Qualifying Examination

Summer 2010

Examination Committee: Bernard Deconinck, Ulrich Hetmaniuk, Randy Leveque, Hong Qian

Day 1. September 20, 2010

You have three hours to complete this exam. Work all problems. Start each problem on a new page. You must show all steps and prove all claims, unless you rely on a standard theorem or result, in which case you should refer to it. You are allowed to use a computer for any part of any problem, but you are not allowed to access the internet or any application file existing on your computer. Any application file you use must be created from scratch.

1. Determine the Taylor series of $y(x) = e \ln \ln(e + x) - xe^{-x/e}$ around $x = 0$, up to and including terms of order x^3 (*i.e.*, your error is order x^4). You may verify your calculations using a computer, but you should show the methodology you use and any intermediate steps leading to your final answer.

2. Consider the initial-value problem

$$\left\{ \begin{array}{rcl} u_t + tuu_x & = & 0, \\ u(x,0) & = & e^{-x^2}, x \in \mathbb{R}. \end{array} \right. .$$

   Solve this initial-value problem for $t \geq 0$ (An implicit solution is fine if no explicit one can be found). Draw the characteristics. Do shocks form? If so, when? You do not have to solve for the solution past the shock formation time if there is one.

3. Consider the following nonlinear system:

$$\left\{ \begin{array}{rcl} x' & = & -y + x^2 y \\ y' & = & x - xy^2 \end{array} \right. .$$

   Analytically determine an expression for the graphs of the trajectories of this system. Using this expression, examine the behavior of these graphs and plot some indicative ones. Note that you do not have to determine expressions for $x(t)$ and $y(t)$.

4. Consider the two-step Adams-Bashforth method to solve the scalar equation $y' = f(t, y)$:

$$y_{n+2} = y_{n+1} + h \left[\frac{3}{2} f(t_{n+1}, y_{n+1}) - \frac{1}{2} f(t_n, y_n) \right].$$

   Show that this method is convergent, find its order, and sketch its region of absolute stability. In particular, determine where this region intersects the real and imaginary axes.

5. Consider the equation

$$x^2 y'' - x(x+4)y' + (2x+6)y = x^4 e^x.$$

   (a) Check that $y_1 = x^2$ is a solution of the homogeneous problem.
   (b) Find the general solution of the nonhomogeneous equation.
   (c) Assume initial conditions are given: $y(0) = 0$, $y'(0) = 0$. Can you solve the corresponding initial-value problem? Why or why not?

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Day 2. September 21, 2010

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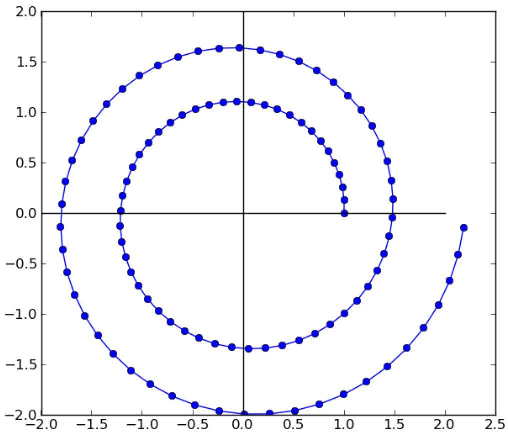
1. Suppose we wish to compute a set of points (x_j, y_j) that can be connected to form a circle of radius 1. We could simply set $x_j = \cos(\theta_j)$ and $y_j = \sin(\theta_j)$ for some points $\theta_j = j\Delta\theta$. But that would be too easy.

   Instead, suppose we decide to numerically solve the system of ODEs

$$\begin{aligned} x'(\theta) &= -y(\theta), \qquad x(0) = 1, \\ y'(\theta) &= x(\theta), \qquad y(0) = 0. \end{aligned} \tag{1}$$

   The exact solution, when plotted in the x-y plane, traces out the desired circle. If we use the Forward Euler method, however, we obtain a figure like this:



   This shows the points computed with $\Delta\theta = 2\pi/50$ for $j = 0, 1, 2, \ldots, 100$. The computed solution spirals outwards rather than tracing a circle.

