Part of what we're left with, namely $\sqrt{n_{\boldsymbol{q}}}|n_{\boldsymbol{p}}(n_{\boldsymbol{q}}-1)n_{\boldsymbol{r}}\rangle$, is exactly the same as if we'd just acted on the original state with $\hat{a}_{\boldsymbol{q}}$. The effect of dynamicizing this operator has just been to multiply by a factor $e^{-iE_{\boldsymbol{q}}t}$, so we conclude that the operator we seek is $\hat{a}_{\boldsymbol{q}}e^{-i(E_{\boldsymbol{q}}t-\boldsymbol{q}\cdot\boldsymbol{x})} = \hat{a}_{\boldsymbol{q}}e^{-iq\cdot x}$.

In summary, what we call the **mode expansion** of the scalar field is given by[4]

$$\hat{\phi}(x) = \int \frac{d^3p}{(2\pi)^{\frac{3}{2}}} \frac{1}{(2E_{\boldsymbol{p}})^{\frac{1}{2}}} \left(\hat{a}_{\boldsymbol{p}}e^{-ip\cdot x} + \hat{a}^{\dagger}_{\boldsymbol{p}}e^{ip\cdot x}\right), \tag{11.12}$$

$$\text{with } E_{\boldsymbol{p}} = +(\boldsymbol{p}^2+m^2)^{\frac{1}{2}}.$$

[4] This looks a little different from what we had in Chapter 4 when we introduced field operators. We will explain the reason for the difference in Section 11.5.

Note that by expanding out the position field we'll get the momentum expansion for free, since for our scalar field example $\Pi^{\mu}(x) = \partial^{\mu}\phi(x)$. Also note that because the field in our classical Lagrangian is a real quantity, the field operator $\hat{\phi}(x)$ should be Hermitian. By inspection $\hat{\phi}^{\dagger}(x) = \hat{\phi}(x)$, so this is indeed the case.

11.2 Normalizing factors

This section can be skipped if you are happy to take eqn 11.12 on trust. The purpose here is simply to justify the factors $(2\pi)^{3/2}(2E_{\boldsymbol{p}})^{1/2}$ which otherwise seem to appear by magic.

Before proceeding, we will justify the normalization factors in eqn 11.12. In evaluating integrals over momentum states we have the problem that d^3p is not a Lorentz-invariant quantity. We can use d^4p where $p=(p^0,\boldsymbol{p})$ is the four-momentum, but for a particle of mass m then only values of the four-momentum which satisfy[5] $p^2=m^2$ need to be considered. This is known as the **mass shell condition** (see Fig. 11.1). Consequently we can write our integration measure

$$d^4p\,\delta(p^2-m^2)\,\theta(p^0). \tag{11.13}$$

[5] The condition $p^2=m^2$ means that $(p^0)^2-\boldsymbol{p}^2 = E_{\boldsymbol{p}}^2-\boldsymbol{p}^2=m^2$.

We have included a Heaviside step function $\theta(p^0)$ to select only positive mass particles.

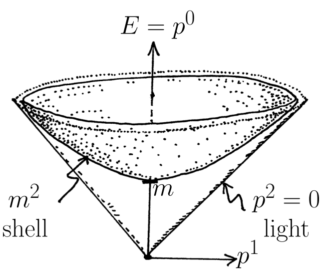

Fig. 11.1 The mass shell $p^2=m^2$ is a hyperboloid in four-momentum space. Also shown is the equivalent surface for light, $p^2=0$.

Example 11.2

Show that $\delta(p^2-m^2)\,\theta(p_0) = \frac{1}{2E_{\boldsymbol{p}}}\delta(p_0-E_{\boldsymbol{p}})\theta(p_0)$.

