

**QIC 890 - Intro. to Noise Processes**

**Qubit Dynamics in Presence of Thermal Noise**

Dr. Na Young Kim

**John Rinehart**

August 5, 2016

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# 1 Noise of Finite-Length Resistor Subject to Thermal Gradient

Consider the problem of a resistor  $R$  of length  $l$  subject to a thermal gradient induced by the resistor being connected to two thermal baths at each of its ends,  $T_1$  at one end and  $T_2$  at the other (see Fig. 1). This is relevant to the experimentalist working with resistive cabling in a dilution refrigerator. Using the result of the noise generated by a resistor at a single finite temperature we will determine the noise generated by this resistor. In the effort to maintain complete generality we will use the following expression for the spectral density of a noisy resistor derived by Nyquist in 1928 which applies over every frequency and temperature[Nyq28]:

$$S_V(f) = \frac{4hfR}{e^{\frac{hf}{k_B T}} - 1} . \quad (1)$$

The expression for the spectral density has units of  $V^2 \text{ Hz}^{-1}$ . The above expression is applicable to a resistor  $R$  anchored at both ends to a bath at a single temperature. We are considering a resistor anchored to baths at two (possibly) different temperatures. In order to solve this problem, the first thing that we will need to do is determine the distribution of temperature along the resistor. The temperature distribution of the resistor must satisfy the heat diffusion equation

$$\frac{\partial T(x, y, z, t)}{\partial t} - \alpha \nabla^2 T(x, y, z) = 0 . \quad (2)$$

Above,  $T(x, y, z, t)$  is the temperature of the object at the position  $(x, y, z)$  at time  $t$  and  $\alpha$  is some diffusion constant associated with the materials of the object. Now, in this case, the resistor is assumed to be a one-dimensional object. Assuming the resistor to be oriented along a straight line in the x-direction, the above equation can be recast as

$$\frac{\partial T(x, t)}{\partial t} - \alpha \frac{\partial^2 T(x)}{\partial x^2} = 0 . \quad (3)$$

If the resistor is allowed to come to a steady state then all time variations of the temperature vanish. This simplifies Eq. 3 to

$$\frac{\partial^2 T(x)}{\partial x^2} = 0 . \quad (4)$$

The solutions to the above equation are simple and are given by  $T(x) = Ax + B$  with  $A$  and  $B$  being determined by the boundary conditions of the problem. In this case, the boundary conditions of the problem determine that

$$T(0) = T_1$$

and

$$T(l) = T_2 .$$

Applying these boundary conditions to the above expression determine that

$$A = \frac{T_2 - T_1}{l}$$

and

$$B = T_1 .$$

Thus  $T(x) = \frac{T_2 - T_1}{l}x + T_1$ . Now, we will adopt the following approach to determine the noise generated by the entire resistor: Discretize the resistor into many resistive segments connected in series; then, express the noise of the entire resistor in terms of a Riemann sum which will be recast as an integral in the limit of an infinite number of segments. This is a valid approach under the following considerations: The noise of adjacent segments is independent (and, so, are uncorrelated). That is, considering the voltage noise

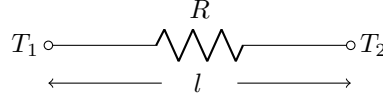


Figure 1: A resistor of length  $l$  and resistance  $R$  anchored at its ends to thermal baths at temperatures  $T_1$  and  $T_2$ .

generated by a segment  $S_a$ ,  $V_{S_a}$ , and another segment  $S_b$ ,  $V_{S_b}$ ,  $\langle V_{S_a} V_{S_b} \rangle = \langle V_{S_a} \rangle \langle V_{S_b} \rangle$  (independent) and  $\langle V_{S_a} V_{S_b} \rangle - \langle V_{S_a} \rangle \langle V_{S_b} \rangle = 0$  (uncorrelated).

Considering the physical origin of such noise (thermal agitation of electrons) these assumptions may seem unreasonable. It may be the case that the agitation of one electron exerts a force on nearby electrons affecting their agitation. This would imply that the position of one electron (a random variable affecting the potential/voltage of the electron) is correlated (alternatively stated, that it shares information) with other electrons. However, for the sake of mathematical tractability, this will be ignored. If physical observations of thermal noise do not correspond to the derived expression, then this should be one of the first assumptions to reconsider. However, assuming for now that this assumption is not unreasonable we can express the noise of the  $i$ th resistor section,  $R_i$ , as

$$S_V(f) = \frac{4hfR_i}{e^{\frac{hf}{k_B T_i}} - 1} . \quad (5)$$

According to the expression for the temperature distribution of the conductor, we have

$$T(R_i) = \frac{T_2 - T_1}{l} x_i + T_1 . \quad (6)$$

If we assume that the resistor is discretized into  $N$  sections of length  $l/N$  and we index each section by  $i$  such that the  $i$ th section is at position  $x_i = li/N$  then we can rewrite Eq. 6 as

$$T(R_i) = \frac{T_2 - T_1}{l} \frac{li}{N} + T_1 = \frac{T_2 - T_1}{N} i + T_1 . \quad (7)$$

Note that we have chosen to index the sections from  $1, \dots, N$  such that there are  $N$  sections and the last section is the  $N$ th section. Thus, the temperature of the first section is

$$T(R_1) = \frac{T_2 - T_1}{N} + T_1 = T_1 \left( 1 - \frac{1}{N} \right) + \frac{T_2}{N} . \quad (8)$$

Clearly the first section is not at temperature  $T_1$  for any finite number of sections. Thus, it is requisite that we take the limit as  $N \rightarrow \infty$ . In order to write an expression for the noise of the entire resistor we must consider how to combine the noise of adjacent resistors. Clearly if we consider the noise of two resistors  $R_1$  and  $R_2$  both at the same temperature  $T$ , the noise generated from the series connection of these two resistors should be the same as that generated from a resistor of value  $R = R_1 + R_2$ .

$$S_{V_R}(f) = \frac{4hfR}{e^{\frac{hf}{k_B T}} - 1} = \frac{4hf(R_1 + R_2)}{e^{\frac{hf}{k_B T}} - 1} = S_{V_{R_1}} + S_{V_{R_2}} . \quad (9)$$

