Qualifying Examination (December 2011)
Committee: Anne Greenbaum, Ulrich Hetmaniuk, Hong Qian
Day 1

Submit solutions to the **five** following problems.

Start each problem on a new page. You must show all steps and prove all claims, unless you rely on a standard theorem or result, in which case you should refer to it. You are allowed to use a computer for any part of any problem, but you are not allowed to access the internet or any application file existing on your computer. Any application file you use must be created from scratch.

1. Find all solutions $y(x)$ to the problem

$$\begin{cases} x(x-1)\dfrac{d^2y}{dx^2} + (3x-1)\dfrac{dy}{dx} + y = 0 \\ y(0) = y_0 \end{cases}$$

2. Consider the trapezoid method to solve the scalar equation $y' = f(t, y)$:

$$y_{n+1} = y_n + \frac{h}{2}(f(t_{n+1}, y_{n+1}) + f(t_n, y_n)).$$

Show that this method is convergent, find its order, and sketch its region of absolute stability. In particular, determine where the region of absolute stability intersects the real and imaginary axes.

3. Noting that

$$\frac{1}{n^2} \sim \frac{1}{n(n+1)} \qquad \text{as } n \to +\infty$$

give an asymptotically closed-form equivalent expression for the remainder

$$\sum_{k=n}^{\infty} \frac{1}{k^2} \equiv \sum_{k=1}^{+\infty} \frac{1}{k^2} - \sum_{k=1}^{n-1} \frac{1}{k^2}$$

that shows how the remainder behaves as $n \to +\infty$.

4. Let $\mathbf{A} \in \mathbb{R}^{m\times n}$ $(m \le n)$. Define the QR factorization of $\mathbf{A} \in \mathbb{R}^{m\times n}$. Describe the algorithm you would recommend to compute the matrices $\mathbf{Q}$ and $\mathbf{R}$?

5. Evaluate the following integral

$$\int_0^\infty \frac{\sin x}{x(x^2+1)}dx.$$

Show details of your work.

Qualifying Examination (December 2011)
Committee: Anne Greenbaum, Ulrich Hetmaniuk, Hong Qian
Day 2

Submit solutions to the **five** following problems.

Start each problem on a new page. You must show all steps and prove all claims, unless you rely on a standard theorem or result, in which case you should refer to it. You are allowed to use a computer for any part of any problem, but you are not allowed to access the internet or any application file existing on your computer. Any application file you use must be created from scratch.

1. Solve the following problem and determine the limiting solution as $\varepsilon \to 0$ through positive values

$$\varepsilon y'' + xy' = x$$

with the conditions $y(0) = y(1) = 1$.

2. Consider the function

$$F(y) = \int_0^{+\infty} \frac{\sin xy}{x(x^2+1)} dx$$

if $y > 0$. Show that the function F is twice-differentiable and each derivative is continuous. Find the limits of F as y approaches 0 through positive values and as y goes to $+\infty$. Plot the function F. You may use

$$\int_0^{+\infty} \frac{\sin x}{x} dx = \frac{\pi}{2}.$$

3. A general description of a network of N nonlinearly coupled units is given by

$$\frac{du_i}{dt} = -u_i + \sum_{j=1}^{N} w_{ij} g(u_j)$$

Here, u_i is the activity of the i^{th} unit, and the matrix $\mathbf{W}$ gives the connection weights among these units; in particular, w_{ij} is the connection weight between unit j and unit i. Finally, g describes how the strength of interaction between units depends on their activities.

- Consider linear interactions: $g(y) = y$. For what assumption on the matrix $\mathbf{W}$, will the solutions be stable for all initial conditions? For what assumption on the matrix $\mathbf{W}$, will the solutions be aymptotically stable for all initial conditions? For what assumption on the matrix $\mathbf{W}$, will the solutions produce oscillations for all initial conditions? Justify your reasoning.
- For nonlinear interactions, choose $g(y) = \tanh(\beta y)$ with $\beta = 1$. Consider the "energy function"

$$H = -1/2 \sum_{i,j=1}^{N} w_{ij} V_i V_j + \sum_{i=1}^{N} \int_0^{V_i} g^{-1}(s) ds$$

  where $V_i = g(u_i)$. For what assumptions on the matrix $\mathbf{W}$, can you show the following bound on the time evolution of the energy:

$$\frac{dH}{dt} \leq 0 \ \ ?$$

  What does this imply about the system dynamics?