We use the identity

$$\delta[f(x)] = \sum_{\{x|f(x)=0\}} \frac{1}{|f'(x)|}\delta(x), \tag{11.14}$$

where the notation tells us that the sum is evaluated for those values of x that make $f(x)=0$. We take $x=p^0$ and $f(p^0)=p^2-m^2=(p^0)^2-\boldsymbol{p}^2-m^2$. This gives us that $|f'(p^0)|=2|p^0|$ and we use the fact that the zeros of $f(p^0)$ occur for $p^0=\pm(\boldsymbol{p}^2+m^2)^{\frac{1}{2}}=\pm E_{\boldsymbol{p}}$ to write

$$\delta(p^2-m^2)\,\theta(p_0) = \frac{1}{2E_{\boldsymbol{p}}}[\delta(p_0-E_{\boldsymbol{p}})\theta(p_0)+\delta(p_0+E_{\boldsymbol{p}})\theta(p_0)], \tag{11.15}$$

and so the result follows (since the second term in eqn 11.15 is zero).

Thus we will write our Lorentz-invariant measure as

$$\frac{\mathrm{d}^3 p}{(2\pi)^3 2E_{\boldsymbol{p}}}, \tag{11.16}$$

where the additional factor $\delta(p_0 - E_{\boldsymbol{p}})\theta(p_0)$ is there in every calculation and so we suppress it, and we have included the factor $1/(2\pi)^3$ because the mode expansion is essentially an inverse Fourier transform (we have one factor of $1/(2\pi)$ for every component of three-momentum). We are now in a position to write down integrals, for example:

$$1 = \int \frac{\mathrm{d}^3 p}{(2\pi)^3 2E_{\boldsymbol{p}}} |p\rangle\langle p|. \tag{11.17}$$

This requires us to have Lorentz-covariant four-momentum states $|p\rangle$. We previously normalized momentum states according to

$$\langle \boldsymbol{p}|\boldsymbol{q}\rangle = \delta^{(3)}(\boldsymbol{p} - \boldsymbol{q}), \tag{11.18}$$

so our new four-momentum states $|p\rangle$ will need to be related to the three-momentum states $|\boldsymbol{p}\rangle$ by

$$|p\rangle = (2\pi)^{3/2}(2E_{\boldsymbol{p}})^{1/2}|\boldsymbol{p}\rangle, \tag{11.19}$$

and then their normalization can be written

$$\langle p|q\rangle = (2\pi)^3 2E_{\boldsymbol{p}}\delta^{(3)}(\boldsymbol{p} - \boldsymbol{q}). \tag{11.20}$$

Similarly, to make creation operators $\hat{\alpha}^\dagger$ appropriately normalized so that they create Lorentz-covariant states, we must define them by

$$\hat{\alpha}^\dagger_{\boldsymbol{p}} = (2\pi)^{3/2}(2E_{\boldsymbol{p}})^{1/2}\hat{a}^\dagger_{\boldsymbol{p}}, \tag{11.21}$$

so that $\hat{\alpha}^\dagger_{\boldsymbol{p}}|0\rangle = |p\rangle$. In this case our mode expansion would take the form of a simple inverse Fourier transform using our Lorentz-invariant measure

$$\hat{\phi}(x) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{(2E_{\boldsymbol{p}})} \left(\hat{\alpha}_{\boldsymbol{p}} \mathrm{e}^{-\mathrm{i}p\cdot x} + \hat{\alpha}^\dagger_{\boldsymbol{p}} \mathrm{e}^{\mathrm{i}p\cdot x}\right), \tag{11.22}$$

or writing in terms of $\hat{a}^\dagger_{\boldsymbol{p}}$ and $\hat{a}_{\boldsymbol{p}}$ rather than $\hat{\alpha}^\dagger_{\boldsymbol{p}}$ and $\hat{\alpha}_{\boldsymbol{p}}$:

$$\hat{\phi}(x) = \int \frac{\mathrm{d}^3 p}{(2\pi)^{\frac{3}{2}}} \frac{1}{(2E_{\boldsymbol{p}})^{\frac{1}{2}}} \left(\hat{a}_{\boldsymbol{p}} \mathrm{e}^{-\mathrm{i}p\cdot x} + \hat{a}^\dagger_{\boldsymbol{p}} \mathrm{e}^{\mathrm{i}p\cdot x}\right), \tag{11.23}$$

which is identical to eqn 11.12.

11.3 What becomes of the Hamiltonian?

We can now substitute our expansion of the field operator $\hat{\phi}(x)$ into the Hamiltonian to complete our programme of canonical quantization. This will provide us with an expression for the energy operator in terms