Quantum Field Theory

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Agenda

Special Relativity

Punchline: SR is about events in spacetime and the invariant spacetime interval.

An event in spacetime has coordinates $x^\mu = (t, x, y, z)$, where $\mu = 0, 1, 2, 3$ is a spacetime index.

Postulate: 2 events in spacetime separated by Δx^μ has an invariant spacetime interval

$$\Delta s^2 = \eta_{\mu\nu}\Delta x^\mu \Delta x^\nu = \Delta t^2 - \Delta\vec{\mathbf{x}}^2 \tag{1}$$

where $\eta_{\mu\nu} = diag(1, -1, -1, -1)$ can be viewed as a 4x4 matrix, which can be inverted. We call the inverse $\eta^{\rho\mu}$ because then we can conveniently do $\eta^{\rho\mu}\eta_{\mu\nu} = \delta^\rho{}_\nu$, which lets use use η as a tool to "raise and lower indices".

Special Relativity

Δs^2 is to be interpreted as a symbol itself, it never makes sense to write Δs alone. Physicists just use square because it has units of length squared.

Question: What are the coordinate transformations that leave Δs^2 unchanged?
Answer: Poincare Group = Lorentz Transformations + Spacetime Translation

Lorentz Transformations: Acting on Position

Lorentz transformations $\Lambda^{\mu}{}_{\nu}$ act on spacetime coordinates x^{μ} as

$$x^{\mu} \mapsto w^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} \tag{2}$$

Similar to the metric, we construct the inverse Lorentz transformation $(\Lambda^{-1})^{\rho}{}_{\mu}$

$$\begin{aligned} w^{\mu} &= \Lambda^{\mu}{}_{\nu} x^{\nu} && (3) \\ (\Lambda^{-1})^{\rho}{}_{\mu} w^{\mu} &= (\Lambda^{-1})^{\rho}{}_{\mu} (\Lambda^{\mu}{}_{\nu}) x^{\nu} && (4) \\ &= \delta^{\rho}{}_{\nu} x^{\nu} && (5) \\ &= x^{\rho} && (6) \end{aligned}$$

Lorentz Invariants

One can show that (in Appendix 2) that all covectors transform as

$$A_\mu \mapsto (\Lambda^{-1})^\rho{}_\mu A_\rho \tag{7}$$

And all vectors transform as

$$v^\mu \mapsto \Lambda^\mu{}_\nu v^\nu \tag{8}$$

This implies that any quantity like $A_\mu v^\mu$ that doesn't have any leftover uncontracted Lorentz indices is invariant under Lorentz transformations.

This lets us build Lorentz invariant actions.

Lorentz Covariance

An equation is Lorentz covariant if it "respects indices rules", which would imply that the equation still holds in all frames, even after quantities in the equation are Lorentz transformed. An example is electromagnetism

$$\partial_\alpha F^{\alpha\beta} = J^\beta \tag{9}$$

This lets us build Lorentz covariant equations of motion (EOM).

Lagrangian Mechanics

In Lagrangian mechanics, we minimize the action

$$S = \int_{t_A}^{t_B} dt\ \mathcal{L}(q_i, \dot{q}_i) \tag{10}$$

which yields Euler Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) = 0 \tag{11}$$

for every degree of freedom q_i. Euler-Lagrange is an EOM for $q_i(t)$.

Classical Field Theory

$$S = \int_{\mathcal{M}} d^4x \; \mathcal{L}(\phi, \partial_\mu \phi) \tag{12}$$

In field theory, there are many more degrees of freedom ($\phi(x)$ $\forall x \in M$), so Euler-Lagrange becomes

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \phi\right)} - \frac{\partial \mathcal{L}}{\partial \phi} = 0 \tag{13}$$

which is an EOM for $\phi(x)$ which **holds at every point in space**.

Classical Field Theory

Let's try an example

$$S = \int_{\mathcal{M}} d^4x \; \frac{1}{2}(\partial^\mu \phi \partial_\mu \phi + m^2 \phi^2) \tag{14}$$

Apply Euler-Lagrange equations yields

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \frac{\partial \mathcal{L}}{\partial \phi} \tag{15}$$

$$\partial_\mu \partial^\mu \phi - m^2 \phi = 0 \tag{16}$$

This is the Klein-Gordan equation. The solutions for $\phi(x)$ can be better described when expanded in Fourier space

$$\phi(x) = \int d^4k \; e^{-ik\cdot x} \tilde{\phi}(k) \tag{17}$$

where $k^\mu = (\omega_k, k_x, k_y, k_z)$.

