Qualifying Examination

Summer 2016

Examination Committee: Anne Greenbaum, Ulrich Hetmaniuk, Eric Shea-Brown

Day 1. Monday, September 12, 2016

You have three hours to complete this exam. Work all problems. Start each problem on a new page. **You must show all steps and prove all claims, unless you rely on a standard theorem or result, in which case you should refer to it. You may use a computer to check your work, but all derivations must be your own.** You are not allowed to access the internet or any application file existing on your computer. Any application file you use must be created from scratch.

1. Let

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 4 & 5 & 6 \\ 7 & 8 & 9 & 10 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

   (a) Find the orthogonal projection of $\mathbf{b}$ onto the span of $\{A\mathbf{b}, A^2\mathbf{b}, A^3\mathbf{b}\}$.

   (b) Find a 4 by 4 matrix M whose eigenvalues are all ones such that $\mathbf{b} \perp \operatorname{span}(\{M\mathbf{b}, M^2\mathbf{b}, M^3\mathbf{b}\})$. [Hint: You might want to use a companion matrix, which has the form

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ * & * & * & * \end{bmatrix},$$

   where the $*$'s can be nonzero.] Could such a matrix be diagonalizable? Explain why or why not.

2. Consider the predictor-corrector scheme

$$\hat{U}_{n+2} = U_{n+1} + hf(t_{n+1}, U_{n+1}),$$

$$U_{n+2} = U_n + hf(t_n, U_n) + hf(t_{n+2}, \hat{U}_{n+2}),$$

   for solving the ODE initial value problem $u' = f(t, u)$, $u(0) = u_0$, using time step size h. Assume that f is C^∞ in both variables.

   (a) Determine the order of the local truncation error of the method.

   (b) Suppose this scheme is applied to the test equation $u' = \lambda u$. Which, if any, of the values $h\lambda = -1, -2, -3$ lie in the *region of absolute stability* of the method (i.e., values of $h\lambda$ for which $U_n \to 0$ as $n \to \infty$, for all initial values U_0 and U_1). Justify your answer.

3. Determine if the following definite integral exists. If it does, find its value, and if it does not, explain why and find its principal value.

$$\int_{-\pi/4}^{\pi} \tan(x)\, dx.$$

4. Consider the vector-valued function $\mathbf{F}(x, y, z) = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$.

   (a) Can this be written as the gradient of a scalar-valued function $\psi(x, y, z)$? If so, find the function ψ; if not, show why not.

   (b) Compute the line integral

   $$\int_C \mathbf{F} \cdot d\mathbf{r},$$

   where C is the curve given parametrically by

   $$\mathbf{r}(t) = e^t \sin(t)\mathbf{i} + e^t\mathbf{j} + t\mathbf{k}, \quad 0 \leq t \leq \frac{\pi}{2}.$$

5. Consider a system of N ODE's in variables $\mathbf{x} = [x_1, \ldots, x_N]^T$:

   $$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}(t)), \tag{1}$$

   where $\mathbf{f}(x_1, \ldots, x_N)$ is a smooth function with continuous derivatives.

   (a) Assume that $\bar{\mathbf{x}}$ is an equilibrium for this system. Write down the linearization of (1) around this steady state, in terms of a (Jacobian) matrix. What are the conditions on this matrix for the equilibrium to be *asymptotically stable* – that is, so that initial conditions sufficiently near to the equilibrium will tend to it as $t \to \infty$?

   (b) Now assume that $\mathbf{f}(\mathbf{x}(t))$ has the form

   $$\mathbf{f}(\mathbf{x}(t)) = B\vec{\nabla}g(\mathbf{x}(t)),$$

   where $g(x_1, \ldots, x_N)$ is a scalar-valued function with continuous second derivatives and B is an $N \times N$ real skew-symmetric matrix. What can you say about the trace of the Jacobian matrix for this system? Can the conditions for this system to have an asymptotically stable equilibrium be met? Explain why or why not.

Qualifying Examination

Summer 2016

Examination Committee: Anne Greenbaum, Ulrich Hetmaniuk, Eric Shea-Brown

Day 2. Tuesday, September 13, 2016

You have three hours to complete this exam. Work all problems. Start each problem on a new page. **You must show all steps and prove all claims, unless you rely on a standard theorem or result, in which case you should refer to it. You may use a computer to check your work, but all derivations must be your own.** You are not allowed to access the internet or any application file existing on your computer. **For problem 4(b), however, use a builtin routine in Matlab, Mathematica, or Python.** Otherwise, any application file you use must be created from scratch.

1. Let $A \in \mathbb{R}^{m\times m}$ be a given matrix whose real singular value decomposition is $A = U\Sigma V^T$. Show that $Q = UV^T$ is a nearest orthogonal matrix to A in the Frobenius norm:
$$\arg \min_{Q^TQ=I} \|A - Q\|_F = UV^T.$$
Discuss circumstances in which the nearest orthogonal matrix is or is not unique. In particular, consider the case where the singular values of A are all positive and distinct.

