Stochastics Methods and Problems

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2.6 Generating Functions

Sample Problems:

- Exercise 4.6: Find invariant distribution of mistakes in editions of a book by computing limit of generating function

Exercise 4.1

A six-sided die is rolled repeatedly. Which of the following a Markov chains? For those that are, find the one-step transition matrix.

(a) X_n is the largest number rolled up to the nth roll.
(b) X_n is the number of sixes rolled in the first n rolls.
(c) At time n, X_n is the time since the last six was rolled.
(d) At time n, X_n is the time until the next six is rolled.

Solution

(a) Yes.

$$P = \begin{bmatrix} 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ & 2/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ & & 36 & 1/6 & 1/6 & 1/6 \\ & & & 4/6 & 1/6 & 1/6 \\ & & & & 5/6 & 1/6 \\ & & & & & 1 \end{bmatrix}$$

(b) Yes.

$$P = \begin{bmatrix} 5/6 & 1/6 & & \\ & 5/6 & 1/6 & \\ & & \ddots & \ddots \end{bmatrix}$$

(c) Yes. Suppose $X_n = i$. The next roll is either a 6, in which case $X_{n+1} = 0$. Otherwise $X_{n+1} = i + 1$.

$$P = \begin{bmatrix} 1/6 & 5/6 & & & \\ 1/6 & & 5/6 & & \\ 1/6 & & & 5/6 & \\ \vdots & & & & \ddots \end{bmatrix}$$

(d) Yes. Suppose $X_n = 0$. The probability of $X_{n+1} = j$ is $(1/6)(5/6)^j$ as you must not roll a 6 for j turns, and then must roll a 6 on the j-th. Suppose $X_n = i > 0$. Then the next step you will be on turn closer to rolling a 6.

That is, $X_{n+1} = i - 1$.

$$P = \begin{bmatrix} \frac{1}{6} & \frac{1}{6}\left(\frac{5}{6}\right) & \frac{1}{6}\left(\frac{5}{6}^2\right) & \frac{1}{6}\left(\frac{5}{6}\right)^3 & \cdots \\ 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & \ddots \end{bmatrix}$$

Exercise 4.2

Let $Y_n = X_{2n}$. Compute the transition matrix for Y when

(a) X is a simple random walk (i.e., X increases by one with probability p and decreases by 1 with probability q)

(b) X is a branching process where G is the generating function of the number of offspring from each individual

Solution

(a) In each step we can go down with probability q and then down again with probability q or up with probability p. Alternatively we can go up with probability p and then down with probability q or up again with probability p.

Therefore we will end up two spaces down with probability q^2, in the same position with probability $qp + pq = 2pq$, or up two spaces with probability p^2. Thus,

$$p(i,j) = \begin{cases} p^2 & j = i+2 \\ 2pq & i = j \\ q^2 & j = i-2 \\ 0 & \text{otherwise} \end{cases}$$

(b) We can obtain the exponents of a generating function $G(s) = a_0 + a_1 s + a_2 s^2 + \ldots$ by,

$$a_n = \frac{1}{n!}\frac{d^n}{ds^n}\Big[G(s)\Big]_{s=0}$$

The coefficient of the s^k term is the value of the probability mass function of X evaluated at k.

The generating function of Y is $G(G(s)) = G_2(s)$ from the notes.

For a branching process with current population k, the population of the next generation will be $X_1 + X_2 + \ldots + X_k$, where each X_i is iid with distribution X. Therefore,

$$p(i,j) = \frac{1}{j!}\frac{d^n}{ds^n}\Big[G_2(s)^i\Big]_{s=0}$$

Exercise 4.3

Let X be a Markov chain with state space S and absorbing state k (i.e., $p(k, j) = 0$ for all $j \in S$). Suppose $j \to k$ for all $j \in S$. Show that all states other than k are transient.

Solution

Fix a state $j \in S$. By definition of $j \to k$, $\exists N \geq 0 : p_N(j, k) \geq 0$. Since $\{X_N = k | X_0 = j\} \subseteq \{\forall n, X_n \neq j | X_0 = j\}$ we have,

$$0 < p_N(j, k) = \mathbb{P}(X_N = k | X_0 = j) \leq \mathbb{P}(\forall n, X_n \neq j | X_0 = j)$$

Therefore,

$$\mathbb{P}(\exists n \geq 0 : X_n = j | X_0 = j) = 1 - \mathbb{P}(\forall n, X_n \neq j | X_0 = j) < 1$$

This proves state j istransient. $\square$

Exercise 4.4

Suppose two distinct states i, j satisfy

$$\mathbb{P}(\tau_j < \tau_i | X_0 = i) = \mathbb{P}(\tau_i < \tau_j | X_0 = j)$$

where $\tau_j = \inf\{n \geq 1 : X_n = j\}$. Show that, if $X_0 = i$, the expected value of visits to j prior to returning to i is one.

Solution

Write

$$p = \mathbb{P}(\tau_j < \tau_i | X_0 = i) = \mathbb{P}(\tau_i < \tau_j | X_0 = j)$$

That is, p is the probability that we go to state j before state i give we are in state i, and p is also the probability that we go to state i before state j given we are in state j.

Then $1 - p$ is the probability that we do not go to state i before returning state j,0 given we start in state j.

So $(1-p)^k$ is the probability that we return to state j exactly k times before moving to state i, given we start in state j.

Let N be the number of visits to j prior to returning to i given we start in state i.

The probability that $N = k \in \mathbb{Z}_{\geq 0}$ is the probability that starting from state i we go to state j, return to state j $(k-1)$ times without returning to state i, and then return to state i without going to returning to state j.

So $\mathbb{P}(N = k | X_0 = i) = p(1-p)^{k-1}p$. This is the probability mass function for N so,

$$\mathbb{E}[N] = \sum_{n=0}^{\infty} np^2(1-p)^{k-1} = p\sum_{n=0}^{\infty} n(1-p)^n = p\frac{p}{(1-(1-p))^2} = 1$$

Exercise 4.5

Let X be a Markov chain with transition matrix,

$$P = \begin{bmatrix} 1-2p & 2p & 0 \\ p & 1-2p & p \\ 0 & 2p & 1-2p \end{bmatrix}, \qquad\qquad p \in (0,1)$$

Find P^n, the invariant distribution π, and the mean-recurrence times $\overline{\tau}_j$ for $j = 1, 2, 3$.

Solution

Note that P has eigendecomposition $P = V \Lambda V^{-1}$ where,

$$\Lambda = \begin{bmatrix} 1 & & \\ & 1-4p & \\ & & 1-2p \end{bmatrix}, \qquad\qquad V = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Therefore, $P^n = V \Lambda^n V^{-1}$. Explicitly,

$$P^n = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1-4p & \\ & & 1-2p \end{bmatrix} \begin{bmatrix} 1/4 & 1/2 & 1/4 \\ 1/4 & -1/2 & 1/4 \\ -1/2 & 0 & 1/2 \end{bmatrix}$$

Invariant distributions are linear combinations of left eigenvectors corresponding to eigenvalues of 1. In this case that is the first row of V^{-1}. That is,

$$\pi = \begin{bmatrix} frac14^{\frac{1}{4} \;\; \frac{1}{2}} \end{bmatrix}$$

Finally, since the invariant distribution is unique, by Theorem we have,

$$\overline{\tau}_i = \frac{1}{\pi(i)}$$

Exercise 4.6

Let X_n be the number of mistakes in the n-th addition of a book. Between the n-th and the $(n+1)$-th addition an editor corrects each mistake independently with probability p and introduces Y_n new mistakes where the(Y_n) are iid and Poisson distributed with parameter λ. Find the invariant distribution π of the number of mistakes in the book.