Classical Field Theory

Substituting Equation 17 into 16 gives the dispersion relation for relativistic particles

$$k^\mu k_\mu - m^2 = 0 \tag{18}$$

$$\omega_k^2 - \vec{k} \cdot \vec{k} - m^2 = 0 \tag{19}$$

$$\omega_k^2 = \vec{k}^2 + m^2 \tag{20}$$

which parallels the relativistic energy relation

$$E^2 = \vec{p}^2 + m^2 \tag{21}$$

So in summary, $\phi(x)$ can be decomposed into different modes, and each mode must satisfy the dispersion relation. Dispersion relation $=$ EOM in momentum space.

Classical Field Theory: Adding a source

One can add a source term $J(x)$

$$S = \int_{\mathcal{M}} d^4x \; \frac{1}{2}(\partial^\mu \phi \partial_\mu \phi + m^2 \phi^2) + J(x)\phi(x) \tag{22}$$

Which would turn the EOM into

$$\partial_\mu \partial^\mu \phi - m^2 \phi = J \tag{23}$$

Later we will learn how to solve this using Green's Functions.

Quantum Mechanics: Operators

Observables are operators (e.g. $\hat{x}, \hat{p}, \hat{p^2}$), we are interested in their expectation values

$$\langle \hat{x} \rangle_\Psi \text{ tells us where particle } |\Psi\rangle \text{ is} \tag{24}$$
$$\langle \hat{x}^2 \rangle_\Psi \text{ tells us how sure we are} \tag{25}$$

where $\langle \hat{A} \rangle_\Psi \equiv \langle \Psi | \hat{A} | \Psi \rangle$.

Operators can have (explicit) time dependence too, e.g.

$$\hat{H}_S(t) = B_x(t)\hat{\sigma}_x + B_y(t)\hat{\sigma}_y + B_z(t)\hat{\sigma}_z \tag{26}$$

We used the subscript S to denote "time dependence in Schrodinger picture". It is important to understand **time dependence** vs **time evolution**.

Quantum Mechanics

Let's denote $|\Psi_0\rangle \equiv |\Psi(t=0)\rangle$.

Expectation values can change over time

$$\langle\hat{A}\rangle_\Psi(t) \equiv \langle\Psi_0|\,\hat{U}^\dagger(t)\hat{A}_S(t)\hat{U}(t)\,|\Psi_0\rangle \tag{27}$$

$$\text{(Schrodinger)} \quad = \left(\langle\Psi_0|\,\hat{U}^\dagger(t)\right)\hat{A}_S(t)\left(\hat{U}(t)\,|\Psi_0\rangle\right) \tag{28}$$

$$\text{(Heisenberg)} \quad = \langle\Psi_0|\left(\hat{U}^\dagger(t)\hat{A}_S(t)\hat{U}(t)\right)|\Psi_0\rangle \tag{29}$$

$$\text{(Interaction)} \quad = \left(\langle\Psi_0|\,\hat{S}^\dagger(t)\right)\left(\hat{U}_0^\dagger(t)\hat{A}_S(t)\hat{U}_0(t)\right)\left(\hat{S}(t)\,|\Psi_0\rangle\right) \tag{30}$$

Schrodinger Picture: Deriving Time Evolution Operator

Putting time evolution of states in the Schrodinger picture into Schrodinger's equation lets us find $\hat{U}(t)$,

$$|\Psi(t)\rangle = \hat{U}(t)\,|\Psi_0\rangle \tag{31}$$

$$i\hbar\frac{\partial}{\partial t}\,|\Psi(t)\rangle = \hat{H}(t)\,|\Psi(t)\rangle \tag{32}$$

gives us a differential equation for $\hat{U}(t)$

$$i\hbar\frac{\partial}{\partial t}\left(\hat{U}(t)\,|\Psi_0\rangle\right) = \hat{H}(t)\left(\hat{U}(t)\,|\Psi_0\rangle\right) \tag{33}$$

Since this holds for any initial state $|\Psi_0\rangle$, it's a differential equation for the $\hat{U}(t)$ operator.

$$i\hbar\frac{\partial}{\partial t}\hat{U}(t) = \hat{H}(t)\hat{U}(t) \tag{34}$$

Time Evolution Operator

$$i\hbar\frac{\partial}{\partial t}\hat{U}(t) = \hat{H}(t)\hat{U}(t) \tag{35}$$

can be solved "recursively" as

$$\hat{U}(t_1) = 1 + \left(\frac{-i}{\hbar}\right)\int_0^{t_1} dt_2 \hat{H}(t_2)\hat{U}(t_2) \tag{36}$$

$$\begin{aligned}\hat{U}(t_1) = 1 &+ \left(\frac{-i}{\hbar}\right)\int_0^{t_1} dt_2 \hat{H}(t_2) \\ &+ \left(\frac{-i}{\hbar}\right)^2\int_0^{t_1} dt_2 \int_0^{t_2} dt_3 \hat{H}(t_2)\hat{H}(t_3) \\ &+ \left(\frac{-i}{\hbar}\right)^3\int_0^{t_1} dt_2 \int_0^{t_2} dt_3 \int_0^{t_3} dt_4 \hat{H}(t_2)\hat{H}(t_3)\hat{H}(t_4) \\ &+ \dots\end{aligned} \tag{37}$$

