Stochastics Methods and Problems

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Contents

1 Random Varibles and Distributions

1.1 Basic Definitions

1.1.1 Probability Mass Function (discrete)

$$p(k) = \mathbb{P}(X = k)$$

1.1.2 Probability Density Function (continuous)

$$f(x)\mathrm{d}x = \mathbb{P}(X \in [x, x + \mathrm{d}x))$$

1.1.3 Cumulative Density Function

$$F(x) = \mathbb{P}(X < x) = \begin{cases} \sum_{k=0}^{\lfloor x \rfloor} p(k) & \text{discrete} \\ \int_{-\infty}^{x} f(x)\mathrm{d}x & \text{continuous} \end{cases}$$

Can obtain probability density function by,

$$f(x) = \frac{\mathrm{d}}{\mathrm{d}x} F(x)$$

1.1.4 Probability Generating Function

$$G(z) = \mathbb{E}\left[z^X\right] = p(0) + p(1)z + p(2)z^2 + p(3)z^3 + \cdots$$

Can obtain probability mass function by,

$$p(k) = \frac{1}{k!} \left[\frac{\mathrm{d}^k}{\mathrm{d}z^k} G(z)\right]_{z=0}$$

1.1.5 Characteristic Function

$$\phi(t) = \mathbb{E}\left[e^{itX}\right]$$

Note: WHAT DO WE USE THIS FOR??

1.2 Bernoulli

Models if a heads is flipped for a biased coin.

Parameters	$p \in [0, 1]$
Support	$\{0, 1\}$
PMF	$\begin{cases} 1-p & k=0 \\ p & k=1 \end{cases}$
Mean	p
Variance	$p(1-p)$
PGF	$(1-p)+pz$
CF	$(1-p)+pe^{it}$

1.3 Binomial

Models the number of heads when flipping a biased coin n times.

Parameters	$p \in [0, 1], n \in \mathbb{N}_{\geq 0}$
Support	$\{0, 1, \ldots, n\}$
PMF	$\binom{n}{k} p^k (1-p)^{n-k}$
Mean	np
Variance	$np(1-p)$
PGF	$[(1-p)+pz]^n$
CF	$[(1-p)+pe^{it}]^n$

1.4 Geometric

Models the number of flips of a biased coin required to flip a heads.

Parameters	$p \in [0, 1]$
Support	$\{1, \ldots, n\}$
PMF	$p(1-p)^{k-1}$
CDF	$1-(1-p)^k$
Mean	$1/p$
Variance	$(1-p)/p^2$
PGF	$ps/(1-(1-p)s)$
CF	$pe^{it}/(1-(1-p)e^{it})$

1.5 Poisson

Expresses the probability of a given number of events occurring in a fixed interval of time or space if these events occur with a known constant rate and independently of the time since the last event.

Parameters	$\lambda > 0$
Support	$\{0, 1, 2, \ldots\}$
PMF	$\lambda^k e^{-\lambda}/k!$
CDF	$e^{-\lambda} \sum_{j=0}^{k} \lambda^j/j!$
Mean	λ
Variance	λ
PGF	$\exp(\lambda(z-1))$
CF	$\exp(\lambda(e^{it}-1))$

1.6 Exponential

Describes times between events in a Poisson point process.

Parameters	$\lambda > 0$
Support	$[0, \infty)$
PDF	$\lambda e^{-\lambda x}$
CDF	$1 - e^{-\lambda x}$
Mean	$1/\lambda$
Variance	$1/\lambda^2$
CF	$\lambda/(\lambda - it)$

2 Table of Random (COUNTING??????) Processes

2.1 Poisson Point Process

A process in which events occur continuously and independently at a constant average rate
Note: IS THIS ENOUGH TO DESCRIBE PPP UNIQUELY?

2.1.1 Viewed as a Counting Process

A counting process $N = (N_t)_{t \geq 0}$ is a Poisson process with parameter λ if it has the properties,

1. $N_0 = 0$
2. independent increments
3. the number of points in any time interval of length t is a Poisson random variable with parameter λt

In other words, a Poisson point process has probability mass function,

$$\mathbb{P}(N_t = n) = \frac{(\lambda t)^n}{n!} e^{-\lambda t}$$

2.1.2 Memoryless property

The distance between two consecutive points will be an exponential random variable with parameter λ (mean $1/\lambda$).

2.1.3 Probability of Jump

$$\mathbb{P}(N_{t+s} = n + m | N_t = n) = \begin{cases} 1 - \lambda s + \mathcal{O}(s^2) & m = 0 \\ \lambda s + \mathcal{O}(s^2) & m = 1 \\ \mathcal{O}(s^2) & m \geq 1 \end{cases}$$

Note: Something about exponentially distributed counting process

3 Generating and Characteristic functions

how to get density from gen function

4 Discrete Time Markov Chains

4.1 Transition Matrix

Sample Problems:

- Exercise 4.1: Write down transition matrices for processes based on rolling a dice
- Exercise 4.2: Write down transition matrices for $Y_n = X_{2n}$
- Exercise 4.7: Give example of transition matrix with multiple stationary distributions

4.2 Classification of States

Sample Problems:

- Exercise 4.3: Show if all states communicate with an absorbing state they must all be transient

4.3 Mean Recurence Time

Sample Problems:

- Exercise 4.4: Find expected visits to a state given some properties
- Exercise 4.5: Find mean-recurrence times using invariant distribution

4.4 Reversibility

Sample Problems:

- Exercise 4.8: Show process is reversible in equilibrium

4.5 Stationary/Invariant distribution

Note: TALK ABOUT VARIOUS METHODS FOR FINDING THIS

Sample Problems:

- Exercise 4.5: Find invariant distribution
- Exercise 4.6: Find invariant distribution of mistakes in editions of a book by computing limit of generating function
- Exercise 4.7: Give example of transition matrix with multiple stationary distributions

4.6 Generating Functions

Sample Problems:

- Exercise 4.6: Find invariant distribution of mistakes in editions of a book by computing limit of generating function

5 Continuous Time Markov Chains

5.1 Transition Matrix

5.2 Stationary/Invariant distribution

Sample Problems:

- Exercise 5.1: Find invariant distribution and conditions for existence
- Exercise 5.2: Show two processes have the same stationary distribution
- Exercise 5.3: Indirectly find stationary distribution by solving KFE, finding generating function for the chain, and computing the distribution of X_t as $t \to \infty$

5.3 Generator

Sample Problems:

- Exercise 5.1: Write down generator
- Exercise 5.3: Given generator solve KFE
- Exercise 5.4: Write down generator and solve KFE/KBE

5.4 Generating Functions

Sample Problems:

- Exercise 5.3: Use KBE to find PDE for generating function of X
- Exercise 5.4: Use KBE to find PDE for generating function of X
- Exercise 5.5: Compute generating function of Poisson process with random intensity. Use generating function to compute mean and variance.

5.5 KFE AND KBE

Sample Problems:

- Exercise 5.3: Given generator solve KFE
- Exercise 5.4: Write down KFE and KBE and solve

Note: ????? Practice Exam 6, Problem ??: fdsaf sad sa??

5.6 Birth Death Processes

General description of birth death processes

5.6.1 General Form for infinite queue

Description:

- Process either jumps up one or down one or stay the same
- Expected wait time in state i is exponentially distributed $\tau \sim \mathcal{E}(\lambda_i + \mu_i)$
- When the process does jump, the probability of an up jump is $\lambda_i/(\lambda_i + \mu_i)$, and the probability of a down jump is $\mu_i/(\lambda_i + \mu_i)$.
- if $\lambda_0 > 0$ the chain is irreducible.

State space: $S = \{1, 2, 3 \ldots\}$.

Generator:

$$G = \begin{bmatrix} -\lambda_0 & \lambda_0 & & & \\ \mu_1 & -(\mu_1+\lambda_1) & \lambda_1 & & \\ & \mu_2 & -(\mu_2+\lambda_2) & \lambda_2 & \\ & & \mu_3 & -(\mu_3+\lambda_3) & \lambda_3 \\ & & & & \ddots \end{bmatrix}$$

Invariant distribution:

$$\pi(k) = \frac{\lambda_0\lambda_1\cdots\lambda_{k-1}}{\mu_1\mu_2\cdots\mu_k}\pi(0), \qquad \pi(0) = \left(1 + \sum_{k=1}^{\infty} \frac{\lambda_0\lambda_1\cdots\lambda_{k-1}}{\mu_1\mu_2\cdots\mu_k}\right)^{-1}$$

Sample Problems: Example 5.2.9

5.6.2 M/M/1 queue

Description:

- Models infinite queue.
- Arrivals occur at a rate λ according to a Poisson process.
- Service times have exponential distribution with rate parameter μ, where $1/\mu$ is the mean service time.
- A single server serves customers one at a time from front of queue, first come first serve

State space: $S = \{1, 2, 3 \ldots\}$.

Generator:

$$G = \begin{bmatrix} -\lambda & \lambda & & \\ \mu & -(\mu+\lambda) & \lambda & \\ & \mu & -(\mu+\lambda) & \lambda \\ & & & \ddots \end{bmatrix}$$

Invariant distribution:

$$\pi(k) = (1 - \lambda/\mu)(\lambda/\mu)^k$$

Expected Response Time: For customers who arrive and find the queue as a stationary process, the response time (sum of waiting and services times) has density function,

$$f(t) = \begin{cases} (\mu - \lambda)e^{-(\mu-\lambda)t}, & t > 0 \\ 0 & \text{ow.} \end{cases}$$

This has mean,

$$\int_0^\infty tf(t)\mathrm{d}t = \frac{1}{\mu - \lambda}$$

Sample Problems:

- Exercise 5.1: fdsaf sad

5.6.3 M/M/∞

Description:

- Arrivals occur at a rate λ according to a Poisson process.
- Service times have exponential distribution with rate parameter μ, where $1/\mu$ is the mean service time.
- There are always enough servers that every arriving job is serviced immediately.

State space: $S = \{1, 2, 3, \ldots\}$.

Generator:

$$G = \begin{bmatrix} -\lambda & \lambda & & & \\ \mu & -(\mu+\lambda) & \lambda & & \\ & 2\mu & -(2\mu+\lambda) & \lambda & \\ & & 3\mu & -(3\mu+\lambda) & \lambda \\ & & & & \ddots \end{bmatrix}$$

Invariant Distribution:

$$\pi(k) = \frac{(\lambda/\mu)^k e^{-\lambda/\mu}}{k!}$$

Sample Problems: Exercise 5.3, Final Problem ??, Practice Exam #? Problem 1

5.6.4 M/M/1/K queue

State space: $S = \{1, 2, \ldots, n\}$.

Generator:

$$G = \begin{bmatrix} -\lambda & \lambda & & & & & \\ \mu & -(\mu+\lambda) & \lambda & & & & \\ & \mu & -(\mu+\lambda) & \lambda & & & \\ & & & & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & & & & \\ & & & \mu & -(\mu+\lambda) & \lambda \\ & & & & \mu & -\mu \end{bmatrix}$$

6 Brownian Motion

Note: add examples from class notes

6.1 Martingale

Sample Problems:

- Exercise 7.1: Show a process is a Martingale using definition
- Exercise 7.4: Show a process is a Martingale using definition

6.2 Characteristic Functions

Sample Problems:

- Exercise 7.2: Compute characteristic function of $W(N(t))$, where $N \sim \text{Pois}(\lambda)$

7.3: n-th variation time

6.3 Laplace Transform

Sample Problems:

- *Note:* Example ??? from book
- Exercise 7.4: Compute Laplace transform of first hitting time.

7 Stochastic Calculus

Note: ITO FORMULA AND STUFF

7.1 Itô's Formula

One Dimension:

$$\mathrm{d}f(X_t) = f'(X_t)\mathrm{d}X_t + \frac{1}{2}f''(X_t)\mathrm{d}[X,X]_t$$

Two Dimensions:

$$\mathrm{d}f(t,X_t) = f_t(t,X_t)\mathrm{d}t + f_x(t,X_t)\mathrm{d}X_t + \frac{1}{2}f_{xx}(t,X_t)\mathrm{d}[X,X]_t$$

Two Dimensions:

$$\begin{aligned}\mathrm{d}f(X_t,Y_t) = f_x(X_t,Y_t)\mathrm{d}X_t + f_y(X_t,Y_t)\mathrm{d}Y_t&\\ + \frac{1}{2}\Big(f_{xx}(X_t,Y_t)\mathrm{d}[X,X]_t + f_{xy}(X_t,Y_t)\mathrm{d}[X,Y]_t&\\ + f_{yx}(X_t,Y_t)\mathrm{d}[Y,X]_t + f_{yy}(X_t,Y_t)\mathrm{d}[Y,Y]_t\Big)&\end{aligned}$$

8 SDEs and PDEs

Note: ADD ASSOCIATED PDEs

8.1 Geometric Brownian Motion

8.2 Ornstein–Uhlenbeck (OU) process

SDE:

$$\mathrm{d}X_t = \kappa(\theta - X_t)\mathrm{d}t + \mathrm{d}W_t$$

Solution:

$$X_t = \theta + e^{-\kappa t}(X_0 - \theta) + \int_0^t e^{-\kappa(t-s)}\mathrm{d}W_s$$

9 Jump Diffusions

10 Practice Qualification Exams

Practice Exam 1, Problem 1

Let $X = (X_n)_{n\in\mathbb{N}_0}$ be a discrete time Markov chain with X_n representig the amount of water in a reservoir at noon on day n. Assume $X_0 \in \mathbb{N}_0$. Let $Y = (Y_n)_{n\in\mathbb{N}_0}$ be a sequence of iid random variables with Y_n representing the aount of water that flows into the reservoir during the n-th day. The state space of Y is $\{0, 1, 2, \ldots\}$. The resevoir has a maximum capacity of $K \in \mathbb{N}$. When the resevoir is filled to level K, all excssive inflows are lost.

(a) Write the one-step transition matrix P of X in terms of the probability generating function G_Y of Y.
(b) Find an expression for the stationary distribution π of X in terms of the probability generating function G_Y of Y.

Solution

(a) We assume all the water comes in the afternoon. That is, $X_{n+1} = X_n + Y_n$.

Suppose on day n the resevoir is not full. That is, $X_n = k < K$. If it is not filled completely by the incoming water, then some amount of water $j < K - k$ was added. In this case $X_{n+1} = k + j$ with probability,

$$\mathbb{P}(Y_n = j) = f_Y(j) = \left[\frac{1}{j!}\frac{\mathrm{d}^j G_Y(s)}{\mathrm{d}s^j}\right]_{s=0}$$

Otherwise, $X_{n+1} = K$ with probability,

$$1 - \sum_{j<K-k} f_Y(j) = 1 - \sum_{j<K-k}\left[\frac{1}{j!}\frac{\mathrm{d}^j G_Y(s)}{\mathrm{d}s^j}\right]_{s=0}$$

Suppose $X_n = K$. Then since no water leaves the resevoir, $X_{n+1} = K$ with probability one.

We can write this as,

$$X_{n+1} = \begin{cases} \left[\frac{1}{j!}\frac{\mathrm{d}^j G_Y(s)}{\mathrm{d}s^j}\right]_{s=0} & j < K - X_n \\ 1 - \sum_{j<K-X_n}\left[\frac{1}{j!}\frac{\mathrm{d}^j G_Y(s)}{\mathrm{d}s^j}\right]_{s=0} & \text{otherwise} \end{cases}$$

(b) Note that $\pi = [0, 0, \ldots, 0, 1]$ is a stationary distribution.

Note: argue the distributoin is unique?

Note: alternative approach?? Clearly $X_n \to K$ as $n \to \infty$.

Note: in what sense?

Practice Exam 1, Problem 2

Let $(X,Y)=(X_t,Y_t)_{t\geq 0}$ satisfy the following SDE,

$$\mathrm{d}X_t = \mathrm{d}W_t^1, \qquad \mathrm{d}Y_t = \mathrm{d}W_t^2, \qquad (X_0,Y_0)=(x,y)$$

where $W=(W_t^1,W_t^2)_{t\geq 0}$ is a two-dimensinoal Brownian motion with independent components. Define a process $(R,\Phi)=(R_t,\Phi_t)_{t\geq 0}$ as follows,

$$\Phi_t = \arctan(Y_t/X_t), \qquad R_t^2 = X_t^2+Y_t^2$$

(a) Derive the SDEs satisfied by (R,Φ).
(b) Define,

$$u(r,\phi)=\mathbb{E}\left[e^{-\lambda\tau}f(R_\tau)|R_0=r,\Phi_0=\phi\right], \quad \tau=\inf\{t\geq 0:\Phi_t\notin(0,\pi/2)\}, \quad \phi\in(0,\pi/2)$$

Derive a PDE satisfied by u.
(c) Desribe with pseudo-code how you would find $u(r,\phi)$ using Monte Carlo simulation.

Solution

(a) Define $f(x,y)=\arctan(y/x)$ and $g(x,y)=\sqrt{x^2+y^2}$. Now note that,

$$\Phi_t = f(X_t,Y_t), \qquad R_t = g(X_t,Y_t)$$

Appying Itô's formula we find,

$$\begin{aligned}\mathrm{d}\Phi_t &= f_x(X_t,Y_t)\mathrm{d}X_t + f_y(X_t,Y_t)\mathrm{d}Y_t \\ &\qquad + \frac{1}{2}\big(f_{xx}(X_t,Y_t)\mathrm{d}[X,X]_t + f_{xy}(X_t,Y_t)\mathrm{d}[X,Y]_t \\ &\qquad\qquad + f_{yx}(X_t,Y_t)\mathrm{d}[Y,X]_t + f_{yy}(X_t,Y_t)\mathrm{d}[Y,Y]_t\big)\end{aligned}$$

Using our Heuristics we have,

$$\mathrm{d}[X,X]_t=\mathrm{d}[Y,Y]_t=\mathrm{d}t, \qquad \mathrm{d}[X,Y]_T=\mathrm{d}[Y,X]_t=0$$

We compute,

$$\begin{aligned}
f_x(x,y) &= -\frac{y}{x^2+y^2} = -\frac{\sin(\arctan(y/x)}{\sqrt{x^2+y^2}} \\
f_y(x,y) &= \frac{x}{x^2+y^2} = \frac{\cos(\arctan(y/x))}{\sqrt{x^2+y^2}} \\
f_{xx}(x,y) &= \frac{2xy}{(x^2+y^2)^2} \\
f_{yy}(x,y) &= -\frac{2xy}{(x^2+y^2)^2}
\end{aligned}$$

Therefore, maxing the substitutions, $\Phi_t = \arctan(Y_t/X_t)$, and $R_t = \sqrt{X_t^2+Y_t^2}$,

$$\mathrm{d}\Phi_t = -\frac{\sin(\Phi_t)}{R_t}\mathrm{d}W_t^1 + \frac{\cos(\Phi_t)}{R_t}\mathrm{d}W_t^2$$

Similarly,

$$\begin{aligned}
\mathrm{d}R_t = g_x(X_t,Y_t)\mathrm{d}X_t + g_y(X_t,Y_t)\mathrm{d}Y_t \\
+ \frac{1}{2}\big(g_{xx}(X_t,Y_t)\mathrm{d}[X,X]_t + g_{xy}(X_t,Y_t)\mathrm{d}[X,Y]_t \\
+ g_{yx}(X_t,Y_t)\mathrm{d}[Y,X]_t + g_{yy}(X_t,Y_t)\mathrm{d}[Y,Y]_t\big)
\end{aligned}$$

We compute,

$$\begin{aligned}
g_x(x,t) &= \frac{x}{\sqrt{x^2+y^2}} = \cos(\arctan(y/x)) \\
g_y(x,t) &= \frac{y}{\sqrt{x^2+y^2}} = \sin(\arctan(y/x)) \\
g_{xx}(x,t) &= \frac{y^2}{(x^2+y^2)^{3/2}} \\
g_{yy}(x,t) &= \frac{x^2}{(x^2+y^2)^{3/2}}
\end{aligned}$$

Therefore, maxing the substitutions, $\Phi_t = \arctan(Y_t/X_t)$, and $R_t = \sqrt{X_t^2+Y_t^2}$,

$$\mathrm{d}R_t = \cos(\Phi_t)\mathrm{d}W_t^1 + \sin(\Phi_t)\mathrm{d}W_t^2 + \frac{1}{2R_t}\mathrm{d}t$$

(b)

(c)

Practice Exam 2, Problem 1

Let $Y = (Y_n)_{n\in\mathbb{N}_0}$ be a sequence of iid random variables with $Y_n \sim \text{Pois}(\lambda)$ representing the number of particles entering a chamer at time n. The lifetimes of the particles are iid geometric random variables with parameter p. Let X_n represent the number of particles in the chamber at time n.

