Qualifying Examination

Summer 2015

Examination Committee: Chris Bretherton, Anne Greenbaum, Randy LeVeque

Day 1. Monday, September 14, 2015

You have three hours to complete this exam. Work all problems. Start each problem on a new page. **You must show all steps and prove all claims, unless you rely on a standard theorem or result, in which case you should refer to it. You may use a computer to check your work, but all derivations must be your own.** You are not allowed to access the internet or any application file existing on your computer. Any application file you use must be created from scratch.

1. Find strict upper and lower bounds for $|\sin(x+iy)|$ that depend on y only. In other words, find functions $f(y)$ and $g(y)$ such that $f(y) \le |\sin(x+iy)| \le g(y)$. Show that your bounds are attained for certain values of $x+iy$.

2. Let A denote a real square matrix with m rows.

   (a) Show that the sequence $\left(\sum_{n=0}^{N} \frac{A^n}{n!}\right)_{N\ge 0}$ is a Cauchy sequence in $\mathbb{R}^{m\times m}$ equipped with the matrix 2-norm. We recall that, for any matrix $A \in \mathbb{R}^{m\times m}$, A^0 is the identity matrix, I.

   (b) Which property of $\mathbb{R}^{m\times m}$ guarantees that the series converges? The limit of the series defines the exponential of a matrix,

   $$e^A = \sum_{n=0}^{+\infty} \frac{A^n}{n!}.$$

   (c) Derive another expression for the matrix exponential by applying the forward Euler method to the equation $\mathbf{y}'(t) = A\mathbf{y}$, $\mathbf{y}(0) = \mathbf{y}_0$, whose solution at time $t = 1$ is $\mathbf{y}(1) = e^A \mathbf{y}_0$. That is, using step size $h = 1/N$, write the approximate solution $\mathbf{y}_N$ as an Nth degree polynomial in A times the initial vector $\mathbf{y_0}$ and show that as $N \to \infty$, $\mathbf{y}_N \to e^A \mathbf{y}_0$.

   (d) Suppose the 2-step method

   $$\mathbf{y}_{n+2} = \mathbf{y}_{n+1} + \frac{k}{2}(3A\mathbf{y}_{n+1} - A\mathbf{y}_n)$$

   with stepsize k is applied to the equation $\mathbf{y}'(t) = A\mathbf{y}$, $\mathbf{y}(0) = \mathbf{y}_0$. Determine if the method is convergent and find the order of accuracy and the leading term in the local truncation error.

3. Consider the ODE boundary value problem

   $$\epsilon x^2 y'' = y, \quad y(0) \text{ bounded}, \ y(1) = 1, \ 0 < \epsilon \ll 1.$$

   (a) For a fixed ϵ, is $x = 0$ a regular or irregular singular point?

   (b) Find an exact solution to this equation and show how you found your answer. (No credit for an answer from Mathematica or other computer program.)

   (c) Sketch the behavior of this solution for some small value of ϵ.

(d) As $\epsilon \to 0$, where does the solution develop a boundary layer? What is the approximate inner solution within this boundary layer?

4. The one dimensional wave equation is

$$u_{tt} - c^2 u_{xx} = 0, \quad c \neq 0.$$

Make a change of variable to put this into the form $u_{\alpha\beta} = 0$ and use this form to find the general solution (as a function of x and t).

Qualifying Examination

Summer 2015

Examination Committee: Chris Bretherton, Anne Greenbaum, Randy LeVeque

Day 2. Tuesday, September 15, 2015

You have three hours to complete this exam. Work all problems. Start each problem on a new page. **You must show all steps and prove all claims, unless you rely on a standard theorem or result, in which case you should refer to it. You may use a computer to check your work, but all derivations must be your own.** You are not allowed to access the internet or any application file existing on your computer. Any application file you use must be created from scratch.

1. Let A be a complex square matrix. A *coneigenvalue* of A is a number $\lambda \in \mathbb{C}$ such that

$$A\bar{\mathbf{x}} = \lambda \mathbf{x},$$

for some nonzero vector $\mathbf{x}$, where $\bar{\mathbf{x}}$ denotes the complex conjugate of $\mathbf{x}$. The vector $\mathbf{x}$ is called a *coneigenvector.*

(a) Show that if λ is a coneigenvalue of A then so is $e^{i\theta}\lambda$ for any $\theta \in [0, 2\pi)$. (Here $i = \sqrt{-1}$.)

(b) Show that if λ is a coneigenvalue of A then $|\lambda|^2$ is an eigenvalue of $\bar{A}A$.

(c) Write down a 2 by 2 matrix that has no coneigenvalues and explain how you know this.

2. Consider the difference equation

$$a_{n+1} = -a_{n-1} + \epsilon a_n, \quad a_0 = 0, \quad a_1 = 1,$$

where ϵ is real.

(a) If $\epsilon = 0$, find

$$\lim_{n\to\infty} |a_{2n+1} - a_{2n}|.$$

(b) If $|\epsilon| \ll 1$, find the leading two nonzero terms in a perturbation series for $|a_3 - a_2|$.

(c) If $|\epsilon| \ll 1$, find the leading nonzero term in a perturbation series for a_{12}.

(d) If $|\epsilon| < 2$, show that a_n remains bounded as $n \to \infty$.

3. Let A be the $n \times n$ matrix of the form

$$A = \frac{1}{4}\begin{bmatrix} 2 & 1 & & & & 1 \\ 1 & 2 & 1 & & & \\ & 1 & 2 & 1 & & \\ & & 1 & 2 & 1 & \\ & & & 1 & 2 & 1 \\ 1 & & & & 1 & 2 \end{bmatrix}.$$

Note that the diagonal entries are $+2$, not -2, so this is not a difference operator, but has similar eigenstructure.