Time Evolution Operator

$$
\begin{aligned}
\hat{U}(t_1) = 1 &+ \left(\frac{-i}{\hbar}\right) \int_0^{t_1} dt_2 \hat{H}(t_2) \\
&+ \left(\frac{-i}{\hbar}\right)^2 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 \hat{H}(t_2)\hat{H}(t_3) \\
&+ \left(\frac{-i}{\hbar}\right)^3 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 \int_0^{t_3} dt_4 \hat{H}(t_2)\hat{H}(t_3)\hat{H}(t_4) \\
&+ \ldots
\end{aligned}
\tag{38}
$$

We define the **Time-Ordered Exponential** of an operator for convenience, and write the above as

$$
\hat{U}(t) = \mathcal{T} \exp\left(\frac{-i}{\hbar} \int_0^t dt' \hat{H}(t')\right) \tag{39}
$$

Time Evolution Operator

$$\hat{U}(t) = \mathcal{T} \exp\left(\frac{-i}{\hbar}\int_0^t dt' \hat{H}(t')\right)$$

means "expand it in Taylor series and time-order the operators appropriately (earliest ones on the right)."

This time-ordered exponential will appear again when talking about the Interaction picture. It will be used in the Dyson Series, and this taylor expansion is represented as Feynman diagrams.

In the special case that $\frac{d}{dt}\hat{H}(t) = 0$,

$$\hat{U}(t) = \exp\left(\frac{-i\hat{H}t}{\hbar}\right) \tag{40}$$

Heisenberg Picture

$$\hat{A}_H(t) \equiv \hat{U}^\dagger(t)\hat{A}_S(t)\hat{U}(t) \tag{41}$$

$$\begin{aligned}
\frac{d}{dt}\hat{A}_H(t) &= \frac{dU^\dagger}{dt}\hat{A}_S(t)\hat{U} + \hat{U}^\dagger\hat{A}_S(t)\frac{d\hat{U}}{dt} + \hat{U}^\dagger\frac{\partial\hat{A}_S(t)}{\partial t}\hat{U} \qquad (42)\\
&= \left(\frac{d\hat{U}^\dagger}{dt}\hat{U}\right)\hat{U}^\dagger\hat{A}_S(t)\hat{U} + \hat{U}^\dagger\hat{A}_S(t)\hat{U}\left(\hat{U}^\dagger\frac{d\hat{U}}{dt}\right) + \left(\frac{\partial\hat{A}_S}{\partial t}\right)_H\\
&= \left(\frac{d\hat{U}^\dagger}{dt}\hat{U}\right)\hat{A}_H(t) + \hat{A}_H(t)\left(\hat{U}^\dagger\frac{d\hat{U}}{dt}\right) + \left(\frac{\partial\hat{A}_S}{\partial t}\right)_H
\end{aligned}$$

Differentiating $\hat{U}^\dagger\hat{U} = \mathbb{I}$ gives

$$\frac{d\hat{U}^\dagger}{dt}\hat{U} + \hat{U}^\dagger\frac{d\hat{U}}{dt} = 0 \tag{43}$$

$$\frac{d\hat{U}^\dagger}{dt}\hat{U} = -\hat{U}^\dagger\frac{d\hat{U}}{dt} \tag{44}$$

Heisenberg Picture

This lets us write a differential equation for the **time evolution** of operators in the Heisenberg picture

$$\frac{d}{dt}\hat{A}_H(t) = -\left[\hat{U}^\dagger \frac{d\hat{U}}{dt}, \hat{A}_H(t)\right] + \left(\frac{\partial \hat{A}_H}{\partial t}\right)_H$$

$$= \frac{i}{\hbar}\left[\hat{U}^\dagger \hat{H}_S(t)\hat{U}, \hat{A}_H(t)\right] + \left(\frac{\partial \hat{A}_S}{\partial t}\right)_H \tag{45}$$

$$= \frac{i}{\hbar}\left[\hat{H}_H(t), \hat{A}_H(t)\right] + \left(\frac{\partial \hat{A}_S}{\partial t}\right)_H \tag{46}$$

Remember this ODE is just another way of expressing

$$\hat{A}_H(t) \equiv \hat{U}^\dagger(t)\hat{A}_S(t)\hat{U}(t) \tag{47}$$

Example: QHO in Heisenberg Picture

$$H = \frac{\hat{p}^2}{2m} + \frac{m\omega^2\hat{x}^2}{2} \tag{48}$$

Differential equation for the operators are

$$\frac{d}{dt}\hat{x}(t) = \frac{i}{\hbar}[H, \hat{x}(t)] = \frac{\hat{p}}{m} \tag{49}$$

$$\frac{d}{dt}\hat{p}(t) = \frac{i}{\hbar}[H, \hat{p}(t)] = -m\omega^2\hat{x} \tag{50}$$

