Freshman Physics

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Updated notes are available at `https://bit.ly/3E6SuwA`. Please inform me of any errors. Feel free to message me to explain something in greater detail :) I also run a weekly SJPO / SPhO class if you're interested.

Contents

1 Math: Functions

Lecture: `https://youtu.be/-Ia9Wv5USaE`

Functions are maps from one set to another set. We denote them as $f : A \to B$ or $f(x) \in B$ where $x \in A$.

Examples of functions $f : \mathbb{R} \to \mathbb{R}$:

- Particle's x coordinate $x(t) = 5\sin(2t)$
- Particle's x component of velocity $v_x(t) = 10\cos(2t)$
- Particle's x component of acceleration $a_x(t) = -20\sin(2t)$
- Potential Energy of spring mass system $U(x) = \frac{1}{2}k(x - x_0)^2$
- Current flowing through Resistor Capacitor circuit $I(t) = I_0 e^{-t/RC}$
- Kinetic Energy in terms of speed $T(|\vec{v}|) = \frac{1}{2}m|\vec{v}|^2$

Examples of functions $f : \mathbb{R} \to \mathbb{R}^3$:

- Particle path in 3D $\vec{s}(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}$

Examples of functions $f : \mathbb{R}^3 \to \mathbb{R}$:

- Gravitational Potential $\phi(x, y, z) = -\frac{GM}{\sqrt{x^2+y^2+z^2}}$
- Electric Potential Energy $\phi(\vec{r}) = -\frac{kQq}{|\vec{r}|}$. The arrow above $\vec{r}$ means it is a vector (more on this later)
- Temperature $T(\vec{r})$

Examples of functions $f : \mathbb{R}^3 \to \mathbb{R}^3$:

- Gravitational Force $\vec{F}(x,y,z) = -\frac{GMm}{(x^2+y^2+z^2)^{3/2}} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

- Electrostatic Force $\vec{F}(\vec{r}) = \frac{kQq}{|\vec{r}|^2}\hat{r}$. The hat above $\hat{r}$ means it is a unit vector (scaled $\vec{r}$ to have length 1).

2 Math: Differentiation

You probably know how to calculate the gradient of a linear (aka straight) graph.

$$m = \frac{y_1 - y_0}{x_1 - x_0}$$

This works if the graph is linear, i.e. $y = mx + c$. But what happens if the graph is not linear? e.g. $y = x^2$. How can we calculate the gradient of this curvy graph? Answer is **Differentiation**.

(Most) functions can be differentiated **with respect to** their parameters. **Algebraically**, differentiation involves following a set of rules. **Geometrically**, differentiation is the slope of the tangent line to the function's graph.

2.1 Geometric Intuition

Lecture: `https://youtu.be/JYnXMoB288Q`

`https://www.desmos.com/calculator/b6ts3ls1zf` Desmos visualization. Given a function $f(x)$, here's what the derivative $\frac{d}{dx}f = \frac{df}{dx} = f'(x)$ means:

- Draw the graph $y = f(x)$.
- For each $x = x_0$ value, find the point on the graph $(x_0, f(x_0))$.
- Draw a (straight) tangent line to the graph at that point.
- Calculate the gradient of that tangent line.
- This gradient is the "derivative of f **at** $x = x_0$".
- If you chose a different $x = x_1$ value, you would get a different value for gradient, and that would be "derivative of f **at** $\mathbf{x = x_1}$".
- So the derivative of a function $f(x)$ is another function $f'(x)$.

2.2 Algebraic Calculations

Lecture: `https://youtu.be/QpSQmggia74`

From the above geometric explanation, one can calculate the derivative of

$f(x) = x^2$ to be $f'(x) = 2x$. This method of differentiation is called "from first principles".

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{\left((x+h)^2\right) - \left(x^2\right)}{h} \\
&= \lim_{h\to 0} \frac{x^2 + 2hx + h^2 - x^2}{h} \\
&= \lim_{h\to 0} \frac{2hx + h^2}{h} \\
&= \lim_{h\to 0} \frac{h(2x+h)}{h} \\
&= \lim_{h\to 0} 2x + h \\
&= 2x
\end{aligned}$$

Lecture: `https://youtu.be/Zg_6w0urW5Q`

One can use first principles to derive the following rules of differentiation for common functions:

- Linearity (Adding)

$$\frac{d}{dx}[f(x) + g(x)] = \frac{df}{dx} + \frac{dg}{dx} \tag{1}$$

$$\frac{d}{dx}[cf(x)] = c\frac{df}{dx} \tag{2}$$

- Polynomial

$$\frac{d}{dx}x^n = nx^{n-1}$$

- Trigonometry

$$\frac{d}{dx}\sin(x) = \cos(x) \tag{3}$$

$$\frac{d}{dx}\cos(x) = -\sin(x) \tag{4}$$

- Exponential

$$\frac{d}{dx}e^x = e^x$$

- Product Rule

$$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{dg}{dx} + g(x)\frac{df}{dx}$$

- Chain Rule

$$\frac{d}{dx}f(y(x)) = \frac{df}{dy}\frac{dy}{dx}$$

2.2.1 Finding Maximum / Minimum

Lecture: `https://youtu.be/McI7tyS_BCo`

When the function $f(x)$ has a maximum $x_{\max}$, the derivative at that maximum point is zero

$$\left.\frac{df}{dx}\right|_{x_{\max}} = 0$$

Likewise for minimum.

So if the function has 0 derivative at some $x = x_0$, how do we determine if it's a maximum or minimum point? We can perform the 2nd derivative test.

$$\left.\frac{d^2 f}{dx^2}\right|_{x_0} > 0 : \text{Minimum} \tag{5}$$

$$\left.\frac{d^2 f}{dx^2}\right|_{x_0} < 0 : \text{Maximum} \tag{6}$$

$$\left.\frac{d^2 f}{dx^2}\right|_{x_0} = 0 : \text{Not enough information to conclude} \tag{7}$$

2.3 Exercises

Try differentiating the following functions with respect to x or t. You can check your answer against Wolfram Derivative Calculator `https://www.wolframalpha.com/calculators/derivative-calculator/`.

- $\frac{d}{dx} x^4 + x^2$
- $\frac{d}{dt} 5t + 3$
- $\frac{d}{dx} \frac{1}{x}$
- $\frac{d}{dt} \sin(2t)$
- $\frac{d}{dt} e^{-5t}$
- $\frac{d}{dx} \tan x$

Extra: If the function of **multiple variables** is differentiated, it's called **multivariate calculus**. Multivariate calculus is used in Electromagnetism (Maxwell's Equations).

3 Math: Integration / Anti-Differentiation

Geometrically, (definite) integration gives you the area under the graph. Algebraically, there are a few techniques for common functions but integration is tricky in general.

3.1 Indefinite Integration / Antiderivative

Lecture: `https://youtu.be/Nm8WVmlnxN8`

Q: If I give you a function $f(x)$ and told you that it's derivative is $f'(x) = 2x+1$, can you find out what $f(x)$ is?
A: $f(x) = x^2 + x + C$ where C is an arbitrary constant. Why are there multiple answers?

Mathematically, we say

$$\int 2x + 1 \; dx = x^2 + x + C$$

More generally,

$$\int f'(x) dx = f(x) + C$$

3.2 Exercises

You can check whether you are correct by putting your answer in the derivative calculator and checking if you get the function to be integrated!

- $\int 5 \; dx$
- $\int \left(\int -10 dt \right) dt$
- $\int \sin(5t) \; dt$
- $\int (1/x^2) dx$

Extra: Some functions have antiderivatives that cannot be even expressed analytically, such as

$$\int e^{-x^2} dx$$

3.3 Definite Integration / Area under graph

Lecture: `https://youtu.be/0IHvAyIaY44`

Lecture (Side Note about Signed Area): `https://youtu.be/g_tr0sqJxM8`

You can calculate the area under a curve by performing a **definite** integral.

$$\begin{aligned} \text{Area under } f'(x) \text{ from } (x = a) \text{ to } (x = b) &= \int_a^b f'(x) \; dx && (8) \\ &= [f(x) + C]_a^b && (9) \\ &= f(b) - f(a) && (10) \end{aligned}$$

Q: Why does the arbitrary constant C not appear in the formula for area under the graph?
A: It cancels out: $[f(b)+C] - [f(a)+C]$. Can you imagine this graphically?

3.4 Exercises

Lecture: `https://youtu.be/ZlvM1BZRFXo`

- Energy in Spring
$$\int_0^x kr\ dr$$
- Energy in Capacitor
$$\int_0^Q \frac{q}{C} dq$$
- Gravitational Potential
$$\int_r^\infty \frac{1}{x^2} dx$$

4 Physics: Kinematics

Lecture: `https://youtu.be/c527UZUQM0s`

After picking a direction which you define as "increasing x direction" as well as an origin for x, you can start describing 1D motion $x(t)$.

