Qualifying Examination

December 2012

Examination Committee: Bernard Deconinck, Anne Greenbaum, Hong Qian

Day 1. Monday, December 10, 2012

You have three hours to complete this exam. Work all problems. Start each problem on a new page. You must show all steps and prove all claims, unless you rely on a standard theorem or result, in which case you should refer to it. You are allowed to use a computer for any part of any problem, but you are not allowed to access the internet or any application file existing on your computer. Any application file you use must be created from scratch.

1. Consider the power series
$$\sum_{n=0}^{\infty} c_n x^n,$$
where
$$c_n = \sum_{k=0}^{n} \frac{1}{k!}.$$
What is its radius of convergence?

2. Consider the following initial boundary value problem for the wave equation
$$\begin{aligned} 4u_{tt} - u_{xx} &= 0, \quad x \in (0,1), t > 0 \\ u(x,0) &= \begin{cases} x^2 & x \in [0,1/2] \\ (x-1)^2 & x \in [1/2,1] \end{cases}, \\ u_t(x,0) &= 0 \\ u(0,t) &= 0 \\ u(1,t) &= 0 \end{aligned}$$
What is the value of $u(3/4, 2)$?

3. Derive a two-point integration formula to approximate
$$\int_{-1}^{1} (1+x^2) f(x) dx$$
that is exact when $f(x)$ is a polynomial of degree 3. Next, compute the error in this approximation when $f(x) = x^4$.

4. Consider the differential equation
$$\frac{dy}{dx} = \frac{1}{e^y - x},$$
with $y(0) = 1$. What is $y(1)$? For which values of x is $y(x)$ defined?

5. Let B and C be real $m \times n$ matrices. Relate the singular values and vectors of $B + iC$ to those of
$$\begin{pmatrix} B & -C \\ C & B \end{pmatrix}.$$

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Day 2. Tuesday, December 11, 2012

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1. Denote by $\lambda_j(M)$ the j-th eigenvalue of the real symmetric $N \times N$ matrix M: $\lambda_1(M) \leq \lambda_2(M) \leq \ldots \leq \lambda_N(M)$. In this list, eigenvalues are repeated according to their algebraic multiplicity. Let S and T be $N \times N$ real symmetric matrices. How are $\lambda_1(S+T)$, $\lambda_1(S) + \lambda_1(T)$, and $\lambda_1(S) + \lambda_N(T)$ related (*i.e.*, is one equal to another, is one less than another, or greater, *etc.*)

2. Consider the differential equation

$$(x + \epsilon y)y' + y = 1, \quad x \in [0, 1],$$

with $y(1) = 2$. Here $\epsilon > 0$ is a small parameter. Find an approximation for $y(x; \epsilon)$, valid for all x, as $\epsilon \to 0$.

3. **For this question: make sure to attempt all parts, even if you're stuck on an earlier part. Not all parts depend on all previous parts. Please make sure to indicate which part is answered where.**

   Consider the scheme

$$u_{n+1} = u_n + hf(t_n + (1-\theta)h, \theta u_n + (1-\theta)u_{n+1})$$

   for solving the ODE $u' = f(t, u)$. Here u_n and u_{n+1} are meant to approximate $u(t_n)$ and $u(t_{n+1}) = u(t_n + h)$, respectively.

   (a) For all $\theta \in [0, 1]$, find the order of this scheme.

   (b) Determine for which $\theta \in [0, 1]$ the scheme is convergent.

   (c) For $\theta_0 = 0$ and $\theta_1 = 1$ determine the stability domain of the scheme.

   (d) Consider the system

$$\frac{d}{dt}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} \epsilon^2 u_1(u_1^2 + u_2^2) \\ -u_2(u_1^2 + u_2^2) \end{pmatrix},$$

   where $\epsilon > 0$ is a parameter. Let $\epsilon = 0.5$. Find the equilibrium points of this system and determine their linear stability. Draw a phase portrait that is qualitatively consistent with your findings. This plot should be big and approximately to scale.

   (e) Choose a suitable value of θ and using the scheme above, produce a plot of the curve $(u_1(t), u_2(t))^T$ with initial conditions $(-1, -1)^T$ in the phase plane. If this initial condition is in the basin of attraction of one of the equilibrium points, run the scheme until you get "reasonably" close to that equilibrium point.

(f) Discuss how your numerical results from (e) agree or disagree with your linear stability results from (d).

4. Consider the three-dimensional wave equation

$$u_{tt} = u_{xx} + u_{yy} + u_{zz},$$

in the domain $x \in \mathbb{R}$, $y \in (0, \pi)$, $z \in \mathbb{R}$, $t > 0$, with initial conditions

$$u(x, y, z, 0) = e^{-(x^2+z^2)} \cos y, \quad u_t(x, y, z, 0) = 0.$$

The boundary conditions are

$$u_y(x, 0, z, t) = 0 = u_y(x, \pi, z, t).$$

Solve this initial-boundary value problem.

Remark. You may need one of the facts below.

- In polar coordinates (r, θ), the Laplacian is written as

$$\Delta u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}.$$

- In spherical coordinates (r, θ, ϕ), the Laplacian is written as

$$\Delta u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial u}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 u}{\partial \phi^2}.$$

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Day 3. Wednesday, December 12, 2012

You have 2 hours to complete this exam. Work all problems. Start each problem on a new page. You must show all steps and prove all claims, unless you rely on a standard theorem or result, in which case you should refer to it. You are allowed to use a computer for any part of any problem, but you are not allowed to access the internet or any application file existing on your computer. Any application file you use must be created from scratch. We recommend you hand in all your work, even if it did not produce any results.

1. Consider the function $f(z)$, $z \in \mathbb{C}$, defined and analytic everywhere in $\mathbb{C}$ except for isolated singularities. In a neighborhood of $z = 0$, it is given by

$$\sum_{n=0}^{\infty} c_n z^n,$$

where

$$c_n = \sum_{k=0}^{n} \frac{1}{k!}.$$

On Day 1, you found that the radius of convergence of this series is 1. What is the numerical value of $f(-1.1)$? Discuss your strategies and findings.

2. Denote by $\lambda_j(M)$ the j-th eigenvalue of the real symmetric $N \times N$ matrix M: $\lambda_1(M) \leq \lambda_2(M) \leq \ldots \leq \lambda_N(M)$. In this list, eigenvalues are repeated according to their algebraic multiplicity. Let S and T be $N \times N$ real symmetric matrices. How are $\lambda_1(S+T)$, $\lambda_1(S) + \lambda_1(T)$, and $\lambda_1(S) + \lambda_N(T)$ related (*i.e.*, is one equal to another, is one less than another, or greater, *etc.*)

3. Consider the differential equation

$$(x + \epsilon y)y' + y = 1, \quad x \in [0, 1],$$

with $y(1) = 2$. Here $\epsilon > 0$ is a small parameter. Find an approximation for $y(x; \epsilon)$, valid for all x, as $\epsilon \to 0$.