Qualifying Examination

Winter 2013

Examination Committee: Randy LeVeque, KK Tung, Chris Bretherton

Day 1. Monday, December 16, 2013

You have three hours to complete this exam. Work all problems. Start each problem on a new page. You must show all steps and prove all claims, unless you rely on a standard theorem or result, in which case you should refer to it. You are allowed to use a computer for any part of any problem, but you are not allowed to access the internet or any application file existing on your computer. Any application file you use must be created from scratch.

1. Let V be a normed linear space. Show that: if a Cauchy sequence contains a convergent subsequence then the entire sequence converges to the same limit.

2. Consider the equation

$$u_t = u_{xx} + \sin(x), \quad x \in \mathbb{R},\ t \geq 0,$$

$$u(x,0) = \begin{cases} 1 & \text{if } 0 < x < \pi, \\ 0 & \text{otherwise.} \end{cases}$$

Determine the function $\bar{u}(x) = \lim_{t\to\infty} u(x,t)$.

3. Consider the advection equation $u_t + au_x = 0$ and the "skewed" upwind method,

$$U_j^{n+1} = U_{j+1}^n - \left(\frac{a\Delta t}{\Delta x} + 1\right)(U_{j+2}^n - U_{j+1}^n).$$

   (a) Show that this method is first order accurate in space and time by computing the local truncation error or by showing that the error after 1 step is $\mathcal{O}(\Delta t^2 + \Delta x^2)$ when applied to a sufficiently smooth function.

   (b) For what values of $\Delta t/\Delta x$ does this method reduce to an "exact" solver, in the sense that if $U_j^n = u(x_j, t_n)$ for all j at time n then $U_j^{n+1} = u(x_j, t_{n+1})$ at the next time as well?

   (c) What restriction must be put on the time step Δt in terms of Δx in order for the CFL condition to be satisfied in the case $a < 0$? In the case $a > 0$?

   (d) Show that the method is in fact stable provided the CFL condition is satisfied, i.e., with the bounds found in part (c).

4. Consider the system

$$\begin{cases} x' &= f(x), \\ y' &= g(x,y), \end{cases}$$

where f and g are real-valued functions, differentiable in all their arguments. Define an oscillatory solution $(x(t), y(t))$ as a trajectory such that $x(t)$ and $y(t)$ are not both constant in t, and, for any integer N we have $x(t+NT) = x(t)$, $y(t+NT) = y(t)$. Here T is called the period of the oscillation.

Can a system of this form have an oscillatory solution? Answer yes or no, and give a simple proof or example.

5. A function $u(x,t)$ is a *similarity solution* to a PDE if it depends on the independent variables x and t only through $\xi = \xi(x,t)$, where $\xi(x,t)$ is invariant (self-similar) under scalings of x and t that leave the PDE invariant. Thus, $u(x,t) = U(\xi)$ with $\xi = \xi(x,t)$, as above.

   Consider the heat equation $u_t(x,t) = u_{xx}(x,t)$ for $t \geq 0$ on the entire real line with boundary conditions of the form

$$\begin{aligned} u(x,t) &\to U_\ell \quad \text{as } x \to -\infty, \\ u(x,t) &\to U_r \quad \text{as } x \to +\infty, \end{aligned}$$

   where U_ℓ and U_r are constants. Show that $\xi = x/\sqrt{4t}$ is a similarity variable and determine the similarity solution satisfying these boundary conditions. You may use the fact that $\int_{-\infty}^{\infty} e^{-z^2}\, dz = \sqrt{\pi}$.

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Day 2. Tuesday, December 17, 2013

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1. Recall that if V is a real Hilbert space and A is a bounded linear operator on V then

$$\|A\| = \sup_{\|u\|=\|v\|=1} (Au, v) = \sup_{u\neq 0, v\neq 0} \frac{(Au, v)}{\|u\| \, \|v\|},$$

where for $v \in V$, $\|v\| = (v, v)^{1/2}$. Now assume additionally that A is a compact, self-adjoint, and bounded linear operator on V such that

$$(Au, u) > 0 \quad \forall u \in V \backslash \{0\}.$$

(a) Prove that $\|A\| = \sup_{\|u\|=1}(Au, u) = \sup_{u\neq 0} \frac{(Au,u)}{\|u\|^2}$.

