Lagrange Basis Functions

- **Basis Functions**

$$L_i^n(u_j) = \begin{cases} 1 & i = j(i, j = 0, 1, \ldots, n) \\ 0 & Otherwise \end{cases}$$

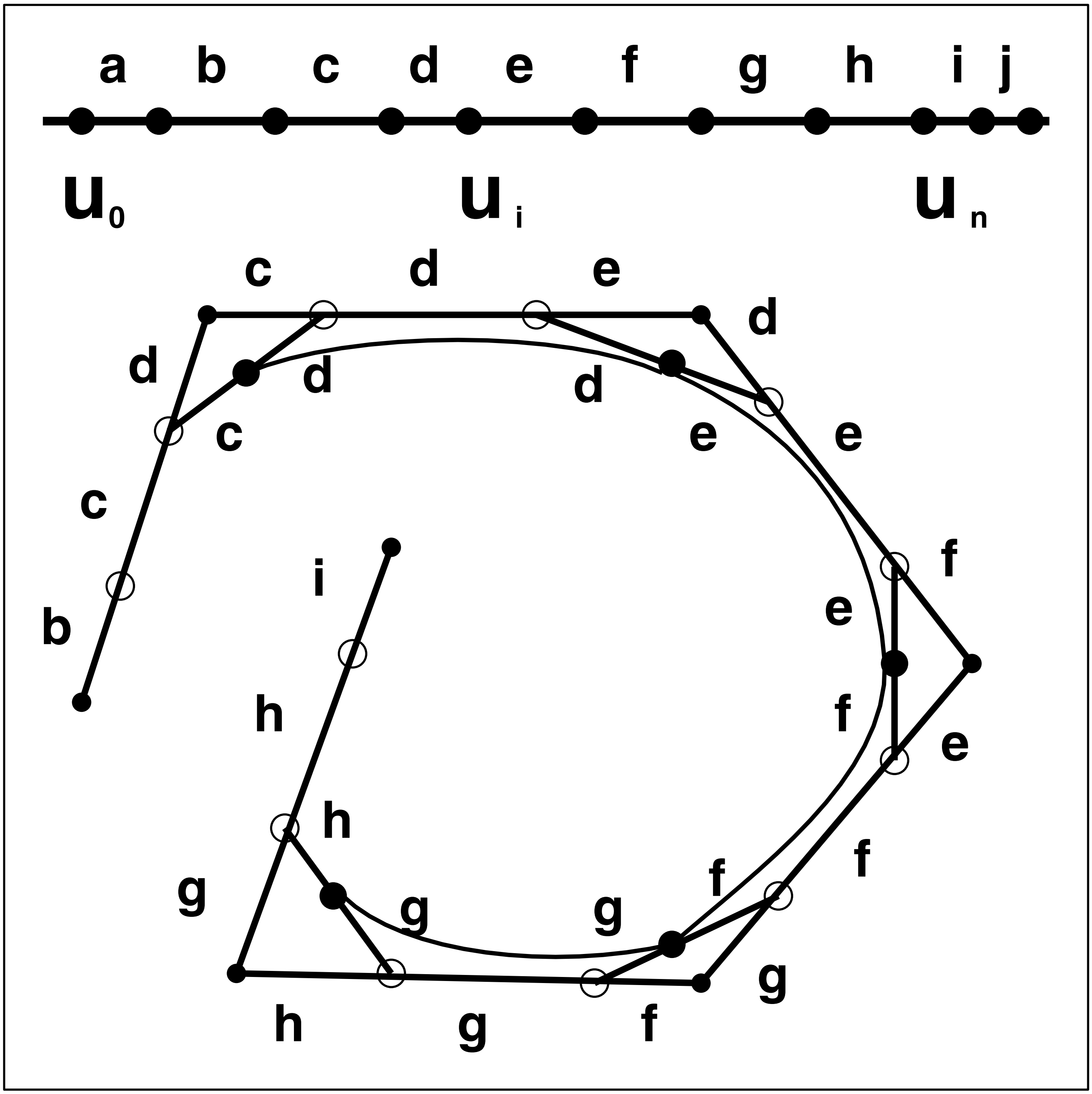
$$L_0^n(u) = \frac{(u - u_1)(u - u_2)\cdots(u - u_n)}{(u_0 - u_1)(u_0 - u_2)\cdots(u_0 - u_n)}$$

$$L_i^n(u) = \frac{(u - u_0)\cdots(u - u_{i-1})(u - u_{i+1})\cdots(u - u_n)}{(u_i - u_0)\cdots(u_i - u_{i-1})(u_i - u_{i+1})\cdots(u_i - u_n)}$$

$$L_n^n(u) = \frac{(u - u_0)(u - u_1)(u - u_2)\cdots(u - u_{n-1})}{(u_n - u_0)(u_n - u_1)(u_n - u_2)\cdots(u_i - u_{n-1})}$$

