Qualifying Examination (September 2011)
Committee: Bernard Deconinck, Anne Greenbaum, Ulrich Hetmaniuk
Day 1

Submit solutions to **five** of the following six problems. If you solve all six problems, the lowest grade will be dropped.

Start each problem on a new page. You must show all steps and prove all claims, unless you rely on a standard theorem or result, in which case you should refer to it. You are allowed to use a computer for any part of any problem, but you are not allowed to access the internet or any application file existing on your computer. Any application file you use must be created from scratch.

1. Find all the pairs (μ, y), where μ is a real number and y is a non-zero function, satisfying the problem
$$-x^2 y'' - 3xy' = \mu y, \quad 1 < x < e, \quad y(1) = y(e) = 0.$$
You may assume that $\mu > 1$.

2. Let
$$f(x) = x, \quad 0 < x < L.$$
   (a) Determine the Fourier sine series for f on $(0, L)$. What can you say about the convergence of this series (pointwise convergence, uniform convergence, absolute convergence)? Motivate your answer.

   (b) In certain cases, a Fourier series may be differentiated term-by-term to obtain the series for $f'(x)$. State a theorem to this effect. Does this theorem apply here? If so, what is the Fourier series for $f'(x)$?

   (c) Write the Parseval identity and verify it for the function $f(x) = x$. (Hint: $\sum_{n=1}^{+\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$)

3. Let $\mathbf{A} \in \mathbb{R}^{n \times n}$.

   - Define the singular value decomposition of $\mathbf{A} \in \mathbb{R}^{n \times n}$.
   - How can you compute the singular values and right and left singular vectors by computing eigenpairs of two operators related to $\mathbf{A}$?

- How is the Frobenius norm of $\mathbf{A}$ related to its singular values?
- Let $\mathbf{b} \in \mathbb{R}^n$. Consider the least-squares problem

  $$\min_{\mathbf{x}\in\mathbb{R}^n} \|\mathbf{b} - \mathbf{A}\mathbf{x}\|_2$$

  (where $\|\cdot\|_2$ is the Euclidian norm). Give an expression for all the minimizer(s) $\mathbf{x}$ using the singular value decomposition of $\mathbf{A}$. (Study all the possible cases.)
- Numerically, if all the singular values of $\mathbf{A}$ are positive, would you use all of them to solve the least-squares problems? Why?

4. Determine

   $$\min_{a,b,c\ \in\ \mathbb{R}} \int_{-1}^{1} \left(x^3 - a - bx - cx^2\right)^2 dx$$

   Explain how to find the miminum and how many minimizers exist. If you use a numerical software to compute the minimum, detail which numerical method is used and why it converges to the correct minimum.

5. Evaluate (counterclockwise)

   $$\int_C \frac{e^z}{\cos(z)} dz, \quad C : \left| z - \frac{\pi i}{2} \right| = 4.5.$$

   Show the details.

6. Let $f \in L^2(\mathbb{R})$ and define

   $$g(t) = \int_{\mathbb{R}} f(x)^2 e^{-t|x|^2} dx, \quad \forall t \geq 0.$$

   Show that g is continuously differentiable on $[0, +\infty)$ if and only if $f \in L^2(\mathbb{R})$ and $xf(x) \in L^2(\mathbb{R})$. Discuss whether g has a limit when $t \to +\infty$. If so, give its value.

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Day 2

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1. Consider the equation

$$\frac{d^2y}{dt^2} + (a + 2\varepsilon \sin 2t)y = 0, \quad t \in \mathbb{R},$$

(where a and ε are parameters). We call a solution of this equation such that $\sup_{t\in\mathbb{R}} |y| < \infty$ stable. Other solutions are called unstable.

- It follows from Floquet theory that we only have to consider fundamental solutions of the form

$$y(t) = e^{\mu t}\phi(t)$$

where μ is a complex number and ϕ is a π-periodic function. Determine for which values of μ the solution y is stable or unstable.

