Stochastics Methods and Problems

Tyler Chen

Contents

1 Generating and Characteristic functions

2 Discrete Time Markov Chains

2.1 Transition Matrix

Sample Problems:

- Exercise 4.1: Write down transition matrices for processes based on rolling a dice
- Exercise 4.2: Write down transition matrices for $Y_n = X_{2n}$
- Exercise 4.7: Give example of transition matrix with multiple stationary distributions

2.2 Classification of States

Sample Problems:

- Exercise 4.3: Show if all states communicate with an absorbing state they must all be transient

2.3 Mean Recurence Time

Sample Problems:

- Exercise 4.4: Find expected visits to a state given some properties
- Exercise 4.5: Find mean-recurrence times using invariant distribution

2.4 Reversibility

Sample Problems:

- Exercise 4.8: Show process is reversible in equilibrium

2.5 Stationary/Invariant distribution

Note: TALK ABOUT VARIOUS METHODS FOR FINDING THIS

Sample Problems:

- Exercise 4.5: Find invariant distribution
- Exercise 4.6: Find invariant distribution of mistakes in editions of a book by computing limit of generating function
- Exercise 4.7: Give example of transition matrix with multiple stationary distributions

2.6 Generating Functions

Sample Problems:

- Exercise 4.6: Find invariant distribution of mistakes in editions of a book by computing limit of generating function

3 Continuous Time Markov Chains

3.1 Transition Matrix

3.2 Stationary/Invariant distribution

Sample Problems:

- Exercise 5.1: Find invariant distribution and conditions for existence
- Exercise 5.2: Show two processes have the same stationary distribution
- Exercise 5.3: Indirectly find stationary distribution by solving KFE, finding generating function for the chain, and computing the distribution of X_t as $t \to \infty$

3.3 Generator

Sample Problems:

- Exercise 5.1: Write down generator
- Exercise 5.3: Given generator solve KFE
- Exercise 5.4: Write down generator and solve KFE/KBE

3.4 Generating Functions

Sample Problems:

- Exercise 5.3: Use KBE to find PDE for generating function of X
- Exercise 5.4: Use KBE to find PDE for generating function of X
- Exercise 5.5: Compute generating function of Poisson process with random intensity. Use generating function to compute mean and variance.

3.5 KFE AND KBE

Sample Problems:

- Exercise 5.3: Given generator solve KFE
- Exercise 5.4: Write down KFE and KBE and solve

3.6 Birth Death Processes

General description of birth death processes

3.6.1 General Form for infinite queue

Description:

- Process either jumps up one or down one or stay the same
- Expected wait time in state i is exponentially distributed $\tau \sim \mathcal{E}(\lambda_i + \mu_i)$
- When the process does jump, the probability of an up jump is $\lambda_i/(\lambda_i + \mu_i)$, and the probability of a down jump is $\mu_i/(\lambda_i + \mu_i)$.
- if $\lambda_0 > 0$ the chain is irreducible.

State space: $S = \{1, 2, 3 \ldots\}$.

Generator:

$$G = \begin{bmatrix} -\lambda_0 & \lambda_0 & & & \\ \mu_1 & -(\mu_1 + \lambda_1) & \lambda_1 & & \\ & \mu_2 & -(\mu_2 + \lambda_2) & \lambda_2 & \\ & & \mu_3 & -(\mu_3 + \lambda_3) & \lambda_3 \\ & & & & \ddots \end{bmatrix}$$

Invariant distribution:

$$\pi(k) = \frac{\lambda_0 \lambda_1 \cdots \lambda_{k-1}}{\mu_1 \mu_2 \cdots \mu_k} \pi(0), \qquad \pi(0) = \left(1 + \sum_{k=1}^{\infty} \frac{\lambda_0 \lambda_1 \cdots \lambda_{k-1}}{\mu_1 \mu_2 \cdots \mu_k} \right)^{-1}$$

Sample Problems: Example 5.2.9

3.6.2 M/M/1 queue

Description:

- Models infinite queue.
- Arrivals occur at a rate λ according to a Poisson process.
- Service times have exponential distribution with rate parameter μ, where $1/\mu$ is the mean service time.
- A single server serves customers one at a time from front of queue, first come first serve

State space: $S = \{1, 2, 3 \ldots\}$.

Generator:

$$G = \begin{bmatrix} -\lambda & \lambda & & \\ \mu & -(\mu + \lambda) & \lambda & \\ & \mu & -(\mu + \lambda) & \lambda \\ & & & \ddots \end{bmatrix}$$

Invariant distribution:

$$\pi(k) = (1 - \lambda/\mu)(\lambda/\mu)^k$$

Expected Response Time: For customers who arrive and find the queue as a stationary process, the response time (sum of waiting and services times) has density function,

$$f(t) = \begin{cases} (\mu - \lambda)e^{-(\mu-\lambda)t}, & t > 0 \\ 0 & \text{ow.} \end{cases}$$

This has mean,

$$\int_0^\infty t f(t) \mathrm{d}t = \frac{1}{\mu - \lambda}$$

Sample Problems: Exercise 5.1

3.6.3 M/M/∞

Description:

- Arrivals occur at a rate λ according to a Poisson process.
- Service times have exponential distribution with rate parameter μ, where $1/\mu$ is the mean service time.
- There are always enough servers that every arriving job is serviced immediately.

State space: $S = \{1, 2, 3, \ldots\}$.

Generator:

$$G = \begin{bmatrix} -\lambda & \lambda & & & \\ \mu & -(\mu+\lambda) & \lambda & & \\ & 2\mu & -(2\mu+\lambda) & \lambda & \\ & & 3\mu & -(3\mu+\lambda) & \lambda \\ & & & & \ddots \end{bmatrix}$$

Invariant Distribution:

$$\pi(k) = \frac{(\lambda/\mu)^k e^{-\lambda/\mu}}{k!}$$

Sample Problems: Exercise 5.3, Final Problem ??, Practice Exam #? Problem 1

3.6.4 M/M/1/K queue

State space: $S = \{1, 2, \ldots, n\}$.

Generator:

$$G = \begin{bmatrix} -\lambda & \lambda & & & & \\ \mu & -(\mu+\lambda) & \lambda & & & \\ & \mu & -(\mu+\lambda) & \lambda & & \\ & & \ddots & \ddots & \ddots & \\ & & & \mu & -(\mu+\lambda) & \lambda \\ & & & & \mu & -\mu \end{bmatrix}$$

4 Brownian Motion

Note: add examples from class notes

4.1 Martingale

Sample Problems:

- Exercise 7.1: Show a process is a Martingale using definition
- Exercise 7.4: Show a process is a Martingale using definition

4.2 Characteristic Functions

Sample Problems:

- Exercise 7.2: Compute characteristic function of $W(N(t))$, where $N \sim \text{Pois}(\lambda)$

7.3: n-th variation time

4.3 Laplace Transform

Sample Problems:

- *Note:* Example ??? from book
- Exercise 7.4: Compute Laplace transform of first hitting time.

5 Stochastic Calculus

Note: ITO FORMULA AND STUFF

6 SDEs and PDEs

7 Jump Diffusions

8 Practice Qualification Exams

Practice Exam 1, Problem 1

Let $X = (X_n)_{n\in\mathbb{N}_0}$ be a discrete time Markov chain with X_n representig the amount of water in a reservoir at noon on day n. Assume $X_0 \in \mathbb{N}_0$. Let $Y = (Y_n)_{n\in\mathbb{N}_0}$ be a sequence of iid random variables with Y_n representing the aount of water that flows into the reservoir during the n-th day. The state space of Y is $\{0, 1, 2, \ldots\}$. The resevoir has a maximum capacity of $K \in \mathbb{N}$. When the resevoir is filled to level K, all excssive inflows are lost.

(a) Write the one-step transition matrix P of X in terms of the probability generating function G_Y of Y.

(b) Find an expression for the stationary distribution π of X in terms of the probability generating function G_Y of Y.

Solution

(a) We assume all the water comes in the afternoon. That is, $X_{n+1} = X_n + Y_n$.

Suppose on day n the resevoir is not full. That is, $X_n = k < K$. If it is not filled completely by the incoming water, then some amount of water $j < K - k$ was added. In this case $X_{n+1} = k + j$ with probability,

$$\mathbb{P}(Y_n = j) = f_Y(j) = \left[\frac{1}{j!}\frac{\mathrm{d}^j G_Y(s)}{\mathrm{d}s^j}\right]_{s=0}$$

Otherwise, $X_{n+1} = K$ with probability,

$$1 - \sum_{j<K-k} f_Y(j) = 1 - \sum_{j<K-k}\left[\frac{1}{j!}\frac{\mathrm{d}^j G_Y(s)}{\mathrm{d}s^j}\right]_{s=0}$$

Suppose $X_n = K$. Then since no water leaves the resevoir, $X_{n+1} = K$ with probability one.

We can write this as,

$$X_{n+1} = \begin{cases} \left[\frac{1}{j!}\frac{\mathrm{d}^j G_Y(s)}{\mathrm{d}s^j}\right]_{s=0} & j < K - X_n \\ \\ 1 - \sum_{j<K-X_n}\left[\frac{1}{j!}\frac{\mathrm{d}^j G_Y(s)}{\mathrm{d}s^j}\right]_{s=0} & \text{otherwise} \end{cases}$$

(b) Note that $\pi = [0, 0, \ldots, 0, 1]$ is a stationary distribution.

Note: argue the distributoin is unique?

Note: alternative approach?? Clearly $X_n \to K$ as $n \to \infty$.

Note: in what sense?

Practice Exam 1, Problem 2

Let $(X, Y) = (X_t, Y_t)_{t\geq 0}$ satisfy the following SDE,

$$\mathrm{d}X_t = \mathrm{d}W_t^1, \qquad \mathrm{d}Y_t = \mathrm{d}W_t^2, \qquad (X_0, Y_0) = (x, y)$$

where $W = (W_t^1, W_t^2)_{t\geq 0}$ is a two-dimensinoal Brownian motion with independent components. Define a process $(R, \Phi) = (R_t, \Phi_t)_{t\geq 0}$ as follows,

$$\Phi_t = \arctan(Y_t/X_t), \qquad R_t^2 = X_t^2 + Y_t^2$$

(a) Derive the SDEs satisfied by (R, Φ).
(b) Define,

$$u(r, \phi) = \mathbb{E}\left[e^{-\lambda\tau} f(R_\tau) | R_0 = r, \Phi_0 = \phi\right], \quad \tau = \inf\{t \geq 0 : \Phi_t \notin (0, \pi/2)\}, \quad \phi \in (0, \pi/2)$$

Derive a PDE satisfied by u.
(c) Desribe with pseudo-code how you would find $u(r, \phi)$ using Monte Carlo simulation.

Solution

(a) Define $f(x, y) = \arctan(y/x)$ and $g(x, y) = \sqrt{x^2 + y^2}$. Now note that,

$$\Phi_t = f(X_t, Y_t), \qquad R_t = g(X_t, Y_t)$$

Appying Itô's formula we find,

$$\begin{aligned}
\mathrm{d}\Phi_t = f_x(X_t, Y_t)\mathrm{d}X_t + f_y(X_t, Y_t)\mathrm{d}Y_t \\
+ \frac{1}{2}\big(f_{xx}(X_t, Y_t)\mathrm{d}[X, X]_t + f_{xy}(X_t, Y_t)\mathrm{d}[X, Y]_t \\
+ f_{yx}(X_t, Y_t)\mathrm{d}[Y, X]_t + f_{yy}(X_t, Y_t)\mathrm{d}[Y, Y]_t\big)
\end{aligned}$$

Using our Heuristics we have,

$$\mathrm{d}[X, X]_t = \mathrm{d}[Y, Y]_t = \mathrm{d}t, \qquad \mathrm{d}[X, Y]_T = \mathrm{d}[Y, X]_t = 0$$

We compute,

$$\begin{aligned}
f_x(x,y) &= -\frac{y}{x^2+y^2} = -\frac{\sin(\arctan(y/x)}{\sqrt{x^2+y^2}} \\
f_y(x,y) &= \frac{x}{x^2+y^2} = \frac{\cos(\arctan(y/x))}{\sqrt{x^2+y^2}} \\
f_{xx}(x,y) &= \frac{2xy}{(x^2+y^2)^2} \\
f_{yy}(x,y) &= -\frac{2xy}{(x^2+y^2)^2}
\end{aligned}$$

Therefore, maxing the substitutions, $\Phi_t = \arctan(Y_t/X_t)$, and $R_t = \sqrt{X_t^2 + Y_t^2}$,

$$\mathrm{d}\Phi_t = -\frac{\sin(\Phi_t)}{R_t}\mathrm{d}W_t^1 + \frac{\cos(\Phi_t)}{R_t}\mathrm{d}W_t^2$$

Similarly,

$$\begin{aligned}
\mathrm{d}R_t = g_x(X_t,Y_t)\mathrm{d}X_t &+ g_y(X_t,Y_t)\mathrm{d}Y_t \\
&+ \frac{1}{2}\big(g_{xx}(X_t,Y_t)\mathrm{d}[X,X]_t + g_{xy}(X_t,Y_t)\mathrm{d}[X,Y]_t \\
&\qquad + g_{yx}(X_t,Y_t)\mathrm{d}[Y,X]_t + g_{yy}(X_t,Y_t)\mathrm{d}[Y,Y]_t\big)
\end{aligned}$$

We compute,

$$\begin{aligned}
g_x(x,t) &= \frac{x}{\sqrt{x^2+y^2}} = \cos(\arctan(y/x)) \\
g_y(x,t) &= \frac{y}{\sqrt{x^2+y^2}} = \sin(\arctan(y/x)) \\
g_{xx}(x,t) &= \frac{y^2}{(x^2+y^2)^{3/2}} \\
g_{yy}(x,t) &= \frac{x^2}{(x^2+y^2)^{3/2}}
\end{aligned}$$

Therefore, maxing the substitutions, $\Phi_t = \arctan(Y_t/X_t)$, and $R_t = \sqrt{X_t^2 + Y_t^2}$,

$$\mathrm{d}R_t = \cos(\Phi_t)\mathrm{d}W_t^1 + \sin(\Phi_t)\mathrm{d}W_t^2 + \frac{1}{2R_t}\mathrm{d}t$$

(b)

(c)

(d)

9 Homework Problems

Exercise 3.1

Let $X \sim \text{Bin}(n, U)$ where $U \sim \mathcal{U}((0,1))$. What is the probability Generating function $G_X(s)$ of X? What is $\mathbb{P}(X = k)$ where $k \in \{0, 1, 2, ..., n\}$?

Solution

Using iterated conditioning, since a Binomial random variable is the sum of n iid Bernioulli random variables,

$$G_X(s) = \mathbb{E}[s^X] = \mathbb{E}\mathbb{E}[s^X|U] = \mathbb{E}[((1-U)s^0 + Us^1)^n]$$

We calculate this by integrating with Mathematica as,

```
Integrate[((1 - x) + x s)^n, {x, 0, 1}, Assumptions -> {s > 0}]
```

This yields,

$$\mathbb{E}[((1-U) + Us)^n] = \int_{\mathbb{R}} \mathbb{1}_{(0,1)}((1-x) + xs)^n \mathrm{d}x = \int_0^1 ((1-x) + xs)^n \mathrm{d}x = \frac{1 - s^{n+1}}{(n+1)(1-s)}$$

This is a finite geometric progression which we simplify so,

$$G_X(s) = \sum_{k=0}^{n} \frac{s^k}{n+1}$$

Therefore $\mathbb{P}(X = k) = 1/(1+n)$ for $k = 0, 1, 2, ..., n$.

Exercise 3.2

Let Z_n be the size of the n-th generation in an ordinary branching process with $Z_0 = 1$, $\mathbb{E}Z_1 = \mu$ and $\mathbb{V}Z_1 > 0$. Show that $\mathbb{E}Z_nZ_m = \mu^{n-m}\mathbb{E}Z_m^2$ for $m \leq n$. Use this to find the correlation coefficient $\rho(Z_m, Z_n)$ in terms of μ, n and m. Consider the case $\mu = 1$ and the case $\mu \neq 1$.

Solution

Let $Y_{m,i}$ denote the number of offspring in the n-th generation that descends from the i-th member of the m-th generation. Then the $(Y_{m,i})$ are iid with distribution Z_{n-m} and $Z_n = Y_{m,1} + Y_{m,2} + ... + Y_{m,Z_m}$.

Then, since $(Y_{m,i})$ are iid with distribution Z_{n-m},

$$\mathbb{E}\left[Z_n|Z_m\right] = \mathbb{E}\left[Y_{m,1} + Y_{m,2} + ... + Y_{m,Z_m}|Z_m\right] = Z_m\mathbb{E}[Z_{m-n}] = Z_m\mu^{n-m}$$

Therefore, by taking out what is known,

$$\mathbb{E}\left[Z_mZ_n\right] = \mathbb{E}\left[\mathbb{E}\left[Z_mZ_n|Z_m\right]\right] = \mathbb{E}\left[Z_m^2\mathbb{E}\left[Z_n|Z_m\right]\right] = \mathbb{E}\left[Z_m^2\mu^{n-m}\right] = \mu^{n-m}\mathbb{E}\left[Z_m^2\right]$$

Observing that $\mathbb{E}[Z_mZ_n] = \mu^{n-m}\mathbb{E}[Z_m^2] = \mu^{n-m}(\mathbb{V}[Z_m] + \mathbb{E}[Z_m]^2) = \mu^{n-m}(\mathbb{V}[Z_m] + \mu^{2m})$, write,

$$\rho(Z_m, Z_n) = \frac{\mathrm{Cov}(Z_n, Z_m)}{(\mathbb{V}[Z_n]\mathbb{V}[Z_m])^{1/2}} = \frac{\mathbb{E}[Z_nZ_m] - \mathbb{E}[Z_n]\mathbb{E}[Z_m]}{(\mathbb{V}[Z_n]\mathbb{V}[Z_m])^{1/2}} = \frac{\mu^{n-m}(\mathbb{V}[Z_m] + \mu^{2m}) - \mu^{n+m}}{(\mathbb{V}[Z_n]\mathbb{V}[Z_m])^{1/2}}$$

Denote $\mathbb{V}[Z_1]$ by σ.

