Finite Element Method (FEM)

GeePs'N Talks special session

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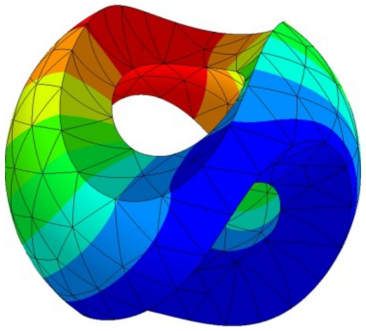
Preliminaries



- Interactive course, using NGSolve (Python FEM toolbox)

- Go to website : https://github.com/tcherrie/tutorial_fem

And click on the yellow button try lite now

- The code should run in your browser without installation required.

- If strange bugs: **reload the webpage** (virtual memory overflow)
- GeePs clusters in backup
- For local installation: ask after the tutorial

Outlines

- 1) Lengthy introduction
  - Function spaces & interpolation
  - Integral formulation
  - Linear system
- 2) Academic Poisson problem
  - Variational formulation
  - Boundary conditions
- 3) Non-linear Magnetostatics (2D)
  - Realistic problem
  - Newton method
- 4) 3D Magnetostatics
  - Iterative solver
  - Gauge

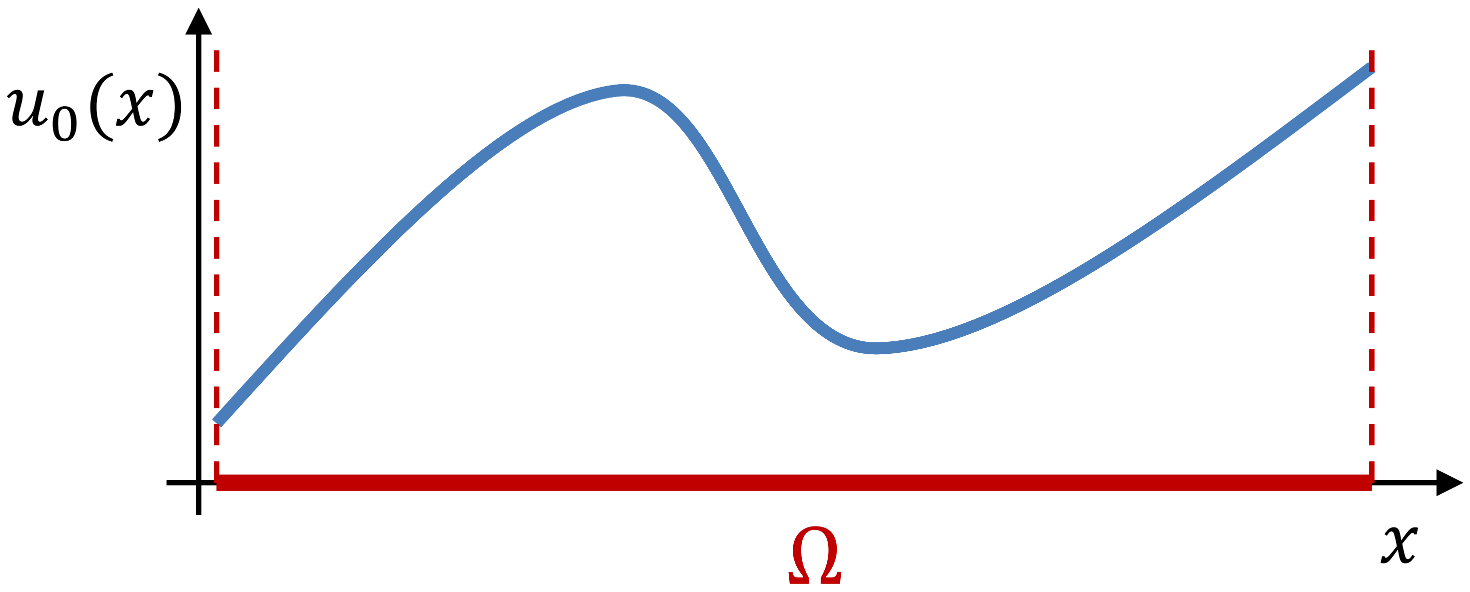
Not addressed in this tutorial: time-harmonic and time-dependent problem.

1) LENGTHY INTRODUCTION

Idea of the method, not boring I promise (hope)

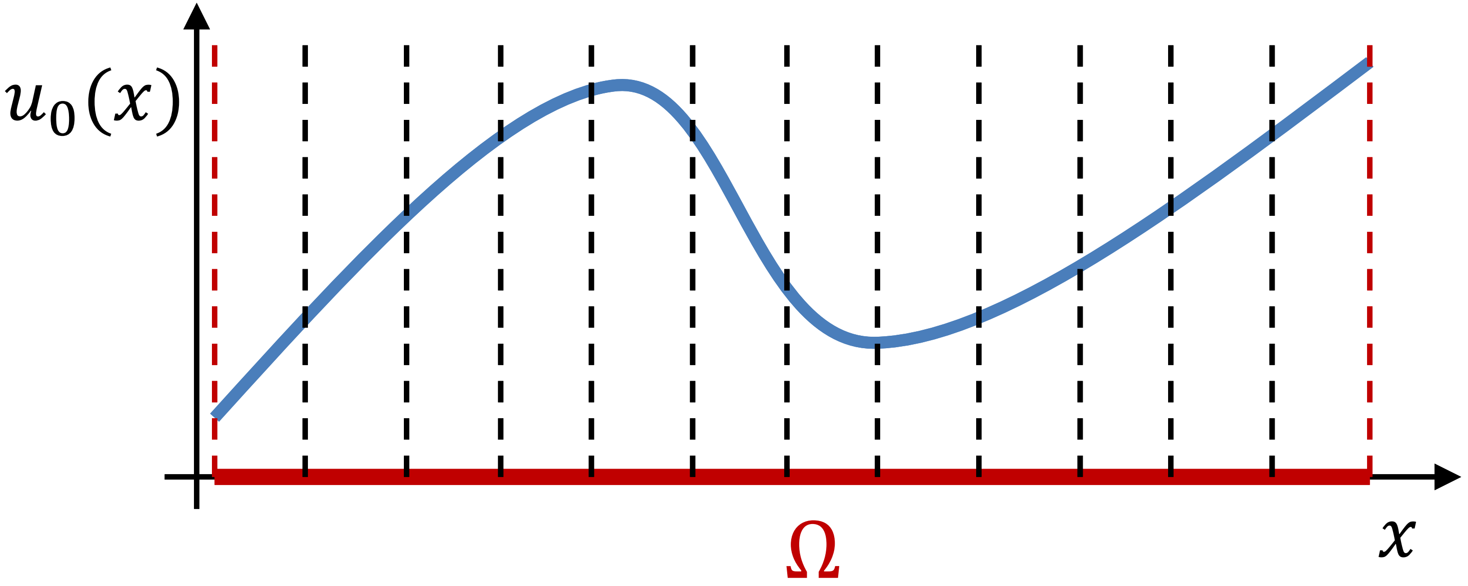
Idea of FEM

How to approximate a function on a finite-dimensional space?



Idea of FEM

How to approximate a function on a finite-dimensional space?

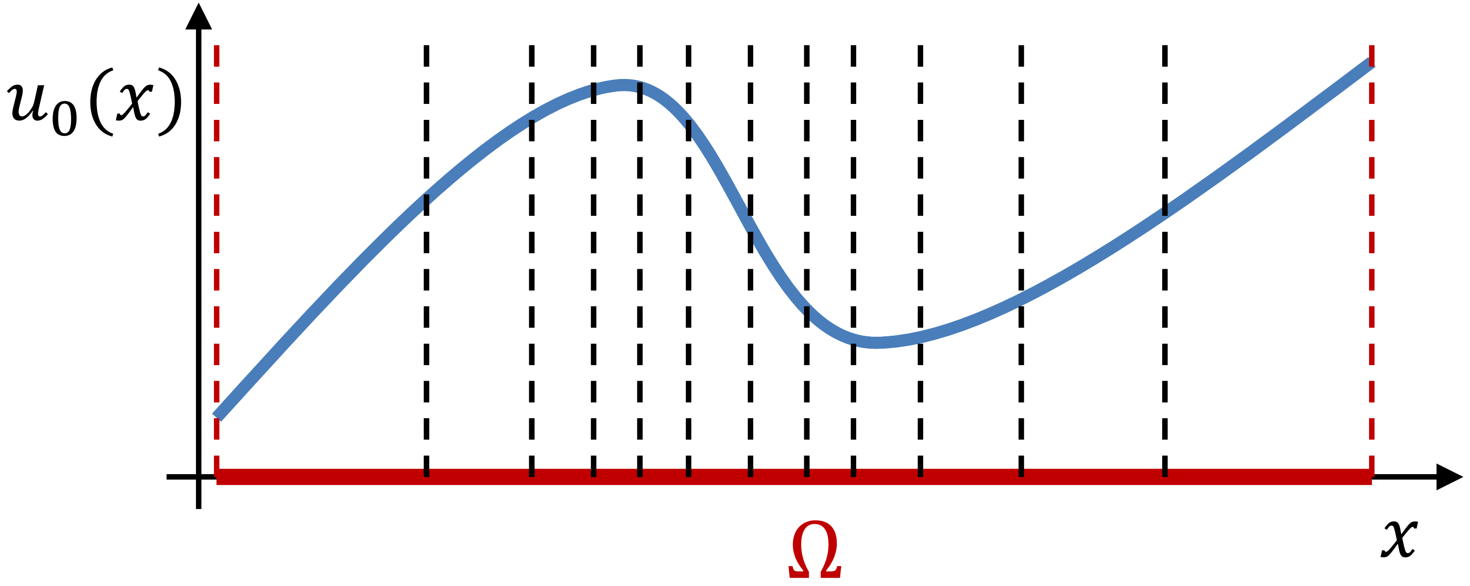


Discretization of geometric space Ω

(uniform mesh)

Idea of FEM

How to approximate a function on a finite-dimensional space?



Discretization of geometric space Ω

(irregular *mesh*)

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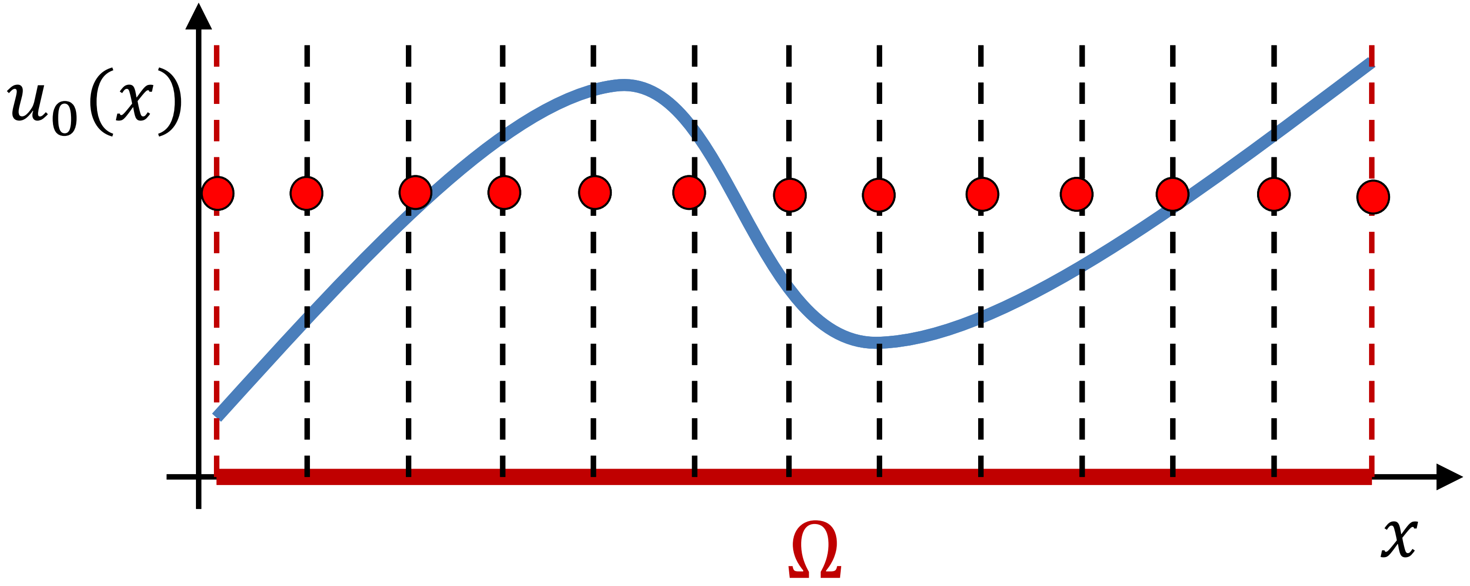
Idea of FEM

How to approximate a function on a finite-dimensional space?