If you want to describe 2D motion, pick a perpendicular y-axis.

If you want to describe 3D motion, z-axis is defined using RHR (for the cross product).

4.1 Path of Particle / Object

Lecture: `https://youtu.be/vszG98TSd6Q`

Mathematically, paths are functions $\vec{s}(t)$ of a parameter t representing time.

- In 1D motion, $x(t)$
- In 2D motion, $x(t),\ y(t)$
- In 3D motion, $x(t),\ y(t),\ z(t)$

To understand an object's behaviour, our goal is to solve for the 3 functions $x(t),\ y(t),\ z(t)$, meaning we obtain a formula like $y(t) = 2t - 5t^2$. Knowing where the object is at every snapshot in time allows us to calculate it's velocity $\vec{v}(t)$, it's acceleration $\vec{a}(t)$, how long it'll take to travel from A to B ($\Delta t = t_B - t_A$), the forces $\vec{F}(\vec{s}(t))$ acting on it at any time, etc.

4.2 SUVAT

$$v(t) = u + at \tag{11}$$
$$s(t) = ut + \frac{1}{2}at^2 \tag{12}$$
$$v(s)^2 = u^2 + 2as \tag{13}$$
$$s(t) = \frac{v(t)+u}{2}t \tag{14}$$
$$s(t) = v(t)t - \frac{1}{2}at^2 \tag{15}$$

SUVAT laws can be derived from calculus with the following definitions.

Let $s(t)$ be the function representing the particle's (1D) coordinate.

$$v(t) := \frac{ds}{dt} \tag{16}$$
$$a(t) := \frac{dv}{dt} = \frac{d^2s}{dt^2} \tag{17}$$

If acceleration is constant, i.e.

$$a(t) = \frac{d^2s}{dt^2} = a_0$$

for some constant a_0, then one can integrate the above equation once and twice to get 2 of the SUVAT laws

$$v(t) = a_0t + v_0 \tag{18}$$
$$s(t) = \frac{1}{2}a_0t^2 + v_0t + s_0 \tag{19}$$

where we identify $a_0 \equiv a$, $v_0 \equiv u$, $s_0 \equiv 0$ to match 11 and 12.

The other 3 equations can be obtained from the first 2 with a bit of algebra.

4.3 Geometric Intuition

SUVAT can be visualized as an area of a trapezium in the v-t graph. [Demonstrate in class]

4.4 Exercises

4.4.1 SJPO 2015 General Round Q13

Lecture: `https://youtu.be/j9qWyWCTjrM`

An object is travelling on a straight path and exhibiting a constant acceleration a starts off with an initial velocity $v = 2.0$ ms^{-1}. It has traversed 4.5 m in the third second (from $t = 2$ to $t = 3$), its acceleration a is

(A) 0.5 ms^{-2}

(B) 1.0 ms^{-2}

(C) 1.5 ms^{-2}

(D) 2.0 ms^{-2}

(E) 2.5 ms^{-2}

Ans:

4.4.2 SJPO 2016 General Round Q8

Lecture: `https://youtu.be/Q3ZbXLegdos`

A train moving on straight horizontal tracks slows down from 66 ms^{-1} to 22 ms^{-1} at a constant rate of 2.0 ms^{-2}. What distance does it travel while slowing down?

(A) 490 m

(B) 650 m

(C) 740 m

(D) 970 m

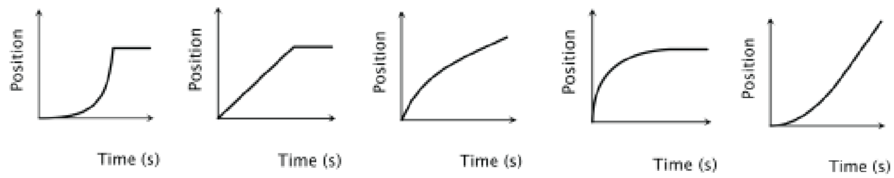
(E) 1100 m

Ans:

4.4.3 SJPO 2018 General Round Q10

Lecture: `https://youtu.be/gLHMK0g8JYc`

The position of an object moving along a linear track is plotted as a function of time. It started from rest and underwent a positive acceleration for some time, followed by a constant velocity. Which of the following graphs correctly shows this situation?



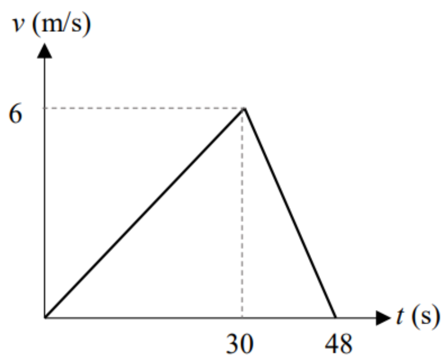
A. B. C. D. E.

Ans:

4.4.4 SJPO 2018 General Round Q16 & Q17

Lecture: `https://youtu.be/5M1mQjZJWI4`

A car travels along a straight road with the speed shown by the v-t graph.



16. What is the acceleration of the car from $t = 30$ to $t = 48$ s ?

    (A) -54 m/s^2

    (B) 48 m/s^2

    (C) -3.0 m/s^2

    (D) 3.0 m/s^2

    (E) -0.33 m/s^2

Ans:

17. What is the total displacement of the car after 48 s ?

(A) 36 m

(B) 48 m

(C) 144 m

(D) 180 m

(E) 210 m

Ans:

4.4.5 SJPO 2017 General Round Q1

Lecture: `https://youtu.be/TW3sIS0lprI`

An electron in a vacuum, starting from rest falls 5 cm near the surface of the earth. Considering only the gravitational force acting on the electron, how long does it take for the electron to travel 5 cm?

(A) 0.1 s

(B) 0.03 s

(C) 0.01 s

(D) 0.001 s

(E) 0.0001 s

Ans:

4.5 1D Dynamics

Lecture: `https://youtu.be/xWJjy_5M4lA`

If we know the acceleration as a function of time $a(t)$, as well as the initial velocity $v(t=0) \equiv v_0$ and position $s(t=0) \equiv s_0$, then we can integrate $a(t)$ twice to get the position as a function of time

$$s(t) = \int \left(\int a(t)\ dt \right)\ dt$$

To obtain this $a(t)$, we use Newton's 2nd law

$$F_{\text{net}}(t) = \frac{d(mv)}{dt} = m\frac{dv}{dt} + v\frac{dm}{dt}$$

In (most) cases where $m(t)$ is a constant wrt time,

$$F_{\text{net}}(t) = ma(t)$$

4.6 Extra: $v^2 = u^2 + 2as$ Connection with Work Energy Theorem

Lecture: `https://youtu.be/1Z9V-1STA_Y`

$v^2 = u^2 + 2as$ is slightly special because it is related to "work energy theorem" in dynamics. One can derive it by integrating $a(t)$ wrt $s(t)$ instead of t

$$\frac{d^2s}{dt^2} = a_0 \tag{20}$$

$$\frac{d}{dt}\left(\frac{ds}{dt}\right)\frac{ds}{dt}dt = a_0 ds \tag{21}$$

$$\frac{1}{2}\frac{d}{dt}\left(v(t)^2\right)dt = a_0 ds \tag{22}$$

$$\int_{t_A}^{t_B}\frac{1}{2}\frac{d}{dt}\left(v(t)^2\right)dt = \int_{s_A}^{s_B} a_0 ds \tag{23}$$

$$\int_{v_A}^{v_B}\frac{1}{2}d\left(v^2\right) = a_0(s_B - s_A) \tag{24}$$

$$\frac{1}{2}(v_B^2 - v_A^2) = a_0(s_B - s_A) \tag{25}$$

$$v_B^2 = v_A^2 + 2a_0(s_B - s_A) \tag{26}$$

where we identify $v_B \equiv v$, $v_A \equiv u$, $s_B \equiv s$, $s_A \equiv 0$, $a_0 \equiv a$ to match 13.

5 Math: 3D Vectors

I made some H2 math videos on vectors

- `https://youtu.be/zohpKrmHkc0` Adding, scaling, subtraction of vectors
- `https://youtu.be/LhXac_HUw-0` Dot product
- `https://youtu.be/1qruXfQRQJU` Cross product

In order to describe 3D motion we have 3 functions $x(t), y(t), z(t)$. To avoid writing three equations, we often package them into a position vector.

$$\vec{r}(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} \tag{27}$$

Geometrically, a vector can be thought of as an arrow. It is the "displacement between 2 points in 3D space", and points in a particular **direction** and has a **length/magnitude**.

