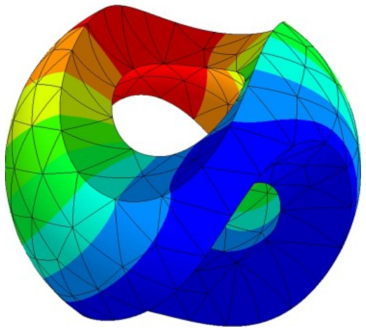
Finite Element Method (FEM)

GeePs'N Talks special session

T. Cherrière, A. El Gode, T. Gauthey

Preliminaries



- Interactive course, using NGSolve (Python FEM toolbox)

- Go to website : https://github.com/tcherrie/tutorial_fem

And click on the yellow button try lite now

- The code should run in your browser without installation required!

- If strange bugs: **reload the webpage** (virtual memory overflow)
- GeePs clusters in backup
- For local installation: ask after the tutorial

Outlines

- 1) Lengthy introduction
  - Function spaces & interpolation
  - Integral formulation
  - Linear system
- 2) Academic Poisson problem
  - Variational formulation
  - Boundary conditions
- 3) Non-linear Magnetostatics (2D)
  - Realistic problem
  - Newton method
- 4) 3D Magnetostatics
  - Iterative solver
  - Gauge
  - Symmetries

Not addressed in this tutorial: time-harmonic and time-dependent problem.

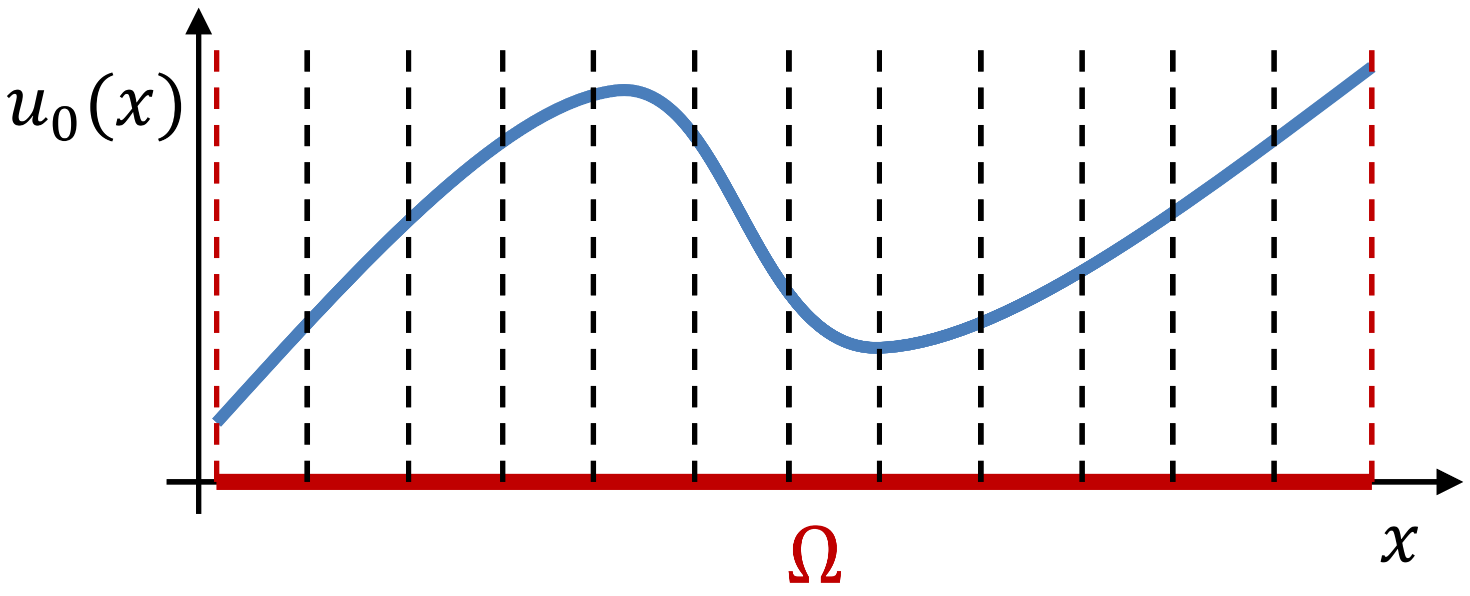
GeePS Génie électrique et électronique de Paris

1) LENGTHY INTRODUCTION

Idea of the method, not boring I promise (hope)

Idea of FEM

How to approximate a function on a finite-dimensional space?

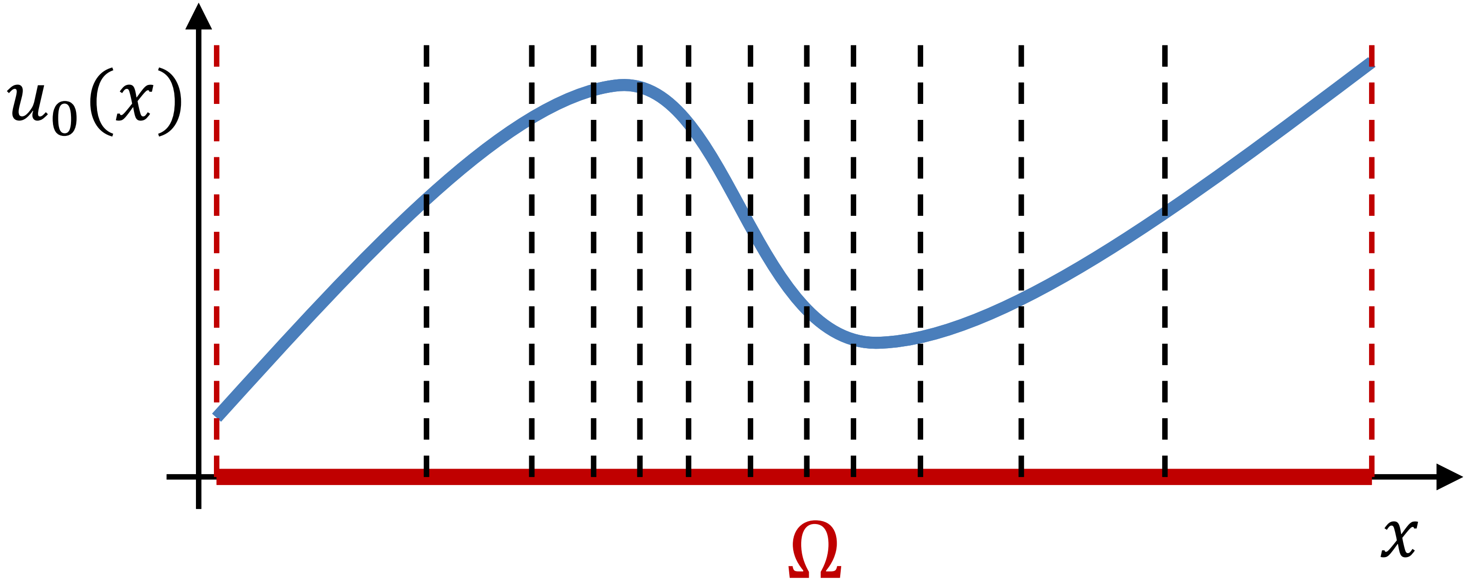


Discretization of geometric space $\boldsymbol{\Omega}$

(uniform mesh)

Idea of FEM

How to approximate a function on a finite-dimensional space?

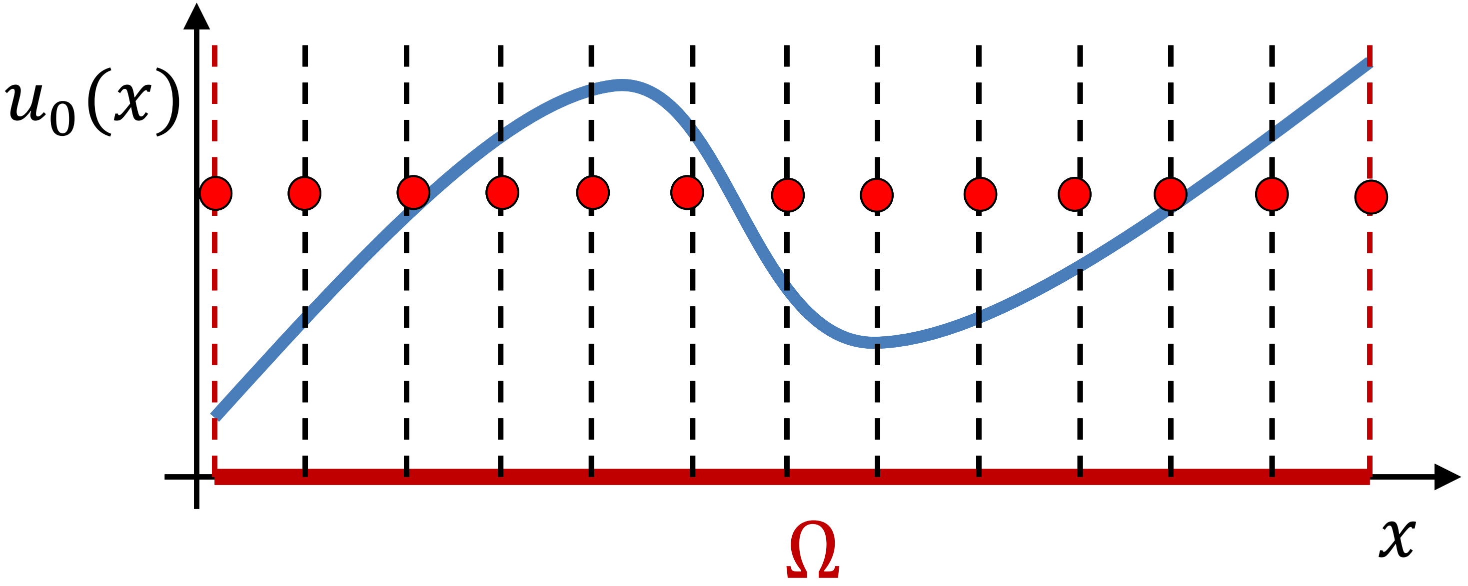


Discretization of geometric space Ω

(irregular *mesh*)

Idea of FEM

How to approximate a function on a finite-dimensional space?