Maybe unsurprisingly, the power spectral densities of individual resistors are additive. So, the noise spectral density from the entire resistor is just the sum of all of the spectral densities of each of the sections:

$$S_{V_R}(f) = \lim_{N \rightarrow \infty} \sum_{i=1}^N S_{V_{R_i}}(f) = \lim_{N \rightarrow \infty} \sum_{i=1}^N \frac{4hfR_i}{e^{\frac{hf}{k_B T_i}} - 1} = \lim_{N \rightarrow \infty} \sum_{i=1}^N \frac{1}{N} \frac{4hfR}{e^{\frac{hf}{k_B T_i}} - 1} \quad (10)$$

where, above, we made the assumption that all  $N$  resistive sections are of value  $R/N$ . Now, in order to express this in terms of an integral we need to take the limit as some quantity (representing some interval)

gets very small. In this case, we can imagine the temperature difference across each resistive section shrinking to zero as the number of sections increases to infinity. Considering this, then, the temperature difference across the  $i$ th resistive section,  $\Delta T$  is

$$\Delta T = \left( \frac{i+1}{N} (T_2 - T_1) + T_1 \right) - \left( \frac{i}{N} (T_2 - T_1) + T_1 \right) = \frac{T_2 - T_1}{N} . \quad (11)$$

This allows us to replace instances of  $\frac{1}{N}$  in Eq. 10 with  $\Delta T$  as

$$S_{V_R}(f) = \lim_{\Delta T \rightarrow 0} \sum_{i=1}^N \frac{\Delta T}{T_2 - T_1} \frac{4hfR}{e^{\frac{hf}{k_B T_i}} - 1} . \quad (12)$$

Rewriting our expression for the temperature of the  $i$ th resistive section in Eq. 7 terms of  $\Delta T$ , now, yields

$$T(R_i) = i\Delta T + T_1 . \quad (13)$$

Substituting the above expression into Eq. 12 yields

$$S_{V_R}(f) = \lim_{\Delta T \rightarrow 0} \sum_{i=1}^N \frac{\Delta T}{T_2 - T_1} \frac{4hfR}{\exp\left(\frac{hf}{k_B(T_1 + i\Delta T)}\right) - 1} = \frac{4hfR}{T_2 - T_1} \lim_{\Delta T \rightarrow 0} \sum_{i=1}^N \frac{\Delta T}{\exp\left(\frac{hf}{k_B(T_1 + i\Delta T)}\right) - 1} . \quad (14)$$

Now, the definition of an integral in terms of a Riemann sum is exactly

$$\int_a^b f(x)dx = \lim_{\Delta x \rightarrow 0} \sum_{i=1}^N f(a + i\Delta x)\Delta x . \quad (15)$$

So, the expression Eq. 14 can be cast as

$$S_{V_R}(f) = \frac{4hfR}{T_2 - T_1} \int_{T_1}^{T_2} \left( \exp\left(\frac{hf}{k_B T}\right) - 1 \right)^{-1} dT . \quad (16)$$

In the event that the resistor  $R$  has the property that the resistance changes as a function of temperature, it is easy to modify the above expression to account for this. Assuming the resistor has a temperature dependence  $R(T)$ , the spectral density of the entire resistor can be calculated as

$$S_{V_R}(f) = \frac{4hf}{T_2 - T_1} \int_{T_1}^{T_2} R(T) \left( \exp\left(\frac{hf}{k_B T}\right) - 1 \right)^{-1} dT . \quad (17)$$

In the appendix, the expression of the noise generated by a resistor at one temperature in the classical limit is obtained. If we want the autocorrelation of this noisy process, the Wiener-Khintchine theorem allows us to obtain this through a (inverse) Fourier transform.

$$\phi_{V_R}(t) = \int_{-\infty}^{\infty} S_{V_R}(f) e^{i2\pi ft} df \quad (18)$$

Note that the spectral density diverges in the limit of large  $f$ . Thus, the proper Fourier integral can not be taken. Rather, it must be windowed for any application.

## 2 Deriving the Spin-Boson Master Equation

One stochastic model of particular interest to experimentalists building quantum computers is the spin-boson model. It describes the interaction of a two-level system (the spin) with a bath of harmonic oscillators (the

bosons). In the present context, the spin is meant to take the place of a quantum bit, or qubit, and the bath of harmonic oscillators is the thermal bath that is presented to the qubit by the resistive network. Having knowledge of the resistive network it may be possible to describe the time evolution of a state including decoherence (loss of phase information) and dissipation (loss of energy).

This noisy process is of particular interest to the developers of superconducting quantum computers which are designed to interface with (approximate) two-level system through the use of resistive cabling. The resistive cables introduce thermal noise. Coming to a detailed understanding of how the resistive network affects the two-level system is of particular interest to the designers of such systems in order that they can reduce the effects the environment induces on the qubit (the spin or the two-level system).

### 3 Deriving the Born-Markov Equation

The problem of solving for the time evolution of the state of the two-level system coupled to a thermal bath has been solved (see [ICJ<sup>+</sup>05], [WSHG06]) but remains a problem of significant pedagogical interest. It is the goal of this work for the author to elucidate the dynamics of the two-level system subject to a thermal environment.

The dynamics of the two-level system coupled to a thermal environment relies on the derivation of a master equation (differential equation describing a quantum state subject to noise) called the Born-Markov equation. It describes the evolution a quantum system (of any size) coupled to a noisy environment. Once that has been determined it will be applied to the specific case of a qubit (two-level system) coupled to a thermal environment (thermal bath of bosons). The approach shown will be very similar to the work of [God] as it was the primary reference for this work.

It is a known principle of quantum mechanics (Stinespring's dilation theorem) that if the state space is chosen to be "large enough" (by adding ancilla systems) that it evolves unitarily (this is the quantum version of the statement "closed systems neither gain nor lose energy"). Unitary evolution of a quantum state  $\rho$ , initialized as  $\rho(0)$ , would then be given by the unitary evolution of that state:  $\rho(t) = \hat{U}(t)\rho(0)U^\dagger(t)$ . This is easily justified by considering the Schrödinger equation in the context of a density matrix evolving according to the Hamiltonian of the system represented by  $\rho$ . Now, imagine that  $\rho$  is a state which exists in a space existing as a combination of a "system" whose dynamics we are interested in and an "environment" whose dynamics are immaterial. We can obtain the evolution of only the state,  $\rho_{\text{sys}}(t)$  by tracing out the environment as

$$\rho_{\text{sys}}(t) = \text{Tr}_{\text{env}} (\rho_{\text{sys} \otimes \text{env}}(t)) \quad . \quad (1)$$

Henceforth, sys and env will be abbreviated in favor of the unambiguous form  $s$  and  $e$ . The details of how to numerically/computationally perform a partial trace are immaterial to the following discussion and, as such, will not be described here. The above notation of  $\text{sys} \otimes \text{env}$  describes a tensor product space (a mathematical space determined by the combination of the system and the environment). The reader is referred to [NC11] for more information regarding elementary quantum information operations like computing the partial trace and the tensor product.

