Qualifying Examination Summer 2017

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Day 1, Monday, September 11, 9:30-12:30, LEW 208

You have three hours to complete this exam. Work all problems. Start each problem on a new page. You must show all steps and prove all claims, unless you rely on a standard theorem or result, in which case you should refer to it. You may use a computer to check your work, but all derivations must be your own. You are not allowed to access the internet or any application file existing on your computer. Any application file you use must be created from scratch.

1. Write down two different functions of a real variable x that have the same asymptotic series in terms of the powers $\{x^{-n}\}$, $n = 0, 1, \cdots$, as $x \to \infty$.

2. Let $A \in \mathbb{C}^{n \times n}$ be an n by n matrix and let $v \in \mathbb{C}^n$ be a nonzero n-vector.

   (a) Show that if $v \in \text{span}\{Av, A^2v\}$, then $v \in \text{span}\{A^2v, A^3v\}$.

   (b) Give an example to show that the converse is *not* necessarily true; that is $v \in \text{span}\{A^2v, A^3v\}$ does *not* necessarily imply that $v \in \text{span}\{Av, A^2v\}$.

3. Consider a $2N$ dimensional ODE in variables $x_1(t), ..., x_{2N}(t)$ defined via a function $H(x_1, ..., x_{2N})$ as follows:

$$\dot{x}_1 = \frac{\partial H}{\partial x_{N+1}}, \quad \dot{x}_2 = \frac{\partial H}{\partial x_{N+2}}, \quad \ldots, \quad \dot{x}_N = \frac{\partial H}{\partial x_{2N}},$$

$$\dot{x}_{N+1} = -\frac{\partial H}{\partial x_1}, \quad \dot{x}_{N+2} = -\frac{\partial H}{\partial x_2}, \quad \ldots, \quad \dot{x}_{2N} = -\frac{\partial H}{\partial x_N}.$$

   (a) Is there a system of this form with an equilibrium that is asymptotically stable: that is, any initial condition that begins near the equilibrium will tend "back" to the equilibrium as time tends to ∞? If yes, give an example; if no, explain why not.

   (b) Is there a system of this form that has a non-constant oscillatory solution? If yes, give an example; if no, explain why not.

   (c) Now consider a system defined via:

$$\dot{x}_1 = -\frac{\partial H}{\partial x_1}, \quad \dot{x}_2 = -\frac{\partial H}{\partial x_2}, \quad \ldots, \quad \dot{x}_N = -\frac{\partial H}{\partial x_N},$$

$$\dot{x}_{N+1} = -\frac{\partial H}{\partial x_{N+1}}, \quad \dot{x}_{N+2} = -\frac{\partial H}{\partial x_{N+2}}, \quad \ldots, \quad \dot{x}_{2N} = -\frac{\partial H}{\partial x_{2N}}.$$

Answer the same questions (a) and (b) above.

4. Consider the two-point boundary-value problem

$$u''(x) = f(x), \quad 0 < x < 1; \quad u(0) = u(1) = 0.$$

Assume that f is as smooth as you like.

(a) Show that

$$u(x) = \int_0^1 G(x, \xi) f(\xi) \, d\xi,$$

where

$$G(x, \xi) := \begin{cases} \xi(x-1), & 0 \le \xi \le x \le 1, \\ x(\xi - 1), & 0 \le x \le \xi \le 1. \end{cases}$$

(b) Replacing $f(x)$ by $f(x) + \Delta f(x)$, where $|\Delta f(x)| \le \epsilon$ for all x changes the solution $u(x)$ to $u(x) + \Delta u(x)$. Prove that

$$|\Delta u(x)| \le \frac{\epsilon}{2} x(1-x), \quad 0 < x < 1.$$

5. Consider the advection equation on an infinite domain

$$u_t + a u_x = 0, \quad -\infty < x < \infty,$$

and finite difference schemes of the form

$$u_j^{n+1} = \alpha u_{j-1}^n + \beta u_{j+1}^n,$$

where u_j^n is the approximation to $u(x_j, t_n)$ and $x_j = jh$, $j = 0, \pm 1, \ldots$, and $t_n = nk$, $n = 0, 1, \ldots$, where u_j^0 is given.