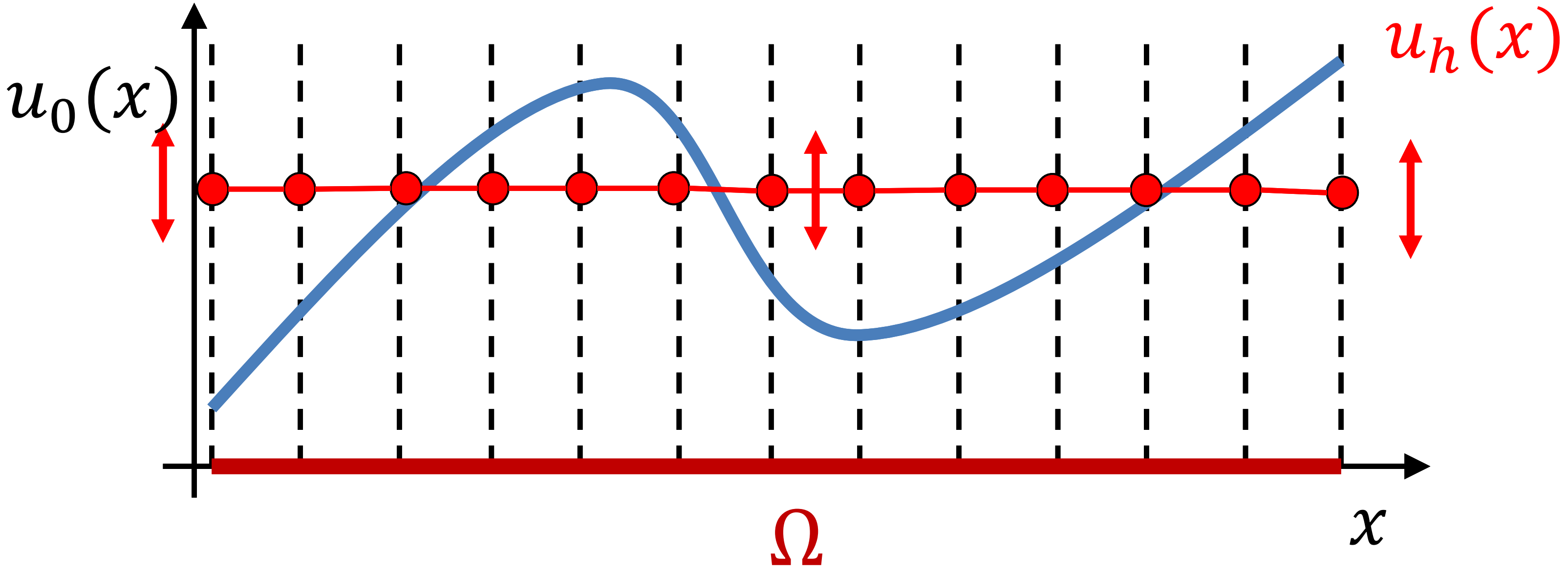


Degrees of freedom (DoFs)

(unknowns of the problem)

Idea of FEM

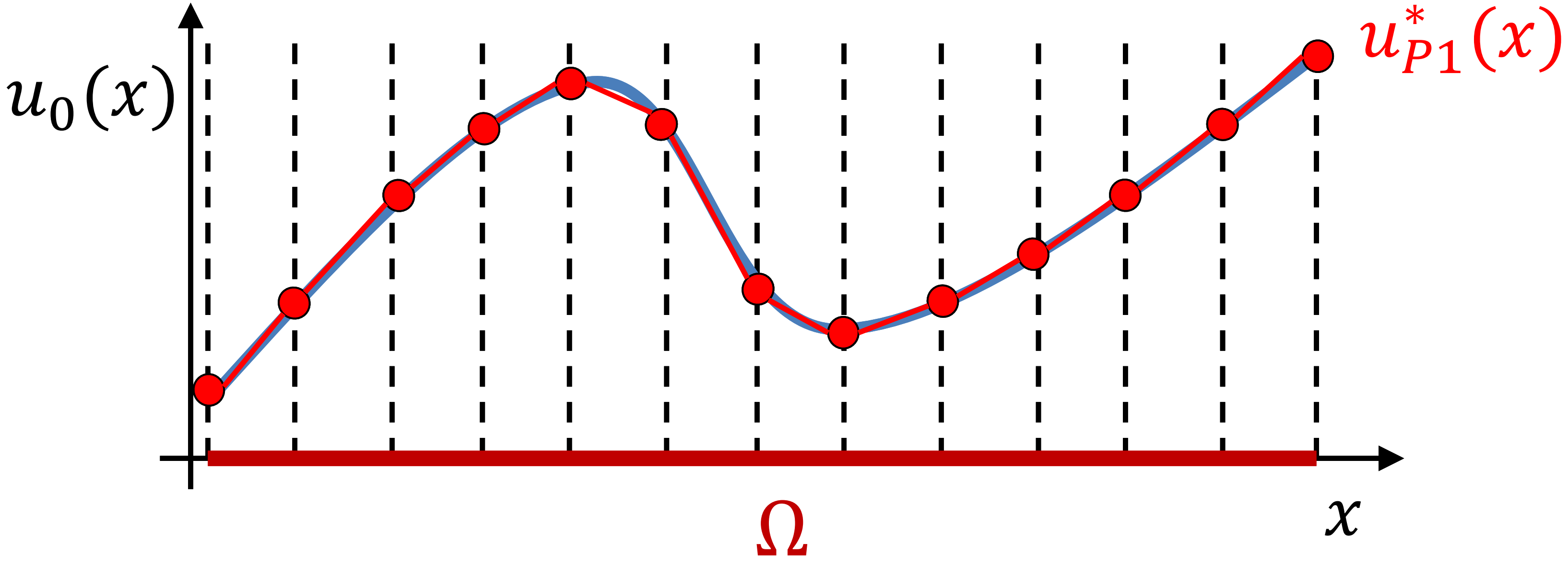
How to approximate a function on a finite-dimensional space?



Interpolation defined from the DoFs

Idea of FEM

How to approximate a function on a finite-dimensional space?



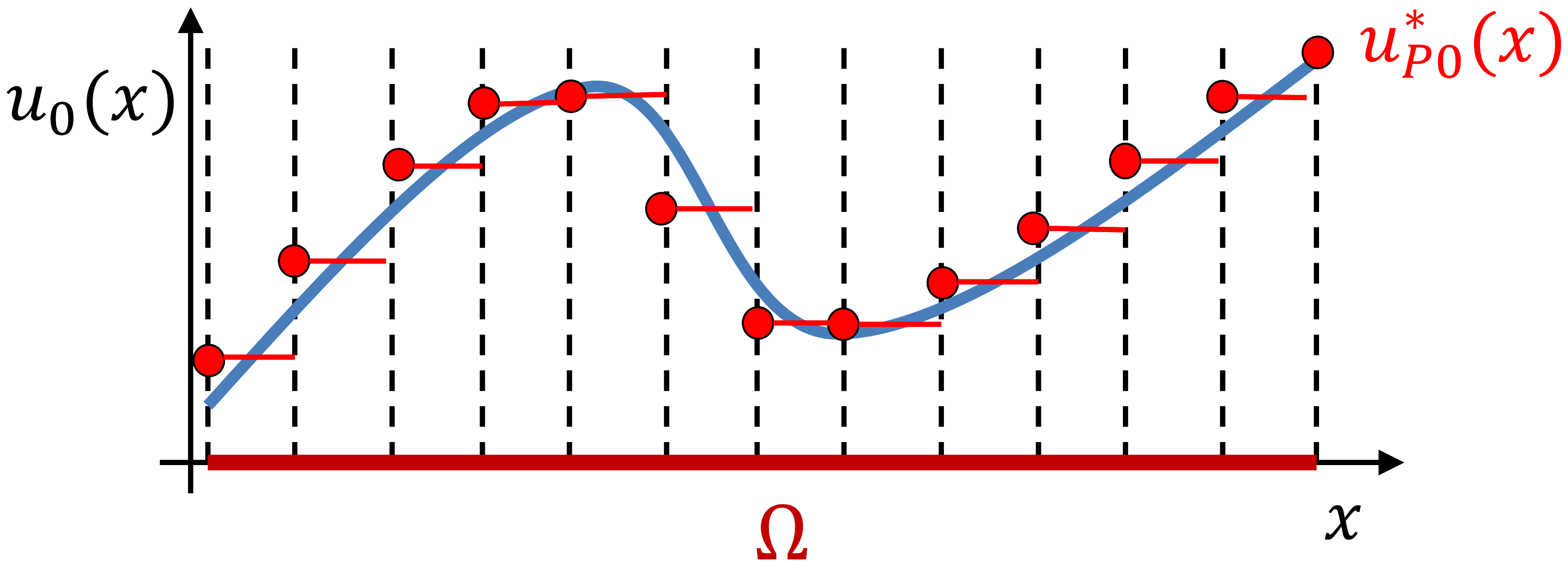
Interpolation defined from the DoFs

Best possible ***linear*** interpolation

Idea of FEM

How to approximate a function on a finite-dimensional space?