(a) Give an expression for $p(i,j) = \mathbb{P}(X_{n+1} = j|X_n = i)$.
(b) Find the stationary distribution π of $X = (X_n)_{n\geq 0}$.

Solution

(a) If the lifetime of a particle is a geometric random variable with parameter p, then at each step there is a probability p that the particle will decay and a probability $1-p$ that the particle will not decay.

Let Z_n represent the number of particles which no *not* decay during the n-th step. That is,

$$X_{n+1} = Z_n + Y_n$$

Since each of the X_n particles no not decay with probability $1-p$ we have $Z_n \sim \text{Bin}(X_n, 1-p)$ and $Y_n \sim \text{Pois}(\lambda)$.

Denote the generating functions of Y_n and Z_n by, $G_{Y_n}(s)$ and $G_{Z_n}(s)$ respectively. Explicitly,

$$G_{Y_n}(s) = G_Y(s) = e^{\lambda(s-1)}, \quad G_{Z_n}(s) = (p+(1-p)s)^{X_n} = G_{X_n}(p+(1-p)s)$$

Assume Y_n is independent of X_n and therefore of Z_n. If $X_n = i$ the generating function is $G_{X_n} = s^i$. We can then write,

$$G_{X_{n+1}}(s) = G_{Y_n}(s)G_{Z_n}(s) = G_Y(s)(p+(1-p)s)^i = e^{\lambda(s-1)}(p+(1-p)s)^i$$

Therefore,

$$\begin{aligned}
p(i,j) &= \mathbb{P}(X_{n+1} = j|X_n = i) \\
&= \left[\frac{1}{j!}\frac{\mathrm{d}^j G_{X_{n+1}}(s)}{\mathrm{d}s^j}\right]_{s=0} \\
&= \left[\frac{1}{j!}\frac{\mathrm{d}^j}{\mathrm{d}s^j}\left[e^{\lambda(s-1)}(p+(1-p)s)^i\right]\right]_{s=0}
\end{aligned}$$

(b) More generally, the generating function $G_{X_{n+1}}(s)$ of X_{n+1} is then,

$$G_{X_{n+1}}(s) = G_{Y_n}(s)G_{Z_n}(s) = G_Y(s)G_{X_n}(p+(1-p)s)$$

This gives a recurrence relation. We assume $X_0 = 0$ so that $G_{X_0}(s) = 1$. For convencience write $q = 1 - p$. Then,

$$1 + q^k(s-1)|_{s=(1+q(s-1))} = 1 + q^k((1+q(s-1)) - 1) = 1 + q^{k+1}(s-1)$$

Therefore,

$$\begin{aligned}
G_{X_n}(s) &= G_Y(s)G_{X_{n-1}}(1+q(s-1)) \\
&= G_Y(s)G_Y(1+q(s-1))G_{x_{n-2}}(1+q^2(s-1)) \\
&\vdots \\
&= \prod_{k=0}^{n} G_Y(1+q^k(s-1))
\end{aligned}$$

We can rewrite this as,

$$G_{X_n}(s) = \exp\left(\sum_{k=0}^{n} \lambda((1+q^k(s-1)) - 1)\right) = \exp\left(\lambda(s-1)\sum_{k=0}^{n} q^k\right)$$

Taking the limit as $n \to \infty$ we find,

$$\begin{aligned}
G_{X_\infty}(s) &= \lim_{n\to\infty} G_{X_n}(s) \\
&= \exp\left(\lambda(s-1)\sum_{k=0}^{\infty} q^k\right) \\
&= \exp\left(\frac{\lambda(s-1)}{1-q}\right) \\
&= \exp\left(\frac{\lambda}{p}(s-1)\right)
\end{aligned}$$

Therefore, by the continuity theorem, X_∞ is distributed like a Poisson random variable with parameter λ/p.

This means the invariant distribution of X is the density function of a Poisson random variable with parameter λ/p. That is,

$$\pi(k) = \left(\frac{\lambda}{p}\right)^k \frac{e^{-\lambda/p}}{k!}$$

Practice Exam 2, Problem 2

Fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a filtration $\mathbb{F} = (\mathcal{F}_t)_{0 \le t \le T}$ where $T < \infty$. Consider a process $P = (P_t)_{0 \le t \le T}$ defined as,

$$P_t = \mathbb{E}[\mathbb{1}_{X_T \le a} | \mathcal{F}_t], \qquad \mathrm{d}X_t = \mathrm{d}W_t$$

where W is a $(\mathbb{P}, \mathbb{F})$-Brownian motion. Derive an SDE for the process P. Your answer should not involve X. You may find it useful to use the CDF Φ of a standard normal random variable,

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-x^2/2} \mathrm{d}x$$

its inverse Φ^{-1} and its derivative $\phi := \Phi'$.

Solution

Solution

(a)

(b)

Practice Exam 3, Problem 1

Let $X = (X_t)_{t\geq 0}$ be a continuous time Markov chain with X_t representing the number of individuals in a population at time t. Individuals do not reproduce. However, immigrants join the population as a Poisson process with parameter λ. The lifetimes of individuals are iid exponentially distributed random varaibles with parameter μ.

(a) Write the generator G of X
(b) Find the stationary distribution π of X.

Solution

(a)

(b)

Practice Exam 3, Problem 2

Fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a filtratoin $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$. Consider a process $X = (X_t)_{t\geq 0}$ that satisfies the following SDE,

$$\mathrm{d}X_t = b\mathrm{d}t + a\mathrm{d}W_t$$

where W is a $(\mathbb{P}, \mathbb{F})$-Brownian motion. Suppose that $X_0 \in (L, R)$ and that $\{L\}$ and $\{R\}$ are reflecting boundaries.

(a) Derive an expression for the invariant distribution of X.
(b) Derive an expression for the transition density $\Gamma(t, x; T, y)\mathrm{d}y := \mathbb{P}(X_t \in \mathrm{d}y | X_t = x)$.
(c) Show that $\Gamma(t, x; T, y) \to \pi(y)$ as $T \to \infty$.

Solution

(a)

(b)

Practice Exam 4, Problem 1

A transition probablity matrix P for a Markov chain with N states is said be doubly stochastic if the entries in each of its columns add up to one.

(a) Show that the uniform distribution given by $q_i = 1/N$ for all j is a stationary distribution for such a Markov chain.

(b) Consider the following random walk on the sets of integers $\{0, 1, \ldots, L\}$. The walk jumps to the right or left at each step with probability $1/2$ subject to the rule that if it tries to go to the left from 0 or to the right from L it stays put. Compute the stationary distribution of this random walk.

(c) Consider the following random walk on state-space $\{0, 1, 2, \ldots, L\}$ of numbers arranged on a ring. At each step, the walk goes to the right with probability a or to the left with probability $1 - a$ subject to the rules if it tries to go to the left from 0 it ends up at L or if it tries to go to the right from L it ends up at 0. Compute the stationary distribution of this chain.

Solution

(a)

(b)

(c)

Practice Exam 4, Problem 2

Fix a probability space $(\Omega, \mathcal{F}, \mathbb{P}$ and a filtration $\mathbb{F} = (\mathcal{F}_t)_{0\le t\le T}$. Suppose that $X = (X_t)_{t\ge 0}$ satisfies the following SDE,

$$\mathrm{d}X_t = -\kappa X_t \mathrm{d}t + \sigma \mathrm{d}W_t$$

where W is a $(\mathbb{P}, \mathbb{F})$-Brownian motion. Now consider a chance of measure,

$$\frac{\mathrm{d}\tilde{\mathbb{P}}}{\mathrm{d}\mathbb{P}} = \exp\left(-\frac{1}{2}\gamma^2 T - \gamma W_T\right).$$

(a) Derive an SDE for theh process X under $\tilde{\mathbb{P}}$. Your answer should be given in terms of a process $\tilde{W}$ which is a $(\tilde{\mathbb{P}}, \mathbb{F})$-Brownian motion.

(b) Define functions $u : [0, T] \times \mathbb{R} \to \mathbb{R}$ and $\tilde{u} : [0, T] \times \mathbb{R} \to \mathbb{R}$ as follows

$$u(t, x) := \mathbb{E}[h(X_T)|X_t = x], \qquad \tilde{u}(t, x) := \tilde{\mathbb{E}}[h(X_T)|X_t = x]$$

Provide the PDEs satisfied by u and $\tilde{u}$, respectively.

Solution

(a)

(b)

Practice Exam 5, Problem 1

Consider a Markov chain with state space $\{0, 1, 2, \ldots\}$ and transition probabilities,

$$\begin{aligned} p(i, i+1) &= p_i, && i \geq 0 \\ p(i, i-1) &= q_i, && i > 0 \\ p(i, i) &= r_i, && i \geq 0 \end{aligned}$$

For $N > 0$ and satet i, let $a_N(i)$ be the probability that the tiem of first visit to state N is strictly less than the time of first visit to state 0 if we start at state i. Note that $a_N(0) = 0$ and $a_N(N) = 1$.

(a) Write a recursive relation for $a_N(i)$ by consider what happens on the first transition out of state i.
(b) Solve the above equation to compute $a_N(i)$.
(c) Use (b) above to show that state 0 is recurrent if and only if,

$$\sum_{j=1}^{\infty} \prod_{i=1}^{j-1} \frac{q_i}{p_i} = \infty$$

(d) Analyze the situation where $p_i = p$, $q_i = 1 - p$, $r_i = 0$ for $x \geq 1$, and $r_0 = 1 - p$.

Solution

(a)

(b)

Practice Exam 5, Problem 2

Fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$. Consider a mean-repelling OU process,

$$\mathrm{d}X_t = X_t \mathrm{d}t + \sqrt{2}\mathrm{d}W_t$$

where W is a $(\mathbb{P}, \mathbb{F})$-Brownian motion.

(a) Derive two representations of the transition density $\Gamma(t, x, T, y)\mathrm{d}y := \mathbb{P}(X_t \in \mathrm{d}y | X_t = x)$ of X. One of the representations should involve Hermite polynomials.

(b) Does the process X have an invariant distribution? If so, provide it. If not, explain why not.

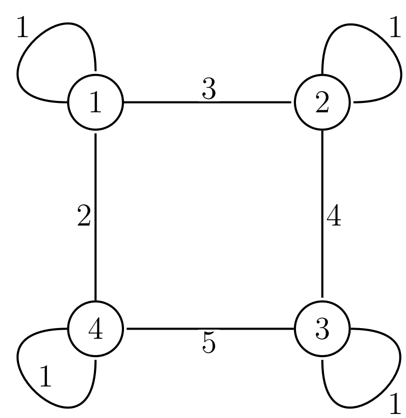
Solution

(a)

(b)

Practice Exam 6, Problem 1

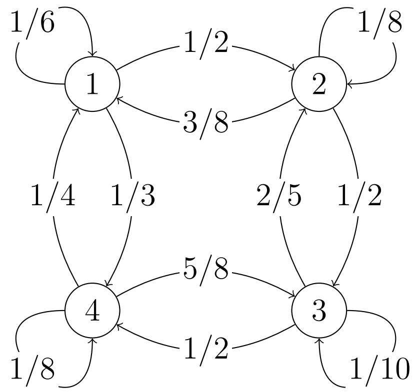
Consider a game which is played on a network with nodes numbered $1, 2, 3, 4$ and edges that connect these nodes as show below. Note that each node in this network has self-loop edges. Every edge in the network has an associated weight. For example, the weight associated with the edge connecting nodes 1 and 2 is 3. Similarly the weight associated with the self-loop edge on node 1 is 1. The game proceeds as follows. You have a token that you move randomly from one node to another with probabilities propotional to the corresponding edge weights. For example, the probability that your token moves from node 1 to node 2 is $3/(1 + 2 + 3) = 1/2$. Let X_n be the position of your token after n moves, for $n = 0, 1, 2, \ldots$.



(a) Model the stochastic process X_n as a Markov chain by drawing its state transition diagram and write the corresponding one-step transition probability matrix.
(b) Calculate the stationary distribution that your token is at node j, for $j = 1, 2, 3, 4$.
(c) Find the expected number of token moves between two consecutive visits to node 2.

Solution

(a) We have transition diagram,



The corresponding probability transition matrix P is,

$$P = \begin{bmatrix} 1/6 & 1/2 & 0 & 1/3 \\ 3/8 & 1/8 & 1/2 & 0 \\ 0 & 2/5 & 1/10 & 1/2 \\ 1/4 & 0 & 5/8 & 1/8 \end{bmatrix}$$

(b) We easily compute the right eigenvector corresponding to eigenvalue 1 as,

$$\pi = \frac{1}{4}[3/4, 1, 5/4, 1] = [3/16, 1/4, 5/16, 1/4]$$

(c) The expected number of moves is the mean recurrence time. This chain is irreducible so by theorem we have,

$$\tau_2 = 1/\pi(2) = 4$$

Practice Exam 6, Problem 2

Fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$. Let X be an OU process, and S a strictly increasing Lévy process (also known as a subordinator), whose dynamics are given by,

$$\mathrm{d}X_t = -X_t \mathrm{d}t + \sqrt{2}\mathrm{d}W_t, \qquad \mathrm{d}S_t = \gamma \mathrm{d}t + \int_0^\infty z N(\mathrm{d}t, \mathrm{d}z), \qquad S_0 = 0$$

where W is a $(\mathbb{P}, \mathbb{F})$-Brownian motion and N is a poisson random measure with associated Lévy measure ν. Consider a Subordianted OU process Y defined as follows,

$$Y_t = X_{S_t}$$

Define the two-parameter semigroup $\mathcal{P}(t,T)$ associated with the Y process,

$$\mathcal{P}(t,T)f(y) := \mathbb{E}[f(Y_T)|Y_t = y)$$

For a fixed $0 \leq t \leq T < \infty$, what are the eigenfunctions and associated eigenvalues of the operator $\mathcal{P}(t,T)$?

Solution

(a)

(b)

Practice Exam 7, Problem 1

(a) (Weather chain) The weather can be either sunny, smoggy, or rainy. The weather stays sunny for an exponentially distributed amount of time with mean 3 days and then turns smoggy. It stays smoggy for an exponentially distributed amount of time with mean 4 days and then turns rainy. Finally, it rains for an exponentially distributed amount of time with mean 1 and then it is sunny. Model the weather system as a continuous time Markov chain and compute its stationary distribution.

(b) (Barbershop) Consider a barbershop with one barber and two waiting chairs. The barber cuts hair at rate 3 customers per hour (exponentially distributed hair-cutting time). Customers arrive according to a Poisson process with rate 2 per hour. Arriving customers leave immediately if they find that the two waiting chairs are occupied. Model this system as a continuous time Markov chain and derive its stationary distribution.

Solution

(a)

(b)

Practice Exam 7, Problem 2

Fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a filtration $\mathbb{F} = (\mathcal{F}_t)_{0 \le t \le T}$. Consider two processes,

$$\mathrm{d}X_t = \sigma \mathrm{d}W_t, \qquad \mathrm{d}S_t = \sigma_t S_t \mathrm{d}W_t$$

where W is a $(\mathbb{P}, \mathbb{F})$-Brownian motino and the process $\sigma = (\sigma_t)_{0 \le t \le T}$ evolves independently of W.

(a) Show that,

$$\mathbb{E}[G(X_T - X_t)|\mathcal{F}_t] = \mathbb{E}[G(X_t - X_T)|\mathcal{F}_t]$$

(b) Show that,

$$\mathbb{E}[G(S_T)|\mathcal{F}_t] = \mathbb{E}[(S_T/S_t)G(S_t^2/S_T)|\mathcal{F}_t]$$

Solution

(a)

(b)

Practice Exam 8, Problem 1

A bakery uses a two-step process to make chocolate cakes. The first step involves baking the cake and the second step involves frosting the cake. Baking takes an exponentially distributed amount of time with rate μ_1 . After a cake is baked, it goes to the frosting machine. Frosting takes an exponentially distributed amount of time with rate μ_2. The processing times at the oven and the frosting machine are independent random variables. Potential cakes arrive according to a Poisson process at rate λ, however, a cake goes to the baking oven only if both the oven and the frosting machine are idle. If any of the two is busy, the cake simply exits the system. We wish to model this system as a continuous-time Markov chain.

(a) Precisely define the states of your continuous-time Markov chain (Hint: your model should have three states and note that there will never be two cakes in this system).

(b) Draw a transition diagram for your continuous-time Markov chain.

(c) Derive stationary distribution for your continuous-time Markov chain.

(d) Find the expected number of cakes in this system in steady-state.

Solution

(a)

(b)

Practice Exam 8, Problem 2

Fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a filtration $\mathbb{F} = (\mathcal{F}_t)_{0\le t\le T}$. Define a procecss X and stoppting time τ as follows,

$$\mathrm{d}X_t = \mu \mathrm{d}t + \sigma \mathrm{d}W_t, \qquad \tau = \inf\{t \ge 0 : X_t \notin (a,b)\}$$

where W is a $(\mathbb{P}, \mathbb{F})$-Brownian motion. Define the following Laplace Transform,

$$L(x;\lambda) := \mathbb{E}[e^{-\lambda\tau} | X_0 = 0), \qquad x \in (a,b)$$

Derive the PDE satisfied by L and use this to find L explicitly.

Solution

(a)

(b)

11 Homework Problems

Exercise 3.1

Let $X \sim \text{Bin}(n, U)$ where $U \sim \mathcal{U}((0,1))$. What is the probability Generating function $G_X(s)$ of X? What is $\mathbb{P}(X = k)$ where $k \in \{0, 1, 2, ..., n\}$?

Solution

Using iterated conditioning, since a Binomial random variable is the sum of n iid Bernioulli random variables,

$$G_X(s) = \mathbb{E}[s^X] = \mathbb{E}\mathbb{E}[s^X|U] = \mathbb{E}[((1-U)s^0 + Us^1)^n]$$

We calculate this by integrating with Mathematica as,

```
Integrate[((1 - x) + x s)^n, {x, 0, 1}, Assumptions -> {s > 0}]
```

This yields,

$$\mathbb{E}[((1-U) + Us)^n] = \int_{\mathbb{R}} \mathbb{1}_{(0,1)}((1-x) + xs)^n \mathrm{d}x = \int_0^1 ((1-x) + xs)^n \mathrm{d}x = \frac{1 - s^{n+1}}{(n+1)(1-s)}$$

This is a finite geometric progression which we simplify so,

$$G_X(s) = \sum_{k=0}^{n} \frac{s^k}{n+1}$$

Therefore $\mathbb{P}(X = k) = 1/(1+n)$ for $k = 0, 1, 2, ..., n$.

Exercise 3.2

Let Z_n be the size of the n-th generation in an ordinary branching process with $Z_0 = 1$, $\mathbb{E}Z_1 = \mu$ and $\mathbb{V}Z_1 > 0$. Show that $\mathbb{E}Z_n Z_m = \mu^{n-m}\mathbb{E}Z_m^2$ for $m \le n$. Use this to find the correlation coefficient $\rho(Z_m, Z_n)$ in terms of μ, n and m. Consider the case $\mu = 1$ and the case $\mu \neq 1$.

Solution

Let $Y_{m,i}$ denote the number of offspring in the n-th generation that descends from the i-th member of the m-th generation. Then the $(Y_{m,i})$ are iid with distribution Z_{n-m} and $Z_n = Y_{m,1} + Y_{m,2} + ... + Y_{m,Z_m}$.

Then, since $(Y_{m,i})$ are iid with distribution Z_{n-m},

$$\mathbb{E}\left[Z_n|Z_m\right] = \mathbb{E}\left[Y_{m,1} + Y_{m,2} + ... + Y_{m,Z_m}|Z_m\right] = Z_m\mathbb{E}[Z_{m-n}] = Z_m\mu^{n-m}$$

Therefore, by taking out what is known,

$$\mathbb{E}\left[Z_m Z_n\right] = \mathbb{E}\left[\mathbb{E}\left[Z_m Z_n|Z_m\right]\right] = \mathbb{E}\left[Z_m^2\mathbb{E}\left[Z_n|Z_m\right]\right] = \mathbb{E}\left[Z_m^2\mu^{n-m}\right] = \mu^{n-m}\mathbb{E}\left[Z_m^2\right]$$

Observing that $\mathbb{E}[Z_m Z_n] = \mu^{n-m}\mathbb{E}[Z_m^2] = \mu^{n-m}(\mathbb{V}[Z_m] + \mathbb{E}[Z_m]^2) = \mu^{n-m}(\mathbb{V}[Z_m] + \mu^{2m})$, write,

$$\rho(Z_m, Z_n) = \frac{\mathrm{Cov}(Z_n, Z_m)}{(\mathbb{V}[Z_n]\mathbb{V}[Z_m])^{1/2}} = \frac{\mathbb{E}[Z_n Z_m] - \mathbb{E}[Z_n]\mathbb{E}[Z_m]}{(\mathbb{V}[Z_n]\mathbb{V}[Z_m])^{1/2}} = \frac{\mu^{n-m}(\mathbb{V}[Z_m] + \mu^{2m}) - \mu^{n+m}}{(\mathbb{V}[Z_n]\mathbb{V}[Z_m])^{1/2}}$$

Denote $\mathbb{V}[Z_1]$ by σ.