Determining the time evolution of the density matrix can be expressed as above but, often, it is difficult, if not impossible, to determine the unitary operator that generates time translations of the system-environment state space. Plus, if we are only interested in the system, the computation of such a time evolution is wasteful. It would be convenient to describe the time evolution of the state in terms of something resembling the Schrödinger equation. However, the dissipation and decoherence introduced by the environment has an effect on the system dynamics. Simply put, it would be nice to find an evolution of the form

$$\frac{d}{dt}\rho_s(t) = -i[H, \rho_s(t)] + D(\rho_s(t)) \quad . \quad (2)$$

The form, above, describes unitary evolution according to *some* Hamiltonian (not necessarily, and probably not, the free system's Hamiltonian). The Hamiltonian above only acts on the system state and ignores the state of the environment. The evolution is described purely in terms of some effective system Hamiltonian, the state of the system and an operator  $D$  which clearly must contain information regarding the environment. Note that in the absence of  $D$  the dynamics are that of a closed system under the influence of its Hamiltonian (the quantum Liouville equation). Clearly, the operator  $D$  addresses the dissipation and decoherence (the choice of the letter  $D$  now being clear) and includes the information regarding the environment. Another noteworthy feature of the above expression is the system evolution's independence of the history of  $\rho_s(t' < t)$ . This can only be accomplished if the system and environment have no "memory". Such an assumption implies the use of the Markov assumption which is exactly this: A system is Markovian if its dynamics depend on nothing other than the current time (in the case of continuous time) or the immediately previous time (in the discrete time case).

To introduce a form from which this (and the Born) assumption can be made we will consider, first, the time evolution of the combined system-environment state. Then, we will trace out the environment making the Born and Markov approximations, as needed. We will then obtain the Born-Markov master equation. In general, the Hamiltonian of the combined system and environment,  $H_{\text{total}}$  can be expressed as

$$H_{\text{total}} = H_s \otimes \mathbb{1}_e + \mathbb{1}_s \otimes H_e + H_{s \otimes e} \quad (3)$$

where  $H_s$  represents the system Hamiltonian absent the environment (similarly with  $H_e$ ) and  $H_{s \otimes e}$  represents the Hamiltonian exchanging information between the system and the environment. The notation  $\mathbb{1}_{\text{space}}$  will be used to denote the identity operator over some space. It can be helpful to work in a domain (picture) where the system and environment Hamiltonians are absent and only the interaction Hamiltonian is relevant. This can be accomplished through the transformation

$$\rho_{s \otimes e}^{\text{int}}(t) = e^{\frac{i}{\hbar}(H_s \otimes \mathbb{1}_e + \mathbb{1}_s \otimes H_e)t} \underbrace{\left( e^{-\frac{i}{\hbar}H_{\text{total}}t} \rho_{s \otimes e}(0) e^{\frac{i}{\hbar}H_{\text{total}}t} \right)}_{\rho_{s \otimes e}(t)} e^{-\frac{i}{\hbar}(H_s \otimes \mathbb{1}_e + \mathbb{1}_s \otimes H_e)t} .$$

The above transformation transforms  $\rho_{s \otimes e}(0)$  (of the combined system and environment) according to the combined system and environment Hamiltonian; then, this is transformed according to both the system and environment Hamiltonian (the so-called "free part" of the Hamiltonian). Taking the time derivative of this expression leaves us with a sum of four terms, each of which is a product similar to the above (the time derivative of  $\rho_{s \otimes e}(0)$  is zero). The derivative of the first and last terms determine a commutation relationship between the free part of the Hamiltonian and the state in the interaction picture. The derivative of the second and fourth terms determine a commutation relationship between the total Hamiltonian and the time-evolved state  $\rho_{s \otimes e}(t)$ . So, we have

$$\frac{d}{dt} \rho_{s \otimes e}^{\text{int}}(t) = \frac{i}{\hbar} [H_s \otimes \mathbb{1}_e + \mathbb{1}_s \otimes H_e, \rho_{s \otimes e}^{\text{int}}(t)] - \frac{i}{\hbar} e^{\frac{i}{\hbar}(H_s \otimes \mathbb{1}_e + \mathbb{1}_s \otimes H_e)t} [H_{\text{total}}, \rho_{s \otimes e}(t)] e^{-\frac{i}{\hbar}(H_s \otimes \mathbb{1}_e + \mathbb{1}_s \otimes H_e)t} . \quad (4)$$

Expanding  $H_{\text{total}}$  into its two parts, the free and interaction parts, and inserting identity in the form of  $e^{-\frac{i}{\hbar}(H_s \otimes \mathbb{1}_e + \mathbb{1}_s \otimes H_e)t} e^{\frac{i}{\hbar}(H_s \otimes \mathbb{1}_e + \mathbb{1}_s \otimes H_e)t}$  yields

$$\frac{d}{dt} \rho_{s \otimes e}^{\text{int}}(t) = -\frac{i}{\hbar} [H_{s \otimes e}^{\text{int}}(t), \rho_{s \otimes e}^{\text{int}}(t)] \quad (5)$$

where  $H_{s \otimes e}^{\text{int}}(t)$  is the time evolution of  $H_{s \otimes e}$  according to the interaction picture (that is, it is evolved just like  $\rho_{s \otimes e}(t) \rightarrow \rho_{s \otimes e}^{\text{int}}(t)$ ). We can integrate Eq. 5 to solve for  $\rho_{s \otimes e}^{\text{int}}(t)$  and back-substitute this solution for  $\rho_{s \otimes e}^{\text{int}}(t)$  into Eq. 5 to obtain

$$\frac{d}{dt} \rho_{s \otimes e}^{\text{int}}(t) = -\frac{i}{\hbar} [H_{s \otimes e}^{\text{int}}(t), \rho_{s \otimes e}^{\text{int}}(0)] - \frac{1}{\hbar^2} \int_0^t dt' [H_{s \otimes e}^{\text{int}}(t), [H_{s \otimes e}^{\text{int}}(t'), \rho_{s \otimes e}^{\text{int}}(t')]] . \quad (6)$$