- **Unwanted oscillation: WHY?**

Nonuniform B-Spline



B-Spline Example

- **Knot vector:** $u_0, u_1, u_2, u_3, u_4, u_5, u_6, u_7$
- **Cubic basis functions:** $B_{0,4}, B_{1,4}, B_{2,4}, B_{3,4}$
- $B_{0,4}$ **is defined over** $[u_0, u_4]$
- $B_{1,4}$ **is defined over** $[u_1, u_5]$
- $B_{2,4}$ **is defined over** $[u_2, u_6]$
- $B_{3,4}$ **is defined over** $[u_3, u_7]$
- **The curve can be defined as**

$$\mathbf{c}(u) = \mathbf{p}_0 B_{0,4} + \mathbf{p}_1 B_{1,4} + \mathbf{p}_2 B_{2,4} + \mathbf{p}_3 B_{3,4} =$$

$$\mathbf{p}_0 \{ \frac{u - u_0}{u_3 - u_0} B_{0,3} + \frac{u_4 - u}{u_4 - u_1} B_{1,3} \} +$$

$$\mathbf{p}_1 \{ \frac{u - u_1}{u_4 - u_1} B_{1,3} + \frac{u_5 - u}{u_5 - u_2} B_{2,3} \} + \ldots + \ldots =$$

$$\mathbf{p}_0 \frac{u-u_0}{u_3-u_0} B_{0,3} + (\mathbf{p}_0 \frac{u_4-u}{u_4-u_1} + \mathbf{p}_1 \frac{u-u_1}{u_4-u_1}) B_{1,3} + \ldots + \ldots$$

- **The B-spline curve of order k (components)...**

$$\mathbf{p}_i B_{i,k}(u) =$$

$$\mathbf{p}_i (\frac{u-u_i}{u_{i+k-1}-u_i} B_{i,k-1}(u) + \frac{u_{i+k}-u}{u_{i+k}-u_{i+1}} B_{i+1,k-1}(u))$$

$$\mathbf{p}_{i+1} B_{i+1,k}(u) =$$

$$\mathbf{p}_{i+1} (\frac{u-u_{i+1}}{u_{i+k}-u_{i+1}} B_{i+1,k-1} + \frac{u_{i+k+1}-u}{u_{i+k+1}-u_{i+2}} B_{i+2,k-1})$$

- **Finally, the curve ...**

$$\ldots + (\mathbf{p}_i \frac{u_{i+k}-u}{u_{i+k}-u_{i+1}} + \mathbf{p}_{i+1} \frac{u-u_{i+1}}{u_{i+k}-u_{i+1}}) B_{i+1,k-1} + \ldots$$

B-Spline Facts

- $n + 1$ **control points:** $\mathbf{p}_i$
- $n + 1$ **basis functions:** $B_{i,k}(u)$
- **Linear combination:** $\mathbf{c}(u) = \sum_{i=0}^{n} \mathbf{p}_i B_{i,k}(u)$
- **Important variables:** i **is index,** k **is the order,** $k - 1$ **is degree**
- **Knots:** $\{u_0, \ldots, u_{k-1}, \ldots, u_{n+1}, \ldots, u_{n+k}\}$
- **The first** k **and last** k **knots do NOT contribute to th parametric domain**
- **Basis function** $B_{i,k}$ **is defined recursively!**

B-Spline Discretization

- Parametric domain: $[u_{k-1}, u_{n+1}]$
- There are $n+2-k$ curve spans (pieces)
- Assume $m+1$ points per span (uniform sampling)
- Total sampling points: $m(n+2-k)+1 = l$
- B-spline discretization: $\mathbf{q}_0, \ldots, \mathbf{q}_{l-1}$
- Corresponding parametric values: $v_0, \ldots, v_{l-1}$