Suppose $\mu = 1$ so that $\mathbb{V}[Z_m] = m\sigma^2$. We use Mathematica to simplify the above expression as,

```
FullSimplify[
 PowerExpand[(\[Mu]^(n - m) (Vzm + \[Mu]^(2 m)) - \[Mu]^(
    n + m))/(Vzn Vzm)^(
   1/2) /. {Vzm -> m \[Sigma]^2, Vzn -> n \[Sigma]^2, \[Mu] ->
      1}],
 Assumptions -> {{m, n, \[Sigma], \[Mu]} > 0}]
```

This yields,

$$\rho(Z_m, Z_n) = \sqrt{\frac{m}{n}}$$

Now suppose $\mu \neq 1$ so that $\mathbb{V}[Z_m] = \sigma^2(\mu^n - 1)\mu^{n-1}/(\mu - 1)$. We use Mathematica to simplify the above expression as,

```
FullSimplify[
 PowerExpand[(\[Mu]^(n - m) (Vzm + \[Mu]^(2 m)) - \[Mu]^(
    n + m))/(Vzn Vzm)^(
   1/2) /. {Vzm -> \[Sigma]^2 (\[Mu]^m - 1) \[Mu]^(m - 1)/(\[Mu] -
       1),
     Vzn -> \[Sigma]^2 (\[Mu]^n - 1) \[Mu]^(n - 1)/(\[Mu] - 1) }],
 Assumptions -> {\[Mu] != 1, {m, n, \[Sigma], \[Mu]} > 0}]
```

This yields,

$$\rho(Z_m, Z_n) = \sqrt{\frac{\mu^n(\mu^m - 1)}{\mu^m(\mu^n - 1)}}$$

Observe that in the limit $\mu \to 1$ this coincides with the previous value.

Exercise 3.3

Solution

Exercise 3.4

Consider a branching process with immigration

$$Z_0 = 1 \qquad\qquad Z_{n+1} = \sum_{i=1}^{Z_n} X_{n,i} + Y_n$$

where the $(X_{n,i})$ are iid with common distribution X, the (Y_n) are iid with common distribution Y, and the $(X_{n,i})$ and (Y_n) are independent. What is $G_{Z_{n+1}}(s)$ in terms of $G_{Z_n}(s)$, $G_X(s)$, and $G_Y(s)$? Write $G_{Z_2}(s)$ explicitly in terms of $G_X(s)$ and $G_Y(s)$.

Solution

Define:

$$G_{Z_n}(s) = s^{Z_n} \qquad\qquad G_X(s) = \mathbb{E}s^X \qquad\qquad G_Y(s) = \mathbb{E}s^Y$$

Write $S_n = \sum_{i=1}^{Z_n} X_{n,i}$ so that, $Z_{n+1} = S_n + Y_n$.
First observe that since the $(X_{n,i})$ are iid with common distribution X,

$$G_{S_n}(s) = \mathbb{E}\left[s^{S_n}\right] = \mathbb{E}\left[\mathbb{E}\left[s^{S_n}|Zn\right]\right] = \mathbb{E}\left[\mathbb{E}[s^X]^{Z_n}\right] = \mathbb{E}\left[G_X(s)^{Z_n}\right] = G_{Z_n}(G_X(s))$$

Since the $(X_{n,i})$ and (Y_n) are independent, S_n and Y_n are independent. Therefore,

$$G_{Z_{n+1}}(s) = G_{S_n+Y_n}(s) = G_{S_n}(s)G_Y(s) = G_{Z_n}(G_X(s))G_Y(s)$$

We calculate,

$$G_{Z_0}(s) = \mathbb{E}\left[s^{Z_0}\right] = \mathbb{E}[s] = s$$

Similarly,

$$G_{Z_1}(s) = G_{Z_0}(G_X(s))G_Y(s) = G_X(s)G_Y(s)$$

Therefore,

$$G_{Z_2}(s) = G_{Z_1}(G_X(s))G_Y(s) = G_X(G_X(s))G_Y(G_X(s))G_Y(s)$$

Exercise 3.5

Find $\phi_{X^2}(t) := \mathbb{E}\exp(itX^2)$ where $X \sim \mathcal{N}(\mu, \sigma)$.

Solution

We have,

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-(x-\mu)^2}{2\sigma^2}\right)$$

Thus,

$$\phi_{X^2}(t) = \mathbb{E}\exp(itX^2) = \int_{-\infty}^{\infty} e^{itx^2} f_X(x)\mathrm{d}x$$

We evaluate with Mathematica as,

```
Integrate[Exp[I t x^2] PDF[NormalDistribution[\[Mu], \[Sigma]], x
   ], {x, -\[Infinity], \[Infinity]},
 Assumptions -> {\[Mu] \[Element] Reals, t \[Element] Reals, \[
    Sigma] > 0}]
```

This yields,

$$\phi_{X^2}(t) = \frac{\exp(it\mu^2/(1-2it\sigma^2))}{\sqrt{1-2it\sigma^2}}$$

Exercise 3.6

Let X_n have cumulative distribution function

$$F_{X_n}(x) = \left(x - \frac{\sin(2n\pi x)}{2n\pi}\right) \mathbb{1}_{0\leq x\leq 1} + \mathbb{1}_{x>1}$$

(a) Show that F_{X_n} is a distribution function and find the corresponding density function f_{X_n}.

(b) Show that F_{X_n} converges to the uniform distribution function F_U as $n \to \infty$, but that the density function f_{X_n} does NOT converge to f_U. Here, $U \sim \mathcal{U}((0,1))$.

Solution

(a) Clearly $F_{X_n}(x) = 0$ for $x \leq 0$ and $F_{X_n}(x) = 1$ for $x \geq 1$. Observe, $x - \sin(2n\pi x)/2n\pi$ is non-decreasing and continuous on $(0,1)$, since the derivative, calculated below is non-negative on this interval. Moreover, $x - \sin(2n\pi x)/2n\pi$ is equal to zero at $x = 0$, and equal to one at $x = 1$.

Therefore $F_{X_n}(x)$ is a non-decreasing continuous function with $F_{X_n}(x) \to 0$ as $x \to -\infty$ and $F_{X_n}(x) \to 1$ as $x \to \infty$. So $F_{X_n}(x)$ is a distribution function.

It is straightforward to compute the density function as,

$$f_{X_n}(x) = \frac{d}{dx} F_{X_n}(x) = (1 - \cos(2n\pi x))\mathbb{1}_{0\leq x\leq 1}$$

(b) The uniform distribution on $(0,1)$ is given by,

$$F_U(x) = x\mathbb{1}_{0\leq x\leq 1} + \mathbb{1}_{x>1}$$

Obviously outside of $(0,1)$ both F_U and F_{X_n} agree exactly. Consider a point $x \in (0,1)$. Then, since $|\sin(u)| \leq 1$ for all u,

$$\lim_{n\to\infty} \left[x - \frac{\sin(2n\pi x)}{2n\pi}\right] = x - 0 = x$$

Therefore F_X converges pointwise on to F_U on $(0,1)$, and therefore on all of $\mathbb{R}$.

It is clear that $f_{X_n}(x)$ does not converge to $f_U(x)$ as $f_U(x)$ is constant on $(0,1)$ while $f_{X_n}(x)$ oscillates between zero and two. In particular, fix a rational number $x = p/q$. Then for $n = qk, k \in \mathbb{N}$, $f_{X_n}(x) = 0$.

Exercise 3.7

A coin is tossed repeatedly, with heads turning up with probability p on each toss. Let N be the minimum number of tosses required to obtain k heads. Show that, as $p \to 0$, the distribution function of $2Np$ converges to that of a gamma distribution. Note that, if $X \sim \Gamma(\lambda, r)$ then,

$$f_X(x) = \frac{1}{\Gamma(r)}\lambda^r x^{r-1}e^{-\lambda x}\mathbb{1}_{x\geq 0}$$

Solution

We have $\Gamma(r) = \int_0^\infty x^{r-1}e^{-x}\mathrm{d}x$. Thus, making the substitution $u = (\lambda - it)x$,

$$\begin{aligned}\phi_X(t) &= \mathbb{E}\left[e^{itx}f_X(x)dx\right]\\ &= \int_0^\infty e^{itx}\frac{1}{\Gamma(r)}\lambda^r x^{r-1}e^{-\lambda x}\mathrm{d}x\\ &= \int_0^\infty \frac{\lambda^r}{\Gamma(r)}e^{-u}\frac{u^{r-1}}{(\lambda-it)^{r-1}}\frac{\mathrm{d}u}{(\lambda-it)}\\ &= \frac{\lambda^r}{\Gamma(r)(\lambda-it)^r}\int_0^\infty e^{-u}u^{r-1}\mathrm{d}u\\ &= \frac{\lambda^r}{(\lambda-it)^r}\end{aligned}$$

Let $(X_i)_{i=1}^k$ be idd with $X, X_i \sim \mathrm{Geo}(p)$. Then $N = \sum_{i=1}^k X_i$ so, since the X_i are iid,

$$\varphi_{2Np}(t) = \mathbb{E}[\exp(it2Np)] = \mathbb{E}[\exp(2itp(X_1 + ... + X_k))] = \mathbb{E}[\exp(2itpX)]^k$$

Therefore, since $|e^{2itp}(1-p)| < 1$ if $p \in (0,1)$,

$$\mathbb{E}[\exp(2itpX)]^k = \left[\sum_{m=1}^\infty e^{2itpm}p(1-p)^{m-1}\right]^k = \left[pe^{2itp}\sum_{m=1}^\infty \left(e^{2itp}(1-p)\right)^{m-1}\right]^k = \left[\frac{pe^{2itp}}{1-(1-p)e^{2itp}}\right]^k$$

With Mathematica we evaluate,

```
Limit[((p Exp[2 I t p])/(1 - (1 - p) Exp[2 I t p]))^k, {p -> 0},
sumptions -> {k \[Element] Integers, k > 0}] // FullSimplify
```

This yields,

$$\lim_{p\to 0}\varphi_{2Np} = \frac{1}{(1-2it)^k} = \frac{(1/2)^k}{(1/2-it)^k}$$

> Thus, for a random variable $X \sim \Gamma(1/2, k)$, by the continuity theorem, $\lim_{p \to 0} f_{2Np}(x) = f_X(x)$

Exercise 4.1

A six-sided die is rolled repeatedly. Which of the following a Markov chains? For those that are, find the one-step transition matrix.

(a) X_n is the largest number rolled up to the nth roll.
(b) X_n is the number of sixes rolled in the first n rolls.
(c) At time n, X_n is the time since the last six was rolled.
(d) At time n, X_n is the time until the next six is rolled.

Solution

(a) Yes.

$$P = \begin{bmatrix} 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ & 2/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ & & 36 & 1/6 & 1/6 & 1/6 \\ & & & 4/6 & 1/6 & 1/6 \\ & & & & 5/6 & 1/6 \\ & & & & & 1 \end{bmatrix}$$

(b) Yes.

$$P = \begin{bmatrix} 5/6 & 1/6 & & \\ & 5/6 & 1/6 & \\ & & \ddots & \ddots \end{bmatrix}$$

(c) Yes. Suppose $X_n = i$. The next roll is either a 6, in which case $X_{n+1} = 0$. Otherwise $X_{n+1} = i + 1$.

$$P = \begin{bmatrix} 1/6 & 5/6 & & & \\ 1/6 & & 5/6 & & \\ 1/6 & & & 5/6 & \\ \vdots & & & & \ddots \end{bmatrix}$$

(d) Yes. Suppose $X_n = 0$. The probability of $X_{n+1} = j$ is $(1/6)(5/6)^j$ as you must not roll a 6 for j turns, and then must roll a 6 on the j-th. Suppose $X_n = i > 0$. Then the next step you will be on turn closer to rolling a 6. That is, $X_{n+1} = i - 1$.

$$P = \begin{bmatrix} \frac{1}{6} & \frac{1}{6}\left(\frac{5}{6}\right) & \frac{1}{6}\left(\frac{5}{6}^2\right) & \frac{1}{6}\left(\frac{5}{6}\right)^3 & \cdots \\ 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & \ddots \end{bmatrix}$$

Exercise 4.2

Let $Y_n = X_{2n}$. Compute the transition matrix for Y when

(a) X is a simple random walk (i.e., X increases by one with probability p and decreases by 1 with probability q)
(b) X is a branching process where G is the generating function of the number of offspring from each individual

Solution

(a) In each step we can go down with probability q and then down again with probability q or up with probability p. Alternatively we can go up with probability p and then down with probability q or up again with probability p.

Therefore we will end up two spaces down with probability q^2, in the same position with probability $qp + pq = 2pq$, or up two spaces with probability p^2. Thus,

$$p(i,j) = \begin{cases} p^2 & j = i+2 \\ 2pq & i = j \\ q^2 & j = i-2 \\ 0 & \text{otherwise} \end{cases}$$

(b) We can obtain the exponents of a generating function $G(s) = a_0 + a_1 s + a_2 s^2 + ...$ by,

$$a_n = \frac{1}{n!}\frac{d^n}{ds^n}\Big[G(s)\Big]_{s=0}$$

The coefficient of the s^k term is the value of the probability mass function of X evaluated at k.

The generating function of Y is $G(G(s)) = G_2(s)$ from the notes.

For a branching process with current population k, the population of the next generation will be $X_1 + X_2 + ... + X_k$, where each X_i is iid with distribution X. Therefore,

$$p(i,j) = \frac{1}{j!}\frac{d^n}{ds^n}\Big[G_2(s)^i\Big]_{s=0}$$

Exercise 4.3

Let X be a Markov chain with state space S and absorbing state k (i.e., $p(k,j) = 0$ for all $j \in S$). Suppose $j \to k$ for all $j \in S$. Show that all states other than k are transient.

Solution

Fix a state $j \in S$. By definition of $j \to k$, $\exists N \geq 0 : p_N(j,k) \geq 0$. Since $\{X_N = k | X_0 = j\} \subseteq \{\forall n, X_n \neq j | X_0 = j\}$ we have,

$$0 < p_N(j,k) = \mathbb{P}(X_N = k | X_0 = j) \leq \mathbb{P}(\forall n, X_n \neq j | X_0 = j)$$

Therefore,

$$\mathbb{P}(\exists n \geq 0 : X_n = j | X_0 = j) = 1 - \mathbb{P}(\forall n, X_n \neq j | X_0 = j) < 1$$

This proves state j istransient. $\square$

Exercise 4.4

Suppose two distinct states i, j satisfy

$$\mathbb{P}(\tau_j < \tau_i | X_0 = i) = \mathbb{P}(\tau_i < \tau_j | X_0 = j)$$

where $\tau_j = \inf\{n \geq 1 : X_n = j\}$. Show that, if $X_0 = i$, the expected value of visits to j prior to returning to i is one.

Solution

Write

$$p = \mathbb{P}(\tau_j < \tau_i | X_0 = i) = \mathbb{P}(\tau_i < \tau_j | X_0 = j)$$

That is, p is the probability that we go to state j before state i give we are in state i, and p is also the probability that we go to state i before state j given we are in state j.

Then $1 - p$ is the probability that we do not go to state i before returning state j,0 given we start in state j.

So $(1 - p)^k$ is the probability that we return to state j exactly k times before moving to state i, given we start in state j.

Let N be the number of visits to j prior to returning to i given we start in state i.

The probability that $N = k \in \mathbb{Z}_{\geq 0}$ is the probability that starting from state i we go to state j, return to state j $(k - 1)$ times without returning to state i, and then return to state i without going to returning to state j.

So $\mathbb{P}(N = k | X_0 = i) = p(1 - p)^{k-1}p$. This is the probability mass function for N so,

$$\mathbb{E}[N] = \sum_{n=0}^{\infty} np^2(1 - p)^{k-1} = p \sum_{n=0}^{\infty} n(1 - p)^n = p\frac{p}{(1 - (1 - p))^2} = 1$$

Exercise 4.5

Let X be a Markov chain with transition matrix,

$$P = \begin{bmatrix} 1-2p & 2p & 0 \\ p & 1-2p & p \\ 0 & 2p & 1-2p \end{bmatrix}, \qquad\qquad p \in (0,1)$$

Find P^n, the invariant distribution π, and the mean-recurrence times $\overline{\tau}_j$ for $j = 1, 2, 3$.

