Qualifying Examination

Winter 2015

Examination Committee: Chris Bretherton, Anne Greenbaum, Randy LeVeque

Day 1. Saturday, December 19, 2015

You have three hours to complete this exam. Work all problems. Start each problem on a new page. **You must show all steps and prove all claims, unless you rely on a standard theorem or result, in which case you should refer to it. You may use a computer to check your work, but all derivations must be your own.** You are not allowed to access the internet or any application file existing on your computer. Any application file you use must be created from scratch.

1. If c is a complex number, the expression $i^c = e^{c \log i}$ may be multivalued. Suppose all the values $|i^c|$ are identical. What are these values and what can be said about the number c? Justify your assertions.

2. Let A be an $n \times n$ matrix and define $\cos(A)$ by the Taylor series

$$\cos(A) = I - \frac{1}{2!}A^2 + \frac{1}{4!}A^4 - \frac{1}{6!}A^6 + \cdots$$

   (a) Show that this series converges in $\|\cdot\|_\infty$ for any matrix.

   (b) If A is diagonalizable, show how $\cos(A)$ can be expressed in terms of the eigenvalues and eigenvectors of A.

   (c) In particular, calculate $\cos(A)$ for

$$A = \frac{2\pi}{5}\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}.$$

   (d) Write out explicitly the matrix $\cos(A)$ if A is a 2×2 Jordan block with eigenvalue λ.

3. Consider the linear multistep method

$$U^{n+2} = U^{n+1} + \frac{k}{2}(3f(U^{n+1}) - f(U^n)),$$

   as a method for solving the ODE initial value problem $u'(t) = f(u(t))$ with step size k.

   (a) Is the method *convergent*? Explain what this term means and justify your answer.

   (b) Determine the order of accuracy and the leading term in the local truncation error.

   (c) Does the point $z = -1$ lie in the region of absolute stability for this method? Justify your answer.

4. If water drains out of a cylindrical bucket with a hole in the bottom, the water depth $y(t)$ can be modeled by an ODE of the form $y'(t) = -\sqrt{y(t)}$ (with various assumptions and choices of paramaters).

   (a) Suppose we know the depth $y(1) = y_1 > 0$ at $t = 1$. What ODE "initial value" problem can be solved to determine the water depth at $t = 0$?

   (b) If $y(1) = 2$, what was $y(0)$?

(c) If $y(1) = 0$, what limits can be put on $y(0)$? Mathematically, why is there no unique solution $y(0)$ for the initial depth in this case?

5. Consider the PDE

$$u_t + xuu_x = 0, \qquad \text{for } 0 < x, t < \infty$$

with initial conditions $u(x, 0) = u_0(x) = \min(x, 1)$.

(a) Determine equations that the characteristic curves satisfy and sketch these curves in the x-t plane.

(b) In what region of the domain is $u(x, t) = 1$?

(c) Explain why no boundary condition on $u(0, t)$ is required.

(d) Suppose you want to determine $u(x, t)$ for some particular values of $x > 0$ and $t > 0$. Explain how you would do this based on the characteristic structure.

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Day 2. Sunday, December 20, 2015

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1. Let V be the function space consisting of all functions of the form $g(x) = c_1 + c_2 x^2$ on $0 \le x \le 1$ where $c_1,\ c_2 \in \mathbb{R}$. Define the L_2 inner product as usual by $(u, v) = \int_0^1 u(x)v(x)\,dx$ and let $\|\cdot\|$ be the corresponding norm.

   (a) Determine an orthonormal basis for V.

   (b) Let P denote the orthogonal projection operator from functions $L_2(0,1)$ to the subspace V. Determine $\|P\|$ and $\|I - P\|$.

   (c) Let $f(x) = x$ and determine the projection $g = Pf$.

   (d) Plot the functions f and g on the same axes.

2. (a) Let A and B be real n by n matrices and define the commutator $C = AB - BA$. Show that trace$(C) = 0$.

   (b) Suppose C has all diagonal entries equal to 0. Find a diagonal matrix X and a matrix Y such that C is the commutator for X and Y: $C = XY - YX$.

3. Consider the two-point boundary value problem

$$u_x - \epsilon u u_{xx} = 0$$

   for $0 \le x \le 1$ with $u(0) = 2$ and $u(1) = 1$.

   (a) What do you expect the solution to look like for small ϵ? In particular, is there a boundary layer, and where is it?

   (b) Suggest a finite difference method to solve this equation *on a uniform grid* with grid spacing $\Delta x = 1/N$ for some integer N. Be sure to discuss boundary conditions.

   (c) Determine the local truncation error and (local) order of accuracy for the method you proposed.

   (d) Explain how you would solve the resulting system of equations that result from your method. You do **not** need to implement it, but explain what is required.

   (e) Suppose you wanted to use this method to obtain a decent approximation to the solution with $\epsilon = 10^{-6}$, e.g. a couple digits of accuracy at all points. Roughly how large must N be taken? Justify your answer.

4. Again consider the PDE

$$u_t + x u u_x = 0, \qquad \text{for } 0 < x, t < \infty$$

with initial conditions $u(x, 0) = u_0(x) = \min(x, 1)$. On Day 1 you should have found that the characteristic curves $X(t)$ satisfy the ODE

$$X'(t) = X(t)u_0(X(0))$$

and hence have the form $X(t) = X(0)\exp(u_0(X(0))t)$.

(a) From this, determine (using some numerical root finding if necessary) the numerical value of $u(20, 10)$.

(b) From the form of the characteristics, determine a series expansion for $u(x, \epsilon)$ that is $\mathcal{O}(\epsilon^3)$ accurate as $\epsilon \to 0$, which can be used to estimate the solution for small times $t = \epsilon$.

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Day 3. Monday, December 21, 2015

You have 2 hours to complete this exam. Work both problems. Start each problem on a new page. You must show all steps and prove all claims, unless you rely on a standard theorem or result, in which case you should refer to it. You are allowed to use a computer for any part of any problem, but you are not allowed to access the internet or any application file existing on your computer. Any application file you use must be created from scratch. We recommend you hand in all your work, even if it did not produce any results.

1. Show that the integral
$$\int_0^\infty \sin\left(\frac{1}{x^2}\right)\,dx$$
is convergent, and use an appropriate numerical approach to calculate it.

2. Consider the partial differential equation
$$u_t(x,t) + v(x,t)u_x(x,t) = 0$$
defined on the interval $0 \le x \le 1$, for time $t \ge 0$, where at each time t the function $v(x,t)$ is to solution to the boundary value problem
$$v_{xx}(x,t) = u(x,t), \qquad v(0,t) = v(1,t) = 0.$$
Suppose we want to solve this equation with initial data $u(x,0) = 1 - x$ and obtain a good estimate of the solution at time $t = 50$. What can you say about how the solution will look, and how would you go about solving it?