Given an interpolation, how can we determine the ***optimal*** *DoF values?*



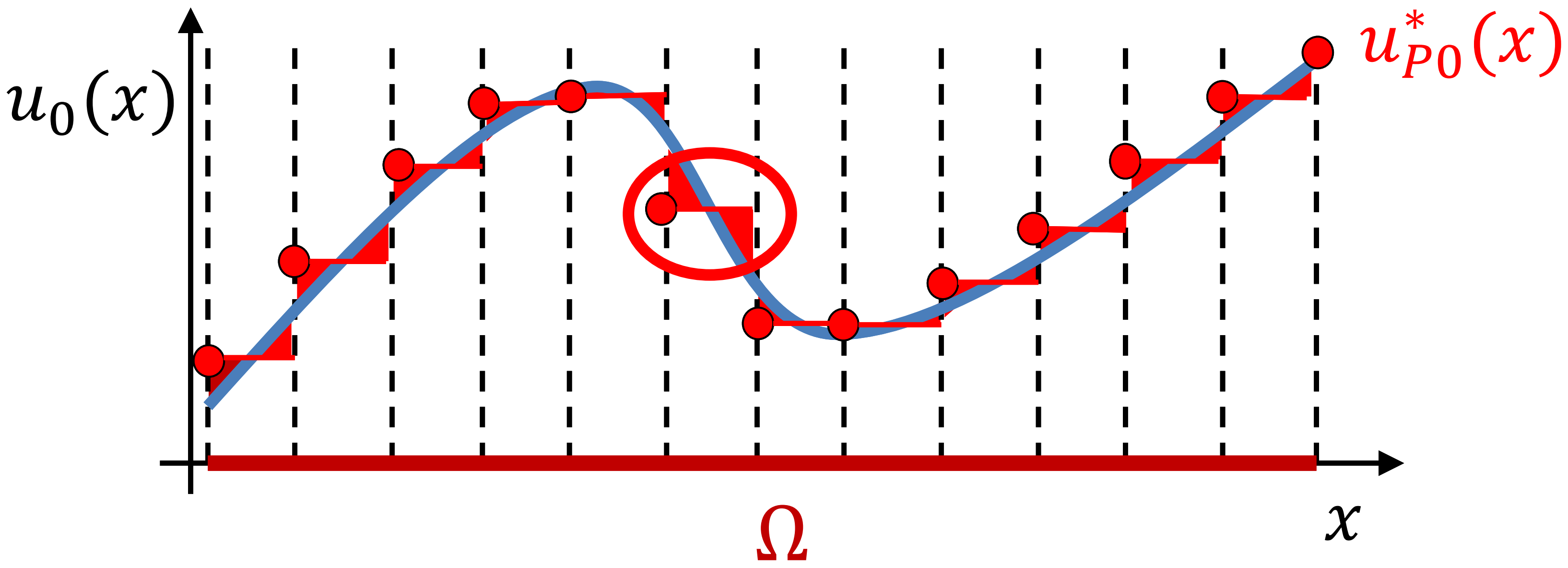
Interpolation defined from the DoFs

Best possible ***piecewise constant*** interpolation

Mean squared error minimization

Illustration

*Given an interpolation, how can we determine the **optimal** DoF values?*



Interpolation defined from the DoFs

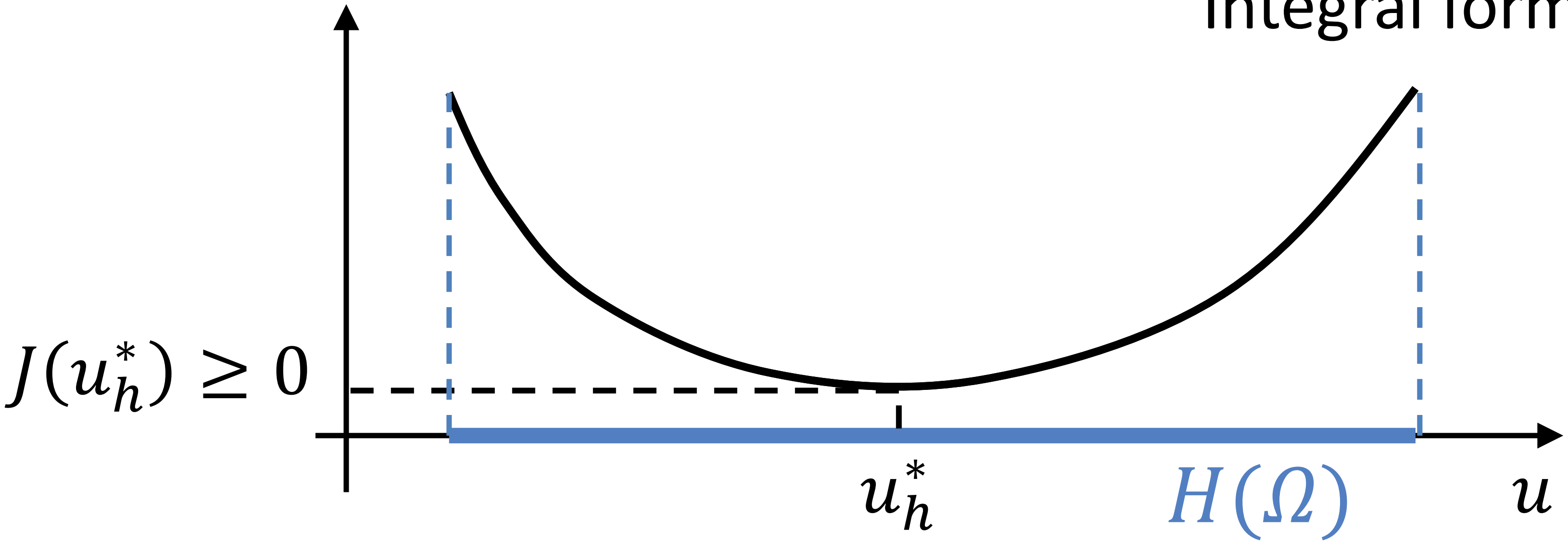
Best possible ***piecewise constant*** interpolation

Mean squared error minimization

Mathematical formulation

$$u_h^* = \arg \min_{u_h \in H(\Omega)} J(u_h) = \underbrace{\frac{1}{2} \int_\Omega \underbrace{(u_h(x) - u_0(x))^2}_{\text{Squared error}} \, \mathrm{d}x}_{\text{Integral formulation}}$$

Admissible function space (continuous or discretized)



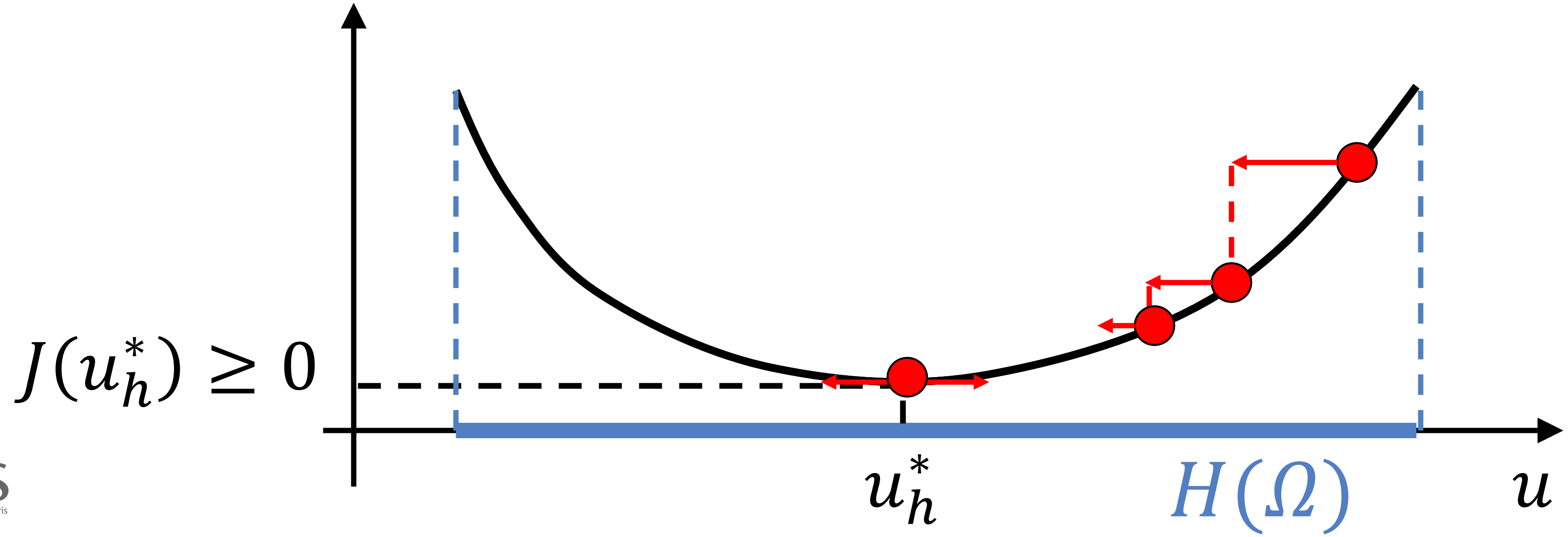
Mean squared error minimization

Algorithms

- Naïve idea: **gradient descent** : $u_h^{k+1} = u_h^k - \alpha\, J'(u_h^k)$
- Better idea: **convex problem** ⇒ unique minimum satisfying

$$J'(u_h^*) = 0$$

How to compute the derivative J' ?



Mean squared error minimization

Directional derivative

- **Directional derivative** in the direction v:

$$\underbrace{J'(u;v) = J'(u)(v) = \langle J'(u), v\rangle}_{} = \lim_{t\to 0} \frac{J(u+tv) - J(u)}{t} \in \mathbb{R}$$

Different notations exist ; all highlighting that v (« test function ») plays a different role than u (point where the derivative is computed).

- We can define a *linear application* $J'(u)\colon v \mapsto J'(u;v) \in L(H,\mathbb{R})$

<u>**Exercise**</u> : compute the directional derivative of the MSE

$$J(u) = \frac{1}{2}\int_\Omega (u(x) - u_0(x))^2 \, \mathrm{d}x$$

Mean squared error minimization

Directional derivative & algorithms

$$J'(u;v) = \int_\Omega (u - u_0)v \ \mathrm{d}x$$

For $d \propto -(u - u_0)$, $J'(u;d) \leq 0$

$\Rightarrow \boxed{d = -(u - u_0)}$ is a ***descent direction***

Gradient descent

$$u_h^{k+1} = u_h^k + \alpha\, d(u_h^k)$$

We can also find $u_h^* \in H(\Omega)$ such that

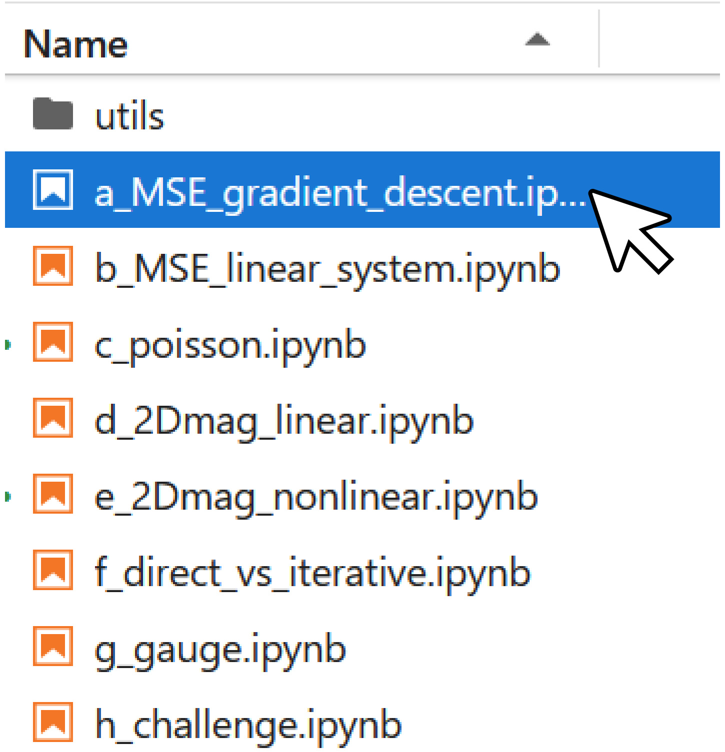
$$\forall v \in H(\Omega), \qquad J'(u_h^*; v) = 0$$

Linear system

To assemble and solve!

Application 1 : gradient descent on MSE

Jupyter Notebook « a_MSE_gradient_descent »



Try out different interpolations:

- Function spaces

  – $L^2(\Omega) = \left\{v\colon \Omega \to \mathbb{R}, \int_\Omega v(x)\,\mathrm{d}x < \infty\right\}$

  (discretized by element-wise ***discontinuous*** functions)

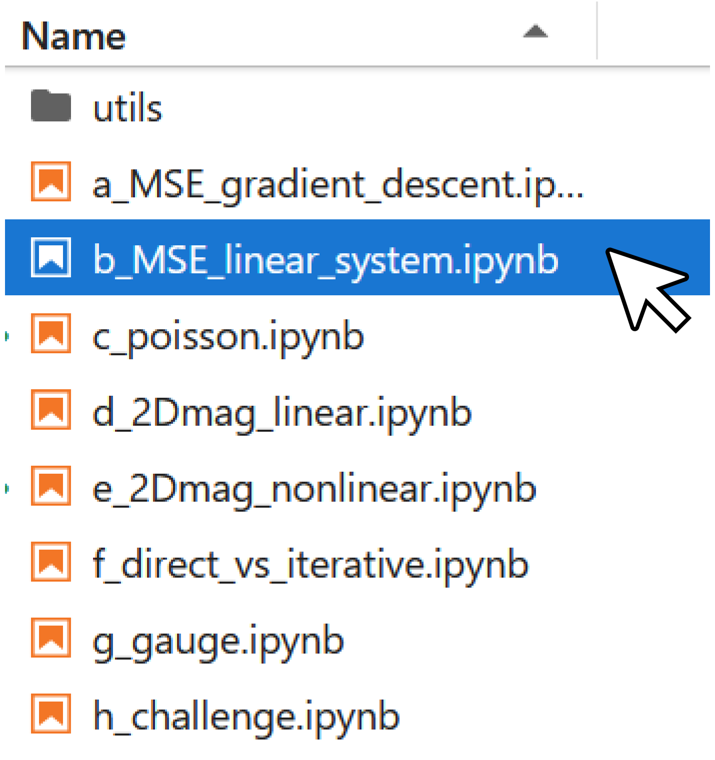
  – $H^1(\Omega) = \{v \in L^2(\Omega), \nabla v \in L^2(\Omega)\}$

  (discretized by nodal ***continuous*** functions)

- Polynomial degrees / order

Application 2 : linear system assembly

Jupyter Notebook « b_MSE_linear_system »



- Gradient descent is generally inefficient and sometimes inapplicable

- From the optimality condition, one can assemble a linear system. So

$$\forall v \in H(\Omega), \qquad \int_\Omega u\, v = \int_\Omega u_0\, v$$

Becomes

$$Ku = f$$

2) ACADEMIC POISSON PROBLEM

Now let’s solve partial differential equations

Variational formulation

General method

- The finite element method is based on variational formulations

- **Main objective of the session**: obtain a variational formulation from the strong equations.