Solution

Let $M_{n,k}$ be distributed as $\mathrm{Ber}(1-p)$ so that M_k is 0 if this mistake is corrected, and 1 otherwise. Let Y_n be Poisson distributed with parameter λ. Then,

$$X_{n+1} = Y_n + \sum_{k-1}^{X_n} M_k$$

Each $M_{n,k}$ has generating function,

$$G_{M_{n,k}} = p + (1-p)s = 1 - q + qs = 1 - q(1-s)$$

Similarly. Y_n has generating function,

$$G_Y(s) = \sum_{k=0}^{\infty} e^{-\lambda}\lambda^k/k!s^k = e^{-\lambda}e^{s\lambda} = e^{\lambda(s-1)}$$

Therefore X_{n+1} has generating function,

$$\begin{aligned} G_{n+1}(s) &= G_Y(s)\mathbb{E}\left[s^{M_{k,1}+M_{k,2}+\ldots+M_{k,X_n}}\right] \\ &= G_Y(s)\mathbb{E}\left[\mathbb{E}\left[s^{M_{k,1}+M_{k,2}+\ldots+M_{k,X_n}}\right]|X_n\right] \\ &= G_Y(s)\mathbb{E}\left[(1-q(1-s))^{X_n}\right] \\ &= G_Y(s)G_n(1-q(1-s)) \end{aligned}$$

First observe $1-q^i(1-(1-q(1-s))) = 1-q^{i+1}(1-s)$. We now use the relation

$G_{n+1}(s) = G_Y(s)G_n(1-q(1-s))$ and the fact that $G_0(s) = 1$ to calculate,

$$\begin{aligned}
G_{n+1}(s) &= G_Y(s)G_n(1-q(1-s)) \\
&= G_Y(s)G_Y(1-q(1-s))G_{n-1}(1-q^2(1-s)) \\
&= G_Y(s)G_Y(1-q(1-s))G_Y(1-q^2(1-s))G_{n-2}(1-q^3(1-s)) \\
&\vdots \\
&= \prod_{i=0}^{n} G_Y(1-q^i(1-s))
\end{aligned}$$

Then,

$$\begin{aligned}
\lim_{n\to\infty} G_n(s) &= \lim_{n\to\infty} G_{n+1}(s) \\
&= \lim_{n\to\infty} \prod_{i=0}^{n} G_Y(1-q^i(1-s)) \\
&= \lim_{n\to\infty} \prod_{i=0}^{n} \exp\left(\lambda(-q^i(1-s))\right) \\
&= \exp\left(\sum_{i=0}^{\infty} \lambda(-q^i(1-s)\right) \\
&= \exp\left(\lambda(s-1)\frac{1}{1-q}\right) \\
&= \exp\left(\frac{\lambda}{p}(s-1)\right)
\end{aligned}$$

Thus, $G_n(S)$ converges to the generating function of a Poisson random variable with parameter λ/p.

Then X_n converges to a random variable distributed like a Poisson random variable with parameter λ/p. The random variable for which X_n converges to must be the variable corresponding to the stationary distribution. Therefore, the stationary distribution is distributed like the probability mass function of this random variable. That is,

$$\pi(k) = e^{-\lambda/p}\frac{(\lambda/p)^k}{k!}$$

In the limit $p \to 1$, where we correct all mistakes, the stationary distribution looks like a Poisson distribution with parameter λ. In the limit $\lambda \to 0$ so we do not make any new mistakes, $\pi(0) \to 1$ as expected.

Exercise 4.7

Give an example of a transition matrix P that admits multiple stationary distributions π.

Solution

Define P to be the identity matrix. Then any distribution is a stationary distribution.

Exercise 4.8

A Markov chain on $S = \{0, 1, 2, ..., n\}$ has transition probabilities $p(0, 0) = 1 - \lambda_0$, $p(i, i+1) = \lambda_i$ and $p(i+1, i) = \mu_{i+1}$ for $i = 0, 1, ..., n-1$, and $p(n, n) = 1-\mu_n$. Show that the process is reversible in equilibrium.

Solution

We assume all entries not specified are zero. (I heard this is the intent, however I wonder why we are given μ_j when $\mu_j = 1 - \lambda_j$ for all j). We write the matrix P as,

Write $\mu_n = 1 - \lambda_n$. Thus, $\mu_i = 1 - \lambda_i$ for $i = 1, ..., n$ as the sum of each row must be 1 (making the assumption that all entries not specified at zero).

$$P = \begin{bmatrix} 1-\lambda_0 & \lambda_0 & & & & \\ \mu_1 & & \lambda_1 & & & \\ & \mu_2 & & \lambda_2 & & \\ & & \mu_3 & & & \\ & & & & & \lambda_{n-1} \\ & & & & \mu_n & 1-\mu_n \end{bmatrix} = \begin{bmatrix} 1-\lambda_0 & \lambda_0 & & & & \\ 1-\lambda_1 & & \lambda_1 & & & \\ & 1-\lambda_2 & & \lambda_2 & & \\ & & 1-\lambda_3 & & & \\ & & & & & \lambda_{n-1} \\ & & & & 1-\lambda_n & \lambda_n \end{bmatrix}$$

This chain is irreducible and finite so a unique invariant distribution π exists. Write $\pi = [\pi_0, \pi_1, ..., \pi_n]$. Then $\pi P = \pi$. That is,

$$\pi P = \begin{bmatrix} \pi_0(1-\lambda_0) + \pi_1(1-\lambda_1) \\ \pi_0\lambda_0 + \pi_2(1-\lambda_2) \\ \pi_1\lambda_1 + \pi_3(1-\lambda_3) \\ \vdots \\ \vdots \\ \pi_{n-1}\lambda_{n-1} + \pi_n\lambda_n \end{bmatrix}^T = \begin{bmatrix} \pi_0 \\ \pi_1 \\ \pi_2 \\ \vdots \\ \pi_j \\ \vdots \\ \pi_n \end{bmatrix}^T$$

$$\begin{aligned} \pi_1 &= \lambda_0\pi_0/(1-\lambda_1) & \lambda_0\pi_0 &= \pi_1(1-\lambda_1) \\ \pi_2 &= (\pi_1 - \pi_0\lambda_0)/(1-\lambda_2) = \pi_1\lambda_1/(1-\lambda_2) & \lambda_1\pi_1 &= \pi_2(1-\lambda_2) \\ \pi_3 &= (\pi_2 - \pi_1\lambda_1)/(1-\lambda_3) = \pi_2\lambda_2/(1-\lambda_3) & \lambda_2\pi_2 &= \pi_3(1-\lambda_3) \\ &\vdots \\ \pi_{j+1} &= (\pi_j - \pi_{j-1}\lambda_{j-1})/(1-\lambda_{j+1}) = \pi_j\lambda_j/(1-\lambda_{j+1}) & \lambda_j\pi_j &= \pi_{j+1}(1-\lambda_{j+1}) \\ &\vdots \\ \pi_n &= (\pi_{n-1}\lambda_{n-1})/(1-\lambda_n) & \pi_{n-1}\lambda_{n-1} &= \pi_n(1-\lambda_n) \end{aligned}$$

Observing the equations on the right hand side we have that for $i = 1, 2, ..., n-1$,

$$\pi_i p(i, i+p) = \pi_{i+1} p(i+1, i)$$

We now show the detail balance condition. In particular, we must show,

$$\pi_i p(i, j) = \pi_j p(j, i) \qquad \text{for all } i, j$$

However, for $j \notin \{i-1, i+1\}$ we have $p(i, j) = 0$. Therefore, for this matrix the previous condition is equivalent to

$$\pi_i p(i, i+1) = \pi_{i+1} p(i+1, i) \qquad \text{for } i = 1, 2, ..., n-1$$

We have shown that these equations hold for all $i = 1, 2, ..., n-1$.

This proves π is in detailed balance with P, and so this process is reversible in equilibrium. $\square$

3 Continuous Time Markov Chains

3.1 Transition Matrix

3.2 Stationary/Invariant distribution

Sample Problems:

- Exercise 5.1: Find invariant distribution and conditions for existence
- Exercise 5.2: Show two processes have the same stationary distribution
- Exercise 5.3: Indirectly find stationary distribution by solving KFE, finding generating function for the chain, and computing the distribution of X_t as $t \to \infty$

3.3 Generator

Sample Problems:

- Exercise 5.1: Write down generator
- Exercise 5.3: Given generator solve KFE
- Exercise 5.4: Write down generator and solve KFE/KBE

3.4 Generating Functions

Sample Problems:

- Exercise 5.3: Use KBE to find PDE for generating function of X
- Exercise 5.4: Use KBE to find PDE for generating function of X
- Exercise 5.5: Compute generating function of Poisson process with random intensity. Use generating function to compute mean and variance.

3.5 KFE AND KBE

Sample Problems:

- Exercise 5.3: Given generator solve KFE
- Exercise 5.4: Write down KFE and KBE and solve

3.6 Birth Death Processes

General description of birth death processes

3.6.1 General Form for infinite queue

Description:

- Process either jumps up one or down one or stay the same
- Expected wait time in state i is exponentially distributed $\tau \sim \mathcal{E}(\lambda_i + \mu_i)$
- When the process does jump, the probability of an up jump is $\lambda_i/(\lambda_i + \mu_i)$, and the probability of a down jump is $\mu_i/(\lambda_i + \mu_i)$.
- if $\lambda_0 > 0$ the chain is irreducible.

State space: $S = \{1, 2, 3 \ldots\}$.

Generator:

$$G = \begin{bmatrix} -\lambda_0 & \lambda_0 & & & & \\ \mu_1 & -(\mu_1 + \lambda_1) & \lambda_1 & & & \\ & \mu_2 & -(\mu_2 + \lambda_2) & \lambda_2 & & \\ & & \mu_3 & -(\mu_3 + \lambda_3) & \lambda_3 & \\ & & & & \ddots & \end{bmatrix}$$

Invariant distribution:

$$\pi(k) = \frac{\lambda_0 \lambda_1 \cdots \lambda_{k-1}}{\mu_1 \mu_2 \cdots \mu_k} \pi(0), \qquad \pi(0) = \left(1 + \sum_{k=1}^{\infty} \frac{\lambda_0 \lambda_1 \cdots \lambda_{k-1}}{\mu_1 \mu_2 \cdots \mu_k} \right)^{-1}$$

Sample Problems: Example 5.2.9

3.6.2 M/M/1 queue

Description:

- Models infinite queue.
- Arrivals occur at a rate λ according to a Poisson process.
- Service times have exponential distribution with rate parameter μ, where $1/\mu$ is the mean service time.
- A single server serves customers one at a time from front of queue, first come first serve

State space: $S = \{1, 2, 3 \ldots\}$.

