Qualifying Examination

Summer 2014

Examination Committee: Loyce Adams, Bernard Deconinck, Matt Lorig

Day 1. Monday, September 15, 2014

You have three hours to complete this exam. Work all problems. Start each problem on a new page. You must show all steps and prove all claims, unless you rely on a standard theorem or result, in which case you should refer to it. You are allowed to use a computer for any part of any problem, unless otherwise specified. You are not allowed to access the internet or any application file existing on your computer. Any application file you use must be created from scratch.

All problems are 20 points. Please email any electronic files in support of your exam to `lma3@uw.edu`, at the conclusion of the exam.

1. Consider the complex-valued expression

$$f(z) = z^{1/2},$$

where $z = x + iy$, with $x, y \in \mathbb{R}$. Derive explicit expressions for the real and imaginary part(s) of $f(z)$ in terms of x and y. If you make any choices (*e.g.*, for branch cuts), show how they impact your answer. Your answer should not contain any trig functions.

2. Let A and B be $n \times n$ Hermitian matrices. Further, assume that A is non-negative definite, while B is positive definite. Define

$$\lambda = \sup_{v \neq 0} \frac{v^* A v}{v^* B v}.$$

Show that λ is the largest generalized eigenvalue of the pair (A, B): λ is the largest scalar satisfying $Ax = \lambda Bx$, for some nonzero vector x.

3. Write down the solution of the following initial-value problem:

$$\begin{aligned} iu_t &= u_{xx}, && x \in \mathbb{R},\ t > 0, \\ u(x,0) &= f(x), && x \in \mathbb{R}, \\ \lim_{x \to -\infty} u(x,t) &= \alpha \neq 0, && t > 0, \\ \lim_{x \to \infty} u(x,t) &= \beta \neq 0, && t > 0. \end{aligned}$$

You may assume that the boundary and initial conditions are compatible.

4. Consider the equation

$$y'' + x^2 y' - y = 0.$$

Find the first four terms of an expansion of the appropriate type (asymptotic or convergent), for the solution satisfying $y(2) = 1, y'(2) = 1$.

5. The Matlab code **poisson.m** provided to you solves the problem

$$u_{xx} + u_{yy} = -5\pi^2 \sin(\pi x) \sin(2\pi y)$$

with $u(x, y) = 0$ on the boundary of the unit square $0 \le x \le 1$, $0 \le y \le 1$.

The mesh spacing h is taken the same in both coordinate directions and $h = 1/(m+1)$ where there are m interior unknowns in each row and each column. This code uses sparse storage to create a matrix problem $A^h u^h = F^h$ where the interior unknowns are by rows, bottom to top, and within each row, from left to right using the natural row-wise ordering. The code is set up to solve this problem using $m = 20$. It uses Matlab's backslash command to solve the linear system. The true solution to this PDE is known and used in the code to set the correct boundary conditions. The norm $||u^h - u_{pde}||_\infty$ measuring the max error in the discrete solution relative to the PDE at the nodal points is printed at the end.

Recall, the eigenvalues λ_{pq} of A^h are $(2/h^2)(\cos(p\pi h) + \cos(q\pi h) - 2)$ where $p = 1, 2, ..., m$, $q = 1, 2, ..., m$.

(a) Modify the **poisson.m** code to solve the Helmholtz problem

$$u_{xx} + u_{yy} + k^2 u = (k^2 - 5\pi^2)\sin(\pi x)\sin(2\pi y)$$

with $u(x, y) = 0$ on the boundary of the unit square $0 \leq x \leq 1$, $0 \leq y \leq 1$. You will need to know the exact solution to this new PDE to set the boundary conditions using the code's approach.

(b) Solve the new problem using your modified code which still uses backslash as the linear system solver. Verify it works for $k = 3$, $k = 10$, and $k = 60$. Do you think the backslash command did any pivoting for any of these problems?

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Day 2. Tuesday, September 16, 2014

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All problems are 20 points. Please email any electronic files in support of your exam to `lma3@uw.edu`, at the conclusion of the exam.

1. Consider the advection equation $u_t + au_x = 0$ with periodic boundary conditions.

   (a) Derive the Lax-Wendroff method below where $U_j^n \approx u(x_j, t_n)$ with $\Delta t = k$, and $\Delta x = h$:

   $$U_j^{n+1} = U_j^n - \frac{ak}{2h}(U_{j+1}^n - U_{j-1}^n) + \frac{a^2k^2}{2h^2}(U_{j-1}^n - 2U_j^n + U_{j+1}^n).$$

   (b) What is the order of the method?

   (c) Derive the stability condition.

   (d) If you program this method, would you expect to see dissipative behavior? Dispersive behavior? Justify your answer by finding a modified PDE on which the method is 3rd order accurate.

2. Consider the function

   $$f(x) = \left(\sin^2[\pi x/2]\right)^{1/\ln x}, \qquad 0 \le x < \infty,$$

   defined at $x = 0$ and $x = 1$ by continuity.

   (a) What is $f(1)$?