4. Suppose you wish to compute the *discrete Fourier transform* (DFT) $\boldsymbol{F} = [F_0, F_1, \ldots, F_{N-1}]^T$ of a vector $\boldsymbol{f} = [f_0, f_1, \ldots, f_{N-1}]^T$, defined by

$$F_k = \sum_{j=0}^{N-1} e^{2\pi ijk/N} f_j, \quad k = 0, 1, \ldots, N-1.$$

Here $i = \sqrt{-1}$, and you may assume that N is a power of 2.

(a) Suppose you do not know $f_0, \ldots, f_{N-1}$, but you know the DFT $\boldsymbol{F}^{(e)}$ of the even numbered terms and the DFT $\boldsymbol{F}^{(o)}$ of the odd numbered terms; i.e.,

$$F_k^{(e)} = \sum_{j=0}^{N/2-1} e^{2\pi ijk/(N/2)} f_{2j}, \quad F_k^{(o)} = \sum_{j=0}^{N/2-1} e^{2\pi ijk/(N/2)} f_{2j+1},$$

where $k = 0, 1, \ldots, N/2-1$. Explain how you can compute $\boldsymbol{F}$ from $\boldsymbol{F}^{(e)}$ and $\boldsymbol{F}^{(o)}$. Be sure to show how you determine the entries $F_{N/2}, \ldots, F_{N-1}$, as well as $F_0, \ldots, F_{N/2-1}$.

(b) About how many operations (additions, subtractions, multiplications, divisions) are required to compute $\boldsymbol{F}$, given $\boldsymbol{F}^{(e)}$ and $\boldsymbol{F}^{(o)}$? Suppose this process is repeated and the length $N/2$ transforms $\boldsymbol{F}^{(e)}$ and $\boldsymbol{F}^{(o)}$ are computed from the DFT's of their even and old entries. How many operations would be required for this computation? Suppose the process is repeated until one reaches vectors of length 1 (for which the DFT is the identity). About how many total operations would be required?

5. Consider the Helmholtz problem

$$u_{xx} + u_{yy} + k^2 u = f(x, y) = (k^2 - 5\pi^2)\sin(\pi x)\sin(2\pi y) \tag{1}$$

with $u(x, y) = 0$ on the boundary of the unit square $0 \le x \le 1$, $0 \le y \le 1$.

(a) Solve the Helmholtz problem using the 5-point Laplacian (second-order finite difference) and the backslash as linear system solver. Verify your code works for $k = 5$, $k = 10$, and $k = 60$ by giving log-log plots for the maximum nodal error as a function of h.

(b) Suppose we change the solver to Jacobi (say take 200 iterations of Jacobi's method). Program this in your code using the matrix version of Jacobi. Derive the spectral radius of Jacobi's iteration matrix in terms of h and k. Recall the eigenvalues λ_{pq} of the 5-point Laplacian are

$$\frac{2}{h^2}\left(\cos(p\pi h) + \cos(q\pi h) - 2\right)$$

where $h = 1/(m+1)$, $p = 1, 2, ..., m$, $q = 1, 2, ..., m$.

(c) If we fix $h = 1/21$, for what values of k will Jacobi's method converge? Verify this in your code by trying $k = 5$, $k = 10$, and $k = 60$, and any other k values, say $k = 0$ for example, you deem appropriate.

Qualifying Examination (December 2011)
Committee: Anne Greenbaum, Ulrich Hetmaniuk, Hong Qian
Day 3

Submit solutions to the following problems.

Start each problem on a new page. You must show all steps and prove all claims, unless you rely on a standard theorem or result, in which case you should refer to it. You are allowed to use a computer for any part of any problem, but you are not allowed to access the internet or any application file existing on your computer. Any application file you use must be created from scratch.

1. Consider the Helmholtz problem

$$u_{xx} + u_{yy} + k^2 u = f(x, y) = (k^2 - 5\pi^2)\sin(\pi x)\sin(2\pi y) \tag{1}$$

with $u(x, y) = 0$ on the boundary of the unit square $0 \le x \le 1$, $0 \le y \le 1$.

(a) Find the exact solution to this Helmholtz problem.

(b) Solve the Helmholtz problem using the 5-point Laplacian (second-order finite difference) and the backslash as linear system solver.

(c) Plot the maximum nodal error as a function of h for $k = 5$, $k = 10$, and $k = 60$ (using a log-log scale).