Generator:

$$G = \begin{bmatrix} -\lambda & \lambda & & & \\ \mu & -(\mu + \lambda) & \lambda & & \\ & \mu & -(\mu + \lambda) & \lambda & \\ & & & \ddots & \end{bmatrix}$$

Invariant distribution:

$$\pi(k) = (1 - \lambda/\mu)(\lambda/\mu)^k$$

Expected Response Time: For customers who arrive and find the queue as a stationary process, the response time (sum of waiting and services times) has density function,

$$f(t) = \begin{cases} (\mu - \lambda)e^{-(\mu-\lambda)t}, & t > 0 \\ 0 & \text{ow.} \end{cases}$$

This has mean,

$$\int_0^\infty t f(t) \mathrm{d}t = \frac{1}{\mu - \lambda}$$

Sample Problems: Exercise 5.1

3.6.3 M/M/∞

Description:

- Arrivals occur at a rate λ according to a Poisson process.
- Service times have exponential distribution with rate parameter μ, where $1/\mu$ is the mean service time.
- There are always enough servers that every arriving job is serviced immediately.

State space: $S = \{1, 2, 3, \ldots\}$.

Generator:

$$G = \begin{bmatrix} -\lambda & \lambda & & & \\ \mu & -(\mu+\lambda) & \lambda & & \\ & 2\mu & -(2\mu+\lambda) & \lambda & \\ & & 3\mu & -(3\mu+\lambda) & \lambda \\ & & & & \ddots \end{bmatrix}$$

Invariant Distribution:

$$\pi(k) = \frac{(\lambda/\mu)^k e^{-\lambda/\mu}}{k!}$$

Sample Problems: Exercise 5.3, Final Problem ??, Practice Exam #? Problem 1

3.6.4 M/M/1/K queue

State space: $S = \{1, 2, \ldots, n\}$.

Generator:

$$G = \begin{bmatrix} -\lambda & \lambda & & & & & \\ \mu & -(\mu+\lambda) & \lambda & & & & \\ & \mu & -(\mu+\lambda) & \lambda & & & \\ & & & & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & & & & \\ & & & \mu & -(\mu+\lambda) & \lambda \\ & & & & \mu & -\mu \end{bmatrix}$$

Exercise 5.1

Patients arrive at an emergency room as a Poisson process with intensity λ. The time to treat each patient is an independent exponential random variable with parameter μ. Let $X = (X_t)_{t\geq 0}$ be the number of patients in the system (either being treated or waiting). Write down the generator of X. Show that X has an invariant distribution π if and only if $\lambda < \mu$. Find π. What is the total expected time (waiting + treatment) a patient waits when the system is in its invariant distribution?

Solution

In some small time interval s there is probability $\lambda s + \mathcal{O}(s^2)$ that a patient arrives, probability $1 - \lambda s + \mathcal{O}^2$ that a patient does not arrive, and probability $\mathcal{O}(s^2)$ that multiple patients arrive.

If there are patients, in this times there is also probability $\mu s + \mathcal{O}(s^2)$ that a patient is treated, probability $1 - \mu s + \mathcal{O}(s^2)$ that a patient is not treated, and probability $\mathcal{O}(s^2)$ that more than one (if possible) patients are treated.

Note that any moves which have more than one transition such as a patient arriving, and a patient being treated are all $\mathcal{O}(s^2)$.

Suppose there are no patients at time t. The probability of transitioning to j patients after a short time s is given by,

$$\mathbb{P}(X_{t+s} = j | X_t = 0) = \begin{cases} \lambda s + \mathcal{O}(s^2) & j = 1 \\ 1 - \lambda s + \mathcal{O}(s^2) & j = 0 \\ \mathcal{O}(s^2) & \text{otherwise} \end{cases}$$

Now suppose there are $i > 0$ patients at time t. The probability of transitioning to j patients after a short time s is given by,

$$\mathbb{P}(X_{t+s} = j | X_t = i) = \begin{cases} (\lambda s + \mathcal{O}(s^2))(1 - \mu s + \mathcal{O}(s^2)) & j = i + 1 \\ (1 - \lambda s + \mathcal{O}(s^2))(1 - \mu s + \mathcal{O}(s^2)) + \mathcal{O}(s^2) & j = i \\ (1 - \lambda s + \mathcal{O}(s^2))(\mu s + \mathcal{O}(s^2)) & j = i - 1 \\ \mathcal{O}(s^2) & \text{otherwise} \end{cases}$$

This is simplified as,

$$\mathbb{P}(X_{t+s} = j | X_t = i) = \begin{cases} \lambda s + \mathcal{O}(s^2) & j = i + 1 \\ 1 - \lambda s - \mu s + \mathcal{O}(s^2) & j = i \\ \mu s + \mathcal{O}(s^2) & j = i - 1 \\ \mathcal{O}(s^2) & \text{otherwise} \end{cases}$$

This gives,

$$G = \begin{bmatrix} -\lambda & \lambda & & & & \\ \mu & -(\lambda+\mu) & \lambda & & & \\ & \mu & -(\lambda+\mu) & \lambda & & \\ & & \mu & -(\lambda+\mu) & \lambda & \cdots \\ & & & \vdots & \vdots & \ddots \end{bmatrix}$$

We recognize this as a birth-death process (a bit ironic in the context of an emergency room) with $\lambda_i = \lambda$ and $\mu_i = \mu$.

Then if a stationary distribution π exists, for $n \in \mathbb{Z}_{>0}$,

$$\pi(n > 0) = \left(\frac{\lambda}{\mu}\right)^n \pi(0)$$

and

$$\pi(0) = \left(1 + \sum_{n=1}^{\infty} \left(\frac{\lambda}{\mu}\right)^n\right)^{-1} = \left(\sum_{n=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^n\right)^{-1}$$

This is a geometric series which is convergent exactly when $\lambda/\mu < 1$. That is, when $\lambda < \mu$. In this case,

$$\pi(0) = \left(\sum_{n=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^n\right)^{-1} = \left(\frac{\mu}{\mu-\lambda}\right)^{-1} = \frac{\mu-\lambda}{\mu}$$

We condition on knowing the number of people on the queue. Suppose there are n people in the queue when a patient arrives. Then the patient will have to wait a random time distributed as the sum of n exponential random variables with parameter μ to be treated and one more to finish treatment. The expectation of each of each exponential random variable is $1/\mu$, so the patient waits an expected time of $(n+1)/\mu$.

In equilibrium, the probability that there are n people in the queue when a patient arrives is $\pi(n)$.

Therefore, the expected wait time is,

$$\sum_{n=0}^{\infty} \pi(n)\frac{(n+1)}{\mu} = \frac{\mu-\lambda}{\mu^2}\sum_{n=0}^{\infty}\left(\frac{\lambda}{\mu}\right)^n (n+1) = \frac{\mu-\lambda}{\mu^2}\left(\frac{\mu\lambda}{(\mu-\lambda)^2} + \frac{\mu}{\mu-\lambda}\right) = \frac{1}{\mu-\lambda}$$

Exercise 5.2

Let $X = (X_t)_{t \geq 0}$ be a Markov chain with stationary distribution π. Let N be an independent Poisson process with intensity λ and denote by τ_n the time of the n-th arrival of N. Define $Y_n := X_{\tau_n +}$ (i.e., Y_n is the value of X immediately after the n-th jump). Show that Y is a discrete time Markov chain with the same stationary distribution as X.

It is obvious that Y is Markov, as given the present, the future is independent of the past. We add a bit more rigor below.

Fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. By hypothesis X_t is a Markov process. That is, for a filtration $(\mathcal{F}_s)_{s \in [0,T]}$, for $0 \leq s \leq t \leq T$, and for every non-negative Borel measurable function f,

$$\mathbb{E}[f(X_t)|\mathcal{F}_s] = \mathbb{E}[f(X_t)|X_s]$$

Let $\mathcal{F}'_n = \mathcal{F}_{\tau_n +}$ be a sub-σ-algebra of $\mathcal{F}$. Then clearly $(\mathcal{F}'_n)$ is a filtration. Let f be any non-negative Borel measurable function. Then,

$$\mathbb{E}[f(Y_n)|\mathcal{F}'_m] = \mathbb{E}[f(X_{\tau_n +})|\mathcal{F}_{\tau_m +}] = \mathbb{E}[f(X_{\tau_n +})|X_{\tau_m +}] = \mathbb{E}[f(Y_n)|Y_m]$$

This means Y is Markov, and clearly Y is discrete time. Therefore Y is a discrete time Markov chain.

