Qualifying Examination

Summer 2013

Examination Committee: Anne Greenbaum, Randy LeVeque, KK Tung, Chris Bretherton, Ulrich Hetmaniuk

Day 1. Monday, September 16, 2013

You have three hours to complete this exam. Work all problems. Start each problem on a new page. You must show all steps and prove all claims, unless you rely on a standard theorem or result, in which case you should refer to it. You are allowed to use a computer for any part of any problem, but you are not allowed to access the internet or any application file existing on your computer. Any application file you use must be created from scratch.

1. Consider the variable coefficient advection problem $u_t + (1+x)u_x = 0$ for $x \geq 0$ and $t \geq 0$ with initial and boundary data

$$u(x,0) = \begin{cases} 1 & \text{when } 0 < x < 2, \\ 0 & \text{otherwise} \end{cases} \qquad u(0,t) = \begin{cases} 2 & \text{when } 0 < t < 1, \\ 0 & \text{when } t \geq 1. \end{cases}$$

   (a) The solution is piecewise constant. Sketch the structure of the solution in the $x - t$ plane, i.e., the curves $X(t)$ that separate constant regions, indicating the constant value in each.

   (b) Determine $u(x, \ln(2))$ explicitly as a function of x at $t = \ln(2)$.

2. Consider the advection equation $u_t + au_x = 0$ and the "skewed" upwind method,

$$U_j^{n+1} = U_{j-1}^n - \left(\frac{a\Delta t}{\Delta x} - 1\right)(U_{j-1}^n - U_{j-2}^n).$$

   (a) Show that this method is first order accurate in space and time by computing the local truncation error.

   (b) What restriction must be put on the time step Δt in terms of Δx in order for the CFL condition to be satisfied in the case $a > 0$? In the case $a < 0$?

   (c) Show that the method is in fact stable provided the CFL condition is satisfied, i.e., with the bounds found in part (b).

3. Solve

$$u_t = u_{xx} + xu, \quad x \in \mathbb{R}, \; t \geq 0,$$
$$u(x,0) = f(x).$$

   You may assume that $f(x)$ has all the smoothness and decay properties that you find convenient. HINT: You may want to use Fourier transforms.

4. Call a vector $y \in \mathbb{R}^n$ a *palindrome* if it reads the same way forwards and back; i.e., if $y_i = y_{n+1-i}$ for $i = 1, 2, \ldots, n$. Any vector $x \in \mathbb{R}^n$ can be mapped to a palindrome y by defining $y_i = \frac{1}{2}(x_i + x_{n+1-i})$. This mapping defines a matrix P.

   (a) Write down P for $n = 4$ and $n = 5$.

   (b) Determine all the eigenvalues and a basis for each eigenspace of P for general n. (Note: Consider both odd and even n.)

   (c) Does P define an orthogonal projection? Justify your answer.

(d) The SVD of P can be written as $P = \sum_{i=1}^{r} \sigma_i u_i v_i^T$ where r is the rank of P. Determine r, σ_i, u_i, and v_i for general n.

5. Let V be a real Hilbert space and A a bounded linear operator on V. Recall that

$$\|A\| = \sup_{u \in V \setminus \{0\}} \frac{\|Au\|}{\|u\|} = \sup_{\|u\|=1} \|Au\|,$$

where for $v \in V$, $\|v\| = (v, v)^{1/2}$.

(a) Prove that $\|A\| = \sup_{\|u\|=\|v\|=1}(Au, v) = \sup_{u \neq 0, v \neq 0} \frac{(Au,v)}{\|u\| \, \|v\|}$.

(b) Suppose $V = \mathbb{R}^n$ with the usual Euclidean inner product. What property of the unit ball in $\mathbb{R}^n$ ($\{u \in \mathbb{R}^n : \|u\| = 1\}$ guarantees that the supremum in (a) is actually achieved by some vectors $u_*, v_* \in \mathbb{R}^n$ with $\|u_*\| = \|v_*\| = 1$? If the linear operator A is represented by an n by n matrix, what are the vectors u_* and v_* that achieve this supremum called? Are they necessarily unique? Explain why or why not.