- When ε is small, for which values of a do stable solutions y exist?
- Investigate the behavior of solutions when $a = 1$ and ε is sufficiently small. Give an expression, up to, but not including, $\mathcal{O}(\varepsilon^2)$, for the boundary between stable and unstable regions.

2. Consider the quasi-linear system

$$\begin{aligned} u_t + uu_x + 2vv_x &= 0 \\ 2v_t + vu_x + 2uv_x &= 0 \end{aligned}$$

for $x \in \mathbb{R}$ and $t > 0$.

(a) Show that these equations can be written in the form

$$\left(\frac{\partial}{\partial t} + \alpha_{\pm}\frac{\partial}{\partial x}\right)\beta_{\pm} = 0$$

Find $\alpha_{\pm}$ and $\beta_{\pm}$ and explain their meaning.

(b) Suppose $v(x,0) = 0$ and $u(x,0) = \frac{1}{1+x^2}$. Find the solution and discuss any special feature of the solution.

(c) Suppose instead of $u = u(x,t)$ and $v = v(x,t)$, we consider $x = x(u,v)$ and $t = t(u,v)$. Obtain the equations governing x and t and explain their significance. Find an equation for t alone and discuss any special properties.

3. Solve the heat equation in the infinite domain in two ways:

$$\begin{cases} \frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2}, \quad x \in \mathbb{R},\ t > 0 \\ u(x,t) \to 0 \text{ as } x \to \pm\infty,\ t > 0 \\ u(x,0) = \delta(x), \quad x \in \mathbb{R} \end{cases}$$

(where δ is the usual delta function).

- Use a Fourier transform in x
- Use a Laplace transform in t. You need to perform the inverse Laplace transform using contour integration (i.e. do not use tables of transforms).

4. Consider the integral equation

$$u(x) - \int_0^x (u(t))^2\, dt = x$$

- Prove that, for some positive number α, the integral equation has a unique solution in $C([0,\alpha])$.
- For the α determined previously, how smooth is the function u in $[0,\alpha]$?

5. Consider the boundary value problem

$$\frac{d^2u}{dx^2} + k^2 u = 0 \quad \text{on } (0,1), \quad u(0) = 1 \text{ and } \frac{du}{dx}(1) = iku(1)$$

- Write a second order finite difference scheme for this boundary value problem. (Denoting h the mesh size, the system of algebraic equations, $A_h u_h = f_h$, should involve a symmetric matrix).
- Prove the existence and uniqueness of a discrete solution.
- Investigate numerically whether the discrete solution u_h converges or not when $k = 75$. Write a simple Matlab script that solves the discrete problem. Do log-log plots for $||\,(A_h)^{-1}\,||_\infty$, $||A_h||_\infty$, and the maximum nodal error as a function of h. Use this information to discuss the stability of the method, the convergence, and the convergence rate.
- Assume that you use the Jacobi iterative method to solve the linear system. Under which condition does the Jacobi method converge? State the general convergence condition. Study numerically the convergence by doing a log-log plot of h vs. the quantity in the general convergence condition. Discuss when or whether this condition is satisfied. Your plot should exhibit a vertical asymptote. Explain why.

6. Consider the following linear multistep method:

$$y_{n+3} + (2b-3)(y_{n+2} - y_{n+1}) - y_n = hb(f_{n+2} + f_{n+1})$$

to approximate the ordinary differential equation

$$\frac{dy}{dx}(x) = f(x, y(x)) \quad y(0) = y_0$$

(where f is a smooth function).

(a) Determine all values of the real parameter b, $b \neq 0$, for which the method is zero-stable.

(b) Show that the truncation error,

$$T_n = \frac{y(x_{n+3}) + (2b-3)(y(x_{n+2}) - y(x_{n+1})) - y(x_n) - hb(f(x_{n+2}) + f(x_{n+1}))}{2hb},$$

(where y is a solution to the ordinary differential equation) is $O(h^2)$ when the method is zero-stable.