Degrees of freedom (DoFs)

(unknowns of the problem)

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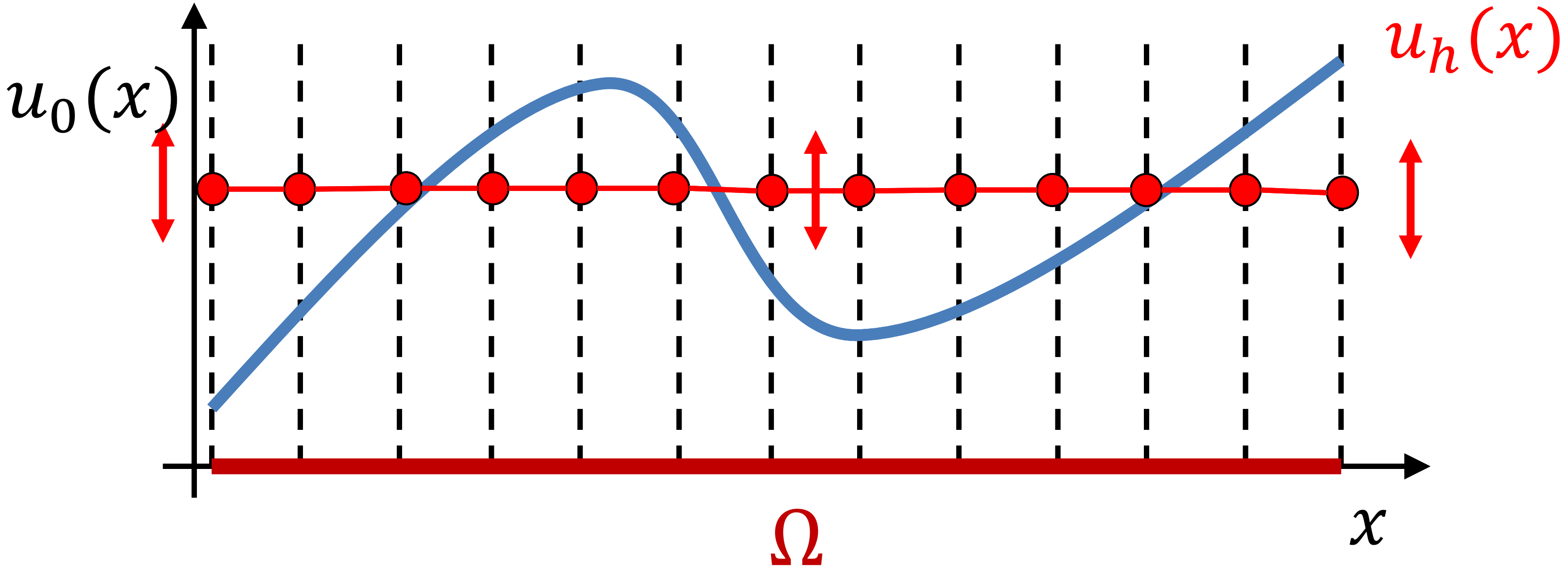
Idea of FEM

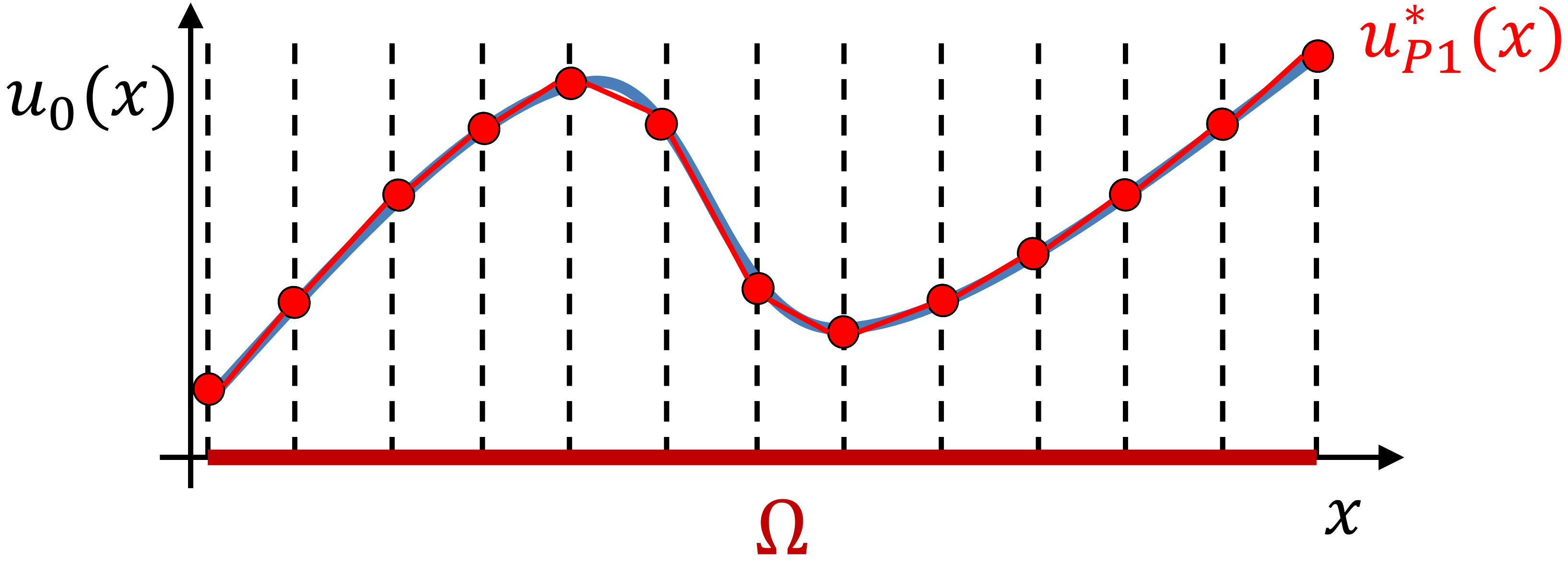
How to approximate a function on a finite-dimensional space?



Interpolation defined from the DoFs

Idea of FEM

How to approximate a function on a finite-dimensional space?



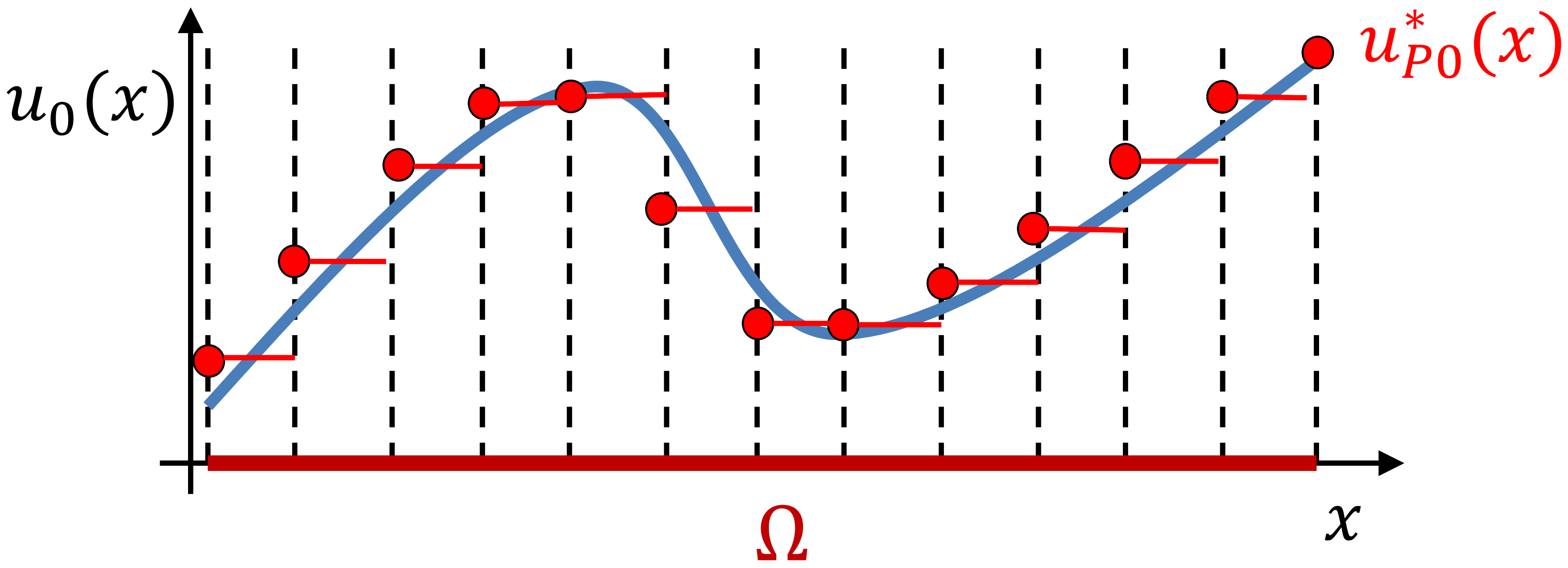
Interpolation defined from the DoFs

Best possible ***linear*** interpolation

Idea of FEM

How to approximate a function on a finite-dimensional space?