Note we assume X is time homogeneous.

Suppose X has stationary distribution π. Then for all $0 \leq t \leq T$, $\pi P_t = \pi$, where,

$$(P_t)_{i,j} = \mathbb{P}(X_t = j | X_0 = i)$$

Thus, the one step probability transition matrix, denoted $\tilde{P}$, for Y is,

$$\tilde{P}_{i,j} = \mathbb{P}(Y_1 = j | Y_0 = i) = \mathbb{P}(X_{\tau_1 +} = j | X_0 = i) = (P_{\tau_1})_{i,j}$$

This means $\pi \tilde{P} = \pi$, so π is a stationary distribution of Y.

Exercise 5.3

Let $X = (X_t)_{t\geq 0}$ be a Markov chain with state space $S = \{0, 1, 2, ...\}$ and generator G whose i-th row has entries

$$g_{i,i-1} = i\mu \qquad g_{i,i} = -i\mu - \lambda \qquad g_{i,i+1} = \lambda,$$

with all other entries being zero (the zeroth row has only two entries: $g_{0,0}$ and $g_{0,1}$). Assume $X_0 = j$. Find $G_{X_t}(s) := \mathbb{E}s^{X_t}$. What is the distribution of X_t as $t \to \infty$?

Solution

We have G in matrix form,

$$G = \begin{bmatrix} -\lambda & \lambda & & & & \\ \mu & -(\mu+\lambda) & \lambda & & & \\ & 2\mu & -(2\mu+\lambda) & \lambda & & \\ & & 3\mu & -3(\mu+\lambda) & \lambda & \cdots \\ & & & \vdots & \vdots & \ddots \end{bmatrix}$$

We wish to find the transition semi group P_t. We know this can be derived from the Kolmogorov forward equations. That is,

$$\frac{d}{dt}P_t = P_t G$$

With the assumption that $X_0 = i$ (*I am using* i *rather than* j *like the problem statement since this is the standard way of doing things*) we have,

$$\frac{d}{dt}p_t(i,0) = \sum_{k=0}^{\infty} p_t(i,k)g(k,0) = -\lambda p_t(i,0) + \mu p_t(i,1)$$

$$\frac{d}{dt}p_t(i,j) = \sum_{k=0}^{\infty} p_t(i,k)g_t(k,j) = \lambda p_t(i,j-1) - (j\mu+\lambda)p_t(i,j) + (j+1)\mu p_t(i,j+1) \qquad j \geq 1$$

We multiply the j-th equation by s^j. This gives,

$$\sum_{j=0}^{\infty} \frac{\partial}{\partial t}p_t(i,j)s^j = \sum_{j=1}^{\infty}\left[\lambda p_t(i,j-1)s^j\right] - \sum_{j=0}^{\infty}\left[(j\mu-\lambda)p_t(i,j)s^j\right] + \sum_{j=0}^{\infty}\left[(j+1)\mu p_t(i,j+1)s^j\right]$$

Summing the left hand sides gives,

$$\sum_{j=0}^{\infty}\frac{\partial}{\partial t}p_t(i,j)s^j = \frac{\partial}{\partial t}\sum_{j=0}^{\infty}p_t(i,j)s^j = \frac{\partial}{\partial t}G_{X_t}(s)$$

The first term of the right hand side gives,

$$\sum_{j=1}^{\infty}\lambda p_t(i,j-1)s^j = \lambda s\sum_{j=1}^{\infty}p_t(i,j-1)s^{j-1} = \lambda s\sum_{j=0}^{\infty}p_t(i,j)s^j = \lambda s G_{X_t}(s)$$

The negative of the first part of the second term of the right hand side gives,

$$\sum_{j=0}^{\infty}j\mu p_t(i,j)s^j = s\mu\sum_{j=0}^{\infty}jp_t(i,j)s^{j-1} = s\mu\sum_{j=0}^{\infty}\frac{\partial}{\partial s}p_t(i,j)s^j = s\mu\frac{\partial}{\partial s}\sum_{j=0}^{\infty}p_t(i,j)s^j = s\mu\frac{\partial}{\partial s}G_{X_t}(s)$$

The negative of the second part of the second term of the right hand side gives,

$$\sum_{j=0}^{\infty}\lambda p_t(i,j)s^j = \lambda\sum_{j=0}^{\infty}p_t(i,j)s^j = \lambda G_{X_t}(s)$$

The third term of the right hand side gives,

$$\sum_{j=1}^{\infty}(j+1)\mu p_t(i,j+1)s^j = \mu\sum_{j=1}^{\infty}\frac{\partial}{\partial s}p_t(i,j+1)s^{j+1} = \mu\frac{\partial}{\partial s}\sum_{j=0}^{\infty}p_t(i,j)s^j = \mu\frac{\partial}{\partial s}G_{X_t}(s)$$

Putting these results together we have,

$$\frac{\partial}{\partial t}G_{X_t}(s) = \left[\lambda s - s\mu\frac{\partial}{\partial s} - \lambda + \mu\frac{\partial}{\partial s}\right]G_{X_t}(s)$$

Since $X_0 = j$ we have initial condition,

$$G_{X_0}(s) = s^j$$

We solve with Mathematica by,

```
DSolve[{
    D[G[s,t],t]==\[Lambda] s G[s,t]-s \[Mu] D[G[s,t],s]-\[
       Lambda] G[s,t]+\[Mu] D[G[s,t],s],
    G[s,0]==s^j
    },G[s,t],{s,t}]//FullSimplify
```

This yields,

$$G_{X_t}(s) = \left((s-1)e^{-\mu t} + 1\right)^j \exp\left[\frac{\lambda(s-1)e^{\mu(-t)}\left(e^{\mu t}-1\right)}{\mu}\right]$$

We find the limit as $t \to \infty$ with Mathematica by,

```
Limit[E^((E^(-t \[Mu]) (-1+E^(t \[Mu])) (-1+s) \[Lambda])/\[
   Mu]) (1+E^(-t \[Mu]) (-1+s))^j,{t->\[Infinity]},
   Assumptions->{\[Lambda]>0,\[Mu]>0}]
```

This yields,

$$G_{X_\infty}(s) = \lim_{t\to\infty} G_{X_t}(s) = e^{\frac{\lambda}{\mu}(s-1)}$$

So $X_\infty = \lim_{t\to\infty} X_t$ is a Poission random variable with parameter λ/μ.

Exercise 5.4

Let N be a time-inhomogeneous Poisson process with intensity function $\lambda(t)$. That is, the probability of a jump of size one in the time interval $(t, t + dt)$ is $\lambda(t)dt$ and the probability of two jumps in that interval of time is $\mathcal{O}(dt^2)$. Write down the Kolmogorov forward and backward equations of N and solve them. Let $N_0 = 0$ and let τ_1 be the time of the first jump of N. If $\lambda(t) = c/(1 + t)$ show that $\mathbb{E}\tau_1 < \infty$ if and only if $c > 1$.

Solution

Based on the definition of the generator and the given transition probabilities we have,

$$G(t) = \begin{bmatrix} -\lambda(t) & \lambda(t) & & & \\ & -\lambda(t) & \lambda(t) & & \\ & & -\lambda(t) & \lambda(t) & \cdots \\ & & \vdots & \vdots & \ddots \end{bmatrix}$$

For $t \geq s$ we define,

$$p_{s,t}(i,j) = \mathbb{P}(N_t = j | N_s = i)$$

We first derive the Kolmogorov forward equations. We consider,

$$\begin{aligned} p_{s,t+\Delta t} &= \mathbb{P}(N_{t+\Delta t} = j | N_s = i) \\ &= \sum_k \mathbb{P}(N_{t+\Delta t} = j | N_t = k)\mathbb{P}(N_t = k | N_s = i) \\ &= \begin{cases} \lambda(t)\Delta t p_{s,t}(i, j-1) + (1 - \lambda(t)\Delta t)p_{s,t}(i,j) + \mathcal{O}(\Delta t^2) & j > i \\ (1 - \lambda(t)\Delta t)p_{s,t}(i,j) + \mathcal{O}(\Delta t^2) & j = i \\ 0 & j < i \end{cases} \end{aligned}$$

Therefore,

$$\frac{p_{s,t+\Delta t}(i,j) - p_{s,t}(i,j)}{\Delta t} = \begin{cases} \lambda(t)\Delta t p_{s,t}(i, j-1) - \lambda(t)\Delta t p_{s,t}(i,j) + \mathcal{O}(\Delta t^2) & j > i \\ -\lambda(t)\Delta t p_{s,t}(i,j) + \mathcal{O}(\Delta t^2) & j = i \\ 0 & j < i \end{cases}$$

Taking the limit as $\Delta t \to 0$ we have,

$$\frac{\partial}{\partial t} p_{s,t}(i,j) = \begin{cases} \lambda(t)p_{s,t}(i,j-1) - \lambda(t)p_{s,t}(i,j) & j > i \\ -\lambda(t)p_{s,t}(i,j) & j = i \\ 0 & j < i \end{cases}$$

