Practice Qualifying Examination

Summer 2010

Examination Committee: Bernard Deconinck, Anne Greenbaum, Randy LeVeque

Day 1.

You have three hours to complete this exam. Work all problems. Start each problem on a new page. You must show all steps and prove all claims, unless you rely on a standard theorem or result, in which case you should refer to it. You are allowed to use a computer for any part of any problem, but you are not allowed to access the internet or any application file existing on your computer. Any application file you use must be created from scratch.

1. Consider the following nonlinear system:

$$\left\{ \begin{array}{rcl} x' & = & y + x^2 y \\ y' & = & -x + 2xy \end{array} \right. .$$

   Find the fixed points of this system. Determine their stability. Assuming that there are no limit cycles, give a sketch of the phase portrait of the nonlinear system (Put this sketch on a separate page, and make it BIG). If you encounter marginal cases for the fixed points, assume that the linearization gives you the correct result.

2. Define the inner product of two functions $f(x)$ and $g(x)$ defined on the interval $[0, 1]$ by

$$\langle f, g \rangle = \int_0^1 f(x) g(x)\, dx.$$

   We say f and g are orthogonal if $\langle f, g \rangle = 0$.

   Let $\mathcal{P}$ be the space of cubic polynomials $p(x)$ satisfying $p(1) = p'(1) = 0$. This is a two-dimensional linear function space. Determine an orthogonal basis for this space with respect to the inner product above. Note: Orthogonal is enough, it need not be orthonormal.

3. Consider the ordinary differential equation

$$\frac{dy}{dt} = -100y.$$

   Your unsophisticated friend is using the forward Euler method to solve this equation over the time interval $[0, 5]$. He observes a discrete approximation to the solution $y(t)$ whose absolute value increases over time. Is this possible? If so, characterize the time step that he is using. Now answer the same question using the backward Euler method.

4. Assume that $k : [0, 1] \times [0, 1] \to \mathbf{R}$ is continuously differentiable and that $f : [0, 1] \to \mathbf{R}$ is continuous. Using the definition of the derivative, find the derivative with respect to x of

$$F(x) = \int_0^x k(x, y) f(y)\, dy, \quad x \in (0, 1).$$

   As an example, let $k(x, y) = x^2 y$ and $f(x) = x$. Calculate $F'(x)$ two different ways: (i) using the formula you found, and (ii) by calculating $F(x)$ and taking a derivative.

5. Define the Weierstrass $\wp$-function as

$$\wp(z) = \frac{1}{z^2} + \sum_{j,k=-\infty}^{\infty} \left(\frac{1}{(z - j\omega_1 - k\omega_2)^2} - \frac{1}{(j\omega_1 + k\omega_2)^2} \right),$$

where $(j, k) = (0, 0)$ is excluded from the double sum. Also, you may assume that ω_1 is a positive real number, and that ω_2 is on the positive imaginary axis. All considerations below are meant for the entire complex plane, except the poles of $\wp(z)$.

(a) Where are these poles of $\wp(z)$?

(b) Show that $\wp(z + M\omega_1 + N\omega_2) = \wp(z)$, for any two integers M, N. In other words, $\wp(z)$ is a doubly-periodic function: it has two independent periods in the complex plane. Doubly periodic functions are called *elliptic* functions.

(c) Establish that $\wp(z)$ is an even function: $\wp(-z) = \wp(z)$.

(d) Find Laurent expansions for $\wp(z)$ and $\wp'(z)$ in a neighborhood of the origin in the form

$$\wp(z) = \frac{1}{z^2} + \frac{\alpha_{-1}}{z} + \alpha_0 + \alpha_1 z + \alpha_2 z^2 + \alpha_3 z^3 + \alpha_4 z^4 + \ldots,$$

and

$$\wp'(z) = -\frac{2}{z^3} + \frac{\beta_{-2}}{z^2} + \frac{\beta_{-1}}{z} + \beta_0 + \beta_1 z + \beta_2 z^2 + \beta_3 z^3 + \ldots.$$

Give expressions for the coefficients introduced above. Make sure to justify all steps.

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Day 2.

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1. Given that
$$\int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi},$$
find the value of
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+(y-x)^2+y^2)}dxdy.$$

2. Consider the partial differential equation
$$\phi_{tt}\phi_x^2 - 2\phi_{xt}\phi_x\phi_t + \phi_t^2\phi_{xx} = 0,$$
for $x \in \mathbb{R}$ and $t > 0$. This equation is a special case of the so-called Monge-Ampere equation. Show that in terms of the auxiliary function $u = \phi_t/\phi_x$ this equation may be reduced to a first-order equation. Assuming initial conditions $\phi(x,0) = 1 + 2e^{3x}$ and $\phi_t(x,0) = 4e^{3x}$, find $\phi(x,t)$.

3. Consider the boundary-value problem
$$-u_{xx} + 2u = f, \quad 0 \le x \le 1, \quad u(0) = 0, u(1) = 0.$$
Assume that $f \in C^\infty([0,1])$.
   (a) On a uniform grid with spacing $h = 1/n$, show that the following set of difference equations has *local truncation error* $O(h^2)$:
   $$\frac{2u_i - u_{i+1} - u_{i-1}}{h^2} + 2u_i = f(x_i), \quad i = 1, \dots, n-1.$$
   Here u_i is the approximate solution at node $x_i = ih$, and the local truncation error $\tau = (\tau_1, \dots, \tau_{n-1})^T$ is defined as the amount by which the true solution u fails to satisfy the difference equation at each node; i.e.,
   $$\tau_i = \frac{2u(x_i) - u(x_{i+1}) - u(x_{i-1})}{h^2} + 2u(x_i) - f(x_i), i = 1, \dots, n-1.$$
   (b) Use Gerschgorin's Theorem to determine upper and lower bounds on the eigenvalues of the coefficient matrix for this set of difference equations.
   (c) Show that the L_2-norm of the *global error* (the difference between the true solution and the approximate solution at the nodes) is of the same order as the local truncation error; i.e., $O(h^2)$.