*Given an interpolation, how can we determine the **optimal** DoF values?*

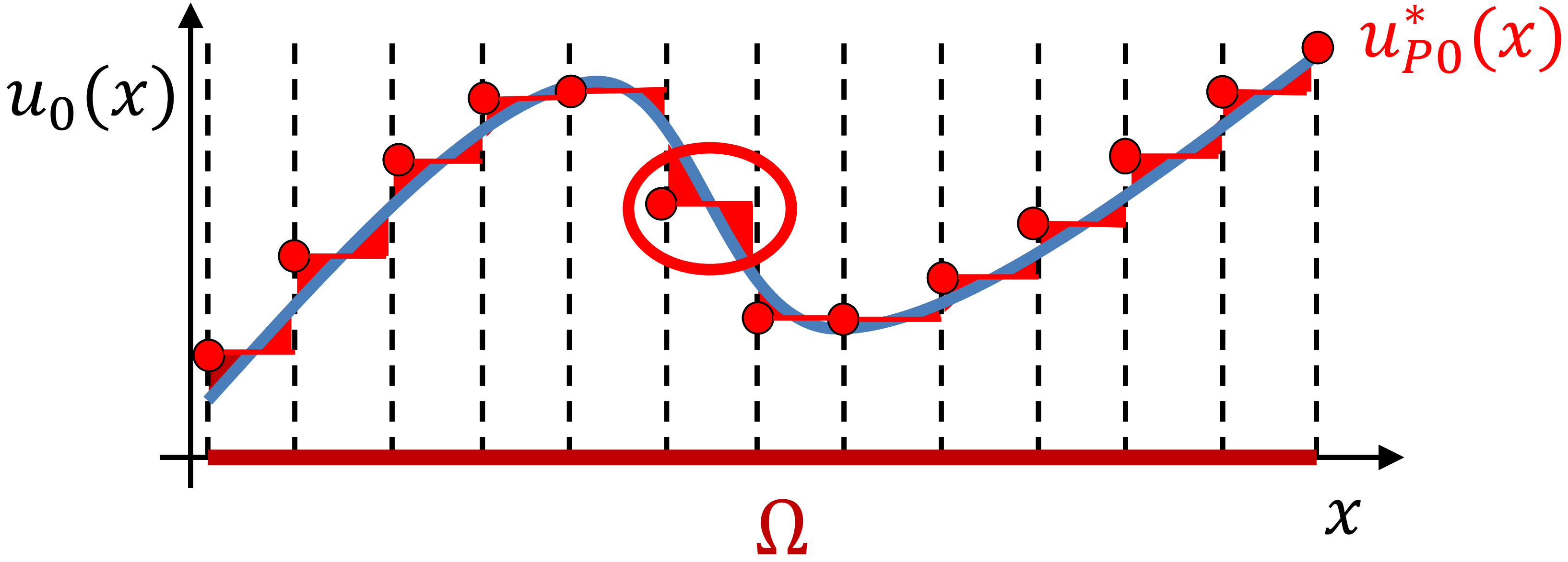


Interpolation defined from the DoFs

Best possible ***constant*** interpolation

Mean squared error minimization

Illustration



Interpolation defined from the DoFs

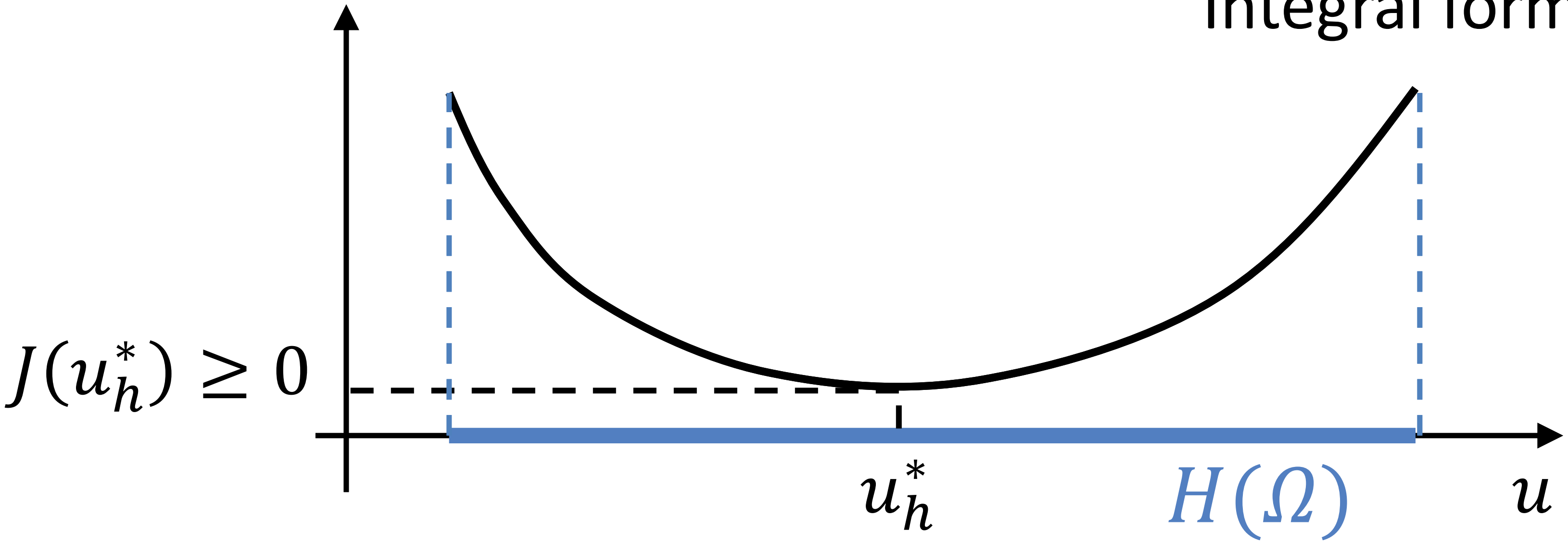
Best possible ***constant*** interpolation

Mean squared error minimization

Mathematical formulation

$$u_h^* = \arg\min_{u_h \in H(\Omega)} J(u_h) = \underbrace{\frac{1}{2}\int_\Omega \underbrace{\left(u_h(x) - u_0(x)\right)^2}_{\text{Squared error}} \,\mathrm{d}x}_{\text{Integral formulation}}$$

Admissible function space
(continuous or discretized)



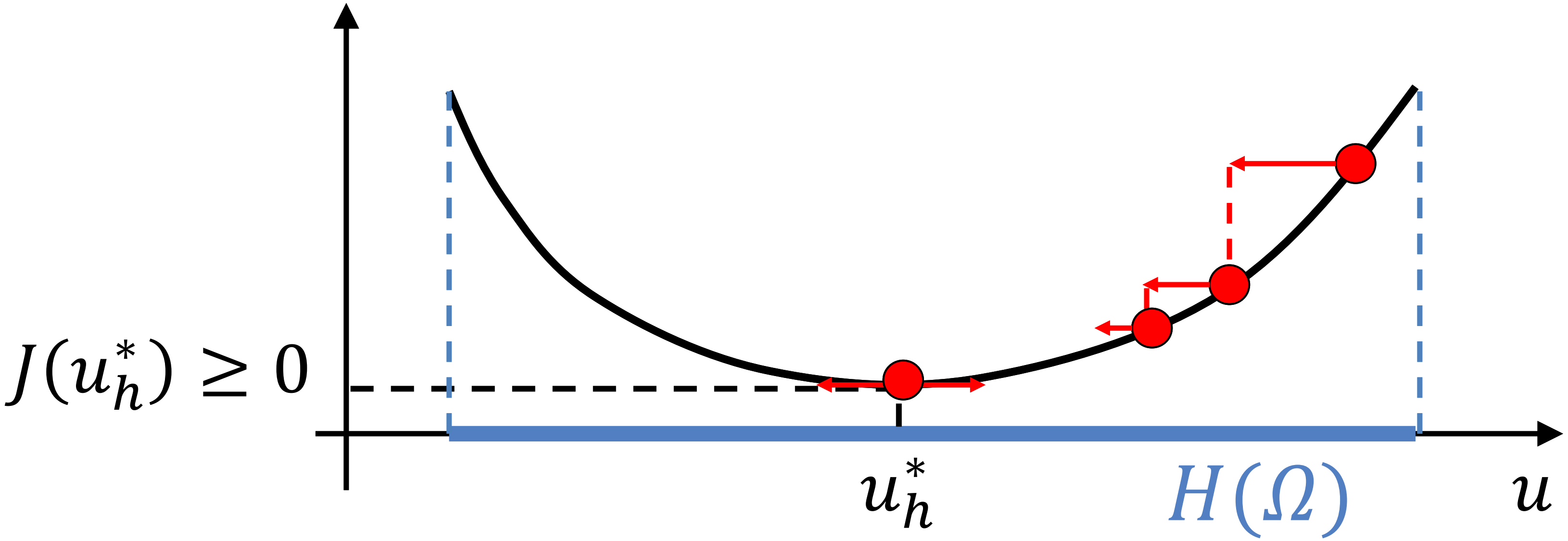
Mean squared error minimization

Algorithms

- Naïve idea: **gradient descent** : $u_h^{k+1} = u_h^k - \alpha\, J'(u_h^k)$
- Better idea: **convex problem** ⇒ unique minimum satisfying

$$J'(u_h^*) = 0$$

How to compute the derivative J' ?



Mean squared error minimization

Directional derivative

- **Directional derivative** in the direction v:

$$\underbrace{J'(u;v) = J'(u)(v) = \langle J'(u), v\rangle}_{} = \lim_{t\to 0} \frac{J(u+tv) - J(u)}{t} \in \mathbb{R}$$

Different notations exist ; all highlighting that v (« test function ») plays a different role than u (point where the derivative is computed).

- We can define a *linear application* $J'(u): v \mapsto J'(u;v) \in L(H, \mathbb{R})$

<u>**Exercise**</u> : compute the directional derivative of the MSE

$$J(u) = \frac{1}{2}\int_{\Omega} (u(x) - u_0(x))^2 \,\mathrm{d}x$$

Mean squared error minimization

Directional derivative & algorithms

$$J'(u;v) = \int_\Omega (u - u_0)v \ \mathrm{d}x$$

For $d \propto -(u - u_0)$, $J'(u;d) \leq 0$

$\Rightarrow \boxed{d = -(u - u_0)}$ is a ***descent direction***

Gradient descent

$$u_h^{k+1} = u_h^k + \alpha\, d(u_h^k)$$

We can also find $u_h^* \in H(\Omega)$ such that

$$\forall v \in H(\Omega), \qquad J'(u_h^*; v) = 0$$

Linear system

To assemble and solve!

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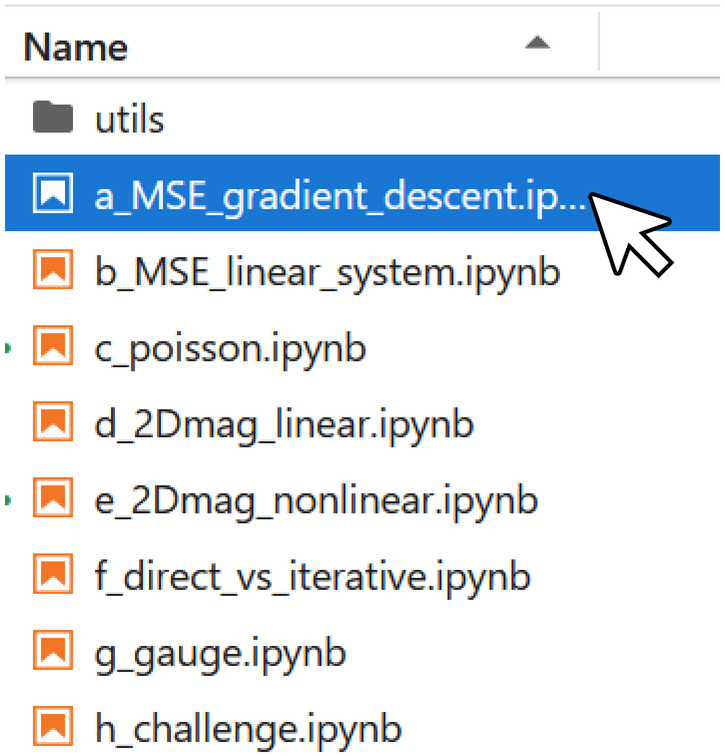
Application 1 : gradient descent on MSE

Jupyter Notebook « a_MSE_gradient_descent »



Try out different interpolations:

- Function spaces
  - $L^2(\Omega) = \left\{v\colon \Omega \to \mathbb{R}, \int_\Omega v(x)\,\mathrm{d}x < \infty\right\}$

  (discretized by element-wise ***discontinuous*** functions)