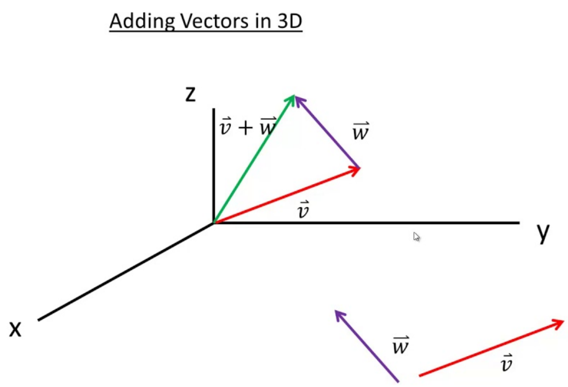
5.1 Vector Operations

5.1.1 Adding

Algebraically, just add each component individually

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{pmatrix}$$

Geometrically,

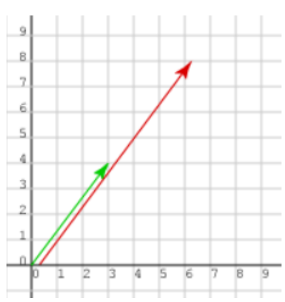


5.1.2 Scaling

Algebraically, scale each component individually

$$\lambda \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \lambda x \\ \lambda y \\ \lambda z \end{pmatrix}$$

Geometrically, if you scale a vector by a positive real number, direction stays the same but length is changed.



If scaled by a negative number, the direction flips (and length changes too).

5.1.3 Length

Algebraically, length is calculated using Pythagoras' theorem.

$$\left| \begin{pmatrix} x \\ y \\ z \end{pmatrix} \right| = \sqrt{x^2 + y^2 + z^2}$$

5.1.4 Dot Product

Dot product takes 2 vectors and outputs a single real number.

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \cdot \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = x_1 x_2 + y_1 y_2 + z_1 z_2$$

Geometrically, dot product is a measure of how similar the direction of the 2 vectors are.

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta$$

where θ is the angle between the 2 vectors.

The dot product is used to define

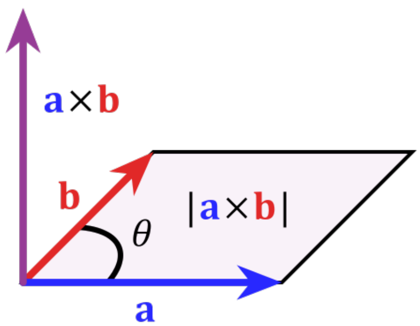
- Work Done $W.(D) = \int \vec{F} \cdot d\vec{r}$
- Magnetic Flux $\Phi = \int \vec{B} \cdot d\vec{A}$

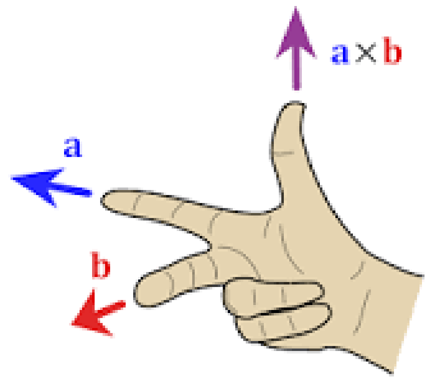
5.1.5 Cross Product

Cross product takes 2 vectors and outputs another vector.

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \times \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} y_1 z_2 - z_1 y_2 \\ z_1 x_2 - x_1 z_2 \\ x_1 y_2 - y_1 x_2 \end{pmatrix}$$

Geometrically, the length of the cross product is the area of the parallelogram. The direction of the cross product is perpendicular (following right hand rule).





The cross product is used to define

- Torque $\vec{\tau} = \vec{r} \times \vec{F}$
- Angular Momentum $\vec{L} = \vec{r} \times \vec{p}$
- Lorentz Force $\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$
- Poynting Vector $\vec{S} = \frac{1}{\mu_0}\vec{E} \times \vec{B}$

5.1.6 Derivatives

Also done component wise

$$\frac{d\vec{r}}{dt} = \frac{d}{dt}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} dx/dt \\ dy/dt \\ dz/dt \end{pmatrix}$$

5.2 Basis Vectors

We sometimes write a vector as

$$\vec{r} = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} \tag{28}$$

$$= x(t)\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + y(t)\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + z(t)\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{29}$$

$$\equiv x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k} \tag{30}$$

Common synonyms for $\hat{\imath}$ are $\hat{x}$ and $\hat{e}_x$. The collection of vectors $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$ are called a set of **basis vectors**, which means that linear combinations of these vectors make up / span the set of all possible vectors. In particular, this set $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$ is called the **Cartesian coordinate basis**. In future, we will learn about other coordinate systems and other basis vectors.

5.3 Exercises

5.3.1 SJPO 2015 General Round Q11

A force $\vec{F} = \vec{F_1} + \vec{F_2}$ can be decomposed as the sum of 2 vectors $\vec{F_1}$ and $\vec{F_2}$. Only the magnitude of $\vec{F_1}$ and the direction of $\vec{F_2}$ are known. Which of the following is the most accurate statement?

(A) Only one combination of $\vec{F_1}$ and $\vec{F_2}$ exists.

(B) There exists exactly two combinations of $\vec{F_1}$ and $\vec{F_2}$.

(C) There exists infinite combinations of $\vec{F_1}$ and $\vec{F_2}$.

(D) At least three combinations of $\vec{F_1}$ and $\vec{F_2}$ exist but the total number of combinations is finite.

(E) Only one or two combinations of $\vec{F_1}$ and $\vec{F_2}$ exist.

Ans:

5.3.2 SJPO 2016 General Round Q21

Initially, a 1 kg box was sliding on frictionless surface at a constant velocity of 4 ms^{-1} in the x direction. A constant force of 1 N was applied on the box in a fixed direction for a time duration of 5 s. After 5 s the speed of the box is 3 ms^{-1}. What is the magnitude of the change in momentum of the box?

(A) 1kgms^{-1}

(B) 2kgms^{-1}

(C) 3kgms^{-1}

(D) 4kgms^{-1}

(E) 5kgms^{-1}

Ans:

Extra: What are the possible directions of the applied force?

6 Physics: Newtonian Mechanics

The following quantities are scalars (real numbers)

- mass m
- speed $|\vec{v}|$
- kinetic energy $E = \frac{1}{2}m|\vec{v}|^2$
- potential energy U

The following quantities are vectors

- position $\vec{r}(t)$
- velocity $\vec{v}(t) \equiv \frac{d\vec{r}}{dt}$
- acceleration $\vec{a}(t) \equiv \frac{d^2\vec{r}}{dt^2}$
- force $\vec{F}(t)$
- momentum $\vec{p}(t) \equiv m\vec{v}(t)$

6.1 Newton's Three Laws

`https://youtu.be/M6uYiOlcOvU` Newton's 1st law defines what an inertial reference frame is: **In an inertial reference frame**, an object at rest remains at rest, or if in motion, remains in motion at a constant velocity unless acted on by a net external force.

In an inertial reference frame, Newton's 2nd law

$$\vec{F}_{\text{net}} = \frac{d(m\vec{v})}{dt}$$

Newton's 3rd law

$$\vec{F}_{\text{A on B}}(t) = -\vec{F}_{\text{B on A}}(t)$$

6.2 Momentum & Impulse

`https://youtu.be/TiOSscc8TKw` Change in a particle's momentum is equal to the impulse it experiences.

Newton's 2nd law: Let $\vec{F}(t)$ be the net force on a particle, then

$$\vec{F}(t) = \frac{d\vec{p}}{dt}$$

Let's focus on one component of the vector equation (say F_x, p_x)

$$F_x(t) = \frac{dp_x}{dt}$$

Integrating both sides wrt time t from $t = t_A$ to $t = t_B$ yields

$$\int_{t_A}^{t_B} F_x(t)dt = \int_{t_A}^{t_B} \frac{dp_x}{dt}dt \tag{31}$$

$$\int_{t_A}^{t_B} F_x(t)dt = \int_{p_x(t_A)}^{p_x(t_B)} dp_x \tag{32}$$

$$= p_x(t_B) - p_x(t_A) \tag{33}$$

$$= \Delta p_x \tag{34}$$

If a particle experiences a force $\vec{F}(t)$ over a time period t_A to t_B, it's (x-component of) momentum will change by $\Delta p_x = \int_{t_A}^{t_B} F_x(t)dt$.

This is true for each component x, y, z so actually it is a vector equation

$$\Delta\vec{p} = \int_{t_A}^{t_B} \vec{F}(t)dt$$

The quantity on the right hand side is called **impulse**.

6.3 Kinetic Energy & Work

`https://youtu.be/DYRDr_ADIAM` Often in physics, it is unnecessary to know the exact path $x(t)$ of a particle. Sometimes, the question only requires us to know $v(t)$.

Example: A particle of mass m experiences a force $F(x(t)) = \frac{A}{x(t)^2}$. It starts at $x(0) = x_0,\ x_0 > 0$ with an initial speed of v_0 toward $x = 0$. Where will the particle come to a rest?