Now,  $[H_{s\otimes e}^{\text{int}}(t), \rho_{s\otimes e}^{\text{int}}(0)]$  is an operator. Allow the result of the commutator to be some operator  $A$ . If  $A$  were added to the interaction part of the total Hamiltonian and subtracted from the interaction part of the total Hamiltonian (such that  $H_{\text{total}}$  remains unchanged) then the previous argument would still hold and the first commutator of Eq. 6 would be zero. Thus, without loss of generality, that term can be dropped from the previous expression. Now, we can trace over the environment in the above expression to obtain  $\rho_s^{\text{int}}$  (the state of only the system in the interaction picture) as

$$\frac{d}{dt}\rho_s^{\text{int}}(t) = -\frac{1}{\hbar^2}\text{Tr}_e\left(\int_0^t dt' [H_{s\otimes e}^{\text{int}}(t), [H_{s\otimes e}^{\text{int}}(t'), \rho_{s\otimes e}^{\text{int}}(t')]]\right). \quad (7)$$

So, before any approximations, we have the master equation of  $\rho_s^{\text{int}}(t)$ . But, this expression has two undesirable properties:

1. It depends on the combined state of the system and environment.
2. It depends on the history of both the interaction Hamiltonian and the combined state of the system and environment.

These problems can be alleviated by introducing the Born-Markov assumptions. The Born assumption claims that the system is weakly coupled to the environment. This has two effects:

1. The initial combined state can be expressed as a product state:  $\rho_{s\otimes e}(0) = \rho_s(0) \otimes \rho_e(0)$ .
2. The time evolution of the system is independent of that of the environment which does not evolve:  $\rho_{s\otimes e}(t) = \rho_s(t) \otimes \rho_e(0)$ .

Because the free Hamiltonian is composed of parts which affect the system and environment separately, it is clear that the time evolution of the combined state in the interaction picture is the same as that in the Schrödinger picture; so,  $\rho_{s\otimes e}^{\text{int}} = \rho_s^{\text{int}}(t) \otimes \rho_e^{\text{int}}(0)$ . Another tool that we will use to simplify the above expression is the diagonal block expansion of the interaction Hamiltonian. It can easily be shown that for some operator  $A \in \mathcal{L}(S_1 \otimes S_2)$  has a block decomposition in terms of the canonical basis of  $S_1$ . The canonical basis of  $S_1 = E_{a,b}$ , where  $E_{a,b}$  is represented as a matrix in  $\mathcal{L}(S_1)$  with a 1 in the  $(a,b)$  entry and 0 elsewhere. The decomposition can be expressed as

$$A = \sum_{a=1}^N \sum_{b=1}^N E_{a,b} \otimes P_{a,b} \quad (8)$$

where  $E_{a,b} \in \mathcal{L}(S_1)$  and  $P_{a,b} \in \mathcal{L}(S_2)$ . Analogously, this can be written as

$$B = \sum_{a=1}^N \sum_{b=1}^N P_{a,b} \otimes E_{a,b} \quad (9)$$

where  $P_{a,b} \in \mathcal{L}(S_2)$  and  $E_{a,b} \in \mathcal{L}(S_1)$ . The tensor product is not commutative; in general  $A \neq B$ . However, all normal operators are diagonalizable; Hermitian matrices are diagonalizable. Thus, there must exist some decomposition of  $H_{s\otimes e}$  as

$$H_{s\otimes e} = \sum_{a=1}^N S_a \otimes E_a \quad (10)$$

where  $S_a \in \mathcal{L}(\text{system})$  and  $E_a \in \mathcal{L}(\text{environment})$ . The time evolution of this Hamiltonian in the interaction picture is easily determined by the independent actions of the system and environment Hamiltonians,

$$H_{s\otimes e}^{\text{int}}(t) = e^{\frac{i}{\hbar}(H_s \otimes \mathbb{I}_e + \mathbb{I}_s \otimes H_e)t} \left( \sum_{a=1}^N S_a \otimes E_a \right) e^{-\frac{i}{\hbar}(H_s \otimes \mathbb{I}_e + \mathbb{I}_s \otimes H_e)t} = \sum_{a=1}^N S_a(t) \otimes E_a(t) \quad (11)$$

with  $S_a(t) = e^{\frac{i}{\hbar}H_s t} S_a e^{-\frac{i}{\hbar}H_s t}$  and  $E_a(t) = e^{\frac{i}{\hbar}H_e t} E_a e^{-\frac{i}{\hbar}H_e t}$ .



We can now use the Born assumption and the block expansion of the composite system-environment Hamiltonian ( $H_{s \otimes e}$ ) to rewrite Eq. 7 as

$$\frac{d}{dt} \rho_s^{\text{int}}(t) = -\frac{1}{\hbar^2} \text{Tr}_e \left( \int_0^t dt' \left[ \sum_{i=1}^N S_i(t) \otimes E_i(t), \left[ \sum_{j=1}^N S_j(t') \otimes E_j(t'), \rho_s^{\text{int}}(t') \otimes \rho_e^{\text{int}}(0) \right] \right] \right) \quad (12)$$

$$= -\frac{1}{\hbar^2} \text{Tr}_e \left( \int_0^t dt' \sum_{i,j} [S_i(t) \otimes E_i(t), [S_j(t') \otimes E_j(t'), \rho_s^{\text{int}}(t') \otimes \rho_e^{\text{int}}(0)]] \right) \quad (13)$$