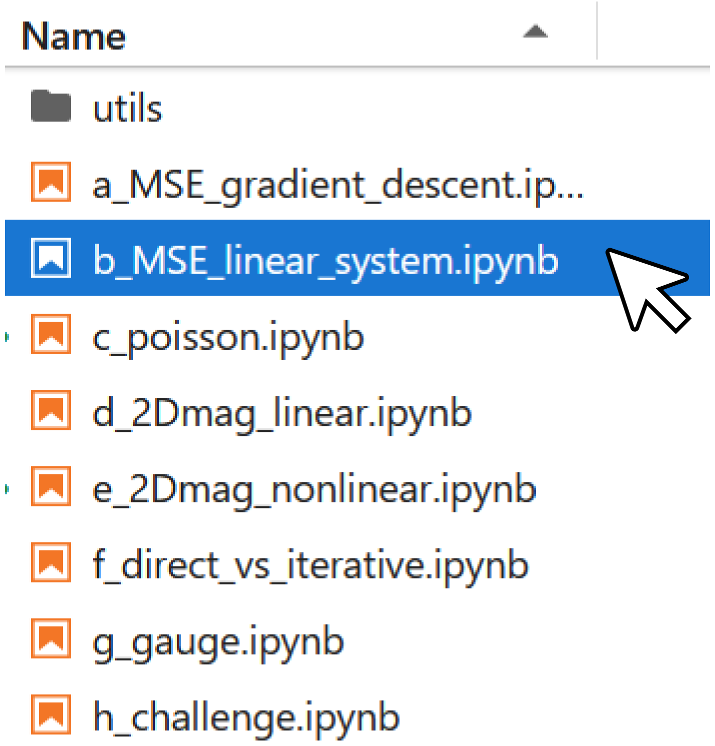
  - $H^1(\Omega) = \{v \in L^2(\Omega), \nabla v \in L^2(\Omega)\}$

  (discretized by nodal ***continuous*** functions)

- Polynomial degrees / order

Application 2 : linear system assembly

Jupyter Notebook « b_MSE_linear_system »



- Gradient descent is generally inefficient and sometimes inapplicable
- From the optimality condition, one can assemble a linear system. So

$$\forall v \in H(\Omega), \qquad \int_\Omega u\, v = \int_\Omega u_0\, v$$

Becomes

$$Ku = f$$

2) ACADEMIC POISSON PROBLEM

Now let’s solve partial differential equations

Variational formulation

General method

- The finite element method is based on variational formulations

- **Main objective of the session**: obtain a variational formulation from the strong equations.

- <u>**Methodology**</u>

1. Choice of relevant variables → *not trivial*… see literature!
2. Choice of the function space H → *often* easy
3. **Projection of the equation on $\boldsymbol{H}$ → *often* easy**

Variational formulation

Choice of relevant variables

- We consider electrostatics

- Maxwell equations

$$\begin{cases} \nabla \cdot \boldsymbol{D} = \rho & \text{(Maxwell–Gauss)} \\ \nabla \times \boldsymbol{E} = 0 & \text{(Maxwell–Faraday)} \end{cases}$$

- Material constitutive law

$$\boldsymbol{D} = \epsilon_0 \epsilon_r \, \boldsymbol{E}$$

With $\epsilon_0 = 8.85 \times 10^{-12}\ F/m$, ϵ_r depending on material

Material	Dielectric constant εr
Vacuum	1
Air	1,0006
Reinforced concrete	1,51
Teflon	2,1
Paper	3,85
Silicon dioxide	3,9
FR-4	4
Mica	5,6 - 8
Marble	8,3
Silicon	11,7
Calcium titanate	150

What is electric permittivity? - Electrical e-Library.com

Variational formulation

Choice of relevant variables

- Many formulations are possible. We usually use scalar electric potential :

$$\boldsymbol{E} = -\nabla u$$

$\Rightarrow \nabla \times \boldsymbol{E} = 0$ is automatically verified (curl of grad is always 0) ; but u is now *defined up to a constant* that should be fixed.

- From the other equations we obtain

Poisson equation $\boxed{-\nabla \cdot (\epsilon_0 \epsilon_r \nabla u) = \rho}$

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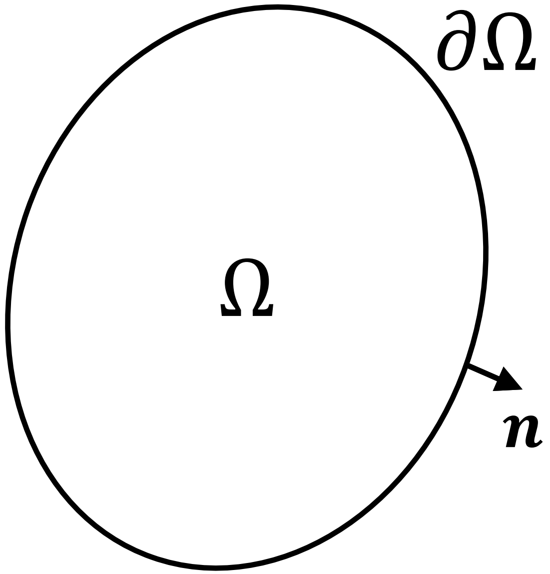
Variational formulation

Formal projection

- We consider a geometric space Ω and a function space $H(\Omega)$, detailed later.

1. **Multiplication** by any test function $v \in H(\Omega)$ and **integration** over Ω :

$$-\int_{\Omega} \nabla \cdot (\epsilon_0 \epsilon_r \nabla u)\, v \;\mathrm{d}x \;=\; \int_{\Omega} \rho\, v \;\mathrm{d}x$$



2. **Integration by part**; using the following formulae :

Leibniz: $b\nabla \cdot \boldsymbol{A} = \nabla \cdot (b\boldsymbol{A}) - \boldsymbol{A} \cdot \nabla b$

Green-Ostrogradski *:* $\int_{\Omega} \nabla \cdot \boldsymbol{A} = \int_{\partial\Omega} \boldsymbol{A} \cdot \boldsymbol{n}$

Outward normal to $\partial\Omega$

Boundary of Ω

Variational formulation

Formal projection

- We obtain:

$$\underbrace{\int_\Omega \epsilon_0\epsilon_r \nabla u \cdot \nabla v \;\mathrm{d}x \; - \overbrace{\int_{\partial\Omega} \epsilon_0\epsilon_r \nabla u \cdot \boldsymbol{n} \; v \,\mathrm{ds}}^{\text{Boundary term}}}_{\text{Bilinear form}} = \underbrace{\int_\Omega v\, \rho}_{\text{Linear form}}$$

<u>Boundary value problem</u> (BVP) with a boundary term on the normal component of electrical displacement:

$$\epsilon_0\epsilon_r \nabla u \cdot \boldsymbol{n} = \boldsymbol{D} \cdot \boldsymbol{n}$$

Homogeneous to a surface charge density ρ_s.

Variational formulation

Flashback to the function space

- We should find $u \in H(\Omega)$, such that

$$\forall v \in H, \qquad \int_\Omega \epsilon_0 \epsilon_r \nabla u \cdot \nabla v \ \mathrm{d}x \ - \int_{\partial\Omega} \epsilon_0 \epsilon_r \nabla u \cdot \boldsymbol{n} \ v \, \mathrm{d}s = \int_\Omega v\rho \, \mathrm{d}x$$

Function space

To have well defined integrals, $\underline{H(\Omega) = \{u \in L^2(\Omega), \nabla u \in L^2(\Omega)\} = H^1(\Omega)}$

Boundary conditions

Natural boundary conditions

- We should find $u \in H(\Omega)$, such that

$$\forall v \in H, \qquad \int_{\Omega} \epsilon_0 \epsilon_r \nabla u \cdot \nabla v \;\mathrm{d}x \;-\int_{\partial\Omega} \epsilon_0 \epsilon_r \nabla u \cdot \boldsymbol{n}\; v \,\mathrm{d}s = \int_{\Omega} v\rho \,\mathrm{d}x$$

- **Natural boundary conditions** : we rewrite the boundary term
  - ***Neumann*** : $\epsilon_0 \epsilon_r \nabla u \cdot \boldsymbol{n} = g \rightarrow$ boundary term becomes a linear form
  - ***Robin*** : $\epsilon_0 \epsilon_r \nabla u \cdot \boldsymbol{n} = g - \alpha\, u \;\rightarrow$ boundary term becomes linear + bilinear form

Since these boundary conditions are part of the variational form, they are « *weakly* » imposed.

Boundary conditions

Natural boundary conditions

- We should find $u \in H(\Omega)$, such that

$$\forall v \in H, \qquad \int_{\Omega} \epsilon_0 \epsilon_r \nabla u \cdot \nabla v \; \mathrm{d}x \; - \int_{\partial\Omega} \epsilon_0 \epsilon_r \nabla u \cdot \boldsymbol{n} \;\; v \, \mathrm{ds} = \int_{\Omega} v\rho \, \mathrm{d}x$$

- **Neumann** (special case of Robin) :

$$\forall v \in H, \qquad \int_{\Omega} \epsilon_0 \epsilon_r \nabla u \cdot \nabla v \; \mathrm{d}x \; = \int_{\Omega} v\rho \, \mathrm{d}x + \int_{\partial\Omega} g \, v \, \mathrm{ds}$$

- **Robin** :

$$\forall v \in H, \qquad \int_{\Omega} \epsilon_0 \epsilon_r \nabla u \cdot \nabla v \; \mathrm{d}x + \int_{\partial\Omega} \alpha \, u \, v \; \mathrm{ds} = \int_{\Omega} v\rho \, \mathrm{d}x + \int_{\partial\Omega} g \, v \, \mathrm{ds}$$

Boundary conditions

Essential boundary conditions

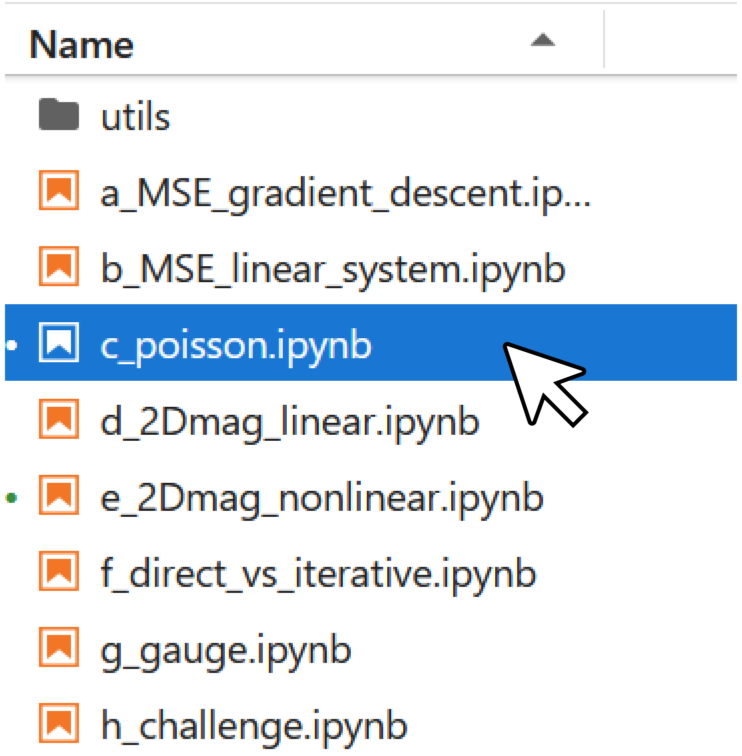
- We should find $u \in H(\Omega)$, such that