Fix i. Noting that $G_F(x)$ is also a function of s, t and j, we have,

$$G_F(x) = \sum_{j=0}^{\infty} \mathbb{P}(N_t = j | N_s = i)x^j = \sum_{j=i}^{\infty} p_{s,t}(i,j)x^j$$

Thus, multiplying the j-th KFE by x^j and summing, we have,

$$\begin{aligned} \frac{\partial}{\partial t} \sum_{j=i}^{\infty} p_{s,t}(i,j)x^j = \sum_{j=i}^{\infty} \frac{\partial}{\partial t} p_{s,t}(i,j)x^j &= \sum_{j=i+1}^{\infty} \lambda(t)p_{s,t}(i,j-1)x^j + \sum_{j=i}^{\infty}(-\lambda(t))p_{s,t}(i,j)x^j \\ &= \lambda(t)x \sum_{j=i}^{\infty} p_{s,t}(i,j)x^j - \lambda(t) \sum_{j=i}^{\infty} p_{s,t}(i,j)x^j \end{aligned}$$

Therefore,

$$\frac{\partial}{\partial t} G_F(x) = \lambda(t)xG_F(x) - \lambda(t)G_F(x) = \lambda(t)(x-1)G_F(x)$$

We have initial condition $N_s = i$, so $G_B(x) = x^i$ when $s = t$.

We solve with Mathematica as,

```
DSolve[{D[G[s, t], t] == \[Lambda][t] (x - 1) G[s, t],
   G[s, s] == x^i
   }, G[s, t], {s, t}] // FullSimplify
```

This gives,

$$G_F(x) = x^i \exp\left((x-1)\int_s^t \lambda(z)\mathrm{d}z\right)$$

Write $I = \int_s^t \lambda(z)dz$. Then,

$$G_F(x) = e^{-I}x^i e^{Ix} = e^{-I}x^i \sum_{k=0}^{\infty} \frac{1}{k!}(Ix)^k = e^{-I} \sum_{k=0}^{\infty} \frac{1}{k!} I^k x^{k+i} = e^{-I} \sum_{j=i}^{\infty} \frac{I^{j-i}}{(j-i)!} x^j$$

Therefore, from the definition of the Generating function we have,

$$P_{s,t}(i,j) = \mathbb{P}(N_t = j | N_s = i) = \frac{1}{(j-i)!}\left[\int_s^t \lambda(z)\mathrm{d}z\right]^{j-i} \exp\left(-\int_s^t \lambda(z)\mathrm{d}z\right)$$

We now derive the Kolmogorov Backward equations. We consider,

$$\begin{aligned}
p_{s-\Delta s,t} &= \mathbb{P}(N_t = j | N_{s-\Delta s} = i) \\
&= \sum_k \mathbb{P}(N_t = j | N_s t = k)\mathbb{P}(N_s = k | N_{s-\Delta s} = i) \\
&= \begin{cases} \lambda(s)\Delta s p_{s,t}(i+1,j) + (1-\lambda(s)\Delta s)p_{s,t}(i,j) + \mathcal{O}(\Delta s^2) & j > i \\ (1-\lambda(s)\Delta s)p_{s,t}(i,j) + \mathcal{O}(\Delta s^2) & j = i \\ 0 & j < i \end{cases}
\end{aligned}$$

Therefore,

$$\frac{p_{s-\Delta s,t}(i,j) - p_{s,t}(i,j)}{\Delta s} = \begin{cases} \lambda(s)\Delta t p_{s,t}(i+1,j) - \lambda(s)\Delta t p_{s,t}(i,j) + \mathcal{O}(\Delta s^2) & j > i \\ -\lambda(s)\Delta t p_{s,t}(i,j) + \mathcal{O}(\Delta s^2) & j = i \\ 0 & j < i \end{cases}$$

Taking the limit as $\Delta s \to 0$ we have,

$$-\frac{\partial}{\partial s} p_{s,t}(i,j) = \begin{cases} \lambda(s)p_{s,t}(i+1,j) - \lambda(s)p_{s,t}(i,j) & j > i \\ -\lambda(s)p_{s,t}(i,j) & j = i \\ 0 & j < i \end{cases}$$

Fix i. Noting that $G_B(x)$ is also a function of s, t and j, we have,

$$G_B(x) = \sum_{j=0}^{\infty} \mathbb{P}(N_t = j | N_s = i)x^j = \sum_{j=i}^{\infty} p_{s,t}(i,j)x^j$$

Thus, multiplying the j-th KBE by x^j and summing, we have,

$$\begin{aligned}
-\frac{\partial}{\partial s}\sum_{j=i}^{\infty} p_{s,t}(i,j)x^j = -\sum_{j=i}^{\infty}\frac{\partial}{\partial s}p_{s,t}(i,j)x^j &= \sum_{j=i+1}^{\infty}\lambda(s)p_{s,t}(i+1,j)x^j + \sum_{j=i}^{\infty}(-\lambda(s))p_{s,t}(i,j)x^j \\
&= \sum_{j=i+1}^{\infty}\lambda(s)p_{s,t}(i,j-1)x^j + \sum_{j=i}^{\infty}(-\lambda(s))p_{s,t}(i,j)x^j \\
&= \lambda(s)x\sum_{j=i}^{\infty}p_{s,t}(i,j)x^j - \lambda(s)\sum_{j=i}^{\infty}p_{s,t}(i,j)x^j
\end{aligned}$$

Therefore,

$$\frac{\partial}{\partial s}G_B(x) = -\lambda(s)xG_B(x) + \lambda(s)G_B(x) = -\lambda(s)(x-1)G_B(x)$$

From the result for $G_F(x)$ we know,

$$G_B(x) = x^i\exp\left(-(x-1)\int_t^s \lambda(z)\mathrm{d}z\right) = x^i\exp\left((x-1)\int_s^t \lambda(z)\mathrm{d}z\right) = G_F(x)$$

We now show that for $\lambda(t) = c/(1+t)$, that $\mathbb{E}\tau_1 < \infty$ if and only if $c < 1$. Indeed,

$$\int_0^t \lambda(z)\mathrm{d}z = \int_0^t \frac{c}{1+z}\mathrm{d}z = c\ln(1+t) - c\ln(1) = c\ln(1+t)$$

Therefore,

$$\mathbb{E}[\tau_1] = \int_0^\infty \mathbb{P}(\tau_1 > t)\mathrm{d}t = \int_0^\infty \mathbb{P}(N_t = 0|N_0 = 0)\mathrm{d}t = \int_0^\infty \exp(-c\ln(1+t))\mathrm{d}t = \int_0^\infty \frac{\mathrm{d}t}{(1+t)^c}$$

This is convergent if and only if $c > 1$.

Exercise 5.5

Let N_t be a Poisson process with a random intensity Λ which is equal to λ_1 with probability p and λ_2 with probability $1-p$. Find $G_{N_t}(s) = \mathbb{E}s^{N_t}$. What is the mean and variance of N_t?

Solution

Recall the generating function for a Poisson process with intensity λ is,

$$G(s) = e^{-\lambda t(1-s)}$$

Therefore,

$$G_{N_t}(s) = \mathbb{E}\left[s^{N_t}\right] = \mathbb{E}\left[\mathbb{E}\left[s^{N_t}\right]\Big|\Lambda\right] = \mathbb{E}\left[e^{-\Lambda t(1-s)}\Big|\Lambda\right] = pe^{-\lambda_1 t(1-s)} + (1-p)e^{-\lambda_2(1-s)}$$

We use Mathematica to caluculate moments,

```
GNt[s_]:=p Exp[-\[Lambda]1 t (1-s)]+(1-p)Exp[-\[Lambda]2 t(1-
   s)]
D[GNt[s],{s,1}]/.{s->1}
D[GNt[s],{s,2}]-D[GNt[s],{s,1}]^2+D[GNt[s],{s,1}]/.{s->1}
```

This yields,

$$\begin{aligned}\mu &= G'_{N_t}(1) = p\lambda_1 t + (1-p)\lambda_2 t\\ \sigma^2 &= G''_{N_t}(1) - [G'_{N_t}(1)]^2 + G'_{N_t}(1) = p(\lambda_1 t)^2 + (1-p)(\lambda_2 t)^2 - \mu^2 + \mu\end{aligned}$$

4 Brownian Motion

Note: add examples from class notes

4.1 Martingale

Sample Problems:

- Exercise 7.1: Show a process is a Martingale using definition
- Exercise 7.4: Show a process is a Martingale using definition

4.2 Characteristic Functions

Sample Problems:

- Exercise 7.2: Compute characteristic function of $W(N(t))$, where $N \sim \text{Pois}(\lambda)$

7.3: n-th variation time

4.3 Laplace Transform

Sample Problems:

- *Note:* Example ??? from book
- Exercise 7.4: Compute Laplace transform of first hitting time.