Initial conditions are

$$\dot{\hat{p}}(0) = -m\omega^2\hat{x}_0, \tag{51}$$

$$\dot{\hat{x}}(0) = \frac{\hat{p}_0}{m} \tag{52}$$

Solving these yield

$$\hat{x}(t) = \hat{x}_0\cos(\omega t) + \frac{\hat{p}_0}{\omega m}\sin(\omega t) \tag{53}$$

$$\hat{p}(t) = \hat{p}_0\cos(\omega t) - m\omega\hat{x}_0\sin(\omega t) \tag{54}$$

Interaction Picture

We split the Hamiltonian $\hat{H}_S(t) = \hat{H}_0(t) + \hat{H}_1(t)$ into a free part and an interacting part. The time evolution operator then splits as follows

$$\hat{U}(t) \equiv \hat{U}_0(t)\hat{S}(t) \tag{55}$$

$$\hat{S}(t) \equiv \hat{U}_0^\dagger(t)\hat{U}(t) \tag{56}$$

We don't know what $\hat{S}(t)$ is at the moment, but will derive an expression for it later.

In the interaction picture, operators evolve using the free part in the Heisenberg picture, while states evolve using $\hat{S}(t)$ in the Schrodinger picture.

$$\langle \hat{A} \rangle(t) = \left\langle \Psi_0 \left| \hat{U}^\dagger(t)\hat{A}_S(t)\hat{U}(t) \right| \Psi_0 \right\rangle \tag{57}$$

$$= \left(\langle \Psi_0 | \, \hat{S}^\dagger(t) \right) \left(\hat{U}_0^\dagger(t)\hat{A}_S(t)\hat{U}_0(t) \right) \left(\hat{S}(t) \, |\Psi_0\rangle \right) \tag{58}$$

Interaction Picture

This $\hat{S}(t)$ will be shown to be equals to the interacting part evolved using free part in the Heisenberg picture

$$\hat{S}(t) = \hat{H}_{1,H}(t) = \hat{U}_0^\dagger(t)\hat{H}_{1,S}(t)\hat{U}_0(t) \tag{59}$$

Proof:

$$i\hbar\frac{\partial}{\partial t}\hat{S}(t)|\Psi_0\rangle = i\hbar\frac{\partial \hat{U}_0^\dagger}{\partial t}\hat{U}(t)\,|\Psi_0\rangle + i\hbar\hat{U}_0^\dagger(t)\frac{\partial \hat{U}(t)}{\partial t}\,|\Psi_0\rangle \tag{60}$$

Then using

$$\hat{U}_0^\dagger(t)\hat{U}_0(t) = \mathbb{I} \tag{61}$$

$$\frac{\partial \hat{U}_0^\dagger}{\partial t}\hat{U}_0(t) + \hat{U}_0^\dagger(t)\frac{\partial \hat{U}_0}{\partial t} = 0 \tag{62}$$

$$i\hbar\frac{\partial \hat{U}_0^\dagger}{\partial t}\hat{U}_0(t) = -\hat{U}_0^\dagger(t)i\hbar\frac{\partial \hat{U}_0}{\partial t} \tag{63}$$

$$= -\hat{U}_0^\dagger(t)\hat{H}_0(t)\hat{U}_0(t) \tag{64}$$

$$i\hbar\frac{\partial \hat{U}_0^\dagger}{\partial t} = -\hat{U}_0^\dagger(t)\hat{H}_0(t) \tag{65}$$

Interaction Picture

We can substitute Equation 65 into 60 to get

$$i\hbar\frac{\partial}{\partial t}\hat{S}(t)\,|\Psi_0\rangle = -\hat{U}_0^\dagger(t)\hat{H}_0(t)\hat{U}(t)\,|\Psi_0\rangle + \hat{U}_0^\dagger(t)\hat{H}_S(t)\hat{U}(t)|\Psi_0\rangle$$

And since $\hat{H}_S(t) = \hat{H}_0(t) + \hat{H}_1(t)$,

$$i\hbar\frac{\partial}{\partial t}\hat{S}(t)|\Psi_0\rangle = \hat{U}_0^\dagger\hat{H}_1(t)\hat{U}(t)\,|\Psi_0\rangle \tag{66}$$

$$= \hat{U}_0^\dagger\hat{H}_1(t)\hat{U}_0(t)\hat{S}(t)\,|\Psi_0\rangle \tag{67}$$

$$i\hbar\frac{\partial}{\partial t}\hat{S}(t) = \left(\hat{U}_0^\dagger(t)\hat{H}_1(t)\hat{U}_0(t)\right)\hat{S}(t) \tag{68}$$

This appeared in the Schrodinger picture before! Solving yields

$$\hat{S}(t) = T\exp\left(-\frac{i}{\hbar}\int_0^t dt'\,\hat{U}_0^\dagger\left(t'\right)\hat{H}_1\left(t'\right)\hat{U}_0(t')\right) \tag{69}$$

$$\text{where } \hat{U}_0(t) = T\exp\left(-\frac{i}{\hbar}\int_0^t dt'\,\hat{H}_0\left(t'\right)\right) \tag{70}$$

Why Interaction Picture?

