Qualifying Examination

Summer 2010

Examination Committee: Bernard Deconinck, Ulrich Hetmaniuk, Randy Leveque, Hong Qian

Day 1. Tuesday, December 14, 2010

You have five hours to complete this exam. Work all problems. Start each problem on a new page. You must show all steps and prove all claims, unless you rely on a standard theorem or result, in which case you should refer to it. You are allowed to use a computer for any part of any problem, but you are not allowed to access the internet or any application file existing on your computer. Any application file you use must be created from scratch.

1. Is it possible for the determinant of a non-triangular matrix to be equal to the product of the diagonal entries? If not, why not. If so, give an example.

2. Consider the initial-value problem

$$\left\{ \begin{array}{rcl} u_t + xuu_x & = & 0, \\ u(x,0) & = & e^{-x^2}, x \in \mathbb{R}. \end{array} \right. .$$

   Solve this initial-value problem for $t \geq 0$ (An implicit solution is fine if no explicit one can be found). Draw the characteristics. Do shocks form? If so, when? You do not have to solve for the solution past the shock formation time if there is one.

3. Consider the one-step Adams-Moulton method (also known as the trapezoidal rule) to solve the scalar equation $y' = f(t, y)$:

$$y_{n+1} = y_n + \frac{h}{2} \left[f(t_{n+1}, y_{n+1}) + f(t_n, y_n) \right].$$

   - Is this an explicit or an implicit method? Why?
   - Show that this method is convergent, find its order, and sketch its region of absolute stability.

4. Consider the equation

$$xy'' - 2(x+1)y' + (x+2)y = xe^x.$$

   (a) Check that $y_1 = e^x$ is a solution of the homogeneous problem.
   (b) Find the general solution of the nonhomogeneous equation.
   (c) Assume initial conditions are given: $y(0) = 0$, $y'(0) = 0$. Can you solve the corresponding initial-value problem? Why or why not?

5. Verify that $y(t) = t^2/4$ solves the initial value problem

$$y' = \sqrt{y}, \quad y(0) = 0.$$

   Apply Euler's method to this problem and explain why the approximation obtained differs from the solution $t^2/4$.

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Day 2. Wednesday, December 15, 2010

You have five hours to complete this exam. Work all problems. Start each problem on a new page. You must show all steps and prove all claims, unless you rely on a standard theorem or result, in which case you should refer to it. You are allowed to use a computer for any part of any problem, but you are not allowed to access the internet or any application file existing on your computer. Any application file you use must be created from scratch.

1. Consider the advection equation $u_t(x,t) + au_x(x,t) = 0$ defined for all x where $a \neq 0$ is a constant.
   (a) What is the solution $u(x,t)$ for $t > 0$ if this equation is solved with initial data $u(x,0) = \eta(x)$?
   (b) Discretize in space with fixed mesh width Δx and time step Δt. Explain why the method
   $$U_j^{n+1} = U_j^n - \frac{a\Delta t}{2\Delta x}(U_{j+1}^n - U_{j-1}^n)$$
   will not converge to the true solution if we refine in space and time with any fixed ratio $\Delta t/\Delta x$.
   (c) Under what conditions will the method
   $$U_j^{n+1} = U_j^n - \frac{a\Delta t}{\Delta x}(U_{j+1}^n - U_j^n)$$
   converge as we refine the grid with fixed ratio $\Delta t/\Delta x$?

2. The Laplace transform of $y(x) = x^\alpha$ $(-1 < \alpha < 0)$ is $Y(s) = \Gamma(\alpha+1)/s^{\alpha+1}$, $\text{Re}(s) > 0$, as you can easily verify. Here $\Gamma(x)$ is the so-called Gamma function
   $$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt.$$
   Using the definition of the inverse Laplace transform, verify that $\mathcal{L}^{-1}[\Gamma(\alpha+1)/s^{\alpha+1}](x) = x^\alpha$ $(-1 < \alpha < 0)$. In other words, calculate
   $$\frac{1}{2\pi i}\lim_{T\to\infty}\int_{p-iT}^{p+iT}\Gamma(\alpha+1)s^{-\alpha-1}e^{sx}ds.$$
   Here p is a constant chosen so that the path of integration lies in the region of convergence for the Laplace transform. In other words, $p > 0$. Hint: you may want to use (near the end of your calculations) that $\Gamma(\alpha+1)\Gamma(-\alpha) = -\pi/\sin(\pi\alpha)$. You do not have to prove it to use it.