Exercise 7.1

Let W be a Brownian motion and let $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ be a filtration for W. Show that $W(t)^2 - t$ is a martingale with respect to the filtration $\mathbb{F}$.

Solution

Suppose $X \sim \mathcal{N}(0, \sigma^2)$. Then,

$$\sigma^2 = \mathbb{V}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \mathbb{E}[X^2] - 0^2 = \mathbb{E}[X^2]$$

Let $0 \leq s \leq t$. By the definition of a filtration, $(W(t) - W(s))$ is independent of $\mathcal{F}_s$. Moreover, by the definition of Brownian Motion we have $W(t) - W(s) \sim \mathcal{N}(0, t-s)$. Thus,

$$\mathbb{E}\left[(W(t) - W(s))^2 \middle| \mathcal{F}_s\right] = \mathbb{E}\left[(W(t) - W(s))^2\right] = (t - s)$$

Since $W(s) \in \mathcal{F}_s$, by "taking out what is known" we have,

$$\begin{aligned} \mathbb{E}\left[W(t)W(s) \middle| \mathcal{F}_s\right] &= W(s)\mathbb{E}\left[W(t) \middle| \mathcal{F}_s\right] = W(s)W(s) = W(s)^2 \\ \mathbb{E}\left[W(s)^2 \middle| \mathcal{F}_2\right] &= W(s)\mathbb{E}\left[W(s) \middle| \mathcal{F}_2\right] = W(s)W(s) = W(s)^2 \end{aligned}$$

Therefore,

$$\begin{aligned} \mathbb{E}\left[W(t)^2 - t \middle| \mathcal{F}_s\right] &= \mathbb{E}\left[(W(t) - W(s) + W(s))^2 - t\right] \\ &= \mathbb{E}\left[(W(t) - W(s))^2 + 2(W(t) - W(s))W(s) + W(s)^2 - t\right] \\ &= \mathbb{E}\left[(W(t) - W(s))^2 \middle| \mathcal{F}_s\right] + 2\mathbb{E}\left[W(t)W(s) \middle| \mathcal{F}_s\right] - \mathbb{E}\left[W(s)^2 \middle| \mathcal{F}_2\right] - \mathbb{E}[t] \\ &= (t - s) + 2W(s)^2 - W(s)^2 - t \\ &= W(s)^2 - s \end{aligned}$$

This proves $W(t) - t$ is a martingale with respect to the filtration $\mathbb{F}$. $\square$

Exercise 7.2

Compute the characteristic function of $W(N(t))$ where N is a Poisson process with intensity λ and the Brownian motion W is independent of the Poisson process N.

Solution

The characteristic function is defined as,

$$\phi(s) = \mathbb{E}e^{isW(N(t))}$$

We condition on $N(t)$ using iterated conditioning,

$$\mathbb{E}\left[e^{isW(N(t))}\right] = \mathbb{E}\left[\mathbb{E}\left[e^{isW(N(t))}\middle|N(t)\right]\right]$$

The characteristic function of $Z \sim \mathcal{N}(\mu, \sigma^2)$ is $\phi_Z(s) = \exp(i\mu s - \sigma^2 s^2/2)$. At time t, $W(t)$ is normally distributed with mean zero and variance t. Thus,

$$\mathbb{E}\left[\mathbb{E}\left[e^{isW(N(t))}\middle|N(t)\right]\right] = \mathbb{E}\left[e^{-N(t)s^2/2}\right]$$

Since $N(t)$ is a Poisson process with parameter λ, then $N(t) = k$ with probability $(\lambda t)^k e^{-\lambda t}/k!$. Thus,

$$\mathbb{E}\left[e^{-N(t)s^2/2}\right] \sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!} e^{-\lambda t} e^{-ks^2/2} = e^{-\lambda t} \sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!} \left(e^{-s^2/2}\right)^k$$

Simplifying yields,

$$e^{-\lambda t} \sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!} \left(e^{-s^2/2}\right)^k = e^{-\lambda t} \sum_{k=0}^{\infty} \frac{1}{k!} \left(\lambda t e^{-s^2/2}\right)^k = e^{-\lambda t} \exp\left(\lambda t e^{-s^2/2}\right) = \exp\left(\lambda t \left(e^{-s^2/2} - 1\right)\right)$$

That is, the characteristic function $\phi(s)$ of $W(N(t))$ is,

$$\phi(s) = \exp\left(\lambda t \left(e^{-s^2/2} - 1\right)\right)$$

Exercise 7.3

The n-th variation of a function f, over the interval $[0, T]$ is defined as,

$$V_T(n, f) := \lim_{\|\Pi\| \to 0} \sum_{j=0}^{m-1} |f(t_{j+1}) - f(t_j)|^n, \quad \Pi = \{0 = t_0, t_1, \ldots, t_m = T\}, \quad \|\Pi\| = \max_j (t_{j+1} - t_j)$$

Show that $V_T(1, W) = \infty$ and $V_T(3, W) = 0$, where W is a Brownian motion.

Solution

We first prove the following useful lemma.

Lemma.

If $f_n \to 0$ and $|g_n| \le M$ for some $|M| < \infty$ then $(f_n g_n) \to 0$.

Fix, $\varepsilon > 0$. Then, by convergence of f_n there is some $N \in \mathbb{N}$ such that $|f_n| < \varepsilon/M$ for all $n \ge N$. Then,

$$|f_n g_n| = |f_n||g_n| \le |f_n|M < (\varepsilon/M)M = \varepsilon$$

This proves $f_n g_n \to 0$. $\square$

Write,

$$V_T(k+1, W) = \lim_{\|\Pi\| \to 0} \sum_{j=0}^{m-1} |W(t_{j+1}) - W(t_j)|^{k+1} = \lim_{\|\Pi\| \to 0} \sum_{j=0}^{m-1} |W(t_{j+1}) - W(t_j)|^k |W(t_{j+1}) - W(t_j)|$$

Let, $M_\Pi = \max_j |W(t_{j+1}) - W(t_j)|$ for a given partition Π. Then,

$$\begin{aligned}
\lim_{\|\Pi\| \to 0} \sum_{j=0}^{m-1} |W(t_{j+1}) - W(t_j)|^k |W(t_{j+1}) - W(t_j)| &\le \lim_{\|\Pi\| \to 0} \sum_{j=0}^{m-1} |W(t_{j+1}) - W(t_j)|^k M_\Pi \\
&= \lim_{\|\Pi\| \to 0} M_\Pi \sum_{j=0}^{n-1} |W(t_{j+1}) - W(t_j)|^k
\end{aligned}$$

Provided, $|V_T(k, T)| = V_T(k, T)$ is not infinite,

$$\lim_{\|\Pi\| \to 0} M_\Pi \sum_{j=0}^{m-1} |W(t_{j+1}) - W(t_j)|^k = \left(\lim_{\|\Pi\| \to 0} M_\Pi \right) \left(\lim_{\|\Pi\| \to 0} \sum_{j=0}^{n-1} |W(t_{j+1}) - W(t_j)|^2 \right)$$

Since $W(t)$ is continuous, $|W(t_{j+1})-W(t_j)| \to 0$ as $\|\Pi\| \to 0$ since $t_{j+1}-t_j \to 0$. In particular, this means that $M_\Pi \to 0$ as $\|\Pi\| \to 0$.

Thus,

$$0 \geq V_T(k+1, W) = \left(\lim_{\|\Pi\|\to 0} M_\Pi\right)\left(\lim_{\|\Pi\|\to 0} \sum_{j=0}^{m-1} |W(t_{j+1}) - W(t_j)|^k\right) \leq 0 \cdot N = 0$$

Recall $V_T(2, W) = T < \infty$. Then, by above, $V_T(3, W) = 0$. $\square$

Suppose, for the sake of contradiction that $V_T(1, W) \neq \infty$. Clearly $V_T(1, W) \geq 0$, so $V_T(1, W)$ is bounded above and below by finite constants. Then, by above, $V_T(2, W) = 0$, a contradiction (for $T > 0$). This proves $V_T(1, W) = \infty$. $\square$

Exercise 7.4

Define

$$X_t = \mu t + W_t \qquad \tau_m := \inf\{t \geq 0 : X_t = m\}$$

Show that Z is a martingale where,

$$Z_t = \exp(\sigma X_t - (\sigma\mu + \sigma^2/2)t)$$

Assume $\mu > 0$ and $m \geq 0$. Assume further that $\tau_m < \infty$ with probability one and the stopped process $Z_{t\wedge\tau_m}$ is a martingale. Find the Laplace transform $\mathbb{E}e^{-\alpha\tau_m}$.