4. Recall the Weierstrass $\wp$-function defined by

$$\wp(z) = \frac{1}{z^2} + \sum_{j,k=-\infty}^{\infty} \left(\frac{1}{(z - j\omega_1 - k\omega_2)^2} - \frac{1}{(j\omega_1 + k\omega_2)^2} \right),$$

where $(j,k) = (0,0)$ is excluded from the double sum. As before, you may assume that ω_1 is a positive real number, and that ω_2 is on the positive imaginary axis. All considerations below are meant for the entire complex plane, except the poles of $\wp(z)$. Yesterday, you proved that $\wp(z)$ (i) has poles at $z \in \{M\omega_1 + N\omega_2 | (N,M) \in \mathbb{Z}^2\}$, (ii) is a doubly-periodic function with periods ω_1 and ω_2, (iii) is even, (iv) has a Laurent expansion of the form

$$\wp(z) = \frac{1}{z^2} + \frac{g_2}{20} z^2 + \frac{g_3}{28} z^4 + \dots,$$

where

$$g_2 = 60 \sum_{(M,N) \neq (0,0)} \frac{1}{(M\omega_1 + N\omega_2)^4}, \quad g_3 = 140 \sum_{(M,N) \neq (0,0)} \frac{1}{(M\omega_1 + N\omega_2)^6}.$$

(a) State Liouville's theorem.

(b) Show that $\wp(z)$ satisfies the differential equation

$$(\wp')^2 = a\wp^3 + b\wp^2 + c\wp + d,$$

for suitable choices of a, b, c, d. Find these constants. You may need to invoke Liouville's theorem to obtain this final result. It turns out that the function $\wp(z)$ is determined by the coefficients c and d, implying that it is possible to recover ω_1 and ω_2 from the knowledge of c and d.

5. Suppose we want to estimate

$$\int_0^1 \frac{1}{\sqrt{\sin(x)}} \, dx$$

numerically. A standard formula like the trapezoidal rule will break down since $f(x) = (\sin(x))^{-1/2}$ blows up at $x = 0$. A method that avoids evaluating f at $x = 0$ will often still need to evaluate f at many points to capture the singularity and compute a good approximation.

However, we can write $f(x) = x^{-1/2} g(x)$ where $g(x)$ is a smooth function and use the fact that the singular part $x^{-1/2}$ can be integrated exactly to find accurate formulas that require evaluating g at only a few points, e.g.,

$$\int_0^1 x^{-1/2} g(x) \, dx \approx \sum_{j=1}^{n} w_j g(x_j) \tag{1}$$

where the x_j are, for example, equally spaced points in the interval. To find such a formula we must determine the weights w_j to use, which will depend on the set of points x_j chosen. One way to do this is to require that the formula (1) be **exact** for the n functions $g(x) = 1,\ x,\ x^2,\ \dots,\ x^{n-1}$. Note that for these choices of g the integral in (1) can be computed exactly, and so this will result in a linear system of n equations to solve for the weights.

(a) Determine the weights w_1, w_2, w_3 using this approach for the case $n = 3$, using equally space points $x_1 = 0,\ x_2 = 0.5,\ x_3 = 1$. Requiring that (1) hold for $g(x) = 1,\ x,\ x^2$ will give a linear system of 3 equations for the w's that can easily be solved by hand (or you may use Matlab, etc. to solve the system).

(b) Use these weights to estimate the original integral. It should agree to several digits with the exact value, which is approximately 2.0348053158...

(c) Write a program in Matlab or other language to do the same thing for arbitrary values of n (taking equally spaced points in $[0, 1]$ for the x_j values). In other words, your program should set up and solve a linear system of n equations to compute the weights w_j and then use these to approximate the original integral. Apply your program with $n = 10$ to approximate the original integral.

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Day 3.

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1. Consider the pendulum equation

$$\theta'' + \frac{g}{L}\sin\theta = 0,$$

where θ denotes the angular position of the pendulum away from the downward vertical position.

(a) Rewriting the equation in the phase plane with $x = \theta$ and $y = \theta'$, analyze the dynamics (equilibrium points, stability, phase plane plot).

(b) Find a conserved quantity for the pendulum equation, and rewrite it in terms of your phase plane variables. Call this quantity $H(x, y)$.

(c) Let $g = 1$, $L = 1$. Solve the pendulum equation numerically, using your (self-programmed) numerical solver of choice. Use initial conditions $\theta(0) = \pi/2$, $\theta'(0) = 0$. Provide the solution at $t = n$, $n \in \{0, 1, \ldots, 20\}$.

(d) At each of these times, give the value of $H(x, y)$. Is it conserved? Discuss your results.

2. Consider the equations

$$\sin\phi + (x^2 - y)\phi + \phi^3 + x + \cos y - 1 = 0, \quad \phi(0,0) = 0,$$

for a function $\phi(x, y)$. What can you say about the solution $\phi(x, y)$ of these equations?

3. Solve the heat equation with radiation (Robin) boundary conditions

$$\begin{cases} u_t = \sigma u_{xx} & x > 0, t > 0 \\ u(x,0) = u_0(x) & x > 0 \\ a u_x(0,t) + b u(0,t) = 0 & t \geq 0 \end{cases}.$$

Here σ, a and b are positive constants, and $u_0(x)$ is a prescribed function. If you make any assumptions on $u_0(x)$, state them explicitly.

4. Discuss the solutions of the ordinary differential equation

$$\frac{dx}{dt} = x^2 + t^2.$$