Now, expand the nested commutators and identify  $\text{Tr}_e (E_i(t) E_j(t') \rho_e(0))$  as a function of  $t$  and  $t'$  indexed by  $i$  and  $j$ , say  $F_{i,j}(t, t')$ . Doing all of this yields

$$= -\frac{1}{\hbar^2} \text{Tr}_e \left( \int_0^t dt' \sum_{i,j} \left( (S_i(t) \otimes E_i(t)) (S_j(t') \otimes E_j(t')) (\rho_s^{\text{int}}(t') \otimes \rho_e^{\text{int}}(0)) \right. \right. \quad (14)$$

$$\begin{aligned} & - (S_i(t) \otimes E_i(t)) (\rho_s^{\text{int}}(t') \otimes \rho_e^{\text{int}}(0)) (S_j(t') \otimes E_j(t')) \\ & - (S_j(t') \otimes E_j(t')) (\rho_s^{\text{int}}(t') \otimes \rho_e^{\text{int}}(0)) (S_i(t) \otimes E_i(t)) \\ & \left. + (\rho_s^{\text{int}}(t') \otimes \rho_e^{\text{int}}(0)) (S_j(t') \otimes E_j(t')) (S_i(t) \otimes E_i(t)) \right) \end{aligned}$$

$$= -\frac{1}{\hbar^2} \text{Tr}_e \left( \int_0^t dt' \sum_{i,j} \left( (S_i(t) S_j(t') \rho_s^{\text{int}}(t') \otimes E_i(t) E_j(t') \rho_e^{\text{int}}(0)) \right. \quad (15)$$

$$\begin{aligned} & - (S_i(t) \rho_s^{\text{int}}(t') S_j(t') \otimes E_i(t) \rho_e^{\text{int}}(0) E_j(t')) \\ & - (S_j(t') \rho_s^{\text{int}}(t') S_i(t) \otimes E_j(t') \rho_e^{\text{int}}(0) E_i(t)) \\ & \left. + (\rho_s^{\text{int}}(t') S_j(t') S_i(t) \otimes \rho_e^{\text{int}}(0) E_j(t') E_i(t)) \right) \end{aligned}$$

$$= -\frac{1}{\hbar^2} \left( \int_0^t dt' \sum_{i,j} \left( F_{i,j}(t, t') S_i(t) S_j(t') \rho_s^{\text{int}}(t') - F_{j,i}(t', t) S_i(t) \rho_s^{\text{int}}(t') S_j(t') \right. \right. \quad (16)$$

$$\left. - F_{i,j}(t, t') S_j(t') \rho_s^{\text{int}}(t') S_i(t) + F_{j,i}(t', t) \rho_s^{\text{int}}(t') S_j(t') S_i(t) \right) \right)$$

$$= -\frac{1}{\hbar^2} \left( \int_0^t dt' \sum_{i,j} F_{i,j}(t, t') \left( S_i(t) S_j(t') \rho_s^{\text{int}}(t') - S_j(t') \rho_s^{\text{int}}(t') S_i(t) \right) \right. \quad (17)$$

$$\left. + F_{j,i}(t', t) \left( \rho_s^{\text{int}}(t') S_j(t') S_i(t) - S_i(t) \rho_s^{\text{int}}(t') S_j(t') \right) \right)$$

Note that the transition from Eq. 15 to Eq. 16 relies on the cyclic property of the trace. Now, some assumptions are going to be made about these autocorrelation functions  $F_{i,j}(t, t')$ . First, it will be assumed that the environment is stationary. This means that all of the statistical properties of the system depend not on the particular times  $t_1, t_2, \dots, t_n$  of interest but only on the differences of time under consideration. So,  $F_{i,j}(t, t') = F_{i,j}(t' - t)$ . This is a reasonable assumption often made of noisy systems. Here, the environment stores all of the noisy information and, as such, can reasonably be considered to have this property. Another assumption that will be made is that the noise is relatively uncorrelated. That is,  $F_{i,j}(t' - t) \approx 0$  for  $|t'| \gg |t|$ . This is the Markovian assumption (the environment has no “memory”). It may prove to be an invalid assumption. As always with science, one has to compare theoretical results with experimentation to confirm the theory.

The assumption that the system is memoryless implies the two following important statements. First, the state’s dynamics, which are assumed to change over intervals greater than the memory time of the environment,

are relatively fixed over  $t' \approx t$ . This allows for the replacement of  $t' \rightarrow t$  as the argument for  $\rho_s^{\text{int}}$  in the last expression. Second, the autocorrelation functions are assumed to be negligible for all  $t' \ll t$ . Since the integral runs from 0 up to time  $t$  there is no harm in extending the lower limit of the integral to  $-\infty$ . The integrand contributes little over that domain. Using these ideas to rewrite the last expression we have

$$\begin{aligned} \frac{d}{dt}\rho_s^{\text{int}}(t) = & -\frac{1}{\hbar^2} \left( \int_{-\infty}^t dt' \sum_{i,j}^N F_{i,j}(t'-t) \left( S_i(t) S_j(t') \rho_s^{\text{int}}(t) - S_j(t') \rho_s^{\text{int}}(t) S_i(t) \right) \right. \\ & \left. + F_{j,i}(t-t') \left( \rho_s^{\text{int}}(t) S_j(t') S_i(t) - S_i(t) \rho_s^{\text{int}}(t) S_j(t') \right) \right) \end{aligned} \quad (18)$$

It will be useful to express everything in terms of the time difference  $t' - t \equiv \tau$ . Note that,  $dt' = d\tau$ . Now,

$$\begin{aligned} \frac{d}{dt}\rho_s^{\text{int}}(t) = & -\frac{1}{\hbar^2} \left( \int_{-\infty}^0 d\tau \sum_{i,j}^N F_{i,j}(\tau) \left( S_i(t) S_j(\tau+t) \rho_s^{\text{int}}(t) - S_j(\tau+t) \rho_s^{\text{int}}(t) S_i(t) \right) \right. \\ & \left. + F_{j,i}(-\tau) \left( \rho_s^{\text{int}}(t) S_j(\tau+t) S_i(t) - S_i(t) \rho_s^{\text{int}}(t) S_j(\tau+t) \right) \right) \end{aligned} \quad (19)$$

This is exactly what we wanted! We have an expression for the time rate of change of our state in terms of operators of the Hamiltonian we would be provided (time advanced in the interaction picture). Crucially, the above result does not depend on the combined state of the system and environment and, also, the above result is no longer in integrodifferential form. The history of the state is no longer being integrated over. However, we have an expression for the state in the interaction picture. It would be nice to have this in the Schrödinger picture. As derived in the appendix, we will now use the transformation of Eq. B.11. Performing the transformation on the expression in Eq. 19 has the effect of shifting the arguments back by  $t$ . The interaction transformation is being undone by an amount  $t$ . So, the final form of the Born-Markov master equation is

$$\begin{aligned} \frac{d}{dt}\rho_s(t) = & -\frac{i}{\hbar} [H_s, \rho_s(t)] - \frac{1}{\hbar^2} \left( \int_{-\infty}^0 d\tau \sum_{i,j}^N F_{i,j}(\tau) \left( S_i S_j(\tau) \rho_s(t) - S_j(\tau) \rho_s(t) S_i \right) \right. \\ & \left. + F_{j,i}(-\tau) \left( \rho_s(t) S_j(\tau) S_i - S_i \rho_s(t) S_j(\tau) \right) \right) . \end{aligned} \quad (20)$$

## 4 Spin-Boson Master Equation

We have now derived a master equation for a system weakly coupled to an environment. The validity of this master equation holds under some conditions. One of those conditions was that of weak coupling or, alternatively said, of environment persistence (the absence of environmental dynamics). This allowed us to express the state of the system as the tensor product of the initial state of the environment and the (dynamic) state of the system. Let us imagine, now, that we are presented with the Hamiltonian of a qubit (a transmon; refer to [Bis10]) coupled to a bath of harmonic oscillators in a thermal state (a thermal bath, like a noisy resistor). This can be written as

$$H_{\text{transmon, bath}} = \hbar\omega_q \sigma_z \otimes \mathbb{1}_{\text{bath}} + \hbar \mathbb{1}_{\text{qubit}} \otimes \sum_{i=1}^N \omega_i a_i^\dagger a_i + \hbar \sum_{i=1}^N g_i \sigma_x \otimes (a_i + a_i^\dagger) . \quad (1)$$

The master equation derived in Eq. 20 required knowledge of the autocorrelation function of the environment operators  $F_{i,j}(\tau)$ . In addition to that, it required knowledge of the time-evolution of the system operators.

We will establish both of those quantities, now. We will begin with the autocorrelation function of the environment operators.

The autocorrelation of the environment operators only involves the environment operators used to describe the interaction part of the Hamiltonian. These are  $g_i (a_i + a_i^\dagger)$ . They were calculated as an average over the measurable quantities and were assumed to rely only on the operators used to represent the environment part of the interaction part of the Hamiltonian and the (initial) state of the environment. Thus, calculating  $F_{i,j}(t, t')$  looks like

$$F_{i,j}(t, t') = \hbar^2 g_i g_j \text{Tr}_e \left( \left( a_i(t) + a_i^\dagger(t) \right) \left( a_j(t') + a_j^\dagger(t') \right) \rho_e \right) . \quad (2)$$

The time evolution of the operators is calculated as  $E_i(t) = e^{\frac{i}{\hbar}(\mathbf{1}_s \otimes H_e)t} E_i e^{-\frac{i}{\hbar}(\mathbf{1}_s \otimes H_e)t}$ . Substituting in the operators for this case

$$a_i(t) = e^{\frac{i}{\hbar}\hbar\omega_i \sum_{i=1}^N a_i^\dagger a_i t} a_i e^{-\frac{i}{\hbar}\hbar\omega_i \sum_{i=1}^N a_i^\dagger a_i t} . \quad (3)$$

But, the above is just the evolution of the annihilation operator in the Heisenberg picture. So, we can say that

$$\frac{d}{dt} a_i(t) = \frac{i}{\hbar} [H_e, a_i] \quad (4)$$

Since different modes commute ( $[a_i, a_j] = 0$ ), we can say

$$\frac{d}{dt} a_i(t) = \frac{i}{\hbar} \omega_i [a_i^\dagger a_i, a_i] \quad (5)$$

$$= \frac{i}{\hbar} \omega_i (a_i^\dagger a_i a_i - a_i a_i^\dagger a_i) . \quad (6)$$

Using the fact that  $[a_i, a_i^\dagger] = 1$ ,

$$= \frac{i}{\hbar} \omega_i (a_i^\dagger a_i a_i - (1 + a_i^\dagger a_i) a_i) \quad (7)$$

$$= -\frac{i}{\hbar} \omega_i a_i . \quad (8)$$

Solving this first order differential equation yields

$$a_i(t) = a_i e^{-i\omega_i t} . \quad (9)$$

Performing a similar analysis for  $a_i^\dagger(t)$  yields

$$a_i^\dagger(t) = a_i^\dagger e^{i\omega_i t} . \quad (10)$$

Note that the fact that  $[a_i, a_j]$  commute also implies that  $F_{i,j}(t, t') = 0$  for all  $i \neq j$ . We are now prepared to solve for  $F_{i,i}(t, t')$  assuming that  $\rho_e$  is given. Assume that  $\rho_e$  is a thermal state. It can be shown that the value of  $F_{i,j}(t, t')$  under these conditions is exactly

$$F_{i,i}(t, t') = \hbar^2 g_i^2 \left( \coth \left( \frac{\hbar\omega_i}{2k_b T} \right) \cos(\omega_i(t' - t)) - i \sin(\omega_i(t' - t)) \right) \quad (11)$$

This is exactly what we need. However, this form has the disadvantage of requiring knowledge of  $g_i$ , the system-environment coupling parameters. A better approach to determining the autocorrelation functions is to determine, in the context of a quantized electrodynamics situation, what the thermal noise generated by a thermal state is. Once the spectral density is determined in the context of quantum electrodynamics, the result can be transformed, using the Wiener-Khintchine theorem, to determine the autocorrelation function,

in time. Now, it has been shown (see, for example, [CW51]) that the quantum spectral density of a noisy resistor is

$$S_{V_R}(\omega) = \frac{4R(\omega, T)\hbar\omega}{e^{\hbar\omega/k_B T} - 1} . \quad (12)$$

This is very similar to the expression that was derived earlier. However, it lacks an integral over temperature because the system is assumed to be at a given temperature in this form.

It would be great to relate this to the autocorrelation function just derived. However, this spectral density, like the one that I derived in the classical case, also diverges without bound with increasing  $\omega$ . Thus, a proper Fourier integral does not exist. However, if an assumption is made that only the frequencies that are “close” to the frequency of the qubit affect the qubit dynamics, then the Fourier integral can be windowed and a result can be obtained. Concretely,

$$\phi_{V_R}(\tau) = \frac{1}{2\pi} \int_{-\omega_b}^{-\omega_a} S_{V_R}(\omega) e^{i\omega\tau} d\omega + \frac{1}{2\pi} \int_{\omega_a}^{\omega_b} S_{V_R}(\omega) e^{i\omega\tau} d\omega . \quad (13)$$