Suppose $\mu = 1$ so that $\mathbb{V}[Z_m] = m\sigma^2$. We use Mathematica to simplify the above expression as,

```
FullSimplify[
 PowerExpand[(\[Mu]^(n - m) (Vzm + \[Mu]^(2 m)) - \[Mu]^(
    n + m))/(Vzn Vzm)^(
   1/2) /. {Vzm -> m \[Sigma]^2, Vzn -> n \[Sigma]^2, \[Mu] ->
      1}],
 Assumptions -> {{m, n, \[Sigma], \[Mu]} > 0}]
```

This yields,

$$\rho(Z_m, Z_n) = \sqrt{\frac{m}{n}}$$

Now suppose $\mu \neq 1$ so that $\mathbb{V}[Z_m] = \sigma^2(\mu^n - 1)\mu^{n-1}/(\mu - 1)$. We use Mathematica to simplify the above expression as,

```
FullSimplify[
 PowerExpand[(\[Mu]^(n - m) (Vzm + \[Mu]^(2 m)) - \[Mu]^(
    n + m))/(Vzn Vzm)^(
   1/2) /. {Vzm -> \[Sigma]^2 (\[Mu]^m - 1) \[Mu]^(m - 1)/(\[Mu] -
       1),
     Vzn -> \[Sigma]^2 (\[Mu]^n - 1) \[Mu]^(n - 1)/(\[Mu] - 1) }],
 Assumptions -> {\[Mu] != 1, {m, n, \[Sigma], \[Mu]} > 0}]
```

This yields,

$$\rho(Z_m, Z_n) = \sqrt{\frac{\mu^n(\mu^m - 1)}{\mu^m(\mu^n - 1)}}$$

Observe that in the limit $\mu \to 1$ this coincides with the previous value.

Exercise 3.3

Solution

Exercise 3.4

Consider a branching process with immigration

$$Z_0 = 1 \qquad\qquad Z_{n+1} = \sum_{i=1}^{Z_n} X_{n,i} + Y_n$$

where the $(X_{n,i})$ are iid with common distribution X, the (Y_n) are iid with common distribution Y, and the $(X_{n,i})$ and (Y_n) are independent. What is $G_{Z_{n+1}}(s)$ in terms of $G_{Z_n}(s)$, $G_X(s)$, and $G_Y(s)$? Write $G_{Z_2}(s)$ explicitly in terms of $G_X(s)$ and $G_Y(s)$.

Solution

Define:

$$G_{Z_n}(s) = s^{Z_n} \qquad\qquad G_X(s) = \mathbb{E}s^X \qquad\qquad G_Y(s) = \mathbb{E}s^Y$$

Write $S_n = \sum_{i=1}^{Z_n} X_{n,i}$ so that, $Z_{n+1} = S_n + Y_n$.
First observe that since the $(X_{n,i})$ are iid with common distribution X,

$$G_{S_n}(s) = \mathbb{E}\left[s^{S_n}\right] = \mathbb{E}\left[\mathbb{E}\left[s^{S_n}|Zn\right]\right] = \mathbb{E}\left[\mathbb{E}[s^X]^{Z_n}\right] = \mathbb{E}\left[G_X(s)^{Z_n}\right] = G_{Z_n}(G_X(s))$$

Since the $(X_{n,i})$ and (Y_n) are independent, S_n and Y_n are independent. Therefore,

$$G_{Z_{n+1}}(s) = G_{S_n+Y_n}(s) = G_{S_n}(s)G_Y(s) = G_{Z_n}(G_X(s))G_Y(s)$$

We calculate,

$$G_{Z_0}(s) = \mathbb{E}\left[s^{Z_0}\right] = \mathbb{E}[s] = s$$

Similarly,

$$G_{Z_1}(s) = G_{Z_0}(G_X(s))G_Y(s) = G_X(s)G_Y(s)$$

Therefore,

$$G_{Z_2}(s) = G_{Z_1}(G_X(s))G_Y(s) = G_X(G_X(s))G_Y(G_X(s))G_Y(s)$$

Exercise 3.5

Find $\phi_{X^2}(t) := \mathbb{E}\exp(itX^2)$ where $X \sim \mathcal{N}(\mu, \sigma)$.

Solution

We have,

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-(x-\mu)^2}{2\sigma^2}\right)$$

Thus,

$$\phi_{X^2}(t) = \mathbb{E}\exp(itX^2) = \int_{-\infty}^{\infty} e^{itx^2} f_X(x) \mathrm{d}x$$

We evaluate with Mathematica as,

```
Integrate[Exp[I t x^2] PDF[NormalDistribution[\[Mu], \[Sigma]], x
   ], {x, -\[Infinity], \[Infinity]},
 Assumptions -> {\[Mu] \[Element] Reals, t \[Element] Reals, \[
    Sigma] > 0}]
```

This yields,

$$\phi_{X^2}(t) = \frac{\exp(it\mu^2/(1-2it\sigma^2))}{\sqrt{1-2it\sigma^2}}$$

Exercise 3.6

Let X_n have cumulative distribution function

$$F_{X_n}(x) = \left(x - \frac{\sin(2n\pi x)}{2n\pi}\right) \mathbb{1}_{0 \le x \le 1} + \mathbb{1}_{x>1}$$

(a) Show that F_{X_n} is a distribution function and find the corresponding density function f_{X_n}.

(b) Show that F_{X_n} converges to the uniform distribution function F_U as $n \to \infty$, but that the density function f_{X_n} does NOT converge to f_U. Here, $U \sim \mathcal{U}((0,1))$.

Solution

(a) Clearly $F_{X_n}(x) = 0$ for $x \le 0$ and $F_{X_n}(x) = 1$ for $x \ge 1$. Observe, $x - \sin(2n\pi x)/2n\pi$ is non-decreasing and continuous on $(0,1)$, since the derivative, calculated below is non-negative on this interval. Moreover, $x - \sin(2n\pi x)/2n\pi$ is equal to zero at $x = 0$, and equal to one at $x = 1$.

Therefore $F_{X_n}(x)$ is a non-decreasing continuous function with $F_{X_n}(x) \to 0$ as $x \to -\infty$ and $F_{X_n}(x) \to 1$ as $x \to \infty$. So $F_{X_n}(x)$ is a distribution function.

It is straightforward to compute the density function as,

$$f_{X_n}(x) = \frac{d}{dx} F_{X_n}(x) = (1 - \cos(2n\pi x)) \mathbb{1}_{0 \le x \le 1}$$

(b) The uniform distribution on $(0,1)$ is given by,

$$F_U(x) = x \mathbb{1}_{0 \le x \le 1} + \mathbb{1}_{x>1}$$

Obviously outside of $(0,1)$ both F_U and F_{X_n} agree exactly. Consider a point $x \in (0,1)$. Then, since $|\sin(u)| \le 1$ for all u,

$$\lim_{n \to \infty} \left[x - \frac{\sin(2n\pi x)}{2n\pi}\right] = x - 0 = x$$

Therefore F_X converges pointwise on to F_U on $(0,1)$, and therefore on all of $\mathbb{R}$.

It is clear that $f_{X_n}(x)$ does not converge to $f_U(x)$ as $f_U(x)$ is constant on $(0,1)$ while $f_{X_n}(x)$ oscillates between zero and two. In particular, fix a rational number $x = p/q$. Then for $n = qk, k \in \mathbb{N}$, $f_{X_n}(x) = 0$.

Exercise 3.7

A coin is tossed repeatedly, with heads turning up with probability p on each toss. Let N be the minimum number of tosses required to obtain k heads. Show that, as $p \to 0$, the distribution function of $2Np$ converges to that of a gamma distribution. Note that, if $X \sim \Gamma(\lambda, r)$ then,

$$f_X(x) = \frac{1}{\Gamma(r)} \lambda^r x^{r-1} e^{-\lambda x} \mathbb{1}_{x \geq 0}$$

Solution

We have $\Gamma(r) = \int_0^\infty x^{r-1} e^{-x} \mathrm{d}x$. Thus, making the substitution $u = (\lambda - it)x$,

$$\begin{aligned}
\phi_X(t) &= \mathbb{E}\left[e^{itx} f_X(x) dx\right] \\
&= \int_0^\infty e^{itx} \frac{1}{\Gamma(r)} \lambda^r x^{r-1} e^{-\lambda x} \mathrm{d}x \\
&= \int_0^\infty \frac{\lambda^r}{\Gamma(r)} e^{-u} \frac{u^{r-1}}{(\lambda - it)^{r-1}} \frac{\mathrm{d}u}{(\lambda - it)} \\
&= \frac{\lambda^r}{\Gamma(r)(\lambda - it)^r} \int_0^\infty e^{-u} u^{r-1} \mathrm{d}u \\
&= \frac{\lambda^r}{(\lambda - it)^r}
\end{aligned}$$

Let $(X_i)_{i=1}^k$ be idd with $X, X_i \sim \mathrm{Geo}(p)$. Then $N = \sum_{i=1}^k X_i$ so, since the X_i are iid,

$$\varphi_{2Np}(t) = \mathbb{E}[\exp(it2Np)] = \mathbb{E}[\exp(2itp(X_1 + ... + X_k))] = \mathbb{E}[\exp(2itpX)]^k$$

Therefore, since $|e^{2itp}(1-p)| < 1$ if $p \in (0, 1)$,

$$\mathbb{E}[\exp(2itpX)]^k = \left[\sum_{m=1}^\infty e^{2itpm} p(1-p)^{m-1}\right]^k = \left[pe^{2itp} \sum_{m=1}^\infty \left(e^{2itp}(1-p)\right)^{m-1}\right]^k = \left[\frac{pe^{2itp}}{1-(1-p)e^{2itp}}\right]^k$$

With Mathematica we evaluate,

```
Limit[((p Exp[2 I t p])/(1 - (1 - p) Exp[2 I t p]))^k, {p -> 0},
sumptions -> {k \[Element] Integers, k > 0}] // FullSimplify
```

This yields,

$$\lim_{p \to 0} \varphi_{2Np} = \frac{1}{(1-2it)^k} = \frac{(1/2)^k}{(1/2 - it)^k}$$

> Thus, for a random variable $X \sim \Gamma(1/2, k)$, by the continuity theorem, $\lim_{p \to 0} f_{2Np}(x) = f_X(x)$

Exercise 4.1

A six-sided die is rolled repeatedly. Which of the following a Markov chains? For those that are, find the one-step transition matrix.

(a) X_n is the largest number rolled up to the nth roll.
(b) X_n is the number of sixes rolled in the first n rolls.
(c) At time n, X_n is the time since the last six was rolled.
(d) At time n, X_n is the time until the next six is rolled.

Solution

(a) Yes.

$$P = \begin{bmatrix} 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ & 2/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ & & 36 & 1/6 & 1/6 & 1/6 \\ & & & 4/6 & 1/6 & 1/6 \\ & & & & 5/6 & 1/6 \\ & & & & & 1 \end{bmatrix}$$

(b) Yes.

$$P = \begin{bmatrix} 5/6 & 1/6 & & \\ & 5/6 & 1/6 & \\ & & \ddots & \ddots \end{bmatrix}$$

(c) Yes. Suppose $X_n = i$. The next roll is either a 6, in which case $X_{n+1} = 0$. Otherwise $X_{n+1} = i + 1$.

$$P = \begin{bmatrix} 1/6 & 5/6 & & & \\ 1/6 & & 5/6 & & \\ 1/6 & & & 5/6 & \\ \vdots & & & & \ddots \end{bmatrix}$$

(d) Yes. Suppose $X_n = 0$. The probability of $X_{n+1} = j$ is $(1/6)(5/6)^j$ as you must not roll a 6 for j turns, and then must roll a 6 on the j-th. Suppose $X_n = i > 0$. Then the next step you will be on turn closer to rolling a 6. That is, $X_{n+1} = i - 1$.

$$P = \begin{bmatrix} \frac{1}{6} & \frac{1}{6}\left(\frac{5}{6}\right) & \frac{1}{6}\left(\frac{5}{6}^2\right) & \frac{1}{6}\left(\frac{5}{6}\right)^3 & \cdots \\ 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & \ddots \end{bmatrix}$$

Exercise 4.2

Let $Y_n = X_{2n}$. Compute the transition matrix for Y when

(a) X is a simple random walk (i.e., X increases by one with probability p and decreases by 1 with probability q)

(b) X is a branching process where G is the generating function of the number of offspring from each individual

Solution

(a) In each step we can go down with probability q and then down again with probability q or up with probability p. Alternatively we can go up with probability p and then down with probability q or up again with probability p.

Therefore we will end up two spaces down with probability q^2, in the same position with probability $qp + pq = 2pq$, or up two spaces with probability p^2. Thus,

$$p(i,j) = \begin{cases} p^2 & j = i+2 \\ 2pq & i = j \\ q^2 & j = i-2 \\ 0 & \text{otherwise} \end{cases}$$

(b) We can obtain the exponents of a generating function $G(s) = a_0 + a_1 s + a_2 s^2 + ...$ by,

$$a_n = \frac{1}{n!}\frac{d^n}{ds^n}\Big[G(s)\Big]_{s=0}$$

The coefficient of the s^k term is the value of the probability mass function of X evaluated at k.

The generating function of Y is $G(G(s)) = G_2(s)$ from the notes.

For a branching process with current population k, the population of the next generation will be $X_1 + X_2 + ... + X_k$, where each X_i is iid with distribution X. Therefore,

$$p(i,j) = \frac{1}{j!}\frac{d^n}{ds^n}\Big[G_2(s)^i\Big]_{s=0}$$

Exercise 4.3

Let X be a Markov chain with state space S and absorbing state k (i.e., $p(k,j) = 0$ for all $j \in S$). Suppose $j \to k$ for all $j \in S$. Show that all states other than k are transient.

Solution

Fix a state $j \in S$. By definition of $j \to k$, $\exists N \geq 0 : p_N(j,k) \geq 0$. Since $\{X_N = k|X_0 = j\} \subseteq \{\forall n, X_n \neq j|X_0 = j\}$ we have,

$$0 < p_N(j,k) = \mathbb{P}(X_N = k|X_0 = j) \leq \mathbb{P}(\forall n, X_n \neq j|X_0 = j)$$

Therefore,

$$\mathbb{P}(\exists n \geq 0 : X_n = j|X_0 = j) = 1 - \mathbb{P}(\forall n, X_n \neq j|X_0 = j) < 1$$

This proves state j istransient. □

Exercise 4.4

Suppose two distinct states i, j satisfy

$$\mathbb{P}(\tau_j < \tau_i | X_0 = i) = \mathbb{P}(\tau_i < \tau_j | X_0 = j)$$

where $\tau_j = \inf\{n \geq 1 : X_n = j\}$. Show that, if $X_0 = i$, the expected value of visits to j prior to returning to i is one.

Solution

Write

$$p = \mathbb{P}(\tau_j < \tau_i | X_0 = i) = \mathbb{P}(\tau_i < \tau_j | X_0 = j)$$

That is, p is the probability that we go to state j before state i give we are in state i, and p is also the probability that we go to state i before state j given we are in state j.

Then $1 - p$ is the probability that we do not go to state i before returning state j,0 given we start in state j.

So $(1 - p)^k$ is the probability that we return to state j exactly k times before moving to state i, given we start in state j.

Let N be the number of visits to j prior to returning to i given we start in state i.

The probability that $N = k \in \mathbb{Z}_{\geq 0}$ is the probability that starting from state i we go to state j, return to state j $(k - 1)$ times without returning to state i, and then return to state i without going to returning to state j.

So $\mathbb{P}(N = k | X_0 = i) = p(1 - p)^{k-1} p$. This is the probability mass function for N so,

$$\mathbb{E}[N] = \sum_{n=0}^{\infty} np^2(1 - p)^{k-1} = p \sum_{n=0}^{\infty} n(1 - p)^n = p \frac{p}{(1 - (1 - p))^2} = 1$$

Exercise 4.5

Let X be a Markov chain with transition matrix,

$$P = \begin{bmatrix} 1-2p & 2p & 0 \\ p & 1-2p & p \\ 0 & 2p & 1-2p \end{bmatrix}, \qquad p \in (0,1)$$

Find P^n, the invariant distribution π, and the mean-recurrence times $\overline{\tau}_j$ for $j = 1,2,3$.

Solution

Note that P has eigendecomposition $P = V\Lambda V^{-1}$ where,

$$\Lambda = \begin{bmatrix} 1 & & \\ & 1-4p & \\ & & 1-2p \end{bmatrix}, \qquad V = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Therefore, $P^n = V\Lambda^n V^{-1}$. Explicitly,

$$P^n = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1-4p & \\ & & 1-2p \end{bmatrix} \begin{bmatrix} 1/4 & 1/2 & 1/4 \\ 1/4 & -1/2 & 1/4 \\ -1/2 & 0 & 1/2 \end{bmatrix}$$

Invariant distributions are linear combinations of left eigenvectors corresponding to eigenvalues of 1. In this case that is the first row of V^{-1}. That is,

$$\pi = \begin{bmatrix} \frac{1}{4} & \frac{1}{2} & \\ \frac{1}{4} & & \end{bmatrix}$$

Finally, since the invariant distribution is unique, by Theorem we have,

$$\overline{\tau}_i = \frac{1}{\pi(i)}$$

Exercise 4.6

Let X_n be the number of mistakes in the n-th addition of a book. Between the n-th and the $(n+1)$-th addition an editor corrects each mistake independently with probability p and introduces Y_n new mistakes where the(Y_n) are iid and Poisson distributed with parameter λ. Find the invariant distribution π of the number of mistakes in the book.

Solution

Let $M_{n,k}$ be distributed as $\mathrm{Ber}(1-p)$ so that M_k is 0 if this mistake is corrected, and 1 otherwise. Let Y_n be Poisson distributed with parameter λ. Then,

$$X_{n+1} = Y_n + \sum_{k-1}^{X_n} M_k$$

Each $M_{n,k}$ has generating function,

$$G_{M_{n,k}} = p + (1-p)s = 1 - q + qs = 1 - q(1-s)$$

Similarly. Y_n has generating function,

$$G_Y(s) = \sum_{k=0}^{\infty} e^{-\lambda}\lambda^k/k!s^k = e^{-\lambda}e^{s\lambda} = e^{\lambda(s-1)}$$

Therefore X_{n+1} has generating function,

$$\begin{aligned}
G_{n+1}(s) &= G_Y(s)\mathbb{E}\left[s^{M_{k,1}+M_{k,2}+\ldots+M_{k,X_n}}\right] \\
&= G_Y(s)\mathbb{E}\left[\mathbb{E}\left[s^{M_{k,1}+M_{k,2}+\ldots+M_{k,X_n}}\right]|X_n\right] \\
&= G_Y(s)\mathbb{E}\left[(1-q(1-s))^{X_n}\right] \\
&= G_Y(s)G_n(1-q(1-s))
\end{aligned}$$

First observe $1 - q^i(1-(1-q(1-s))) = 1 - q^{i+1}(1-s)$. We now use the relation

$G_{n+1}(s) = G_Y(s)G_n(1 - q(1 - s))$ and the fact that $G_0(s) = 1$ to calculate,

$$\begin{aligned}
G_{n+1}(s) &= G_Y(s)G_n(1 - q(1 - s)) \\
&= G_Y(s)G_Y(1 - q(1 - s))G_{n-1}(1 - q^2(1 - s)) \\
&= G_Y(s)G_Y(1 - q(1 - s))G_Y(1 - q^2(1 - s))G_{n-2}(1 - q^3(1 - s)) \\
&\vdots \\
&= \prod_{i=0}^{n} G_Y(1 - q^i(1 - s))
\end{aligned}$$

Then,

$$\begin{aligned}
\lim_{n\to\infty} G_n(s) &= \lim_{n\to\infty} G_{n+1}(s) \\
&= \lim_{n\to\infty} \prod_{i=0}^{n} G_Y(1 - q^i(1 - s)) \\
&= \lim_{n\to\infty} \prod_{i=0}^{n} \exp\left(\lambda(-q^i(1 - s))\right) \\
&= \exp\left(\sum_{i=0}^{\infty} \lambda(-q^i(1 - s)\right) \\
&= \exp\left(\lambda(s - 1)\frac{1}{1 - q}\right) \\
&= \exp\left(\frac{\lambda}{p}(s - 1)\right)
\end{aligned}$$

Thus, $G_n(S)$ converges to the generating function of a Poisson random variable with parameter λ/p.