One way of solving this is to find the function $x(t)$ that satisfies $F = ma$

$$\frac{A}{x^2} = m\frac{d^2x}{dt^2}$$

This is a differential equation, which we will cover later. We realise we cannot integrate wrt t because the LHS itself is a function of t we do not know the expression for. We will actually integrate wrt $x(t)$. We need to massage the equation into a different form first.

By chain rule,

$$\frac{d}{dt}\left[\left(\frac{dx}{dt}\right)^2\right] = 2\frac{dx}{dt}\frac{d^2x}{dt^2} \tag{35}$$

Substituting this back in and simplifying

$$\frac{A}{x^2} = \frac{m}{2\frac{dx}{dt}} \frac{d}{dt} \left[\left(\frac{dx}{dt} \right)^2 \right] \tag{36}$$

$$= \frac{m}{2} \frac{d}{dx} \left[\left(\frac{dx}{dt} \right)^2 \right] \tag{37}$$

$$= \frac{m}{2} \frac{d}{dx} v^2 \tag{38}$$

Then integrating from the starting to the stopping point wrt x instead of t,

$$\int_{x_0}^{x_{\text{stop}}} \frac{A}{x^2} dx = \frac{m}{2} \int_{v_0^2}^{v_{\text{stop}}^2 = 0} d \left[\left(\frac{dx}{dt} \right)^2 \right] \tag{39}$$

$$\left[-\frac{A}{x} \right]_{x_0}^{x_{\text{stop}}} = \left[\frac{1}{2} m v^2 \right]_{v^2 = v_0^2}^{v^2 = 0} \tag{40}$$

$$\frac{A}{x_0} - \frac{A}{x_{\text{stop}}} = -\frac{1}{2} m v_0^2 \tag{41}$$

$$\frac{A}{x_0} + \frac{1}{2} m v_0^2 = \frac{A}{x_{\text{stop}}} \tag{42}$$

$$x_{\text{stop}} = A \left(\frac{A}{x_0} + \frac{1}{2} m v_0^2 \right)^{-1} \tag{43}$$

The left hand side of Equation 42 is actually potential energy + kinetic energy. We will talk about potential energy in future, but for now we observe that integrating F with respect to x instead of t was a useful trick. Generalizing this to a general force F,

$$F = ma \tag{44}$$

$$\int_{x_A}^{x_B} F dx = m \int_{x_A}^{x_B} a dx \tag{45}$$

Claim: $a\ dx = v\ dv$. Proof:

$$v\ dv = \frac{1}{2} d \left(v^2 \right) \tag{46}$$

$$= \frac{1}{2} \frac{d}{dt} \left(v^2 \right) dt \tag{47}$$

$$= \frac{1}{2} 2va\ dt \tag{48}$$

$$= a\ v\ dt \tag{49}$$

$$= a\ dx \tag{50}$$

So by a change in variables, $a\ dx = v\ dv$.

$$\int_{x_A}^{x_B} F dx = m \int_{x_A}^{x_B} a dx \tag{51}$$

$$= m \int_{v_A}^{v_B} v dv \tag{52}$$

$$= \frac{1}{2} m \left[v^2\right]_{v_A}^{v_B} \tag{53}$$

$$= \frac{1}{2} m v_B^2 - \frac{1}{2} m v_A^2 \tag{54}$$

$$= \Delta K.E. \tag{55}$$

In other words, if a particle experiences a force $F(x)$ over a distance interval from x_A to x_B, it's change of kinetic energy $\Delta\left(\frac{1}{2}mv^2\right)$ is given by

$$\Delta K.E. = \int_{x_A}^{x_B} F(x)\ dx$$

which we call the **work done** on the particle. This is the **Work Energy theorem**.

Extra: Mathematically, what we have actually done is we have transformed a 2nd order ODE $a(x)$ into a 1st order ODE $v(x)$, which should be easier to integrate in general.

7 Physics: Projectile Motion

`https://youtu.be/-Yq6wzXTU84`

$$\vec{F}_{\text{net}} = m\vec{a} \tag{56}$$

$$\Rightarrow \begin{pmatrix} 0 \\ -mg \\ 0 \end{pmatrix} = m \begin{pmatrix} d^2x/dt^2 \\ d^2y/dt^2 \\ d^2z/dt^2 \end{pmatrix} \tag{57}$$

Integrating twice with respect to time t,

$$\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} u_x t + r_{0x} \\ -\frac{1}{2}gt^2 + u_y t + r_{0y} \\ u_z t + r_{0z} \end{pmatrix} \tag{58}$$

We need to know the initial position $\vec{r}_0 \equiv \vec{r}(t=0)$ and initial velocity $\vec{u} \equiv \vec{v}(t=0)$. For example,

$$\vec{r}(t=0) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{59}$$

$$\vec{v}(t=0) = \begin{pmatrix} u\cos\theta \\ u\sin\theta \\ 0 \end{pmatrix} \tag{60}$$

Then that gives us the path of the projectile with respect to time

$$x(t) = u \cos\theta \; t \tag{61}$$

$$y(t) = u \sin\theta \; t - \frac{1}{2} g t^2 \tag{62}$$

$$z(t) = 0 \tag{63}$$

This is also known as a parametric curve, with time t being the parameter. Every value of t gives us a point in space $(x(t), y(t), z(t))$.

7.1 From parametric to $y(x)$

https://youtu.be/IfFNoepeMow If we want to convert parametric curve into a formula for the graph $y(x)$, then we need to invert Equation 61 to yield

$$t(x) = \frac{x}{u \cos\theta}$$

Substituting $t(x)$ into $y(t)$ gives us $y(x)$

$$y(x) = \tan\theta \; x - \frac{g}{2u^2 \cos^2\theta} x^2 \tag{64}$$

7.2 Finding Range

There are 2 solutions to $y(x) = 0$: the starting one being $x_{\text{start}} = 0$. The other is

$$x_{\text{end}} = \frac{2u^2}{g} \sin\theta \cos\theta = \frac{u^2}{g} \sin 2\theta$$

We now have range as a function of angle $x_{\text{end}}(\theta)$. We can differentiate this with respect to θ and set the derivative $dx_{\text{end}}/d\theta = 0$ to find the maximum range. Strictly speaking, we have to check the second derivative too.

7.3 Exercises

7.3.1 SJPO 2016 General Round Q9

`https://youtu.be/gBjapzk5Tws` A fruit drops from a tree. A boy, 1.5 m tall, stands on the flat ground just under the fruit. The fruit was initially 10 m above the boy's head. A woman standing on the level ground 10 m from the boy immediately throws a ball from a height of 1.5 m above the ground, and deflected the fruit from its path towards the boy's head. Assume that air resistance and her reaction time are negligible. Calculate the minimum speed of the ball?

(A) 10ms^{-1}

(B) 15ms^{-1}

(C) 20ms^{-1}

(D) 25ms^{-1}

(E) 30ms^{-1}

Ans:

7.3.2 SJPO 2016 General Round Q11: Region of Reachability

`https://youtu.be/axU4CSp8UqI` A projectile is launched at velocity v_0 into an ideal ball istic trajectory from the origin of a coordinate system. Given that: when the launch angle is varied, all the possible points that can be hit by the projectile are exactly contained within a parabola with equation $y = a + bx^2$ where y is the vertical height, x is the horizontal displacement from the origin, while a and b are constants. What could be the expression for a and b ?

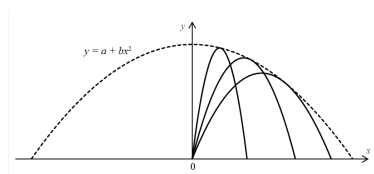
(A) $a = \frac{{v_0}^2}{2g}, b = \frac{g}{{v_0}^2}$

(B) $a = \frac{{v_0}^2}{2g}, b = \frac{g}{2{v_0}^2}$

(C) $a = \frac{{v_0}^2}{2g}, b = \frac{2g}{{v_0}^2}$

(D) $a = \frac{{v_0}^2}{g}, b = -\frac{g}{{v_0}^2}$

(E) $a = \frac{{v_0}^2}{2g}, b = -\frac{g}{2{v_0}^2}$



Ans:

Extra: Prove that the envelope is a parabola. `https://en.wikipedia.org/wiki/Envelope_(mathematics)`

7.3.3 SJPO 2014 General Round Q12: Projectile on Slope

`https://youtu.be/l4lq6dXTug0` As shown in the figure below, a ball is thrown out horizontally from a slope. The slope makes an angle θ with the ground. At the first throw, the ball is ejected with a speed v_1 and at the second throw, it is ejected with a speed v_2. The angles that the ball made with the slope are measured to be α_1 and α_2 respectively. If v_1 is greater than v_2,

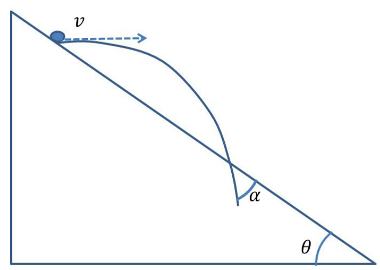
(A) $\alpha_1 = \alpha_2$

(B) $\alpha_1 > \alpha_2$

(C) $\alpha_1 < \alpha_2$

(D) $\alpha < \theta$

(E) It is not possible to infer much as the mass of the ball is not given.