Solution

Let $0 \leq s \leq t$. Rewrite,

$$\mathbb{E}\left[Z_t \middle| \mathcal{F}_s\right] = \mathbb{E}\left[e^{\sigma X_t - (\sigma\mu+\sigma^2/2)t} \middle| \mathcal{F}_s\right] = \mathbb{E}\left[e^{\sigma(\mu t + W_t) - (\sigma\mu+\sigma^2/2)t} \middle| \mathcal{F}_s\right] = \mathbb{E}\left[e^{\sigma W_t - (\sigma^2/2)t} \middle| \mathcal{F}_s\right]$$

Now, pulling out what is known,

$$\mathbb{E}\left[e^{\sigma W_t - (\sigma^2/2)t} \middle| \mathcal{F}_s\right] = \mathbb{E}\left[e^{\sigma(W_t - W_s) + \sigma W_s - (\sigma^2/2)t)} \middle| \mathcal{F}_s\right] = e^{\sigma W_s - (\sigma^2/2)t}\mathbb{E}\left[e^{\sigma(W_t - W_s)} \middle| \mathcal{F}_s\right]$$

By the property of independent increments,

$$e^{\sigma W_s - (\sigma^2/2)t}\mathbb{E}\left[e^{\sigma(W_t - W_s)} \middle| \mathcal{F}_s\right] = e^{\sigma W_s - (\sigma^2/2)t}\mathbb{E}\left[e^{\sigma(W_t - W_s)}\right] = e^{\sigma W_s - (\sigma^2/2)t}e^{\sigma^2(t-s)/2}$$

Finally,

$$e^{\sigma W_s - (\sigma^2/2)t}e^{\sigma^2(t-s)/2} = e^{\sigma W_s - (\sigma^2/2)s} = e^{\sigma(\mu s + W_s) - (\sigma\mu+\sigma^2/2)s} = e^{\sigma X_2 - (\sigma\mu+\sigma^2/2)s}$$

This proves Z_t is a martingale. $\square$

Define $s = \min\{t, \tau_m\}$. Fix $m \geq 0$ and define,

$$Z^{(m)} = \left(Z_t^{(m)}\right)_{t\geq 0}, \qquad Z_t^{(m)} = Z_s$$

Then, using the fact that Z_t is a martingale we have,

$$1 = Z_0^{(m)} = \mathbb{E}\left[Z_t^{(m)}\right] = \mathbb{E}\left[e^{\sigma X_s - (\sigma\mu+\sigma^2/2)s}\right]$$

If $\tau_m = \infty$ then $X_t < m$ for all t. Thus, since $\sigma \geq 0, \mu > 0$,

$$e^{\sigma X_t - (\sigma\mu + \sigma^2/2)t} \leq e^{\sigma m - (\sigma\mu + \sigma^2/2)t} < \infty$$

Therefore, since $\mathbb{P}(\tau_m < \infty) = 0$,

$$\begin{aligned}
\mathbb{E}\left[e^{\sigma X_s - (\sigma\mu+\sigma^2/2)s}\right] &= \mathbb{E}\left[\mathbb{1}_{\{\tau_m=\infty\}}\left(e^{\sigma X_s-(\sigma\mu+\sigma^2/2)s}\right) + \mathbb{1}_{\{\tau_m<\infty\}}\left(e^{\sigma X_s-(\sigma\mu+\sigma^2/2)s}\right)\right] \\
&= \mathbb{E}\left[\mathbb{1}_{\{\tau_m=\infty\}}\left(e^{\sigma X_t-(\sigma\mu+\sigma^2/2)t}\right)\right] + \mathbb{E}\left[\mathbb{1}_{\{\tau_m<\infty\}}\left(e^{\sigma X_{\tau_m}-(\sigma\mu+\sigma^2/2)\tau_m}\right)\right] \\
&= 0 + \mathbb{E}\left[\mathbb{1}_{\{\tau_m<\infty\}}\left(e^{\sigma m-(\sigma\mu+\sigma^2/2)\tau_m}\right)\right]
\end{aligned}$$

Similarly, since $\sigma \geq 0, \mu > 0$, $e^{\sigma m - (\sigma\mu+\sigma^2/2)\tau_m)} < \infty$. Therefore,

$$\begin{aligned}
\mathbb{E}\left[\mathbb{1}_{\{\tau_m<\infty\}}\left(e^{\sigma m-(\sigma\mu+\sigma^2/2)\tau_m}\right)\right] &= \mathbb{E}\left[\mathbb{1}_{\{\tau_m=\infty\}}\left(e^{\sigma m-(\sigma\mu+\sigma^2/2)\tau_m}\right)\right] + \mathbb{E}\left[\mathbb{1}_{\{\tau_m<\infty\}}\left(e^{\sigma m-(\sigma\mu+\sigma^2/2)\tau_m}\right)\right] \\
&= \mathbb{E}\left[\mathbb{1}_{\{\tau_m=\infty\}}\left(e^{\sigma m-(\sigma\mu+\sigma^2/2)\tau_m}\right) + \mathbb{1}_{\{\tau_m<\infty\}}\left(e^{\sigma m-(\sigma\mu+\sigma^2/2)\tau_m}\right)\right] \\
&= \mathbb{E}\left[e^{\sigma m-(\sigma\mu+\sigma^2/2)\tau_m}\right]
\end{aligned}$$

Then, setting $\alpha = (\sigma\mu + \sigma^2/2)$,

$$e^{-\sigma m} = \mathbb{E}\left[e^{-(\sigma\mu+\sigma^2/2)\tau_m}\right] = \mathbb{E}\left[e^{-\alpha\tau_m}\right]$$

We solve the equation, $\alpha = (\sigma\mu + \sigma^2/2)$ for σ using the quadratic equation, yielding,

$$\sigma = -\mu \pm \sqrt{\mu^2 + 2\alpha}$$

However, $\sigma, \alpha \geq 0$ so we must take $\sigma = -\mu + \sqrt{\mu^2 + 2\alpha}$. Thus,

$$\mathbb{E}\left[e^{-\alpha\tau_m}\right] = e^{\left(\mu - \sqrt{\mu^2+2\alpha}\right)m}$$

5 Stochastic Calculus

Exercise 8.1

Compute $\mathrm{d}(W_t^4)$. Write W_T^4 as an integral with respect to W plus an integral with respect to t. Use this representation of W_T^4 to show that $\mathbb{E}W_T^4 = 3T^2$. Compute $\mathbb{E}W_T^6$ using the same technique.

Solution

Write $f(x) = x^4$ so that $f(W_t) = W_t^4$. Then, $f'(x) = 4x^3$ and $f''(x) = 12x^2$. Therefore, Itô's formula gives,

$$\mathrm{d}W_t^4 = f'(W_t)\mathrm{d}W_t + \frac{1}{2}f''(W_t)\mathrm{d}[W,W]_t = 4W_t^3\mathrm{d}W_t + \frac{12}{2}W_t^2\mathrm{d}[W,W]_t$$

Thus, writing $\mathrm{d}[W,W]_t = \mathrm{d}t$ we have,

$$\mathrm{d}W_t^4 = 4W_t^3\mathrm{d}W_t + 6W_t^2\mathrm{d}t$$

Thus, since $W_0 = 0$,

$$W_T^4 = W_T^4 - W_0^4 = 4\int_0^T W_t^3\mathrm{d}W_t + 6\int_0^T W_t^2\mathrm{d}t$$

Recall Itô integrals are martingales so that,

$$\mathbb{E}\left[\int_0^T W_t^3\mathrm{d}W_t\right] = 0$$

Note also that since $\mathbb{E}\left[W_t^2\right] = t$,

$$\mathbb{E}\left[\int_0^T W_t^2\mathrm{d}t\right] = \int_0^T \mathbb{E}\left[W_t^2\right]\mathrm{d}t = \int_0^T t\mathrm{d}t = \frac{T^2}{2}$$

Therefore,

$$\mathbb{E}\left[W_T^4\right] = 4\mathbb{E}\left[\int_0^T W_t^3\mathrm{d}W_t\right] + 6\mathbb{E}\left[\int_0^T W_t^2\mathrm{d}t\right] = 6\frac{T^2}{2} = 3T^2$$

Similarly, we have,

$$W_T^6 = 6\int_0^T W_t^5\mathrm{d}W_t + \frac{6\cdot 5}{2}\int_0^T W_t^4\mathrm{d}t$$

Therefore, since $\mathbb{E}\left[W_t^4\right] = 3t^2$,

$$\mathbb{E}\left[W_T^6\right] = 6\mathbb{E}\left[\int_0^T W_t^5 \mathrm{d}W_t\right] + 15\mathbb{E}\left[\int_0^T W_t^4 \mathrm{d}t\right] = 15\int_0^T \mathbb{E}\left[W_t^4\right]\mathrm{d}t = 15\int_0^T 3t^2 \mathrm{d}t = 15T^3$$

Exercise 8.2

Find an explicit expression for Y_T where,

$$\mathrm{d}Y_t = r\mathrm{d}t + \alpha Y_t \mathrm{d}W_t$$

Hint: Multiply the above equation by $F_t := \exp(-\alpha W_t + \frac{1}{2}\alpha^2 t)$.