Solution

Note that P has eigendecomposition $P = V \Lambda V^{-1}$ where,

$$\Lambda = \begin{bmatrix} 1 & & \\ & 1-4p & \\ & & 1-2p \end{bmatrix}, \qquad\qquad V = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Therefore, $P^n = V \Lambda^n V^{-1}$. Explicitly,

$$P^n = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1-4p & \\ & & 1-2p \end{bmatrix} \begin{bmatrix} 1/4 & 1/2 & 1/4 \\ 1/4 & -1/2 & 1/4 \\ -1/2 & 0 & 1/2 \end{bmatrix}$$

Invariant distributions are linear combinations of left eigenvectors corresponding to eigenvalues of 1. In this case that is the first row of V^{-1}. That is,

$$\pi = \begin{bmatrix} frac14^{\frac{1}{4} \quad \frac{1}{2}} \end{bmatrix}$$

Finally, since the invariant distribution is unique, by Theorem we have,

$$\overline{\tau}_i = \frac{1}{\pi(i)}$$

Exercise 4.6

Let X_n be the number of mistakes in the n-th addition of a book. Between the n-th and the $(n+1)$-th addition an editor corrects each mistake independently with probability p and introduces Y_n new mistakes where the(Y_n) are iid and Poisson distributed with parameter λ. Find the invariant distribution π of the number of mistakes in the book.

Solution

Let $M_{n,k}$ be distributed as $\text{Ber}(1-p)$ so that M_k is 0 if this mistake is corrected, and 1 otherwise. Let Y_n be Poisson distributed with parameter λ. Then,

$$X_{n+1} = Y_n + \sum_{k-1}^{X_n} M_k$$

Each $M_{n,k}$ has generating function,

$$G_{M_{n,k}} = p + (1-p)s = 1 - q + qs = 1 - q(1-s)$$

Similarly. Y_n has generating function,

$$G_Y(s) = \sum_{k=0}^{\infty} e^{-\lambda}\lambda^k/k!s^k = e^{-\lambda}e^{s\lambda} = e^{\lambda(s-1)}$$

Therefore X_{n+1} has generating function,

$$\begin{aligned}
G_{n+1}(s) &= G_Y(s)\mathbb{E}\left[s^{M_{k,1}+M_{k,2}+\ldots+M_{k,X_n}}\right] \\
&= G_Y(s)\mathbb{E}\left[\mathbb{E}\left[s^{M_{k,1}+M_{k,2}+\ldots+M_{k,X_n}}\right]|X_n\right] \\
&= G_Y(s)\mathbb{E}\left[(1-q(1-s))^{X_n}\right] \\
&= G_Y(s)G_n(1-q(1-s))
\end{aligned}$$

First observe $1 - q^i(1-(1-q(1-s))) = 1 - q^{i+1}(1-s)$. We now use the relation

$G_{n+1}(s) = G_Y(s)G_n(1-q(1-s))$ and the fact that $G_0(s) = 1$ to calculate,

$$\begin{aligned}
G_{n+1}(s) &= G_Y(s)G_n(1-q(1-s)) \\
&= G_Y(s)G_Y(1-q(1-s))G_{n-1}(1-q^2(1-s)) \\
&= G_Y(s)G_Y(1-q(1-s))G_Y(1-q^2(1-s))G_{n-2}(1-q^3(1-s)) \\
&\vdots \\
&= \prod_{i=0}^{n} G_Y(1-q^i(1-s))
\end{aligned}$$

Then,

$$\begin{aligned}
\lim_{n\to\infty} G_n(s) &= \lim_{n\to\infty} G_{n+1}(s) \\
&= \lim_{n\to\infty} \prod_{i=0}^{n} G_Y(1-q^i(1-s)) \\
&= \lim_{n\to\infty} \prod_{i=0}^{n} \exp\left(\lambda(-q^i(1-s))\right) \\
&= \exp\left(\sum_{i=0}^{\infty} \lambda(-q^i(1-s)\right) \\
&= \exp\left(\lambda(s-1)\frac{1}{1-q}\right) \\
&= \exp\left(\frac{\lambda}{p}(s-1)\right)
\end{aligned}$$

Thus, $G_n(S)$ converges to the generating function of a Poisson random variable with parameter λ/p.

Then X_n converges to a random variable distributed like a Poisson random variable with parameter λ/p. The random variable for which X_n converges to must be the variable corresponding to the stationary distribution. Therefore, the stationary distribution is distributed like the probability mass function of this random variable. That is,

$$\pi(k) = e^{-\lambda/p}\frac{(\lambda/p)^k}{k!}$$

In the limit $p \to 1$, where we correct all mistakes, the stationary distribution looks like a Poisson distribution with parameter λ. In the limit $\lambda \to 0$ so we do not make any new mistakes, $\pi(0) \to 1$ as expected.

Exercise 4.7

Give an example of a transition matrix P that admits multiple stationary distributions π.

Solution

Define P to be the identity matrix. Then any distribution is a stationary distribution.

Exercise 4.8

A Markov chain on $S = \{0, 1, 2, ..., n\}$ has transition probabilities $p(0, 0) = 1 - \lambda_0$, $p(i, i+1) = \lambda_i$ and $p(i+1, i) = \mu_{i+1}$ for $i = 0, 1, ..., n-1$, and $p(n, n) = 1 - \mu_n$. Show that the process is reversible in equilibrium.

Solution

We assume all entries not specified are zero. (I heard this is the intent, however I wonder why we are given μ_j when $\mu_j = 1 - \lambda_j$ for all j). We write the matrix P as,

Write $\mu_n = 1 - \lambda_n$. Thus, $\mu_i = 1 - \lambda_i$ for $i = 1, ..., n$ as the sum of each row must be 1 (making the assumption that all entries not specified at zero).

$$P = \begin{bmatrix} 1-\lambda_0 & \lambda_0 & & & & \\ \mu_1 & & \lambda_1 & & & \\ & \mu_2 & & \lambda_2 & & \\ & & \mu_3 & & & \\ & & & & & \lambda_{n-1} \\ & & & & \mu_n & 1-\mu_n \end{bmatrix} = \begin{bmatrix} 1-\lambda_0 & \lambda_0 & & & & \\ 1-\lambda_1 & & \lambda_1 & & & \\ & 1-\lambda_2 & & \lambda_2 & & \\ & & 1-\lambda_3 & & & \\ & & & & & \lambda_{n-1} \\ & & & & 1-\lambda_n & \lambda_n \end{bmatrix}$$

This chain is irreducible and finite so a unique invariant distribution π exists. Write $\pi = [\pi_0, \pi_1, ..., \pi_n]$. Then $\pi P = \pi$. That is,

$$\pi P = \begin{bmatrix} \pi_0(1-\lambda_0) + \pi_1(1-\lambda_1) \\ \pi_0\lambda_0 + \pi_2(1-\lambda_2) \\ \pi_1\lambda_1 + \pi_3(1-\lambda_3) \\ \vdots \\ \vdots \\ \pi_{n-1}\lambda_{n-1} + \pi_n\lambda_n \end{bmatrix}^T = \begin{bmatrix} \pi_0 \\ \pi_1 \\ \pi_2 \\ \vdots \\ \pi_j \\ \vdots \\ \pi_n \end{bmatrix}^T$$

$$\begin{aligned}
\pi_1 &= \lambda_0\pi_0/(1-\lambda_1) & \lambda_0\pi_0 &= \pi_1(1-\lambda_1) \\
\pi_2 &= (\pi_1 - \pi_0\lambda_0)/(1-\lambda_2) = \pi_1\lambda_1/(1-\lambda_2) & \lambda_1\pi_1 &= \pi_2(1-\lambda_2) \\
\pi_3 &= (\pi_2 - \pi_1\lambda_1)/(1-\lambda_3) = \pi_2\lambda_2/(1-\lambda_3) & \lambda_2\pi_2 &= \pi_3(1-\lambda_3) \\
&\vdots \\
\pi_{j+1} &= (\pi_j - \pi_{j-1}\lambda_{j-1})/(1-\lambda_{j+1}) = \pi_j\lambda_j/(1-\lambda_{j+1}) & \lambda_j\pi_j &= \pi_{j+1}(1-\lambda_{j+1}) \\
&\vdots \\
\pi_n &= (\pi_{n-1}\lambda_{n-1})/(1-\lambda_n) & \pi_{n-1}\lambda_{n-1} &= \pi_n(1-\lambda_n)
\end{aligned}$$

Observing the equations on the right hand side we have that for $i = 1, 2, ..., n-1$,

$$\pi_i p(i, i+p) = \pi_{i+1} p(i+1, i)$$

We now show the detail balance condition. In particular, we must show,

$$\pi_i p(i, j) = \pi_j p(j, i) \qquad \text{for all } i, j$$

However, for $j \notin \{i-1, i+1\}$ we have $p(i, j) = 0$. Therefore, for this matrix the previous condition is equivalent to

$$\pi_i p(i, i+1) = \pi_{i+1} p(i+1, i) \qquad \text{for } i = 1, 2, ..., n-1$$

We have shown that these equations hold for all $i = 1, 2, ..., n-1$.

This proves π is in detailed balance with P, and so this process is reversible in equilibrium. $\square$

Exercise 5.1

Patients arrive at an emergency room as a Poisson process with intensity λ. The time to treat each patient is an independent exponential random variable with parameter μ. Let $X = (X_t)_{t\geq 0}$ be the number of patients in the system (either being treated or waiting). Write down the generator of X. Show that X has an invariant distribution π if and only if $\lambda < \mu$. Find π. What is the total expected time (waiting + treatment) a patient waits when the system is in its invariant distribution?

Solution

In some small time interval s there is probability $\lambda s + \mathcal{O}(s^2)$ that a patient arrives, probability $1 - \lambda s + \mathcal{O}^2$ that a patient does not arrive, and probability $\mathcal{O}(s^2)$ that multiple patients arrive.

If there are patients, in this times there is also probability $\mu s + \mathcal{O}(s^2)$ that a patient is treated, probability $1 - \mu s + \mathcal{O}(s^2)$ that a patient is not treated, and probability $\mathcal{O}(s^2)$ that more than one (if possible) patients are treated.

Note that any moves which have more than one transition such as a patient arriving, and a patient being treated are all $\mathcal{O}(s^2)$.

Suppose there are no patients at time t. The probability of transitioning to j patients after a short time s is given by,

$$\mathbb{P}(X_{t+s} = j | X_t = 0) = \begin{cases} \lambda s + \mathcal{O}(s^2) & j = 1 \\ 1 - \lambda s + \mathcal{O}(s^2) & j = 0 \\ \mathcal{O}(s^2) & \text{otherwise} \end{cases}$$

Now suppose there are $i > 0$ patients at time t. The probability of transitioning to j patients after a short time s is given by,

$$\mathbb{P}(X_{t+s} = j | X_t = i) = \begin{cases} (\lambda s + \mathcal{O}(s^2))(1 - \mu s + \mathcal{O}(s^2)) & j = i+1 \\ (1 - \lambda s + \mathcal{O}(s^2))(1 - \mu s + \mathcal{O}(s^2)) + \mathcal{O}(s^2) & j = i \\ (1 - \lambda s + \mathcal{O}(s^2))(\mu s + \mathcal{O}(s^2)) & j = i-1 \\ \mathcal{O}(s^2) & \text{otherwise} \end{cases}$$

This is simplified as,

$$\mathbb{P}(X_{t+s} = j | X_t = i) = \begin{cases} \lambda s + \mathcal{O}(s^2) & j = i+1 \\ 1 - \lambda s - \mu s + \mathcal{O}(s^2) & j = i \\ \mu s + \mathcal{O}(s^2) & j = i-1 \\ \mathcal{O}(s^2) & \text{otherwise} \end{cases}$$

This gives,

$$G = \begin{bmatrix} -\lambda & \lambda & & & & \\ \mu & -(\lambda+\mu) & \lambda & & & \\ & \mu & -(\lambda+\mu) & \lambda & & \\ & & \mu & -(\lambda+\mu) & \lambda & \cdots \\ & & & \vdots & \vdots & \ddots \end{bmatrix}$$

We recognize this as a birth-death process (a bit ironic in the context of an emergency room) with $\lambda_i = \lambda$ and $\mu_i = \mu$.

Then if a stationary distribution π exists, for $n \in \mathbb{Z}_{>0}$,

$$\pi(n > 0) = \left(\frac{\lambda}{\mu}\right)^n \pi(0)$$

and

$$\pi(0) = \left(1 + \sum_{n=1}^{\infty} \left(\frac{\lambda}{\mu}\right)^n\right)^{-1} = \left(\sum_{n=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^n\right)^{-1}$$

This is a geometric series which is convergent exactly when $\lambda/\mu < 1$. That is, when $\lambda < \mu$. In this case,

$$\pi(0) = \left(\sum_{n=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^n\right)^{-1} = \left(\frac{\mu}{\mu-\lambda}\right)^{-1} = \frac{\mu-\lambda}{\mu}$$

We condition on knowing the number of people on the queue. Suppose there are n people in the queue when a patient arrives. Then the patient will have to wait a random time distributed as the sum of n exponential random variables with parameter μ to be treated and one more to finish treatment. The expectation of each of each exponential random variable is $1/\mu$, so the patient waits an expected time of $(n+1)/\mu$.

In equilibrium, the probability that there are n people in the queue when a patient arrives is $\pi(n)$.

Therefore, the expected wait time is,

$$\sum_{n=0}^{\infty} \pi(n)\frac{(n+1)}{\mu} = \frac{\mu-\lambda}{\mu^2}\sum_{n=0}^{\infty}\left(\frac{\lambda}{\mu}\right)^n (n+1) = \frac{\mu-\lambda}{\mu^2}\left(\frac{\mu\lambda}{(\mu-\lambda)^2} + \frac{\mu}{\mu-\lambda}\right) = \frac{1}{\mu-\lambda}$$

Exercise 5.2

Let $X = (X_t)_{t\geq 0}$ be a Markov chain with stationary distribution π. Let N be an independent Poisson process with intensity λ and denote by τ_n the time of the n-th arrival of N. Define $Y_n := X_{\tau_n +}$ (i.e., Y_n is the value of X immediately after the n-th jump). Show that Y is a discrete time Markov chain with the same stationary distribution as X.

It is obvious that Y is Markov, as given the present, the future is independent of the past. We add a bit more rigor below.

Fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. By hypothesis X_t is a Markov process. That is, for a filtration $(\mathcal{F}_s)_{s\in[0,T]}$, for $0 \leq s \leq t \leq T$, and for every non-negative Borel measurable function f,

$$\mathbb{E}[f(X_t)|\mathcal{F}_s] = \mathbb{E}[f(X_t)|X_s]$$

Let $\mathcal{F}'_n = \mathcal{F}_{\tau_n +}$ be a sub-σ-algebra of $\mathcal{F}$. Then clearly $(\mathcal{F}'_n)$ is a filtration. Let f be any non-negative Borel measurable function. Then,

$$\mathbb{E}[f(Y_n)|\mathcal{F}'_m] = \mathbb{E}[f(X_{\tau_n +})|\mathcal{F}_{\tau_m +}] = \mathbb{E}[f(X_{\tau_n +})|X_{\tau_m +}] = \mathbb{E}[f(Y_n)|Y_m]$$

This means Y is Markov, and clearly Y is discrete time. Therefore Y is a discrete time Markov chain.

Note we assume X is time homogeneous.

Suppose X has stationary distribution π. Then for all $0 \leq t \leq T$, $\pi P_t = \pi$, where,

$$(P_t)_{i,j} = \mathbb{P}(X_t = j|X_0 = i)$$

Thus, the one step probability transition matrix, denoted $\tilde{P}$, for Y is,

$$\tilde{P}_{i,j} = \mathbb{P}(Y_1 = j|Y_0 = i) = \mathbb{P}(X_{\tau_1 +} = j|X_0 = i) = (P_{\tau_1})_{i,j}$$

This means $\pi\tilde{P} = \pi$, so π is a stationary distribution of Y.

Exercise 5.3

Let $X = (X_t)_{t\geq 0}$ be a Markov chain with state space $S = \{0, 1, 2, ...\}$ and generator G whose i-th row has entries

$$g_{i,i-1} = i\mu \qquad g_{i,i} = -i\mu - \lambda \qquad g_{i,i+1} = \lambda,$$

with all other entries being zero (the zeroth row has only two entries: $g_{0,0}$ and $g_{0,1}$). Assume $X_0 = j$. Find $G_{X_t}(s) := \mathbb{E}s^{X_t}$. What is the distribution of X_t as $t \to \infty$?