Then X_n converges to a random variable distributed like a Poisson random variable with parameter λ/p. The random variable for which X_n converges to must be the variable corresponding to the stationary distribution. Therefore, the stationary distribution is distributed like the probability mass function of this random variable. That is,

$$\pi(k) = e^{-\lambda/p}\frac{(\lambda/p)^k}{k!}$$

In the limit $p \to 1$, where we correct all mistakes, the stationary distribution looks like a Poisson distribution with parameter λ. In the limit $\lambda \to 0$ so we do not make any new mistakes, $\pi(0) \to 1$ as expected.

Exercise 4.7

Give an example of a transition matrix P that admits multiple stationary distributions π.

Solution

Define P to be the identity matrix. Then any distribution is a stationary distribution.

Exercise 4.8

A Markov chain on $S = \{0, 1, 2, ..., n\}$ has transition probabilities $p(0, 0) = 1 - \lambda_0$, $p(i, i+1) = \lambda_i$ and $p(i+1, i) = \mu_{i+1}$ for $i = 0, 1, ..., n-1$, and $p(n, n) = 1 - \mu_n$. Show that the process is reversible in equilibrium.

Solution

We assume all entries not specified are zero. (I heard this is the intent, however I wonder why we are given μ_j when $\mu_j = 1 - \lambda_j$ for all j). We write the matrix P as,

Write $\mu_n = 1 - \lambda_n$. Thus, $\mu_i = 1 - \lambda_i$ for $i = 1, ..., n$ as the sum of each row must be 1 (making the assumption that all entries not specified at zero).

$$P = \begin{bmatrix} 1-\lambda_0 & \lambda_0 & & & & \\ \mu_1 & & \lambda_1 & & & \\ & \mu_2 & & \lambda_2 & & \\ & & \mu_3 & & & \\ & & & & & \lambda_{n-1} \\ & & & & \mu_n & 1-\mu_n \end{bmatrix} = \begin{bmatrix} 1-\lambda_0 & \lambda_0 & & & & \\ 1-\lambda_1 & & \lambda_1 & & & \\ & 1-\lambda_2 & & \lambda_2 & & \\ & & 1-\lambda_3 & & & \\ & & & & & \lambda_{n-1} \\ & & & & 1-\lambda_n & \lambda_n \end{bmatrix}$$

This chain is irreducible and finite so a unique invariant distribution π exists. Write $\pi = [\pi_0, \pi_1, ..., \pi_n]$. Then $\pi P = \pi$. That is,

$$\pi P = \begin{bmatrix} \pi_0(1-\lambda_0) + \pi_1(1-\lambda_1) \\ \pi_0\lambda_0 + \pi_2(1-\lambda_2) \\ \pi_1\lambda_1 + \pi_3(1-\lambda_3) \\ \vdots \\ \vdots \\ \pi_{n-1}\lambda_{n-1} + \pi_n\lambda_n \end{bmatrix}^T = \begin{bmatrix} \pi_0 \\ \pi_1 \\ \pi_2 \\ \vdots \\ \pi_j \\ \vdots \\ \pi_n \end{bmatrix}^T$$

$$\begin{aligned}
\pi_1 &= \lambda_0\pi_0/(1-\lambda_1) & \lambda_0\pi_0 &= \pi_1(1-\lambda_1) \\
\pi_2 &= (\pi_1 - \pi_0\lambda_0)/(1-\lambda_2) = \pi_1\lambda_1/(1-\lambda_2) & \lambda_1\pi_1 &= \pi_2(1-\lambda_2) \\
\pi_3 &= (\pi_2 - \pi_1\lambda_1)/(1-\lambda_3) = \pi_2\lambda_2/(1-\lambda_3) & \lambda_2\pi_2 &= \pi_3(1-\lambda_3) \\
&\vdots & & \\
\pi_{j+1} &= (\pi_j - \pi_{j-1}\lambda_{j-1})/(1-\lambda_{j+1}) = \pi_j\lambda_j/(1-\lambda_{j+1}) & \lambda_j\pi_j &= \pi_{j+1}(1-\lambda_{j+1}) \\
&\vdots & & \\
\pi_n &= (\pi_{n-1}\lambda_{n-1})/(1-\lambda_n) & \pi_{n-1}\lambda_{n-1} &= \pi_n(1-\lambda_n)
\end{aligned}$$

Observing the equations on the right hand side we have that for $i = 1, 2, ..., n-1$,

$$\pi_i p(i, i+p) = \pi_{i+1} p(i+1, i)$$

We now show the detail balance condition. In particular, we must show,

$$\pi_i p(i, j) = \pi_j p(j, i) \qquad \text{for all } i, j$$

However, for $j \notin \{i-1, i+1\}$ we have $p(i, j) = 0$. Therefore, for this matrix the previous condition is equivalent to

$$\pi_i p(i, i+1) = \pi_{i+1} p(i+1, i) \qquad \text{for } i = 1, 2, ..., n-1$$

We have shown that these equations hold for all $i = 1, 2, ..., n-1$.

This proves π is in detailed balance with P, and so this process is reversible in equilibrium. $\square$

Exercise 5.1

Patients arrive at an emergency room as a Poisson process with intensity λ. The time to treat each patient is an independent exponential random variable with parameter μ. Let $X = (X_t)_{t\geq 0}$ be the number of patients in the system (either being treated or waiting). Write down the generator of X. Show that X has an invariant distribution π if and only if $\lambda < \mu$. Find π. What is the total expected time (waiting + treatment) a patient waits when the system is in its invariant distribution?

Solution

In some small time interval s there is probability $\lambda s + \mathcal{O}(s^2)$ that a patient arrives, probability $1 - \lambda s + \mathcal{O}^2$ that a patient does not arrive, and probability $\mathcal{O}(s^2)$ that multiple patients arrive.

If there are patients, in this times there is also probability $\mu s + \mathcal{O}(s^2)$ that a patient is treated, probability $1 - \mu s + \mathcal{O}(s^2)$ that a patient is not treated, and probability $\mathcal{O}(s^2)$ that more than one (if possible) patients are treated.

Note that any moves which have more than one transition such as a patient arriving, and a patient being treated are all $\mathcal{O}(s^2)$.

Suppose there are no patients at time t. The probability of transitioning to j patients after a short time s is given by,

$$\mathbb{P}(X_{t+s} = j|X_t = 0) = \begin{cases} \lambda s + \mathcal{O}(s^2) & j = 1 \\ 1 - \lambda s + \mathcal{O}(s^2) & j = 0 \\ \mathcal{O}(s^2) & \text{otherwise} \end{cases}$$

Now suppose there are $i > 0$ patients at time t. The probability of transitioning to j patients after a short time s is given by,

$$\mathbb{P}(X_{t+s} = j|X_t = i) = \begin{cases} (\lambda s + \mathcal{O}(s^2))(1 - \mu s + \mathcal{O}(s^2)) & j = i + 1 \\ (1 - \lambda s + \mathcal{O}(s^2))(1 - \mu s + \mathcal{O}(s^2)) + \mathcal{O}(s^2) & j = i \\ (1 - \lambda s + \mathcal{O}(s^2))(\mu s + \mathcal{O}(s^2)) & j = i - 1 \\ \mathcal{O}(s^2) & \text{otherwise} \end{cases}$$

This is simplified as,

$$\mathbb{P}(X_{t+s} = j|X_t = i) = \begin{cases} \lambda s + \mathcal{O}(s^2) & j = i + 1 \\ 1 - \lambda s - \mu s + \mathcal{O}(s^2) & j = i \\ \mu s + \mathcal{O}(s^2) & j = i - 1 \\ \mathcal{O}(s^2) & \text{otherwise} \end{cases}$$

This gives,

$$G = \begin{bmatrix} -\lambda & \lambda & & & & \\ \mu & -(\lambda+\mu) & \lambda & & & \\ & \mu & -(\lambda+\mu) & \lambda & & \\ & & \mu & -(\lambda+\mu) & \lambda & \cdots \\ & & & \vdots & \vdots & \ddots \end{bmatrix}$$

We recognize this as a birth-death process (a bit ironic in the context of an emergency room) with $\lambda_i = \lambda$ and $\mu_i = \mu$.

Then if a stationary distribution π exists, for $n \in \mathbb{Z}_{>0}$,

$$\pi(n > 0) = \left(\frac{\lambda}{\mu}\right)^n \pi(0)$$

and

$$\pi(0) = \left(1 + \sum_{n=1}^{\infty} \left(\frac{\lambda}{\mu}\right)^n\right)^{-1} = \left(\sum_{n=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^n\right)^{-1}$$

This is a geometric series which is convergent exactly when $\lambda/\mu < 1$. That is, when $\lambda < \mu$. In this case,

$$\pi(0) = \left(\sum_{n=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^n\right)^{-1} = \left(\frac{\mu}{\mu-\lambda}\right)^{-1} = \frac{\mu-\lambda}{\mu}$$

We condition on knowing the number of people on the queue. Suppose there are n people in the queue when a patient arrives. Then the patient will have to wait a random time distributed as the sum of n exponential random variables with parameter μ to be treated and one more to finish treatment. The expectation of each of each exponential random variable is $1/\mu$, so the patient waits an expected time of $(n+1)/\mu$.

In equilibrium, the probability that there are n people in the queue when a patient arrives is $\pi(n)$.

Therefore, the expected wait time is,

$$\sum_{n=0}^{\infty} \pi(n)\frac{(n+1)}{\mu} = \frac{\mu-\lambda}{\mu^2}\sum_{n=0}^{\infty}\left(\frac{\lambda}{\mu}\right)^n (n+1) = \frac{\mu-\lambda}{\mu^2}\left(\frac{\mu\lambda}{(\mu-\lambda)^2} + \frac{\mu}{\mu-\lambda}\right) = \frac{1}{\mu-\lambda}$$

Exercise 5.2

Let $X = (X_t)_{t\geq 0}$ be a Markov chain with stationary distribution π. Let N be an independent Poisson process with intensity λ and denote by τ_n the time of the n-th arrival of N. Define $Y_n := X_{\tau_n +}$ (i.e., Y_n is the value of X immediately after the n-th jump). Show that Y is a discrete time Markov chain with the same stationary distribution as X.

It is obvious that Y is Markov, as given the present, the future is independent of the past. We add a bit more rigor below.

Fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. By hypothesis X_t is a Markov process. That is, for a filtration $(\mathcal{F}_s)_{s\in[0,T]}$, for $0 \leq s \leq t \leq T$, and for every non-negative Borel measurable function f,

$$\mathbb{E}[f(X_t)|\mathcal{F}_s] = \mathbb{E}[f(X_t)|X_s]$$

Let $\mathcal{F}'_n = \mathcal{F}_{\tau_n+}$ be a sub-σ-algebra of $\mathcal{F}$. Then clearly $(\mathcal{F}'_n)$ is a filtration. Let f be any non-negative Borel measurable function. Then,

$$\mathbb{E}[f(Y_n)|\mathcal{F}'_m] = \mathbb{E}[f(X_{\tau_n+})|\mathcal{F}_{\tau_m+}] = \mathbb{E}[f(X_{\tau_n+})|X_{\tau_m+}] = \mathbb{E}[f(Y_n)|Y_m]$$

This means Y is Markov, and clearly Y is discrete time. Therefore Y is a discrete time Markov chain.

Note we assume X is time homogeneous.

Suppose X has stationary distribution π. Then for all $0 \leq t \leq T$, $\pi P_t = \pi$, where,

$$(P_t)_{i,j} = \mathbb{P}(X_t = j | X_0 = i)$$

Thus, the one step probability transition matrix, denoted $\tilde{P}$, for Y is,

$$\tilde{P}_{i,j} = \mathbb{P}(Y_1 = j|Y_0 = i) = \mathbb{P}(X_{\tau_1+} = j|X_0 = i) = (P_{\tau_1})_{i,j}$$

This means $\pi\tilde{P} = \pi$, so π is a stationary distribution of Y.

Exercise 5.3

Let $X = (X_t)_{t\geq 0}$ be a Markov chain with state space $S = \{0, 1, 2, ...\}$ and generator G whose i-th row has entries

$$g_{i,i-1} = i\mu \qquad g_{i,i} = -i\mu - \lambda \qquad g_{i,i+1} = \lambda,$$

with all other entries being zero (the zeroth row has only two entries: $g_{0,0}$ and $g_{0,1}$). Assume $X_0 = j$. Find $G_{X_t}(s) := \mathbb{E}s^{X_t}$. What is the distribution of X_t as $t \to \infty$?

Solution

We have G in matrix form,

$$G = \begin{bmatrix} -\lambda & \lambda & & & & \\ \mu & -(\mu+\lambda) & \lambda & & & \\ & 2\mu & -(2\mu+\lambda) & \lambda & & \\ & & 3\mu & -3(\mu+\lambda) & \lambda & \cdots \\ & & & \vdots & \vdots & \ddots \end{bmatrix}$$

We wish to find the transition semi group P_t. We know this can be derived from the Kolmogorov forward equations. That is,

$$\frac{d}{dt}P_t = P_tG$$

With the assumption that $X_0 = i$ (*I am using i rather than j like the problem statement since this is the standard way of doing things*) we have,

$$\frac{d}{dt}p_t(i,0) = \sum_{k=0}^{\infty} p_t(i,k)g(k,0) = -\lambda p_t(i,0) + \mu p_t(i,1)$$
$$\frac{d}{dt}p_t(i,j) = \sum_{k=0}^{\infty} p_t(i,k)g_t(k,j) = \lambda p_t(i,j-1) - (j\mu+\lambda)p_t(i,j) + (j+1)\mu p_t(i,j+1) \qquad j \geq 1$$

We multiply the j-th equation by s^j. This gives,

$$\sum_{j=0}^{\infty} \frac{\partial}{\partial t} p_t(i,j)s^j = \sum_{j=1}^{\infty}\left[\lambda p_t(i,j-1)s^j\right] - \sum_{j=0}^{\infty}\left[(j\mu-\lambda)p_t(i,j)s^j\right] + \sum_{j=0}^{\infty}\left[(j+1)\mu p_t(i,j+1)s^j\right]$$

Summing the left hand sides gives,

$$\sum_{j=0}^{\infty} \frac{\partial}{\partial t} p_t(i,j) s^j = \frac{\partial}{\partial t} \sum_{j=0}^{\infty} p_t(i,j) s^j = \frac{\partial}{\partial t} G_{X_t}(s)$$

The first term of the right hand side gives,

$$\sum_{j=1}^{\infty} \lambda p_t(i,j-1) s^j = \lambda s \sum_{j=1}^{\infty} p_t(i,j-1) s^{j-1} = \lambda s \sum_{j=0}^{\infty} p_t(i,j) s^j = \lambda s G_{X_t}(s)$$

The negative of the first part of the second term of the right hand side gives,

$$\sum_{j=0}^{\infty} j \mu p_t(i,j) s^j = s\mu \sum_{j=0}^{\infty} j p_t(i,j) s^{j-1} = s\mu \sum_{j=0}^{\infty} \frac{\partial}{\partial s} p_t(i,j) s^j = s\mu \frac{\partial}{\partial s} \sum_{j=0}^{\infty} p_t(i,j) s^j = s\mu \frac{\partial}{\partial s} G_{X_t}(s)$$

The negative of the second part of the second term of the right hand side gives,

$$\sum_{j=0}^{\infty} \lambda p_t(i,j) s^j = \lambda \sum_{j=0}^{\infty} p_t(i,j) s^j = \lambda G_{X_t}(s)$$

The third term of the right hand side gives,

$$\sum_{j=1}^{\infty} (j+1) \mu p_t(i,j+1) s^j = \mu \sum_{j=1}^{\infty} \frac{\partial}{\partial s} p_t(i,j+1) s^{j+1} = \mu \frac{\partial}{\partial s} \sum_{j=0}^{\infty} p_t(i,j) s^j = \mu \frac{\partial}{\partial s} G_{X_t}(s)$$

Putting these results together we have,

$$\frac{\partial}{\partial t} G_{X_t}(s) = \left[\lambda s - s\mu \frac{\partial}{\partial s} - \lambda + \mu \frac{\partial}{\partial s} \right] G_{X_t}(s)$$

Since $X_0 = j$ we have initial condition,

$$G_{X_0}(s) = s^j$$

We solve with Mathematica by,

```
DSolve[{
    D[G[s,t],t]==\[Lambda] s G[s,t]-s \[Mu] D[G[s,t],s]-\[Lambda]
       G[s,t]+\[Mu] D[G[s,t],s],
    G[s,0]==s^j
    },G[s,t],{s,t}]//FullSimplify
```

This yields,

$$G_{X_t}(s) = \left((s-1)e^{-\mu t} + 1\right)^j \exp\left[\frac{\lambda(s-1)e^{\mu(-t)}\left(e^{\mu t}-1\right)}{\mu}\right]$$

We find the limit as $t \to \infty$ with Mathematica by,

```
Limit[E^((E^(-t \[Mu]) (-1+E^(t \[Mu])) (-1+s) \[Lambda])/\[Mu])
   (1+E^(-t \[Mu]) (-1+s))^j,{t->\[Infinity]},Assumptions->{\[
   Lambda]>0,\[Mu]>0}]
```

This yields,

$$G_{X_\infty}(s) = \lim_{t\to\infty} G_{X_t}(s) = e^{\frac{\lambda}{\mu}(s-1)}$$

So $X_\infty = \lim_{t\to\infty} X_t$ is a Poission random variable with parameter λ/μ.

Exercise 5.4

Let N be a time-inhomogeneous Poisson process with intensity function $\lambda(t)$. That is, the probability of a jump of size one in the time interval $(t, t+dt)$ is $\lambda(t)dt$ and the probability of two jumps in that interval of time is $\mathcal{O}(dt^2)$. Write down the Kolmogorov forward and backward equations of N and solve them. Let $N_0 = 0$ and let τ_1 be the time of the first jump of N. If $\lambda(t) = c/(1+t)$ show that $\mathbb{E}\tau_1 < \infty$ if and only if $c > 1$.

Solution

Based on the definition of the generator and the given transition probabilities we have,

$$G(t) = \begin{bmatrix} -\lambda(t) & \lambda(t) & & & \\ & -\lambda(t) & \lambda(t) & & \\ & & -\lambda(t) & \lambda(t) & \cdots \\ & & \vdots & \vdots & \ddots \end{bmatrix}$$

For $t \geq s$ we define,

$$p_{s,t}(i,j) = \mathbb{P}(N_t = j | N_s = i)$$

We first derive the Kolmogorov forward equations. We consider,

$$\begin{aligned} p_{s,t+\Delta t} &= \mathbb{P}(N_{t+\Delta t} = j | N_s = i) \\ &= \sum_k \mathbb{P}(N_{t+\Delta t} = j | N_t = k)\mathbb{P}(N_t = k | N_s = i) \\ &= \begin{cases} \lambda(t)\Delta t p_{s,t}(i, j-1) + (1 - \lambda(t)\Delta t) p_{s,t}(i,j) + \mathcal{O}(\Delta t^2) & j > i \\ (1 - \lambda(t)\Delta t) p_{s,t}(i,j) + \mathcal{O}(\Delta t^2) & j = i \\ 0 & j < i \end{cases} \end{aligned}$$

Therefore,

$$\frac{p_{s,t+\Delta t}(i,j) - p_{s,t}(i,j)}{\Delta t} = \begin{cases} \lambda(t)\Delta t p_{s,t}(i, j-1) - \lambda(t)\Delta t p_{s,t}(i,j) + \mathcal{O}(\Delta t^2) & j > i \\ -\lambda(t)\Delta t p_{s,t}(i,j) + \mathcal{O}(\Delta t^2) & j = i \\ 0 & j < i \end{cases}$$

Taking the limit as $\Delta t \to 0$ we have,

$$\frac{\partial}{\partial t} p_{s,t}(i,j) = \begin{cases} \lambda(t)p_{s,t}(i,j-1) - \lambda(t)p_{s,t}(i,j) & j > i \\ -\lambda(t)p_{s,t}(i,j) & j = i \\ 0 & j < i \end{cases}$$

Fix i. Noting that $G_F(x)$ is also a function of s, t and j, we have,

$$G_F(x) = \sum_{j=0}^{\infty} \mathbb{P}(N_t = j | N_s = i)x^j = \sum_{j=i}^{\infty} p_{s,t}(i,j)x^j$$

Thus, multiplying the j-th KFE by x^j and summing, we have,

$$\begin{aligned} \frac{\partial}{\partial t} \sum_{j=i}^{\infty} p_{s,t}(i,j)x^j = \sum_{j=i}^{\infty} \frac{\partial}{\partial t} p_{s,t}(i,j)x^j &= \sum_{j=i+1}^{\infty} \lambda(t)p_{s,t}(i,j-1)x^j + \sum_{j=i}^{\infty} (-\lambda(t))p_{s,t}(i,j)x^j \\ &= \lambda(t)x \sum_{j=i}^{\infty} p_{s,t}(i,j)x^j - \lambda(t) \sum_{j=i}^{\infty} p_{s,t}(i,j)x^j \end{aligned}$$

Therefore,

$$\frac{\partial}{\partial t} G_F(x) = \lambda(t)xG_F(x) - \lambda(t)G_F(x) = \lambda(t)(x-1)G_F(x)$$

We have initial condition $N_s = i$, so $G_B(x) = x^i$ when $s = t$.