(d) Suppose we change the solver to Jacobi (say take 200 iterations of Jacobi's method). Program this in your code using the matrix version of Jacobi. Derive the spectral radius of Jacobi's iteration matrix in terms of h and k. Recall the eigenvalues λ_{pq} of the 5-point Laplacian are

$$\frac{2}{h^2}\left(\cos(p\pi h) + \cos(q\pi h) - 2\right)$$

where $h = 1/(m+1)$, $p = 1, 2, ..., m$, $q = 1, 2, ..., m$.

(e) If we fix $h = 1/21$, for what values of k will Jacobi's method converge? Verify this in your code by trying $k = 5$, $k = 10$, and $k = 60$, and any other k values, say $k = 0$ for example, you deem appropriate.

2. A general description of a network of N nonlinearly coupled units is given by

$$\frac{du_i}{dt} = -u_i + \sum_{j=1}^{N} w_{ij} g(u_j)$$

Here, u_i is the activity of the i^{th} unit, and the matrix $\mathbf{W}$ gives the connection weights among these units; in particular, w_{ij} is the connection weight between unit j and unit i. Choose $g(y) = \tanh(\beta y)$ with $\beta > 0$. Assume that each unit u_i is located at spatial location x_i in $[-L, L]$.

- Consider the "energy function"

$$H = -1/2 \sum_{i,j=1}^{N} w_{ij} V_i V_j + \sum_{i=1}^{N} \int_0^{V_i} g^{-1}(s) ds$$

where $V_i = g(u_i)$. For what assumptions on the matrix $\mathbf{W}$, can you show the following bound on the time evolution of the energy:

$$\frac{dH}{dt} \leq 0 \text{ ?}$$

- As $N \to +\infty$, under what if any conditions on $\mathbf{W}$ and x_i will the system be described or approximated by an integral equation of the form

$$\frac{\partial u(x,t)}{\partial t} = -u(x,t) + \int_{-L}^{L} W(x - x') g(u(x', t)) dx' \text{ ?}$$

Justify your answer.

- Make a natural-seeming choice for $W(\cdot)$: that closest neighbor units have positive weights, and further away ones have negative weights, and very far away units do not interact at all (weights $\to 0$). Perform numerical simulations of the integral equation with such a $W(\cdot)$ and describe the dynamics you find and / or the appropriate algorithms.

- Can you do any analysis to explain what you find? The limit $\beta \to +\infty$ might help.

3. Consider linear partial differential equation for $u(x,t)$:

$$\frac{\partial}{\partial t}u = D\frac{\partial^2}{\partial x^2}u + \frac{\partial}{\partial x}(kxu), \tag{2}$$

in which constants D and k are positive, $x \in \mathbb{R}$, and the boundary condition for $u(x,t)$ is

$$\lim_{x\to\infty} Du_x(\pm x,t) + kxu(\pm x,t) = 0. \tag{3}$$

(a) Show that the solution to Eq. (2) with initial data

$$u(x,0) = A\delta(x - x_0),$$

$\delta(x)$ being the Dirac-delta function, is never negative.

(b) Show that

$$\int_{-\infty}^{+\infty} u(x,t)dx$$

as a function of t is a conserved quantity.

(c) Show that $u(x,t)$ can be expressed as a Gaussian distribution with time-dependent mean $\mu(t)$ and variance $\sigma^2(t)$. Determine the functional form of $\mu(t)$ and $\sigma^2(t)$ in terms of D, k, and x_0.

(d) Generalize the Eq. (2) to higher dimensional case yields

$$\frac{\partial}{\partial t}u(\vec{x},t) = \sum_{i=1}^{n}\sum_{j=1}^{n}\left[D_{ij}\frac{\partial^2}{\partial x_i \partial x_j}u + \frac{\partial}{\partial x_i}(k_{ij}x_j u)\right], \quad \vec{x} = (x_1, x_2, \cdots, x_n), \tag{4}$$

in which $n \times n$ matrix $\{D_{ij}\}$ is symmetric and positive definit, $\vec{x} \in \mathbb{R}^n$. What conditions are required for the matrix $K = \{k_{ij}\}$ so that the solution to (4) is a multi-variate Gaussian distribution?

(e) How does one approach to solve the Eq. (4)?

(f) Show that

$$u^{ss}(\vec{x}) = \exp\left[-\frac{1}{2}\vec{x}\Xi^{-1}\vec{x}\right] \tag{5}$$

is a steady state solution to Eq. (4). In (5), Ξ is a non-singular $n \times n$ matrix satisfying matrix equation

$$K\Xi + \Xi K^T = 2D. \tag{6}$$