Solution

We have G in matrix form,

$$G = \begin{bmatrix} -\lambda & \lambda & & & & \\ \mu & -(\mu+\lambda) & \lambda & & & \\ & 2\mu & -(2\mu+\lambda) & \lambda & & \\ & & 3\mu & -3(\mu+\lambda) & \lambda & \cdots \\ & & & \vdots & \vdots & \ddots \end{bmatrix}$$

We wish to find the transition semi group P_t. We know this can be derived from the Kolmogorov forward equations. That is,

$$\frac{d}{dt}P_t = P_t G$$

With the assumption that $X_0 = i$ (*I am using i rather than j like the problem statement since this is the standard way of doing things*) we have,

$$\frac{d}{dt}p_t(i,0) = \sum_{k=0}^{\infty} p_t(i,k)g(k,0) = -\lambda p_t(i,0) + \mu p_t(i,1)$$
$$\frac{d}{dt}p_t(i,j) = \sum_{k=0}^{\infty} p_t(i,k)g_t(k,j) = \lambda p_t(i,j-1) - (j\mu+\lambda)p_t(i,j) + (j+1)\mu p_t(i,j+1)$$
$$j \geq 1$$

We multiply the j-th equation by s^j. This gives,

$$\sum_{j=0}^{\infty} \frac{\partial}{\partial t}p_t(i,j)s^j = \sum_{j=1}^{\infty}\left[\lambda p_t(i,j-1)s^j\right] - \sum_{j=0}^{\infty}\left[(j\mu-\lambda)p_t(i,j)s^j\right] + \sum_{j=0}^{\infty}\left[(j+1)\mu p_t(i,j+1)s^j\right]$$

Summing the left hand sides gives,

$$\sum_{j=0}^{\infty} \frac{\partial}{\partial t} p_t(i,j)s^j = \frac{\partial}{\partial t} \sum_{j=0}^{\infty} p_t(i,j)s^j = \frac{\partial}{\partial t} G_{X_t}(s)$$

The first term of the right hand side gives,

$$\sum_{j=1}^{\infty} \lambda p_t(i,j-1)s^j = \lambda s \sum_{j=1}^{\infty} p_t(i,j-1)s^{j-1} = \lambda s \sum_{j=0}^{\infty} p_t(i,j)s^j = \lambda s G_{X_t}(s)$$

The negative of the first part of the second term of the right hand side gives,

$$\sum_{j=0}^{\infty} j\mu p_t(i,j)s^j = s\mu \sum_{j=0}^{\infty} jp_t(i,j)s^{j-1} = s\mu \sum_{j=0}^{\infty} \frac{\partial}{\partial s} p_t(i,j)s^j = s\mu \frac{\partial}{\partial s} \sum_{j=0}^{\infty} p_t(i,j)s^j = s\mu \frac{\partial}{\partial s} G_{X_t}(s)$$

The negative of the second part of the second term of the right hand side gives,

$$\sum_{j=0}^{\infty} \lambda p_t(i,j)s^j = \lambda \sum_{j=0}^{\infty} p_t(i,j)s^j = \lambda G_{X_t}(s)$$

The third term of the right hand side gives,

$$\sum_{j=1}^{\infty} (j+1)\mu p_t(i,j+1)s^j = \mu \sum_{j=1}^{\infty} \frac{\partial}{\partial s} p_t(i,j+1)s^{j+1} = \mu \frac{\partial}{\partial s} \sum_{j=0}^{\infty} p_t(i,j)s^j = \mu \frac{\partial}{\partial s} G_{X_t}(s)$$

Putting these results together we have,

$$\frac{\partial}{\partial t} G_{X_t}(s) = \left[\lambda s - s\mu \frac{\partial}{\partial s} - \lambda + \mu \frac{\partial}{\partial s} \right] G_{X_t}(s)$$

Since $X_0 = j$ we have initial condition,

$$G_{X_0}(s) = s^j$$

We solve with Mathematica by,

```
DSolve[{
    D[G[s,t],t]==\[Lambda] s G[s,t]-s \[Mu] D[G[s,t],s]-\[Lambda]
       G[s,t]+\[Mu] D[G[s,t],s],
    G[s,0]==s^j
    },G[s,t],{s,t}]//FullSimplify
```

This yields,

$$G_{X_t}(s) = \left((s-1)e^{-\mu t}+1\right)^j \exp\left[\frac{\lambda(s-1)e^{\mu(-t)}\left(e^{\mu t}-1\right)}{\mu}\right]$$

We find the limit as $t \to \infty$ with Mathematica by,

```
Limit[E^((E^(-t \[Mu]) (-1+E^(t \[Mu])) (-1+s) \[Lambda])/\[Mu])
   (1+E^(-t \[Mu]) (-1+s))^j,{t->\[Infinity]},Assumptions->{\[
   Lambda]>0,\[Mu]>0}]
```

This yields,

$$G_{X_\infty}(s) = \lim_{t\to\infty} G_{X_t}(s) = e^{\frac{\lambda}{\mu}(s-1)}$$

So $X_\infty = \lim_{t\to\infty} X_t$ is a Poission random variable with parameter λ/μ.

Exercise 5.4

Let N be a time-inhomogeneous Poisson process with intensity function $\lambda(t)$. That is, the probability of a jump of size one in the time interval $(t, t+dt)$ is $\lambda(t)dt$ and the probability of two jumps in that interval of time is $\mathcal{O}(dt^2)$. Write down the Kolmogorov forward and backward equations of N and solve them. Let $N_0 = 0$ and let τ_1 be the time of the first jump of N. If $\lambda(t) = c/(1+t)$ show that $\mathbb{E}\tau_1 < \infty$ if and only if $c > 1$.

Solution

Based on the definition of the generator and the given transition probabilities we have,

$$G(t) = \begin{bmatrix} -\lambda(t) & \lambda(t) & & & \\ & -\lambda(t) & \lambda(t) & & \\ & & -\lambda(t) & \lambda(t) & \cdots \\ & & \vdots & \vdots & \ddots \end{bmatrix}$$

For $t \geq s$ we define,

$$p_{s,t}(i,j) = \mathbb{P}(N_t = j | N_s = i)$$

We first derive the Kolmogorov forward equations. We consider,

$$\begin{aligned} p_{s,t+\Delta t} &= \mathbb{P}(N_{t+\Delta t} = j | N_s = i) \\ &= \sum_k \mathbb{P}(N_{t+\Delta t} = j | N_t = k)\mathbb{P}(N_t = k | N_s = i) \\ &= \begin{cases} \lambda(t)\Delta t p_{s,t}(i,j-1) + (1-\lambda(t)\Delta t)p_{s,t}(i,j) + \mathcal{O}(\Delta t^2) & j > i \\ (1-\lambda(t)\Delta t)p_{s,t}(i,j) + \mathcal{O}(\Delta t^2) & j = i \\ 0 & j < i \end{cases} \end{aligned}$$

Therefore,

$$\frac{p_{s,t+\Delta t}(i,j) - p_{s,t}(i,j)}{\Delta t} = \begin{cases} \lambda(t)\Delta t p_{s,t}(i,j-1) - \lambda(t)\Delta t p_{s,t}(i,j) + \mathcal{O}(\Delta t^2) & j > i \\ -\lambda(t)\Delta t p_{s,t}(i,j) + \mathcal{O}(\Delta t^2) & j = i \\ 0 & j < i \end{cases}$$

Taking the limit as $\Delta t \to 0$ we have,

$$\frac{\partial}{\partial t} p_{s,t}(i,j) = \begin{cases} \lambda(t)p_{s,t}(i,j-1) - \lambda(t)p_{s,t}(i,j) & j > i \\ -\lambda(t)p_{s,t}(i,j) & j = i \\ 0 & j < i \end{cases}$$

Fix i. Noting that $G_F(x)$ is also a function of s, t and j, we have,

$$G_F(x) = \sum_{j=0}^{\infty} \mathbb{P}(N_t = j | N_s = i)x^j = \sum_{j=i}^{\infty} p_{s,t}(i,j)x^j$$

Thus, multiplying the j-th KFE by x^j and summing, we have,

$$\begin{aligned}\frac{\partial}{\partial t} \sum_{j=i}^{\infty} p_{s,t}(i,j)x^j = \sum_{j=i}^{\infty} \frac{\partial}{\partial t} p_{s,t}(i,j)x^j &= \sum_{j=i+1}^{\infty} \lambda(t)p_{s,t}(i,j-1)x^j + \sum_{j=i}^{\infty}(-\lambda(t))p_{s,t}(i,j)x^j \\ &= \lambda(t)x \sum_{j=i}^{\infty} p_{s,t}(i,j)x^j - \lambda(t) \sum_{j=i}^{\infty} p_{s,t}(i,j)x^j\end{aligned}$$

Therefore,

$$\frac{\partial}{\partial t} G_F(x) = \lambda(t)xG_F(x) - \lambda(t)G_F(x) = \lambda(t)(x-1)G_F(x)$$

We have initial condition $N_s = i$, so $G_B(x) = x^i$ when $s = t$.

We solve with Mathematica as,

```
DSolve[{D[G[s, t], t] == \[Lambda][t] (x - 1) G[s, t],
   G[s, s] == x^i
   }, G[s, t], {s, t}] // FullSimplify
```

This gives,

$$G_F(x) = x^i \exp\left((x-1)\int_s^t \lambda(z)\mathrm{d}z\right)$$

Write $I = \int_s^t \lambda(z)dz$. Then,

$$G_F(x) = e^{-I}x^i e^{Ix} = e^{-I}x^i \sum_{k=0}^{\infty} \frac{1}{k!}(Ix)^k = e^{-I}\sum_{k=0}^{\infty} \frac{1}{k!} I^k x^{k+i} = e^{-I} \sum_{j=i}^{\infty} \frac{I^{j-i}}{(j-i)!} x^j$$

Therefore, from the definition of the Generating function we have,

$$P_{s,t}(i,j) = \mathbb{P}(N_t = j|N_s = i) = \frac{1}{(j-i)!}\left[\int_s^t \lambda(z)\mathrm{d}z\right]^{j-i} \exp\left(-\int_s^t \lambda(z)\mathrm{d}z\right)$$

We now derive the Kolmogorov Backward equations. We consider,

$$\begin{aligned}
p_{s-\Delta s,t} &= \mathbb{P}(N_t = j|N_{s-\Delta s} = i) \\
&= \sum_k \mathbb{P}(N_t = j|N_s t = k)\mathbb{P}(N_s = k|N_{s-\Delta s} = i) \\
&= \begin{cases} \lambda(s)\Delta s p_{s,t}(i+1,j) + (1-\lambda(s)\Delta s)p_{s,t}(i,j) + \mathcal{O}(\Delta s^2) & j > i \\ (1-\lambda(s)\Delta s)p_{s,t}(i,j) + \mathcal{O}(\Delta s^2) & j = i \\ 0 & j < i \end{cases}
\end{aligned}$$

Therefore,

$$\frac{p_{s-\Delta s,t}(i,j) - p_{s,t}(i,j)}{\Delta s} = \begin{cases} \lambda(s)\Delta t p_{s,t}(i+1,j) - \lambda(s)\Delta t p_{s,t}(i,j) + \mathcal{O}(\Delta s^2) & j > i \\ -\lambda(s)\Delta t p_{s,t}(i,j) + \mathcal{O}(\Delta s^2) & j = i \\ 0 & j < i \end{cases}$$

Taking the limit as $\Delta s \to 0$ we have,

$$-\frac{\partial}{\partial s} p_{s,t}(i,j) = \begin{cases} \lambda(s)p_{s,t}(i+1,j) - \lambda(s)p_{s,t}(i,j) & j > i \\ -\lambda(s)p_{s,t}(i,j) & j = i \\ 0 & j < i \end{cases}$$

Fix i. Noting that $G_B(x)$ is also a function of s, t and j, we have,

$$G_B(x) = \sum_{j=0}^{\infty} \mathbb{P}(N_t = j|N_s = i)x^j = \sum_{j=i}^{\infty} p_{s,t}(i,j)x^j$$

Thus, multiplying the j-th KBE by x^j and summing, we have,

$$
\begin{aligned}
-\frac{\partial}{\partial s}\sum_{j=i}^{\infty} p_{s,t}(i,j)x^j = -\sum_{j=i}^{\infty}\frac{\partial}{\partial s}p_{s,t}(i,j)x^j &= \sum_{j=i+1}^{\infty}\lambda(s)p_{s,t}(i+1,j)x^j + \sum_{j=i}^{\infty}(-\lambda(s))p_{s,t}(i,j)x^j \\
&= \sum_{j=i+1}^{\infty}\lambda(s)p_{s,t}(i,j-1)x^j + \sum_{j=i}^{\infty}(-\lambda(s))p_{s,t}(i,j)x^j \\
&= \lambda(s)x\sum_{j=i}^{\infty}p_{s,t}(i,j)x^j - \lambda(s)\sum_{j=i}^{\infty}p_{s,t}(i,j)x^j
\end{aligned}
$$

Therefore,

$$
\frac{\partial}{\partial s}G_B(x) = -\lambda(s)xG_B(x) + \lambda(s)G_B(x) = -\lambda(s)(x-1)G_B(x)
$$

From the result for $G_F(x)$ we know,

$$
G_B(x) = x^i\exp\left(-(x-1)\int_t^s\lambda(z)\mathrm{d}z\right) = x^i\exp\left((x-1)\int_s^t\lambda(z)\mathrm{d}z\right) = G_F(x)
$$

We now show that for $\lambda(t) = c/(1+t)$, that $\mathbb{E}\tau_1 < \infty$ if and only if $c < 1$. Indeed,

$$
\int_0^t\lambda(z)\mathrm{d}z = \int_0^t\frac{c}{1+z}\mathrm{d}z = c\ln(1+t) - c\ln(1) = c\ln(1+t)
$$

Therefore,

$$
\mathbb{E}[\tau_1] = \int_0^{\infty}\mathbb{P}(\tau_1 > t)\mathrm{d}t = \int_0^{\infty}\mathbb{P}(N_t = 0|N_0 = 0)\mathrm{d}t = \int_0^{\infty}\exp(-c\ln(1+t))\mathrm{d}t = \int_0^{\infty}\frac{\mathrm{d}t}{(1+t)^c}
$$

This is convergent if and only if $c > 1$.

Exercise 5.5

Let N_t be a Poisson process with a random intensity Λ which is equal to λ_1 with probability p and λ_2 with probability $1-p$. Find $G_{N_t}(s) = \mathbb{E}s^{N_t}$. What is the mean and variance of N_t?

Solution

Recall the generating function for a Poisson process with intensity λ is,

$$G(s) = e^{-\lambda t(1-s)}$$

Therefore,

$$G_{N_t}(s) = \mathbb{E}\left[s^{N_t}\right] = \mathbb{E}\left[\mathbb{E}\left[s^{N_t}\right]\middle|\Lambda\right] = \mathbb{E}\left[e^{-\Lambda t(1-s)}\middle|\Lambda\right] = pe^{-\lambda_1 t(1-s)} + (1-p)e^{-\lambda_2(1-s)}$$

We use Mathematica to caluculate moments,

```
GNt[s_]:=p Exp[-\[Lambda]1 t (1-s)]+(1-p)Exp[-\[Lambda]2 t(1-s)]
D[GNt[s],{s,1}]/.{s->1}
D[GNt[s],{s,2}]-D[GNt[s],{s,1}]^2+D[GNt[s],{s,1}]/.{s->1}
```

This yields,

$$\begin{aligned} \mu &= G'_{N_t}(1) = p\lambda_1 t + (1-p)\lambda_2 t \\ \sigma^2 &= G''_{N_t}(1) - [G'_{N_t}(1)]^2 + G'_{N_t}(1) = p(\lambda_1 t)^2 + (1-p)(\lambda_2 t)^2 - \mu^2 + \mu \end{aligned}$$

Exercise 7.1

Let W be a Brownian motion and let $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ be a filtration for W. Show that $W(t)^2 - t$ is a martingale with respect to the filtration $\mathbb{F}$.

Solution

Suppose $X \sim \mathcal{N}(0, \sigma^2)$. Then,

$$\sigma^2 = \mathbb{V}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \mathbb{E}[X^2] - 0^2 = \mathbb{E}[X^2]$$

Let $0 \leq s \leq t$. By the definition of a filtration, $(W(t) - W(s))$ is independent of $\mathcal{F}_s$. Moreover, by the definition of Brownian Motion we have $W(t) - W(s) \sim \mathcal{N}(0, t-s)$. Thus,

$$\mathbb{E}\left[(W(t) - W(s))^2 \middle| \mathcal{F}_s\right] = \mathbb{E}\left[(W(t) - W(s))^2\right] = (t-s)$$

Since $W(s) \in \mathcal{F}_s$, by "taking out what is known" we have,

$$\begin{aligned}
\mathbb{E}\left[W(t)W(s) \middle| \mathcal{F}_s\right] &= W(s)\mathbb{E}\left[W(t) \middle| \mathcal{F}_s\right] = W(s)W(s) = W(s)^2 \\
\mathbb{E}\left[W(s)^2 \middle| \mathcal{F}_2\right] &= W(s)\mathbb{E}\left[W(s) \middle| \mathcal{F}_2\right] = W(s)W(s) = W(s)^2
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathbb{E}\left[W(t)^2 - t \middle| \mathcal{F}_s\right] &= \mathbb{E}\left[(W(t) - W(s) + W(s))^2 - t\right] \\
&= \mathbb{E}\left[(W(t) - W(s))^2 + 2(W(t) - W(s))W(s) + W(s)^2 - t\right] \\
&= \mathbb{E}\left[(W(t) - W(s))^2 \middle| \mathcal{F}_s\right] + 2\mathbb{E}\left[W(t)W(s) \middle| \mathcal{F}_s\right] - \mathbb{E}\left[W(s)^2 \middle| \mathcal{F}_2\right] - \mathbb{E}[t] \\
&= (t-s) + 2W(s)^2 - W(s)^2 - t \\
&= W(s)^2 - s
\end{aligned}$$

This proves $W(t) - t$ is a martingale with respect to the filtration $\mathbb{F}$. $\square$

Exercise 7.2

Compute the characteristic function of $W(N(t))$ where N is a Poisson process with intensity λ and the Brownian motion W is independent of the Poisson process N.