- **Methodology**

1. Choice of relevant variables → *not trivial*… see literature!
2. Choice of the function space H → *often* easy
3. **Projection of the equation on $\boldsymbol{H}$ → *often* easy**

Variational formulation

Choice of relevant variables

- We consider electrostatics
- Maxwell equations

$$\begin{cases} \nabla \cdot \boldsymbol{D} = \rho & \text{(Maxwell–Gauss)} \\ \nabla \times \boldsymbol{E} = 0 & \text{(Maxwell–Faraday)} \end{cases}$$

- Material constitutive law

$$\boldsymbol{D} = \epsilon_0 \epsilon_r \, \boldsymbol{E}$$

With $\epsilon_0 = 8.85 \times 10^{-12}\ F/m$, ϵ_r depending on material

Material	Dielectric constant εr
Vacuum	1
Air	1,0006
Reinforced concrete	1,51
Teflon	2,1
Paper	3,85
Silicon dioxide	3,9
FR-4	4
Mica	5,6 - 8
Marble	8,3
Silicon	11,7
Calcium titanate	150

What is electric permittivity? - Electrical e-Library.com

Variational formulation

Choice of relevant variables

- Many formulations are possible. We usually use scalar electric potential :

$$\boldsymbol{E} = -\nabla u$$

$\Rightarrow \nabla \times \boldsymbol{E} = 0$ is automatically verified (curl of grad is always 0) ; but u is now *defined up to a constant* that should be fixed.

- From the other equations we obtain

Poisson equation $$\boxed{-\nabla \cdot (\epsilon_0 \epsilon_r \nabla u) = \rho}$$

GeePS Génie électrique et électronique de Paris

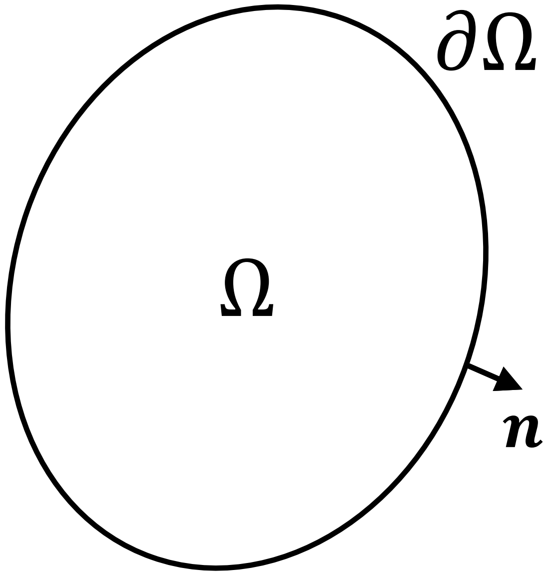
Variational formulation

Formal projection

- We consider a geometric space Ω and a function space $H(\Omega)$, detailed later.

1. **Multiplication** by any test function $v \in H(\Omega)$ and **integration** over Ω :

$$-\int_{\Omega} \nabla \cdot (\epsilon_0 \epsilon_r \nabla u)\, v \; \mathrm{d}x \; = \int_{\Omega} \rho\, v \; \mathrm{d}x$$



2. **Integration by part**; using the following formulae :

Leibniz: $b\nabla \cdot \boldsymbol{A} = \nabla \cdot (b\boldsymbol{A}) - \boldsymbol{A} \cdot \nabla b$

Green-Ostrogradski *:* $\int_{\Omega} \nabla \cdot \boldsymbol{A} = \int_{\partial\Omega} \boldsymbol{A} \cdot \boldsymbol{n}$

Outward normal to $\partial\Omega$

Boundary of Ω

Variational formulation

Formal projection

- We obtain:

$$\underbrace{\int_{\Omega} \epsilon_0 \epsilon_r \nabla u \cdot \nabla v \, \mathrm{d}x - \overbrace{\int_{\partial\Omega} \epsilon_0 \epsilon_r \nabla u \cdot \boldsymbol{n} \; v \, \mathrm{ds}}^{\text{Boundary term}}}_{\text{Bilinear form}} = \underbrace{\int_{\Omega} v \, \rho}_{\text{Linear form}}$$

<u>Boundary value problem</u> (BVP) with a boundary term on the normal component of electrical displacement:

$$\epsilon_0 \epsilon_r \nabla u \cdot \boldsymbol{n} = \boldsymbol{D} \cdot \boldsymbol{n}$$

Homogeneous to a surface charge density ρ_s.

Variational formulation

Flashback to the function space

- We should find $u \in H(\Omega)$, such that

$$\forall v \in H, \qquad \int_\Omega \epsilon_0 \epsilon_r \nabla u \cdot \nabla v \;\mathrm{d}x \;-\int_{\partial\Omega} \epsilon_0 \epsilon_r \nabla u \cdot \boldsymbol{n}\;\, v \,\mathrm{d}s = \int_\Omega v\rho \,\mathrm{d}x$$

Function space

To have well defined integrals, $\underline{H(\Omega) = \{u \in L^2(\Omega), \nabla u \in L^2(\Omega)\} = H^1(\Omega)}$

GeePS Génie électrique et électronique de Paris

Boundary conditions

Natural boundary conditions

- We should find $u \in H(\Omega)$, such that

$$\forall v \in H, \qquad \int_{\Omega} \epsilon_0 \epsilon_r \nabla u \cdot \nabla v \; \mathrm{d}x \; - \int_{\partial\Omega} \epsilon_0 \epsilon_r \nabla u \cdot \boldsymbol{n} \;\; v \, \mathrm{d}s = \int_{\Omega} v\rho \, \mathrm{d}x$$

- **Natural boundary conditions** : we rewrite the boundary term
  - ***Neumann*** : $\epsilon_0 \epsilon_r \nabla u \cdot \boldsymbol{n} = g \rightarrow$ boundary term becomes a linear form
  - ***Robin*** : $\epsilon_0 \epsilon_r \nabla u \cdot \boldsymbol{n} = g - \alpha \, u \; \rightarrow$ boundary term becomes linear + bilinear form

Since these boundary conditions are part of the variational form, they are « *weakly* » imposed.

Boundary conditions

Natural boundary conditions

- We should find $u \in H(\Omega)$, such that

$$\forall v \in H, \qquad \int_{\Omega} \epsilon_0 \epsilon_r \nabla u \cdot \nabla v \;\mathrm{d}x \;-\int_{\partial\Omega} \epsilon_0 \epsilon_r \nabla u \cdot \boldsymbol{n}\; v \,\mathrm{ds} = \int_{\Omega} v\rho \,\mathrm{d}x$$

- **Neumann** (special case of Robin) :

$$\forall v \in H, \qquad \int_{\Omega} \epsilon_0 \epsilon_r \nabla u \cdot \nabla v \;\mathrm{d}x \;= \int_{\Omega} v\rho \,\mathrm{d}x + \int_{\partial\Omega} g\, v \,\mathrm{ds}$$

- **Robin** :

$$\forall v \in H, \qquad \int_{\Omega} \epsilon_0 \epsilon_r \nabla u \cdot \nabla v \;\mathrm{d}x + \int_{\partial\Omega} \alpha\, u\, v \;\mathrm{ds} = \int_{\Omega} v\rho \,\mathrm{d}x + \int_{\partial\Omega} g\, v \,\mathrm{ds}$$

GeePS Génie électrique et électronique de Paris

Boundary conditions

Essential boundary conditions

- We should find $u \in H(\Omega)$, such that

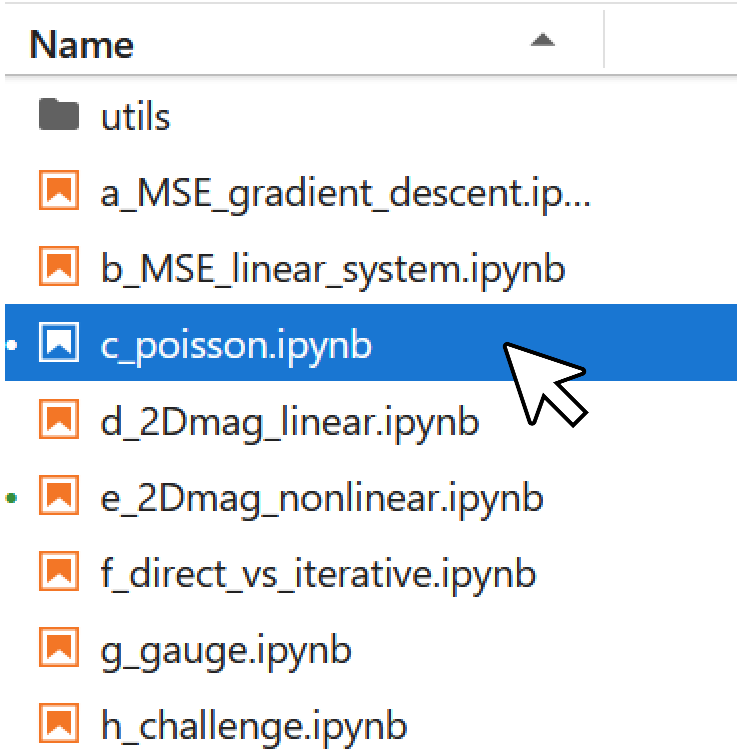
$$\forall v \in H, \qquad \int_\Omega \epsilon_0 \epsilon_r \nabla u \cdot \nabla v \;\mathrm{d}x \;-\; \cancel{\int_{\partial\Omega} \epsilon_0 \epsilon_r \nabla u \cdot \boldsymbol{n}\; v \,\mathrm{ds}} = \int_\Omega v\rho \,\mathrm{d}x$$

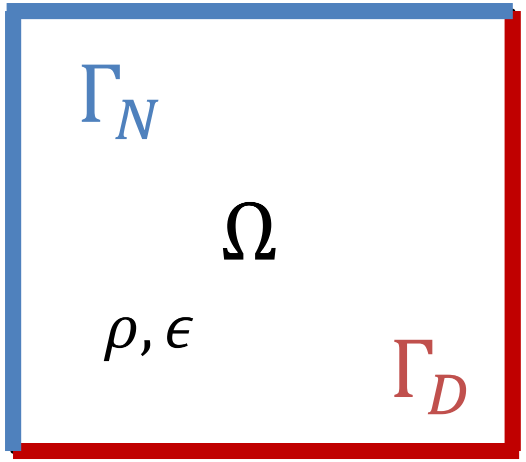
- **Essential boundary conditions**: not appearing directly in the boundary term; therefore should be imposed (exactly!) in the **function space**
  - ***Dirichlet*** : $u = u_d$ on the boundary
  - ***Periodicity / anti-periodicity*** : $u_1 = \pm u_2$

Sets the boundary term to 0

Application 3 : Poisson problem

Jupyter Notebook « c_Poisson »





Find $u \in H(\Omega), \forall v \in H,$

$$\int_\Omega \epsilon_0 \epsilon_r \nabla u \cdot \nabla v \; \mathrm{d}x = \int_\Omega v\rho \, \mathrm{d}x$$

- Homogeneous Dirichlet : $u|_{\partial\Omega} = 0$

$\Rightarrow \vec{E}$ orthogonal to the boundary (gradient is orthogonal to the isolines of u).
$\Rightarrow \partial\Omega$ is an anti-symmetry plane
Can also troncate infinity $(u(\infty) = 0)$

- Homogeneous Neumann : $\vec{D} \cdot \vec{n} = 0$

$\Rightarrow \vec{D}$ tangential to the boundary
$\Rightarrow \partial\Omega$ is a symmetry plane

GeePS Génie électrique et électronique de Paris

3) 2D MAGNETOSTATICS

Non-linearity and Newton method

Magnetostatics

Equations

- We give the equations

$\boldsymbol{B} = \nabla \times \boldsymbol{a}$	Magnetic vector potential (unknown)
$\nabla \times \boldsymbol{H} = \boldsymbol{j}$	Maxwell Ampère
$\boldsymbol{H} = \nu(\lvert\boldsymbol{B}\rvert^2)\boldsymbol{B}$	Constitutive law of iron
$\boldsymbol{a} = \boldsymbol{0}$ on $\partial\Omega$	Dirichlet boundary condition (boundary term →0)

$\boxed{\text{What is the strong formulation?}}$

Magnetostatics

Strong form

- The b-conform strong equation reads

$$\nabla \times \left(\nu(|\nabla \times \boldsymbol{a}|^2)\, \nabla \times \boldsymbol{a}\right) = \boldsymbol{j}$$

Or

$$\mathbf{curl}\left(\nu(|\mathbf{curl}\ \boldsymbol{a}|^2)\ \mathbf{curl}\ (\boldsymbol{a})\right) = \boldsymbol{j}$$

What is the variational formulation?

Leibniz: $\boldsymbol{A} \cdot \mathbf{curl}(\boldsymbol{B}) = \boldsymbol{B} \cdot \mathbf{curl}(\boldsymbol{A}) - \operatorname{div}(\boldsymbol{A} \times \boldsymbol{B})$

Green-Ostrogradski : $\int_{\Omega} \operatorname{div}(\boldsymbol{A}) = \int_{\partial\Omega} \boldsymbol{A} \cdot \boldsymbol{n}$

GeePS Génie électrique et électronique de Paris

Magnetostatics

Variational formulation

- We find $\boldsymbol{a} \in H_0(\mathbf{curl};\Omega) = \{\boldsymbol{a} \in L^2(\Omega), \mathbf{curl}(\boldsymbol{a}) \in L^2(\Omega),\ \ \boldsymbol{a} \times \boldsymbol{n} = 0 \text{ on } \partial\Omega\}$

$$\forall \boldsymbol{v} \in H_0(\mathbf{curl};\Omega), \qquad \int_\Omega \mathbf{curl}\,\boldsymbol{v} \cdot \left(\nu\left(|\mathbf{curl}\,\boldsymbol{a}|^2\right)\mathbf{curl}\,\boldsymbol{a}\right) = \int_\Omega \boldsymbol{v} \cdot \boldsymbol{j}$$

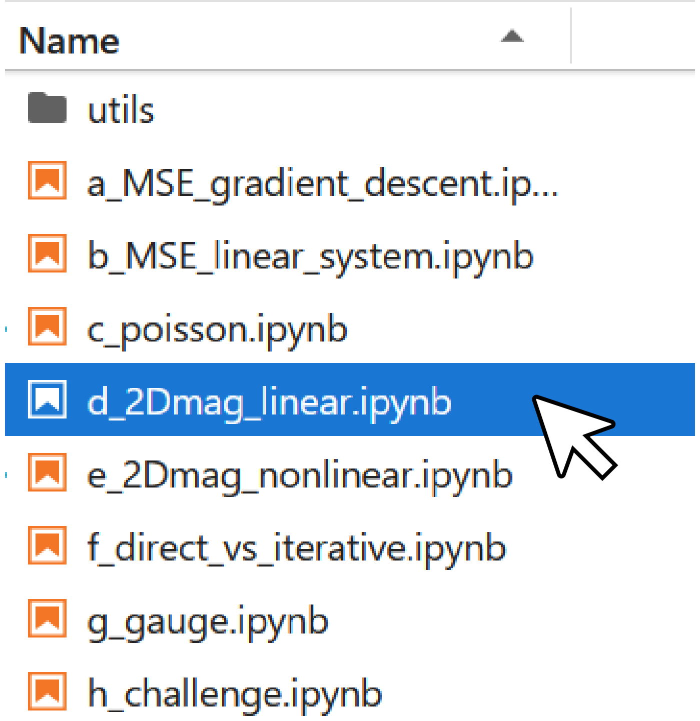
- In 2D, we have

$$\boldsymbol{j} = \begin{bmatrix} 0 \\ 0 \\ j_z \end{bmatrix}, \qquad \boldsymbol{a} = \begin{bmatrix} 0 \\ 0 \\ a_z \end{bmatrix}, \qquad \mathrm{v} = \begin{bmatrix} 0 \\ 0 \\ v_z \end{bmatrix}, \qquad \mathbf{curl}(\boldsymbol{a}) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \mathbf{grad}(a_z)$$

So that we can rewrite the equation w.r.t the z-components only.

Application 4 : 2D linear magnetostatics

Jupyter Notebook « d_nonlinear.ipynb »



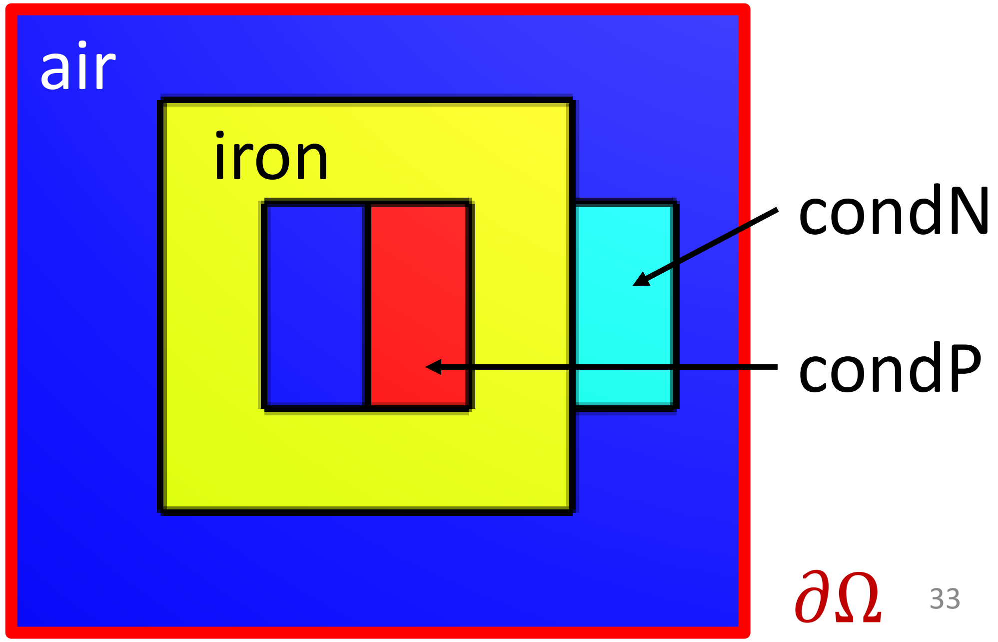
- Find

$$a_z \in H_0^1(\Omega) = \{a \in L^2(\Omega), \nabla a \in L^2(\Omega), a = 0 \text{ on } \partial\Omega\}$$

$$\forall v \in H_0^1(\Omega), \qquad \int_\Omega \mathbf{curl}\, v \cdot \nu\, \mathbf{curl}\, a_z = \int_\Omega v\, j_z$$

For now, we assume **iron is linear** :

- $\mu_{iron} = 1000\mu_0 \Rightarrow \nu_{iron} = \frac{1}{1000\mu_0}$
- $J = 10\ A/mm^2$



GeePS
Génie électrique et électronique de Paris

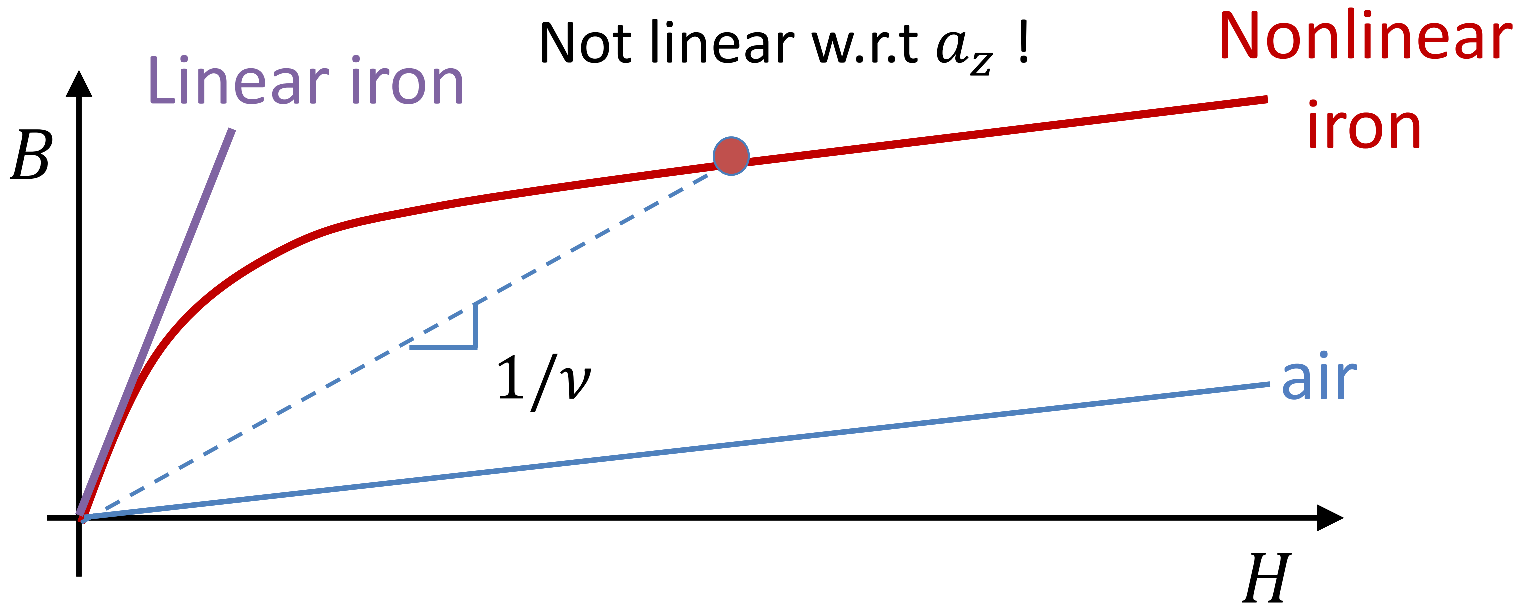
Nonlinearity

2D Variational formulation

- We should find $a_z \in H_0^1(\Omega) = \{a \in L^2(\Omega), \nabla a \in L^2(\Omega),\ \ a = 0 \text{ on } \partial\Omega\}$