In QFT, we work with operator-valued fields. Free theories (without any interaction) have exact solutions. Interacting theories are hard to solve exactly. We split an interacting theory $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1$ into free and interacting parts. We will solve for the free part, and since we know the free field operator evolves under the free theory, it remains unchanged! All that remains is to expand $\hat{S}(t)$ perturbatively in a Taylor series, which will lead to Feynman diagrams!

Propagators

Propagators play a big part in QFT, they are

- Green's functions (Inverses) of differential operators
- calculated with Path Integrals
- calculated with contour integrals
- Lines in Feynman diagrams

Propagators are Green's Functions

We first inspect Electromagnetism to learn Green's functions. In the Lorenz gauge,

$$\vec{\nabla}^2\phi - \mu_0\epsilon_0\frac{\partial^2\phi}{\partial t^2} = -\frac{\rho}{\epsilon_0} \tag{71}$$

$$\vec{\nabla}^2\vec{A} - \mu_0\epsilon_0\frac{\partial^2\vec{A}}{\partial t^2} = -\mu_0\vec{J} \tag{72}$$

We wish to solve for $f(\vec{x}, t)$ given any $h(\vec{x}, t)$.

$$\left(\vec{\nabla}^2 - \mu_0\epsilon_0\frac{\partial^2}{\partial t^2}\right) f(\vec{x}, t) = h(\vec{x}, t) \tag{73}$$

There are infinitely many $h(\vec{x}, t)$, so how do we even start?

Borrowing the Inverse from Linear Algebra

$$\left(\vec{\nabla}^2 - \mu_0\epsilon_0 \frac{\partial^2}{\partial t^2}\right) f(\vec{x}, t) = h(\vec{x}, t) \tag{74}$$

This has the structure of $\mathbf{A}\vec{x} = \vec{b}$. In linear algebra, we don't even care what $\vec{b}$ is, we just need to find the inverse A^{-1}, and whatever $\vec{b}$ comes along just slap it to get $\vec{x} = A^{-1}\vec{b}$.

It would be nice if we can "invert" the differential operator $L := \left(\vec{\nabla}^2 - \mu_0\epsilon_0 \frac{\partial^2}{\partial t^2}\right)$ too. One needs to be careful in cases where the kernel (i.e. functions f that satisfy $Lf = 0$) of L is non-trivial (such as in gauge theories), because then the inverse would not be well defined. But for now our goal is to find Green's functions for L satisfying

$$LG(\vec{x}, t, \vec{x}', t') = \delta^3(\vec{x} - \vec{x}')\delta(t - t') \tag{75}$$

Green's Function is a Kernel

$$LG(\vec{x}, t, \vec{x}', t') = \delta^3(\vec{x} - \vec{x}')\delta(t - t') \tag{76}$$

The RHS is "identity" in the vector space of functions and G is the "inverse" of L.

Knowing green's functions allow us to find solutions to arbitrary $h(\vec{x}, t)$.

$$\left(\vec{\nabla}^2 - \mu_0\epsilon_0 \frac{\partial^2}{\partial t^2}\right) f(\vec{x}, t) = h(\vec{x}, t) \tag{77}$$

$$f(x, t) = \int d^3\vec{x}' \int dt' h(\vec{x}', t') G(\vec{x}, t, \vec{x}', t') \tag{78}$$

Solving for Green's Function

Proof:

$$
\begin{aligned}
Lf(x,t) &= \int d^3\vec{x}' \int dt' h(\vec{x}', t') G(\vec{x}, t, \vec{x}', t') && (79)\\
&= \int d^3\vec{x}' \int dt' h(\vec{x}', t') LG(\vec{x}, t, \vec{x}', t') && (80)\\
&= \int d^3\vec{x}' \int dt' h(\vec{x}', t') \delta^3(\vec{x} - \vec{x}') \delta(t - t') && (81)\\
&= h(\vec{x}, t) && (82)
\end{aligned}
$$

Solving for Green's Function

Awesome! Now what is this magical $G(\vec{x}, t, \vec{x}', t')$?