$$\mathbf{q}_i = \mathbf{c}(v_i) = \sum_{j=0}^{n} \mathbf{p}_j B_{j,k}(v_i)$$

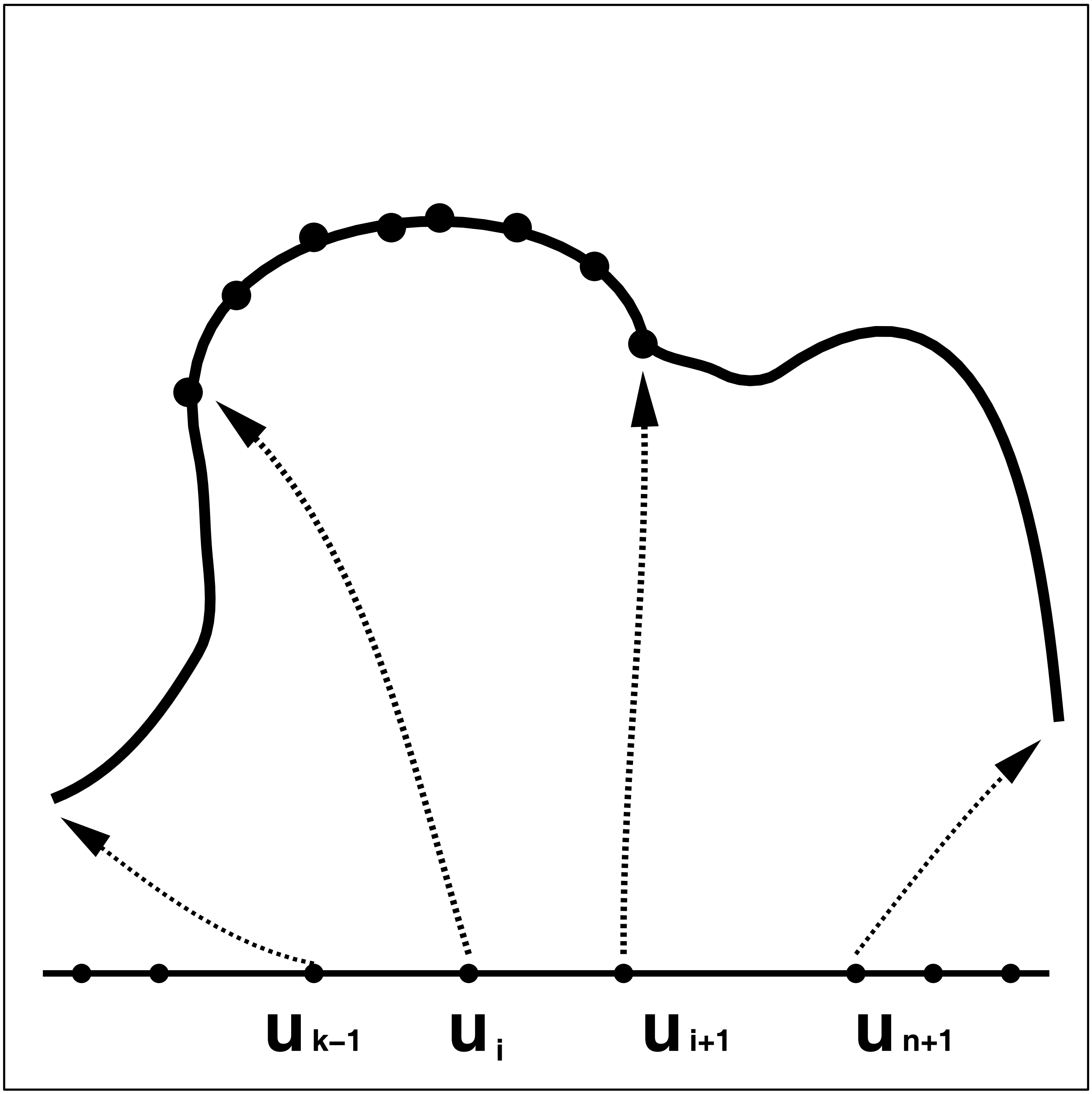
- Matrix equation

$$\begin{bmatrix} \mathbf{q}_0 \\ \vdots \\ \mathbf{q}_{l-1} \end{bmatrix} = \begin{bmatrix} B_{0,k}(v_0) & \cdots & B_{n,k}(v_0) \\ \vdots & \ddots & \vdots \\ B_{0,k}(v_{l-1}) & \cdots & B_{n,k}(v_{l-1}) \end{bmatrix} \begin{bmatrix} \mathbf{p}_0 \\ \vdots \\ \mathbf{p}_n \end{bmatrix}$$

- $\mathbf{A}$ is $(l) \times (n+1)$ matrix

- In general, l is much larger than $n+1$, A is sparse
- The linear discretization for both modeling and rendering

B-Spline Discretization



Another Discretization

- From B-spline to Bezier spline
- B-spline control points: $\mathbf{p}_0, \ldots, \mathbf{p}_n$
- Control points of piecewise Bezier curves

$$\mathbf{v}_0, \ldots, \mathbf{v}_3, \mathbf{v}_4, \ldots, \mathbf{v}_7, \ldots, \mathbf{v}_{4(n-3)}, \ldots, \mathbf{v}_{4(n-3)+3}$$

- Matrix expression

$$\begin{bmatrix} \mathbf{v}_0 \\ \vdots \\ \mathbf{v}_{4(n-3)+3} \end{bmatrix} = \mathrm{B} \begin{bmatrix} \mathbf{p}_0 \\ \vdots \\ \mathbf{p}_n \end{bmatrix}$$

- The matrix structure and components of B ???

$$\mathbf{q} = \mathbf{A}\mathbf{v} = \mathbf{A}\mathbf{B}\mathbf{p}$$

- The matrix structure and components of A ???