Tutorial 2 - 2: Play with gradient descent in Python

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1.1 Basic implementation

· Objective function: $f(x) = x^2$
· Gradient: $\nabla f(x) = 2x$
· Convergence criterion: $|f(x^*) - f(x)| \leq 0.01$
· Learning rate: 0.4

```
import numpy as np

f = lambda x: x ** 2    # Objective function
df = lambda x: 2 * x    # Gradient

epsilon = 0.01    # Convergence threshold
alpha = 0.4    # Learning rate

# Initialization
x = 10
x_last = np.inf * np.ones_like(x)
print('Initialize: x =', x)
print('')

# Start gradient descent
iters = 0
while round(abs(f(x)), 10) > epsilon:
	x_last = x
	x = round(x - alpha * df(x), 10)
	iters += 1
	print('==> Iter %s, x=%s, f(x)=%s' % (iters, x, round(f(x), 10)))
print('------------- The model is converged -------------')

print("\nOptimal solution = ", x)
print("Minimum f(x) = x^2 = ", f(x))
```

Output:
```
Initialize: x = 10
==> Iter 1, x=2.0, f(x)=4.0
==> Iter 2, x=0.4, f(x)=0.16
==> Iter 3, x=0.08, f(x)=0.0064
------------- The model is converged ------------
Optimal solution = 0.008
Minimum f(x) = x^2 = 0.0064
```

1.2 Encapsulation of gradient descent

· Objective function: $f(x) = x^2$
· Gradient: $\nabla f(x) = 2x$
· Convergence criterion: $|f(x^*) - f(x)| \leq 0.01$
· Learning rate: 0.4

```
import numpy as np

def grad_descent(obj_fn, grad_fn, alpha, is_converge, init_x, max_iters = 100):
    # initialize x and x_lst
    x = init_x
    x_last = np.inf * np.ones_like(init_x)
    # Start gradient descent
    iters = 0
    while not is_converge(obj_fn(x_last), obj_fn(x)):
        x_last = x
        x = x - alpha * grad_fn(x)
        iters += 1
        if iters >= max_iters:
            print('Not converged in %s steps' % iters)
            return x
    print('Converged in %s steps' % iters)
    print("Optimal solution = ", x)
    print("Minimum f(x) = x^2 = ", obj_fn(x))

    # Return optimal solution
    return x

if __name__ == '__main__':
    f = lambda x: x ** 2    # Objective function
    df = lambda x: 2 * x    # Gradient

    epsilon = 0.01    # Convergence threshold
    alpha = 0.4    # Learning rate
    x0 = 10    # Initialize

    converge_criterion = lambda last, curr: abs(curr) <= epsilon
    grad_descent(f, df, alpha, converge_criterion, x0)
```

Output:
Converged in 3 steps
Optimal solution = x - a * df(x) = 0.08
Minimum f(x) = x^2 = 0.0064

1.3 Verify the example (not converged)

- Objective function: $f(x) = x^2$
- Gradient: $\nabla f(x) = 2x$
- Convergence criterion: $|f(x^*) - f(x)| \leq 0.01$
- Learning rate: 1

```
import numpy as np

def grad_descent(obj_fn, grad_fn, alpha, is_converge, init_x, max_iters = 100):
    # initialize x and x_lst
    x = init_x
    x_last = np.inf * np.ones_like(init_x)
    # Start gradient descent
    iters = 0
    while not is_converge(obj_fn(x_last), obj_fn(x)):
        x_last = x
        x = x - alpha * grad_fn(x)
        iters += 1
        if iters >= max_iters:
            print('Not converged in %s steps' % iters)
            return x
    print('Converged in %s steps' % iters)
    print("Optimal solution = ", x)
    print("Minimum f(x) = x^2 = ", obj_fn(x))

    # Return optimal solution
    return x

if __name__ == '__main__':
    f = lambda x: x ** 2    # Objective function
    df = lambda x: 2 * x    # Gradient

    epsilon = 0.01    # Convergence threshold
    alpha = 1    # Learning rate
    x0 = 10    # Initialize

    converge_criterion = lambda last, curr: abs(curr) <= epsilon
    grad_descent(f, df, alpha, converge_criterion, x0)
```

Output:

Not converged in 100 steps

1.4 Gradient descent for logistic regression

1.4.1 Data generation

```
import numpy as np
import matplotlib.pyplot as plt

###################### 1.4.1 Data generation ######################
# Feature 1
raw_x1 = [0.959, 0.750, 0.395, 0.823, 0.761, 0.844]
# Feature 2
raw_x2 = [0.382, 0.306, 0.760, 0.764, 0.874, 0.435]
# Observed labels
raw_labels = [0, 0, 0, 1, 1, 1]

data_X = np.stack([raw_x1, raw_x2], axis=1)
# Rows: samples, columns: features
print("data_X", data_X)

gt_Y = np.array(raw_labels, dtype=np.float32)
gt_Y = gt_Y[:, np.newaxis]    # Make Y a column vector
print("gt_Y", gt_Y)
```

Output:

```
data_X:
array([[0.959, 0.382],
       [0.75, 0.306],
       [0.395, 0.76 ],
       [0.823, 0.764],
       [0.761, 0.874],
       [0.844, 0.435]])
gt_Y:
array([[0.],
       [0.],
       [0.],
       [1.],
       [1.],
       [1.]], dtype=float32)
```