Changing the sign of  $\omega$  in the first integral,

$$= -\frac{1}{2\pi} \int_{\omega_b}^{\omega_a} S_{V_R}(-\omega) e^{-i\omega\tau} d\omega + \frac{1}{2\pi} \int_{\omega_a}^{\omega_b} S_{V_R}(\omega) e^{i\omega\tau} d\omega \quad (14)$$

$$= \frac{\hbar}{\pi} \int_{\omega_b}^{\omega_a} \frac{R(\omega)\omega e^{-i\omega\tau}}{e^{-\hbar\omega/k_B T} - 1} d\omega + \frac{\hbar}{\pi} \int_{\omega_a}^{\omega_b} \frac{R(\omega)\omega e^{i\omega\tau}}{e^{\hbar\omega/k_B T} - 1} d\omega \quad (15)$$

$$= -\frac{\hbar}{\pi} \int_{\omega_a}^{\omega_b} \frac{R(\omega)\omega e^{-i\omega\tau}}{e^{-\hbar\omega/k_B T} - 1} d\omega + \frac{\hbar}{\pi} \int_{\omega_a}^{\omega_b} \frac{R(\omega)\omega e^{i\omega\tau}}{e^{\hbar\omega/k_B T} - 1} d\omega . \quad (16)$$

Unfortunately, this integral does not have a nice closed-form solution. However, this integral can be numerically approximated for small values of  $\tau$ . In fact, depending on the temperature,  $T$ ,  $\frac{\hbar\omega}{e^{\hbar\omega/k_B T} - 1} \approx k_B T$ . This is the case at room temperature for frequencies below  $\approx 1$  THz. However, at 10 mK (the coldest temperature in a state-of-the-art dilution refrigerator) the approximation breaks validity at  $\approx 1$  MHz. Accounting for temperature variations in the resistive network which generates the noise, the final expression for the windowed spectral density can be expressed as

$$\phi_{V_R}(\tau) = F(\tau) = -\frac{\hbar}{\pi} \int_{\omega_a}^{\omega_b} \omega e^{-i\omega\tau} \left( \frac{1}{T_{\text{high}} - T_{\text{low}}} \int_{T_{\text{low}}}^{T_{\text{high}}} \frac{R(\omega, T)}{e^{-\hbar\omega/k_B T} - 1} dT \right) d\omega \quad (17)$$

$$+ \frac{\hbar}{\pi} \int_{\omega_a}^{\omega_b} \omega e^{i\omega\tau} \left( \frac{1}{T_{\text{high}} - T_{\text{low}}} \int_{T_{\text{low}}}^{T_{\text{high}}} \frac{R(\omega, T)}{e^{\hbar\omega/k_B T} - 1} dT \right) d\omega \quad (18)$$

So, were a simulation of the noise delivered to a two-level system to be desired, it is now only necessary to evaluate this integral over the appropriate  $\omega$  and  $T$ . This should be evaluated for  $\tau$  smaller than the scale of the system dynamics. Then, all that needs to be done is to evolve the system operators under the system Hamiltonian. In this case (of the transmon) there is only one system operator, the Pauli-Z spin operator, which does not time evolve according to its own Hamiltonian. Thus,  $\sigma_z(\tau) = \sigma_z$ . Since  $\sigma_z \sigma_z = \mathbb{1}_2$ , the Born-Markov master equation which governs a transmon's decoherence and dissipation is given by the following expression:

$$\frac{d}{dt} \rho_{\text{qubit}}(t) = -i\omega_q [\sigma_z, \rho_s(t)] - \frac{1}{\hbar^2} \int_{-\infty}^0 \left( F(\tau) \left( \rho_s(t) - \sigma_z \rho_s \sigma_z \right) + F(-\tau) \left( \rho_s(t) - \sigma_z \rho_s(t) \sigma_z(t) \right) \right) d\tau . \quad (19)$$

This concludes the analysis of the dynamics of a two-level system exposed to a thermal bath.

## 5 Acknowledgements

This work could not have been accomplished without the “live” interaction with the following three people: T.C. Fraser, Hammam Qassim and Arnaud Carignan-Dugas.

T.C. Fraser took much of his time to develop and walk me through the noise generated by a resistor anchored to two thermal baths. He also spent much of his time reading and understanding Timothy Goddard’s work “Quantum Decoherence and the Spin-Boson Model”. His help in understanding Mr. Goddard’s work was invaluable to the creation of this work.

Hammam Qassim was helpful in (very painstakingly) explaining the block decomposition of an operator living in a tensor-product space.

Arnaud Carignan-Dugas was instrumental in understanding the time derivative of the tensor product of two (time-varying) operators.

The author would also like to thank Timothy Goddard for his time spent in writing a very pedagogical introduction to quantum decoherence and dissipation.

## A Classical Thermal Noise of a Resistor at One Temperature

Considering this expression in the limit that  $T_2 \rightarrow T_1$ :

$$S_{V_R}(f) = \lim_{T_2 \rightarrow T_1} \frac{4hf}{T_2 - T_1} \int_{T_1}^{T_2} R(T) \left( \exp\left(\frac{hf}{k_B T}\right) - 1 \right)^{-1} dT \quad (\text{A.1})$$

$$= \lim_{T_2 \rightarrow T_1} \frac{4hf R(T_1)}{T_2 - T_1} \int_{T_1}^{T_2} \left( \exp\left(\frac{hf}{k_B T}\right) - 1 \right)^{-1} dT \quad (\text{A.2})$$

For  $hf \ll k_B T$  the integrand,  $(\exp(hf/k_B T) - 1)^{-1}$  is approximately  $\frac{k_B T}{hf}$ . In this regime, we are only integrating over  $T$  and the result is,

$$= \lim_{T_2 \rightarrow T_1} \frac{4k_B R(T_1)}{T_2 - T_1} \frac{T_2^2 - T_1^2}{2} \quad (\text{A.3})$$

$$= \lim_{T_2 \rightarrow T_1} 4k_B R(T_1) \frac{T_2 + T_1}{2} \quad (\text{A.4})$$

$$= 4k_B R(T_1) T_1 \quad (\text{A.5})$$

This is the classical noise temperature of a resistor at temperature  $T_1$ .