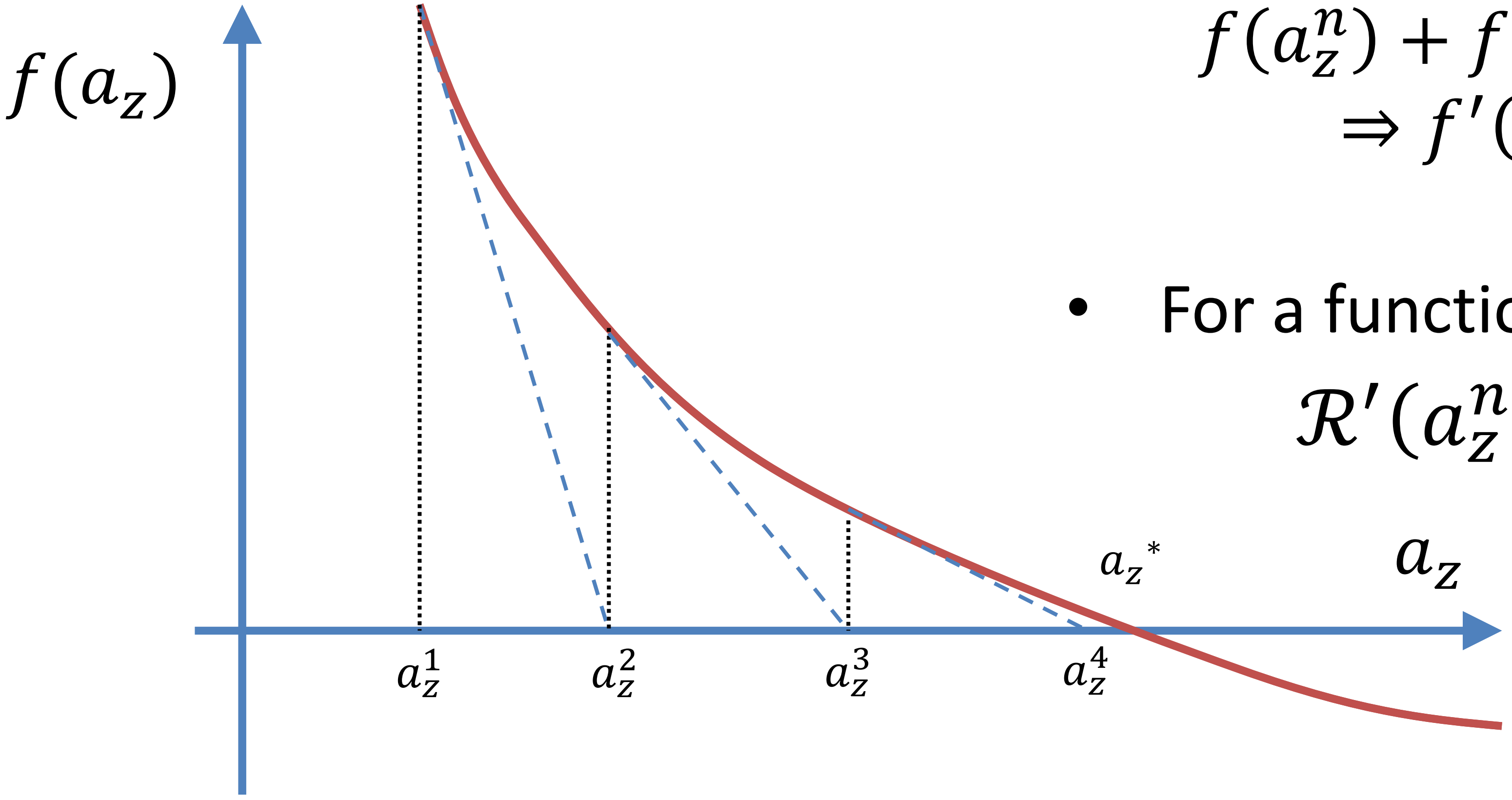
$$\forall v \in H_0^1(\Omega), \qquad \underbrace{\int_\Omega \mathbf{curl}\, v \cdot \nu\left(|\mathbf{curl}\, a_z|^2\right)\mathbf{curl}\, a_z}_{\text{Not linear w.r.t } a_z \text{ !}} = \int_\Omega v\, j_z$$

BH curves:



Nonlinearity

Newton method



- For a usual function:

$$f(a_z^n) + f'(a_z^n)\,\overbrace{(a_z^{n+1} - a_z^n)}^{\delta a} = 0$$
$$\Rightarrow f'(a_z^n)\,\delta a = -f(a_z^n)$$

- For a functional $\mathcal{R}$: find δa, such that

$$\mathcal{R}'(a_z^n; \delta a) = -\mathcal{R}(a_z^n)$$

Nonlinearity

Newton method

- We should now define the functional $\mathcal{R}$ to cancel.
- **Residual**

$$\mathcal{R}(a_z, v) = \int_\Omega \mathbf{curl}\, v \cdot \nu(|\mathbf{curl}\, a_z|^2)\mathbf{curl}\, a_z - \int_\Omega v j_z$$

- **Directional derivative**:

$$\begin{aligned}&\mathcal{R}'(a_z, v; \delta a) \\ &= \int_\Omega \mathbf{curl}\, v \cdot \nu(|\mathbf{curl}\, a_z|^2)\mathbf{curl}\, \delta a \\ &+ 2\int_\Omega \mathbf{curl}\, v \cdot (\nu'(|\mathbf{curl}\, a_z|^2)\mathbf{curl}\, a_z \cdot \mathbf{curl}\, \delta a)\mathbf{curl}\, a_z\end{aligned}$$

Nonlinearity

Newton method

- <u>**Typical algorithm**</u>:

1) Solve the linearized problem

$$\forall v \in H_0^1(\Omega), \qquad \mathcal{R}'(a_z, v; \delta a) = -\mathcal{R}(a_z, v)$$

2) Adapt step size α and update

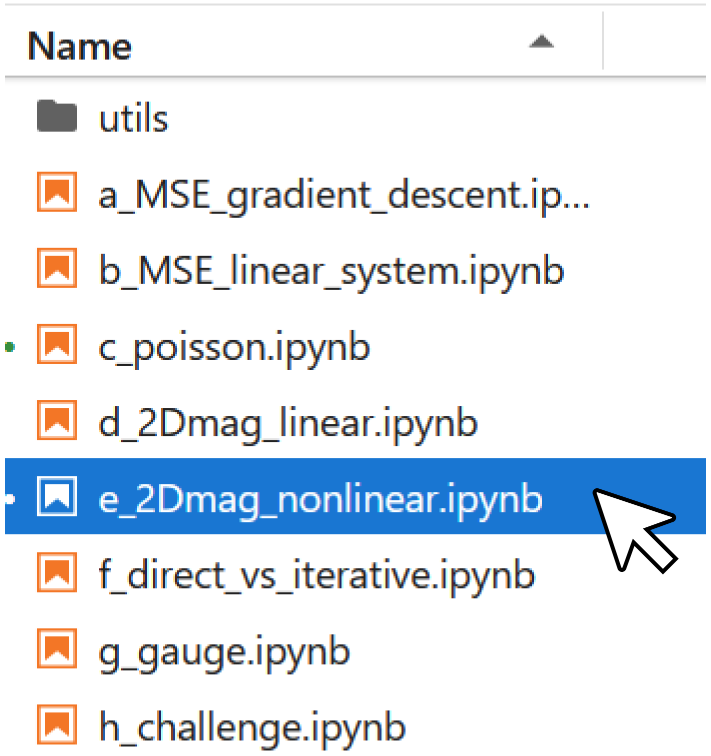
$$a_z^{n+1} = a_z^n + \alpha\,\delta a$$

3) Stop criterion (several possibilities…)

$|\mathcal{R}(a_z^n, \delta a)| \leq \mathrm{tol}$, or $|\mathcal{R}(a_z^n, v_i)| \leq \mathrm{tol}$ … and always $n > n_{max}$

Application 5 : non-linear 2D magnetostatics

Jupyter Notebook « e_nonlinear_2D_magnetostatics »



- **Residual**

$$\mathcal{R}(a_z, v) = \int_\Omega \mathbf{curl}\, v \cdot \nu(|\mathbf{curl}\, a_z|^2)\mathbf{curl}\, a_z - \int_\Omega v j_z$$

- **Directional derivative**:

$$\mathcal{R}'(a_z, v; {\color{red}\delta a})$$
$$= \int_\Omega \mathbf{curl}\, v \cdot \nu(|\mathbf{curl}\, a_z|^2)\mathbf{curl}\, {\color{red}\delta a}$$
$$+ 2 \int_\Omega \mathbf{curl}\, v \cdot (\nu'(|\mathbf{curl}\, a_z|^2)\mathbf{curl}\, a_z \cdot \mathbf{curl}\, {\color{red}\delta a})\mathbf{curl}\, a_z$$

1) Find $\delta a \in H_0^1(\Omega)$, such that $\forall v \in H_0^1(\Omega)$, $\mathcal{R}'(a_z, v; \delta a) = -\mathcal{R}(a_z, v)$
2) Update $a_z \leftarrow a_z + \alpha\, \delta a$
3) Stop criterion

4) 3D MAGNETOSTATICS

Iterative solver, edge elements and gauge

GeePS
Génie électrique et électronique de Paris

Preliminaries

Iterative solver VS direct solver

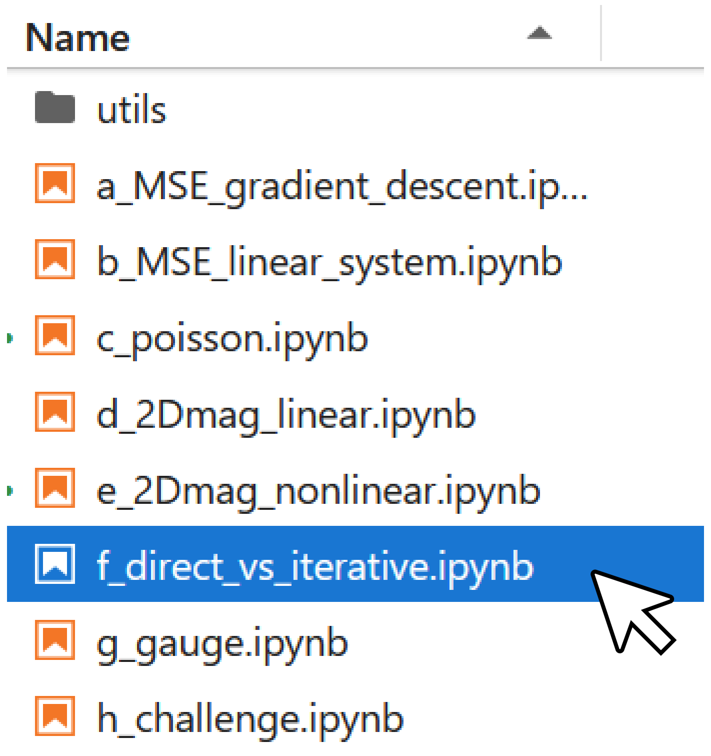
Time complexity
Provide matrix decomposition
Exact
Sensitive to matrix bandwith
Sensitive to condition number
Typical use