We solve with Mathematica as,

```
DSolve[{D[G[s, t], t] == \[Lambda][t] (x - 1) G[s, t],
   G[s, s] == x^i
   }, G[s, t], {s, t}] // FullSimplify
```

This gives,

$$G_F(x) = x^i \exp\left((x-1) \int_s^t \lambda(z) \mathrm{d}z\right)$$

Write $I = \int_s^t \lambda(z)dz$. Then,

$$G_F(x) = e^{-I}x^i e^{Ix} = e^{-I}x^i \sum_{k=0}^{\infty} \frac{1}{k!}(Ix)^k = e^{-I} \sum_{k=0}^{\infty} \frac{1}{k!} I^k x^{k+i} = e^{-I} \sum_{j=i}^{\infty} \frac{I^{j-i}}{(j-i)!} x^j$$

Therefore, from the definition of the Generating function we have,

$$P_{s,t}(i,j) = \mathbb{P}(N_t = j | N_s = i) = \frac{1}{(j-i)!}\left[\int_s^t \lambda(z)\mathrm{d}z\right]^{j-i} \exp\left(-\int_s^t \lambda(z)\mathrm{d}z\right)$$

We now derive the Kolmogorov Backward equations. We consider,

$$\begin{aligned}
p_{s-\Delta s,t} &= \mathbb{P}(N_t = j | N_{s-\Delta s} = i) \\
&= \sum_k \mathbb{P}(N_t = j | N_s t = k)\mathbb{P}(N_s = k | N_{s-\Delta s} = i) \\
&= \begin{cases} \lambda(s)\Delta s p_{s,t}(i+1,j) + (1-\lambda(s)\Delta s)p_{s,t}(i,j) + \mathcal{O}(\Delta s^2) & j > i \\ (1-\lambda(s)\Delta s)p_{s,t}(i,j) + \mathcal{O}(\Delta s^2) & j = i \\ 0 & j < i \end{cases}
\end{aligned}$$

Therefore,

$$\frac{p_{s-\Delta s,t}(i,j) - p_{s,t}(i,j)}{\Delta s} = \begin{cases} \lambda(s)\Delta t p_{s,t}(i+1,j) - \lambda(s)\Delta t p_{s,t}(i,j) + \mathcal{O}(\Delta s^2) & j > i \\ -\lambda(s)\Delta t p_{s,t}(i,j) + \mathcal{O}(\Delta s^2) & j = i \\ 0 & j < i \end{cases}$$

Taking the limit as $\Delta s \to 0$ we have,

$$-\frac{\partial}{\partial s}p_{s,t}(i,j) = \begin{cases} \lambda(s)p_{s,t}(i+1,j) - \lambda(s)p_{s,t}(i,j) & j > i \\ -\lambda(s)p_{s,t}(i,j) & j = i \\ 0 & j < i \end{cases}$$

Fix i. Noting that $G_B(x)$ is also a function of s, t and j, we have,

$$G_B(x) = \sum_{j=0}^{\infty} \mathbb{P}(N_t = j | N_s = i)x^j = \sum_{j=i}^{\infty} p_{s,t}(i,j)x^j$$

Thus, multiplying the j-th KBE by x^j and summing, we have,

$$
\begin{aligned}
-\frac{\partial}{\partial s}\sum_{j=i}^{\infty} p_{s,t}(i,j)x^j = -\sum_{j=i}^{\infty}\frac{\partial}{\partial s}p_{s,t}(i,j)x^j &= \sum_{j=i+1}^{\infty}\lambda(s)p_{s,t}(i+1,j)x^j + \sum_{j=i}^{\infty}(-\lambda(s))p_{s,t}(i,j)x^j \\
&= \sum_{j=i+1}^{\infty}\lambda(s)p_{s,t}(i,j-1)x^j + \sum_{j=i}^{\infty}(-\lambda(s))p_{s,t}(i,j)x^j \\
&= \lambda(s)x\sum_{j=i}^{\infty}p_{s,t}(i,j)x^j - \lambda(s)\sum_{j=i}^{\infty}p_{s,t}(i,j)x^j
\end{aligned}
$$

Therefore,

$$
\frac{\partial}{\partial s}G_B(x) = -\lambda(s)xG_B(x) + \lambda(s)G_B(x) = -\lambda(s)(x-1)G_B(x)
$$

From the result for $G_F(x)$ we know,

$$
G_B(x) = x^i\exp\left(-(x-1)\int_t^s \lambda(z)\mathrm{d}z\right) = x^i\exp\left((x-1)\int_s^t \lambda(z)\mathrm{d}z\right) = G_F(x)
$$

We now show that for $\lambda(t) = c/(1+t)$, that $\mathbb{E}\tau_1 < \infty$ if and only if $c < 1$. Indeed,

$$
\int_0^t \lambda(z)\mathrm{d}z = \int_0^t \frac{c}{1+z}\mathrm{d}z = c\ln(1+t) - c\ln(1) = c\ln(1+t)
$$

Therefore,

$$
\mathbb{E}[\tau_1] = \int_0^\infty \mathbb{P}(\tau_1 > t)\mathrm{d}t = \int_0^\infty \mathbb{P}(N_t = 0|N_0 = 0)\mathrm{d}t = \int_0^\infty \exp(-c\ln(1+t))\mathrm{d}t = \int_0^\infty \frac{\mathrm{d}t}{(1+t)^c}
$$

This is convergent if and only if $c > 1$.

Exercise 5.5

Let N_t be a Poisson process with a random intensity Λ which is equal to λ_1 with probability p and λ_2 with probability $1-p$. Find $G_{N_t}(s) = \mathbb{E}s^{N_t}$. What is the mean and variance of N_t?

Solution

Recall the generating function for a Poisson process with intensity λ is,

$$G(s) = e^{-\lambda t(1-s)}$$

Therefore,

$$G_{N_t}(s) = \mathbb{E}\left[s^{N_t}\right] = \mathbb{E}\left[\mathbb{E}\left[s^{N_t}\right]\middle|\Lambda\right] = \mathbb{E}\left[e^{-\Lambda t(1-s)}\middle|\Lambda\right] = pe^{-\lambda_1 t(1-s)} + (1-p)e^{-\lambda_2(1-s)}$$

We use Mathematica to caluculate moments,

```
GNt[s_]:=p Exp[-\[Lambda]1 t (1-s)]+(1-p)Exp[-\[Lambda]2 t(1-s)]
D[GNt[s],{s,1}]/.{s->1}
D[GNt[s],{s,2}]-D[GNt[s],{s,1}]^2+D[GNt[s],{s,1}]/.{s->1}
```

This yields,

$$\mu = G'_{N_t}(1) = p\lambda_1 t + (1-p)\lambda_2 t$$
$$\sigma^2 = G''_{N_t}(1) - [G'_{N_t}(1)]^2 + G'_{N_t}(1) = p(\lambda_1 t)^2 + (1-p)(\lambda_2 t)^2 - \mu^2 + \mu$$

Exercise 7.1

Let W be a Brownian motion and let $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ be a filtration for W. Show that $W(t)^2 - t$ is a martingale with respect to the filtration $\mathbb{F}$.

Solution

Suppose $X \sim \mathcal{N}(0, \sigma^2)$. Then,

$$\sigma^2 = \mathbb{V}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \mathbb{E}[X^2] - 0^2 = \mathbb{E}[X^2]$$

Let $0 \leq s \leq t$. By the definition of a filtration, $(W(t) - W(s))$ is independent of $\mathcal{F}_s$. Moreover, by the definition of Brownian Motion we have $W(t) - W(s) \sim \mathcal{N}(0, t-s)$. Thus,

$$\mathbb{E}\left[(W(t) - W(s))^2 \middle| \mathcal{F}_s\right] = \mathbb{E}\left[(W(t) - W(s))^2\right] = (t-s)$$

Since $W(s) \in \mathcal{F}_s$, by "taking out what is known" we have,

$$\begin{aligned}
\mathbb{E}\left[W(t)W(s) \middle| \mathcal{F}_s\right] &= W(s)\mathbb{E}\left[W(t) \middle| \mathcal{F}_s\right] = W(s)W(s) = W(s)^2 \\
\mathbb{E}\left[W(s)^2 \middle| \mathcal{F}_2\right] &= W(s)\mathbb{E}\left[W(s) \middle| \mathcal{F}_2\right] = W(s)W(s) = W(s)^2
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathbb{E}\left[W(t)^2 - t \middle| \mathcal{F}_s\right] &= \mathbb{E}\left[(W(t) - W(s) + W(s))^2 - t\right] \\
&= \mathbb{E}\left[(W(t) - W(s))^2 + 2(W(t) - W(s))W(s) + W(s)^2 - t\right] \\
&= \mathbb{E}\left[(W(t) - W(s))^2 \middle| \mathcal{F}_s\right] + 2\mathbb{E}\left[W(t)W(s) \middle| \mathcal{F}_s\right] - \mathbb{E}\left[W(s)^2 \middle| \mathcal{F}_2\right] - \mathbb{E}[t] \\
&= (t-s) + 2W(s)^2 - W(s)^2 - t \\
&= W(s)^2 - s
\end{aligned}$$

This proves $W(t) - t$ is a martingale with respect to the filtration $\mathbb{F}$. $\square$

Exercise 7.2

Compute the characteristic function of $W(N(t))$ where N is a Poisson process with intensity λ and the Brownian motion W is independent of the Poisson process N.

Solution

The characteristic function is defined as,

$$\phi(s) = \mathbb{E}e^{isW(N(t))}$$

We condition on $N(t)$ using iterated conditioning,

$$\mathbb{E}\left[e^{isW(N(t))}\right] = \mathbb{E}\left[\mathbb{E}\left[e^{isW(N(t))}\middle|N(t)\right]\right]$$

The characteristic function of $Z \sim \mathcal{N}(\mu, \sigma^2)$ is $\phi_Z(s) = \exp(i\mu s - \sigma^2 s^2/2)$. At time t, $W(t)$ is normally distributed with mean zero and variance t. Thus,

$$\mathbb{E}\left[\mathbb{E}\left[e^{isW(N(t))}\middle|N(t)\right]\right] = \mathbb{E}\left[e^{-N(t)s^2/2}\right]$$

Since $N(t)$ is a Poisson process with parameter λ, then $N(t) = k$ with probability $(\lambda t)^k e^{-\lambda t}/k!$. Thus,

$$\mathbb{E}\left[e^{-N(t)s^2/2}\right] \sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!} e^{-\lambda t} e^{-ks^2/2} = e^{-\lambda t} \sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!} \left(e^{-s^2/2}\right)^k$$

Simplifying yields,

$$e^{-\lambda t} \sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!} \left(e^{-s^2/2}\right)^k = e^{-\lambda t} \sum_{k=0}^{\infty} \frac{1}{k!} \left(\lambda t e^{-s^2/2}\right)^k = e^{-\lambda t} \exp\left(\lambda t e^{-s^2/2}\right) = \exp\left(\lambda t \left(e^{-s^2/2} - 1\right)\right)$$

That is, the characteristic function $\phi(s)$ of $W(N(t))$ is,

$$\phi(s) = \exp\left(\lambda t \left(e^{-s^2/2} - 1\right)\right)$$

Exercise 7.3

The n-th variation of a function f, over the interval $[0,T]$ is defined as,

$$V_T(n,f) := \lim_{\|\Pi\|\to 0} \sum_{j=0}^{m-1} |f(t_{j+1}) - f(t_j)|^n, \quad \Pi = \{0 = t_0, t_1, \dots, t_m = T\}, \quad \|\Pi\| = \max_j (t_{j+1} - t_j)$$

Show that $V_T(1,W) = \infty$ and $V_T(3,W) = 0$, where W is a Brownian motion.

Solution

We first prove that if $f_n \to 0$ and $|g_n| \le M$ for some $|M| < \infty$ then $(f_n g_n) \to 0$.

Indeed, fix $\varepsilon > 0$. Then, by convergence of f_n there is some $N \in \mathbb{N}$ such that $|f_n| < \varepsilon/M$ for all $n \ge N$. Then,

$$|f_n g_n| = |f_n||g_n| \le |f_n| M < (\varepsilon/M)M = \varepsilon$$

This proves $f_n g_n \to 0$. $\square$

Write,

$$V_T(k+1,W) = \lim_{\|\Pi\|\to 0} \sum_{j=0}^{m-1} |W(t_{j+1}) - W(t_j)|^{k+1} = \lim_{\|\Pi\|\to 0} \sum_{j=0}^{m-1} |W(t_{j+1}) - W(t_j)|^k |W(t_{j+1}) - W(t_j)|$$

Let, $M_\Pi = \max_j |W(t_{j+1}) - W(t_j)|$ for a given partition Π. Then,

$$\begin{aligned}
\lim_{\|\Pi\|\to 0} \sum_{j=0}^{m-1} |W(t_{j+1}) - W(t_j)|^k |W(t_{j+1}) - W(t_j)| &\le \lim_{\|\Pi\|\to 0} \sum_{j=0}^{m-1} |W(t_{j+1}) - W(t_j)|^k M_\Pi \\
&= \lim_{\|\Pi\|\to 0} M_\Pi \sum_{j=0}^{n-1} |W(t_{j+1}) - W(t_j)|^k
\end{aligned}$$

Provided, $|V_T(k,T)| = V_T(k,T)$ is not infinite,

$$\lim_{\|\Pi\|\to 0} M_\Pi \sum_{j=0}^{m-1} |W(t_{j+1}) - W(t_j)|^k = \left(\lim_{\|\Pi\|\to 0} M_\Pi\right) \left(\lim_{\|\Pi\|\to 0} \sum_{j=0}^{n-1} |W(t_{j+1}) - W(t_j)|^2\right)$$

Since $W(t)$ is continuous, $|W(t_{j+1}) - W(t_j)| \to 0$ as $\|\Pi\| \to 0$ since $t_{j+1} - t_j \to 0$. In particular, this means that $M_\Pi \to 0$ as $\|\Pi\| \to 0$.

Thus,

$$0 \geq V_T(k+1, W) = \left(\lim_{\|\Pi\|\to 0} M_\Pi\right)\left(\lim_{\|\Pi\|\to 0} \sum_{j=0}^{m-1} |W(t_{j+1}) - W(t_j)|^k\right) \leq 0 \cdot N = 0$$

Recall $V_T(2, W) = T < \infty$. Then, by above, $V_T(3, W) = 0$. ☐

Suppose, for the sake of contradiction that $V_T(1, W) \neq \infty$. Clearly $V_T(1, W) \geq 0$, so $V_T(1, W)$ is bounded above and below by finite constants. Then, by above, $V_T(2, W) = 0$, a contradiction (for $T > 0$). This proves $V_T(1, W) = \infty$. ☐

Exercise 7.4

Define

$$X_t = \mu t + W_t \qquad\qquad \tau_m := \inf\{t \geq 0 : X_t = m\}$$

Show that Z is a martingale where,

$$Z_t = \exp(\sigma X_t - (\sigma\mu + \sigma^2/2)t)$$

Assume $\mu > 0$ and $m \geq 0$. Assume further that $\tau_m < \infty$ with probability one and the stopped process $Z_{t\wedge\tau_m}$ is a martingale. Find the Laplace transform $\mathbb{E}e^{-\alpha\tau_m}$.

Solution

Let $0 \leq s \leq t$. Rewrite,

$$\mathbb{E}\left[Z_t\middle|\mathcal{F}_s\right] = \mathbb{E}\left[e^{\sigma X_t-(\sigma\mu+\sigma^2/2)t}\middle|\mathcal{F}_s\right] = \mathbb{E}\left[e^{\sigma(\mu t+W_t)-(\sigma\mu+\sigma^2/2)t}\middle|\mathcal{F}_s\right] = \mathbb{E}\left[e^{\sigma W_t-(\sigma^2/2)t}\middle|\mathcal{F}_s\right]$$

Now, pulling out what is known,

$$\mathbb{E}\left[e^{\sigma W_t-(\sigma^2/2)t}\middle|\mathcal{F}_s\right] = \mathbb{E}\left[e^{\sigma(W_t-W_s)+\sigma W_s-(\sigma^2/2)t)}\middle|\mathcal{F}_s\right] = e^{\sigma W_s-(\sigma^2/2)t}\mathbb{E}\left[e^{\sigma(W_t-W_s)}\middle|\mathcal{F}_s\right]$$

By the property of independent increments,

$$e^{\sigma W_s-(\sigma^2/2)t}\mathbb{E}\left[e^{\sigma(W_t-W_s)}\middle|\mathcal{F}_s\right] = e^{\sigma W_s-(\sigma^2/2)t}\mathbb{E}\left[e^{\sigma(W_t-W_s)}\right] = e^{\sigma W_s-(\sigma^2/2)t}e^{\sigma^2(t-s)/2}$$

Finally,

$$e^{\sigma W_s-(\sigma^2/2)t}e^{\sigma^2(t-s)/2} = e^{\sigma W_s-(\sigma^2/2)s} = e^{\sigma(\mu s+W_s)-(\sigma\mu+\sigma^2/2)s} = e^{\sigma X_2-(\sigma\mu+\sigma^2/2)s}$$

This proves Z_t is a martingale. $\square$

Define $s = \min\{t, \tau_m\}$. Fix $m \geq 0$ and define,

$$Z^{(m)} = \left(Z_t^{(m)}\right)_{t\geq 0}, \qquad\qquad Z_t^{(m)} = Z_s$$

Then, using the fact that Z_t is a martingale we have,

$$1 = Z_0^{(m)} = \mathbb{E}\left[Z_t^{(m)}\right] = \mathbb{E}\left[e^{\sigma X_s-(\sigma\mu+\sigma^2/2)s}\right]$$

If $\tau_m = \infty$ then $X_t < m$ for all t. Thus, since $\sigma \geq 0, \mu > 0$,

$$e^{\sigma X_t - (\sigma\mu + \sigma^2/2)t} \leq e^{\sigma m - (\sigma\mu + \sigma^2/2)t} < \infty$$

Therefore, since $\mathbb{P}(\tau_m < \infty) = 0$,

$$\begin{aligned}
\mathbb{E}\left[e^{\sigma X_s - (\sigma\mu+\sigma^2/2)s}\right] &= \mathbb{E}\left[\mathbb{1}_{\{\tau_m = \infty\}}\left(e^{\sigma X_s - (\sigma\mu+\sigma^2/2)s}\right) + \mathbb{1}_{\{\tau_m < \infty\}}\left(e^{\sigma X_s - (\sigma\mu+\sigma^2/2)s}\right)\right] \\
&= \mathbb{E}\left[\mathbb{1}_{\{\tau_m = \infty\}}\left(e^{\sigma X_t - (\sigma\mu+\sigma^2/2)t}\right)\right] + \mathbb{E}\left[\mathbb{1}_{\{\tau_m < \infty\}}\left(e^{\sigma X_{\tau_m} - (\sigma\mu+\sigma^2/2)\tau_m}\right)\right] \\
&= 0 + \mathbb{E}\left[\mathbb{1}_{\{\tau_m < \infty\}}\left(e^{\sigma m - (\sigma\mu+\sigma^2/2)\tau_m}\right)\right]
\end{aligned}$$

Similarly, since $\sigma \geq 0, \mu > 0$, $e^{\sigma m - (\sigma\mu+\sigma^2/2)\tau_m)} < \infty$. Therefore,

$$\begin{aligned}
\mathbb{E}\left[\mathbb{1}_{\{\tau_m < \infty\}}\left(e^{\sigma m - (\sigma\mu+\sigma^2/2)\tau_m}\right)\right] &= \mathbb{E}\left[\mathbb{1}_{\{\tau_m = \infty\}}\left(e^{\sigma m - (\sigma\mu+\sigma^2/2)\tau_m}\right)\right] + \mathbb{E}\left[\mathbb{1}_{\{\tau_m < \infty\}}\left(e^{\sigma m - (\sigma\mu+\sigma^2/2)\tau_m}\right)\right] \\
&= \mathbb{E}\left[\mathbb{1}_{\{\tau_m = \infty\}}\left(e^{\sigma m - (\sigma\mu+\sigma^2/2)\tau_m}\right) + \mathbb{1}_{\{\tau_m < \infty\}}\left(e^{\sigma m - (\sigma\mu+\sigma^2/2)\tau_m}\right)\right] \\
&= \mathbb{E}\left[e^{\sigma m - (\sigma\mu+\sigma^2/2)\tau_m}\right]
\end{aligned}$$

Then, setting $\alpha = (\sigma\mu + \sigma^2/2)$,

$$e^{-\sigma m} = \mathbb{E}\left[e^{-(\sigma\mu+\sigma^2/2)\tau_m}\right] = \mathbb{E}\left[e^{-\alpha\tau_m}\right]$$

We solve the equation, $\alpha = (\sigma\mu + \sigma^2/2)$ for σ using the quadratic equation, yielding,

$$\sigma = -\mu \pm \sqrt{\mu^2 + 2\alpha}$$

However, $\sigma, \alpha \geq 0$ so we must take $\sigma = -\mu + \sqrt{\mu^2 + 2\alpha}$. Thus,

$$\mathbb{E}\left[e^{-\alpha\tau_m}\right] = e^{\left(\mu - \sqrt{\mu^2+2\alpha}\right)m}$$

Exercise 8.1

Compute $\mathrm{d}(W_t^4)$. Write W_T^4 as an integral with respect to W plus an integral with respect to t. Use this representation of W_T^4 to show that $\mathbb{E}W_T^4 = 3T^2$. Compute $\mathbb{E}W_T^6$ using the same technique.