Ans:

8 Physics: Collisions

Collisions between multiple bodies (e.g. dropping a stack of balls) are analysed as a sequence of collisions between 2 bodies at a time. When 2 bodies collide, their **total momentum is always conserved** / invariant before and after the collision (invariant means doesn't change). Their total energy, however, has 3 possibilities:

- Total energy conserved (**elastic**)
- Total energy decreased (**inelastic**) e.g. billiard balls
- Total energy increased (**superelastic**) e.g. compressed spring, explosion

8.1 Momentum Conservation

Q: Why is momentum invariant before and after the collision?

A: Momentum conservation is due to Newton's 3rd Law. To see this, recall that an object's momentum changes by impulse

$$\Delta\vec{p} \text{ from } t_0 \text{ to } t_1 = \int_{t_0}^{t_1} \vec{F}(t)\ dt$$

By Newton's 3rd Law, when 2 bodies interact, the forces they exert on each other are equal in magnitude and opposite in direction.

$$\vec{F}_{\text{A on B}} = -\vec{F}_{\text{B on A}}$$

Integrating both sides wrt time t, one sees that the impulse they experience is also equal in magnitude and opposite in direction.

$$\Delta\vec{p}_B = -\Delta\vec{p}_A$$

As such, the sum of their momentum remains unchanged.

$$\Delta(\vec{p}_A + \vec{p}_B) = \vec{0}$$

This remains true for all forces that obey N3L, including collisions (normal contact force).

8.2 1D Elastic Collisions

Let the masses be m_A, m_B, the initial velocities be u_A, u_B and final velocities be v_A, v_B. Conservation of momentum and energies yields

$$m_A u_A + m_B u_B = m_A v_A + m_B v_B \tag{65}$$
$$\frac{1}{2}m_A u_A^2 + \frac{1}{2}m_B u_B^2 = \frac{1}{2}m_A v_A^2 + \frac{1}{2}m_B v_B^2 \tag{66}$$

Rearranging both equations gives us (respectively)

$$m_A(u_A - v_A) = -m_B(u_B - v_B) \tag{67}$$
$$m_A(u_A - v_A)(u_A + v_A) = -m_B(u_B - v_B)(u_B + v_B) \tag{68}$$

If we divide equation 68 by 67 we obtain

$$u_A + v_A = u_B + v_B \tag{69}$$
$$\Rightarrow u_A - u_B = -(v_A - v_B) \tag{70}$$

This equation is sometimes said verbally as

velocity of approach = velocity of separation

This is because if object A is positioned to the left of object B (and our choice of axis direction is rightward positive), then a positive $u_A - u_B > 0$ would imply that A is getting closer to B, and $u_A - u_B$ describes how fast the distance between them is decreasing. Hence, $u_A - u_B > 0$ is called the **velocity of approach**.

After the collision, $v_A - v_B = -(u_A - u_B) < 0$ implies that B is getting further and further away from A. $-(v_A - v_B) > 0$ quantifies how fast the distance between them is increasing. Hence, $-(v_A - v_B) > 0$ is called the **velocity of**

separation.

Anyway, previously we had one linear equation (momentum) and one quadratic equation (energy). Now we have 2 linear equations (momentum and "rate of approach = separation")

$$m_A u_A + m_B u_B = m_A v_A + m_B v_B \tag{71}$$
$$u_A - u_B = -(v_A - v_B) \tag{72}$$

Linear simultaneous equations are easy to solve, rearrange equation (72) and substitute $v_A = v_B - u_A + u_B$ into (71) to obtain

$$m_A u_A + m_B u_B = m_A(v_B - u_A + u_B) + m_B v_B \tag{73}$$
$$= (m_A + m_B)v_B - m_A u_A + m_A u_B \tag{74}$$
$$v_B = \frac{2m_A}{m_A + m_B} u_A + \frac{m_B - m_A}{m_A + m_B} u_B \tag{75}$$

Repeating this with $v_B = u_A - u_B + v_A$ gives us

$$m_A u_A + m_B u_B = m_A v_A + m_B(u_A - u_B + v_A) \tag{76}$$
$$= (m_A + m_B)v_A + m_B u_A - m_B u_B \tag{77}$$
$$v_A = \frac{m_A - m_B}{m_A + m_B} u_A + \frac{2m_B}{m_A + m_B} u_B \tag{78}$$

One sees that the final velocities almost look like they have a nice symmetry to them. You can memorise these formulas but personally I just derive it every time I need it because I don't like memorising.

8.2.1 Exercises

8.3 Center of Momentum Frame

The Center of Momentum frame (CoM) is very useful because it simplifies the momentum equation greatly. The CoM frame is defined to be the frame in which the total momentum is zero. So, in any dimension,

$$m_A \mathbf{u}_{A,CM} + m_B \mathbf{u}_{B,CM} = m_A \mathbf{v}_{A,CM} + m_B \mathbf{v}_{B,CM} = \mathbf{0} \tag{79}$$

where the bolded quantities **u** emphasise that this is a vector equation. We can show that such a frame exists by proof by construction

$$\mathbf{u}_A \equiv \mathbf{u}_{CM} + \mathbf{u}_{A,CM} \tag{80}$$
$$\mathbf{u}_B \equiv \mathbf{u}_{CM} + \mathbf{u}_{B,CM} \tag{81}$$
$$\text{Define/Choose } \mathbf{u}_{CM} = \frac{m_A \mathbf{u}_A + m_B \mathbf{u}_B}{m_A + m_B} \tag{82}$$
$$m_A \mathbf{u}_{A,CM} + m_B \mathbf{u}_{B,CM} = m_A(\mathbf{u}_A - \mathbf{u}_{CM}) + m_B(\mathbf{u}_B - \mathbf{u}_{CM}) \tag{83}$$
$$= m_A \mathbf{u}_A + m_B \mathbf{u}_B - (m_A + m_B)\mathbf{u}_{CM} \tag{84}$$
$$= \mathbf{0} \tag{85}$$

Rearranging the zero total momentum equation (79) yields A's velocities in terms of B's velocities

$$\mathbf{u}_{A,CM} = -\frac{m_B}{m_A}\mathbf{u}_{B,CM} \tag{86}$$

$$\mathbf{v}_{A,CM} = -\frac{m_B}{m_A}\mathbf{v}_{B,CM} \tag{87}$$

which can be substituted into the conservation of energy equation to eliminate A's initial and final velocities, leaving us with an equation purely relating B's initial and final velocities

$$\frac{1}{2}m_A\mathbf{u}_A^2 + \frac{1}{2}m_B\mathbf{u}_B^2 = \frac{1}{2}m_A\mathbf{v}_A^2 + \frac{1}{2}m_B\mathbf{v}_B^2 \tag{88}$$

$$\Rightarrow \left(\frac{m_B^2}{m_A} + m_B\right)\mathbf{u}_{B,CM}^2 = \left(\frac{m_B^2}{m_A} + m_B\right)\mathbf{v}_{B,CM}^2 \tag{89}$$

$$\Rightarrow \mathbf{u}_{B,CM}^2 = \mathbf{v}_{B,CM}^2 \tag{90}$$

8.3.1 1D Elastic Collision Revisited using CoM Frame

The CoM equations become

$$m_A u_{A,CM} + m_B u_{B,CM} = 0 \tag{91}$$

$$m_A v_{A,CM} + m_B v_{B,CM} = 0 \tag{92}$$

$$u_{B,CM}^2 = v_{B,CM}^2 \tag{93}$$

If B is to the right of A (and our axis is defined as rightward positive), then $u_{B,CM} < 0$ because B is moving left in the CM frame before the collision. $v_{B,CM} > 0$ because B is moving right in the CM frame after the collision. Hence

$$u_{B,CM}^2 = v_{B,CM}^2 \tag{94}$$

$$\Rightarrow v_{B,CM} = -u_{B,CM} \tag{95}$$

$$v_{A,CM} = -\frac{m_B}{m_A}v_{B,CM} \tag{96}$$

$$= \frac{m_B}{m_A}u_{B,CM} \tag{97}$$

$$= -u_{A,CM} \tag{98}$$

Wow! What a remarkably neat result. In the CoM frame, the velocities simply flip direction during the collision. Returning to the lab frame, we obtain

$$\begin{aligned}
v_A &= u_{CM} + v_{A,CM} && (99)\\
&= u_{CM} - u_{A,CM} && (100)\\
&= u_{CM} - (u_A - u_{CM}) && (101)\\
&= 2u_{CM} - u_A && (102)\\
&= 2\frac{m_A u_A + m_B u_B}{m_A + m_B} - u_A && (103)\\
v_A &= \frac{m_A - m_B}{m_A + m_B}u_A + \frac{2m_B}{m_A + m_B}u_B && (104)\\
v_B &= \frac{2m_A}{m_A + m_B}u_A + \frac{m_B - m_A}{m_A + m_B}u_B && (105)
\end{aligned}$$

which matches our 1D elastic collision result. Calculation wise, one might not see the merits of CoM frame in 1D. But intuition wise, the fact that the velocities just flip during the collision gives us the intuition that the balls are just hitting an imaginary wall at the point of collision.

