Qualifying Examination Winter 2017

Examination Committee: Anne Greenbaum, Hong Qian, Eric Shea-Brown

Day 1, Tuesday, December 12, 9:30-12:30, LEW 208

You have three hours to complete this exam. Work all problems. Start each problem on a new page. You must show all steps and prove all claims, unless you rely on a standard theorem or result, in which case you should refer to it. You may use a computer to check your work, but all derivations must be your own. You are not allowed to access the internet or any application file existing on your computer. Any application file you use must be created from scratch.

1. An undirected graph consists of N nodes and a set of edges that connect given nodes. The adjacency matrix for the graph, $\mathbf{A}$, has entries $A_{ij} = 1$ if there is an edge between nodes i and j and $A_{ij} = 0$ otherwise. Note that $A_{ij} = A_{ji}$, because we say if there is an edge between nodes i and j, then there is also an edge between nodes j and i. We do not allow self-connections, so that diagonal entries A_{ii} are always 0.

   The degree of a node is the number of edges that connect to it. A d-regular random graph is a graph of which every node has degree d, for some integer $d \leq N$, but the edges are otherwise in random positions. For the adjacency matrix A of a d-regular, undirected random graph with N nodes, answer the following questions.

   (a) Can $\mathbf{A}$ ever have negative eigenvalues? If yes, give an example, if not, explain why not.

   (b) Can $\mathbf{A}$ ever have imaginary eigenvalues? If yes, give an example, if not, explain why not.

   (c) Identify a vector $\mathbf{v}$ and value λ that is an eigenvector-eigenvalue pair for the adjacency matrix of every d-regular undirected random graph with N nodes?

   (d) Give a lower bound on the eigenvalues of the adjacency matrix for any d-regular undirected random graph.

2. Consider difference equations of the form

$$u_{n+2} + a_1 u_{n+1} + a_0 u_n = kbf(u_{n+1}),$$

   for the initial value problem $u'(t) = f(u(t))$, where $k = t_{n+1} - t_n$ is the timestep.

(a) Determine the coefficients a_0, a_1, and b that give the highest order local truncation error and say what that order is.

(b) Is the resulting method convergent? Explain why or why not.

(c) Determine which, if any, of the points $-1, \pm i, \pm\frac{1}{2}i$ lie in the region of absolute stability.

3. Let $f(z) = |z+1|^2$ and let γ be the unit circle oriented counterclockwise.

   (a) Show that f is *not* holomorphic on any domain that contains γ.

   (b) Find a function g that is holomorphic on some domain containing γ and such that $g(z) = f(z)$ on γ. [Hint: Write $f(z) = (z+1)(\bar{z}+1)$.]

   (c) Use g to evaluate $\int_\gamma |z+1|^2\, dz$.

4. Find the appropriate boundary value $y(0,\epsilon)$ such that the two-point problem

$$\begin{cases} \epsilon y''(x) + y'(x) + y(x) = 0 \\ y(0,\epsilon) \;\; \text{as determined}, \quad y(1) = 1, \end{cases}$$

has a smooth solution with an asymptotic power series expansion

$$y(x,\epsilon) \sim \sum_{j=0}^{\infty} a_j(x)\epsilon^j,$$

that is uniformly valid for $x \in [0,1]$.

2017 Winter Qual Day 2

Wednesday, December 13, 9:30-12:30, LEW 208

You have three hours to complete this exam. Work all problems. Start each problem on a new page. You must show all steps and prove all claims, unless you rely on a standard theorem or result, in which case you should refer to it. You may use a computer to check your work, but all derivations must be your own. You are not allowed to access the internet or any application file existing on your computer. Any application file you use must be created from scratch.

1. Let $\mathbf{A}$ be an n by n real symmetric matrix with eigenvalues $\lambda_1 \leq \ldots \leq \lambda_n$. Let $\mathbf{x}$ be any real n-vector with $\|\mathbf{x}\|_2 = 1$. Show that the Rayleigh quotient

$$r(\mathbf{x}) = \mathbf{x}^T \mathbf{A} \mathbf{x}$$

satisfies $\lambda_1 \leq r(\mathbf{x}) \leq \lambda_n$. Show that, by varying $\mathbf{x}$ while keeping $\|\mathbf{x}\|_2 = 1$, the Rayleigh quotient can take on every value in the interval $[\lambda_1, \lambda_n]$.

2. Consider a general description of a system of N nonlinearly coupled units, as given by

$$\frac{du_i}{dt} = -u_i + \sum_{j=1}^{N} w_{ij} g(u_j), \quad 1 \leq i \leq N. \tag{1}$$

Here, u_i is the *activity* of the i^{th} unit, and the matrix $\mathbf{W}$ gives the connection *weights* among these units; in particular, its element w_{ij} is the connection weight between unit j and unit i. Finally, $g(\cdot)$ is a smooth monotonically increasing function that describes how the strength of interaction between units depends on their activities.

A system is said to be ***bi-directional*** and ***full coupling*** if $w_{ij} = w_{ji}$, and $w_{ij} \neq 0$ for $i \neq j$. For such a system, answer the following questions.