Derivation for Fourier Space Green's Function

$$\left(\vec{\nabla}^2 - \mu_0\epsilon_0 \frac{\partial^2}{\partial t^2}\right) G\left(\vec{x}, t; \vec{x}', t'\right) = \delta^3\left(\vec{x} - \vec{x}'\right)\delta\left(t - t'\right) \tag{83}$$

Dirac Delta in Fourier space:

$$\delta^3\left(\vec{x} - \vec{x}'\right) = \int_{-\infty}^{\infty} \frac{d^3\vec{k}}{(2\pi)^3} e^{i\vec{k}\cdot(\vec{x}-\vec{x}')} \tag{84}$$

$$\delta\left(t - t'\right) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega(t-t')} \tag{85}$$

And for the Green's function (note we transform both r and t),

$$G = \int_{-\infty}^{\infty} d^3\vec{k} \int_{-\infty}^{\infty} d\omega g(\vec{k}, \omega) e^{i\vec{k}\cdot(\vec{x}-\vec{x}')} e^{-i\omega(t-t')} \tag{86}$$

Substituting Equation 33 into 83 gives

$$\left(\vec{\nabla}^2 - \mu_0\epsilon_0\frac{\partial^2}{\partial t^2}\right)\int_{-\infty}^{\infty} d^3\vec{k}\int_{-\infty}^{\infty} d\omega g(\vec{k},\omega)e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}e^{-i\omega(t-t')} \tag{87}$$

$$= \int_{-\infty}^{\infty} d^3\vec{k}\int_{-\infty}^{\infty} d\omega g(\vec{k},\omega)\left(\vec{\nabla}^2 - \mu_0\epsilon_0\frac{\partial^2}{\partial t^2}\right)e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}e^{-i\omega(t-t')} \tag{88}$$

$$= \int_{-\infty}^{\infty} d^3\vec{k}\int_{-\infty}^{\infty} d\omega g(\vec{k},\omega)\left(-\vec{k}^2 + \frac{\omega^2}{c^2}\right)e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}e^{-i\omega(t-t')} \tag{89}$$

And comparing this with Equation yields

$$g(\vec{k},\omega) = -\frac{1}{(2\pi)^4}\frac{1}{k^2 - \frac{\omega^2}{c^2}} \tag{90}$$

Contour Integral

So the Green's function is given by this integral,

$$G\left(\vec{x}, t; \vec{x}', t'\right) = -\frac{1}{(2\pi)^4}\int_{-\infty}^{\infty} d^3\vec{k}\int_{-\infty}^{\infty} d\omega \frac{1}{k^2 - \frac{\omega^2}{c^2}} e^{i\vec{k}\cdot(\vec{x}-\vec{x}')} e^{-i\omega(t-t')} \tag{91}$$

but one can see that when integrating ω over $(-\infty, +\infty)$ we hit divergences at $\omega = \pm kc$ where the integrand is undefined. Oops!

Hadamard Regularization comes to the rescue! We can choose to go around the poles in 4 different ways, each way giving us a different solution.

Retarded Green's Function

One choice is to go over both poles,

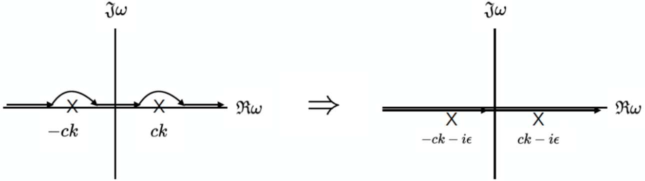

Figure: Retarded Contour

Note: We can freely "push" the poles and contour around (without passing through any poles) because in complex analysis, poles are basically at the center of attention. Contour integrals' values are solely determined by the poles they encircle. This results in

$$G\left(\vec{x}, t; \vec{x}', t'\right) = G^R\left(\vec{x}, t; \vec{x}', t'\right) \tag{92}$$

$$= \lim_{\epsilon \to 0} \frac{-1}{(2\pi)^4} \int_{-\infty}^{\infty} d^3\vec{k} \int_{-\infty}^{\infty} d\omega \frac{1}{k^2 - \frac{(\omega + i\epsilon)^2}{c^2}} e^{i\vec{k}\cdot(\vec{x}-\vec{x}')} e^{-i\omega(t-t')}$$

Retarded Green's Function

Using $(ck)^2 - (\omega + i\epsilon)^2 = -(\omega - (ck - i\epsilon))(\omega - (-ck - i\epsilon))$,

$$
\begin{aligned}
& G\left(\vec{x}, t; \vec{x}', t'\right) && (93) \\
&= \lim_{\epsilon \to 0} \frac{1}{(2\pi)^4} \int_{-\infty}^{\infty} d^3\vec{k} \int_{-\infty}^{\infty} d\omega \frac{e^{i\vec{k}\cdot(\vec{x}-\vec{x}')} e^{-i\omega(t-t')} c^2}{(\omega - (ck - i\epsilon))(\omega - (-ck - i\epsilon))} \\
&=: \lim_{\epsilon \to 0} \frac{1}{(2\pi)^4} \int_{-\infty}^{\infty} d^3\vec{k} \int_{-\infty}^{\infty} d\omega I_\omega(\omega, k) && (94)
\end{aligned}
$$

Jordan's lemma (or decay argument) dictates that how we close the contour depends on whether $t - t' < 0$ or > 0.