Solution

The characteristic function is defined as,

$$\phi(s) = \mathbb{E}e^{isW(N(t))}$$

We condition on $N(t)$ using iterated conditioning,

$$\mathbb{E}\left[e^{isW(N(t))}\right] = \mathbb{E}\left[\mathbb{E}\left[e^{isW(N(t))}\middle|N(t)\right]\right]$$

The characteristic function of $Z \sim \mathcal{N}(\mu, \sigma^2)$ is $\phi_Z(s) = \exp(i\mu s - \sigma^2 s^2/2)$. At time t, $W(t)$ is normally distributed with mean zero and variance t. Thus,

$$\mathbb{E}\left[\mathbb{E}\left[e^{isW(N(t))}\middle|N(t)\right]\right] = \mathbb{E}\left[e^{-N(t)s^2/2}\right]$$

Since $N(t)$ is a Poisson process with parameter λ, then $N(t) = k$ with probability $(\lambda t)^k e^{-\lambda t}/k!$. Thus,

$$\mathbb{E}\left[e^{-N(t)s^2/2}\right] \sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!} e^{-\lambda t} e^{-ks^2/2} = e^{-\lambda t} \sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!} \left(e^{-s^2/2}\right)^k$$

Simplifying yields,

$$e^{-\lambda t} \sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!} \left(e^{-s^2/2}\right)^k = e^{-\lambda t} \sum_{k=0}^{\infty} \frac{1}{k!} \left(\lambda t e^{-s^2/2}\right)^k = e^{-\lambda t} \exp\left(\lambda t e^{-s^2/2}\right) = \exp\left(\lambda t \left(e^{-s^2/2} - 1\right)\right)$$

That is, the characteristic function $\phi(s)$ of $W(N(t))$ is,

$$\phi(s) = \exp\left(\lambda t \left(e^{-s^2/2} - 1\right)\right)$$

Exercise 7.3

The n-th variation of a function f, over the interval $[0,T]$ is defined as,

$$V_T(n,f) := \lim_{\|\Pi\|\to 0} \sum_{j=0}^{m-1} |f(t_{j+1}) - f(t_j)|^n, \quad \Pi = \{0 = t_0, t_1, \ldots, t_m = T\}, \quad \|\Pi\| = \max_j (t_{j+1} - t_j)$$

Show that $V_T(1,W) = \infty$ and $V_T(3,W) = 0$, where W is a Brownian motion.

Solution

We first prove that if $f_n \to 0$ and $|g_n| \le M$ for some $|M| < \infty$ then $(f_n g_n) \to 0$.

Indeed, fix $\varepsilon > 0$. Then, by convergence of f_n there is some $N \in \mathbb{N}$ such that $|f_n| < \varepsilon/M$ for all $n \ge N$. Then,

$$|f_n g_n| = |f_n||g_n| \le |f_n| M < (\varepsilon/M)M = \varepsilon$$

This proves $f_n g_n \to 0$. $\square$

Write,

$$V_T(k+1,W) = \lim_{\|\Pi\|\to 0} \sum_{j=0}^{m-1} |W(t_{j+1}) - W(t_j)|^{k+1} = \lim_{\|\Pi\|\to 0} \sum_{j=0}^{m-1} |W(t_{j+1}) - W(t_j)|^k |W(t_{j+1}) - W(t_j)|$$

Let, $M_\Pi = \max_j |W(t_{j+1}) - W(t_j)|$ for a given partition Π. Then,

$$\begin{aligned}\lim_{\|\Pi\|\to 0} \sum_{j=0}^{m-1} |W(t_{j+1}) - W(t_j)|^k |W(t_{j+1}) - W(t_j)| &\le \lim_{\|\Pi\|\to 0} \sum_{j=0}^{m-1} |W(t_{j+1}) - W(t_j)|^k M_\Pi \\ &= \lim_{\|\Pi\|\to 0} M_\Pi \sum_{j=0}^{n-1} |W(t_{j+1}) - W(t_j)|^k\end{aligned}$$

Provided, $|V_T(k,T)| = V_T(k,T)$ is not infinite,

$$\lim_{\|\Pi\|\to 0} M_\Pi \sum_{j=0}^{m-1} |W(t_{j+1}) - W(t_j)|^k = \left(\lim_{\|\Pi\|\to 0} M_\Pi\right)\left(\lim_{\|\Pi\|\to 0} \sum_{j=0}^{n-1} |W(t_{j+1}) - W(t_j)|^2\right)$$

Since $W(t)$ is continuous, $|W(t_{j+1}) - W(t_j)| \to 0$ as $\|\Pi\| \to 0$ since $t_{j+1} - t_j \to 0$. In particular, this means that $M_\Pi \to 0$ as $\|\Pi\| \to 0$.

Thus,

$$0 \geq V_T(k+1, W) = \left(\lim_{\|\Pi\| \to 0} M_\Pi \right) \left(\lim_{\|\Pi\| \to 0} \sum_{j=0}^{m-1} |W(t_{j+1}) - W(t_j)|^k \right) \leq 0 \cdot N = 0$$

Recall $V_T(2, W) = T < \infty$. Then, by above, $V_T(3, W) = 0$. □

Suppose, for the sake of contradiction that $V_T(1, W) \neq \infty$. Clearly $V_T(1, W) \geq 0$, so $V_T(1, W)$ is bounded above and below by finite constants. Then, by above, $V_T(2, W) = 0$, a contradiction (for $T > 0$). This proves $V_T(1, W) = \infty$. □

Exercise 7.4

Define

$$X_t = \mu t + W_t \qquad \tau_m := \inf\{t \geq 0 : X_t = m\}$$

Show that Z is a martingale where,

$$Z_t = \exp(\sigma X_t - (\sigma\mu + \sigma^2/2)t)$$

Assume $\mu > 0$ and $m \geq 0$. Assume further that $\tau_m < \infty$ with probability one and the stopped process $Z_{t\wedge\tau_m}$ is a martingale. Find the Laplace transform $\mathbb{E}e^{-\alpha\tau_m}$.

Solution

Let $0 \leq s \leq t$. Rewrite,

$$\mathbb{E}\left[Z_t \middle| \mathcal{F}_s\right] = \mathbb{E}\left[e^{\sigma X_t - (\sigma\mu+\sigma^2/2)t} \middle| \mathcal{F}_s\right] = \mathbb{E}\left[e^{\sigma(\mu t + W_t) - (\sigma\mu+\sigma^2/2)t} \middle| \mathcal{F}_s\right] = \mathbb{E}\left[e^{\sigma W_t - (\sigma^2/2)t} \middle| \mathcal{F}_s\right]$$

Now, pulling out what is known,

$$\mathbb{E}\left[e^{\sigma W_t - (\sigma^2/2)t} \middle| \mathcal{F}_s\right] = \mathbb{E}\left[e^{\sigma(W_t - W_s) + \sigma W_s - (\sigma^2/2)t)} \middle| \mathcal{F}_s\right] = e^{\sigma W_s - (\sigma^2/2)t}\mathbb{E}\left[e^{\sigma(W_t - W_s)} \middle| \mathcal{F}_s\right]$$

By the property of independent increments,

$$e^{\sigma W_s - (\sigma^2/2)t}\mathbb{E}\left[e^{\sigma(W_t - W_s)} \middle| \mathcal{F}_s\right] = e^{\sigma W_s - (\sigma^2/2)t}\mathbb{E}\left[e^{\sigma(W_t - W_s)}\right] = e^{\sigma W_s - (\sigma^2/2)t}e^{\sigma^2(t-s)/2}$$

Finally,

$$e^{\sigma W_s - (\sigma^2/2)t}e^{\sigma^2(t-s)/2} = e^{\sigma W_s - (\sigma^2/2)s} = e^{\sigma(\mu s + W_s) - (\sigma\mu+\sigma^2/2)s} = e^{\sigma X_2 - (\sigma\mu+\sigma^2/2)s}$$

This proves Z_t is a martingale. □

Define $s = \min\{t, \tau_m\}$. Fix $m \geq 0$ and define,

$$Z^{(m)} = \left(Z_t^{(m)}\right)_{t\geq 0}, \qquad Z_t^{(m)} = Z_s$$

Then, using the fact that Z_t is a martingale we have,

$$1 = Z_0^{(m)} = \mathbb{E}\left[Z_t^{(m)}\right] = \mathbb{E}\left[e^{\sigma X_s - (\sigma\mu+\sigma^2/2)s}\right]$$

If $\tau_m = \infty$ then $X_t < m$ for all t. Thus, since $\sigma \geq 0, \mu > 0$,

$$e^{\sigma X_t - (\sigma\mu + \sigma^2/2)t} \leq e^{\sigma m - (\sigma\mu + \sigma^2/2)t} < \infty$$

Therefore, since $\mathbb{P}(\tau_m < \infty) = 0$,

$$\begin{aligned}
\mathbb{E}\left[e^{\sigma X_s - (\sigma\mu+\sigma^2/2)s}\right] &= \mathbb{E}\left[\mathbb{1}_{\{\tau_m = \infty\}}\left(e^{\sigma X_s - (\sigma\mu+\sigma^2/2)s}\right) + \mathbb{1}_{\{\tau_m < \infty\}}\left(e^{\sigma X_s - (\sigma\mu+\sigma^2/2)s}\right)\right] \\
&= \mathbb{E}\left[\mathbb{1}_{\{\tau_m = \infty\}}\left(e^{\sigma X_t - (\sigma\mu+\sigma^2/2)t}\right)\right] + \mathbb{E}\left[\mathbb{1}_{\{\tau_m < \infty\}}\left(e^{\sigma X_{\tau_m} - (\sigma\mu+\sigma^2/2)\tau_m}\right)\right] \\
&= 0 + \mathbb{E}\left[\mathbb{1}_{\{\tau_m < \infty\}}\left(e^{\sigma m - (\sigma\mu+\sigma^2/2)\tau_m}\right)\right]
\end{aligned}$$

Similarly, since $\sigma \geq 0, \mu > 0$, $e^{\sigma m - (\sigma\mu+\sigma^2/2)\tau_m)} < \infty$. Therefore,

$$\begin{aligned}
\mathbb{E}\left[\mathbb{1}_{\{\tau_m < \infty\}}\left(e^{\sigma m - (\sigma\mu+\sigma^2/2)\tau_m}\right)\right] &= \mathbb{E}\left[\mathbb{1}_{\{\tau_m = \infty\}}\left(e^{\sigma m - (\sigma\mu+\sigma^2/2)\tau_m}\right)\right] + \mathbb{E}\left[\mathbb{1}_{\{\tau_m < \infty\}}\left(e^{\sigma m - (\sigma\mu+\sigma^2/2)\tau_m}\right)\right] \\
&= \mathbb{E}\left[\mathbb{1}_{\{\tau_m = \infty\}}\left(e^{\sigma m - (\sigma\mu+\sigma^2/2)\tau_m}\right) + \mathbb{1}_{\{\tau_m < \infty\}}\left(e^{\sigma m - (\sigma\mu+\sigma^2/2)\tau_m}\right)\right] \\
&= \mathbb{E}\left[e^{\sigma m - (\sigma\mu+\sigma^2/2)\tau_m}\right]
\end{aligned}$$

Then, setting $\alpha = (\sigma\mu + \sigma^2/2)$,

$$e^{-\sigma m} = \mathbb{E}\left[e^{-(\sigma\mu+\sigma^2/2)\tau_m}\right] = \mathbb{E}\left[e^{-\alpha\tau_m}\right]$$

We solve the equation, $\alpha = (\sigma\mu + \sigma^2/2)$ for σ using the quadratic equation, yielding,

$$\sigma = -\mu \pm \sqrt{\mu^2 + 2\alpha}$$

However, $\sigma, \alpha \geq 0$ so we must take $\sigma = -\mu + \sqrt{\mu^2 + 2\alpha}$. Thus,

$$\mathbb{E}\left[e^{-\alpha\tau_m}\right] = e^{\left(\mu - \sqrt{\mu^2+2\alpha}\right)m}$$

Exercise 8.1

Compute $\mathrm{d}(W_t^4)$. Write W_T^4 as an integral with respect to W plus an integral with respect to t. Use this representation of W_T^4 to show that $\mathbb{E}W_T^4 = 3T^2$. Compute $\mathbb{E}W_T^6$ using the same technique.

Solution

Write $f(x) = x^4$ so that $f(W_t) = W_t^4$. Then, $f'(x) = 4x^3$ and $f''(x) = 12x^2$. Therefore, Itô's formula gives,

$$\mathrm{d}W_t^4 = f'(W_t)\mathrm{d}W_t + \frac{1}{2}f''(W_t)\mathrm{d}[W,W]_t = 4W_t^3\mathrm{d}W_t + \frac{12}{2}W_t^2\mathrm{d}[W,W]_t$$

Thus, writing $\mathrm{d}[W,W]_t = \mathrm{d}t$ we have,

$$\mathrm{d}W_t^4 = 4W_t^3\mathrm{d}W_t + 6W_t^2\mathrm{d}t$$

Thus, since $W_0 = 0$,

$$W_T^4 = W_T^4 - W_0^4 = 4\int_0^T W_t^3\mathrm{d}W_t + 6\int_0^T W_t^2\mathrm{d}t$$

Recall Itô integrals are martingales so that,

$$\mathbb{E}\left[\int_0^T W_t^3\mathrm{d}W_t\right] = 0$$

Note also that since $\mathbb{E}\left[W_t^2\right] = t$,

$$\mathbb{E}\left[\int_0^T W_t^2\mathrm{d}t\right] = \int_0^T \mathbb{E}\left[W_t^2\right]\mathrm{d}t = \int_0^T t\mathrm{d}t = \frac{T^2}{2}$$

Therefore,

$$\mathbb{E}\left[W_T^4\right] = 4\mathbb{E}\left[\int_0^T W_t^3\mathrm{d}W_t\right] + 6\mathbb{E}\left[\int_0^T W_t^2\mathrm{d}t\right] = 6\frac{T^2}{2} = 3T^2$$

Similarly, we have,

$$W_T^6 = 6\int_0^T W_t^5\mathrm{d}W_t + \frac{6\cdot 5}{2}\int_0^T W_t^4\mathrm{d}t$$

Therefore, since $\mathbb{E}\left[W_t^4\right] = 3t^2$,

$$\mathbb{E}\left[W_T^6\right] = 6\mathbb{E}\left[\int_0^T W_t^5 \mathrm{d}W_t\right] + 15\mathbb{E}\left[\int_0^T W_t^4 \mathrm{d}t\right] = 15\int_0^T \mathbb{E}\left[W_t^4\right] \mathrm{d}t = 15\int_0^T 3t^2 \mathrm{d}t = 15T^3$$

Exercise 8.2

Find an explicit expression for Y_T where,

$$\mathrm{d}Y_t = r\mathrm{d}t + \alpha Y_t \mathrm{d}W_t$$

Hint: Multiply the above equation by $F_t := \exp(-\alpha W_t + \frac{1}{2}\alpha^2 t)$.

Solution

Let $f(x, y) = \exp(-\alpha x + \frac{1}{2}\lambda^2 y)$ so that,

$$f_x(W_t, t) = -\alpha F_t \qquad f_y(W_t, t) = \frac{\alpha^2}{2}F_t \qquad f_{xx}(W_t, t) = \alpha^2 F_t$$

Then $F_t = f(W_t, t)$, so by Itô's formula and the heuristic $(\mathrm{d}W_t)^2 = \mathrm{d}t, (\mathrm{d}t)^2 = \mathrm{d}t\mathrm{d}W_t = 0$,

$$\begin{aligned}
\mathrm{d}F_t = \mathrm{d}f(W_t, t) &= f_y(W_t, t)\mathrm{d}t + f_x(W_t, t)\mathrm{d}W_t + \frac{1}{2}f_{xx}(W_t, t)(\mathrm{d}W_t)^2 \\
&= \frac{\alpha^2}{2}F_t\mathrm{d}t - \alpha F_t\mathrm{d}W_t + \frac{\alpha^2}{2}F_t\mathrm{d}t \\
&= \alpha^2 F_t\mathrm{d}t - \alpha F_t\mathrm{d}W_t
\end{aligned}$$

Using our heuristics we have,

$$\mathrm{d}[F, Y]_t = (\mathrm{d}F_t)(\mathrm{d}Y_t) = \left(\alpha^2 F_t\mathrm{d}t - \alpha F_t\mathrm{d}W_t\right)(r\mathrm{d}t + \alpha Y_t\mathrm{d}W_t) = -\alpha^2 F_t Y_t(\mathrm{d}W_t)^2 = -\alpha^2 F_t Y_t\mathrm{d}t$$

By the product rule we have,

$$\begin{aligned}
\mathrm{d}(F_t Y_t) &= F_t\mathrm{d}Y_t + Y_t\mathrm{d}F_t + \mathrm{d}[F, Y]_t \\
&= F_t(r\mathrm{d}t + \alpha Y_t\mathrm{d}W_t) + Y_t(\alpha^2 F_t\mathrm{d}t - \alpha F_t\mathrm{d}W_t) - \alpha^2 F_t Y_t\mathrm{d}t \\
&= rF_t\mathrm{d}t
\end{aligned}$$

In integral form,

$$F_t Y_t - F_0 Y_0 = \int_0^t rF_s\mathrm{d}s = \int_0^t re^{-\alpha W_s + \frac{1}{2}\alpha^2 s}\mathrm{d}s$$

We can add $F_0 Y_0 = Y_0$ and divide by F_t yielding,

$$Y_t = Y_0 + re^{\alpha W_t - \frac{1}{2}\alpha^2 t}\int_0^t e^{-\alpha W_s + \frac{1}{2}\alpha^2 s}\mathrm{d}s$$

Exercise 8.3

Suppose X, Δ, and Π are given by,

$$\mathrm{d}X_t = \sigma X_t \mathrm{d}W_t, \qquad \Delta_t = \frac{\partial f}{\partial x}(t, X_t), \qquad \Pi_t = X_t \Delta_t$$

where f is some smooth function. Show that if f satisfies,

$$\left(\frac{\partial}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2}{\partial x^2} \right) f(t,x) = 0$$

for all (t, x), then Π is a martingale with respect to a filtration $\mathcal{F}_t$ for W.