3. The period of the stationary solutions of a bizarre partial differential equation is proportional to
   $$L(\mu) = \int_0^1 \frac{\mu}{(F(\mu) - F(\mu z))^k}dz,$$
   where $F(x) = x^2/2 - x^3/3$, and $\mu \in [0,1]$. For what values of k is this integral defined? Justify your answer in full detail.

4. Consider the initial-value problem

$$\begin{cases} u''(t) + \frac{2}{t}u'(t) = -e^{u(t)} & \text{for } t \geq 0, \\ u(0) = 0 = u'(0), \end{cases}$$

   Prove, using the Fixed Point Theorem, that there exists a twice differentiable solution to this problem for $t \leq \delta$, for some sufficiently small δ. Hint: rewrite the problem as an integral equation, using $u'' + \frac{2}{t}u' = \frac{1}{t^2}(t^2 u')'$.

5. Consider the generalized Rayleigh oscillator:

$$\epsilon y'' - 2\sin(y') + \sin(y) = \sqrt{3}\cos(t)$$

   with the initial conditions $y(0) = \alpha$ and $y'(0) = \beta$ where α and β can be prescribed as desired. Analyze the dynamics of this equation for $\epsilon \ll 1$.

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Day 3. Thursday, December 16, 2010

You have 3 hours to complete this exam. Work all problems. Start each problem on a new page. You must show all steps and prove all claims, unless you rely on a standard theorem or result, in which case you should refer to it. You are allowed to use a computer for any part of any problem, but you are not allowed to access the internet or any application file existing on your computer. Any application file you use must be created from scratch. We recommend you hand in all your work, even if it did not produce any results.

1. Consider the advection equation $u_t(x,t) - au_x(x,t) = 0$ defined for $x \in [0, L]$ with periodic boundary conditions. Here $a > 0$ is a constant.

   (a) What is the solution $u(x,t)$ for $t > 0$ if this equation is solved with initial data $u(x,0) = \eta(x)$, where $\eta(x+L) = \eta(x)$?

   (b) Discretize in space with fixed mesh width Δx and time step Δt. Under what conditions will the explicit method

   $$U_j^{n+1} = U_j^n + \frac{a\Delta t}{\Delta x}(U_{j+1}^n - U_j^n)$$

   converge as we refine the grid with fixed ratio $\Delta t/\Delta x$?

   (c) Similarly, under what conditions will the implicit method

   $$U_j^{n+1} = U_j^n + \frac{a\Delta t}{\Delta x}(U_{j+1}^{n+1} - U_j^{n+1})$$

   converge as we refine the grid with fixed ratio $\Delta t/\Delta x$?

2. The Laplace transform of $y(x) = x^\alpha$ $(-1 < \alpha < 0)$ is $Y(s) = \Gamma(\alpha+1)/s^{\alpha+1}$, $\mathrm{Re}(s) > 0$, as you can easily verify. Here $\Gamma(x)$ is the so-called Gamma function

   $$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt.$$

   Using the definition of the inverse Laplace transform, verify that $\mathcal{L}^{-1}[\Gamma(\alpha+1)/s^{\alpha+1}](x) = x^\alpha$ $(-1 < \alpha < 0)$. In other words, calculate

   $$\frac{1}{2\pi i}\lim_{T\to\infty}\int_{p-iT}^{p+iT}\Gamma(\alpha+1)s^{-\alpha-1}e^{sx}ds.$$

   Here p is a constant chosen so that the path of integration lies in the region of convergence for the Laplace transform. In other words, $p > 0$. Hint: you may want to use (near the end of your calculations) that $\Gamma(\alpha+1)\Gamma(-\alpha) = -\pi/\sin(\pi\alpha)$. You do not have to prove it to use it.

3. Find the global minimum of the function

   $$\exp(\sin(50x)) + \sin(60\exp(y)) + \sin(70\sin(x)) + \sin(\sin(80y)) - \sin(10(x+y)) + \frac{1}{4}(x^2+y^2),$$

   with at least 10 digits of accuracy. This problem was one of Nick Trefethen's original hundred-dollar, hundred digit challenge problems.

4. The following system of partial differential equations is a special case of one recently proposed by Bona, Chen and Saut (J. Nonlinear Science, vol 12, 283-318, 2002) to describe small (but finite) amplitude waves in water.

$$\left\{ \begin{array}{rcl} \eta_t + w_x + (w\eta)_x - w_{xxx} - 2\eta_{xxt} & = & 0 \\ w_t + \eta_x + ww_x - 2\eta_{xxx} - 2w_{xxt} & = & 0 \end{array} \right. ,$$

where η represents a nondimensional elevation of the water surface, and w is related to the velocity potential evaluated at a certain height (in other words, w_x is a velocity).

What can you say about these equations?