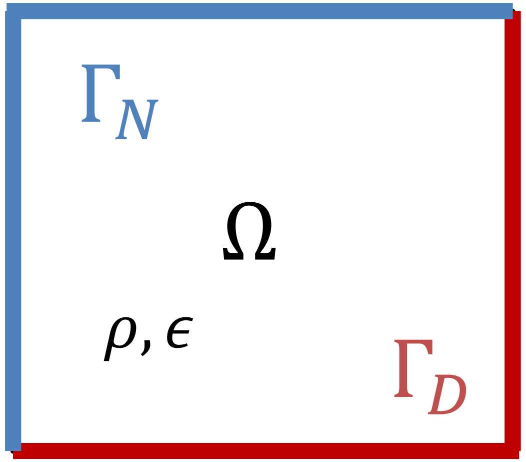
$$\forall v \in H, \qquad \int_\Omega \epsilon_0 \epsilon_r \nabla u \cdot \nabla v \;\mathrm{d}x \; - \cancel{\int_{\partial\Omega} \epsilon_0 \epsilon_r \nabla u \cdot \boldsymbol{n} \; v \,\mathrm{ds}} = \int_\Omega v\rho \,\mathrm{d}x$$

$$\left.\begin{array}{l}\text{• } \textbf{Essential boundary conditions}\text{: not appearing directly in the boundary term;} \\ \quad\text{therefore should be imposed (exactly!) in the } \textbf{function space} \\ \quad\quad \text{• } \textbf{\textit{Dirichlet}} : u = u_d \text{ on the boundary} \\ \quad\quad \text{• } \textbf{\textit{Periodicity / anti-periodicity}} : u_1 = \pm u_2 \end{array}\right\} \text{Sets the boundary term to 0}$$

Application 3 : Poisson problem

Jupyter Notebook « c_Poisson »





Find $u \in H(\Omega), \forall v \in H,$

$$\int_\Omega \epsilon_0 \epsilon_r \nabla u \cdot \nabla v \ \mathrm{d}x = \int_\Omega v\rho \,\mathrm{d}x$$

- Homogeneous Dirichlet : $u|_{\partial\Omega} = 0$

$\Rightarrow \vec{E}$ orthogonal to the boundary (gradient is orthogonal to the isolines of u).
$\Rightarrow \partial\Omega$ is an anti-symmetry plane
Can also troncate infinity $(u(\infty) = 0)$

- Homogeneous Neumann : $\vec{D} \cdot \vec{n} = 0$

$\Rightarrow \vec{D}$ tangential to the boundary
$\Rightarrow \partial\Omega$ is a symmetry plane

3) 2D MAGNETOSTATICS

Non-linearity and Newton method

Magnetostatics

Equations

- We give the equations

$\boldsymbol{B} = \nabla \times \boldsymbol{a}$	Magnetic vector potential (unknown)
$\nabla \times \boldsymbol{H} = \boldsymbol{j}$	Maxwell Ampère
$\boldsymbol{H} = \nu(\lvert\boldsymbol{B}\rvert^2)\boldsymbol{B}$	Constitutive law of iron
$\boldsymbol{a} = \mathbf{0}$ on $\partial\Omega$	Dirichlet boundary condition (boundary term →0)

What is the strong formulation?

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Magnetostatics

Strong form

- The b-conform strong equation reads

$$\nabla \times (\nu(|\nabla \times \boldsymbol{a}|^2)\, \nabla \times \boldsymbol{a}) = \boldsymbol{j}$$

Or

$$\mathbf{curl}\big(\nu(|\mathbf{curl}\, \boldsymbol{a}|^2)\, \mathbf{curl}\, (\boldsymbol{a})\big) = \boldsymbol{j}$$

$$\boxed{\text{What is the variational formulation?}}$$

Leibniz: $\boldsymbol{A} \cdot \mathbf{curl}(\boldsymbol{B}) = \boldsymbol{B} \cdot \mathbf{curl}(\boldsymbol{A}) - \mathrm{div}(\boldsymbol{A} \times \boldsymbol{B})$

Green-Ostrogradski : $\int_\Omega \mathrm{div}(\boldsymbol{A}) = \int_{\partial\Omega} \boldsymbol{A} \cdot \boldsymbol{n}$

Magnetostatics

Variational formulation

- We find $\boldsymbol{a} \in H_0(\mathbf{curl};\Omega) = \{\boldsymbol{a} \in L^2(\Omega), \mathbf{curl}(\boldsymbol{a}) \in L^2(\Omega),\ \ \boldsymbol{a} \times \boldsymbol{n} = 0 \text{ on } \partial\Omega\}$

$$\forall \boldsymbol{v} \in H_0(\mathbf{curl};\Omega), \qquad \int_\Omega \mathbf{curl}\, \boldsymbol{v} \cdot \left(\nu\left(|\mathbf{curl}\, \boldsymbol{a}|^2\right)\mathbf{curl}\, \boldsymbol{a}\right) = \int_\Omega \boldsymbol{v} \cdot \boldsymbol{j}$$

- In 2D, we have

$$\boldsymbol{j} = \begin{bmatrix} 0 \\ 0 \\ j_z \end{bmatrix}, \qquad \boldsymbol{a} = \begin{bmatrix} 0 \\ 0 \\ a_z \end{bmatrix}, \qquad \mathrm{v} = \begin{bmatrix} 0 \\ 0 \\ v_z \end{bmatrix}, \qquad \mathbf{curl}(\boldsymbol{a}) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \mathbf{grad}(a_z)$$

So that we can rewrite the equation w.r.t the z-components only.

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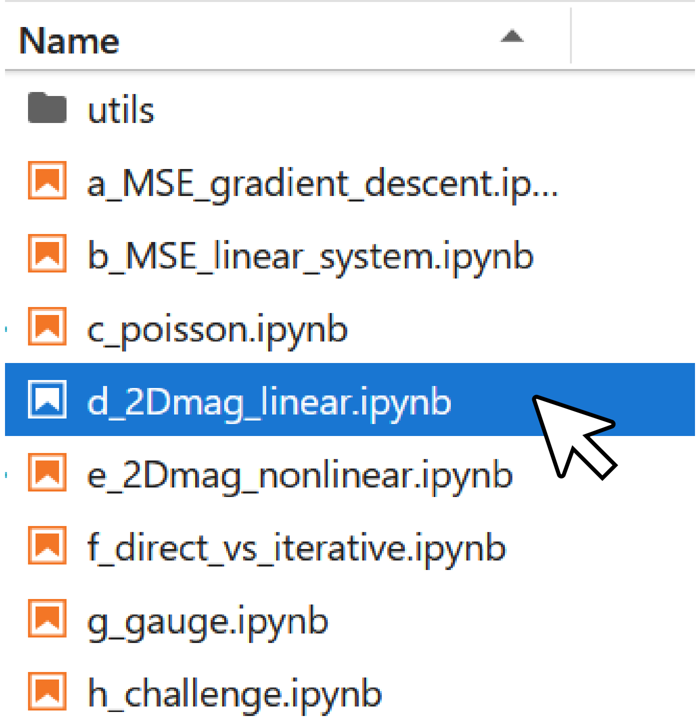
Application 4 : 2D linear magnetostatics

Jupyter Notebook « d_nonlinear.ipynb »



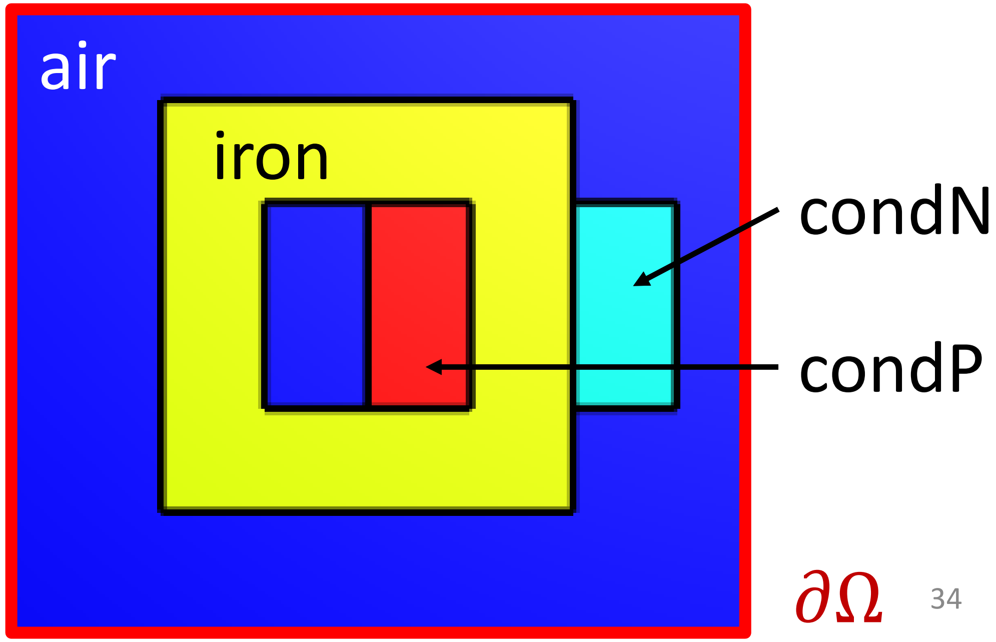
- Find

$$a_z \in H_0^1(\Omega) = \{a \in L^2(\Omega), \nabla a \in L^2(\Omega), a = 0 \text{ on } \partial\Omega\}$$

$$\forall v \in H_0^1(\Omega), \qquad \int_\Omega \mathbf{curl}\, v \cdot \nu\, \mathbf{curl}\, a_z = \int_\Omega v\, j_z$$

For now, we assume iron is linear :

- $\mu_{iron} = 1000\mu_0 \Rightarrow \nu_{iron} = \dfrac{1}{1000\mu_0}$
- $J = 10\ A/mm^2$



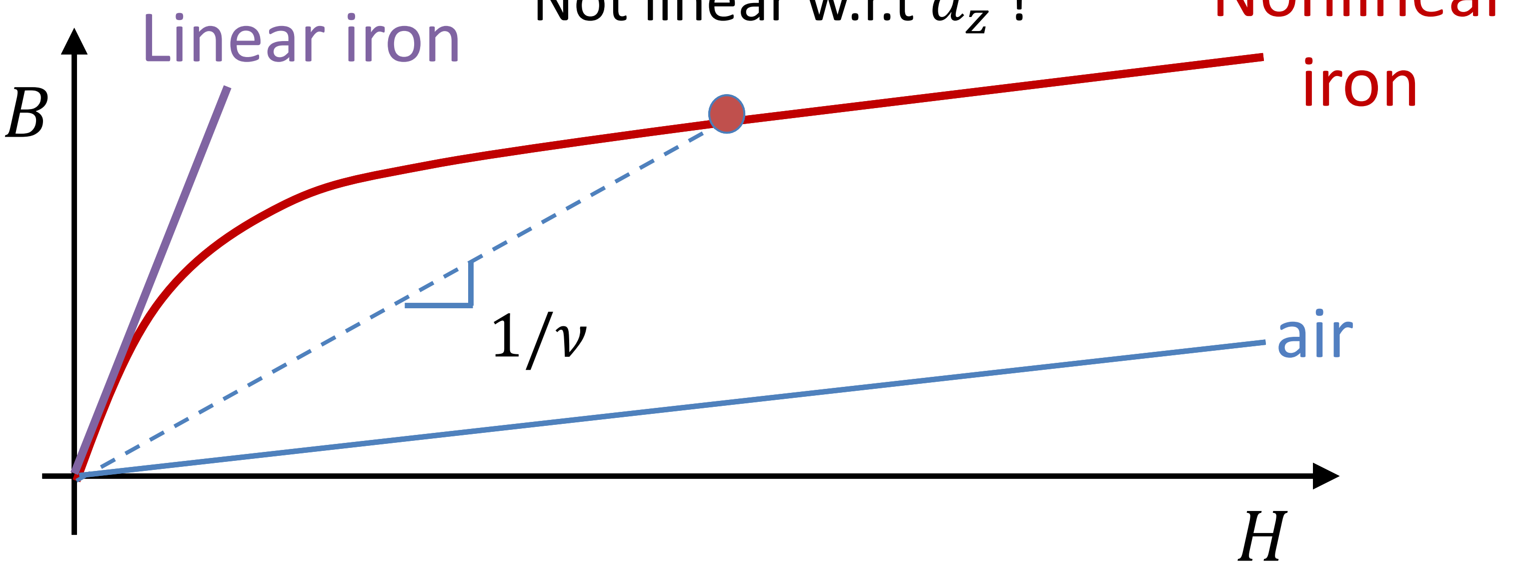
Nonlinearity

2D Variational formulation

- We should find $a_z \in H_0^1(\Omega) = \{a \in L^2(\Omega), \nabla a \in L^2(\Omega),\ \ a = 0 \text{ on } \partial\Omega\}$

