7. Prove that for any natural number n

$$2 + 2^2 + 2^3 + .. + 2^n = 2^{n+1} - 2$$

Proof. Proof by induction. Base case for $n = 1$:

$$\begin{aligned} 2^1 &= 2^{1+1} - 2 \\ &= 2^2 - 2 \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

Substituting $n + 1$, we get:

$$\begin{aligned} LHS &= 2 + 2^2 + 2^3 + .. + 2^{n+1} \\ &= 2 + 2^2 + 2^3 + .. + 2^n + 2^{n+1} && \text{(show second to last term)} \\ &= (2 + 2^2 + 2^3 + .. + 2^n) + 2^{n+1} && \text{(group first terms)} \\ &= (2^{n+1} - 2) + 2^{n+1} && \text{(by inductive statement)} \\ &= 2^{n+1} + 2^{n+1} - 2 && \text{(re-arrange terms)} \\ &= 2(2^{n+1}) - 2 && \text{(group similar terms)} \\ &= 2^{n+2} - 2 \\ &= 2^{(n+1)+1} - 2 \\ &= RHS \end{aligned}$$

$\square$