GeePS Génie électrique et électronique de Paris

https://www.cs.cmu.edu/~quake-papers/painless-conjugate-gradient.pdf

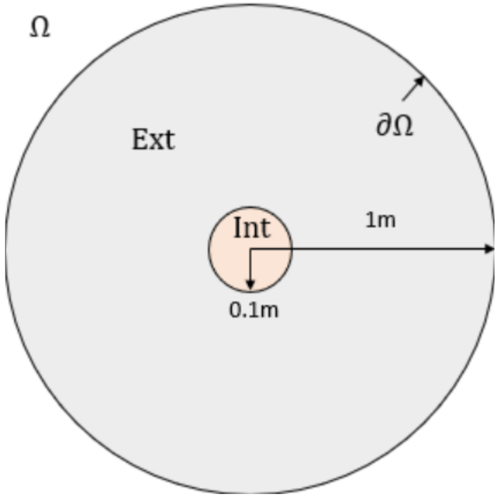
Application 6 : iterative vs direct solver

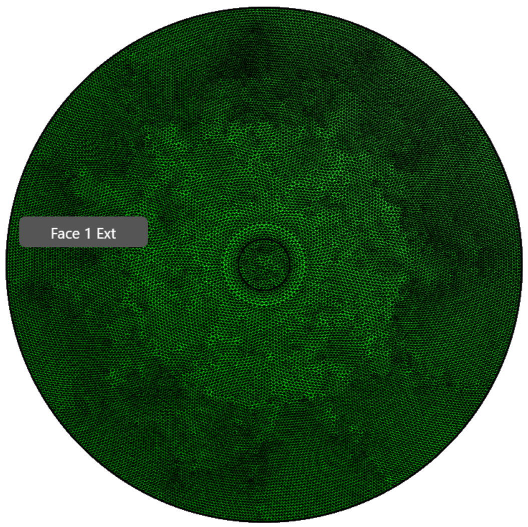
Poisson in 2D vs 3D

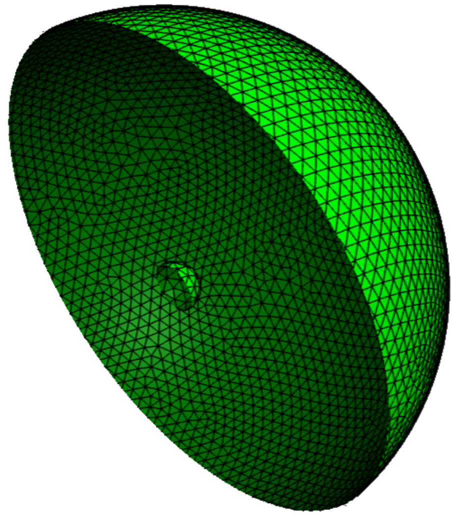


- Poisson equation on a disk and a ball with the same number of DoFs

$$\forall v \in H_0^1(\Omega), \qquad \int_\Omega \nabla v \cdot \nabla u \,\mathrm{d}x = \int_{int} 1\, v \,\mathrm{d}x$$







3D Magnetostatics : variational formulation

Function space

Find $\boldsymbol{a} \in H_0(\mathbf{curl};\Omega) = \{\boldsymbol{a} \in L^2(\Omega), \mathbf{curl}(\boldsymbol{a}) \in L^2(\Omega), \quad \boldsymbol{a} \times \boldsymbol{n} = 0 \text{ on } \partial\Omega\}$

$$\forall \boldsymbol{v} \in H_0(\mathbf{curl};\Omega), \qquad \int_\Omega \mathbf{curl}\, \boldsymbol{v} \cdot \nu\, \mathbf{curl}\, \boldsymbol{a} = \int_\Omega \boldsymbol{v} \cdot \boldsymbol{j}$$

What is $H(\mathbf{curl};\Omega)$ function space?

Function spaces

Space
H^1
$H(\mathrm{curl})$
$H(\mathrm{div})$
L^2

3D Magnetostatics : variational formulation

Gauge

Find $\boldsymbol{a} \in H_0(\mathbf{curl};\Omega) = \{\boldsymbol{a} \in L^2(\Omega), \mathbf{curl}(\boldsymbol{a}) \in L^2(\Omega),\ \ \boldsymbol{a} \times \boldsymbol{n} = 0 \text{ on } \partial\Omega\}$

$$\forall \boldsymbol{v} \in H_0(\mathbf{curl};\Omega), \qquad \int_\Omega \mathbf{curl}\,\boldsymbol{v} \cdot \nu\,\mathbf{curl}\,\boldsymbol{a} = \int_\Omega \boldsymbol{v} \cdot \boldsymbol{j}$$

$\boxed{\text{Is the solution uniquely defined?}}$

No ! Assuming a is solution , then $\widetilde{\boldsymbol{a}} = \boldsymbol{a} + \mathbf{grad}\,u$ is also solution, for u any differentiable scalar field, since $\mathbf{curl}\,\mathbf{grad}(.) = \mathbf{0}$

3D Magnetostatics

Gauge

There are many different ways to obtain uniqueness:

- Add a small « mass » term

$$\int_\Omega \mathbf{curl}\,\boldsymbol{v} \cdot \nu\, \mathbf{curl}\,\boldsymbol{a} + \int_{\boldsymbol{\Omega}} \epsilon\, \boldsymbol{v} \cdot \boldsymbol{a} = \int_\Omega \boldsymbol{v} \cdot \boldsymbol{j}$$

- Add an equation (Coulomb gauge) : $\mathrm{div}(\boldsymbol{a}) = 0$

Weak form : find $\boldsymbol{a}, \lambda \in H_0(\mathbf{curl}; \Omega) \times H_0^1(\Omega)$,

$$\begin{cases} \forall \boldsymbol{v} \in H_0(\mathbf{curl}; \Omega), & \displaystyle\int_\Omega \mathbf{curl}\,\boldsymbol{v} \cdot \nu\, \mathbf{curl}\,\boldsymbol{a} + \int_{\boldsymbol{\Omega}} \nabla\lambda \cdot \boldsymbol{v} = \int_\Omega \boldsymbol{v} \cdot \boldsymbol{j} \\ \forall \mu \in H_0^1(\Omega), & \displaystyle\int_{\boldsymbol{\Omega}} \nabla\mu \cdot \boldsymbol{a} = 0 \end{cases}$$

GeePS Génie électrique et électronique de Paris

Variational formulation

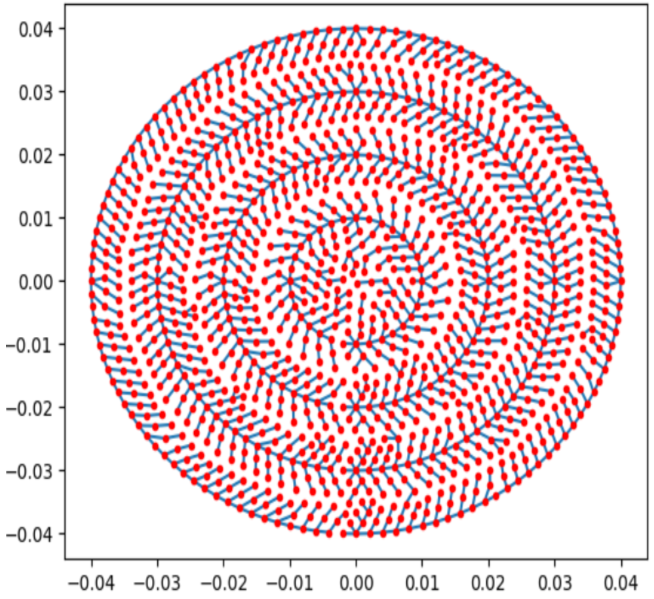
Gauge

- Solve with an **iterative solver** and a compatible right-hand side

1) Find $\boldsymbol{T} \in H(\mathrm{curl}, \Omega)$, s.t. $\int_\Omega \mathbf{curl}\,\boldsymbol{v} \cdot \mathbf{curl}\,\boldsymbol{T} = \int_\Omega \boldsymbol{v} \cdot \boldsymbol{j}\ \forall \boldsymbol{v} \in \boldsymbol{H}(\boldsymbol{curl}, \boldsymbol{\Omega})$

2) Find $\boldsymbol{a} \in H(\mathrm{curl}, \Omega)$,

$$\int_\Omega \mathbf{curl}\,\boldsymbol{v} \cdot \nu\, \mathbf{curl}\,\boldsymbol{a} = \int_\Omega \boldsymbol{v} \cdot \mathbf{curl}\,\boldsymbol{T}$$

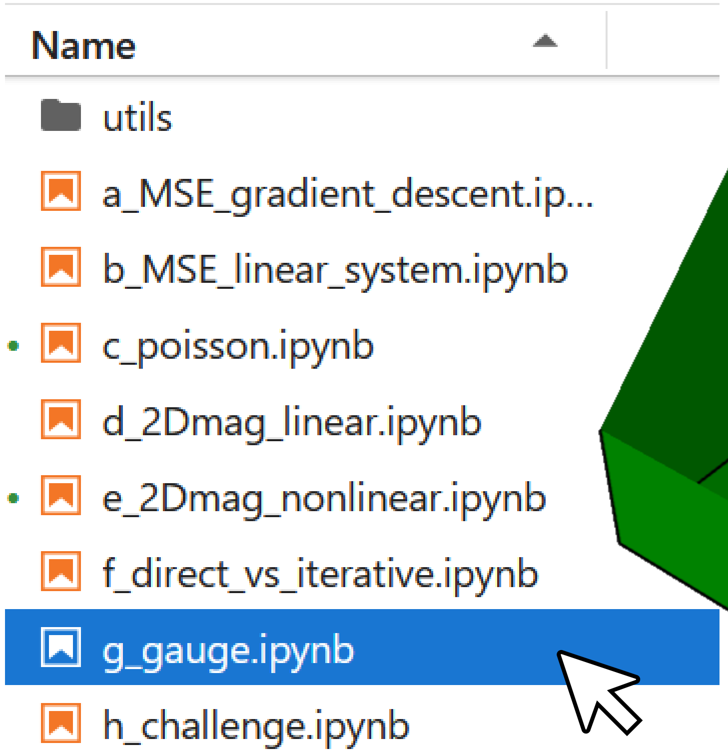


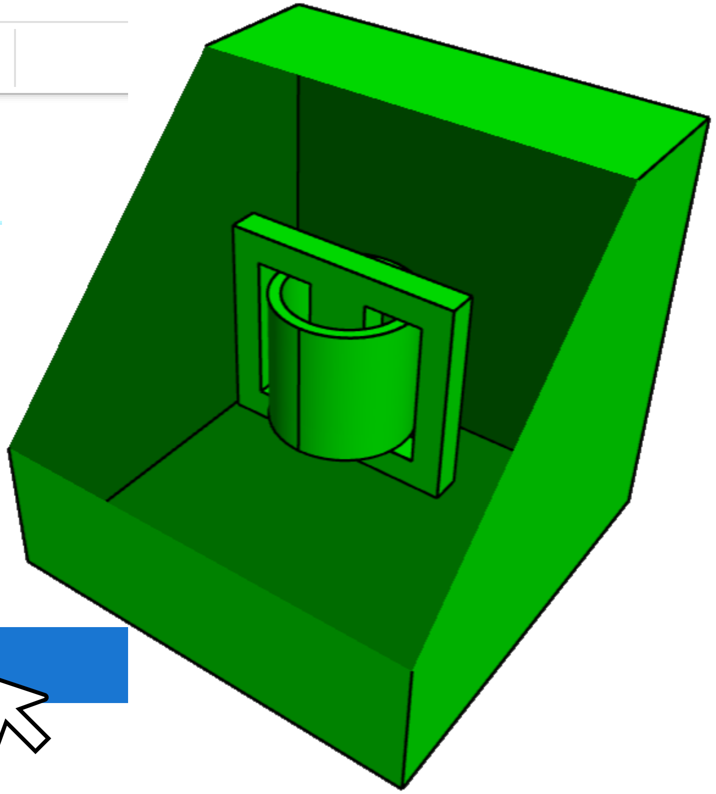
- Remove the redundant DoF (tree-cotree gauge).