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Day 2. Tuesday, September 17, 2013

You have three hours to complete this exam. Work all problems. Start each problem on a new page. You must show all steps and prove all claims, unless you rely on a standard theorem or result, in which case you should refer to it. You are allowed to use a computer for any part of any problem, but you are not allowed to access the internet or any application file existing on your computer. Any application file you use must be created from scratch.

1. On Day 1, you showed that if V is a real Hilbert space and A is a bounded linear operator on V then

$$\|A\| = \sup_{\|u\|=\|v\|=1} (Au, v) = \sup_{u\neq 0, v\neq 0} \frac{(Au, v)}{\|u\|\,\|v\|},$$

where for $v \in V$, $\|v\| = (v, v)^{1/2}$. Now assume additionally that A is a compact, self-adjoint, and bounded linear operator on V such that

$$(Au, u) > 0 \quad \forall u \in V \backslash \{0\}.$$

   (a) Prove that $\|A\| = \sup_{\|u\|=1}(Au, u) = \sup_{u\neq 0} \frac{(Au,u)}{\|u\|^2}$.

   (b) Without assuming that V is finite dimensional (as you did on Day 1), prove that there exists a nonzero vector $v \in V$ such that $Av = \|A\|\, v$; i.e., $\|A\|$ is the largest eigenvalue of A.

2. Solve the integral equation

$$\varphi(x) + \frac{\alpha}{i\pi} \int_{-\infty}^{\infty} \frac{\varphi(\xi)}{\xi - x}\, d\xi = \frac{\sin x}{x}, \quad x \in \mathbb{R}$$

for $\varphi(x)$ by using Fourier transforms as follows:

   (a) First derive an explicit expression for the Fourier transform of $\frac{\sin x}{x}$.

   (b) Use the formula for the Fourier transform of a convolution to determine the Fourier transform of $\int_{-\infty}^{\infty} \frac{\varphi(\xi)}{\xi - x}\, d\xi$.

   (c) Substitute these expressions into the Fourier transformed integral equation and solve for $\hat{\varphi}$, then φ. Be sure to discuss what happens when $\alpha = \pm 1$.

3. Solve the 1-D heat equation $u_t = u_{xx}$, $-\infty < x < \infty$, $t > 0$, $u(x,t) \to 0$ as $x \to \pm\infty$, with a delta function initial condition $u(x, 0) = \delta(x)$ using the Laplace transform in t. Do the inverse Laplace transform as a contour integral in the complex plane.

4. Consider the equation

$$\sqrt{x^2 + \epsilon} + 1 = e^x.$$

Determine how many real roots this equation has when $\epsilon = 10^{-20}$ and for each root estimate its value with a relative error of at most 10^{-6}.

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Day 3. Wednesday, September 18, 2012

You have 2 hours to complete this exam. Work all problems. Start each problem on a new page. You must show all steps and prove all claims, unless you rely on a standard theorem or result, in which case you should refer to it. You are allowed to use a computer for any part of any problem, but you are not allowed to access the internet or any application file existing on your computer. Any application file you use must be created from scratch. We recommend you hand in all your work, even if it did not produce any results.

1. Consider the PDE
$$4u_{xx} + 12u_{xy} + 9u_{yy} = 0$$
on the rectangle $0 \leq x \leq 2$, $0 \leq y \leq 3$ with boundary conditions
$$u(x,0) = 0, \quad u(x,3) = 3, \quad u(0,y) = 2, \quad u(2,y) = 1.$$
   (a) Develop a finite difference method for this problem on a uniform grid with $\Delta x = \Delta y = h$, and use it to solve the problem. Discuss the order of accuracy and convergence of your method.
   (b) Find an analytical solution to the problem.
   (c) Suppose that the coefficient of the cross-term u_{xy} in the PDE were $12 - \epsilon$ instead of 12, where ϵ is a small positive number. Call the solution to this modified problem $u_\epsilon(x,y)$. Let $M(\epsilon)$ be the max-norm of $|u_\epsilon(x,y) - u(x,y)|$ over the rectangle. Can you estimate how $M(\epsilon)$ scales with ϵ?