\hat\rho,k,n-k^2t)$ be given by 4.2. Then $a(\rho,k,n)=a(\hat\rho,k,n-k^2t)$.*

The proof of 4.3 will follow from the following

LEMMA 4.4. *Let $\lambda\in H(k,0;n)$, $\mu\in H(l,0;n)$. By 1.7, all components of $\chi_\lambda\otimes\chi_\mu$ are in $H(kl,0;n)$. Let $\eta\in H(kl,0;n)$ and let $0\leqslant t\leqslant\eta_{kl}$.*

(a) *Assume $\lambda_k\geqslant lt$ and $\mu_l\geqslant kt$. Write $\lambda=(\hat\lambda_1+lt,...,\hat\lambda_k+lt)$, $\mu=(\hat\mu_1+kt,...,\hat\mu_l+kt)$, and $\eta=(\hat\eta_1+t,...,\hat\eta_{kl}+t)$. Then*

$$\langle\chi_\lambda\otimes\chi_\mu,\chi_\eta\rangle_{S_n}=\langle\chi_{\hat\lambda}\otimes\chi_{\hat\mu},\chi_{\hat\eta}\rangle_{S_{n-klt}}.$$

(b) *Let $t\geqslant 1$. If either $\lambda_k<lt$ or $\mu_l<kt$, then*

$$\langle\chi_\lambda\otimes\chi_\mu,\chi_\eta\rangle_{S_n}=0.$$

Proof. (a) Induction on $t\geqslant 0$. Trivial for $t=0$. Assume true for $t-1\geqslant 0$, and apply 2.4′: notice that

$$\lambda\cap\mu'=\underbrace{(l,l,...,l)}_{k}$$

hence $\lambda/\lambda\cap\mu'=(\lambda_1-l,\lambda_2-l,...,\lambda_k-l)$. Similarly, $\mu/\lambda'\cap\mu=(\mu_1-k,$

$\mu_2-k, \dots, \mu_l-k)$. In the notation of 2.4′, $\zeta=\zeta(\eta)=(\eta_1-1, \eta_2-1, \dots, \eta_{kl}-1)$. Thus, by 2.4′

$$\begin{aligned}&\langle \chi_\lambda \otimes \chi_\mu, \chi_\eta \rangle_{S_n}\\ &\qquad = \langle \chi_{(\lambda_1-l, \lambda_2-l, \dots)} \otimes \chi_{(\mu_1-k, \mu_2-k, \dots)}, \chi_{(\eta_1-1, \eta_2-1, \dots, \eta_{lk}-1)} \rangle_{S_{n-kl}}.\end{aligned}$$

By induction the proof is now complete.

(b) Here $|\lambda \cap \mu'| < kl$ but $h(\eta)=kl$; hence by 1.6(b), $\langle \chi_\lambda \otimes \chi_\mu, \chi_\eta \rangle_{S_n} = 0$. ∎

Proof of 4.3. Apply 4.4 with $k=l$. Let

$$\alpha = (\underbrace{kt, kt, \dots, kt}_{k}) \vdash k^2 t \qquad \text{and} \qquad \beta = (\underbrace{t, t, \dots, t}_{k^2}) \vdash k^2 t.$$

Thus

$$\begin{aligned}
a(\rho, k, n) &= \langle \psi(k, 0; n, m), \chi_\rho \rangle_{S_m} \qquad \text{(by Frobenius Reciprocity)}\\
&= \Big\langle \sum_{\lambda \in H(k,0;n)} \chi_\lambda \otimes \chi_\lambda, (\chi_\rho)^{S_n} \Big\rangle_{S_n} \qquad \text{(by 1.4)}\\
&= \Big\langle \sum_{\lambda \in H(k,0;n)} \chi_\lambda \otimes \chi_\lambda, \sum_{\substack{\eta \vdash n\\ \eta \supseteq \rho}} k(\eta, \rho)\, \chi_\eta \Big\rangle_{S_n} \qquad \text{(by 1.7)}\\
&= \sum_{\lambda \in H(k,0;n)} \sum_{\substack{\eta \in H(k^2,0;n)\\ \eta \supseteq \rho}} k(\eta, \rho) \langle \chi_\lambda \otimes \chi_\lambda, \chi_\eta \rangle_{S_n} \qquad \text{(by 4.4 with } l=k)\\
&= \sum_{\substack{\lambda \in H(k,0;n)\\ \lambda \supseteq \alpha}} \sum_{\substack{\eta \in H(k^2,0;n)\\ \eta \supseteq \rho}} k(\eta, \rho) \langle \chi_{\lambda/\alpha} \otimes \chi_{\lambda/\alpha}, \chi_{\eta/\beta} \rangle_{S_n}.
\end{aligned}$$