Let's do some contour integration!

$$\oint_C I_\omega d\omega = \int_{-\infty}^{\infty} I_\omega d\omega + \int_{semi} I_\omega d\omega \qquad (95)$$

$t - t' < 0$ case: $\int_{-\infty}^{\infty} I_\omega d\omega = 0$

If the probe time t comes after source time t', we close the contour in the **Upper** Half plane (infinite semicircle). Then since C_{upper} **doesn't** enclose any poles, $\oint_{C_{upper}} I_\omega d\omega = 0$.

(Decay Argument) Im $\omega > 0$, so $\exp(-i\omega(t - t'))$ is a decay term, so the $\int_{semi} I_\omega d\omega = 0$.

That means $\int_{-\infty}^{\infty} I_\omega d\omega = 0$ so $G^R = 0$ if $t < t'$. This makes sense, as we don't want a source to influence the past.

$t - t' > 0$ case: $\int_{-\infty}^{\infty} I_\omega d\omega \propto \sin\left(ck\left(t - t'\right)\right)$

If the probe time t comes after source time t', we close the contour in the **Lower** Half plane (infinite semicircle). Then since C_{lower} **does** enclose 2 poles,

$$\oint_{C_{lower}} I_\omega d\omega = -2\pi i \sum \text{Res}\left(\frac{e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}e^{-i\omega(t-t')}c^2}{(\omega-(ck-i\epsilon))(\omega-(-ck-i\epsilon))}\right)$$

| (minus sign because contour goes clockwise)

$$= -2\pi i c^2 \left[\frac{e^{-i(-ck-i\epsilon)(t-t')}}{-ck-i\epsilon-ck+i\epsilon} + \frac{e^{-i(ck-i\epsilon)(t-t')}}{ck-i\epsilon+ck+i\epsilon}\right] \tag{96}$$

| take the limit $\epsilon \to 0$

$$= \frac{-2\pi i c^2}{2ck}\left(-e^{ick(t-t')} + e^{-ick(t-t')}\right) \tag{97}$$

$$= \frac{-\pi i c}{k}\left(-2i \sin\left(ck\left(t-t'\right)\right)\right) \tag{98}$$

$$= -\frac{2\pi c}{k} \sin\left(ck\left(t-t'\right)\right) \tag{99}$$

$t - t' > 0$ case: $\int_{-\infty}^{\infty} I_\omega d\omega \propto \sin\left(ck\left(t - t'\right)\right)$

(Decay Argument) Im $\omega < 0$ so $\exp(-i\omega(t - t'))$ is a decay term, so the $\int_{semi} I_\omega d\omega = 0$.

That means $\int_{-\infty}^{\infty} I_\omega d\omega = -\frac{2\pi c}{k} \sin\left(ck\left(t - t'\right)\right)$ and so

$$G^R = -\frac{c}{(2\pi)^3} \int d^3\vec{k} \frac{e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}}{k} \sin\left(ck\left(t - t'\right)\right) \tag{100}$$

if $t > t'$. This makes sense, as a source can influence the future. Doing the integral in spherical coordinates (not so important for this lecture),

$$G^R = -\frac{1}{4\pi|\vec{x} - \vec{x}'|} \delta\left(t - t' - \frac{|\vec{x} - \vec{x}'|}{c}\right) \quad \text{if } t > t' \tag{101}$$

Retarded Green's Function: Putting it all together

This describes how a source at (x, t) can influence the future (x', t')!

$$G^R(x, t, x', t') = \theta\left(t - t'\right) \frac{-1}{4\pi|\vec{x} - \vec{x}'|} \delta\left(t - t' - \frac{|\vec{x} - \vec{x}'|}{c}\right)$$

We see the beauty of complex analysis!!!

- causality $\theta\left(t - t'\right)$
- potential $\frac{-1}{4\pi|\vec{x} - \vec{x}'|}$
- propagation $\delta\left(t - t' - \frac{|\vec{x} - \vec{x}'|}{c}\right)$

I made a handwritten video a while ago (but honestly these slides are better).

Other Green's Functions

What about the other 3 regularization choices for the contour? One will give us the Advanced propagator, one will give us the Feynman propagator. For electrodynamics we reject them all because any information travelling back in time is forbidden, but in QFT we actually use the Feynman propagator! Antiparticles travelling backward in time help cancel amplitudes for particle travelling forward, helping **ensure** causality instead of breaking it!