## B Derivation of Time Evolution of System State in Schrödinger Picture

The time evolution of a state is determined by the Hamiltonian that governs it. For a closed system, a state  $\rho(t)$  evolves according to its Hamiltonian  $H$  according to the Schrödinger equation

$$\frac{d}{dt} \rho(t) = -\frac{i}{\hbar} [H, \rho(t)] \quad .$$

If the Hamiltonian has the form  $H_A \otimes \mathbb{1}_B + \mathbb{1}_A \otimes H_B + H_{AB}$  and the time evolution of  $\rho_A(t) = \text{Tr}_B(\rho(t))$  is desired then it can be found as

$$\frac{d}{dt}\rho_A(t) = \frac{d}{dt}\text{Tr}_B(\rho(t)) = \text{Tr}_B\left(\frac{d}{dt}\rho(t)\right) \quad (\text{B.1})$$

$$= -\frac{i}{\hbar}\text{Tr}_B([H, \rho(t)]) \quad (\text{B.2})$$

$$= -\frac{i}{\hbar}\text{Tr}_B([H_A \otimes \mathbb{1}_B + \mathbb{1}_A \otimes H_B + H_{AB}, \rho(t)]) \quad (\text{B.3})$$

$$= -\frac{i}{\hbar}\text{Tr}_B([H_A \otimes \mathbb{1}_B, \rho(t)] + [\mathbb{1}_A \otimes H_B, \rho(t)] + [H_{AB}, \rho(t)]) \quad (\text{B.4})$$

Now, in the case where  $\rho(t)$  can not be expressed in tensor product form this is all that can be said. However, if  $\rho(t)$  can be expressed as  $\rho_A(t) \otimes \rho_B(t)$  then we have

$$= -\frac{i}{\hbar}\text{Tr}_B([H_A \otimes \mathbb{1}_B, \rho_A(t) \otimes \rho_B(t)] + [\mathbb{1}_A \otimes H_B, \rho_A(t) \otimes \rho_B(t)] + [H_{AB}, \rho_A(t) \otimes \rho_B(t)]) \quad (\text{B.5})$$

Using the fact that the trace of any density matrix is 1, we can simplify the first term above as

$$= -\frac{i}{\hbar}[H_A, \rho_A(t)] - \frac{i}{\hbar}\text{Tr}_B([\mathbb{1}_A \otimes H_B, \rho_A(t) \otimes \rho_B(t)] + [H_{AB}, \rho_A(t) \otimes \rho_B(t)]) \quad (\text{B.6})$$

Factoring out a  $\rho_A$  from the second term, we have

$$= -\frac{i}{\hbar}[H_A, \rho_A(t)] - \frac{i}{\hbar}\rho_A(t) \otimes \text{Tr}_B([H_B, \rho_B(t)]) - \frac{i}{\hbar}\text{Tr}_B([H_{AB}, \rho_A(t) \otimes \rho_B(t)]) \quad (\text{B.7})$$

Now, again, this is all we can say unless we admit more assumptions to shape our mathematics. So, assume that  $[H_B, \rho_B(t)] = 0$  (a statement that  $\rho_B(t)$  doesn't evolve under the action of  $H_B$ ; it is a stationary state of  $H_B$ ). This eliminates the second term of the above. The last term represents the evolution of the combined state subject to the interaction Hamiltonian. We know from Eq. 5 that the commutator can be expressed as

$$[H_{AB}, \rho_A(t) \otimes \rho_B(t)] = i\hbar e^{-i(H_A \otimes \mathbb{1}_B + \mathbb{1}_A \otimes H_B)t} \left( \frac{d}{dt}\rho_{A \otimes B}^{\text{int}}(t) \right) e^{i(H_A \otimes \mathbb{1}_B + \mathbb{1}_A \otimes H_B)t} \quad (\text{B.8})$$

If we assume that the combined state in the interaction picture is separable as  $\rho_{A \otimes B}^{\text{int}}(t) = \rho_A^{\text{int}}(t) \otimes \rho_B^{\text{int}}(t)$  then, by product rule of the tensor product, we have

$$[H_{AB}, \rho_A(t) \otimes \rho_B(t)] = i\hbar e^{-i(H_A \otimes \mathbb{1}_B + \mathbb{1}_A \otimes H_B)t} \left( \frac{d}{dt}\rho_A^{\text{int}}(t) \otimes \rho_B^{\text{int}}(t) + \rho_A^{\text{int}}(t) \otimes \frac{d}{dt}\rho_B^{\text{int}}(t) \right) e^{i(H_A \otimes \mathbb{1}_B + \mathbb{1}_A \otimes H_B)t} \quad (\text{B.9})$$

By commutativity of the Hamiltonians  $H_A \otimes \mathbb{1}_B$  and  $\mathbb{1}_A \otimes H_B$  those can be expressed as

$$\begin{aligned} [H_{AB}, \rho_A(t) \otimes \rho_B(t)] &= i\hbar e^{-i(H_A \otimes \mathbb{1}_B)t} \frac{d}{dt}\rho_A^{\text{int}}(t) e^{i(H_A \otimes \mathbb{1}_B)t} \otimes e^{-i(\mathbb{1}_A \otimes H_B)t} \rho_B^{\text{int}}(t) e^{i(\mathbb{1}_A \otimes H_B)t} \\ &\quad + i\hbar e^{-i(H_A \otimes \mathbb{1}_B)t} \rho_A^{\text{int}}(t) e^{i(H_A \otimes \mathbb{1}_B)t} \otimes e^{-i(\mathbb{1}_A \otimes H_B)t} \frac{d}{dt}\rho_B^{\text{int}}(t) e^{i(\mathbb{1}_A \otimes H_B)t} \end{aligned}$$

Now, using one more assumption, if one of those states, say  $\rho_B^{\text{int}}(t)$  is independent of time, then its time derivative will be zero. This is the case for the environmental bath. It is assumed to be stationary in time. Thus, only the first term of the previous expression would survive. Taking the trace of the first term over space  $B$  would yield 1 since the trace of a density matrix is 1 and, as can be seen by the cyclic property of the trace, the exponentials have no effect on the trace. So, we could now write

$$\text{Tr}_B([H_{AB}, \rho_A(t) \otimes \rho_B(t)]) = i\hbar e^{-i(H_A \otimes \mathbb{1}_B)t} \frac{d}{dt}\rho_A^{\text{int}}(t) e^{i(H_A \otimes \mathbb{1}_B)t} \quad (\text{B.10})$$

Substituting this expression in Eq. B.7 (and throwing away the second term in accordance with the previous expression)

$$\frac{d}{dt}\rho_A(t) = -\frac{i}{\hbar}[H_A, \rho_A(t)] + e^{-i(H_A \otimes \mathbb{1}_B)t} \frac{d}{dt}\rho_A^{\text{int}}(t) e^{i(H_A \otimes \mathbb{1}_B)t} . \quad (\text{B.11})$$

So, armed with the knowledge of the reduced state in the interaction picture allow we can write the time rate of change of the reduced state in the Schrödinger picture.

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