Solution

We have,

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2}{\partial x^2} \right) = \frac{\partial^2}{\partial x \partial t} + \frac{1}{2}\sigma^2 \left[x^2 \frac{\partial^3}{\partial x^3} + 2x \frac{\partial^2}{\partial x^2} \right]$$

Thus, using the condition for f we have,

$$\frac{\partial^2 f}{\partial x \partial t} + \frac{1}{2}\sigma^2 X_t^2 \frac{\partial^3 f}{\partial x^3} = -\sigma^2 X_t \frac{\partial^2 f}{\partial x^2}$$

Using our heuristics we have,

$$\mathrm{d}[X, X] = \sigma^2 X_t^2 (\mathrm{d}W_t)^2 = \sigma^2 X_t^2 \mathrm{d}t$$

Similarly,

$$\mathrm{d}[X, t] = \mathrm{d}[t, X] = \mathrm{d}[t, t] = 0$$

Therefore, by Itô's formula,

$$\begin{aligned}
\mathrm{d}\Delta_t &= \frac{\partial^2 f}{\partial x \partial t}(t, X_t)\mathrm{d}t + \frac{\partial^2 f}{\partial x^2}(t, X_t)\mathrm{d}X_t + \frac{1}{2}\mathrm{d}[X, X] \\
&= \frac{\partial^2 f}{\partial x \partial t}(t, X_t)\mathrm{d}t + \sigma X_t \frac{\partial^2 f}{\partial x^2}(t, X_t)\mathrm{d}W_t + \frac{1}{2}\sigma^2 X_t^2 \frac{\partial^3 f}{\partial x^3}(t, X_t)\mathrm{d}t \\
&= -\sigma^2 X_t \frac{\partial^2 f}{\partial x^2}(t, X_t)\mathrm{d}t + \sigma X_t \frac{\partial^2 f}{\partial x^2}(t, X_t)\mathrm{d}W_t
\end{aligned}$$

Therefore,

$$\mathrm{d}[X, \Delta]_t = (\mathrm{d}X_t)(\mathrm{d}\Delta_t) = \sigma^2 X_t^2 \frac{\partial^2 f}{\partial x^2}(t, X_t)(\mathrm{d}W_t)^2 = \sigma^2 X_t^2 \frac{\partial^2 f}{\partial x^2}(t, X_t)\mathrm{d}t$$

Finally, we have,

$$\begin{aligned}
\mathrm{d}\Pi_t &= \mathrm{d}(X_t \Delta_t) = X_t \mathrm{d}\Delta_t + \Delta_t \mathrm{d}X_t + \mathrm{d}[X, \Delta]_t \\
&= X_t \left(-\sigma^2 X_t \frac{\partial^2 f}{\partial x^2}(t, X_t)\mathrm{d}t + \sigma X_t \frac{\partial^2 f}{\partial x^2}(t, X_t)\mathrm{d}W_t \right) + \sigma X_t \frac{\partial f}{\partial x}(t, X_t)\mathrm{d}W_t + \sigma^2 X_t^2 \frac{\partial^2 f}{\partial x^2}\mathrm{d}t \\
&= \sigma X_t \left(X_t \frac{\partial^2 f}{\partial x^2}(t, X_t) + \frac{\partial f}{\partial x}(t, X_t) \right) \mathrm{d}W_t
\end{aligned}$$

Since there is no $\mathrm{d}t$ dependence this is an Itô integral and therefore a martingale with respect to a filtration for W. (there are probably some technical assumptions we need about X and f, but in class we never dealt with these) $\square$

Exercise 8.4

Suppose X is given by,

$$\mathrm{d}X_t = \mu(t, X_t)dt + \sigma(t, X_t)\mathrm{d}W_t$$

For any smooth function f define,

$$M_t^f := f(t, X_t) - f(0, X_0) - \int_0^t \left(\frac{\partial}{\partial s} + \mu(s, X_s)\frac{\partial}{\partial x} + \frac{1}{2}\sigma^2(s, X_s)\frac{\partial^2}{\partial x^2} \right) f(s, X_s)ds$$

Show that M^f is a martingale with respect to a filtration $\mathcal{F}_t$ for W.

Solution

We first compute,

$$\mathrm{d}[X, X]_t = (\mathrm{d}X_t)(\mathrm{d}X_t) = \sigma^2(t, X_t)(\mathrm{d}W_t)^2 = \sigma^2(t, X_t)\mathrm{d}t$$

We then have,

$$\begin{aligned}
\mathrm{d}f(t, X_t) &= \frac{\partial f}{\partial t}(t, X_t)\mathrm{d}t + \frac{\partial f}{\partial x}(t, X_t)\mathrm{d}X_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\mathrm{d}[X, X]_t \\
&= \frac{\partial f}{\partial t}(t, X_t)\mathrm{d}t + \frac{\partial f}{\partial x}(t, X_t)[\mu(t, X_t)\mathrm{d}t + \sigma(t, X_t)\mathrm{d}W_t] + \frac{1}{2}\sigma^2(t, X_t)\frac{\partial^2 f}{\partial x^2}\mathrm{d}t \\
&= \left(\frac{\partial}{\partial t} + \mu(t, X_t)\frac{\partial}{\partial x} + \frac{1}{2}\sigma^2(t, X_t)\frac{\partial^2}{\partial x^2} \right) f(t, X_t)\mathrm{d}t + \sigma(t, X_t)\frac{\partial f}{\partial x}\mathrm{d}W_t
\end{aligned}$$

Finally, since $f(0, X_0)$ is a constant,

$$\begin{aligned}
\mathrm{d}M_t^f &= \mathrm{d}f(t, X_t) - \left(\frac{\partial}{\partial t} + \mu(t, X_t)\frac{\partial}{\partial x} + \frac{1}{2}\sigma^2(t, X_t)\frac{\partial^2}{\partial x^2} \right) f(t, X_t)\mathrm{d}t \\
&= \sigma(t, X_t)\frac{\partial f}{\partial x}\mathrm{d}W_t
\end{aligned}$$

Since there is no dt dependence this an Itô integral and therefore a martingale with respect to a filtration for W. $\square$

Exercise 9.2

Let X be a solution to the following SDE

$$\mathrm{d}X_t = \kappa(\theta - X_t)\mathrm{d}t + \delta\sqrt{X_t}\mathrm{d}W_t$$

Define

$$u(t,x) = \mathbb{E}\left[\exp\left(-\int_t^T X_s \mathrm{d}s\right)\middle| X_t = x\right]$$

Derive a PDE for the function u. To solve the PDE for u, try a solution of the form

$$u(t,x) = \exp(-xA(t) - B(t)),$$

where A and B are deterministic functions of t. Show that A and B must satisfy a pair of coupled ODEs (with appropriate terminal conditions at time T). Bonus question: solve the ODEs (it may be helpful to note that one of the ODEs is a Riccati equation).

Solution

With $\gamma(u,x) = x$, $\phi(x) = 1$, $g(u,x) = 0$ this is a subcase of an example in the notes. We then know $u(t,x)$ solves,

$$(\partial_t + \mathcal{A})u + g = 0, \qquad u(T,\cdot) = \phi, \qquad \mathcal{A} = \frac{1}{2}\sigma^2\partial_x^2 + \mu\partial_x - \gamma = 0$$

First compute,

$$\partial_t u = (-xA' - B')u \qquad \partial_x u = -Au \qquad \partial_x^2 u = A^2 u$$

This gives,

$$\begin{aligned}
0 &= \left[\partial_t + \frac{1}{2}\delta^2 x\partial_x^2 + \kappa(\theta - x)\partial_x - x\right]u \\
&= \left[-xA' - B' + \frac{1}{2}\delta^2 xA^2 + \kappa(\theta - x)(-A) - x\right]u \\
&= \left[\left(-A' + \frac{1}{2}\delta^2 A^2 + \kappa A - 1\right)x + (-B' - \kappa\theta A)\right]u
\end{aligned}$$

Observe $u(t,x) > 0$ for all t, x. Therefore we require the bracketed term above to be zero for all x, t. Setting the coefficients of the x terms and constant terms to zero gives a coupled pair of ODEs,

$$\begin{cases} -A'(t) + \frac{1}{2}\delta^2 A^2(t) + \kappa A(t) - 1 = 0 \\ -B'(t) - \kappa\theta A(t) = 0 \end{cases}$$

We have,

$$1 = \varphi(x) = u(T,x) = \exp\left(-xA(T) - B(T)\right)$$

This gives terminal condition,

$$A(T) = 0 \qquad\qquad B(T) = 0$$

We solve this in Mathematica without boundary conditions using,

```
DSolve[{-D[A[t],t]+1/2 \[Delta]^2 A[t]^2+\[Kappa] A[t] - 1 ==0 , -
   D[B[t],t]-\[Kappa] \[Theta] A[t]==0},{A,B},t]
```

This gives solution,

$$\begin{aligned} A(t) &= \frac{\sqrt{-2\delta^2-\kappa^2}\tan\left(\frac{1}{2}\left(2c_1\sqrt{-2\delta^2-\kappa^2} + t\sqrt{-2\delta^2-\kappa^2}\right)\right) - \kappa}{\delta^2} \\ B(t) &= \frac{\theta\kappa\left(2\log\left(\cos\left(c_1\sqrt{-2\delta^2-\kappa^2} + \frac{1}{2}t\sqrt{-2\delta^2-\kappa^2}\right)\right) + \kappa t\right)}{\delta^2} + c_2 \end{aligned}$$

where,

$$\begin{aligned} c_1 &= \frac{1}{2\sqrt{-2\delta^2-\kappa^2}}\left[2\arctan\left(\frac{\kappa}{\sqrt{-2\delta^2-\kappa}}\right) - T\sqrt{-2\delta^2-\kappa^2}\right] \\ c_2 &= -\frac{\theta\kappa\left(2\log\left(\cos\left(c_1\sqrt{-2\delta^2-\kappa^2} + \frac{1}{2}T\sqrt{-2\delta^2-\kappa^2}\right)\right) + \kappa T\right)}{\delta^2} \end{aligned}$$

We could have done this by hand by since the first equation is separable but its just as ugly.

Exercise 9.2

Solution

Exercise 9.3

For $i = 1, 2, \ldots, d$ let $X^{(i)}$ satisfy,

$$\mathrm{d}X_t^{(i)} = -\frac{b}{2} X_t^{(i)} \mathrm{d}t + \frac{1}{2}\sigma \mathrm{d}W_t^{(i)}$$

where $(W_t^{(i)})_{i=1}^d$ are independent Brownian motions. Define

$$R_t := \sum_{i=1}^d \left(X_t^{(i)}\right)^2, \qquad B_t := \sum_{i=1}^d \int_0^t \frac{1}{\sqrt{R_s}} X_s^{(i)} \mathrm{d}W_s^{(i)}$$

Show that B is a Brownian motion. Derive an SDE for R that involves only $\mathrm{d}t$ and $\mathrm{d}B_t$ terms (i.e., no $\mathrm{d}W_t^{(i)}$ terms should appear).

Solution

We use the Lévy characterization of Brownian motion. In particular, we must show B is a martingale, B has continuous sample paths, and $B_0 = 0$ with $[B, B]_t = t$ for all $t \geq 0$.

Write,

$$\mathrm{d}B_t = \mathrm{d}\left[\sum_{i=1}^d \int_0^t \frac{1}{\sqrt{R_s}} X_s^{(i)} \mathrm{d}W_s^{(i)}\right] = \sum_{i=1}^d \frac{1}{\sqrt{R_t}} X_t^{(i)} \mathrm{d}W_t^{(i)}$$

As B_t is an Itô integral it is a martingale with respect to a filtration $\mathbb{F} = (\mathcal{F}_\sqcup)_{t\geq 0}$ for $W_t^{(i)}$.

Similarly, B_t has continuous sample paths as $W_t^{(i)}$ have continuous sample paths.

Clearly $B_0 = 0$ as $W_0^{(i)} = 0$.

Now,

$$\begin{aligned}
(\mathrm{d}B_t)(\mathrm{d}B_t) &= \frac{1}{R_t} \sum_{i=1}^d \sum_{j=1}^d X_t^{(i)} X_t^{(j)} \mathrm{d}W_t^{(i)} \mathrm{d}W_t^{(j)} \\
&= \frac{1}{R_t} \left(\sum_{j=1}^d \left(X_t^{(i)} \mathrm{d}W_t^{(i)}\right)^2 + 2 \sum_{i=1}^d \sum_{j=1}^i X_t^{(i)} X_t^{(j)} \mathrm{d}W_t^{(i)} \mathrm{d}W_t^{(j)} \right)
\end{aligned}$$

Using the heuristic, $\mathrm{d}W_t^{(i)}\mathrm{d}W_t^{(j)} = \delta_{ij}\mathrm{d}t$ and the definition of R_t we have,

$$\mathrm{d}[B,B]_t = \frac{1}{R_t}\sum_{i=1}^{d}\left(X_t^{(i)}\right)^2 \mathrm{d}t = \mathrm{d}t$$

Therefore, $[B,B]_t = t$.

This proves B is a Brownian motion. $\square$

Compute, using Itô's formula,

$$\mathrm{d}R_t = \mathrm{d}\left[\sum_{i=1}^{d}\left(X_t^{(i)}\right)^2\right] = \sum_{i=1}^{d} 2X_t^{(i)}\mathrm{d}X_t^{(i)} + \frac{1}{2}2\mathrm{d}[X^{(i)},X^{(i)}]_t = \sum_{i=1}^{d} 2X_t^{(i)}\mathrm{d}X_t^{(i)} + \mathrm{d}[X^{(i)},X^{(i)}]_t$$

Using our heuristics we have,

$$\mathrm{d}[X^{(i)},X^{(i)}]_t = \left(\mathrm{d}X_t^{(i)}\right)\left(\mathrm{d}X_t^{(i)}\right) = \left(-\frac{b}{2}X_t^{(i)}\mathrm{d}t + \frac{1}{s}\sigma\mathrm{d}W_t^{(i)}\right)^2 = \frac{\sigma^2}{4}\mathrm{d}t$$

Now,

$$\begin{aligned}\sum_{i=1}^{d} 2X_t^{(i)}\mathrm{d}X_t^{(i)} + \mathrm{d}[X^{(i)},X^{(i)}]_t &= \sum_{i=1}^{d} 2X_t^{(i)}\left(-\frac{b}{2}X_t^{(i)}\mathrm{d}t + \frac{1}{2}\sigma\mathrm{d}W_t^{(i)}\right) + \frac{\sigma^2}{4}\mathrm{d}t \\ &= \sum_{i=1}^{d}\left(\frac{\sigma^2}{4} - b\left(X_t^{(i)}\right)^2\right)\mathrm{d}t + \sigma\sqrt{R_t}\frac{1}{\sqrt{R_t}}X_t^{(i)}\mathrm{d}W_t^{(i)}\end{aligned}$$

Therefore, simplifying slightly we have,

$$\mathrm{d}R_t = (d\sigma^2/4 - bR_t)\mathrm{d}t + \sigma\sqrt{R_t}\mathrm{d}B_t$$

Exercise 9.4

Solution

Exercise 9.5

Consider a diffusion $X = (X_t)_{t\geq 0}$ that lives on a finite interval (l, r), $0 < l < r < \infty$ and satisfies the SDE

$$\mathrm{d}X_t = \mu X_t \mathrm{d}t + \sigma X_t \mathrm{d}W_t$$

One can easily check that the endpoints l and r are regular (you do not have to prove it here). Assume both endpoints are killing. Find the transition density $\Gamma(t, x; T, y)$ of X.

Solution

We have, $\Gamma(\cdot, \cdot; T, y)$ satisfies,

$$(\partial_t + \mathcal{A}(t))\Gamma(\cdot, t; T, y) = 0 \qquad\qquad \Gamma(T, \cdot; T, y) = \delta_y$$

where the infinitesimal generator $\mathcal{A}$ is,

$$\mathcal{A} = \mu x \partial_x + \frac{1}{2}\sigma^2 x^2 \partial_x^2$$

We seek a spectral representation for $\mathcal{A}$. That is, a basis $\{\Psi_n\}_{n\geq 0}$ for such that $\mathcal{A}\Psi_n = \lambda_n \Psi_n$.

Since the endpoints are killing we also require,

$$\Psi_n(l) = 0, \qquad\qquad \Psi_n(r) = 0$$

We make a change of variables. Let $z = \log(x)$. Then,

$$\partial_x = \frac{1}{x}\partial_z, \qquad\qquad \partial_x^2 = -\frac{1}{x^2}\partial_z + \frac{1}{x}\partial_z^2$$

Then, in terms of z we have generator,

$$\mathcal{A}_z = \left(\mu - \frac{\sigma^2}{2}\right)\partial_z + \frac{1}{2}\sigma^2 \partial_z^2$$

This equation is very similar to a damped harmonic oscillator. We therefore guess that the eigenfunctions have the form,

$$\psi_n(z) = \exp(\gamma_n z)\left[A \sin\left(\frac{n\pi(z - \log(l)}{\log(r) - \log(l)}\right) + B\cos\left(\frac{n\pi(z - \log(l)}{\log(r) - \log(l)}\right)\right]$$

In order to satisfy the boundary conditions listed above we need $B = 0$. The constant A will be determined by the normalization of ψ_n, so we will leave it off until the end.