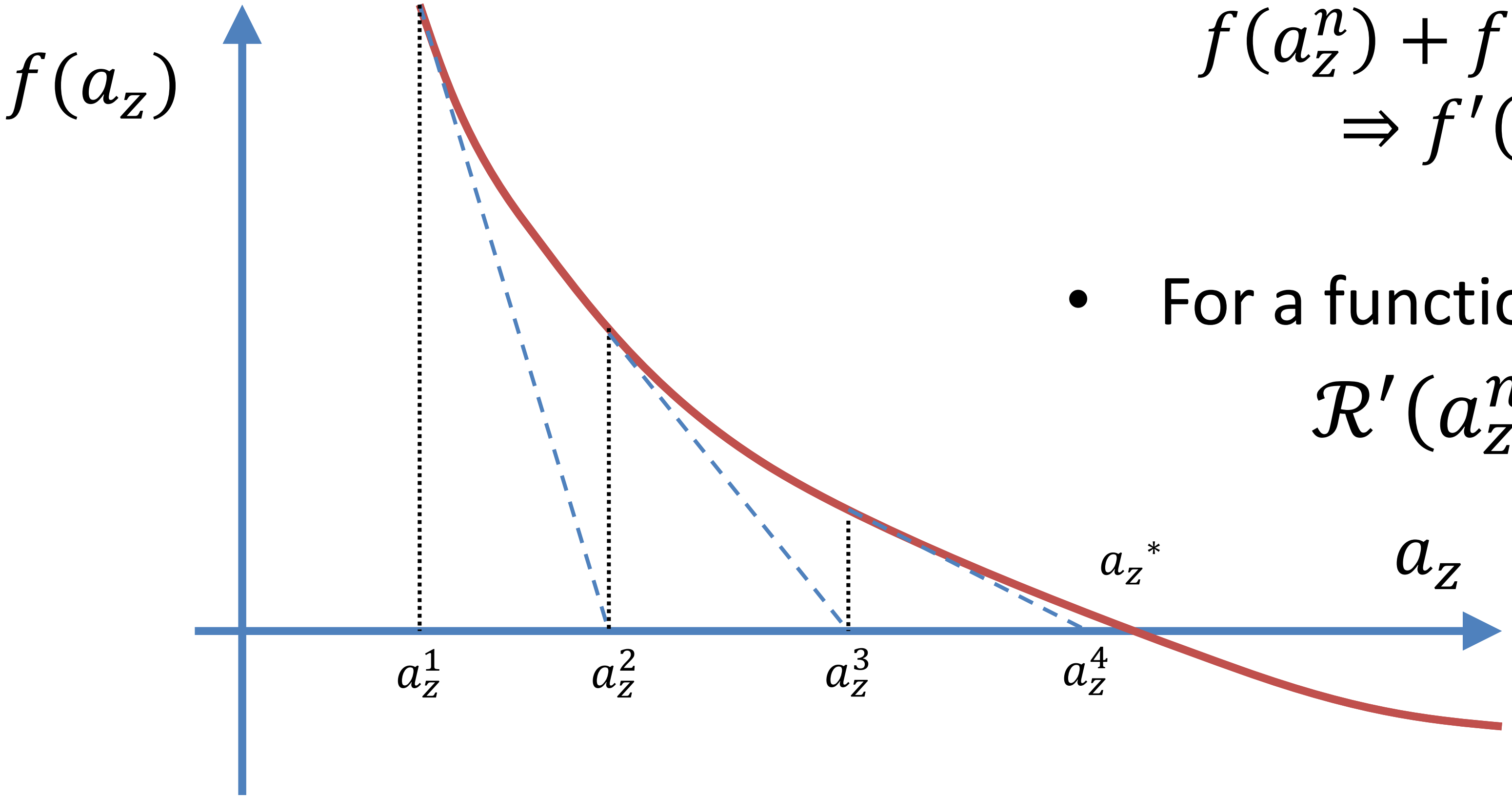
$$\forall v \in H_0^1(\Omega), \qquad \underbrace{\int_\Omega \mathbf{curl}\, v \cdot \nu\left(|\mathbf{curl}\, a_z|^2\right)\mathbf{curl}\, a_z}_{\text{Not linear w.r.t } a_z\ !} = \int_\Omega v\, j_z$$

BH curves:



Nonlinearity

Newton method



- For a usual function:

$$f(a_z^n) + f'(a_z^n)\,\overbrace{(a_z^{n+1} - a_z^n)}^{\delta a} = 0$$
$$\Rightarrow f'(a_z^n)\,\delta a = -f(a_z^n)$$

- For a functional $\mathcal{R}$: find δa, such that

$$\mathcal{R}'(a_z^n; \delta a) = -\mathcal{R}(a_z^n)$$

Nonlinearity

Newton method

- We should now define the functional $\mathcal{R}$ to cancel.
- **Residual**

$$\mathcal{R}(a_z, v) = \int_\Omega \mathbf{curl}\, v \cdot \nu(|\mathbf{curl}\, a_z|^2)\mathbf{curl}\, a_z - \int_\Omega v j_z$$

- **Directional derivative**:

$$\begin{aligned}
&\mathcal{R}'(a_z, v; \delta a) \\
&= \int_\Omega \mathbf{curl}\, v \cdot \nu(|\mathbf{curl}\, a_z|^2)\mathbf{curl}\, \delta a \\
&+ 2\int_\Omega \mathbf{curl}\, v \cdot (\nu'(|\mathbf{curl}\, a_z|^2)\mathbf{curl}\, a_z \cdot \mathbf{curl}\, \delta a)\mathbf{curl}\, a_z
\end{aligned}$$

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Nonlinearity

Newton method

- **Typical algorithm**:

1) Solve the linearized problem

$$\forall v \in H_0^1(\Omega), \qquad \mathcal{R}'(a_z, v; \delta a) = -\mathcal{R}(a_z, v)$$

2) Adapt step size α and update

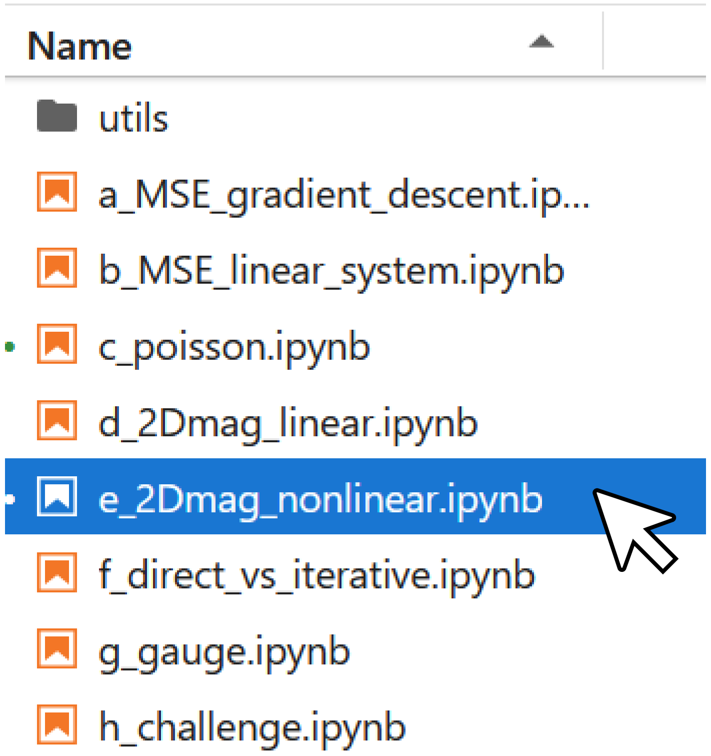
$$a_z^{n+1} = a_z^n + \alpha\, \delta a$$

3) Stop criterion (several possibilities…)

$|\mathcal{R}(a_z^n, \delta a)| \leq \mathrm{tol}$, or $|\mathcal{R}(a_z^n, v_i)| \leq \mathrm{tol}$ … and always $n > n_{max}$

Application 5 : non-linear 2D magnetostatics

Jupyter Notebook « e_nonlinear_2D_magnetostatics »



- **Residual**

$$\mathcal{R}(a_z, v) = \int_\Omega \mathbf{curl}\, v \cdot \nu(|\mathbf{curl}\, a_z|^2)\mathbf{curl}\, a_z - \int_\Omega v j_z$$

- **Directional derivative**:

$$\mathcal{R}'(a_z, v; {\color{red}\delta a})$$
$$= \int_\Omega \mathbf{curl}\, v \cdot \nu(|\mathbf{curl}\, a_z|^2)\mathbf{curl}\, {\color{red}\delta a}$$
$$+ 2\int_\Omega \mathbf{curl}\, v \cdot (\nu'(|\mathbf{curl}\, a_z|^2)\mathbf{curl}\, a_z \cdot \mathbf{curl}\, {\color{red}\delta a})\mathbf{curl}\, a_z$$

1) Find $\delta a \in H_0^1(\Omega)$, such that $\forall v \in H_0^1(\Omega)$, $\mathcal{R}'(a_z, v; \delta a) = -\mathcal{R}(a_z, v)$
2) Update $a_z \leftarrow a_z + \alpha\, \delta a$
3) Stop criterion

4) 3D MAGNETOSTATICS

Iterative solver, edge elements and gauge

Preliminaries

Iterative solver VS direct solver

	Direct solver	Iterative solver
Time complexity	Max $O(bn^2)$ b = bandwith	Each iteration is $O(n_{nz})$ with n_{nz} the number of nonzero terms
Provide matrix decomposition	yes	no
Exact	yes	no
Sensitive to matrix bandwith	yes	no
Sensitive to condition number	no	Yes (need $O(\sqrt{c})$ iterations)
Typical use	1D, 2D	3D