(a) Is there a system of this form such with an equilibrium that is asymptotically stable: that is, any initial condition that begins near the equilibrium will tend "back" to the equilibrium as time tends to ∞? If yes, give an example, numerically or analytically or both; if no, explain why not.

(b) Consider the "energy function"

$$H(\mathbf{u}) = -\frac{1}{2} \sum_{ij} w_{ij} V_i(\mathbf{u}) V_j(\mathbf{u}) + \sum_i \int_0^{V_i} g^{-1}(V) dV$$

where $\mathbf{u} = (u_1, ..., u_N)^T$, $V_i = g(u_i)$. Compute the time evolution of the energy, $\frac{\mathrm{d}}{\mathrm{d}t} H(\mathbf{u}(t))$, along solutions $\mathbf{u}(t)$ to the ODE above. Show that $\frac{\mathrm{d}}{\mathrm{d}t} H(\mathbf{u}(t)) < 0$ in the case where you have "bi-directional" and "full coupling". In this case, is there a system of this form that has a nonconstant periodic solution? If so, give an example; if not, state why not.

3. Consider linear partial differential equation with boundary values:

$$\frac{\partial u(x,t)}{\partial t} = \epsilon \left(\frac{\partial^2 u}{\partial x^2} \right) + b(x) \frac{\partial u(x,t)}{\partial x} + c(x) u(x,t); \quad c(x) < 0. \tag{2a}$$

$$x \in [-1,1], \quad u(-1,t) = u_{-1}, \ u(1,t) = u_1, \tag{2b}$$

in which $b(x), c(x)$ are smooth functions of $x \in [-1,1]$.

(a) Show that if twice differentiable function $u(x,0) > 0$, then $u(x,t) \geq 0$ for $t \geq 0$. Similarly, if $u(x,0) < 0$, then $u(x,t) < 0$ for $t \geq 0$.

(b) Now consider the stationary solution to (2a):

$$\epsilon \left(\frac{\partial^2 u}{\partial x^2} \right) + b(x) \frac{\partial u(x)}{\partial x} + c(x) u(x) = 0 \tag{3}$$

with the boundary conditions in (2b). Because of the ϵ, in addition to the posibility of boundary layers at $x = \pm 1$, there could also be internal layers. Where should an internal layer be located if it has one? Why?

4. Consider the linear ODE system $\mathbf{u}'(t) = \mathbf{A}\mathbf{u}(t)$, with an arbitrary initial vector $\mathbf{u}(0)$.

(a) Show that for a general real, constant matrix $\mathbf{A}$, the necessary and sufficient condition for $\|\mathbf{u}(t)\|_2$ to decrease monotonically (for initial vector $\mathbf{u}(0)$) is that the eigenvalues of $\mathbf{A} + \mathbf{A}^T$ be negative.

(b) Give an example to show that in general it is *not* sufficient to have the eigenvalues of $\mathbf{A}$ in the left half-plane; that is, write down a matrix $\mathbf{A}$ whose eigenvalues are all in the left half-plane but for which the 2-norm of the solution to $\mathbf{u}' = \mathbf{A}\mathbf{u}$ with some initial vector $\mathbf{u}(0)$ does *not* decay monotonically.

2017 Winter Qual Day 3

Thursday, December 14, LEW 208

You have 2 hours to complete this exam. DO 1 OF THE FOLLOWING 2 PROBLEMS. You must show all steps and prove all claims, unless you rely on a standard theorem or result, in which case you should refer to it. You are allowed to use a computer for any part of any problem, but you are not allowed to access the internet or any application file existing on your computer. Any application file you use must be created from scratch. We recommend you hand in all your work, even if it did not produce any results.

1. Let $\mathbf{A} = \{a_{ij}\}$ be a real n by n matrix.

   (a) Suppose
   $$\|\mathbf{A}\|_F \leq \|\mathbf{A} + t\mathbf{I}\|_F \quad \forall t \in \mathbb{R},$$
   where $\|\cdot\|_F$ denotes the Frobenius norm: $\|\mathbf{A}\|_F = \left(\sum_{i,j=1}^n a_{ij}^2\right)^{1/2}$. Show that the trace of $\mathbf{A}$ is 0.

   If you don't see how to do this, write a code to test the result numerically. Perhaps you can get some insight from your numerical experiments.

   (b) Suppose
   $$\|\mathbf{A}\|_2 \leq \|\mathbf{A} + t\mathbf{I}\|_2 \quad \forall t \in \mathbb{R},$$
   where $\|\cdot\|_2$ is the spectral norm; i.e., the largest singular value. Show that the left and right singular vectors of $\mathbf{A}$ corresponding to the largest singular value are orthogonal to each other.

   If you don't see how to do this, write a code to test the result numerically. Perhaps you can get some insight from your numerical experiments.

2. Consider singularly perturbed ordinary differential equation
   $$\epsilon\left(\frac{\partial^2 u}{\partial x^2}\right) + bx\frac{\partial u(x)}{\partial x} + cu(x) = 0,$$
   where b, c are two constants, with two-point boundary conditions
   $$u(0) = u_0, \; u(1) = u_1.$$
   Please describe the asymptotic behavior of the solution $u(x, \epsilon)$ as $\epsilon \to 0$.