(b) Without assuming that V is finite dimensional, prove that there exists a nonzero vector $v \in V$ such that $Av = \|A\| \, v$; i.e., $\|A\|$ is the largest eigenvalue of A.

2. Suppose $A \in \mathbb{C}^{m\times m}$ has an SVD $A = U\Sigma V^*$. Find an eigenvalue decomposition of the form $B = X\Lambda X^{-1}$ for the $2m \times 2m$ matrix

$$B = \begin{bmatrix} 0 & A^* \\ A & 0 \end{bmatrix}.$$

where A^* is the conjugate-transpose matrix $A^* = \bar{A}^T$.

Check that the eigenvectors of B are mutually orthogonal, as expected since this matrix is hermitian.

3. Consider the heat equation

$$u_t(x,t) = u_{xx}(x,t)$$

on the domain $-\infty < x \le b(t)$ with a moving boundary $b(t)$ and the boundary conditions

$$\lim_{x\to-\infty} u(x,t) = 1, \qquad u(b(t),t) = 0$$

and suppose the boundary motion satisfies $b'(t) = -u_x(b(t),t)$ with $b(0) = 0$. (This models temperature in a semi-infinite body of water bounding by melting ice on the right.)

Determine the similarity solution to this problem, recalling from Day 1 that $\xi = x/\sqrt{4t}$ is the appropriate similarity variable.

4. Attempt all parts of this problem (you may use the result of one part in another part even if you don't manage to prove it). Let Δ be the Laplacian operater in two dimensions and Ω the unit square.

   (a) Show the *mean value property*: If $\Delta u = 0$ in $B_r(x_0)$, the disk of radius r centered at some point $x_0 \in \mathbb{R}^2$, then the value of u at the center is equal to the average value of u on the circle $\partial B_r(x_0)$.

   Hint: Let $\phi(r)$ denote the value on the circle of radius r and show that $\phi'(r) = 0$ and then take the limit $r \to 0$.

   (b) Prove the *maximum principle:* If $\Delta u = 0$ in Ω and $u = f$ on the boundary $\partial\Omega$, then the maximum value of u is attained on the boundary (and similarly for the minimum):

   $$\min_{\partial\Omega} f \leq u(x) \leq \max_{\partial\Omega} f$$

   for all $x \in \Omega$.

   (c) Let U_{ij} solve the discrete version of Laplace's equation on the unit square Ω, obtained by using the standard 5-point Laplacian for a finite difference discretization on a Cartesian grid. Show that U_{ij} satisfies an analogous maximum principle.

   (d) Now consider the Poisson problem $\Delta u = g$ in Ω with Dirchlet boundary conditions, where g is a given function of $x \in \mathbb{R}^2$. Suppose u_1 and u_2 are two solutions with different boundary values f_1 and f_2. Show that the solution depends continuously on the boundary data in the max-norm.

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Day 3. Wednesday, December 18, 2012

You have 2 hours to complete this exam. Work all problems. Start each problem on a new page. You must show all steps and prove all claims, unless you rely on a standard theorem or result, in which case you should refer to it. You are allowed to use a computer for any part of any problem, but you are not allowed to access the internet or any application file existing on your computer. Any application file you use must be created from scratch. We recommend you hand in all your work, even if it did not produce any results.

1. Estimate the value of

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left(-\sqrt{x^2+y^2}\right)\, y\, \tanh(y/\epsilon))\, dx\, dy$$

for $\epsilon = 10^{-3}$ and also give an estimate for the size of the error in your approximation.

2. Solve the following PDE with an oscillatory source:

$$\begin{cases} u_{tt} - c^2 u_{xx} = \delta(x)e^{-i\omega_0 t}, & x \in \mathbb{R},\ t > 0 \\ u(x,0) = u_t(x,0) = 0, & x \in \mathbb{R} \\ u(x,t) \to 0 \text{ as } x \to \pm\infty,\ t > 0 \end{cases}$$