   Let $u = \begin{bmatrix} x \\ y \end{bmatrix}$, so that the Forward Euler method can be written as $u_{j+1} = Cu_j$ for some matrix C.

   (a) Explain why $\|u_j\|_2$ increases with j based on the eigenvalues of C.

   (b) Using the eigenstructure of C, determine what x_{1000} and y_{1000} would be, with $\Delta\theta$ as above.

(c) If Backward Euler is used instead, the computed curve will spiral inward instead of outward. Explain this using eigenvalue analysis.

(d) How will the curve behave if the trapezoidal method is used?

(e) Produce plots on the computer analogous to the figure above for the Backward Euler and Trapezoidal methods by programming this in Matlab or another language.

2. Consider

$$F(z) = 1 + z + z^2 + z^3 + \ldots = \sum_{n=0}^{\infty} z^n.$$

Where is this function analytic?

Use the above representation to induce a Taylor representation of $F(z)$ centered at $z = -1/2$. Call this representation $G(z)$. Your final result should be of the form

$$G(z) = \sum_{m=0}^{\infty} c_m \left(z + \frac{1}{2}\right)^m.$$

Without any calculations, where should this series be convergent and why? Verify this answer by using the ratio test directly using the expressions you found for the coefficients c_m.

3. The period of the stationary solutions of a certain reaction-diffusion equation is proportional to

$$L(\mu) = \int_0^1 \frac{\mu}{\sqrt{F(\mu) - F(\mu z)}} dz,$$

where $F(x) = x^2/2 - x^3/3$.

(a) Using your favorite software, plot $L(\mu)$ for $0 \le \mu < 1$.

(b) Your figure should display the following properties:

i. $L(\mu)$ is an increasing function of μ, for $0 \le \mu < 1$.

ii. $L(\mu)$ is concave up, for $0 \le \mu < 1$.

iii. $\lim_{\mu \to 1^-} L(\mu) = \infty$.

iv. $\lim_{\mu \to 0^+} L(\mu) = \frac{\pi}{\sqrt{2}}$.

Prove these properties, justifying all steps.

4. Let X denote the set of all continuously differentiable functions on the interval $[0, 1]$. Define for X the norm

$$||u|| = |u(0)| + \max_{0 \le t \le 1} |u'(t)|.$$

(a) Prove that $|| \cdot ||$ is a norm on X.

(b) Prove that X equipped with this norm is complete. You may use that $C([0, 1])$ is complete with respect to the uniform norm (also known as the max norm or the infinity norm).

(c) Prove that X is not complete with respect to the uniform norm.

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Day 3. September 22, 2010

You have 2.5 hours to complete this exam. Work all problems. Start each problem on a new page. You must show all steps and prove all claims, unless you rely on a standard theorem or result, in which case you should refer to it. You are allowed to use a computer for any part of any problem, but you are not allowed to access the internet or any application file existing on your computer. Any application file you use must be created from scratch. We recommend you hand in all your work, even if it did not produce any results.

1. Prove the identity
$$\sum_{n=1}^{p-1} \operatorname{cosec}^2\left(\frac{n\pi}{p}\right) = \frac{p^2-1}{3},$$
for any integer $p \geq 2$.

2. Consider the equation
$$u(x) + u^2(x) + \int_0^x (1 + \cos(x + u(y)))dy = 0.$$
for $u(x)$. What can you say about $u(x)$?

3. Evaluate
$$\int_0^\infty \int_0^\infty \frac{x^2+y^2}{1+(x^2-y^2)^2} e^{-2xy} dxdy.$$

4. Consider the generalized Rayleigh oscillator:
$$\epsilon y'' - \delta \sin(y') + \sin(y) = \gamma \cos(t)$$
with the initial conditions $y(0) = \alpha$ and $y'(0) = \beta$ where α and β can be prescribed as desired. Analyze the behavior of this equation for various distinguished limits. In particular, what can you say about the solutions when $\epsilon \ll 1$ and when $\epsilon \sim O(1)$ with $\delta, \gamma \ll 1$ and/or $\delta, \gamma \sim O(1)$.