The real power of CoM frame comes in when we consider 2D or 3D collisions. The intuition of the objects hitting a wall still holds true, but in 2D the wall is slanted at an angle. We discuss more about this now.

8.4 2D Elastic Collisions

In 2D elastic collisions, there are 4 unknowns $v_{A,x}, v_{A,y}, v_{B,x}, v_{B,y}$ but only 3 equations from momentum and energy conservation.

$$\begin{aligned}
m_A u_{A,x} + m_B u_{B,x} &= m_A v_{A,x} + m_B v_{B,x} && (106)\\
m_A u_{A,y} + m_B u_{B,y} &= m_A v_{A,y} + m_B v_{B,y} && (107)
\end{aligned}$$

$$\frac{1}{2}m_A(u_{A,x}^2 + u_{A,y}^2) + \frac{1}{2}m_B(u_{B,x}^2 + u_{B,y}^2) = \frac{1}{2}m_A(v_{A,x}^2 + v_{A,y}^2) + \frac{1}{2}m_B(v_{B,x}^2 + v_{B,y}^2) \tag{108}$$

We need 4 equations, but the 4th equation cannot be determined from physics alone. It is a parameter we need to put in by hand, and this parameter can be any (non-trivial) relation between the velocities. An example would be the final angle of velocity of A $\vec{v}_A$

$$v_{A,y} = v_{A,x} \tan\theta_A \tag{109}$$

but really any other constraint could do. Solving these 4 simultaneous equations is possible but extremely tedious. Therefore, it will help alot to consider the

CoM frame. From the previous section, we derived the following result

$$\mathbf{u}_{A,CM}^2 = \mathbf{v}_{A,CM}^2 \tag{110}$$
$$\mathbf{u}_{B,CM}^2 = \mathbf{v}_{B,CM}^2 \tag{111}$$
$$\mathbf{u}_{A,CM} = -\frac{m_B}{m_A}\mathbf{u}_{B,CM} \tag{112}$$
$$\mathbf{v}_{A,CM} = -\frac{m_B}{m_A}\mathbf{v}_{B,CM} \tag{113}$$

In the CoM frame, both objects are bouncing off an imaginary common wall elastically. Conservation of momentum and energy doesn't dictate what the angle of the wall is, we need to specify so by a parameter θ.

Extra: This extends to 3D as well! But this time the wall is a 2D plane, and the normal vector of the wall is described by 2 angles θ, ϕ.

8.4.1 Exercises

In most physics questions, however, we aren't required to solve the general case. Sometimes the CoM frame isn't necessary. Typically these questions are just a test of your simultaneous equation solving skills (that's really most mechanics questions anyway).

8.5 1D Inelastic Collisions

Conservation of momentum still holds, but some energy is lost in the form of heat and sound. Instead of having an equation from energy, we parameterize the inelastic collision with **coefficient of restitution** $0 \leq e \leq 1$, defined by the ratio of the velocity of separation over the velocity of approach, which is information that needs to be provided by the question. The 2 linear equations are

$$m_A u_A + m_B u_B = m_A v_A + m_B v_B \tag{114}$$
$$e(u_A - u_B) = -(v_A - v_B) \tag{115}$$

Solving these 2 simultaneously gives us

$$v_A = \frac{em_B(u_B - u_A) + m_A u_A + m_B u_B}{m_A + m_B} \tag{116}$$
$$v_B = \frac{em_A(u_A - u_B) + m_A u_A + m_B u_B}{m_A + m_B} \tag{117}$$

One can see that substituting $e = 1$ recovers the result for elastic collision. In fact, one can kind of see that these equations are even easier to memorize than the elastic ones.

8.5.1 Exercises

8.6 1D Perfectly Inelastic

Perfectly inelastic collisions are inelastic collisions where $e = 0$. This implies that $v_A = v_B$, which physically means that the 2 bodies stick together after the collision and effectively travel as one.

Maximum Kinetic Energy Loss occurs in inelastic collisions. One can see this clearly in the CoM frame, where both objects collide with each other and stop moving.

8.6.1 Exercises

8.7 Accounting for Rotation

So far the objects were assumed to be point masses, or solid bodies that only move translationally. In reality we know objects can rotate before and after the collision. Rotational dynamics is what we will cover now.

9 Math: Differential Equations

When we studied $F(t) = ma(t)$, we noted that as long as we know the function $F(t)$, we can find $a(t)$ and integrate twice to obtain $x(t)$. However, in most systems, $F(t)$ depends on $x(t)$ or $v(t) \equiv \dot{x}(t)$. Examples include

- Spring Mass $F(x(t)) = -kx(t)$
- Gravitation $F(\vec{r}(t)) = -\frac{GMm}{|r(\vec{t})|^2}\hat{r}$
- Drag $F(v(t)) = \frac{1}{2}\rho v(t)^2 C_D A$
- Lorentz Force $\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$

In these scenarios, we can no longer simply "integrate twice" since we hit a circular dependency. To resolve this, we need to solve the "differential equation". Differential equations are very common in physics. If you solve the Navier-Stokes Differential Equation, you earn a million dollars.

9.1 What is a Differential Equation?

When we first learned the quadratic equation, we were looking for **values** of x that satisfy

$$ax^2 + bx + c = 0$$

We can find 2 complex **solutions** to this **algebraic** equation

$$x = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \quad \text{or} \quad x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

In **differential** equations, we are looking for **functions** $y(x)$ that satisfy (for example)

$$\frac{dy}{dx} = y$$

In this example, $y(x) = e^x$ works! Substitute it into the above to check that LHS = RHS. In fact, one can check that any multiple of e^x works too! So $y(x) = Ae^x$ for any arbitrary constant A is a solution. To determine A, we need to know the "initial condition" $y(0) = A$.

Let's try another example, find the function $y(x)$ that satisfies

$$\frac{dy}{dx} = 5y$$

with initial condition $\left.\frac{dy}{dx}\right|_{x=0} = 10$
Solution: $y(x) = 2e^{5x}$

9.1.1 Physics: RC circuit

Find $q(t)$ that satisfies the following (R, C are constants).

$$\frac{dq}{dt} = -\frac{q}{RC}$$

with initial condition $q(t = 0) = q_0$.
Answer: $q(t) = q_0 \exp(-t/RC)$

What if the initial condition was $I(t = 0) := \left.\frac{dq}{dt}\right|_{t=0} = I_0$ instead?
Answer: $q(t) = I_0 RC \exp(-t/RC)$

9.2 Simple Harmonic Motion ODE

In polynomial equations, the largest power x^n is the degree n of the polynomial. In differential equations, the highest derivative $\frac{d^n y}{dx^n}$ is the order of the differential equation. In this section, we cover a very common class of 2nd order differential equations

$$\frac{d^2 y}{dx^2} = -\omega^2 y$$

which has solution

$$y(x) = A \sin(\omega x + \phi)$$

9.2.1 Derivation by Layman Arguments

$\sin x$ and $\cos x$ are the only (proof involves linear algebra) functions that when differentiated twice, pick up a negative sign. We can afford to put a constant ϕ in the parameter of sin, since constants vanish when differentiated.

9.2.2 Derivation using Complex Exponential

$$\frac{d^2y}{dx^2} = Ay \tag{118}$$

where $-\infty < A < \infty$ is a real constant.

Solution:

$$y = y_0 \exp(\sqrt{A}x)$$

Question: What happens if $A < 0$?
Answer: $\sqrt{A}$ is complex! To be more precise, if $A = -\omega^2$ for a positive real number ω, then

$$y = y_0 \exp(\pm i\omega x)$$

are 2 valid solutions.