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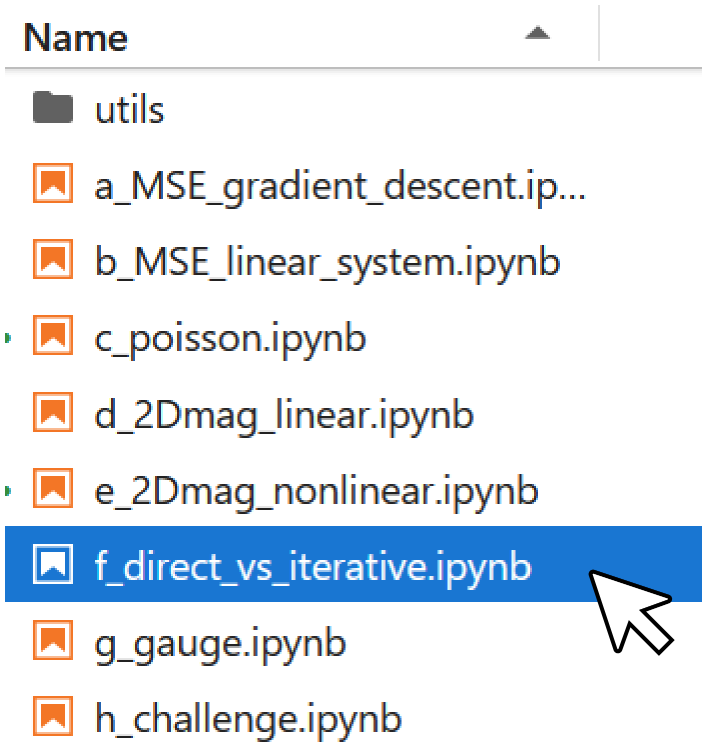
https://www.cs.cmu.edu/~quake-papers/painless-conjugate-gradient.pdf

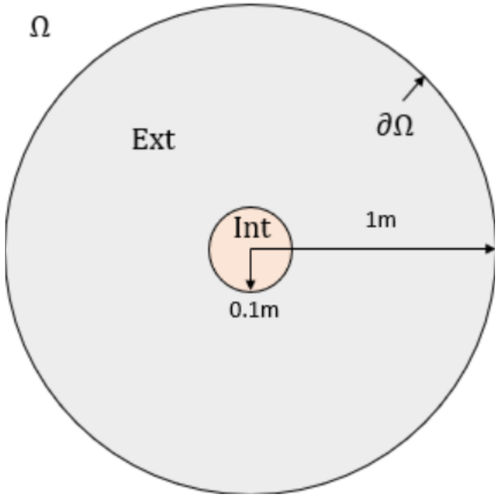
Application 6 : iterative vs direct solver

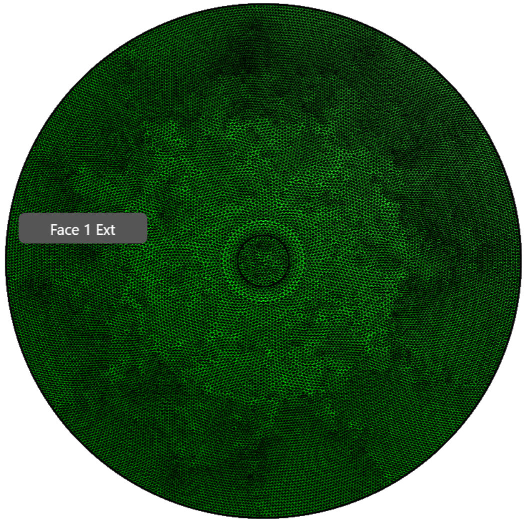
Poisson in 2D vs 3D

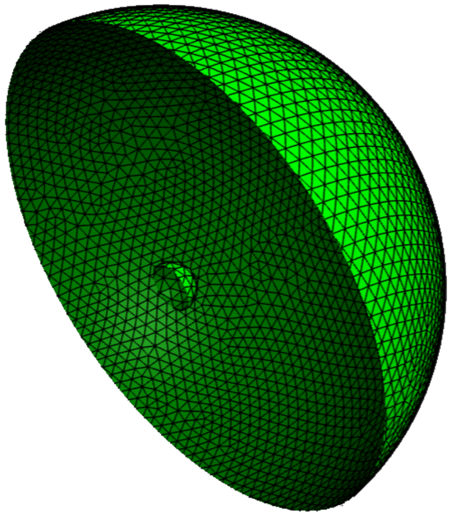


- Poisson equation on a disk and a ball with the same number of DoFs

$$\forall v \in H_0^1(\Omega), \qquad \int_\Omega \nabla v \cdot \nabla u \, \mathrm{d}x = \int_{int} 1 \, v \, \mathrm{d}x$$







3D Magnetostatics : variational formulation

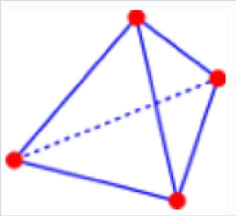
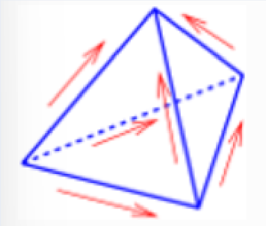
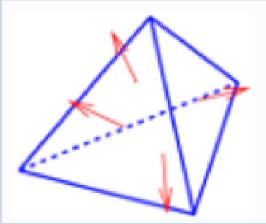
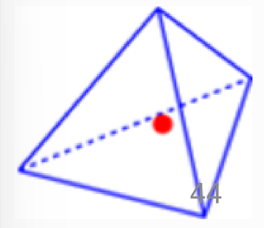
Function space

Find $\boldsymbol{a} \in H_0(\mathbf{curl};\Omega) = \{\boldsymbol{a} \in L^2(\Omega), \mathbf{curl}(\boldsymbol{a}) \in L^2(\Omega), \quad \boldsymbol{a} \times \boldsymbol{n} = 0 \text{ on } \partial\Omega\}$

$$\forall \boldsymbol{v} \in H_0(\mathbf{curl};\Omega), \qquad \int_\Omega \mathbf{curl}\,\boldsymbol{v} \cdot \nu\,\mathbf{curl}\,\boldsymbol{a} = \int_\Omega \boldsymbol{v} \cdot \boldsymbol{j}$$

What is $H(\mathbf{curl};\Omega)$ function space?

Function spaces

Space	Continuity	Differential geometry	Natural DoF	Typical Elements	Illustration
H^1	Full	0-form	Field value	Lagrange	
$H(\mathrm{curl})$	Tangential	1-form	Edge circulation $\int$	Nédélec	
$H(\mathrm{div})$	Normal	2-form	Facet flux $\iint$	Raviart-Thomas	
L^2	None	3-form	Cell integral $\iiint$ or average value	P_0	

$H^1 \xrightarrow{\mathrm{grad}} H(\mathrm{curl}) \xrightarrow{\mathrm{curl}} H(\mathrm{div}) \xrightarrow{\mathrm{div}} L^2$

3D Magnetostatics : variational formulation

Gauge

Find $\boldsymbol{a} \in H_0(\mathbf{curl};\Omega) = \{\boldsymbol{a} \in L^2(\Omega), \mathbf{curl}(\boldsymbol{a}) \in L^2(\Omega),\ \ \boldsymbol{a} \times \boldsymbol{n} = 0 \text{ on } \partial\Omega\}$

$$\forall \boldsymbol{v} \in H_0(\mathbf{curl};\Omega), \qquad \int_\Omega \mathbf{curl}\,\boldsymbol{v} \cdot \nu\,\mathbf{curl}\,\boldsymbol{a} = \int_\Omega \boldsymbol{v} \cdot \boldsymbol{j}$$

$$\boxed{\text{Is the solution uniquely defined?}}$$

No ! Assuming a is solution , then $\tilde{\boldsymbol{a}} = \boldsymbol{a} + \mathbf{grad}\,u$ is also solution, for u any differentiable scalar field, since $\mathbf{curl}\,\mathbf{grad}(.) = \mathbf{0}$

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3D Magnetostatics

Gauge

There are many different ways to obtain uniqueness:

- Add a small « mass » term

$$\int_\Omega \mathbf{curl}\, \boldsymbol{v} \cdot \nu\, \mathbf{curl}\, \boldsymbol{a} + \int_{\boldsymbol{\Omega}} \epsilon\, \boldsymbol{v} \cdot \boldsymbol{a} = \int_\Omega \boldsymbol{v} \cdot \boldsymbol{j}$$

- Add an equation (Coulomb gauge) : $\mathrm{div}(\boldsymbol{a}) = 0$

Weak form : find $\boldsymbol{a}, \lambda \in H_0(\mathbf{curl}; \Omega) \times H_0^1(\Omega)$,

$$\begin{cases} \forall \boldsymbol{v} \in H_0(\mathbf{curl}; \Omega), & \displaystyle\int_\Omega \mathbf{curl}\, \boldsymbol{v} \cdot \nu\, \mathbf{curl}\, \boldsymbol{a} + \int_{\boldsymbol{\Omega}} \nabla\lambda \cdot \boldsymbol{v} = \int_\Omega \boldsymbol{v} \cdot \boldsymbol{j} \\ \forall \mu \in H_0^1(\Omega), & \displaystyle\int_{\boldsymbol{\Omega}} \nabla\mu \cdot \boldsymbol{a} \qquad\qquad\qquad\qquad = 0 \end{cases}$$

Variational formulation

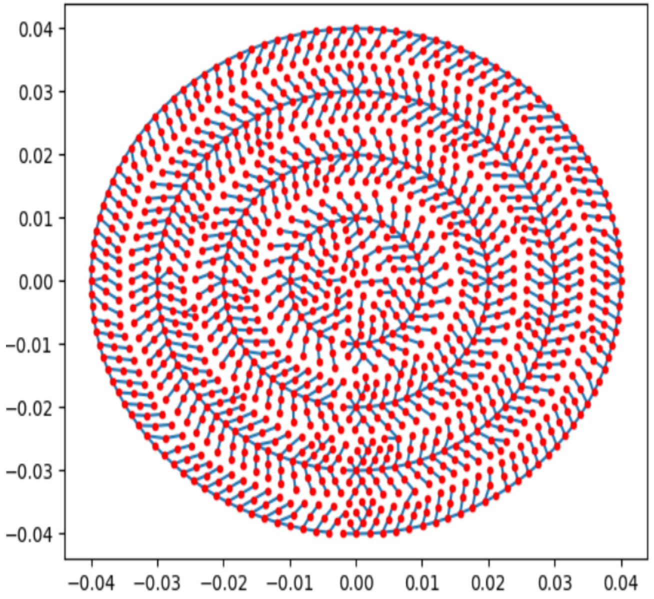
Gauge

- Solve with an **iterative solver** and a compatible right-hand side

1) Find $\boldsymbol{T} \in H(\mathrm{curl}, \Omega)$, s.t. $\int_\Omega \mathbf{curl}\,\boldsymbol{v} \cdot \mathbf{curl}\,\boldsymbol{T} = \int_\Omega \boldsymbol{v} \cdot \boldsymbol{j}$ $\forall \boldsymbol{v} \in \boldsymbol{H}(\boldsymbol{curl}, \boldsymbol{\Omega})$

2) Find $\boldsymbol{a} \in H(\mathrm{curl}, \Omega)$,

$$\int_\Omega \mathbf{curl}\,\boldsymbol{v} \cdot \nu\, \mathbf{curl}\,\boldsymbol{a} = \int_\Omega \boldsymbol{v} \cdot \mathbf{curl}\,\boldsymbol{T}$$