(c) Show that there exists a value of b for which the truncation error is $O(h^4)$.

Examination (September 2011)
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Day 3

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1. Analyze the long time behavior of $(x, y) \in [0,1] \times [0,1]$

$$\begin{aligned} \frac{dx}{d\tau} &= -\frac{\theta_1 x}{1-y} + \frac{(1-x-y)}{1-x}, \\ \frac{dy}{d\tau} &= \chi \left[\frac{\theta_2(1-x-y)}{1-y} - \frac{y}{1-x} \right], \end{aligned}$$

where θ_1, θ_2, and χ are positive. θ_1 and θ_2 are also less than 1.

2. In his book on symmetric eigenvalue problems, B. Partlett proves the result

> Let $\hat{\mathbf{A}} \in \mathbb{R}^{n \times n}$ be a symmetric matrix, $\hat{\mathbf{y}}$ a nonzero vector in $\mathbb{R}^n$, $\hat{\theta}$ a real number, and $\hat{\mathbf{r}}$ the residual vector
>
> $$\hat{\mathbf{r}} = \hat{\mathbf{A}}\hat{\mathbf{y}} - \hat{\mathbf{y}}\hat{\theta}.$$
>
> If $\hat{\alpha}$ is the eigenvalue of $\hat{\mathbf{A}}$ closest to $\hat{\theta}$, where $\hat{\mathbf{A}}\hat{\mathbf{z}} = \hat{\mathbf{z}}\hat{\alpha}$ and $\|\hat{\mathbf{z}}\| = 1$, then
>
> $$\left|\hat{\theta} - \hat{\alpha}\right| \le \frac{\|\hat{\mathbf{r}}\|}{\|\hat{\mathbf{y}}\|}, \quad |\sin \angle(\hat{\mathbf{y}}, \hat{\mathbf{z}})| \le \frac{1}{\min\limits_{\hat{\lambda}_i \neq \hat{\alpha}} \left|\hat{\lambda}_i - \hat{\theta}\right|} \frac{\|\hat{\mathbf{r}}\|}{\|\hat{\mathbf{y}}\|}$$
>
> where $\hat{\lambda}_i$ is an eigenvalue of $\hat{\mathbf{A}}$ and $\|\cdot\|$ is the Euclidian norm.

Derive similar estimates for the eigenvalue problem

$$\begin{bmatrix} \mathbf{K} & \mathbf{B} \\ \mathbf{B}^T & \mathbf{0} \end{bmatrix} \mathbf{x} = \lambda \begin{bmatrix} \mathbf{M} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{x}$$

where the matrices $\mathbf{K}$ and $\mathbf{M}$ belong to $\mathbb{R}^{n\times n}$ and are symmetric positive definite. The matrix $\mathbf{B}$ belongs to $\mathbb{R}^{n\times p}$ $(p < n)$ and is full-rank.

3. Let the function f such that

$$\frac{df}{dx}(x) = \frac{1}{(1+x^3)^{1/3}}$$

Give as much information about f as you can.

4. Consider the boundary value problem

$$\frac{d^2u}{dx^2} + k^2 u = f \quad \text{on } (0,1), \quad u(0) = 0 \text{ and } \frac{du}{dx}(1) = iku(1)$$

where $k \in \mathbb{R}$ and $f \in L^2(0,1)$.

- Prove the existence and uniqueness of a solution.
- Using the Green function to write

$$u(x) = \int_0^1 G(x,s)f(s)ds,$$

prove the bounds

$$\int_0^1 |u(x)|^2\, dx \leq \frac{1}{k^2}\int_0^1 |f(x)|^2\, dx$$

and

$$\int_0^1 \left|\frac{du}{dx}(x)\right|^2 dx \leq \int_0^1 |f(x)|^2\, dx$$

- Extend as much as possible these results to $\Omega = (0,1)^2$ and to $\Omega = (0,1)^3$