A word on linear differential equation: Equation 118 is said to be linear, because if I have 2 solutions $f(x)$ and $g(x)$, then adding them together or scaling them is still a valid solution.

$$\begin{aligned}
\text{Given } \frac{d^2f}{dx^2} &= Af(x) && (119)\\
\text{And } \frac{d^2g}{dx^2} &= Ag(x) && (120)\\
y(x) &= f(x) + g(x) \text{ is a solution too} && (121)\\
\text{Goal: Show } \frac{d^2y}{dx^2} &= Ay(x) && (122)\\
\text{Proof: LHS} &= \frac{d^2}{dx^2}[f(x) + g(x)] && (123)\\
&= \frac{d^2f}{dx^2} + \frac{d^2g}{dx^2} && (124)\\
&= Af(x) + Ag(x) && (125)\\
&= A[f(x) + g(x)] && (126)\\
&= \text{RHS} && (127)
\end{aligned}$$

Examples of Linear Differential Equations:

- Simple Harmonic Motion
- RLC Circuits (linear components)
- (Linear) Wave Equation
- Schrodinger equation (Quantum Mechanics)
- Maxwell's Equation (Electromagnetism)

Since

$$y_+(x) = y_0 \exp(+i\omega x)$$
$$y_-(x) = y_0 \exp(-i\omega x)$$

are 2 solutions to the **linear** DE

$$\frac{d^2 f}{dx^2} = Af(x)$$

any linear combination of them is a valid solution.

$$y(x) = C \exp(+i\omega x) + D \exp(-i\omega x)$$

where C, D are arbitrary complex constants.

But what is a complex exponential? Euler's formula:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Some say it's just a mathematical trick, since "our system doesn't involve complex numbers". It gets philosophical. Richard Feynman famously said: "Shut up and calculate".

So we want our solution $y(x)$ to be real, i.e. Im $[y(x)] = 0$. So that necessitates that $C^* = D$ and so the general solution is

$$\begin{aligned} y(x) &= C \exp(+i\omega x) + C^* \exp(-i\omega x) && (128) \\ &\quad | \quad \text{Polar Form: } C := |C| \exp(i\phi) && (129) \\ &= |C| \mathrm{Re}[\exp(+i(\omega x + \phi))] && (130) \\ &= |C| \cos(\omega x + \phi) && (131) \end{aligned}$$

which is the general solution we heuristically arrived at previously.

Extra: How do we know there are no other functions that solve the ODE? The answer is we can factorise $\left(\frac{d^2}{dx^2} + \omega^2\right) = \left(\frac{d}{dx} - i\omega\right)\left(\frac{d}{dx} + i\omega\right)$ and calculate the kernel of these 2 differential operators.

9.3 Separable ODE

A first order separable differential equation is one of the form

$$\frac{dy}{dx} = \frac{f(x)}{g(y)}$$

One can solve these type of equations in general by rearranging and integrating.

$$\begin{aligned} g(y)\ dy &= f(x)\ dx && (132) \\ \int g(y)\ dy &= \int f(x)\ dx && (133) \end{aligned}$$

9.4 1st Order ODE

A 1st Order ODE takes the following general form

$$\frac{dy}{dx} + P(x)y = Q(x)$$

The solution is

$$y(x) = \frac{\int \left(\exp(\int P(x)\ dx)\right) Q(x)\ dx}{\exp(\int P(x)\ dx)}$$

Derivation: We first multiply both sides by a specific function $\mu(x)$ called the **integrating factor**, which we currently don't know the expression for.

$$\frac{dy}{dx}\mu(x) + [\mu(x)P(x)]y = \mu(x)Q(x)$$

The idea is that we want to make use of the product rule

$$\frac{d}{dx}[y(x)f(x)] = \frac{dy}{dx}f(x) + \frac{df}{dx}y(x) \tag{134}$$

$$\text{which we compare with LHS} \quad \frac{dy}{dx}\mu(x) + [\mu(x)P(x)]y \tag{135}$$

to rewrite the LHS as a total derivative. So we need to choose $\mu(x)$ to satisfy

$$\mu(x) = f(x) \tag{136}$$

$$\mu(x)P(x) = \frac{df}{dx} \tag{137}$$

This is separable, so it's just

$$\frac{d\mu(x)}{dx} = \mu(x)P(x) \tag{138}$$

$$\frac{1}{\mu(x)}\ d\mu(x) = P(x)\ dx \tag{139}$$

$$\int \frac{1}{\mu(x)}\ d\mu(x) = \int P(x)\ dx \tag{140}$$

$$\ln|\mu(x)| = \int P(x)\ dx + C \tag{141}$$

$$\mu(x) = \pm e^C e^{\int P(x)\ dx} \tag{142}$$

After that, we substitute $\mu(x)$ back into the ODE and integrate both sides to solve for $y(x)$

$$\frac{d}{dx}[y(x)\mu(x)] = \mu(x)Q(x) \tag{143}$$

$$y(x)\mu(x) = \int \mu(x)Q(x)\ dx \tag{144}$$

$$y(x) = \frac{\int \left(\exp(\int P(x)\ dx)\right) Q(x)\ dx}{\exp(\int P(x)\ dx)} \tag{145}$$

Side Note: The constant of integration that appears in the integrating factor $\exp(\int P(x)\ dx)$ will cancel out. This makes sense because scaling integrating factor $\mu(x)$ by a constant $\mu(x) \mapsto \lambda\mu(x), \lambda \in \mathbb{R}$ shouldn't affect the solution $y(x)$ since the integrating factor was not in the original ODE.

10 Physics: Simple Harmonic Motion (Part I)

There are a lot of ways Simple Harmonic Motion (SHM) can appear, but one thing that is universal is that the equations of motion always simplify (usually after some approximations) to the form

$$\frac{d^2y}{dt^2} = -\omega^2 y(t) \tag{146}$$

$$y(t) = A\sin(\omega t + \phi) \tag{147}$$

where $\omega = 2\pi/T$ will turn out to be the angular frequency of the oscillation, T being the period of oscillation.

Side note: The ϕ accounts for the cos solution by R-formula $a\sin\theta + b\cos\theta = \sqrt{a^2+b^2}\sin(\theta + \arctan\left(\frac{b}{a}\right))$

10.1 Spring Mass

A mass m lies on a frictionless table, attached to an unstretched spring with spring constant k. The other end of the spring is fixed to a wall. The mass is displaced from it's equilibrium position by x_0 (in a direction perpendicular to the wall) and released. Find the amplitude and period of oscillation.

Hooke's Law: The restoring force F of a spring stretched by x is $\vec{F}(x) = -kx\ \hat{x}$.

Extra: Find the period of oscillation of a mass m hung vertically on a spring with spring constant k in a gravitational field strength g.

10.2 Spring Mass (with a Push)

Same as the above, mass m on frictioness table attached to spring with spring constant k displaced by x_0. But this time, instead of a release, it is pushed, giving the mass an initial speed v_0 (toward the point of equilibrium). Find the new amplitude of oscillation.

This question emphasizes the initial conditions of a differential equation.

Extra: What happens if the initial speed v_0 was directed away from the point of equilibrium instead? Why do we get the same answer for amplitude?

10.3 General Pattern for SHM Questions

I will compile all the SHM questions later on because SHM spans across all the physics topics (from EM to buoyancy), but in general the pattern is just

1. Equations of Motion (EOM)
2. Solve for equilibrium
3. Taylor expand about equilibrium
4. Match with the SHM equation $\ddot{q}(t) = -\omega^2 q(t)$

11 Math: Polar Coordinates (2D)

We previously described the location of a particle using 2 coordinates $x(t), y(t)$. When it comes to rotational questions, it's often more convenient to describe location using polar coordinates $r(t), \theta(t)$. The conversion between the coordinates are given by

$$r = \sqrt{x^2 + y^2} \tag{148}$$
$$\theta = \operatorname{atan2}(y, x) \tag{149}$$
$$x = r\cos\theta \tag{150}$$
$$y = r\sin\theta \tag{151}$$

where atan2 is basically $\arctan(y/x)$ but sensitive to signs.

$$\operatorname{atan2}(y, x) = \begin{cases} \arctan\left(\frac{y}{x}\right) & \text{if } x > 0 \\ \arctan\left(\frac{y}{x}\right) + \pi & \text{if } x < 0 \text{ and } y \geq 0 \\ \arctan\left(\frac{y}{x}\right) - \pi & \text{if } x < 0 \text{ and } y < 0 \\ \frac{\pi}{2} & \text{if } x = 0 \text{ and } y > 0 \\ -\frac{\pi}{2} & \text{if } x = 0 \text{ and } y < 0 \\ \text{undefined} & \text{if } x = 0 \text{ and } y = 0 \end{cases} \tag{152}$$

Side Note: If we used $\theta = \arctan(y/x)$ instead, one might be worried it is unable to distinguish (x, y) and $(-x, -y)$, since both result in the same numerical value for θ. This is a valid mathematical concern. However, for most physics calculations we won't run into issues since we mainly deal with differential forms $ds^2 = dr^2 + r^2 d\theta^2$ and vectors (which are secretly derivatives $\partial/\partial r, \partial/\partial\theta$), both of which are "local objects". Even though the definition uses $\operatorname{atan2}(y, x)$, when performing calculations, we often pretend it is $\arctan(y/x)$ because it results in the same formulas. The reason that it gives the same results is intuitive from the definition (152), but to be mathematically rigorous, the proof will be annoyingly lengthy.