Table of Green's Functions

Path Integrals

Derivation in QM is shown in a lot of places such as Andrew Dotson. Specifically, he shows that

$$\langle x_f, t_f \mid x_i, t_i \rangle = \int \mathcal{D}x(t) e^{iS[x(t)]/\hbar} \tag{102}$$

One can **hypothesise** that this principle extends to all of physics, such as field theory

$$Z = \int D\varphi e^{(i/\hbar)\int d^4x \mathcal{L}(\varphi)} \tag{103}$$

There is a school of thought that says path integral is an axiom of nature, but unfortunately it's not mathematically rigorous. There are many nice features of the path integral though

- The limit $\hbar \to 0$ recovers classical theory through stationary phase approximation
- QFT is quite natural using this framework (approach taken by Anthony Zee's book)

First touch at QFT (Path Integral Quantisation)

$$Z = \int D\varphi e^{i\int d^4x\left\{\frac{1}{2}\left[(\partial\varphi)^2 - m^2\varphi^2\right] + J\varphi\right\}} \tag{104}$$

$$\Big|\ \text{Integral by parts} \tag{105}$$

$$= \int D\varphi e^{i\int d^4x\left[-\frac{1}{2}\varphi\left(\partial^2 + m^2\right)\varphi + J\varphi\right]} \tag{106}$$

To perform this path integral (warning: not mathematically rigorous), we draw an analogy with multivariate gaussian integrals

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty} dq_1 dq_2 \cdots dq_N e^{(i/2)q\cdot A\cdot q + iJ\cdot q} \tag{107}$$

$$= \left(\frac{(2\pi i)^N}{\det[A]}\right)^{\frac{1}{2}} e^{-(i/2)J\cdot A^{-1}\cdot J} \tag{108}$$

Just like how we invert the matrix A^{-1}, we need to invert the differential operator $(\partial^2 + m^2)^{-1} \equiv D(x-y)$. $D(x-y)$ is the green's function!

Green's Functions are Propagators

$$Z(J) = \mathcal{C}e^{-(i/2)\iint d^4x d^4y J(x)D(x-y)J(y)} \equiv \mathcal{C}e^{iW(J)} \quad (109)$$

where $\mathcal{C}$ is a normalization constant (involving the determinant), which one can identify as $Z(J=0)$.

$$Z(J) \equiv Z(J=0)e^{iW(J)} \quad (110)$$

$$W(J) = -\frac{1}{2}\iint d^4x d^4y J(x)D(x-y)J(y) \quad (111)$$

a heck one of an interesting topic for another day!

For now observe that W is quadratic in J, but Z depends on a taylor expansion of W and will depend on arbitrarily high powers of J. This is a step closer to Feynman diagrams!

Assignment: Calculations

Common integrals in QFT Table of Green's Functions

Assignments: Readings

- Anthony Zee QFT up to chap 1.7 on Feynman diagrams (inclusive)
- Frederic Schuller Geometric Anatomy Lectures up to lecture 18 (impt: 4, 6, 9, 10, 13, 17 ,18)

Try to explain the following statements rigorously:

- Manifold M is a collection of coordinate charts (lec 9)
- Tangent space is spanned by partial derivatives (lec 10)
- Lie algebra is the tangent space at the identity (lec 13)
- Representation of a group is an action on some vector space (lec 17)
- Exponential map from Lie algebra to Lie group (lec 18)

These math foundations will be indispensible when we want to grow out of scalar fields to play with spin, vector, and gauge fields.

Appendix

Lorentz Transformations: Acting on Derivatives

(From Manifolds) Spacetime is $\mathbb{R}^{1,3}$. It has a tangent bundle $T\mathbb{R}^{1,3}$, which contains tangent vectors $\partial_\mu \in T\mathbb{R}^{1,3}$

When Lorentz transformations $\Lambda^\mu{}_\nu$ act on tangent vectors

$$(\Lambda^{-1})^\rho{}_\mu w^\mu = x^\rho \tag{112}$$

$$\partial_\mu = \frac{\partial}{\partial x^\mu} \mapsto \frac{\partial}{\partial w^\mu} = \frac{\partial x^\rho}{\partial w^\mu}\frac{\partial}{\partial x^\rho} \tag{113}$$

$$= (\Lambda^{-1})^\rho{}_\mu \frac{\partial}{\partial x^\rho} \tag{114}$$

$$= (\Lambda^{-1})^\rho{}_\mu \partial_\rho \tag{115}$$

Now because (one can check that)

$$\tag{116}$$

Equations 115 and **??** motivates us to define $\Lambda_\mu{}^\rho \equiv (\Lambda^{-1})^\rho{}_\mu$.