   (b) How many local minima and maxima does $f(x)$ have?

   (c) What is $\max_{x\in[0,\infty)} f(x)$?

3. Suppose we want to compute

   $$\int_0^\pi \frac{1}{\sin^{1/4}(x)}\, dx.$$

   A standard formula like the trapezoidal rule will break down since $f(x) = (\sin(x))^{-1/4}$ blows up at $x = 0$ and $x = \pi$. A method that avoids evaluating f at its singularities will often still need to evaluate f at many points to capture the singularity and compute a good approximation.

   Notice that the singularities $x = 0$ and $x = \pi$ are integrable, which is why the question makes sense in the first place. Break up the integral in a sum of integrals,

   $$\int_0^\pi \frac{1}{\sin^{1/4}(x)}\, dx = \int_0^\alpha \frac{1}{\sin^{1/4}(x)}\, dx + \int_\alpha^\pi \frac{1}{\sin^{1/4}(x)}\, dx$$

each one containing only one singularity of the integrand (as an endpoint). Here $\alpha \in (0, \pi)$ is a number you get to pick for your convenience.

Near the singularity $x = 0$ we can write $f(x) = x^{-1/4} g(x)$ where $g(x)$ is a smooth function. We use the fact that the singular part $x^{-1/4}$ can be integrated exactly to find accurate formulas that require evaluating g at only a few points, e.g.,

$$\int_0^\alpha x^{-1/4} g(x)\, dx \approx \sum_{j=1}^{n} w_j g(x_j),$$

where the x_j are, for example, equally spaced points in the interval. To find such a formula we must determine the weights w_j to use, which depends on the set of points x_j chosen. One way to do this is to require that the above integration formula is **exact** for the n functions $g(x) = 1,\ x,\ x^2,\ \ldots,\ x^{n-1}$. Note that for these choices of g the integral on the left-hand side can be computed exactly. Thus this results in a linear system of n equations to solve for the weights.

(a) With your choice of α, determine the weights w_1, w_2, w_3 using this approach for the case $n = 3$, using equally spaced points $x_1 = 0,\ x_2 = \alpha/2,\ x_3 = \alpha$. Requiring that the above equality holds for $g(x) = 1,\ x,\ x^2$ gives a linear system of 3 equations for the w's that can easily be solved however you want (by hand or by computer).

(b) Use these weights to estimate the integral $\int_0^\alpha (\sin(x))^{-1/4} dx$.

(c) Deal with $\int_\alpha^\pi (\sin(x))^{-1/4} dx$ however you want.

(d) Use their combination to estimate the original integral. It should agree to several digits with the exact value, which you can get very accurately using your favorite software.

4. Let A and B be $n \times n$ Hermitian nonnegative definite matrices. Define

$$\lambda = \sup_{v \notin \mathcal{N}(B)} \frac{v^* A v}{v^* B v},$$

where $\mathcal{N}(B)$ denotes the null space of B. What are the necessary and sufficient conditions so that λ is finite?

Note. This problem setting differs from Problem 2 on Day 1, by the fact that B is not positive definite here. As a consequence, the sup above is taken over a different space.

5. Evaluate

$$I(\alpha, \beta) = \int_{-\infty}^{\infty} \frac{\sin(\alpha x)}{\sinh(\beta x)} dx.$$

You may assume that α and β are positive and nonzero.

Note. We are willing to give you a good contour to use, at the cost of 10 points.

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Day 3. Wednesday, September 17, 2014

You have two hours to complete this exam. Work all problems. Start each problem on a new page. You must show all steps and prove all claims, unless you rely on a standard theorem or result, in which case you should refer to it. You are allowed to use a computer for any part of any problem, unless otherwise specified. You are not allowed to access the internet or any application file existing on your computer. Any application file you use must be created from scratch.

1. Find the leading behavior as $x \to \infty$ of
$$I(x) = \int_0^\infty \ln(1+t)e^{-x\sinh^2 t}dt.$$
Next, find the first correction to this leading behavior.

2. Consider the so-called minimal surface equation
$$(1+u_y^2)u_{xx} - 2u_x u_y u_{xy} + (1+u_x^2)u_{yy} = 0.$$
What can you say about this equation and any of its solutions?

3. A heavily used method for solving $Ax = b$ where A is sparse, nonsingular, and nonsymmetric is the Krylov-space method called GMRES. Starting with any initial guess x^1, the method at step k chooses $x^{k+1} = x^k + Qy$ where Q is an $n \times k$ matrix with orthonormal columns and y is a $k \times 1$ vector.
   (a) Derive the method by showing how Q and y are chosen.
   (b) Show how the Arnoldi process is used in the method.
   (c) Discuss the arithmetic complexity and convergence properties. How does the truncated GMRES compare to the untruncated algorithm in both complexity and convergence properties?
   (d) Program as much of the algorithm as you can by either using pseudo code or actual Matlab code. Email your code (or pseudo code, if done electronically) to `lma3@uw.edu` after you finish the written part of the exam.