Suppose we choose a starting vector $y^{[0]} \in \mathbb{R}^n$ and we define $y^{[k+1]} = Ay^{[k]}$ for $k = 0,\ 1,\ 2,\ \ldots$

(a) Do you expect this to converge to a limit as $k \to 0$ in general? How does it depend on the starting vector $y^{[0]}$?

(b) In cases when it does converge, what can you say about the rate of convergence in general?

(c) Take $n = 10$ and $y^{[0]} = [1,\ 2,\ ,\ldots,\ 10]$. What is the limit $\hat{y}$ in this case? Estimate by analysis if possible, not (only) experimentation, how many iterations m would be needed to guarantee that $\|y^{[m]} - \hat{y}\|_\infty < 10^{-8}$.

(d) Now let B be the $n \times n$ matrix of the form

$$B = \frac{1}{2}\begin{bmatrix} 0 & 1 & & & & 1 \\ 1 & 0 & 1 & & & \\ & 1 & 0 & 1 & & \\ & & 1 & 0 & 1 & \\ & & & 1 & 0 & 1 \\ 1 & & & & 1 & 0 \end{bmatrix}.$$

What can you say about convergence of the iteration $y^{[k+1]} = By^{[k]}$ for $k = 0,\ 1,\ 2,\ \ldots$? In particular, why are the cases n even and n odd different?

4. Consider the family of 2-stage Runge-Kutta methods of the form

$$\begin{aligned} Y_1 &= U^n + \alpha k f(U^n) + \beta k f(Y_1) \\ U^{n+1} &= U^n + \gamma k f(Y_1) \end{aligned}$$

for solving the ODE initial value problem $u'(t) = f(u(t))$ with step size k. A particular method is obtained by choosing $\alpha,\ \beta,\ \gamma \in \mathbb{R}$.

(a) What can you say about the consistency, stability, and covergence of methods from this family (for various values of the parameters $\alpha,\ \beta,\ \gamma$)?

(b) By considering the behavior of these methods on the test problem $y' = \lambda y$, what can you say about the order of accuracy for various choices of the parameters?

(c) Are there methods from this family that would be reasonable methods to use for solving stiff ODEs? Justify your answer.

(d) If you were forced to use one of these methods to solve an ODE, what choice of $\alpha,\ \beta,\ \gamma$ would you use and why? (Your answer might depend on what sort of ODE it is.)

Qualifying Examination

Summer 2015

Examination Committee: Chris Bretherton, Anne Greenbaum, Randy LeVeque

Day 3. Wednesday, September 16, 2015

You have 2 hours to complete this exam. **DO 1 OF THE FOLLOWING 2 PROBLEMS**. You must show all steps and prove all claims, unless you rely on a standard theorem or result, in which case you should refer to it. You are allowed to use a computer for any part of any problem, but you are not allowed to access the internet or any application file existing on your computer. Any application file you use must be created from scratch. We recommend you hand in all your work, even if it did not produce any results.

1. Consider the ODE BVP from Day 1 with a particular choice of ϵ:

$$\epsilon x^2 y'' = y, \qquad y(0) \text{ bounded, } y(1) = 1, \epsilon = 0.1.$$

   Implement a finite difference method using $N = 50$ grid points (your choice how to distribute them across the computational domain) for solving this problem and show that your numerical solution at least visually matches the exact solution (which has the form $y(x) = x^p$ for some p). Discuss how your method enforces the boundary condition at $x = 0$ as stated in the problem. [Note that it would be easier to solve the problem numerically if the boundary condition were $y(0) = 0$. You could instead solve this modified problem for partial credit.]

2. A Blaschke product of degree n is a function defined on the unit disk $\mathcal{D}$ that has the form

$$B(z) = \gamma \prod_{j=1}^{n} \frac{z - \alpha_j}{1 - \bar{\alpha}_j z}, \quad |\gamma| = 1, \quad |\alpha_j| < 1.$$

   (a) Show that B is bounded on $\mathcal{D}$ and that $|B(z)| = 1$ for z on the unit circle.

   (b) Show that $B(z)$ also can be written in the form

$$B(z) = \frac{\sum_{j=0}^{n} c_j z^j}{\sum_{j=0}^{n} \bar{c}_j z^{n-j}}, \tag{1}$$

   for some constants $c_0, \ldots, c_n$. [Hint: You might want to use induction on n.] [Note, however, that **not all** functions of this form are Blaschke products because the denominator might have roots inside the unit disk.]

   (c) Given $n+1$ distinct points $\zeta_1, \ldots, \zeta_{n+1}$ inside the open unit disk ($|\zeta_j| < 1$) and given any complex values $w_1, \ldots, w_{n+1}$, it can be shown that there is a real nonnegative scalar μ and a Blaschke product B of degree at most n such that $\mu B(\zeta_k) = w_k$, $k = 1, \ldots, n+1$. Suppose we wish to find nonnegative scalar(s) μ and function(s) B of the form (1) satisfying $\mu B(\zeta_k) = w_k$, $k = 1, \ldots, n+1$. Write this problem as a coneigenvalue problem (something of the form $A\bar{x} = \lambda x$) for the unknown scalar μ and the coefficients $c_0, \ldots, c_n$.

   (d) Devise a numerical method to solve the coneigenvalue problem in (c). Apply your method to the problem:

$$\zeta_1 = 0.5, \quad \zeta_2 = -0.5, \quad w_1 = 1, \quad w_2 = 1 + i,$$

   and determine which of the function(s) of the form (1) that you find is actually a Blaschke product.