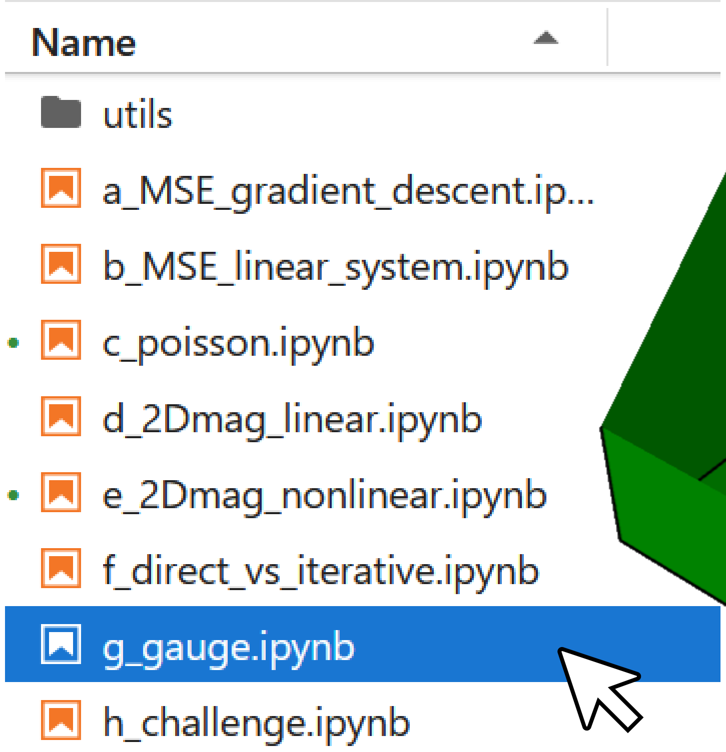


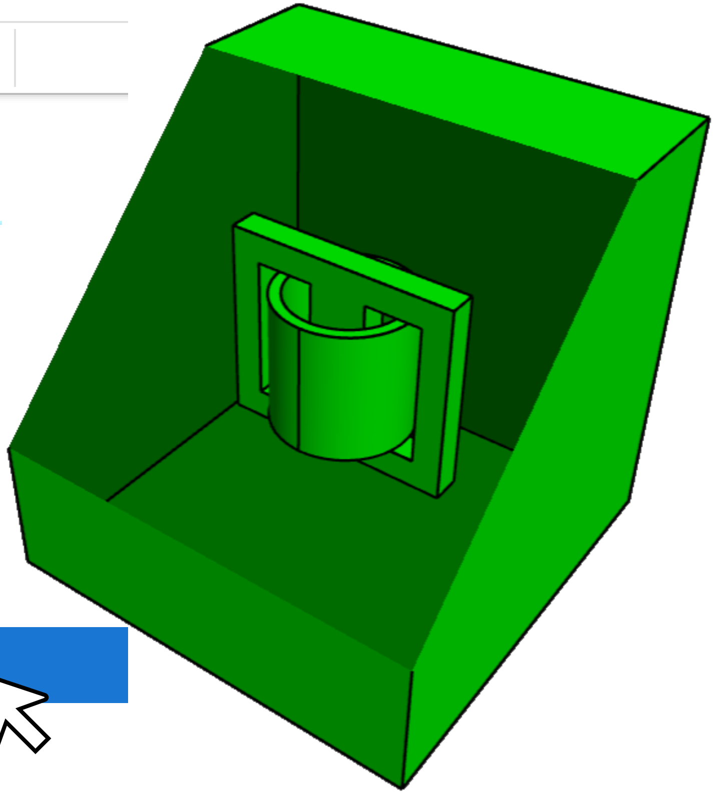
- Remove the redundant DoF (tree-cotree gauge).

You can implement and compare all of these possibilities!

Application 7 : 3D Magnetostatics

Uniqueness of the solution





- **Small mass term**

$$\int_\Omega \mathbf{curl}\,\boldsymbol{v} \cdot \nu\,\mathbf{curl}\,\boldsymbol{a} + \int_\Omega \epsilon\,\boldsymbol{v} \cdot \boldsymbol{a} = \int_\Omega \boldsymbol{v} \cdot \boldsymbol{j}$$

- **Coulomb gauge**: solve simultaneously

$$\begin{cases} \forall \boldsymbol{v} \in H_0(\mathbf{curl};\Omega), & \int_\Omega \mathbf{curl}\,\boldsymbol{v} \cdot \nu\,\mathbf{curl}\,\boldsymbol{a} + \int_\Omega \nabla\lambda \cdot \boldsymbol{v} = \int_\Omega \boldsymbol{v} \cdot \boldsymbol{j} \\ \forall \mu \in H_0^1(\Omega), & \int_\Omega \nabla\mu \cdot \boldsymbol{a} \qquad\qquad = 0 \end{cases}$$

- **Compatible RHS**

1) Find $\mathrm{T} \in H(\mathbf{curl},\Omega),\ s.t.$

$$\forall \boldsymbol{v} \in H_0(\mathbf{curl};\Omega), \int_\Omega \mathbf{curl}\,\boldsymbol{v} \cdot \mathbf{curl}\,\boldsymbol{j} = \int_\Omega \boldsymbol{v} \cdot \boldsymbol{j}$$

2) Find **a** $\in H(\mathbf{curl},\Omega),\ s.t.$

$$\int_\Omega \mathbf{curl}\,\boldsymbol{v} \cdot \nu\,\mathbf{curl}\,\boldsymbol{a} = \int_\Omega \boldsymbol{v} \cdot \mathbf{curl}\,\boldsymbol{T}$$

- **Tree-Cotree gauging**

3D Magnetostatics : symmetries

Boundary conditions

- Homogeneous Neumann :

$$\forall v \in H(\mathbf{curl}, \Omega) \int_{\partial\Omega} (\boldsymbol{h} \times \boldsymbol{n}) \cdot \boldsymbol{v} = \mathbf{0} \Rightarrow \ \boldsymbol{h} \times \boldsymbol{n} = 0$$

⇒ magnetic field orthogonal to the boundary

= **Symmetry**

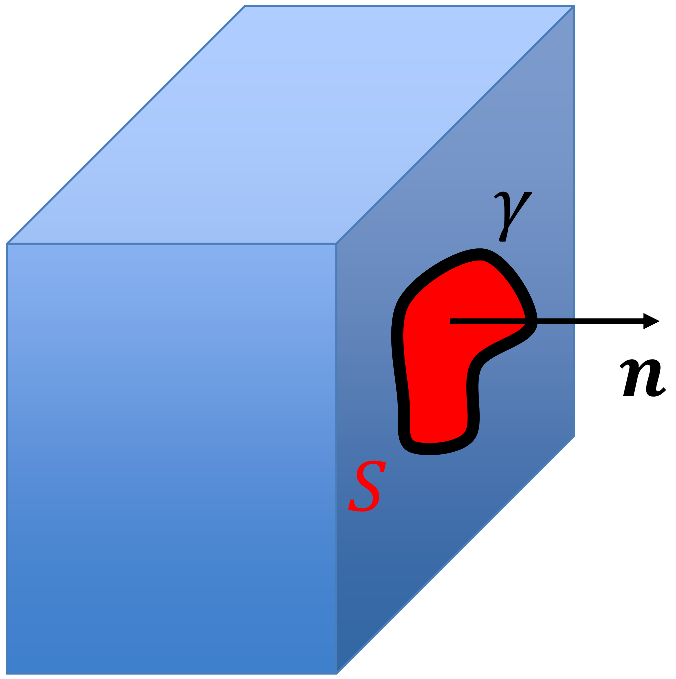
- Homogeneous Dirichlet : $\boldsymbol{a} \times \boldsymbol{n} = \mathbf{0}$

⇒ vector potential orthogonal to the boundary

$$\Rightarrow \forall S \in \partial\Omega, \phi_{out} = \iint_{S_\gamma \in \partial\Omega} \boldsymbol{B} \cdot d\boldsymbol{S} = \oint_{\gamma = \partial S} \boldsymbol{a} \cdot d\boldsymbol{l} = 0$$

⇒ flux density tangential to the boundary

= **Anti-symmetry**

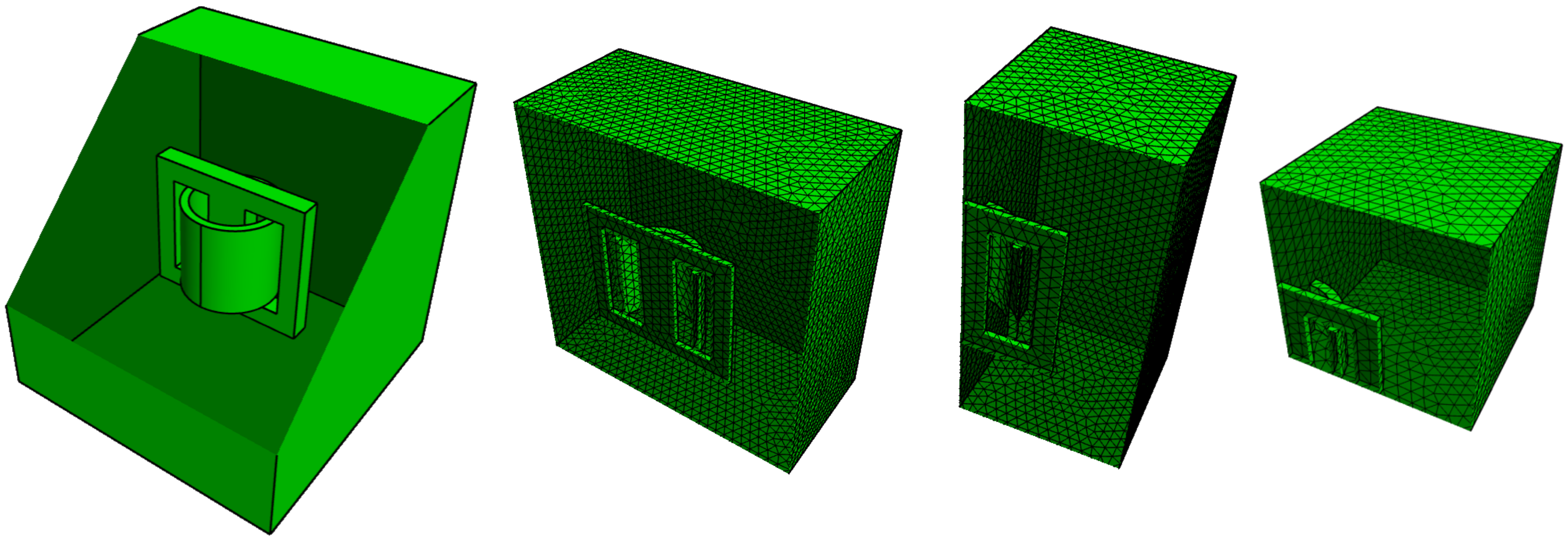


Application 8 : synthesis

Challenge : implement the fastest 3D magnetostatic solver for the inductance problem

Name

- utils
- a_MSE_gradient_descent.ip...
- b_MSE_linear_system.ipynb
- c_poisson.ipynb
- d_2Dmag_linear.ipynb
- e_2Dmag_nonlinear.ipynb
- f_direct_vs_iterative.ipynb
- g_gauge.ipynb
- h_challenge.ipynb



Full model *1/2* *1/4* *1/8*

To discuss on December 10 during the next GeePs'N Talks session!

Outlook

The journey is not over…

- How to mesh / remesh? → GMSH, Netgen…
- How to control the error ? → adaptive mesh refinement
- Advanced solvers (multifrontal, multigrid, etc.)
- Harmonic / Time dependant problems…
- Multiphysics / coupled problems
- What can we put over FEM? → interface tracking, topology optimization…
- Other methods(BEM, FIT, IGA, MoM, hybrid methods…)

References

- J. Schöberl, An Interactive Introduction to the Finite Element Method (https://jschoeberl.github.io/iFEM/intro.html)

- A. Ern, Finite Elements I: Approximation and interpolation https://hal.science/hal-03226049v1

- Z. Ren, “Influence of the R.H.S. on the Convergence Behaviour of the Curl-Curl Equation,” *IEEE Trans. Magn.*, vol. 32, no. 3, pp. 655–658, 1996.

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Thank you for your attention !

GeePs'N Talks special session

T. Cherrière, A. El Gode, T. Gauthey