Obviously, $\lambda \leftrightarrow \lambda/\alpha = \hat{\lambda}$ is a bijection between $\{\lambda \in H(k, 0; n) \mid \alpha \subseteq \lambda\}$ and $H(k, 0; n-k^2t)$. Similarly, $\eta \leftrightarrow \eta/\beta = \hat{\eta}$ is a bijection between $\{\eta \in H(k^2, 0; n) \mid \rho \subseteq \eta\}$ and $\{\hat{\eta} \in H(k^2, 0, n-k^2t) \mid (\rho/\beta) \subseteq \hat{\eta}\}$. Finally $k(\eta, \rho)$, which is the degree of $\chi_{\eta/\rho}$, depends only on the shape η/ρ; hence $k(\eta, \rho) = k(\hat{\eta}, \hat{\rho})$. Thus

$$\begin{aligned}
&a(\rho, k, n)\\
&\quad = \sum_{\hat{\lambda} \in H(k,0;n-k^2t)} \sum_{\substack{\hat{\eta} \in H(k^2,0;n-k^2t)\\ \hat{\eta} \supseteq \hat{\rho}}} k(\hat{\eta}, \hat{\rho}) \langle \chi_{\hat{\lambda}} \otimes \chi_{\hat{\lambda}}, \chi_{\hat{\eta}} \rangle_{S_{n-k^2t}}
\end{aligned}$$

(by linearity and 1.6)

$$= \Big\langle \sum_{\hat{\lambda} \in H(k,0;n-k^2t)} \chi_{\hat{\lambda}} \otimes \chi_{\hat{\lambda}}, \sum_{\substack{\hat{\eta} \vdash n-k^2t\\ \hat{\eta} \supseteq \hat{\rho}}} k(\hat{\eta}, \hat{\rho})\, \chi_{\hat{\eta}} \Big\rangle_{S_{n-k^2t}} \qquad \text{(by 1.4)}$$

$$= \left\langle \sum_{\lambda \in H(k,0;n-k^2t)} \chi_\lambda \otimes \chi_\lambda, (\chi_{\hat\rho})^{S_{n-k^2t}} \right\rangle_{S_{n-k^2t}} \qquad \text{(Frobenius Reciprocity)}$$

$$= \langle \psi(k, 0; n-k^2t, m-k^2t, m-k^2t), \chi_{\hat\rho} \rangle_{S_{m-k^2t}} = a(\hat\rho, k, n-k^2t). \quad ▮$$

COROLLARY 4.5. *Let* $k \geq 1$, $t \geq 0$, $i \in \{0, 1\}$, *and*

$$\phi = (t+i, \underbrace{t, ..., t}_{k^2-1}).$$

Let $a(\phi, k, n)$ *be given by* 4.2. *Then*

(a) $a(\phi, k, n) = \sum_{\lambda \in H(k,0;n-k^2t)} (\deg(\chi_\lambda))^2$.

(b) *The case* $m = |\phi| = n$ *and* $m = |\phi| = n-1$ *is of special importance* (*see* 4.1). *In that case*

$$a(\phi, k, n) = \begin{cases} 2 & \text{if } k \geq 2,\ i=1, \text{ and } |\phi| = n-1; \\ 1 & \text{otherwise.} \end{cases}$$

Proof. (a) By 4.3, $a(\phi, k, n) = a((i), k, n-k^2t)$. Thus

$$a(\phi, k, n) = a((i), k, n-k^2t) = \langle \psi(k, 0; n-k^2t, i), \chi_{(i)} \rangle_{S_i}$$

(since the trivial character is the only irreducible one of S_i, $i = 0, 1$)

$$= \deg(\psi(k, 0; n-k^2t, i))$$

$$= \deg\left(\left(\sum_{\lambda \in H(k,0,n-k^2t)} \chi_\lambda \otimes \chi_\lambda\right)_{S_i}\right) = \deg\left(\sum_{\lambda \in H(k,0,n-k^2t)} \chi_\lambda \otimes \chi_\lambda\right)$$

$$= \sum_{\lambda \in H(k,0;n-k^2t)} (\deg(\chi_\lambda))^2.$$

(b) Easy to check from (a). ▮

Remark 4.6. Let $k \geq 1$, $n \geq 0$, and denote $S_k^{(\beta)}(n) = \sum_{\lambda \in H(k,0,n)} (\deg(\chi_\lambda))^\beta$.

(a) By 4.1, $S_k^{(2)}(n) = \deg(\psi(k, 0; n, m))$, and by 4.5, $a(\phi, k, n) = S_k^{(2)}(n-k^2t)$.

(b) Explicit values of $S_k^{(\beta)}(n)$ are known only in few special cases [Reg 2, Introduction], such as: $S_2^{(2)}(n) = (1/(n+1))\binom{2n}{n}$, and if $k \geq n$ then $S_k^{(2)}(n) = n!$.

(c) The asymptotic behavior of $S_k^{(\beta)}(n)$ is given in [Reg 2]. The algebraicity of the related generating functions is studied in [Bec-Reg]. For example, it is shown there that when $k \geq 3$ is odd, $\sum_{n \geq 0} S_k^{(2)}(n)\, t^n$ is not algebraic.

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