For convenience, write,

$$\psi = \psi_n, \qquad \gamma = \gamma_n, \qquad k = \frac{n\pi}{\log(l/r)}, \qquad \cos(z') = \cos(k(z - \log l))$$

We then have,

$$\partial_z \psi(z) = \gamma\psi + \exp(\gamma z) k \cos(z')$$
$$\partial_z^2 \psi(z) = \gamma^2\psi + \gamma\exp(\gamma z) k \cos(z')) + \gamma\exp(\gamma z) k\cos(z') - k^2\psi = \gamma^2\psi + 2\gamma\exp(\gamma z) k\cos(z') - k^2\psi$$

We seek γ such that $\mathcal{A}_z\psi = \lambda\psi$ for some constant λ. That is, in our expression of $\mathcal{A}_z\psi$ we require the terms not containing a ψ be zero. Thus,

$$0 = \left(\mu - \frac{\sigma^2}{2}\right)\exp(\gamma z) k\cos(z') + \left(\frac{\sigma^2}{2}\right) 2\gamma\exp(\gamma z) k\cos(z') = \left[\left(\mu - \frac{\sigma^2}{2}\right) + \sigma^2\gamma\right]\exp(\gamma z)\cos(z')$$

Suppose $k \neq 0$ (i.e. that the solution is non-trivial). Since $\exp(\gamma z)$ and $\cos(z') \neq 0$ we have,

$$0 = \left(\mu - \frac{\sigma^2}{2}\right) + \sigma^2\gamma$$

Solving for γ we have,

$$\gamma = \frac{1}{2} - \frac{\mu}{\sigma^2}$$

The eigenvalues are,

$$\lambda_n = \left(\mu - \frac{\sigma^2}{2}\right)\gamma + \left(\frac{\sigma^2}{2}\right)\left(\gamma^2 - k^2\right) = -\frac{\sigma^2}{2}[k^2 + \gamma^2]$$

Transforming back to x we have, $\hat{\Psi}_n(x) = \psi_n(\log(x))$ satisfies,

$$\mathcal{A}\hat{\Psi}_n(x) = \lambda_n\hat{\Psi}_n(x), \qquad \mathcal{A} = \mu x\partial_x + \frac{1}{2}\sigma^2 x^2\partial_x^2$$

Define,

$$m(y) = \frac{2}{\sigma^2 y^2}\exp\left(\int \mathrm{d}y\frac{2\mu y}{\sigma^2 y^2}\right) = \frac{2}{\sigma^2 y^2}\exp\left(\frac{2\mu}{\sigma^2}\log(y)\right) = \frac{2}{\sigma^2}y^{2\mu/\sigma^2 - 2} = \frac{2}{\sigma^2}y^{-2\gamma-1}$$

It is clear that the $\hat{\Psi}_n$ are orthogonal (properties of sines). We compute,

$$\langle \hat{\Psi}_n(x), \hat{\Psi}_n(x) \rangle_m = \int_l^r \Psi_n(x)^2 m(x) \mathrm{d}x = \log(r/l)/\sigma^2$$

We then satisfy $\langle \Psi_k, \Psi_l \rangle_m = \delta_{kl}$ by defining,

$$\Psi_n(x) = \frac{\hat{\Psi}_n(x)}{\sqrt{\langle \Psi_n(x), \Psi_n(x) \rangle_m}}$$

Explicitly,

$$\Psi_n(x) = \frac{\sigma}{\sqrt{\log(r/l)}} x^\gamma \sin(k(z - \log l)) = \frac{\sigma}{\sqrt{\log(r/l)}} x^{1/2-\mu/\sigma^2} \sin\left(n\pi \frac{\log(x/l)}{\log(r/l)}\right)$$

Finally,

$$\Gamma(t, x; T, y) = m(y) \sum_n \exp((T-t)\lambda_n) \Psi_n(x) \Psi_n(y)$$

Explicitly,

$$\Gamma(t, x; T, y) = \frac{2}{\log(r/l)} \left(\frac{x}{y}\right)^{1/2-\mu/\sigma^2} y^{-1} \sum_n \exp((T-t)\lambda_n) \sin\left(n\pi \frac{\log(x/l)}{\log(r/l)}\right) \sin\left(n\pi \frac{\log(y/l)}{\log(r/l)}\right)$$

Since the Ψ_n are normalized then Γ is normalized.

We verify in Mathematica that Γ satisfies both the KFE and KBE.

Exercise 9.6

Consider a two-dimensional diffusion processes $X = (X_t)_{t \geq 0}$ and $Y = (Y_t)_{t \geq 0}$ that satisfy the SDEs

$$\mathrm{d}X_t = \mathrm{d}W_t^1 \qquad\qquad \mathrm{d}Y_t = \mathrm{d}W_t^2$$

where W_t^1 and W_t^2 are two independent Brownian motions. Define a function u as follows

$$u(x, y) = \mathbb{E}\left[\phi(X_\tau) | X_t = x, Y_t = y\right], \qquad\qquad \tau = \inf\{s \geq t : Y_s = a\}$$

1. State a PDE and boundary conditions satisfied by the function u.
2. Let us define the Fourier transform and and inverse Fourier transform, respectively, as follows

$$\text{Fourier Transform:} \qquad\qquad \hat{f}(\omega) := \int e^{-i\omega x} f(x)\mathrm{d}x$$

$$\text{Inverse Transform:} \qquad\qquad f(x) := \frac{1}{2\pi} \int e^{i\omega x} \hat{f}(\omega)\mathrm{d}\omega$$

   Use Fourier transforms and a conditioning argument to derive an expression for $u(x, y)$ as an inverse Fourier transform. Use this result to derive an explicit form for $\mathbb{P}(X_\tau \in \mathrm{d}z | X_t = x, Y_t = y)$(i.e., an expression involving no integrals).

3. Show the expression you derived in part 2 for $u(x, y)$ satisfies the PDE and BCs you stated in part 1.

Solution

1. Since there are no $\mathrm{d}t$ terms in either Brownian motion, and since the coefficient in both of the $\mathrm{d}W_t$ term is 1 we have, generator,

$$\mathcal{A} = \frac{1}{2}\partial_x^2 + \frac{1}{2}\partial_y^2$$

   The PDE satisfied by u is,

$$\mathcal{A}u = \left(\frac{1}{2}\partial_x^2 + \frac{1}{2}\partial_y^2\right) u = 0 \qquad \Longleftrightarrow \qquad \left(\partial_x^2 + \partial_y^2\right) u = 0$$

   If $y = a$ then $\tau = t$ so $X_\tau = x$. We therefore have boundary condition,

$$u(x, a) = \phi(x)$$

2. Given starting position (x, y) at time t, and time τ, from the notes we know X_τ is normally distributed with mean x and variance $\tau - t$ by the independent increments property of Brownian motion. We know the characteristic function of a normally distributed random variable with distribution $\mathcal{N}(\mu, \sigma^2)$ is $e^{i\omega x - \sigma^2\omega^2/2}$. Therefore,

$$\mathbb{E}\left[e^{i\omega X_\tau}\middle|\tau, X_t = x, Y_t = y\right] = e^{i\omega x - (\tau - t)\omega^2/2}$$

Thus, using iterated conditioning,

$$\begin{aligned}\mathbb{E}\left[e^{i\omega X_\tau}|X_t = x, Y_t = y\right] &= \mathbb{E}\left[\mathbb{E}[e^{i\omega X_\tau}|\tau, X_t = x, Y_t = y]|X_t = x, Y_t = y\right]\\ &= \mathbb{E}\left[e^{i\omega x - (\tau - t)\omega^2/2}|X_t = x, Y_t = y\right]\\ &= e^{i\omega x}\mathbb{E}\left[e^{-(\tau - t)\omega^2/2}|X_t = x, Y_t = y\right]\end{aligned}$$

We have previously shown that the first hitting time of a Brownian motion τ_m satisfies,

$$\mathbb{E}\left[e^{-\lambda\tau_m}\right] = e^{-|m|\sqrt{2\lambda}}$$

where $\tau_m = \inf\{t \geq 0 : W_t = m\}$ and $W_0 = 0$.

Since we start at position y at time t (rather that position 0 and time 0 as above), we know that,

$$\mathbb{E}\left[e^{-(\omega^2/2)(\tau - t)}|X_t = x, Y_t = y\right] = e^{-|a-y||\omega|}$$

Therefore,

$$\mathbb{E}\left[e^{i\omega X_\tau}|X_t = x, Y_t = y\right] = e^{-|a-y||\omega|}$$

Write,

$$\phi(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\omega x}\hat{\phi}(\omega)\mathrm{d}\omega$$

Then,

$$u(x, y) = \mathbb{E}[\phi(X_\tau)|X_t = x, Y_t = y] = \mathbb{E}\left[\frac{1}{2\pi}\int_{\mathbb{R}} e^{i\omega X_\tau}\hat{\phi}(\omega)\mathrm{d}\omega\middle|X_t = x, Y_t = y\right]$$

Now, bringing the expectation through the integral, and applying the above result,

$$\mathbb{E}\left[\frac{1}{2\pi}\int_{\mathbb{R}} e^{i\omega X_\tau}\hat{\phi}(\omega)\mathrm{d}\omega \middle| X_t = x, Y_t = y\right] = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{\phi}(\omega)\mathbb{E}\left[e^{i\omega X_\tau}|X_t = x, Y_t = y\right]\mathrm{d}\omega$$
$$= \frac{1}{2\pi}\int_{\mathbb{R}} \hat{\phi}(\omega)e^{-|a-y||\omega|}e^{i\omega x}\mathrm{d}\omega$$

First recall, $\mathbb{E}[\phi(X)] = \int \phi(x)f_X(x)\mathrm{d}x$ and $\mathbb{P}(X \in \mathrm{d}z) = f_X(z)\mathrm{d}z$. Then, taking $\phi(x) = \mathbb{1}_{\{x\in\mathrm{d}z\}}$ means $\mathbb{E}[\phi(X)] = f_X(z)\mathrm{d}z = \mathbb{P}(X \in \mathrm{d}z)$. Therefore,

$$u(x, y) = \mathbb{E}[\mathbb{1}_{\{X_\tau\in\mathrm{d}z\}}|X_t = x, Y_t = y] = \mathbb{P}(X_\tau \in \mathrm{d}z|X_t = x, Y_t = y)$$

In this case,

$$\hat{\phi}(\omega) = \int_{\mathbb{R}} e^{-i\omega x}\mathbb{1}_{\{x\in\mathrm{d}z\}}\mathrm{d}x = e^{-i\omega z}\mathrm{d}z$$

Thus, computing this integral by splitting it at 0,

$$u(x, y) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-i\omega z}\mathrm{d}z e^{-|a-y||\omega|}e^{i\omega x}\mathrm{d}\omega = \frac{1}{2\pi}\left[\frac{2|a-y|}{(a-y)^2 + (x-z)^2}\right]\mathrm{d}z = \frac{1}{\pi}\left[\frac{|y-a|}{(y-a)^2 + (x-z)^2}\right] \mathrm{d}$$

3. First observe,

$$u(x, a) = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{\phi}(\omega)e^{-|a-a|}|\omega|e^{i\omega x}\mathrm{d}\omega = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{\phi}(\omega)e^{i\omega x}\mathrm{d}\omega = \phi(x)$$

Define,

$$c = \begin{cases} 1 & y \geq a \\ -1 & y < a \end{cases}$$

Now observe,

$$\partial_x^2 u(x, y) = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{\phi}(\omega)e^{-c(y-a)|\omega|}\partial_x^2 e^{i\omega x}\mathrm{d}\omega = \frac{(i^2\omega^2)}{2\pi}\int_{\mathbb{R}} \hat{\phi}(\omega)e^{-c(y-a)|\omega|}e^{i\omega x}\mathrm{d}\omega$$

Then,

$$\partial_y^2 u(x, y) = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{\phi}(\omega)\partial_y^2 e^{-c(y-a)|\omega|}e^{i\omega x}\mathrm{d}\omega = \frac{c^2\omega^2}{2\pi}\int_{\mathbb{R}} \hat{\phi}(\omega)e^{-c(y-a)|\omega|}e^{i\omega x}\mathrm{d}\omega$$

Thus, since $i^2 = -1$ and $c^2 = 1$,

$$(\partial_x^2 + \partial_y^2)u(x, y) = 0$$

Note there is probably some issue with the partial derivative with respect to y at $y = a$, since $|y - a|$ is not differentiable at this point.

Therefore $u(x, y) = \mathbb{E}[\phi(X_\tau)|X_t = x, Y_t = y]$ satisfies the PDE from 1.

Exercise 10.1

Let $P = (P_t)_{t\geq 0}$ be a Poisson process with intensity λ.

(a) What is the Lévy Measure ν of P.

(b) Let $\mathrm{d}X_t = \mathrm{d}P_t$. Define $u(x,t) := \mathbb{E}\left[\varphi(X_T)|X_t = x\right]$. Find $u(t,x)$ and verify it solves the Kolmogorov Backward equation.

Solution

(a) We have,

$$\nu(U) = \mathbb{E}\left[N(1,U)\right] = \mathbb{E}\left[\sum_{0\leq s\leq 1} \mathbb{1}_{\Delta P_s \in U}\right] = \mathbb{E}\left[\sum_{i=1}^{P_1} \mathbb{1}_{1\in U}\right] = \mathbb{E}\left[P_1\right]\mathbb{1}_{1\in U} = \lambda \mathbb{1}_{1\in U}$$

(b) Integrating $\mathrm{d}X_t = \mathrm{d}P_t$ from 0 to t gives, $X_t - X_0 = P_t - P_0$. Since $P_0 = 0$ we have,

$$X_t = X_0 + P_t$$

First observe,

$$\mathbb{P}(X_T = k|X_t = x) = \mathbb{P}(X_0 + P_T = k|X_0 + P_t = x) = \mathbb{P}(P_T = k - X_0|P_t = x - X_0)$$

Since P has independent increments, and since P is Markov,

$$\mathbb{P}(P_T = k - X_0|P_t = x - X_0) = \mathbb{P}(P_{T-t} = k - x) = \frac{(\lambda(T-t))^{k-x}}{(k-x)!}e^{-\lambda(T-t)}$$

Thus,

$$u(t,x) = \mathbb{E}\left[\varphi(X_T)|X_t = x\right] = \sum_{k=x}^{\infty}\varphi(k)\mathbb{P}(X_T = k|X_t = x) = \sum_{k=x}^{\infty}\varphi(k)\frac{(\lambda(T-t))^{k-x}}{(k-x)!}e^{-\lambda(T-t)}$$

Reindexing with $n = k - x$,

$$u(t,x) = e^{-\lambda(T-t)}\sum_{k=x}^{\infty}\varphi(k)\frac{(\lambda(T-t))^{k-x}}{(k-x)!} = e^{-\lambda(T-t)}\sum_{n=0}^{\infty}\varphi(n+x)\frac{(\lambda(T-t))^{n}}{n!}$$

We now compute the generator $\mathcal{A}(t)$ for P. By definition,

$$\mathcal{A}(t)\varphi(x) = \lim_{s\to t^+} \frac{1}{s-t}\left[\mathcal{P}(t,s)\varphi(x) - \varphi(x)\right] = \lim_{s\to t^+} \frac{1}{s-t}\left[\mathbb{E}\left[\varphi(X_s)|X_t = x\right] - \varphi(x)\right]$$

In a small interval $\mathrm{d}t$ the probability $X_{t+\mathrm{d}t} = X_t + 1$ is $\lambda \mathrm{d}t$ and probability $X_{t+\mathrm{d}t} = X_t$ is $(1-\lambda)\mathrm{d}t$. Therefore,

$$\mathcal{A}(t)\varphi(x) = \frac{1}{\mathrm{d}t}\left[\varphi(x+1)\lambda + \varphi(x)(1-\lambda) - \varphi(x)\right] = \lambda(\varphi(x+1) - \varphi(x))$$

Since the t-derivative of the $n = 0$ term is zero,

$$\begin{aligned}
\sum_{n=0}^{\infty} \varphi(n+x)\partial_t\left[\frac{(\lambda(T-t))^n}{n!}\right] &= \sum_{n=1}^{\infty} \varphi(n+x)\partial_t\left[\frac{(\lambda(T-t))^n}{n!}\right] \\
&= \sum_{n=1}^{\infty} \varphi(n+x)(n)(-\lambda)\frac{(\lambda(T-t))^{n-1}}{n!} \\
&= -\lambda\sum_{n=1}^{\infty} \varphi(n+x)\frac{(\lambda(T-t))^{n-1}}{(n-1)!}
\end{aligned}$$

Observe, by the chain rule and assuming we can bring a derivative through a sum,

$$\begin{aligned}
\partial_t u(t,x) &= \left[\partial_t e^{-\lambda(T-t)}\right]\sum_{n=0}^{\infty}\varphi(n+x)\frac{(\lambda(T-t))^n}{n!} + e^{-\lambda(T-t)}\sum_{n=0}^{\infty}\varphi(n+x)\partial_t\left[\frac{(\lambda(T-t))^n}{n!}\right] \\
&= \lambda e^{-\lambda(T-t)}\sum_{n=0}^{\infty}\varphi(n+x)\frac{(\lambda(T-t))^n}{n!} - \lambda e^{-\lambda(T-t)}\sum_{n=1}^{\infty}\varphi(n+x)\frac{(\lambda(T-t))^{n-1}}{(n-1)!} \\
&= \lambda e^{-\lambda(T-t)}\sum_{n=0}^{\infty}\varphi(n+x)\frac{(\lambda(T-t))^n}{n!} - \lambda e^{-\lambda(T-t)}\sum_{n=m}^{\infty}\varphi(m+1+x)\frac{(\lambda(T-t))^m}{m!} \\
&= \lambda(u(t,x) - u(t,x+1))
\end{aligned}$$

Therefore the KBE is satisfied as

$$[\partial_t + \mathcal{A}]u(t,x) = \lambda(u(t,x) - u(t,x+1)) - \lambda(u(t,x+1) - u(t,x)) = 0, \quad u(T,x) = \varphi(x)$$

Exercise 10.2

Solution

Exercise 10.3

Let $X = (X_t)_{t\geq 0}$ be a process defined by,

$$
\begin{aligned}
\mathrm{d}X_t &= \mu_t X_t \mathrm{d}t + \sigma_t X_t \mathrm{d}W_t + \int_{\mathbb{R}} \left(e^{\gamma_t(z)} - 1\right) X_{t-} \tilde{N}(\mathrm{d}t, \mathrm{d}z) \\
\mathrm{d}Y_t &= b_t Y_t \mathrm{d}t + a_t Y_t \mathrm{d}W_t + \int_{\mathbb{R}} \left(e^{g_t(z)} - 1\right) Y_{t-} \tilde{N}(\mathrm{d}t, \mathrm{d}z)
\end{aligned}
$$

where W is a one-dimensional Brownian motion, $\tilde{N}$ is a one-dimensional compensated Poisson random measure on $\mathbb{R}$, and $\mu, b, \sigma, a, \gamma, g$ are $\mathbb{F}$-adapted stochastic processes.

