2. Say whether the following is true or false and support your answer by a proof: The sum of any five consecutive integers is divisible by 5 (without remainder).

*Proof.* By mathematical induction.

For any integer $x$ the sum ($X$) of the first 5 consecutive integers starting at $x$ can be expressed as:

$$X = x + (x+1) + (x+2) + (x+3) + (x+4)$$

For $x = 0$, we have:

$$\begin{aligned} X &= 0 + (0+1) + (0+2) + (0+3) + (0+4) \\ &= 1 + 2 + 3 + 4 \\ &= 10 \\ &= 5(2) \\ &= 5n && \text{(divisible by five)} \end{aligned}$$

This is the inductive base case. To prove this relation for all $x$ we show that it is valid for both positive and negative $x$.

For positive $x$:

$$\begin{aligned} X &= x + (x+1) + (x+2) + (x+3) + (x+4) \\ &= (x+1) + ((x+1)+1) + ((x+1)+2) + ((x+1)+3) + ((x+1)+4) \\ & \hfill \text{(subsitute x+1)} \\ &= 5 + (x + (x+1) + (x+2) + (x+3) + (x+4)) \\ & \hfill \text{(bring plus ones out the front)} \\ &= 5 + 5n \\ & \hfill \text{(from the inductive statement)} \\ &= 5(1+n) \end{aligned}$$

This proves that the inductive statement is true for as $x$ tends to $\infty$.

For negative $x$:

$$
\begin{aligned}
X &= x + (x-1) + (x-2) + (x-3) + (x-4) \\
&= (x-1) + ((x-1)+1) + ((x-1)+2) + ((x-1)+3) + ((x-1)+4) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{(subsitute x+1)} \\
&= -5 + (x + (x+1) + (x+2) + (x+3) + (x+4)) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{(bring plus ones out the front)} \\
&= -5 + 5n \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{(from the inductive statement)} \\
&= 5(-1+n)
\end{aligned}
$$

This proves that the inductive statement is true for as $x$ tends to $-\infty$, thereby proving the theorem is true for all $x$ (by mathematical induction).

$\square$