Qualifying Examination

Summer 2012

Examination Committee: Bernard Deconinck, Anne Greenbaum, Hong Qian

Day 1. Monday, September 17, 2012

You have three hours to complete this exam. Work all problems. Start each problem on a new page. You must show all steps and prove all claims, unless you rely on a standard theorem or result, in which case you should refer to it. You are allowed to use a computer for any part of any problem, but you are not allowed to access the internet or any application file existing on your computer. Any application file you use must be created from scratch.

1. Let $\epsilon = 0.001$. Find all three roots of $x^3 + \epsilon x^2 - x + \epsilon = 0$, to within at least 0.00001. No preprogrammed routines allowed. Show your work.

2. Consider the following initial boundary value problem for the wave equation

$$\begin{aligned} u_{tt} - u_{xx} &= 0, \quad x \in (0,1), t > 0 \\ u(x,0) &= \begin{cases} 0 & x \in [0,1/4] \text{ or } x \in [3/4,1] \\ x - 1/4 & x \in [1/4,1/2] \\ 3/4 - x & x \in [1/2,3/4] \end{cases}, \\ u_t(x,0) &= 0 \\ u_x(0,t) &= 0 \\ u_x(1,t) &= 0 \end{aligned}$$

What is the value of $u(3/8, 1)$?

3. Consider the following difference scheme:

$$u(x,t+k) - u(x,t) = \frac{\sigma k}{h^2}\left(u(x+h,t) - u(x,t) - u(x,t+k) + u(x-h,t+k)\right).$$

   (a) In the limit $h \to 0$, $k \to 0$, what equation is this scheme consistent with? Are there any conditions you need to impose on h and k for this to be true?

   (b) Discuss the stability of this scheme.

4. Prove that the Fresnel integrals

$$F_1 = \int_0^\infty \cos(x^2)dx, \quad F_2 = \int_0^\infty \sin(x^2)dx$$

are well-defined (*i.e.*, convergent) improper integrals.

5. Let X denote a Hilbert space. Let T denote a compact operator on X with $||T|| < 1/2$. Define $A_0 = I + T$, where I denotes the identity operator. For $n \geq 1$, define

$$A_n = \frac{1}{2}\left(A_{n-1} + A_{n-1}^{-1}\right).$$

Prove that the sequence $(A_n)_{n=1}^\infty$ converges.

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Day 2. Tuesday, September 18, 2012

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1. Consider the wave equation
$$u_{tt} - c^2 \Delta u = 0,$$
with initial conditions $u(x, 0) = 0$, $u_t(x, 0) = \delta(x)$, for $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, $t > 0$. Further, $\Delta = \partial_{x_1}^2 + \partial_{x_2}^2 + \partial_{x_3}^2$ and $c > 0$. Solve this problem and describe its solution.

2. Consider the initial-value problem
$$\left\{ \begin{array}{l} y''' + \alpha y' + \sin(y) = \sin(t), \quad t > 0, \\ y(0) = 1, \; y'(0) = 0, \; y''(0) = \alpha, \end{array} \right.$$
where $\alpha \gg 1$ is a parameter. Tell us about $y(t)$ (please).

3. **For this question: make sure to attempt all parts, even if you're stuck on an earlier part. Not all parts depend on all previous parts. Please make sure to indicate which part is answered where.**

   Consider the scheme
$$u_{n+1} = u_n + hf(t_n + (1-\theta)h, \theta u_n + (1-\theta)u_{n+1})$$
for solving the ODE $u' = f(t, u)$. Here u_n and u_{n+1} are meant to approximate $u(t_n)$ and $u(t_{n+1}) = u(t_n + h)$, respectively.

   (a) For all $\theta \in [0, 1]$, find the order of this scheme.

   (b) Determine for which $\theta \in [0, 1]$ the scheme is convergent.

   (c) For $\theta_0 = 0$ and $\theta_1 = 1$ determine the stability domain of the scheme.

   (d) Consider the system
$$\frac{d}{dt}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} -a & 0 \\ 0 & -1/a \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \frac{\pi}{a}\begin{pmatrix} u_1 u_2 \\ u_2^2 - u_1 \end{pmatrix},$$
where a is a positive parameter. Find its equilibrium points and determine their linear stability. Are your results qualitatively the same for all values of $a > 0$? For $a = 10$, draw a phase portrait that is qualitatively consistent with your findings. This plot does not have to be to scale.

   (e) Choose a suitable value of θ and using the scheme above, produce a plot of the curve $(u_1(t), u_2(t))^T$ with initial conditions $(1, -1)^T$ in the phase plane. Run the scheme until you get "reasonably" close to an equilibrium point.

4. Define the incomplete Fresnel integrals

$$f_1(y) = \int_0^y \cos(x^2)dx, \quad f_2(y) = \int_0^y \sin(x^2)dx.$$

Yesterday you showed that

$$F_1 = \lim_{y\to\infty} f_1(y) \;\; \text{and} \;\; F_2 = \lim_{y\to\infty} f_2(y)$$

are well defined.

(a) For $y \in [-5, 5]$, produce a plot of the so-called Euler or Cornu spiral, which is the graph in the (f_1, f_2) plane of $(f_1(y), f_2(y))$ as y varies. This graph is used in train track design and diffraction optics.

(b) Starting from the origin, what is the arc length of the Cornu spiral as a function of y?

(c) Calculate F_1 and F_2. The value of $\int_0^\infty e^{-x^2}dx = \sqrt{\pi}/2$ may be useful. Does your result (visually) agree with the plot you obtained for the Cornu spiral?

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Day 3. Wednesday, September 19, 2012

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1. Let P be an orthogonal projection operator in $\mathbb{R}^n$, $n > 1$. In other words, every $u \in \mathbb{R}^n$ can be written uniquely as $u = v + w$, where v is in R_P, the range of P; and w is in the null space of P which is the orthogonal complement of R_P. Thus $Pu = v$, $(I - P)u = w$ and $Pv = v$. Let P and Q be two orthogonal projections in $\mathbb{R}^n$. What can you say about the 2-norm of $P - Q$?

2. Let
$$f(x) = \sum_{n=2}^{\infty} \frac{\sin(nx)}{\ln(n)}.$$
What can you say about $f(x)$?

3. Evaluate
$$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}.$$