11.1 Basis Vectors

The basis vectors for polar coordinates are shown in Figure 1. To understand why they are defined as such, we should understand the motivation for defining basis vectors and vectors. The motivation is to define velocity. The essence of velocity is to know how an object's coordinates changes after a small amount of time. If a vector is $\vec{v} = 4\hat{e}_x + 3\hat{e}_y$, it means that after a small amount of time Δt, the x coordinate changes by $4\Delta t$, and the y coordinate changes by $3\Delta t$. This implies that the direction a basis vector (e.g. $\hat{e}_x$) points, is by definition, the direction that the location moves when the coordinate (e.g. x) changes.

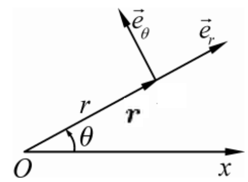


Figure 1: Basis vectors for polar coordinates

For the cartesian coordinate system, the basis vectors are constant: they point in the same direction wherever we are in space. However, the basis vectors for polar coordinates are *not* constant: they point in different directions depending on where we are in space. This is consequential because it implies that the derivative of the basis vectors are not zero, and this mathematical fact gives rise to fictitious/inertial forces such as centrifugal and Coriolis force!

Enough talking, the expression for basis vectors

$$\hat{e}_r = \cos\theta\ \hat{e}_x + \sin\theta\ \hat{e}_y \tag{153}$$

$$\hat{e}_\theta = -\sin\theta\ \hat{e}_x + \cos\theta\ \hat{e}_y \tag{154}$$

11.2 Velocity in Polar Coordinates

The velocity vector can be decomposed into the basis vectors as follows

$$\vec{v} = \dot{r}\ \hat{e}_r + r\dot{\theta}\ \hat{e}_\theta \tag{155}$$

Proof:

$$\vec{v} = \frac{d\vec{r}}{dt} \tag{156}$$

$$= \frac{d}{dt}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \tag{157}$$

$$= \frac{d}{dt}\begin{pmatrix} r(t)\cos\theta(t) \\ r(t)\sin\theta(t) \end{pmatrix} \tag{158}$$

$$= \begin{pmatrix} \dot{r}\cos\theta - r\sin\theta\ \dot{\theta} \\ \dot{r}\sin\theta + r\cos\theta\ \dot{\theta} \end{pmatrix} \tag{159}$$

$$= \dot{r}\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} + r\dot{\theta}\begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} \tag{160}$$

$$\vec{v} = \dot{r}\ \hat{e}_r + r\dot{\theta}\ \hat{e}_\theta \tag{161}$$

There are other ways to prove the above, but the above is simplest.

11.3 Acceleration in Polar Coordinates

Acceleration is the derivative of velocity (Eq 161) wrt time, which can be shown to be equal to

$$\vec{a} = (\ddot{r} - r\dot{\theta}^2)\ \hat{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\ \hat{e}_\theta \tag{162}$$

When we talk about **rotating reference frames** in future, the $r\dot{\theta}^2$ term is responsible for **centrifugal force**, and the $2\dot{r}\dot{\theta}$ term is responsible for **Coriolis force**.

Proof: First we obtain the derivative of the polar basis vectors.

$$\begin{aligned}
\frac{d\hat{e}_r}{dt} &= \frac{d}{dt}\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} && (163)\\
&= \begin{pmatrix} -\sin\theta\ \dot{\theta} \\ \cos\theta\ \dot{\theta} \end{pmatrix} && (164)\\
&= \dot{\theta}\ \hat{e}_\theta && (165)\\
\frac{d\hat{e}_\theta}{dt} &= \frac{d}{dt}\begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} && (166)\\
&= \begin{pmatrix} -\cos\theta\ \dot{\theta} \\ -\sin\theta\ \dot{\theta} \end{pmatrix} && (167)\\
&= -\dot{\theta}\ \hat{e}_r && (168)
\end{aligned}$$

Then we simply apply chain rule

$$\begin{aligned}
\vec{a} &= \frac{d\vec{v}}{dt} && (169)\\
&= \frac{d}{dt}(\dot{r}\ \hat{e}_r + r\dot{\theta}\ \hat{e}_\theta) && (170)\\
&= \ddot{r}\ \hat{e}_r + \dot{r}\frac{d\hat{e}_r}{dt} + \dot{r}\dot{\theta}\ \hat{e}_\theta + r\ddot{\theta}\ \hat{e}_\theta + r\dot{\theta}\frac{d\hat{e}_\theta}{dt} && (171)\\
&= (\ddot{r} - r\dot{\theta}^2)\ \hat{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\ \hat{e}_\theta && (172)
\end{aligned}$$

12 Physics: Rotational Kinematics in 2D (Polar)

Newton's 2nd law is just

$$\begin{aligned}
F_r\ \hat{e}_r + F_\theta\ \hat{e}_\theta = \vec{F}_{net} = m\vec{a} &= m\left[(\ddot{r} - r\dot{\theta}^2)\ \hat{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\ \hat{e}_\theta\right] && (173)\\
\Rightarrow F_r &= m(\ddot{r} - r\dot{\theta}^2) && (174)\\
\text{and } F_\theta &= m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) && (175)
\end{aligned}$$

12.1 Centripetal Acceleration

Often, the radial coordinate function $r(t)$ is constant. Examples include: pendulum, point mass sliding down hemisphere, planet in orbit, cup on rotating Chinese table. When $\dot{r} = 0$, Newton's 2nd law simplifies to

$$F_r = -r\dot{\theta}^2 \tag{176}$$

$$F_\theta = r\ddot{\theta} \tag{177}$$

12.2 Example: Pendulum

A mass m is hung by an inextensible string of length l in a gravitational field strength g. It is displaced by a small angular displacement θ_0 and released. Find the amplitude and period of oscillation.

12.2.1 Exercises

SJPO 2015 General Round Q16 As shown in the diagram, a pendulum of length L is hung from the ceiling and at a point P, a peg is placed. L' denotes the shortened length of the pendulum during part of its oscillation. The period of the pendulums oscillation is now

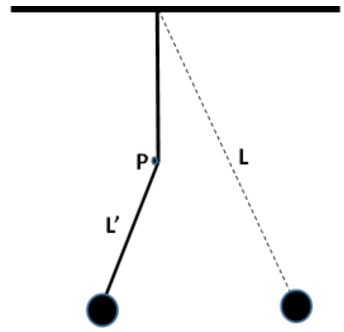
(A) $2\pi\sqrt{\frac{L}{g}}$

(B) $2\pi\sqrt{\frac{L'}{g}}$

(C) $2\pi\left[\sqrt{\frac{L}{g}}+\sqrt{\frac{L'}{g}}\right]$

(D) $\pi\left[\sqrt{\frac{L}{g}}+\sqrt{\frac{L'}{g}}\right]$

(E) $\pi\sqrt{\frac{L+L'}{g}}$



Ans:

SJPO 2018 General Round Q21 The bob of a simple pendulum travels 2 m in one complete oscillation in a time of 2.000 s. Assuming that damping is negligible, when the same pendulum is made to travel 4 m in one complete oscillation, the time taken is

(A) 4.000 s

(B) More than 2.000 s

(C) 2.000 s

(D) Less than 2.000 s

(E) 1.000 s

Ans:

13 Physics: Rotational Dynamics

So far we have been dealing with point masses. But in reality most objects take up some volume.

Before we talk about *continuous* mass distributions, let's derive everything using discrete masses first. If there is a mass m

We usually model these **rigid body** objects as a uniform mass density over some volume. It is made up of many small masses $dm = \rho dV$, where dm is called an infinitesimal mass element and $dV = dx\ dy\ dz$ is an infinitesimal volume element. If we integral dm over volume $\mathcal{V}$ the object occupies, we get the mass

$$\int_{\mathcal{V}} dm = m \tag{178}$$

14 Physics: Statics

Need torque

15 Math: Other Coordinates

16 Math: 2D & 3D Integrals

17 Physics: Moment of Inertia

18 Physics: Electrostatics

19 Math: Vector Calculus

20 Physics: Potentials & Potential Energy

21 Physics: Electromagnetism

22 Physics: AC Circuits

23 Math: Differential Equations

23.1 SHM: Driven and Damping

23.2 Laplace Transform

24 Physics: Constraining Forces - Normal, Tension, Friction

25 Physics: Pulleys

25.1 Friction - Capstan Equation

26 Physics: DC Circuits

27 Physics: Fluid Mechanics

28 Physics: Waves

29 Physics: Optics

30 Physics: Thermodynamics