You can implement and compare all of these possibilities!

Application 7 : 3D Magnetostatics

Uniqueness of the solution





- **Small mass term**

$$\int_\Omega \mathbf{curl}\,\boldsymbol{v}\cdot\nu\,\mathbf{curl}\,\boldsymbol{a} + \int_\Omega \epsilon\,\boldsymbol{v}\cdot\boldsymbol{a} = \int_\Omega \boldsymbol{v}\cdot\boldsymbol{j}$$

- **Coulomb gauge**: solve simultaneously

$$\begin{cases} \forall \boldsymbol{v} \in H_0(\mathbf{curl};\Omega), & \int_\Omega \mathbf{curl}\,\boldsymbol{v}\cdot\nu\,\mathbf{curl}\,\boldsymbol{a} + \int_\Omega \nabla\lambda\cdot\boldsymbol{v} = \int_\Omega \boldsymbol{v}\cdot\boldsymbol{j} \\ \forall \mu \in H_0^1(\Omega), & \int_\Omega \nabla\mu\cdot\boldsymbol{a} \qquad\qquad = 0 \end{cases}$$

- **Compatible RHS**

1) Find $\mathrm{T} \in H(\mathbf{curl},\Omega),\ s.t.$

$$\forall \boldsymbol{v} \in H_0(\mathbf{curl};\Omega), \int_\Omega \mathbf{curl}\,\boldsymbol{v}\cdot\mathbf{curl}\,\boldsymbol{j} = \int_\Omega \boldsymbol{v}\cdot\boldsymbol{j}$$

2) Find **a** $\in H(\mathbf{curl},\Omega),\ s.t.$

$$\int_\Omega \mathbf{curl}\,\boldsymbol{v}\cdot\nu\,\mathbf{curl}\,\boldsymbol{a} = \int_\Omega \boldsymbol{v}\cdot\mathbf{curl}\,\boldsymbol{T}$$

- **Tree-Cotree gauging**

GeePS Génie électrique et électronique de Paris

3D Magnetostatics : symmetries

Boundary conditions

- Homogeneous Neumann :

$$\forall v \in H(\mathbf{curl}, \Omega) \int_{\partial\Omega} (\boldsymbol{h} \times \boldsymbol{n}) \cdot \boldsymbol{v} = \mathbf{0} \Rightarrow \ \boldsymbol{h} \times \boldsymbol{n} = 0$$

⇒ magnetic field orthogonal to the boundary

= **Symmetry**

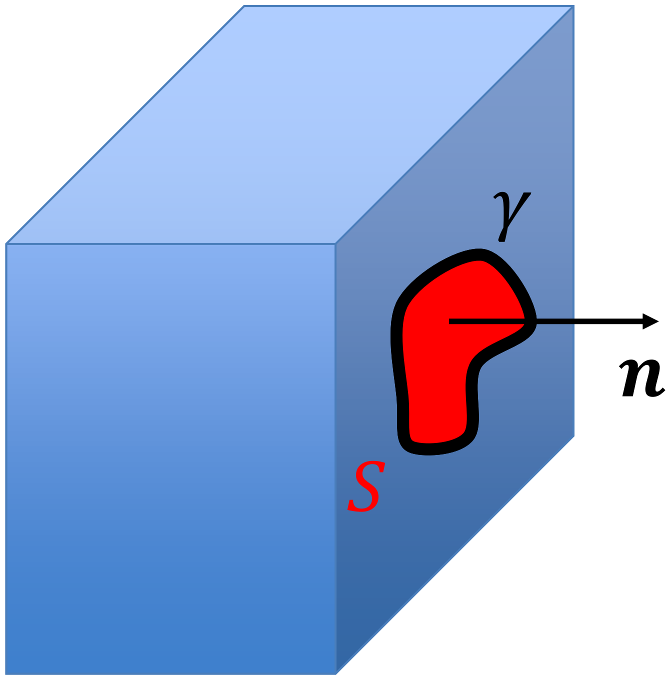
- Homogeneous Dirichlet : $\boldsymbol{a} \times \boldsymbol{n} = \mathbf{0}$

⇒ vector potential orthogonal to the boundary

$$\Rightarrow \forall S \in \partial\Omega, \phi_{out} = \iint_{S_\gamma \in \partial\Omega} \boldsymbol{B} \cdot d\boldsymbol{S} = \oint_{\gamma = \partial S} \boldsymbol{a} \cdot d\boldsymbol{l} = 0$$

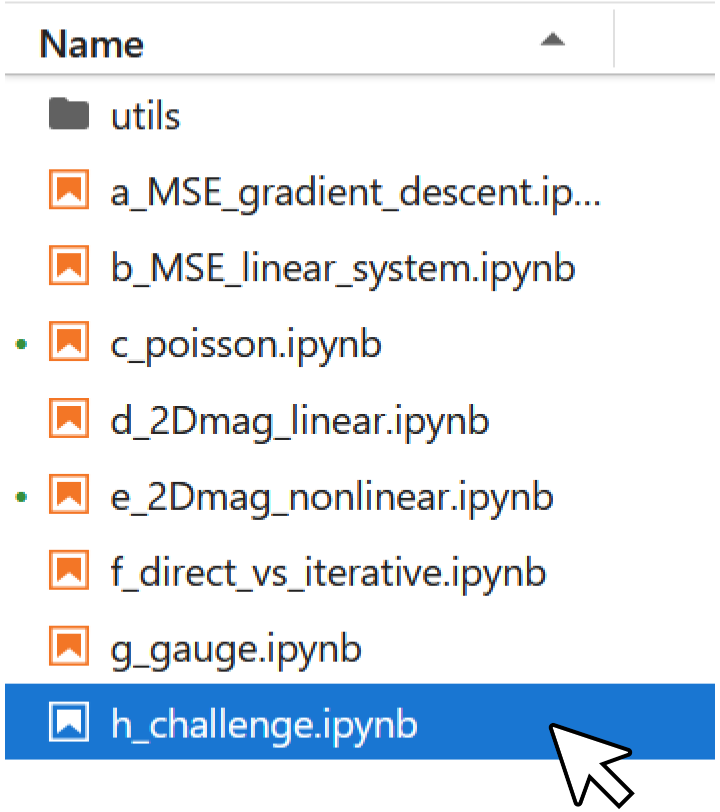
⇒ flux density tangential to the boundary

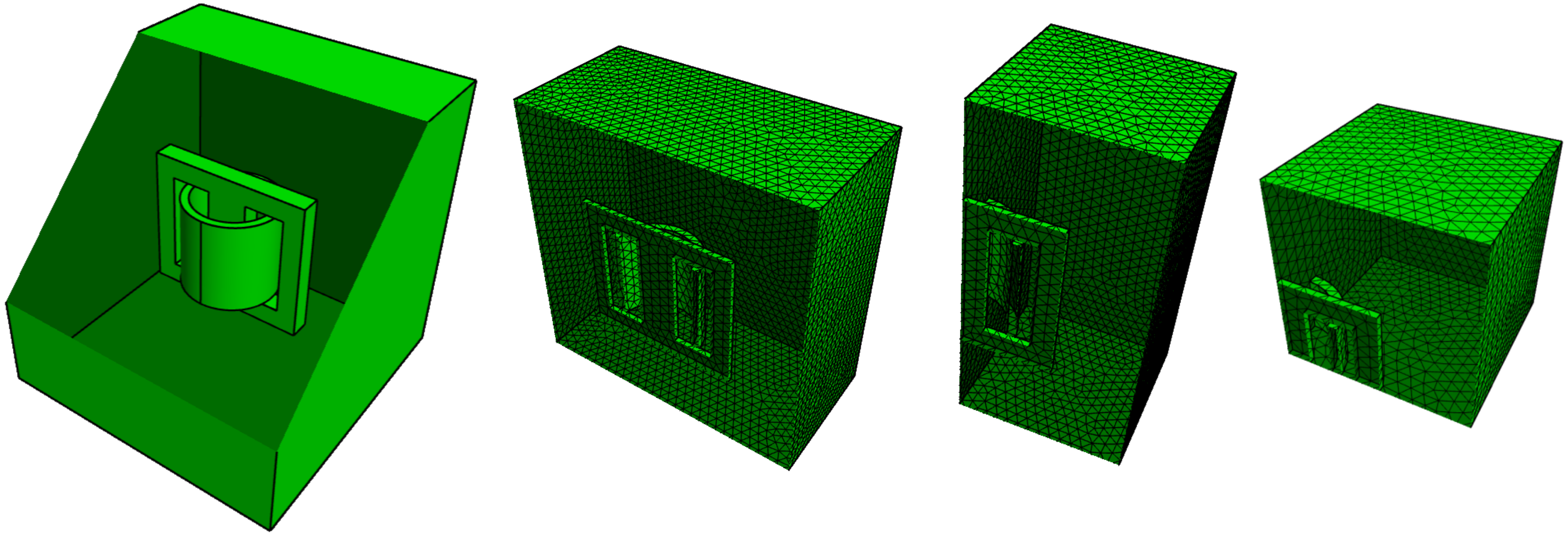
= **Anti-symmetry**



Application 8 : synthesis

Challenge : implement the fastest 3D magnetostatic solver for the inductance problem





Full model *1/2* *1/4* *1/8*

To discuss on December 10 during the next GeePs'N Talks session!

Outlook

The journey is not over…

- How to mesh / remesh? → GMSH, Netgen…
- How to control the error ? → adaptive mesh refinement
- Advanced solvers (multifrontal, multigrid, etc.)
- Harmonic / Time dependant problems…
- Multiphysics / coupled problems
- What can we put over FEM? → interface tracking, topology optimization…
- Other methods(BEM, FIT, IGA, MoM, hybrid methods…)

References

- J. Schöberl, An Interactive Introduction to the Finite Element Method (https://jschoeberl.github.io/iFEM/intro.html)

- A. Ern, Jean-Luc Guermond, Finite Elements I: Approximation and interpolation, 2004, https://hal.science/hal-03226049v1

- Bossavit, A. Whitney forms: a class of finite elements for three-dimensional computations in electromagnetism. IEE Proceedings A Physical Science, Measurement and Instrumentation, Management and Education, Reviews, 135(8), 493, 1988

- Z. Ren, "Influence of the R.H.S. on the Convergence Behaviour of the Curl-Curl Equation," *IEEE Trans. Magn.*, vol. 32, no. 3, pp. 655–658, 1996.

Thank you for your attention !

GeePs'N Talks special session

T. Cherrière, A. El Gode, T. Gauthey