Solution

Let $f(x,y) = \exp(-\alpha x + \frac{1}{2}\lambda^2 y)$ so that,

$$f_x(W_t,t) = -\alpha F_t \qquad f_y(W_t,t) = \frac{\alpha^2}{2}F_t \qquad f_{xx}(W_t,t) = \alpha^2 F_t$$

Then $F_t = f(W_t,t)$, so by Itô's formula and the heuristic $(\mathrm{d}W_t)^2 = \mathrm{d}t, (\mathrm{d}t)^2 = \mathrm{d}t\mathrm{d}W_t = 0$,

$$\begin{aligned}\mathrm{d}F_t = \mathrm{d}f(W_t,t) &= f_y(W_t,t)\mathrm{d}t + f_x(W_t,t)\mathrm{d}W_t + \frac{1}{2}f_{xx}(W_t,t)(\mathrm{d}W_t)^2 \\ &= \frac{\alpha^2}{2}F_t\mathrm{d}t - \alpha F_t\mathrm{d}W_t + \frac{\alpha^2}{2}F_t\mathrm{d}t \\ &= \alpha^2 F_t\mathrm{d}t - \alpha F_t\mathrm{d}W_t\end{aligned}$$

Using our heuristics we have,

$$\mathrm{d}[F,Y]_t = (\mathrm{d}F_t)(\mathrm{d}Y_t) = \left(\alpha^2 F_t\mathrm{d}t - \alpha F_t\mathrm{d}W_t\right)(r\mathrm{d}t + \alpha Y_t\mathrm{d}W_t) = -\alpha^2 F_t Y_t(\mathrm{d}W_t)^2 = -\alpha^2 F_t Y_t\mathrm{d}t$$

By the product rule we have,

$$\begin{aligned}\mathrm{d}(F_tY_t) &= F_t\mathrm{d}Y_t + Y_t\mathrm{d}F_t + \mathrm{d}[F,Y]_t \\ &= F_t(r\mathrm{d}t + \alpha Y_t\mathrm{d}W_t) + Y_t(\alpha^2 F_t\mathrm{d}t - \alpha F_t\mathrm{d}W_t) - \alpha^2 F_t Y_t\mathrm{d}t \\ &= rF_t\mathrm{d}t\end{aligned}$$

In integral form,

$$F_tY_t - F_0Y_0 = \int_0^t rF_s\mathrm{d}s = \int_0^t re^{-\alpha W_s + \frac{1}{2}\alpha^2 s}\mathrm{d}s$$

We can add $F_0Y_0 = Y_0$ and divide by F_t yielding,

$$Y_t = Y_0 + re^{\alpha W_t - \frac{1}{2}\alpha^2 t}\int_0^t e^{-\alpha W_s + \frac{1}{2}\alpha^2 s}\mathrm{d}s$$

Exercise 8.3

Suppose X, Δ, and Π are given by,

$$\mathrm{d}X_t = \sigma X_t \mathrm{d}W_t, \qquad \Delta_t = \frac{\partial f}{\partial x}(t, X_t), \qquad \Pi_t = X_t \Delta_t$$

where f is some smooth function. Show that if f satisfies,

$$\left(\frac{\partial}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2}{\partial x^2}\right) f(t, x) = 0$$

for all (t, x), then Π is a martingale with respect to a filtration $\mathcal{F}_t$ for W.

Solution

We have,

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2}{\partial x^2}\right) = \frac{\partial^2}{\partial x \partial t} + \frac{1}{2}\sigma^2 \left[x^2 \frac{\partial^3}{\partial x^3} + 2x \frac{\partial^2}{\partial x^2}\right]$$

Thus, using the condition for f we have,

$$\frac{\partial^2 f}{\partial x \partial t} + \frac{1}{2}\sigma^2 X_t^2 \frac{\partial^3 f}{\partial x^3} = -\sigma^2 X_t \frac{\partial^2 f}{\partial x^2}$$

Using our heuristics we have,

$$\mathrm{d}[X, X] = \sigma^2 X_t^2 (\mathrm{d}W_t)^2 = \sigma^2 X_t^2 \mathrm{d}t$$

Similarly,

$$\mathrm{d}[X, t] = \mathrm{d}[t, X] = \mathrm{d}[t, t] = 0$$

Therefore, by Itô's formula,

$$\begin{aligned}
\mathrm{d}\Delta_t &= \frac{\partial^2 f}{\partial x \partial t}(t, X_t)\mathrm{d}t + \frac{\partial^2 f}{\partial x^2}(t, X_t)\mathrm{d}X_t + \frac{1}{2}\mathrm{d}[X, X] \\
&= \frac{\partial^2 f}{\partial x \partial t}(t, X_t)\mathrm{d}t + \sigma X_t \frac{\partial^2 f}{\partial x^2}(t, X_t)\mathrm{d}W_t + \frac{1}{2}\sigma^2 X_t^2 \frac{\partial^3 f}{\partial x^3}(t, X_t)\mathrm{d}t \\
&= -\sigma^2 X_t \frac{\partial^2 f}{\partial x^2}(t, X_t)\mathrm{d}t + \sigma X_t \frac{\partial^2 f}{\partial x^2}(t, X_t)\mathrm{d}W_t
\end{aligned}$$

Therefore,

$$d[X,\Delta]_t = (dX_t)(d\Delta_t) = \sigma^2 X_t^2 \frac{\partial^2 f}{\partial x^2}(t,X_t)(dW_t)^2 = \sigma^2 X_t^2 \frac{\partial^2 f}{\partial x^2}(t,X_t)dt$$

Finally, we have,

$$\begin{aligned}
d\Pi_t &= d(X_t\Delta_t) = X_t d\Delta_t + \Delta_t dX_t + d[X,\Delta]_t \\
&= X_t\left(-\sigma^2 X_t \frac{\partial^2 f}{\partial x^2}(t,X_t)dt + \sigma X_t \frac{\partial^2 f}{\partial x^2}(t,X_t)dW_t\right) + \sigma X_t \frac{\partial f}{\partial x}(t,X_t)dW_t + \sigma^2 X_t^2 \frac{\partial^2 f}{\partial x^2}dt \\
&= \sigma X_t\left(X_t \frac{\partial^2 f}{\partial x^2}(t,X_t) + \frac{\partial f}{\partial x}(t,X_t)\right)dW_t
\end{aligned}$$

Since there is no dt dependence this is an Itô integral and therefore a martingale with respect to a filtration for W. (there are probably some technical assumptions we need about X and f, but in class we never dealt with these)$\square$

Exercise 8.4

Suppose X is given by,

$$\mathrm{d}X_t = \mu(t, X_t)dt + \sigma(t, X_t)\mathrm{d}W_t$$

For any smooth function f define,

$$M_t^f := f(t, X_t) - f(0, X_0) - \int_0^t \left(\frac{\partial}{\partial s} + \mu(s, X_s)\frac{\partial}{\partial x} + \frac{1}{2}\sigma^2(s, X_s)\frac{\partial^2}{\partial x^2}\right) f(s, X_s)ds$$

Show that M^f is a martingale with respect to a filtration $\mathcal{F}_t$ for W.

Solution

We first compute,

$$\mathrm{d}[X, X]_t = (\mathrm{d}X_t)(\mathrm{d}X_t) = \sigma^2(t, X_t)(\mathrm{d}W_t)^2 = \sigma^2(t, X_t)\mathrm{d}t$$

We then have,

$$\begin{aligned}
\mathrm{d}f(t, X_t) &= \frac{\partial f}{\partial t}(t, X_t)\mathrm{d}t + \frac{\partial f}{\partial x}(t, X_t)\mathrm{d}X_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\mathrm{d}[X, X]_t \\
&= \frac{\partial f}{\partial t}(t, X_t)\mathrm{d}t + \frac{\partial f}{\partial x}(t, X_t)[\mu(t, X_t)\mathrm{d}t + \sigma(t, X_t)\mathrm{d}W_t] + \frac{1}{2}\sigma^2(t, X_t)\frac{\partial^2 f}{\partial x^2}\mathrm{d}t \\
&= \left(\frac{\partial}{\partial t} + \mu(t, X_t)\frac{\partial}{\partial x} + \frac{1}{2}\sigma^2(t, X_t)\frac{\partial^2}{\partial x^2}\right) f(t, X_t)\mathrm{d}t + \sigma(t, X_t)\frac{\partial f}{\partial x}\mathrm{d}W_t
\end{aligned}$$

Finally, since $f(0, X_0)$ is a constant,

$$\begin{aligned}
\mathrm{d}M_t^f &= \mathrm{d}f(t, X_t) - \left(\frac{\partial}{\partial t} + \mu(t, X_t)\frac{\partial}{\partial x} + \frac{1}{2}\sigma^2(t, X_t)\frac{\partial^2}{\partial x^2}\right) f(t, X_t)\mathrm{d}t \\
&= \sigma(t, X_t)\frac{\partial f}{\partial x}\mathrm{d}W_t
\end{aligned}$$

Since there is no dt dependence this an Itô integral and therefore a martingale with respect to a filtration for W. $\square$