1.4.2 Data visualization

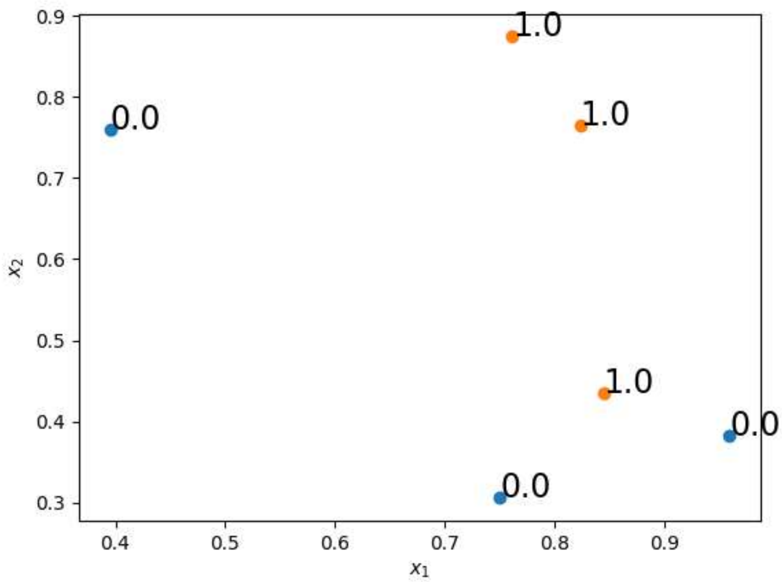
```
###################### 1.4.2 Data visualization ######################
# Plot data points with label = 0
neg_labels_idx = gt_Y[:,0] == 0
plt.scatter(data_X[neg_labels_idx, 0], data_X[neg_labels_idx, 1])

# Plot data points with label = 1
pos_labels_idx = gt_Y[:,0] == 1
plt.scatter(data_X[pos_labels_idx, 0], data_X[pos_labels_idx, 1])

# Annotate each point with its ground truth label
for i, txt in enumerate(gt_Y[:, 0]):
    plt.annotate(txt, data_X[i,0:2], fontsize=17)

plt.xlabel('$x_1$')
plt.ylabel('$x_2$')
plt.show()
```

Output:



1.4.3 Logistic regression

```
##################### 1.4.3 Logistic regression ######################
# Sigmoid function
sigmoid = lambda z: 1.0 / (1 + np.exp(-z))

# Probability P(Y=1|X), parameterized by Theta
P = lambda X, Theta: np.clip(sigmoid(X @ Theta), 1e-3, 1 - 1e-3)

# Cross-entropy loss
ce_loss = lambda X, Theta, Y_true: np.sum(-Y_true * np.log(P(X, Theta)) - (1.0 - Y_true)
* np.log(1 - P(X, Theta)))
d_ce_loss = lambda X, Theta, Y_true: np.sum((P(X, Theta) - Y_true) * X, axis=0,
keepdims=True).T    # Output: column vector

# Probability to category
def prob2category(prob_ls, threshold = 0.5):
    prob_ls = prob_ls.copy()
    prob_ls[prob_ls >= 0.5] = 1
    prob_ls[prob_ls < 0.5] = 0
    return prob_ls

# grad_descent
def grad_descent(obj_fn, grad_fn, alpha, is_converge, init_x, max_iters = 100):
    # initialize x and x_lst
    x = init_x
    x_last = np.inf * np.ones_like(init_x)
    # Start gradient descent
    iters = 0
    while not is_converge(obj_fn(x_last), obj_fn(x)):
        x_last = x
        x = x - alpha * grad_fn(x)
        iters += 1
        if iters >= max_iters:
            print('Not converged in %s steps' % iters)
            return x
    print('Converged in %s steps' % iters)
    # Return optimal solution
    return x

def logistic_regress(X, Y, learning_rate = 0.01):
    # Note that Theta should be in 2-dimensional real space
    Theta_0 = np.random.randn(X.shape[1], 1)
    converge_criterion3 = lambda last, curr: abs(last - curr) < 1e-5
    obj_f = lambda Theta: ce_loss(X, Theta, Y)
    d_obj_f = lambda Theta: d_ce_loss(X, Theta, Y)
    # Use gradient descent to find the best model
    Theta_optim = grad_descent(obj_f, d_obj_f, learning_rate, converge_criterion3,
```

```
Theta_0, max_iters=10000)
        return Theta_optim

Theta_optim_1 = logistic_regress(data_X, gt_Y)
print("Theta_optim_1", Theta_optim_1)

prob_ls = P(data_X, Theta_optim_1)
pred_Y = prob2category(prob_ls)
print("pred_Y", pred_Y)

# Plot data points with label = 0
neg_labels_idx = pred_Y[:,0] == 0
plt.scatter(data_X[neg_labels_idx, 0], data_X[neg_labels_idx, 1])

# Plot data points with label = 1
pos_labels_idx = pred_Y[:,0] == 1
plt.scatter(data_X[pos_labels_idx, 0], data_X[pos_labels_idx, 1])

# Annotate each point with its ground truth label
for i, txt in enumerate(gt_Y[:, 0]):
       plt.annotate(txt, data_X[i,0:2], fontsize=17)

# Draw decision boundary
range_x1 = np.linspace(0.4, 1, 100)
lin_eq = lambda x1: -Theta_optim_1[0] * x1 / Theta_optim_1[1]
plt.plot(range_x1, lin_eq(range_x1))
plt.xlabel('$x_1$')
plt.ylabel('$x_2$')
plt.show()
```

Output: