Qualifying Examination

December 2014

Examination Committee: Loyce Adams, Bernard Deconinck, Matt Lorig

Day 1. Tuesday, December 9, 2014

You have three hours to complete this exam. Work all problems. Start each problem on a new page. You must show all steps and prove all claims, unless you rely on a standard theorem or result, in which case you should refer to it. You are allowed to use a computer for any part of any problem, unless otherwise specified. You are not allowed to access the internet or any application file existing on your computer. Any application file you use must be created from scratch.

All problems are 20 points. Please email any electronic files in support of your exam to `lma3@uw.edu`, at the conclusion of the exam.

1. Let

$$A(t) = \begin{pmatrix} t^2+1 & t \\ -2 & t-1 \end{pmatrix}.$$

Calculate $\frac{d}{dt}A^2(t)$.

2. Suppose we have an $m \times n$ matrix W that has n orthonormal columns and $m > n$. Write Matlab statements to find an $m \times (m-n)$ matrix V that has $m-n$ orthonormal columns so that the matrix $Q = \begin{bmatrix} W & V \end{bmatrix}$ is unitary.

3. Consider the equation

$$x(x-2)y'' + (2-x^2)y' + 2(x-1)y = (x-2)^2.$$

(a) Check that $y_1 = x^2$ is a solution of the homogeneous problem.

(b) Find the general solution of the homogeneous problem.

(c) Find the general solution of the non-homogeneous problem.

For this problem, you are allowed to use a computer *only* to compute anti-derivatives.

4. Let S be a symmetric matrix. The following is an incorrect proof that S is non-negative definite! Find the flaw in the proof.

Pf: Let $S = U\Sigma U^*$ be the SVD of S where U is unitary and Σ is a diagonal matrix of nonzero real elements ordered as $\sigma_1 \geq \sigma_2 \cdots \geq \sigma_n \geq 0$. Then,

$$x^*Sx = x^*U\Sigma U^*x = y^*\Sigma y,$$

where $y = U^*x$. Also, since U is unitary, $x^*x = y^*y$, and we have

$$\frac{x^*Sx}{x^*x} = \frac{y^*\Sigma y}{y^*y} \tag{1}$$

for all nonzero x (and hence nonzero y). Note now that $\sigma_n \leq \frac{y^*\Sigma y}{y^*y} \leq \sigma_1$ and since both σ_n and σ_1 are non-negative, it follows that S is non-negative definite.

5. Write down the solution of the following initial-value problem:

$$
\begin{aligned}
u_t &= u_{xx}, && x \in \mathbb{R},\ t > 0, \\
u(x,0) &= f(x), && x \in \mathbb{R}, \\
\lim_{x \to -\infty} u(x,t) &= \alpha \neq 0, && t > 0, \\
\lim_{x \to \infty} u(x,t) &= \beta \neq 0, && t > 0.
\end{aligned}
$$

You may assume that the boundary and initial conditions are compatible.

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Day 2. Wednesday, December 10, 2014

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1. Consider the advection equation $u_t + au_x = 0$ with periodic boundary conditions.
   (a) Derive the Upwind method below where $U_j^n \approx u(x_j, t_n)$ with $\Delta t = k$, and $\Delta x = h$:
   $$U_j^{n+1} = U_j^n - \frac{ak}{h}(U_j^n - U_{j-1}^n), \;\; a > 0$$
   (b) Show the method is first order accurate in both time and space.
   (c) Use either MOL or Von Neumann analysis to derive the stability condition.
   (d) Show this Upwind method is an $O(k^2)$ accurate method applied to the modified PDE, $v_t + av_x = .5ah(1 - \frac{ak}{h})v_{xx}$ when we keep k/h fixed. Based on this, would you expect to see dissipative or dispersive behavior when the Upwind method is applied to the original advection equation? Explain.

2. Let $f_n(x) = \frac{x^n}{n(n+1)}$.
   (a) Show that the series $\sum_{n=1}^{+\infty} f_n(x)$ converges if $|x| < 1$.
   (b) Show that the function $f(x) = \sum_{n=1}^{+\infty} f_n(x)$ is continuous on $(-1, 1)$.
   (c) Show that the function $f(x) = \sum_{n=1}^{+\infty} f_n(x)$ is differentiable on $(-1, 1)$.
   (d) Give an expression for the derivative $f'(x)$ when $|x| < 1$.

3. Find the possible leading behaviors as $x \to 0$ of
   $$x^3y'' - xy' + 3y = 0.$$

4. Define the incomplete Fresnel integrals
   $$f_1(y) = \int_0^y \cos(x^2)dx, \quad f_2(y) = \int_0^y \sin(x^2)dx.$$
   (a) Prove that $F_1 = \lim_{y\to\infty} f_1(y)$ and $F_2 = \lim_{y\to\infty} f_2(y)$ are well defined.
   (b) For $y \in [-5, 5]$, produce a plot of the so-called Euler or Cornu spiral, which is the graph in the (f_1, f_2) plane of $(f_1(y), f_2(y))$ as y varies. This graph is used in train track design and diffraction optics.
   (c) Starting from the origin, what is the arc length of the Cornu spiral as a function of y?
   (d) Calculate F_1 and F_2. The value of $\int_0^\infty e^{-x^2}dx = \sqrt{\pi}/2$ may be useful. Does your result (visually) agree with the plot you obtained for the Cornu spiral?

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Day 3. Thursday, December 11, 2014

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