(a) Define $Z_t := X_t/Y_t$. Compute the differential $\mathrm{d}Z_t$. Your answer should not involve X_t or Y_t.

(b) Find μ_t so that Z is a martingale.

Solution

(a) Define $f(x, y) = x/y$. Then $Z_t = f(X_t, Y_t)$.

We have,

$$
[(e^{\gamma_t(z)} - 1)X_t; (e^{g_t(z)} - 1)Y_t] \cdot \nabla f(X_{t-}, Y_{t-}) = (e^{\gamma_t(z)} - 1)X_{t-} f_x(X_{t-}, Y_{t-}) + (e^{g_t(z)} - 1)Y_{t-} f_y(X_{t-}, Y
$$

We use Itô's formula to compute,

$$
\begin{aligned}
\mathrm{d}Z_t = \mathrm{d}f(X_t, Y_t) = &\left(\mu_t X_t f_x + b_t Y_t f_y + \frac{1}{2}\left((\sigma_t X_t)^2 f_{xx} + 2(\sigma_t X_t)(a_t Y_) f_{xy} + (a_t Y_t)^2 f_{yy}\right)\right) \mathrm{d}t \\
&+ (\sigma_t X_t f_x + a_t Y_t f_y)\, \mathrm{d}W_t \\
&+ \int_{\mathbb{R}} \left(f\left(X_{t-} + (e^{\gamma_t(z)} - 1)X_{t-}, Y_{t-} + (e^{g_t(z)} - 1)Y_{t-}\right) - f(X_{t-}, Y_{t-})\right) \tilde{N}(\mathrm{d}t, \mathrm{d}z \\
&+ \int_{\mathbb{R}} \Big(f\left(X_{t-} + (e^{\gamma_t(z)} - 1)X_{t-}, Y_{t-} + (e^{g_t(z)} - 1)Y_{t-}\right) - f(X_{t-}, Y_{t-}) \\
&\qquad - (e^{\gamma_t(z)} - 1)X_{t-} f_x(X_{t-}, Y_{t-}) - (e^{g_t(z)} - 1)Y_{t-} f_y(X_{t-}, Y_{t-})\Big)\nu(\mathrm{d}z)\mathrm{d}t
\end{aligned}
$$

Now, using $f_x = 1/y, f_y = -x/y^2, f_{xy} = -1/y^2, f_{xx} = 0, f_{yy} = 2x/y^3$ we have,

$$
\mu_t X_t f_x + b_t Y_t f_y = \mu_t X_t \left(\frac{1}{Y_t}\right) + b_t Y_t \left(\frac{-X_t}{Y_t^2}\right) = \mu_t Z_t - b_t Z_t
$$

$$
(\sigma_t X_t)^2 f_{xx} + 2(\sigma_t X_t)(a_t Y_t) f_{xy} + (a_t Y_t)^2 f_{yy} = 2(\sigma_t X_t)(a_t Y_t) \left(\frac{-1}{Y_t^2}\right) + a_t^2 Y_t^2 \left(\frac{2X_t}{Y_t^3}\right) = -2\sigma_t a_t Z_t + 2
$$

$$\sigma_t X_t f_x + a_t Y_t f_y = \sigma_t X_t \left(\frac{1}{Y_t}\right) + a_t Y_t \left(\frac{-X_t}{Y_t^2}\right) = \sigma_t Z_t - a_t Z_t$$

$$f\left(X_{t^-} + (e^{\gamma_t(z)} - 1)X_{t^-}, Y_{t^-} + (e^{g_t(z)} - 1)Y_{t^-}\right) - f(X_{t^-}, Y_{t^-}) = \frac{e^{\gamma_t(z)}}{e^{g_t(z)}} Z_{t^-} - Z_{t^-}$$

$$\begin{aligned}
&(e^{\gamma_t(z)} - 1)X_{t^-} f_x(X_{t^-}, Y_{t^-}) + (e^{g_t(z)} - 1)Y_{t^-} f_y(X_{t^-}, Y_{t^-}) \\
&= (e^{\gamma_t(z)} - 1)X_{t^-} \left(\frac{1}{Y_{t^-}}\right) + (e^{g_t(z)} - 1)Y_{t^-} \left(\frac{-X_{t^-}}{Y_{t^-}^2}\right) \\
&= (e^{\gamma_t(z)} - 1)Z_{t^-} - (e^{g_t(z)} - 1)Z_{t^-}
\end{aligned}$$

Inserting these evaluated expressions into the original expression for $\mathrm{d}Z_t$ gives,

$$\begin{aligned}
\mathrm{d}Z_t = &\left(\mu_t - b_t - \sigma_t a_t + a_t^2\right) Z_t \mathrm{d}t + (\sigma_t - a_t) Z_t \mathrm{d}W_t \\
&+ \int_{\mathbb{R}} \left(\frac{e^{\gamma_t(z)}}{e^{g_t(z)}} - 1\right) Z_{t^-} \tilde{N}(\mathrm{d}t, \mathrm{d}z) \\
&+ \int_{\mathbb{R}} \left(\frac{e^{\gamma_t(z)}}{e^{g_t(z)}} - e^{\gamma_t(z)} + e^{g_t(z)} - 1\right) Z_{t^-} \nu(\mathrm{d}z)\mathrm{d}t
\end{aligned}$$

(b) We need the $\mathrm{d}t$ term to be zero. Therefore pick,

$$\mu_t = b_t + \sigma_t a_t - a_t^2 - \int_{\mathbb{R}} \left(\frac{e^{\gamma_t(z)}}{e^{g_t(z)}} - e^{\gamma_t(z)} + e^{g_t(z)} - 1\right) \nu(\mathrm{d}z)\mathrm{d}t$$

Exercise 10.4

Let $\eta = (\eta_t)_{t\geq 0}$ be a one-dimensional Lévy Process and define $X = (X_t)_{t\geq 0}$ by

$$\mathrm{d}X_t = \kappa(\theta - X_t)\mathrm{d}t + \mathrm{d}\eta_t$$

(a) Find X_t explicitly as a function of η.

(b) Assume $\eta_t = \sigma W_t + \int_{\mathbb{R}} z\tilde{N}(t, \mathrm{d}z)$. Compute $m(t) := \mathbb{E}X_t$ and $c(t,s) := \mathbb{E}(X_t - m(t))(X_s - m(s))$.

Solution

(a) Let $Y_t = X_t - \theta$ and $Z_t = e^{\kappa t}Y_t = f(t, Y_t)$, where $f(t,y) = e^{\kappa t}y$.

Then,

$$\mathrm{d}Y_t = \mathrm{d}X_t = -\kappa Y_t \mathrm{d}t + \mathrm{d}\eta_t$$

Recall the product rule (which applies to Lévy Itô processes),

$$\mathrm{d}(U_tV_t) = U_{t^-}\mathrm{d}V_t + V_{t^-}\mathrm{d}U_t + \mathrm{d}[U,V]_t$$

Therefore,

$$\mathrm{d}Z_t = \mathrm{d}(e^{\kappa t}Y_t) = e^{\kappa t^-}\mathrm{d}Y_t + Y_{t^-}\mathrm{d}e^{\kappa t} + \mathrm{d}[e^{\kappa t}, Y]_t$$

Using our heuristics we have $\mathrm{d}(e^{\kappa t})\mathrm{d}Y_t = 0$. Therefore, since t^- and t can be "treated the same" on $\mathrm{d}t$ terms which are continuous,

$$\mathrm{d}Z_t = e^{\kappa t^-}\mathrm{d}Y_t + \kappa e^{\kappa t}Y_{t^-} = e^{\kappa t^-}\mathrm{d}\eta_t$$

Integrating we have,

$$Z_t = Z_0 + \int_0^t e^{\kappa s}\mathrm{d}\eta_s$$

Therefore, since $Y_t = e^{-\kappa t}Z_t$, $Z_0 = Y_0$ so,

$$Y_t = e^{-\kappa t}\left(Y_0 + \int_0^t e^{\kappa s}\mathrm{d}\eta_s\right)$$

Finally, since $X_t = \theta + Y_t$, $Y_0 = X_0 - \theta$ so,

$$X_t = \theta + e^{-\kappa t}\left(X_0 - \theta + \int_0^t e^{\kappa s}\mathrm{d}\eta_s\right) = \theta + e^{-\kappa t}(X_0 - \theta) + \int_0^t e^{\kappa(s-t)}\mathrm{d}\eta_s$$

(b) We have,

$$\mathrm{d}\eta_t = \sigma \mathrm{d}W_t + \int_{\mathbb{R}} z\tilde{N}(\mathrm{d}t, \mathrm{d}z)$$

Observe, that since integrals with respect to $\mathrm{d}W_t$ and $\int_{\mathbb{R}} \tilde{N}(\mathrm{d}t, \mathrm{d}z)$ are martingales so,

$$\mathbb{E}\left[\int_0^t e^{\kappa(s-t)}\mathrm{d}\eta_s\right] = \mathbb{E}\left[\int_0^t e^{\kappa(s-t)}\sigma\mathrm{d}W_t + \int_0^t e^{\kappa(s-t)}\int_{\mathbb{R}} z\tilde{N}(\mathrm{d}t, \mathrm{d}z)\right] = 0$$

Therefore,

$$m(t) = \mathbb{E}\left[X_t\right] = \mathbb{E}\left[\theta + e^{-\kappa t}(X_0 - \theta) + \int_0^t e^{\kappa(s-t)}\mathrm{d}\eta_s\right] = \theta + e^{-\kappa t}(X_0 - \theta)$$

Clearly,

$$X_t - m(t) = \int_0^t e^{\kappa(u-t)}\mathrm{d}\eta_u$$

Without loss of generality assume $t \geq s$. Then, using the independent increments property to write the expectation of a product as the product of expectations,

$$\begin{aligned}
\mathbb{E}\left[(X_t - m(t))(X_s - m(s))\right] &= \mathbb{E}\left[\left(\int_0^t e^{\kappa(u-t)}\mathrm{d}\eta_u\right)\left(\int_0^s e^{\kappa(v-s)}\mathrm{d}\eta_v\right)\right] \\
&= \mathbb{E}\left[\left(\int_0^s e^{\kappa(u-t)}\mathrm{d}\eta_u + \int_s^t e^{\kappa(u-t)}\mathrm{d}\eta_u\right)\left(\int_0^s e^{\kappa(v-s)}\mathrm{d}\eta_v\right)\right] \\
&= \mathbb{E}\left[e^{-\kappa(t+s)}\left(\int_0^s e^{\kappa u}\mathrm{d}\eta_u\right)^2 + e^{-\kappa(t+s)}\left(\int_s^t e^{\kappa u}\mathrm{d}\eta_u\right)\left(\int_0^s e^{\kappa v}\mathrm{d}\eta_v\right)\right] \\
&= e^{-\kappa(t+s)}\mathbb{E}\left[\left(\int_0^s e^{\kappa u}\mathrm{d}\eta_u\right)^2\right] + e^{-\kappa(t+s)}\mathbb{E}\left[\int_s^t e^{\kappa u}\mathrm{d}\eta_u\right]\mathbb{E}\left[\int_0^s e^{\kappa v}\mathrm{d}\eta_v\right]
\end{aligned}$$

We now note that, Lévy processes without a dt term are martingales so that,

$$\mathbb{E}\left[\int_0^s e^{\kappa u}\mathrm{d}\eta_u\right] = \mathbb{E}\left[\int_0^s e^{\kappa u}\left(\sigma\mathrm{d}W_u + \int_{\mathbb{R}} z\tilde{N}(\mathrm{d}u,\mathrm{d}z)\right)\right] = 0$$

Define,

$$Z_s = \int_0^s e^{\kappa u}\mathrm{d}\eta_u$$

Then,

$$\mathrm{d}Z_s = e^{\kappa s}\mathrm{d}\eta_s = \sigma e^{\kappa s}\mathrm{d}W_s + \int_{\mathbb{R}} e^{\kappa s} z\tilde{N}(\mathrm{d}s,\mathrm{d}z)$$

Using Itô's isometry we have,

$$\mathbb{E}\left[\left(\int_0^s e^{\kappa u}\mathrm{d}\eta_u\right)^2\right] = \mathbb{E}\left[\int_0^s \left(\sigma^2 e^{2\kappa u} + \int_{\mathbb{R}} e^{2\kappa u} z^2\nu(\mathrm{d}z)\right)\mathrm{d}u\right] = \mathbb{E}\left[\left(\sigma^2 + \int_{\mathbb{R}} z^2\nu(\mathrm{d}z)\right)\frac{e^{2\kappa s}-1}{2\kappa}\right]$$

Therefore,

$$c(t,s) = e^{-\kappa(t+s)}\frac{e^{2\kappa s}-1}{2\kappa}\left(\sigma^2 + \int_{\mathbb{R}} z^2\nu(\mathrm{d}z)\right) = \frac{e^{\kappa(s-t)} - e^{-\kappa(t+s)}}{2\kappa}\left(\sigma^2 + \int_{\mathbb{R}} z^2\nu(\mathrm{d}z)\right)$$

We can remove our assumption that $t \geq s$ and write,

$$c(t,s) = \frac{e^{-\kappa|t-s|} - e^{-\kappa(t+s)}}{2\kappa}\left(\sigma^2 + \int_{\mathbb{R}} z^2\nu(\mathrm{d}z)\right)$$

Exercise 10.5

Let X be the following one-dimensional jump-diffusion

$$\mathrm{d}X_t = \mu(t, X_t)\mathrm{d}t + \sigma(t, X_t)\mathrm{d}W_t + \int_{\mathbb{R}} \gamma(t, X_{t^-}, z)\tilde{N}(t, \mathrm{d}z),$$

where W is a one-dimensional Brownian motion and $\tilde{N}$ is a one-dimensional compensated Poisson random measure on $\mathbb{R}$. Derive using the Lévy-Itô formula the infinitesimal generator $\mathcal{A}(t)$ of the X process,

$$\mathcal{A}(t)\varphi(x) := \lim_{s\to t^+} \frac{\mathbb{E}\left[\varphi(X_s)|X_t = x\right] - \varphi(x)}{s-t}$$

Solution

Since $\mathbb{E}[\varphi(X_t)|X_t = x] = \varphi(x)$,

$$\mathbb{E}\left[\varphi(X_s)|X_t = x\right] - \varphi(x) = \mathbb{E}\left[\varphi(X_t) + \int_t^s \mathrm{d}\varphi(X_u)\right] - \varphi(x) = \mathbb{E}\left[\int_t^s \mathrm{d}\varphi(X_u)\right]$$

From the Lévy-Itô formula we have,

$$\begin{aligned}\mathrm{d}\varphi(X_u) = &\left(\mu(u, X_u)\varphi'(X_u) + \frac{1}{2}\sigma(u, X_u)^2\varphi''(X_u)\right)\mathrm{d}u + \sigma(u, X_u)\varphi'(X_u)\mathrm{d}W_u \\ &+ \int_{\mathbb{R}} \Big(\varphi(X_{u^-} + \gamma(u, X_{u^-}, z)) - \varphi(X_{u^-})\Big)\tilde{N}(\mathrm{d}u, \mathrm{d}z) \\ &+ \int_{\mathbb{R}} \Big(\varphi(X_{u^-} + \gamma(u, X_{u^-}, z)) - \varphi(X_{u^-}) - \gamma(u, X_{u^-}, z)\varphi'(X_{u^-})\Big)\nu(\mathrm{d}z)\mathrm{d}u\end{aligned}$$

We note that as integrals with respect to W and $\tilde{N}$ are martingales that,

$$\begin{aligned}\mathbb{E}\left[\int_t^s \mathrm{d}\varphi(X_u)\right] = \mathbb{E}\Bigg[\int_t^s \Bigg(&\mu(u, X_u)\varphi'(X_u) + \frac{1}{2}\sigma(u, X_u)^2\varphi''(X_u)\mathrm{d}u \\ &+ \int_{\mathbb{R}} \Big(\varphi(X_{u^-} + \gamma(u, X_{u^-}, z)) - \varphi(X_{u^-}) - \gamma(u, X_{u^-}, z)\varphi'(X_{u^-})\Big)\nu(\mathrm{d}z)\Bigg)\mathrm{d}u\Bigg]\end{aligned}$$

Thus, taking the limit as $s \to t^+$,

$$\mathcal{A}(t)\varphi(x) = \left(\mu(t, X_t)\partial_x + \frac{1}{2}\sigma(t, X_t)\partial_x^2 + \int_{\mathbb{R}} \nu(\mathrm{d}z)\left(\theta_{\gamma(t, X_t, z)} - 1 - \gamma(t, X_t, z)\partial_x\right)\right)\varphi(x)$$