(a) For what values of α and β is the scheme *consistent* with the differential equation, assuming that k/h and h/k remain bounded as $k, h \to 0$?

(b) Use von Neumann analysis or a method of your choice to show that the method is stable if $|\alpha| + |\beta| \le 1$.

2017 Qual Day 2, Tuesday, September 12, 9:30-12:30, LEW 208

You have three hours to complete this exam. Work all problems. Start each problem on a new page. You must show all steps and prove all claims, unless you rely on a standard theorem or result, in which case you should refer to it. You may use a computer to check your work, but all derivations must be your own. You are not allowed to access the internet or any application file existing on your computer. Any application file you use must be created from scratch.

1. Let f be an entire function and let γ be the unit circle, parameterized counterclockwise. Prove that for any $a \in \mathbb{C}$, $|a| \neq 1$,
$$\frac{1}{2\pi i}\int_\gamma \frac{\overline{f(z)}}{z-a}\,dz = \begin{cases} \overline{f(0)}, & \text{if } |a|<1, \\ \overline{f(0)} - \overline{f(1/\bar{a})}, & \text{if } |a|>1. \end{cases}$$
Use the following steps.
   (a) First show that $\overline{f(z)}$ is *not* a holomorphic function of $z \in \mathbb{C}$ (unless f is a constant function) but that $f^*(z) := \overline{f(\bar{z})}$ is.
   (b) Now note that $\bar{z} = 1/z$ on the unit circle and make the change of variable $w = 1/z$ in the above integral and evaluate.

2. Write down the solution of the Robin problem to the wave equation on the half line:
$$\begin{aligned} u_{tt} - c^2 u_{xx} &= 0, & x>0, t>0, \\ u(x,0) &= u_0(x), & x>0, \\ u_t(x,0) &= v_0(x), & x>0, \\ u_x(0,t) + \gamma u(0,t) &= h(t), & t>0, \end{aligned}$$
where γ is a real constant, $u_0(x)$, $v_0(x)$, and $h(t)$ are given functions, as differentiable as you need them to be.

3. Consider the n-th order linear ODE
$$y^{(n)}(x) + p_{n-1}(x)y^{(n-1)}(x) + \cdots + p_1(x)y'(x) + p_0(x)y(x) = 0.$$
Give the transformation $y(x) \to u(x)$ that results in a new ODE, for $u(x)$, that is without the second term:
$$u^{(n)}(x) + q_{n-2}(x)u^{(n-2)}(x) + \cdots + q_1(x)u'(x) + q_0(x)u(x) = 0.$$

4. On Day 1, you should have found that the finite difference scheme
$$u_j^{n+1} = \frac{1}{2}(u_{j+1}^n + u_{j-1}^n) - \frac{ak}{2h}(u_{j+1}^n - u_{j-1}^n)$$
is a consistent and stable method for the advection equation
$$u_t + au_x = 0, \quad -\infty < x < \infty, \quad t>0,$$
provided k/h and h/k remain bounded as $k, h \to 0$.

(a) Show that this difference scheme is a second-order in space and first-order in time approximation for the advection-diffusion equation

$$u_t + au_x = \epsilon u_{xx}, \quad \epsilon = \frac{h^2}{2k}. \tag{1}$$

(b) Instead of an infinite domain, assume a finite domain $-1 \le x \le 1$ with periodic boundary conditions $u(-1,t) = u(1,t)$. Show that the above difference scheme is the forward Euler method applied to the system of ODEs resulting from discretizing equation (1) in space. Write out the system of ODEs in the form $U'(t) = A_\epsilon U(t)$ and show that the eigenvalues of A_ϵ lie in the left half-plane. [Hint: It may help to recall that all n by n circulant matrices have the same eigenvectors: $v_j = (1, \omega_j, \omega_j^2, \ldots, \omega_j^{n-1})^T$, $j = 0, 1, \ldots, n-1$, where $\omega_j = \exp(2\pi i j/n)$.]

2017 Qual Day 3, Wednesday, September 13, LEW 208

You have 2 hours to complete this exam. DO 1 OF THE FOLLOWING 3 PROBLEMS. You must show all steps and prove all claims, unless you rely on a standard theorem or result, in which case you should refer to it. You are allowed to use a computer for any part of any problem, but you are not allowed to access the internet or any application file existing on your computer. Any application file you use must be created from scratch. We recommend you hand in all your work, even if it did not produce any results.