Solution

Write $f(x) = x^4$ so that $f(W_t) = W_t^4$. Then, $f'(x) = 4x^3$ and $f''(x) = 12x^2$. Therefore, Itô's formula gives,

$$\mathrm{d}W_t^4 = f'(W_t)\mathrm{d}W_t + \frac{1}{2}f''(W_t)\mathrm{d}[W,W]_t = 4W_t^3\mathrm{d}W_t + \frac{12}{2}W_t^2\mathrm{d}[W,W]_t$$

Thus, writing $\mathrm{d}[W,W]_t = \mathrm{d}t$ we have,

$$\mathrm{d}W_t^4 = 4W_t^3\mathrm{d}W_t + 6W_t^2\mathrm{d}t$$

Thus, since $W_0 = 0$,

$$W_T^4 = W_T^4 - W_0^4 = 4\int_0^T W_t^3\mathrm{d}W_t + 6\int_0^T W_t^2\mathrm{d}t$$

Recall Itô integrals are martingales so that,

$$\mathbb{E}\left[\int_0^T W_t^3\mathrm{d}W_t\right] = 0$$

Note also that since $\mathbb{E}\left[W_t^2\right] = t$,

$$\mathbb{E}\left[\int_0^T W_t^2\mathrm{d}t\right] = \int_0^T \mathbb{E}\left[W_t^2\right]\mathrm{d}t = \int_0^T t\mathrm{d}t = \frac{T^2}{2}$$

Therefore,

$$\mathbb{E}\left[W_T^4\right] = 4\mathbb{E}\left[\int_0^T W_t^3\mathrm{d}W_t\right] + 6\mathbb{E}\left[\int_0^T W_t^2\mathrm{d}t\right] = 6\frac{T^2}{2} = 3T^2$$

Similarly, we have,

$$W_T^6 = 6\int_0^T W_t^5\mathrm{d}W_t + \frac{6\cdot 5}{2}\int_0^T W_t^4\mathrm{d}t$$

Therefore, since $\mathbb{E}\left[W_t^4\right] = 3t^2$,

$$\mathbb{E}\left[W_T^6\right] = 6\mathbb{E}\left[\int_0^T W_t^5 \mathrm{d}W_t\right] + 15\mathbb{E}\left[\int_0^T W_t^4 \mathrm{d}t\right] = 15\int_0^T \mathbb{E}\left[W_t^4\right] \mathrm{d}t = 15\int_0^T 3t^2 \mathrm{d}t = 15T^3$$

Exercise 8.2

Find an explicit expression for Y_T where,

$$\mathrm{d}Y_t = r\mathrm{d}t + \alpha Y_t \mathrm{d}W_t$$

Hint: Multiply the above equation by $F_t := \exp(-\alpha W_t + \frac{1}{2}\alpha^2 t)$.

Solution

Let $f(x,y) = \exp(-\alpha x + \frac{1}{2}\lambda^2 y)$ so that,

$$f_x(W_t,t) = -\alpha F_t \qquad f_y(W_t,t) = \frac{\alpha^2}{2}F_t \qquad f_{xx}(W_t,t) = \alpha^2 F_t$$

Then $F_t = f(W_t,t)$, so by Itô's formula and the heuristic $(\mathrm{d}W_t)^2 = \mathrm{d}t, (\mathrm{d}t)^2 = \mathrm{d}t\mathrm{d}W_t = 0$,

$$\begin{aligned}
\mathrm{d}F_t = \mathrm{d}f(W_t,t) &= f_y(W_t,t)\mathrm{d}t + f_x(W_t,t)\mathrm{d}W_t + \frac{1}{2}f_{xx}(W_t,t)(\mathrm{d}W_t)^2 \\
&= \frac{\alpha^2}{2}F_t\mathrm{d}t - \alpha F_t \mathrm{d}W_t + \frac{\alpha^2}{2}F_t\mathrm{d}t \\
&= \alpha^2 F_t \mathrm{d}t - \alpha F_t \mathrm{d}W_t
\end{aligned}$$

Using our heuristics we have,

$$\mathrm{d}[F,Y]_t = (\mathrm{d}F_t)(\mathrm{d}Y_t) = \left(\alpha^2 F_t\mathrm{d}t - \alpha F_t \mathrm{d}W_t\right)(r\mathrm{d}t + \alpha Y_t \mathrm{d}W_t) = -\alpha^2 F_t Y_t (\mathrm{d}W_t)^2 = -\alpha^2 F_t Y_t \mathrm{d}t$$

By the product rule we have,

$$\begin{aligned}
\mathrm{d}(F_tY_t) &= F_t\mathrm{d}Y_t + Y_t\mathrm{d}F_t + \mathrm{d}[F,Y]_t \\
&= F_t(r\mathrm{d}t + \alpha Y_t\mathrm{d}W_t) + Y_t(\alpha^2 F_t\mathrm{d}t - \alpha F_t\mathrm{d}W_t) - \alpha^2 F_t Y_t \mathrm{d}t \\
&= rF_t\mathrm{d}t
\end{aligned}$$

In integral form,

$$F_tY_t - F_0Y_0 = \int_0^t rF_s\mathrm{d}s = \int_0^t re^{-\alpha W_s + \frac{1}{2}\alpha^2 s}\mathrm{d}s$$

We can add $F_0Y_0 = Y_0$ and divide by F_t yielding,

$$Y_t = Y_0 + re^{\alpha W_t - \frac{1}{2}\alpha^2 t}\int_0^t e^{-\alpha W_s + \frac{1}{2}\alpha^2 s}\mathrm{d}s$$

Exercise 8.3

Suppose X, Δ, and Π are given by,

$$\mathrm{d}X_t = \sigma X_t \mathrm{d}W_t, \qquad \Delta_t = \frac{\partial f}{\partial x}(t, X_t), \qquad \Pi_t = X_t \Delta_t$$

where f is some smooth function. Show that if f satisfies,

$$\left(\frac{\partial}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2}{\partial x^2}\right) f(t,x) = 0$$

for all (t, x), then Π is a martingale with respect to a filtration $\mathcal{F}_t$ for W.

Solution

We have,

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2}{\partial x^2}\right) = \frac{\partial^2}{\partial x \partial t} + \frac{1}{2}\sigma^2\left[x^2 \frac{\partial^3}{\partial x^3} + 2x\frac{\partial^2}{\partial x^2}\right]$$

Thus, using the condition for f we have,

$$\frac{\partial^2 f}{\partial x \partial t} + \frac{1}{2}\sigma^2 X_t^2 \frac{\partial^3 f}{\partial x^3} = -\sigma^2 X_t \frac{\partial^2 f}{\partial x^2}$$

Using our heuristics we have,

$$\mathrm{d}[X, X] = \sigma^2 X_t^2 (\mathrm{d}W_t)^2 = \sigma^2 X_t^2 \mathrm{d}t$$

Similarly,

$$\mathrm{d}[X, t] = \mathrm{d}[t, X] = \mathrm{d}[t, t] = 0$$

Therefore, by Itô's formula,

$$\begin{aligned}
\mathrm{d}\Delta_t &= \frac{\partial^2 f}{\partial x \partial t}(t, X_t)\mathrm{d}t + \frac{\partial^2 f}{\partial x^2}(t, X_t)\mathrm{d}X_t + \frac{1}{2}\mathrm{d}[X, X] \\
&= \frac{\partial^2 f}{\partial x \partial t}(t, X_t)\mathrm{d}t + \sigma X_t \frac{\partial^2 f}{\partial x^2}(t, X_t)\mathrm{d}W_t + \frac{1}{2}\sigma^2 X_t^2 \frac{\partial^3 f}{\partial x^3}(t, X_t)\mathrm{d}t \\
&= -\sigma^2 X_t \frac{\partial^2 f}{\partial x^2}(t, X_t)\mathrm{d}t + \sigma X_t \frac{\partial^2 f}{\partial x^2}(t, X_t)\mathrm{d}W_t
\end{aligned}$$

Therefore,

$$\mathrm{d}[X,\Delta]_t = (\mathrm{d}X_t)(\mathrm{d}\Delta_t) = \sigma^2 X_t^2 \frac{\partial^2 f}{\partial x^2}(t,X_t)(\mathrm{d}W_t)^2 = \sigma^2 X_t^2 \frac{\partial^2 f}{\partial x^2}(t,X_t)\mathrm{d}t$$

Finally, we have,

$$\begin{aligned}
\mathrm{d}\Pi_t &= \mathrm{d}(X_t\Delta_t) = X_t\mathrm{d}\Delta_t + \Delta_t\mathrm{d}X_t + \mathrm{d}[X,\Delta]_t \\
&= X_t\left(-\sigma^2 X_t \frac{\partial^2 f}{\partial x^2}(t,X_t)\mathrm{d}t + \sigma X_t \frac{\partial^2 f}{\partial x^2}(t,X_t)\mathrm{d}W_t\right) + \sigma X_t \frac{\partial f}{\partial x}(t,X_t)\mathrm{d}W_t + \sigma^2 X_t^2 \frac{\partial^2 f}{\partial x^2}\mathrm{d}t \\
&= \sigma X_t\left(X_t \frac{\partial^2 f}{\partial x^2}(t,X_t) + \frac{\partial f}{\partial x}(t,X_t)\right)\mathrm{d}W_t
\end{aligned}$$

Since there is no $\mathrm{d}t$ dependence this is an Itô integral and therefore a martingale with respect to a filtration for W. (there are probably some technical assumptions we need about X and f, but in class we never dealt with these) $\square$

Exercise 8.4

Suppose X is given by,

$$\mathrm{d}X_t = \mu(t, X_t)dt + \sigma(t, X_t)\mathrm{d}W_t$$

For any smooth function f define,

$$M_t^f := f(t, X_t) - f(0, X_0) - \int_0^t \left(\frac{\partial}{\partial s} + \mu(s, X_s)\frac{\partial}{\partial x} + \frac{1}{2}\sigma^2(s, X_s)\frac{\partial^2}{\partial x^2} \right) f(s, X_s)ds$$

Show that M^f is a martingale with respect to a filtration $\mathcal{F}_t$ for W.

Solution

We first compute,

$$\mathrm{d}[X, X]_t = (\mathrm{d}X_t)(\mathrm{d}X_t) = \sigma^2(t, X_t)(\mathrm{d}W_t)^2 = \sigma^2(t, X_t)\mathrm{d}t$$

We then have,

$$\begin{aligned}
\mathrm{d}f(t, X_t) &= \frac{\partial f}{\partial t}(t, X_t)\mathrm{d}t + \frac{\partial f}{\partial x}(t, X_t)\mathrm{d}X_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\mathrm{d}[X, X]_t \\
&= \frac{\partial f}{\partial t}(t, X_t)\mathrm{d}t + \frac{\partial f}{\partial x}(t, X_t)[\mu(t, X_t)\mathrm{d}t + \sigma(t, X_t)\mathrm{d}W_t] + \frac{1}{2}\sigma^2(t, X_t)\frac{\partial^2 f}{\partial x^2}\mathrm{d}t \\
&= \left(\frac{\partial}{\partial t} + \mu(t, X_t)\frac{\partial}{\partial x} + \frac{1}{2}\sigma^2(t, X_t)\frac{\partial^2}{\partial x^2} \right) f(t, X_t)\mathrm{d}t + \sigma(t, X_t)\frac{\partial f}{\partial x}\mathrm{d}W_t
\end{aligned}$$

Finally, since $f(0, X_0)$ is a constant,

$$\begin{aligned}
\mathrm{d}M_t^f &= \mathrm{d}f(t, X_t) - \left(\frac{\partial}{\partial t} + \mu(t, X_t)\frac{\partial}{\partial x} + \frac{1}{2}\sigma^2(t, X_t)\frac{\partial^2}{\partial x^2} \right) f(t, X_t)\mathrm{d}t \\
&= \sigma(t, X_t)\frac{\partial f}{\partial x}\mathrm{d}W_t
\end{aligned}$$

Since there is no dt dependence this an Itô integral and therefore a martingale with respect to a filtration for W. □

Exercise 9.2

Let X be a solution to the following SDE

$$\mathrm{d}X_t = \kappa(\theta - X_t)\mathrm{d}t + \delta\sqrt{X_t}\mathrm{d}W_t$$

Define

$$u(t,x) = \mathbb{E}\left[\exp\left(-\int_t^T X_s \mathrm{d}s\right)\middle| X_t = x\right]$$

Derive a PDE for the function u. To solve the PDE for u, try a solution of the form

$$u(t,x) = \exp(-xA(t) - B(t)),$$

where A and B are deterministic functions of t. Show that A and B must satisfy a pair of coupled ODEs (with appropriate terminal conditions at time T). Bonus question: solve the ODEs (it may be helpful to note that one of the ODEs is a Riccati equation).

Solution

With $\gamma(u,x) = x$, $\phi(x) = 1$, $g(u,x) = 0$ this is a subcase of an example in the notes. We then know $u(t,x)$ solves,

$$(\partial_t + \mathcal{A})u + g = 0, \qquad u(T,\cdot) = \phi, \qquad \mathcal{A} = \frac{1}{2}\sigma^2\partial_x^2 + \mu\partial_x - \gamma = 0$$

First compute,

$$\partial_t u = (-xA' - B')u \qquad \partial_x u = -Au \qquad \partial_x^2 u = A^2 u$$

This gives,

$$\begin{aligned}
0 &= \left[\partial_t + \frac{1}{2}\delta^2 x\partial_x^2 + \kappa(\theta - x)\partial_x - x\right]u \\
&= \left[-xA' - B' + \frac{1}{2}\delta^2 xA^2 + \kappa(\theta - x)(-A) - x\right]u \\
&= \left[\left(-A' + \frac{1}{2}\delta^2 A^2 + \kappa A - 1\right)x + (-B' - \kappa\theta A)\right]u
\end{aligned}$$

Observe $u(t,x) > 0$ for all t, x. Therefore we require the bracketed term above to be zero for all x, t. Setting the coefficients of the x terms and constant terms to zero gives a coupled pair of ODEs,

$$\begin{cases} -A'(t) + \frac{1}{2}\delta^2 A^2(t) + \kappa A(t) - 1 = 0 \\ -B'(t) - \kappa\theta A(t) = 0 \end{cases}$$

We have,

$$1 = \varphi(x) = u(T, x) = \exp\left(-xA(T) - B(T)\right)$$

This gives terminal condition,

$$A(T) = 0 \qquad\qquad B(T) = 0$$

We solve this in Mathematica without boundary conditions using,

```
DSolve[{-D[A[t],t]+1/2 \[Delta]^2 A[t]^2+\[Kappa] A[t] - 1 ==0 , -
   D[B[t],t]-\[Kappa] \[Theta] A[t]==0},{A,B},t]
```

This gives solution,

$$\begin{aligned} A(t) &= \frac{\sqrt{-2\delta^2 - \kappa^2}\tan\left(\frac{1}{2}\left(2c_1\sqrt{-2\delta^2 - \kappa^2} + t\sqrt{-2\delta^2 - \kappa^2}\right)\right) - \kappa}{\delta^2} \\ B(t) &= \frac{\theta\kappa\left(2\log\left(\cos\left(c_1\sqrt{-2\delta^2 - \kappa^2} + \frac{1}{2}t\sqrt{-2\delta^2 - \kappa^2}\right)\right) + \kappa t\right)}{\delta^2} + c_2 \end{aligned}$$

where,

$$\begin{aligned} c_1 &= \frac{1}{2\sqrt{-2\delta^2 - \kappa^2}}\left[2\arctan\left(\frac{\kappa}{\sqrt{-2\delta^2 - \kappa}}\right) - T\sqrt{-2\delta^2 - \kappa^2}\right] \\ c_2 &= -\frac{\theta\kappa\left(2\log\left(\cos\left(c_1\sqrt{-2\delta^2 - \kappa^2} + \frac{1}{2}T\sqrt{-2\delta^2 - \kappa^2}\right)\right) + \kappa T\right)}{\delta^2} \end{aligned}$$

We could have done this by hand by since the first equation is separable but its just as ugly.

Exercise 9.2

Solution

Exercise 9.3

For $i = 1, 2, \ldots, d$ let $X^{(i)}$ satisfy,

$$\mathrm{d}X_t^{(i)} = -\frac{b}{2}X_t^{(i)}\mathrm{d}t + \frac{1}{2}\sigma \mathrm{d}W_t^{(i)}$$

where $(W_t^{(i)})_{i=1}^d$ are independent Brownian motions. Define

$$R_t := \sum_{i=1}^d \left(X_t^{(i)}\right)^2, \qquad B_t := \sum_{i=1}^d \int_0^t \frac{1}{\sqrt{R_s}} X_s^{(i)} \mathrm{d}W_s^{(i)}$$

Show that B is a Brownian motion. Derive an SDE for R that involves only $\mathrm{d}t$ and $\mathrm{d}B_t$ terms (i.e., no $\mathrm{d}W_t^{(i)}$ terms should appear).

Solution

We use the Lévy characterization of Brownian motion. In particular, we must show B is a martingale, B has continuous sample paths, and $B_0 = 0$ with $[B, B]_t = t$ for all $t \geq 0$.

Write,

$$\mathrm{d}B_t = \mathrm{d}\left[\sum_{i=1}^d \int_0^t \frac{1}{\sqrt{R_s}} X_s^{(i)} \mathrm{d}W_s^{(i)}\right] = \sum_{i=1}^d \frac{1}{\sqrt{R_t}} X_t^{(i)} \mathrm{d}W_t^{(i)}$$

As B_t is an Itô integral it is a martingale with respect to a filtration $\mathbb{F} = (\mathcal{F}_\sqcup)_{t \geq 0}$ for $W_t^{(i)}$.

Similarly, B_t has continuous sample paths as $W_t^{(i)}$ have continuous sample paths.

Clearly $B_0 = 0$ as $W_0^{(i)} = 0$.