2. Calculate the integral
$$\int_0^\infty \cos(x^n)\, dx,$$
where n is a natural number greater than 1. Be sure to show how you obtain your answer and to justify each step.

3. Find an asymptotic expansion for
$$C(x) = \int_0^x \cos(t^2)\, dt,$$
for large x, valid on the positive real line, up to but not including the $O(x^{-2})$ term. [You may use the fact (which should follow from your results in the previous problem) that $\int_0^\infty \cos(t^2)\, dt = \sqrt{2\pi}/4$.]

4. Consider a large number of particles that live on a circular state space $\theta \in [0, 2\pi)$. The position $\theta(t)$ of each particle at time t satisfies
$$\dot\theta = \omega + Iz(\theta) = v(\theta)$$
where z is a smooth 2π-periodic function, I and ω are real constants, and we've introduced the notation $v(\theta)$ for convenience. Assume that the solution to this equation from initial condition $\theta(0) = \theta_0$ is a known function $\phi(\theta_0, t)$.

   The density of particles $\rho(\theta, t)$ evolves via the advection equation
$$\frac{\partial \rho(\theta, t)}{\partial t} = -\frac{\partial}{\partial \theta}\left[v(\theta)\, \rho(\theta, t)\right] .$$
   We will consider the uniform initial condition $\rho(\theta, 0) = 1/(2\pi)$.

(a) Use the method of characteristics to solve for $\rho(\theta, t)$ in terms of the functions z and ϕ.

(b) Now make the specific choices $z(\theta) = \cos(\theta)$, $\omega = 2$, $I = 1$, $\theta_0 = 0$. Use a built-in numerical initial-value ODE method (from MATLAB or otherwise) to find $\phi(\theta_0, t)$ for $t \in [0, 10]$, and use this in your expression from the method of characteristics to plot the resulting solution.

(c) Does your solution appear to head to a steady-state? Should it? Give a few words of intuitive explanation, either way.

Qualifying Examination

Summer 2016

Examination Committee: Anne Greenbaum, Ulrich Hetmaniuk, Eric Shea-Brown

Day 3. Wednesday, September 14, 2016

You have 2 hours to complete this exam. **DO 1 OF THE FOLLOWING 2 PROBLEMS**. You must show all steps and prove all claims, unless you rely on a standard theorem or result, in which case you should refer to it. You are allowed to use a computer for any part of any problem, but you are not allowed to access the internet or any application file existing on your computer. Any application file you use must be created from scratch. We recommend you hand in all your work, even if it did not produce any results.

1. Recall from Day 2 that you studied the equation

$$\frac{\partial \rho(\theta,t)}{\partial t} = -\frac{\partial}{\partial \theta}\left[v(\theta)\,\rho(\theta,t)\right]\ , \quad \rho(\theta,0) = \frac{1}{2\pi},$$

where the velocity $v(\theta) = \dot{\theta}$ satisfied

$$\dot{\theta} = \omega + Iz(\theta), \quad \theta(0) = \theta_0.$$

You computed $\theta(t)$ when $z(\theta) = \cos(\theta)$, $\omega = 2$, $I = 1$, and $\theta_0 = 0$, using a built-in ODE solver, and you then used this to evaluate ρ.

(a) Check your result by numerically solving for $\rho(\theta,t)$ a second time, this time using a spectral method or finite differences, and including a variety of initial angles $\theta_0 \in [0, 2\pi)$.

(b) What is a relationship on I and ω that leads to $\rho(\theta,t)$ becoming UNbounded (as a function of θ) as $t \to \infty$.

(c) Add a diffusion term representing noise of strength D in the velocity:

$$\frac{\partial \rho(\theta,t)}{\partial t} = -\frac{\partial}{\partial \theta}\left[v(\theta,t)\rho(\theta,t)\right] + D\frac{\partial^2}{\partial \theta^2}\left[z^2(\theta,t)\rho(\theta,t)\right],$$

and numerically solve for $\rho(\theta,t)$ for two different values of I and ω. Does your solution head to a steady-state in these cases? Give a sentence or two of intuition.

2. Let $f \in C^\infty(\mathbb{R})$ and assume that f and its derivatives decay to 0 as rapidly as you like as $x \to \pm\infty$. Define the Fourier transform of f as

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,dx.$$

Show that

$$\sum_{k=-\infty}^{\infty} f(x + 2\pi k) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} \hat{f}(n)e^{inx}. \tag{2}$$

[Substituting $x = 0$ gives the *Poisson summation formula*: $\sum_{k=-\infty}^{\infty} f(2\pi k) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} \hat{f}(n)$.]

If $f = 0$ outside the interval $[-\pi, \pi]$, show that this implies that f can be reconstructed from equally spaced samples of its Fourier transform; i.e.,

$$f(x) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} \hat{f}(n)e^{inx}.$$

Finally, discuss how much smoothness and decay properties of f and its derivatives are actually needed in order for formula (2) to hold.