1. Consider the non-autonomous, nonlinear ODE

$$\frac{\mathrm{d}x}{\mathrm{d}t} = (\epsilon t - 1) + x^2,$$

with $\epsilon > 0$, $t \geq t_0$, and initial condition $x(t_0) = x_0$, $t_0 \leq 0$. Use various methods – e.g., numerical, exact solutions, perturbation, etc. – to piece together a coherent description, e.g., describing features, of the solution to the problem.

Note that when ϵ is very small, the problem has a "fast" and a "slow" dynamics. Intuitively, if one treats ϵt as a slowly varying *parameter*, this problem is essentially a saddle-node *bifurcation problem*. How can you formalize this intuition mathematically?

Note also that the equation has a Riccati form:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x^2 + a_1(t)x + a_0(t),$$

with $a_1(t) = 0$ and $a_0(t) = \epsilon t - 1$. According to the standard method, this nonlinear first-order equation can be transformed into a linear second-order equation via $x = \frac{1}{u}\left(\frac{\mathrm{d}u}{\mathrm{d}t}\right)$. The solution to the latter equation can be expressed in terms of Airy functions:

$$\begin{aligned}
\mathrm{Ai}(z) &= \frac{1}{\pi}\int_0^\infty \cos\left(\frac{s^3}{3} + sz\right)\mathrm{d}s,\\
\mathrm{Bi}(z) &= \frac{1}{\pi}\int_0^\infty \left[\exp\left(-\frac{s^3}{3} + sz\right) + \sin\left(\frac{s^3}{3} + sz\right)\right]\mathrm{d}s,
\end{aligned}$$

which are the two linearly independent solutions to the ODE $\frac{\mathrm{d}^2 u}{\mathrm{d}z^2} = zu$.

2. The linear PDE with constant coefficients,

$$u_t = u_{xx} + \alpha u_x + \beta u, \quad u(0,x) = u_0(x), \tag{2}$$

can be solved exactly in terms of Fourier analysis, either on the entire real line $x \in \mathbb{R}$ or on an interval $x \in [a,b]$.

(a) Try to develop a transformation which will convert a subclass of the following linear PDE with non-constant diffusion coefficient,

$$u_t = g^2(x)u_{xx}, \quad u(0,x) = f(x),$$

into the form in (2). Give the condition on $g(x)$ for the subclass you have considered. You may assume that the functions $f(x)$ and $g(x)$ are as differentiable as you need them to be.

(b) Give an example of your result.

3. Let A be an n by n matrix and write its Jordan decomposition as

$$A = S \begin{bmatrix} B & 0 \\ 0 & N \end{bmatrix} S^{-1},$$

where S is a nonsingular n by n matrix, B is a nonsingular m by m matrix which is the direct sum of all the nonsingular Jordan blocks, $0 \le m \le n$, and N is a nilpotent $n-m$ by $n-m$ matrix which is the direct sum of all singular Jordan blocks; i.e., $N^\alpha = 0$ for some positive integer α. The smallest integer α for which $N^\alpha = 0$ is called the *index* of A. If A is nonsingular then its index is 0; if it is diagonalizable with one or more eigenvalues equal to 0, then its index is 1; if it has Jordan blocks with eigenvalue 0, then its index is the size of the largest such Jordan block. The *Drazin inverse* A^D of A is defined as

$$A^D = S \begin{bmatrix} B^{-1} & 0 \\ 0 & 0 \end{bmatrix} S^{-1}.$$

Note that $A^D = A^{-1}$ if A is nonsingular.

(a) Show that $AA^D = A^D A$, $A^{\alpha+1}A^D = A^\alpha$, and $A^D A A^D = A^D$.

(b) Show that A^D can be written as a polynomial in A.

(c) Consider a linear system $Ax = b$ that might or might not have a solution x. Use some of the above properties to derive an algorithm to compute the Drazin inverse "solution" $A^D b$. Your algorithm can be direct or iterative. If direct, discuss accuracy and stability issues; if iterative, say something about its rate of convergence. If you have time, program your algorithm and apply it to some inconsistent linear system $Ax = b$ and report on the results.