Now,

$$\begin{aligned}
(\mathrm{d}B_t)(\mathrm{d}B_t) &= \frac{1}{R_t}\sum_{i=1}^d \sum_{j=1}^d X_t^{(i)} X_t^{(j)} \mathrm{d}W_t^{(i)} \mathrm{d}W_t^{(j)} \\
&= \frac{1}{R_t}\left(\sum_{j=1}^d \left(X_t^{(i)}\mathrm{d}W_t^{(i)}\right)^2 + 2\sum_{i=1}^d \sum_{j=1}^i X_t^{(i)} X_t^{(j)} \mathrm{d}W_t^{(i)} \mathrm{d}W_t^{(j)}\right)
\end{aligned}$$

Using the heuristic, $\mathrm{d}W_t^{(i)}\mathrm{d}W_t^{(j)} = \delta_{ij}\mathrm{d}t$ and the definition of R_t we have,

$$\mathrm{d}[B,B]_t = \frac{1}{R_t}\sum_{i=1}^{d}\left(X_t^{(i)}\right)^2 \mathrm{d}t = \mathrm{d}t$$

Therefore, $[B,B]_t = t$.

This proves B is a Brownian motion. $\square$

Compute, using Itô's formula,

$$\mathrm{d}R_t = \mathrm{d}\left[\sum_{i=1}^{d}\left(X_t^{(i)}\right)^2\right] = \sum_{i=1}^{d} 2X_t^{(i)}\mathrm{d}X_t^{(i)} + \frac{1}{2}2\mathrm{d}[X^{(i)},X^{(i)}]_t = \sum_{i=1}^{d} 2X_t^{(i)}\mathrm{d}X_t^{(i)} + \mathrm{d}[X^{(i)},X^{(i)}]_t$$

Using our heuristics we have,

$$\mathrm{d}[X^{(i)},X^{(i)}]_t = \left(\mathrm{d}X_t^{(i)}\right)\left(\mathrm{d}X_t^{(i)}\right) = \left(-\frac{b}{2}X_t^{(i)}\mathrm{d}t + \frac{1}{s}\sigma\mathrm{d}W_t^{(i)}\right)^2 = \frac{\sigma^2}{4}\mathrm{d}t$$

Now,

$$\begin{aligned}\sum_{i=1}^{d} 2X_t^{(i)}\mathrm{d}X_t^{(i)} + \mathrm{d}[X^{(i)},X^{(i)}]_t &= \sum_{i=1}^{d} 2X_t^{(i)}\left(-\frac{b}{2}X_t^{(i)}\mathrm{d}t + \frac{1}{2}\sigma\mathrm{d}W_t^{(i)}\right) + \frac{\sigma^2}{4}\mathrm{d}t \\ &= \sum_{i=1}^{d}\left(\frac{\sigma^2}{4} - b\left(X_t^{(i)}\right)^2\right)\mathrm{d}t + \sigma\sqrt{R_t}\frac{1}{\sqrt{R_t}}X_t^{(i)}\mathrm{d}W_t^{(i)}\end{aligned}$$

Therefore, simplifying slightly we have,

$$\mathrm{d}R_t = (d\sigma^2/4 - bR_t)\mathrm{d}t + \sigma\sqrt{R_t}\mathrm{d}B_t$$

Exercise 9.4

Solution

Exercise 9.5

Consider a diffusion $X = (X_t)_{t\geq 0}$ that lives on a finite interval (l, r), $0 < l < r < \infty$ and satisfies the SDE

$$\mathrm{d}X_t = \mu X_t \mathrm{d}t + \sigma X_t \mathrm{d}W_t$$

One can easily check that the endpoints l and r are regular (you do not have to prove it here). Assume both endpoints are killing. Find the transition density $\Gamma(t, x; T, y)$ of X.

Solution

We have, $\Gamma(\cdot, \cdot; T, y)$ satisfies,

$$(\partial_t + \mathcal{A}(t))\Gamma(\cdot, t; T, y) = 0 \qquad\qquad \Gamma(T, \cdot; T, y) = \delta_y$$

where the infinitesimal generator $\mathcal{A}$ is,

$$\mathcal{A} = \mu x \partial_x + \frac{1}{2}\sigma^2 x^2 \partial_x^2$$

We seek a spectral representation for $\mathcal{A}$. That is, a basis $\{\Psi_n\}_{n\geq 0}$ for such that $\mathcal{A}\Psi_n = \lambda_n \Psi_n$.

Since the endpoints are killing we also require,

$$\Psi_n(l) = 0, \qquad\qquad \Psi_n(r) = 0$$

We make a change of variables. Let $z = \log(x)$. Then,

$$\partial_x = \frac{1}{x}\partial_z, \qquad\qquad \partial_x^2 = -\frac{1}{x^2}\partial_z + \frac{1}{x}\partial_z^2$$

Then, in terms of z we have generator,

$$\mathcal{A}_z = \left(\mu - \frac{\sigma^2}{2}\right)\partial_z + \frac{1}{2}\sigma^2 \partial_z^2$$

This equation is very similar to a damped harmonic oscillator. We therefore guess that the eigenfunctions have the form,

$$\psi_n(z) = \exp(\gamma_n z)\left[A \sin\left(\frac{n\pi(z - \log(l)}{\log(r) - \log(l)}\right) + B\cos\left(\frac{n\pi(z - \log(l)}{\log(r) - \log(l)}\right)\right]$$

In order to satisfy the boundary conditions listed above we need $B = 0$. The constant A will be determined by the normalization of ψ_n, so we will leave it off until the end.

For convenience, write,

$$\psi = \psi_n, \qquad \gamma = \gamma_n, \qquad k = \frac{n\pi}{\log(l/r)}, \qquad \cos(z') = \cos(k(z - \log l))$$

We then have,

$$\partial_z\psi(z) = \gamma\psi + \exp(\gamma z)k\cos(z')$$
$$\partial_z^2\psi(z) = \gamma^2\psi + \gamma\exp(\gamma z)k\cos(z')) + \gamma\exp(\gamma z)k\cos(z') - k^2\psi = \gamma^2\psi + 2\gamma\exp(\gamma z)k\cos(z') - k^2\psi$$

We seek γ such that $\mathcal{A}_z\psi = \lambda\psi$ for some constant λ. That is, in our expression of $\mathcal{A}_z\psi$ we require the terms not containing a ψ be zero. Thus,

$$0 = \left(\mu - \frac{\sigma^2}{2}\right)\exp(\gamma z)k\cos(z') + \left(\frac{\sigma^2}{2}\right)2\gamma\exp(\gamma z)k\cos(z') = \left[\left(\mu - \frac{\sigma^2}{2}\right) + \sigma^2\gamma\right]\exp(\gamma z)\cos(z')$$

Suppose $k \neq 0$ (i.e. that the solution is non-trivial). Since $\exp(\gamma z)$ and $\cos(z') \neq 0$ we have,

$$0 = \left(\mu - \frac{\sigma^2}{2}\right) + \sigma^2\gamma$$

Solving for γ we have,

$$\gamma = \frac{1}{2} - \frac{\mu}{\sigma^2}$$

The eigenvalues are,

$$\lambda_n = \left(\mu - \frac{\sigma^2}{2}\right)\gamma + \left(\frac{\sigma^2}{2}\right)\left(\gamma^2 - k^2\right) = -\frac{\sigma^2}{2}[k^2 + \gamma^2]$$

Transforming back to x we have, $\hat{\Psi}_n(x) = \psi_n(\log(x))$ satisfies,

$$\mathcal{A}\hat{\Psi}_n(x) = \lambda_n\hat{\Psi}_n(x), \qquad \mathcal{A} = \mu x\partial_x + \frac{1}{2}\sigma^2x^2\partial_x^2$$

Define,

$$m(y) = \frac{2}{\sigma^2y^2}\exp\left(\int \mathrm{d}y\frac{2\mu y}{\sigma^2y^2}\right) = \frac{2}{\sigma^2y^2}\exp\left(\frac{2\mu}{\sigma^2}\log(y)\right) = \frac{2}{\sigma^2}y^{2\mu/\sigma^2-2} = \frac{2}{\sigma^2}y^{-2\gamma-1}$$

It is clear that the $\hat{\Psi}_n$ are orthogonal (properties of sines). We compute,

$$\langle \hat{\Psi}_n(x), \hat{\Psi}_n(x) \rangle_m = \int_l^r \Psi_n(x)^2 m(x) \mathrm{d}x = \log(r/l)/\sigma^2$$

We then satisfy $\langle \Psi_k, \Psi_l \rangle_m = \delta_{kl}$ by defining,

$$\Psi_n(x) = \frac{\hat{\Psi}_n(x)}{\sqrt{\langle \Psi_n(x), \Psi_n(x) \rangle_m}}$$

Explicitly,

$$\Psi_n(x) = \frac{\sigma}{\sqrt{\log(r/l)}} x^\gamma \sin(k(z - \log l)) = \frac{\sigma}{\sqrt{\log(r/l)}} x^{1/2-\mu/\sigma^2} \sin\left(n\pi \frac{\log(x/l)}{\log(r/l)} \right)$$

Finally,

$$\Gamma(t, x; T, y) = m(y) \sum_n \exp((T-t)\lambda_n) \Psi_n(x) \Psi_n(y)$$

Explicitly,

$$\Gamma(t, x; T, y) = \frac{2}{\log(r/l)} \left(\frac{x}{y} \right)^{1/2-\mu/\sigma^2} y^{-1} \sum_n \exp((T-t)\lambda_n) \sin\left(n\pi \frac{\log(x/l)}{\log(r/l)} \right) \sin\left(n\pi \frac{\log(y/l)}{\log(r/l)} \right)$$

Since the Ψ_n are normalized then Γ is normalized.

We verify in Mathematica that Γ satisfies both the KFE and KBE.

Exercise 9.6

Consider a two-dimensional diffusion processes $X = (X_t)_{t\geq 0}$ and $Y = (Y_t)_{t\geq 0}$ that satisfy the SDEs

$$\mathrm{d}X_t = \mathrm{d}W_t^1 \qquad \mathrm{d}Y_t = \mathrm{d}W_t^2$$

where W_t^1 and W_t^2 are two independent Brownian motions. Define a function u as follows

$$u(x,y) = \mathbb{E}\left[\phi(X_\tau) | X_t = x, Y_t = y\right], \qquad \tau = \inf\{s \geq t : Y_s = a\}$$

1. State a PDE and boundary conditions satisfied by the function u.
2. Let us define the Fourier transform and and inverse Fourier transform, respectively, as follows

$$\text{Fourier Transform:} \qquad \hat{f}(\omega) := \int e^{-i\omega x} f(x) \mathrm{d}x$$

$$\text{Inverse Transform:} \qquad f(x) := \frac{1}{2\pi} \int e^{i\omega x} \hat{f}(\omega) \mathrm{d}\omega$$

   Use Fourier transforms and a conditioning argument to derive an expression for $u(x,y)$ as an inverse Fourier transform. Use this result to derive an explicit form for $\mathbb{P}(X_\tau \in \mathrm{d}z | X_t = x, Y_t = y)$(i.e., an expression involving no integrals).
3. Show the expression you derived in part 2 for $u(x,y)$ satisfies the PDE and BCs you stated in part 1.

Solution

1. Since there are no $\mathrm{d}t$ terms in either Brownian motion, and since the coefficient in both of the $\mathrm{d}W_t$ term is 1 we have, generator,

$$\mathcal{A} = \frac{1}{2}\partial_x^2 + \frac{1}{2}\partial_y^2$$

   The PDE satisfied by u is,

$$\mathcal{A}u = \left(\frac{1}{2}\partial_x^2 + \frac{1}{2}\partial_y^2\right) u = 0 \qquad \Longleftrightarrow \qquad \left(\partial_x^2 + \partial_y^2\right) u = 0$$

   If $y = a$ then $\tau = t$ so $X_\tau = x$. We therefore have boundary condition,

$$u(x,a) = \phi(x)$$

2. Given starting position (x, y) at time t, and time τ, from the notes we know X_τ is normally distributed with mean x and variance $\tau - t$ by the independent increments property of Brownian motion. We know the characteristic function of a normally distributed random variable with distribution $\mathcal{N}(\mu, \sigma^2)$ is $e^{i\omega x - \sigma^2\omega^2/2}$. Therefore,

$$\mathbb{E}\left[e^{i\omega X_\tau}\middle|\tau, X_t = x, Y_t = y\right] = e^{i\omega x - (\tau - t)\omega^2/2}$$

Thus, using iterated conditioning,

$$\begin{aligned}
\mathbb{E}\left[e^{i\omega X_\tau}|X_t = x, Y_t = y\right] &= \mathbb{E}\left[\mathbb{E}[e^{i\omega X_\tau}|\tau, X_t = x, Y_t = y]|X_t = x, Y_t = y\right] \\
&= \mathbb{E}\left[e^{i\omega x - (\tau - t)\omega^2/2}|X_t = x, Y_t = y\right] \\
&= e^{i\omega x}\mathbb{E}\left[e^{-(\tau - t)\omega^2/2}|X_t = x, Y_t = y\right]
\end{aligned}$$

We have previously shown that the first hitting time of a Brownian motion τ_m satisfies,

$$\mathbb{E}\left[e^{-\lambda\tau_m}\right] = e^{-|m|\sqrt{2\lambda}}$$

where $\tau_m = \inf\{t \geq 0 : W_t = m\}$ and $W_0 = 0$.

Since we start at position y at time t (rather that position 0 and time 0 as above), we know that,

$$\mathbb{E}\left[e^{-(\omega^2/2)(\tau - t)}|X_t = x, Y_t = y\right] = e^{-|a-y||\omega|}$$

Therefore,

$$\mathbb{E}\left[e^{i\omega X_\tau}|X_t = x, Y_t = y\right] = e^{-|a-y||\omega|}$$

Write,

$$\phi(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\omega x}\hat{\phi}(\omega)\mathrm{d}\omega$$

Then,

$$u(x, y) = \mathbb{E}[\phi(X_\tau)|X_t = x, Y_t = y] = \mathbb{E}\left[\frac{1}{2\pi}\int_{\mathbb{R}} e^{i\omega X_\tau}\hat{\phi}(\omega)\mathrm{d}\omega\middle|X_t = x, Y_t = y\right]$$

Now, bringing the expectation through the integral, and applying the above result,

$$\mathbb{E}\left[\frac{1}{2\pi}\int_{\mathbb{R}} e^{i\omega X_\tau}\hat{\phi}(\omega)\mathrm{d}\omega\middle|X_t = x, Y_t = y\right] = \frac{1}{2\pi}\int_{\mathbb{R}}\hat{\phi}(\omega)\mathbb{E}\left[e^{i\omega X_\tau}|X_t = x, Y_t = y\right]\mathrm{d}\omega$$
$$= \frac{1}{2\pi}\int_{\mathbb{R}}\hat{\phi}(\omega)e^{-|a-y||\omega|}e^{i\omega x}\mathrm{d}\omega$$

First recall, $\mathbb{E}[\phi(X)] = \int \phi(x)f_X(x)\mathrm{d}x$ and $\mathbb{P}(X \in \mathrm{d}z) = f_X(z)\mathrm{d}z$. Then, taking $\phi(x) = \mathbb{1}_{\{x\in \mathrm{d}z\}}$ means $\mathbb{E}[\phi(X)] = f_X(z)\mathrm{d}z = \mathbb{P}(X \in \mathrm{d}z)$. Therefore,

$$u(x, y) = \mathbb{E}[\mathbb{1}_{\{X_\tau\in \mathrm{d}z\}}|X_t = x, Y_t = y] = \mathbb{P}(X_\tau \in \mathrm{d}z|X_t = x, Y_t = y)$$

In this case,

$$\hat{\phi}(\omega) = \int_{\mathbb{R}} e^{-i\omega x}\mathbb{1}_{\{x\in \mathrm{d}z\}}\mathrm{d}x = e^{-i\omega z}\mathrm{d}z$$

Thus, computing this integral by splitting it at 0,

$$u(x, y) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-i\omega z}\mathrm{d}z e^{-|a-y||\omega|}e^{i\omega x}\mathrm{d}\omega = \frac{1}{2\pi}\left[\frac{2|a-y|}{(a-y)^2 + (x-z)^2}\right]\mathrm{d}z = \frac{1}{\pi}\left[\frac{|y-a|}{(y-a)^2 + (x-z)^2}\right] \mathrm{d}$$

3. First observe,

$$u(x, a) = \frac{1}{2\pi}\int_{\mathbb{R}}\hat{\phi}(\omega)e^{-|a-a|}|\omega|e^{i\omega x}\mathrm{d}\omega = \frac{1}{2\pi}\int_{\mathbb{R}}\hat{\phi}(\omega)e^{i\omega x}\mathrm{d}\omega = \phi(x)$$

Define,

$$c = \begin{cases} 1 & y \geq a \\ -1 & y < a \end{cases}$$

Now observe,

$$\partial_x^2 u(x, y) = \frac{1}{2\pi}\int_{\mathbb{R}}\hat{\phi}(\omega)e^{-c(y-a)|\omega|}\partial_x^2 e^{i\omega x}\mathrm{d}\omega = \frac{(i^2\omega^2)}{2\pi}\int_{\mathbb{R}}\hat{\phi}(\omega)e^{-c(y-a)|\omega|}e^{i\omega x}\mathrm{d}\omega$$

Then,

$$\partial_y^2 u(x, y) = \frac{1}{2\pi}\int_{\mathbb{R}}\hat{\phi}(\omega)\partial_y^2 e^{-c(y-a)|\omega|}e^{i\omega x}\mathrm{d}\omega = \frac{c^2\omega^2}{2\pi}\int_{\mathbb{R}}\hat{\phi}(\omega)e^{-c(y-a)|\omega|}e^{i\omega x}\mathrm{d}\omega$$

Thus, since $i^2 = -1$ and $c^2 = 1$,

$$(\partial_x^2 + \partial_y^2)u(x, y) = 0$$

Note there is probably some issue with the partial derivative with respect to y at $y = a$, since $|y - a|$ is not differentiable at this point.

Therefore $u(x, y) = \mathbb{E}[\phi(X_\tau)|X_t = x, Y_t = y]$ satisfies the PDE from 1.

Exercise 10.1

Let $P = (P_t)_{t\geq 0}$ be a Poisson process with intensity λ.

(a) What is the Lévy Measure ν of P.

(b) Let $\mathrm{d}X_t = \mathrm{d}P_t$. Define $u(x,t) := \mathbb{E}\left[\varphi(X_T)|X_t = x\right]$. Find $u(t,x)$ and verify it solves the Kolmogorov Backward equation.

Solution

(a) We have,

$$\nu(U) = \mathbb{E}\left[N(1,U)\right] = \mathbb{E}\left[\sum_{0\leq s\leq 1} \mathbb{1}_{\Delta P_s \in U}\right] = \mathbb{E}\left[\sum_{i=1}^{P_1} \mathbb{1}_{1\in U}\right] = \mathbb{E}\left[P_1\right]\mathbb{1}_{1\in U} = \lambda \mathbb{1}_{1\in U}$$

(b) Integrating $\mathrm{d}X_t = \mathrm{d}P_t$ from 0 to t gives, $X_t - X_0 = P_t - P_0$. Since $P_0 = 0$ we have,

$$X_t = X_0 + P_t$$

First observe,

$$\mathbb{P}(X_T = k|X_t = x) = \mathbb{P}(X_0 + P_T = k|X_0 + P_t = x) = \mathbb{P}(P_T = k - X_0|P_t = x - X_0)$$

Since P has independent increments, and since P is Markov,

$$\mathbb{P}(P_T = k - X_0|P_t = x - X_0) = \mathbb{P}(P_{T-t} = k - x) = \frac{(\lambda(T-t))^{k-x}}{(k-x)!}e^{-\lambda(T-t)}$$

Thus,

$$u(t,x) = \mathbb{E}\left[\varphi(X_T)|X_t = x\right] = \sum_{k=x}^{\infty}\varphi(k)\mathbb{P}(X_T = k|X_t = x) = \sum_{k=x}^{\infty}\varphi(k)\frac{(\lambda(T-t))^{k-x}}{(k-x)!}e^{-\lambda(T-t)}$$

Reindexing with $n = k - x$,

$$u(t,x) = e^{-\lambda(T-t)}\sum_{k=x}^{\infty}\varphi(k)\frac{(\lambda(T-t))^{k-x}}{(k-x)!} = e^{-\lambda(T-t)}\sum_{n=0}^{\infty}\varphi(n+x)\frac{(\lambda(T-t))^{n}}{n!}$$

We now compute the generator $\mathcal{A}(t)$ for P. By definition,

$$\mathcal{A}(t)\varphi(x) = \lim_{s\to t^+} \frac{1}{s-t}\left[\mathcal{P}(t,s)\varphi(x) - \varphi(x)\right] = \lim_{s\to t^+} \frac{1}{s-t}\left[\mathbb{E}\left[\varphi(X_s)|X_t = x\right] - \varphi(x)\right]$$

In a small interval dt the probability $X_{t+\mathrm{d}t} = X_t + 1$ is $\lambda\mathrm{d}t$ and probability $X_{t+\mathrm{d}t} = X_t$ is $(1-\lambda)\mathrm{d}t$. Therefore,

$$\mathcal{A}(t)\varphi(x) = \frac{1}{\mathrm{d}t}\left[\varphi(x+1)\lambda + \varphi(x)(1-\lambda) - \varphi(x)\right] = \lambda(\varphi(x+1) - \varphi(x))$$

Since the t-derivative of the $n = 0$ term is zero,

$$\begin{aligned}
\sum_{n=0}^{\infty} \varphi(n+x)\partial_t\left[\frac{(\lambda(T-t))^n}{n!}\right] &= \sum_{n=1}^{\infty} \varphi(n+x)\partial_t\left[\frac{(\lambda(T-t))^n}{n!}\right] \\
&= \sum_{n=1}^{\infty} \varphi(n+x)(n)(-\lambda)\frac{(\lambda(T-t))^{n-1}}{n!} \\
&= -\lambda\sum_{n=1}^{\infty} \varphi(n+x)\frac{(\lambda(T-t))^{n-1}}{(n-1)!}
\end{aligned}$$

Observe, by the chain rule and assuming we can bring a derivative through a sum,

$$\begin{aligned}
\partial_t u(t,x) &= \left[\partial_t e^{-\lambda(T-t)}\right]\sum_{n=0}^{\infty}\varphi(n+x)\frac{(\lambda(T-t))^n}{n!} + e^{-\lambda(T-t)}\sum_{n=0}^{\infty}\varphi(n+x)\partial_t\left[\frac{(\lambda(T-t))^n}{n!}\right] \\
&= \lambda e^{-\lambda(T-t)}\sum_{n=0}^{\infty}\varphi(n+x)\frac{(\lambda(T-t))^n}{n!} - \lambda e^{-\lambda(T-t)}\sum_{n=1}^{\infty}\varphi(n+x)\frac{(\lambda(T-t))^{n-1}}{(n-1)!} \\
&= \lambda e^{-\lambda(T-t)}\sum_{n=0}^{\infty}\varphi(n+x)\frac{(\lambda(T-t))^n}{n!} - \lambda e^{-\lambda(T-t)}\sum_{n=m}^{\infty}\varphi(m+1+x)\frac{(\lambda(T-t))^m}{m!} \\
&= \lambda(u(t,x) - u(t,x+1))
\end{aligned}$$

Therefore the KBE is satisfied as

$$[\partial_t + \mathcal{A}]u(t,x) = \lambda(u(t,x) - u(t,x+1)) - \lambda(u(t,x+1) - u(t,x)) = 0, \quad u(T,x) = \varphi(x)$$

Exercise 10.2

Solution

Exercise 10.3

Let $X = (X_t)_{t\geq 0}$ be a process defined by,

$$\mathrm{d}X_t = \mu_t X_t \mathrm{d}t + \sigma_t X_t \mathrm{d}W_t + \int_{\mathbb{R}} \left(e^{\gamma_t(z)} - 1\right) X_{t^-} \tilde{N}(\mathrm{d}t, \mathrm{d}z)$$
$$\mathrm{d}Y_t = b_t Y_t \mathrm{d}t + a_t Y_t \mathrm{d}W_t + \int_{\mathbb{R}} \left(e^{g_t(z)} - 1\right) Y_{t^-} \tilde{N}(\mathrm{d}t, \mathrm{d}z)$$

where W is a one-dimensional Brownian motion, $\tilde{N}$ is a one-dimensional compensated Poisson random measure on $\mathbb{R}$, and $\mu, b, \sigma, a, \gamma, g$ are $\mathbb{F}$-adapted stochastic processes.

(a) Define $Z_t := X_t/Y_t$. Compute the differential $\mathrm{d}Z_t$. Your answer should not involve X_t or Y_t.

(b) Find μ_t so that Z is a martingale.

Solution

(a) Define $f(x, y) = x/y$. Then $Z_t = f(X_t, Y_t)$.

We have,

$$[(e^{\gamma_t(z)} - 1)X_t; (e^{g_t(z)} - 1)Y_t] \cdot \nabla f(X_{t^-}, Y_{t^-}) = (e^{\gamma_t(z)} - 1)X_{t^-} f_x(X_{t^-}, Y_{t^-}) + (e^{g_t(z)} - 1)Y_{t^-} f_y(X_{t^-}, Y$$

We use Itô's formula to compute,

$$\begin{aligned}
\mathrm{d}Z_t = \mathrm{d}f(X_t, Y_t) = &\left(\mu_t X_t f_x + b_t Y_t f_y + \frac{1}{2}\left((\sigma_t X_t)^2 f_{xx} + 2(\sigma_t X_t)(a_t Y_) f_{xy} + (a_t Y_t)^2 f_{yy}\right)\right) \mathrm{d}t \\
&+ (\sigma_t X_t f_x + a_t Y_t f_y)\, \mathrm{d}W_t \\
&+ \int_{\mathbb{R}} \left(f\left(X_{t^-} + (e^{\gamma_t(z)} - 1)X_{t^-}, Y_{t^-} + (e^{g_t(z)} - 1)Y_{t^-}\right) - f(X_{t^-}, Y_{t^-})\right) \tilde{N}(\mathrm{d}t, \mathrm{d}z \\
&+ \int_{\mathbb{R}} \Big(f\left(X_{t^-} + (e^{\gamma_t(z)} - 1)X_{t^-}, Y_{t^-} + (e^{g_t(z)} - 1)Y_{t^-}\right) - f(X_{t^-}, Y_{t^-}) \\
&\qquad - (e^{\gamma_t(z)} - 1)X_{t^-} f_x(X_{t^-}, Y_{t^-}) - (e^{g_t(z)} - 1)Y_{t^-} f_y(X_{t^-}, Y_{t^-})\Big)\nu(\mathrm{d}z)\mathrm{d}t
\end{aligned}$$

Now, using $f_x = 1/y, f_y = -x/y^2, f_{xy} = -1/y^2, f_{xx} = 0, f_{yy} = 2x/y^3$ we have,

$$\mu_t X_t f_x + b_t Y_t f_y = \mu_t X_t \left(\frac{1}{Y_t}\right) + b_t Y_t \left(\frac{-X_t}{Y_t^2}\right) = \mu_t Z_t - b_t Z_t$$

$$(\sigma_t X_t)^2 f_{xx} + 2(\sigma_t X_t)(a_t Y_t) f_{xy} + (a_t Y_t)^2 f_{yy} = 2(\sigma_t X_t)(a_t Y_t)\left(\frac{-1}{Y_t^2}\right) + a_t^2 Y_t^2 \left(\frac{2X_t}{Y_t^3}\right) = -2\sigma_t a_t Z_t + 2$$

$$\sigma_t X_t f_x + a_t Y_t f_y = \sigma_t X_t \left(\frac{1}{Y_t}\right) + a_t Y_t \left(\frac{-X_t}{Y_t^2}\right) = \sigma_t Z_t - a_t Z_t$$

$$f\left(X_{t^-} + (e^{\gamma_t(z)} - 1)X_{t^-}, Y_{t^-} + (e^{g_t(z)} - 1)Y_{t^-}\right) - f(X_{t^-}, Y_{t^-}) = \frac{e^{\gamma_t(z)}}{e^{g_t(z)}} Z_{t^-} - Z_{t^-}$$

$$\begin{aligned}
&(e^{\gamma_t(z)} - 1)X_{t^-} f_x(X_{t^-}, Y_{t^-}) + (e^{g_t(z)} - 1)Y_{t^-} f_y(X_{t^-}, Y_{t^-}) \\
&= (e^{\gamma_t(z)} - 1)X_{t^-} \left(\frac{1}{Y_{t^-}}\right) + (e^{g_t(z)} - 1)Y_{t^-} \left(\frac{-X_{t^-}}{Y_{t^-}^2}\right) \\
&= (e^{\gamma_t(z)} - 1)Z_{t^-} - (e^{g_t(z)} - 1)Z_{t^-}
\end{aligned}$$

Inserting these evaluated expressions into the original expression for $\mathrm{d}Z_t$ gives,

$$\begin{aligned}
\mathrm{d}Z_t = &\left(\mu_t - b_t - \sigma_t a_t + a_t^2\right) Z_t \mathrm{d}t + (\sigma_t - a_t) Z_t \mathrm{d}W_t \\
&+ \int_{\mathbb{R}} \left(\frac{e^{\gamma_t(z)}}{e^{g_t(z)}} - 1\right) Z_{t^-} \tilde{N}(\mathrm{d}t, \mathrm{d}z) \\
&+ \int_{\mathbb{R}} \left(\frac{e^{\gamma_t(z)}}{e^{g_t(z)}} - e^{\gamma_t(z)} + e^{g_t(z)} - 1\right) Z_{t^-} \nu(\mathrm{d}z)\mathrm{d}t
\end{aligned}$$

(b) We need the $\mathrm{d}t$ term to be zero. Therefore pick,

$$\mu_t = b_t + \sigma_t a_t - a_t^2 - \int_{\mathbb{R}} \left(\frac{e^{\gamma_t(z)}}{e^{g_t(z)}} - e^{\gamma_t(z)} + e^{g_t(z)} - 1\right) \nu(\mathrm{d}z)\mathrm{d}t$$

Exercise 10.4

Let $\eta = (\eta_t)_{t\geq 0}$ be a one-dimensional Lévy Process and define $X = (X_t)_{t\geq 0}$ by

$$\mathrm{d}X_t = \kappa(\theta - X_t)\mathrm{d}t + \mathrm{d}\eta_t$$

(a) Find X_t explicitly as a function of η.

(b) Assume $\eta_t = \sigma W_t + \int_{\mathbb{R}} z\tilde{N}(t, \mathrm{d}z)$. Compute $m(t) := \mathbb{E}X_t$ and $c(t,s) := \mathbb{E}(X_t - m(t))(X_s - m(s))$.

Solution

(a) Let $Y_t = X_t - \theta$ and $Z_t = e^{\kappa t}Y_t = f(t, Y_t)$, where $f(t,y) = e^{\kappa t}y$.

Then,

$$\mathrm{d}Y_t = \mathrm{d}X_t = -\kappa Y_t \mathrm{d}t + \mathrm{d}\eta_t$$

Recall the product rule (which applies to Lévy Itô processes),

$$\mathrm{d}(U_tV_t) = U_{t^-}\mathrm{d}V_t + V_{t^-}\mathrm{d}U_t + \mathrm{d}[U,V]_t$$

Therefore,

$$\mathrm{d}Z_t = \mathrm{d}(e^{\kappa t}Y_t) = e^{\kappa t^-}\mathrm{d}Y_t + Y_{t^-}\mathrm{d}e^{\kappa t} + \mathrm{d}[e^{\kappa t}, Y]_t$$

Using our heuristics we have $\mathrm{d}(e^{\kappa t})\mathrm{d}Y_t = 0$. Therefore, since t^- and t can be "treated the same" on $\mathrm{d}t$ terms which are continuous,

$$\mathrm{d}Z_t = e^{\kappa t^-}\mathrm{d}Y_t + \kappa e^{\kappa t}Y_{t^-} = e^{\kappa t^-}\mathrm{d}\eta_t$$

Integrating we have,

$$Z_t = Z_0 + \int_0^t e^{\kappa s}\mathrm{d}\eta_s$$

Therefore, since $Y_t = e^{-\kappa t}Z_t$, $Z_0 = Y_0$ so,

$$Y_t = e^{-\kappa t}\left(Y_0 + \int_0^t e^{\kappa s}\mathrm{d}\eta_s\right)$$

Finally, since $X_t = \theta + Y_t$, $Y_0 = X_0 - \theta$ so,

$$X_t = \theta + e^{-\kappa t}\left(X_0 - \theta + \int_0^t e^{\kappa s}\mathrm{d}\eta_s\right) = \theta + e^{-\kappa t}(X_0-\theta) + \int_0^t e^{\kappa(s-t)}\mathrm{d}\eta_s$$

(b) We have,

$$\mathrm{d}\eta_t = \sigma \mathrm{d}W_t + \int_{\mathbb{R}} z\tilde{N}(\mathrm{d}t, \mathrm{d}z)$$

Observe, that since integrals with respect to $\mathrm{d}W_t$ and $\int_{\mathbb{R}} \tilde{N}(\mathrm{d}t, \mathrm{d}z)$ are martingales so,

$$\mathbb{E}\left[\int_0^t e^{\kappa(s-t)}\mathrm{d}\eta_s\right] = \mathbb{E}\left[\int_0^t e^{\kappa(s-t)}\sigma\mathrm{d}W_t + \int_0^t e^{\kappa(s-t)}\int_{\mathbb{R}} z\tilde{N}(\mathrm{d}t,\mathrm{d}z)\right] = 0$$

Therefore,

$$m(t) = \mathbb{E}\left[X_t\right] = \mathbb{E}\left[\theta + e^{-\kappa t}(X_0-\theta) + \int_0^t e^{\kappa(s-t)}\mathrm{d}\eta_s\right] = \theta + e^{-\kappa t}(X_0 - \theta)$$

Clearly,

$$X_t - m(t) = \int_0^t e^{\kappa(u-t)}\mathrm{d}\eta_u$$

Without loss of generality assume $t \geq s$. Then, using the independent increments property to write the expectation of a product as the product of expectations,

$$\begin{aligned}
\mathbb{E}\left[(X_t - m(t))(X_s - m(s))\right] &= \mathbb{E}\left[\left(\int_0^t e^{\kappa(u-t)}\mathrm{d}\eta_u\right)\left(\int_0^s e^{\kappa(v-s)}\mathrm{d}\eta_v\right)\right] \\
&= \mathbb{E}\left[\left(\int_0^s e^{\kappa(u-t)}\mathrm{d}\eta_u + \int_s^t e^{\kappa(u-t)}\mathrm{d}\eta_u\right)\left(\int_0^s e^{\kappa(v-s)}\mathrm{d}\eta_v\right)\right] \\
&= \mathbb{E}\left[e^{-\kappa(t+s)}\left(\int_0^s e^{\kappa u}\mathrm{d}\eta_u\right)^2 + e^{-\kappa(t+s)}\left(\int_s^t e^{\kappa u}\mathrm{d}\eta_u\right)\left(\int_0^s e^{\kappa v}\mathrm{d}\eta_v\right)\right] \\
&= e^{-\kappa(t+s)}\mathbb{E}\left[\left(\int_0^s e^{\kappa u}\mathrm{d}\eta_u\right)^2\right] + e^{-\kappa(t+s)}\mathbb{E}\left[\int_s^t e^{\kappa u}\mathrm{d}\eta_u\right]\mathbb{E}\left[\int_0^s e^{\kappa v}\mathrm{d}\eta_v\right]
\end{aligned}$$

We now note that, Lévy processes without a dt term are martingales so that,

$$\mathbb{E}\left[\int_0^s e^{\kappa u}\mathrm{d}\eta_u\right] = \mathbb{E}\left[\int_0^s e^{\kappa u}\left(\sigma \mathrm{d}W_u + \int_{\mathbb{R}} z\tilde{N}(\mathrm{d}u,\mathrm{d}z)\right)\right] = 0$$

Define,

$$Z_s = \int_0^s e^{\kappa u}\mathrm{d}\eta_u$$

Then,

$$\mathrm{d}Z_s = e^{\kappa s}\mathrm{d}\eta_s = \sigma e^{\kappa s}\mathrm{d}W_s + \int_{\mathbb{R}} e^{\kappa s} z\tilde{N}(\mathrm{d}s,\mathrm{d}z)$$

Using Itô's isometry we have,

$$\mathbb{E}\left[\left(\int_0^s e^{\kappa u}\mathrm{d}\eta_u\right)^2\right] = \mathbb{E}\left[\int_0^s \left(\sigma^2 e^{2\kappa u} + \int_{\mathbb{R}} e^{2\kappa u} z^2\nu(\mathrm{d}z)\right)\mathrm{d}u\right] = \mathbb{E}\left[\left(\sigma^2 + \int_{\mathbb{R}} z^2\nu(\mathrm{d}z)\right)\frac{e^{2\kappa s}-1}{2\kappa}\right]$$

Therefore,

$$c(t,s) = e^{-\kappa(t+s)}\frac{e^{2\kappa s}-1}{2\kappa}\left(\sigma^2 + \int_{\mathbb{R}} z^2\nu(\mathrm{d}z)\right) = \frac{e^{\kappa(s-t)} - e^{-\kappa(t+s)}}{2\kappa}\left(\sigma^2 + \int_{\mathbb{R}} z^2\nu(\mathrm{d}z)\right)$$

We can remove our assumption that $t \geq s$ and write,

$$c(t,s) = \frac{e^{-\kappa|t-s|} - e^{-\kappa(t+s)}}{2\kappa}\left(\sigma^2 + \int_{\mathbb{R}} z^2\nu(\mathrm{d}z)\right)$$

Exercise 10.5

Let X be the following one-dimensional jump-diffusion

$$\mathrm{d}X_t = \mu(t, X_t)\mathrm{d}t + \sigma(t, X_t)\mathrm{d}W_t + \int_{\mathbb{R}} \gamma(t, X_{t^-}, z)\tilde{N}(t, \mathrm{d}z),$$

where W is a one-dimensional Brownian motion and $\tilde{N}$ is a one-dimensional compensated Poisson random measure on $\mathbb{R}$. Derive using the Lévy-Itô formula the infinitesimal generator $\mathcal{A}(t)$ of the X process,

$$\mathcal{A}(t)\varphi(x) := \lim_{s \to t^+} \frac{\mathbb{E}\left[\varphi(X_s) | X_t = x\right] - \varphi(x)}{s - t}$$

Solution

Since $\mathbb{E}[\varphi(X_t)|X_t = x] = \varphi(x)$,

$$\mathbb{E}\left[\varphi(X_s)|X_t = x\right] - \varphi(x) = \mathbb{E}\left[\varphi(X_t) + \int_t^s \mathrm{d}\varphi(X_u)\right] - \varphi(x) = \mathbb{E}\left[\int_t^s \mathrm{d}\varphi(X_u)\right]$$

From the Lévy-Itô formula we have,

$$\begin{aligned}
\mathrm{d}\varphi(X_u) = &\left(\mu(u, X_u)\varphi'(X_u) + \frac{1}{2}\sigma(u, X_u)^2\varphi''(X_u)\right)\mathrm{d}u + \sigma(u, X_u)\varphi'(X_u)\mathrm{d}W_u \\
&+ \int_{\mathbb{R}} \Big(\varphi(X_{u^-} + \gamma(u, X_{u^-}, z)) - \varphi(X_{u^-})\Big)\tilde{N}(\mathrm{d}u, \mathrm{d}z) \\
&+ \int_{\mathbb{R}} \Big(\varphi(X_{u^-} + \gamma(u, X_{u^-}, z)) - \varphi(X_{u^-}) - \gamma(u, X_{u^-}, z)\varphi'(X_{u^-})\Big)\nu(\mathrm{d}z)\mathrm{d}u
\end{aligned}$$

We note that as integrals with respect to W and $\tilde{N}$ are martingales that,

$$\begin{aligned}
\mathbb{E}\left[\int_t^s \mathrm{d}\varphi(X_u)\right] = \mathbb{E}\Bigg[\int_t^s \Bigg(&\mu(u, X_u)\varphi'(X_u) + \frac{1}{2}\sigma(u, X_u)^2\varphi''(X_u)\mathrm{d}u \\
&+ \int_{\mathbb{R}} \Big(\varphi(X_{u^-} + \gamma(u, X_{u^-}, z)) - \varphi(X_{u^-}) - \gamma(u, X_{u^-}, z)\varphi'(X_{u^-})\Big)\nu(\mathrm{d}z)\Bigg)\mathrm{d}u\Bigg]
\end{aligned}$$

Thus, taking the limit as $s \to t^+$,

$$\mathcal{A}(t)\varphi(x) = \left(\mu(t, X_t)\partial_x + \frac{1}{2}\sigma(t, X_t)\partial_x^2 + \int_{\mathbb{R}} \nu(\mathrm{d}z)\left(\theta_{\gamma(t, X_t, z)} - 1 - \gamma(t, X_t, z)\partial_x\right)\right)\varphi(x)$$