Box Spline Mesh Algorithm Summary
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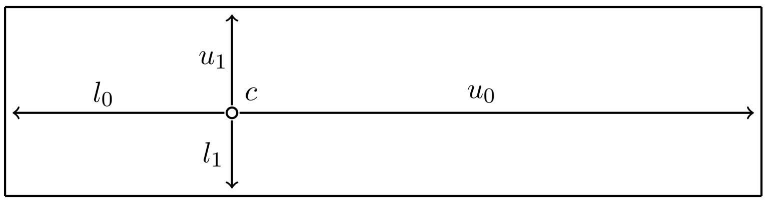
Thomas Lux

Given data $N = \{x_1, x_2, \ldots, x_n\} \mid x_i \in \mathbb{R}^d$, and associated response values $\{y_1, y_2, \ldots, y_n\}$.

1 Construct Box Mesh

Each box in the box mesh will be specified by a center $c \in \mathbb{R}^d$, lower width $l \in \mathbb{R}^d$, and upper width $u \in \mathbb{R}^d$. A point x_i is contained by a box if $(1 \le j \le d)(c_j - l_j < x_{i,j} < c_j + u_j)$. A two dimensional visual example of the containment region of a box, as we refer to it:



The only restriction is that all widths must be greater than zero. This means that $\min_i l_i > 0$ and $\min_i u_i > 0$. First, order the data in N appropriately (discussed later), initialize a single box with $c = x_1$ and lower and upper widths of infinity. Now, proceed to add the rest of the points in N by:

for $i = 2, \cdots, n$ **do**
  Identify boxes $\mathbf{B}$ that contain x_i;
  Initialize box *m* with center x_i that contains all boxes in $\mathbf{B}$;
  for b *in* $\mathbf{B}$ **do**
    split dimension := $\max_j$ |b-center$_j$ − m-center$_j$|;
    Adjust the widths of b and *m* in *split dimension*
    to be the difference between the centers in that dimension.
  end
end

2 Evaluating A Box

Interpolating and approximating requires using a set of boxes as regions of influence for response values. Given a new data point x is contained by a box b (with associated c,l,and u), we evaluate $b(x)$ as:

1. Scale x according to the lower and upper widths of b such that all $x_j < c_j$ map to $(0, 0.5)$ and all $x_j > c_j$ map to $(0.5, 1)$.

2. Scale this box-normalized x to be in the appropriate range for the order of B-Spline chosen ($\times 2$ for order 2,$\times 3$ for order 3, etc.)

3. Take the product of the B-Splines along each dimension for the box spline normalized x value. $\prod_{j=1}^{d} f(x_j)$ where f is the piecewise polynomial defining the b-spline of the desired order.

3 Interpolate y for new x

Identify the set of boxes $\mathbf{B} = \{b_i \mid b_i \text{ contains } x\}$. Calculate the estimated response value by

$$\frac{\sum_{b \in \mathbf{B}} b\text{-response} \times b(x)}{\sum_{b \in \mathbf{B}} b(x)}$$

where b-response is the response value associated with each box. Notice that this is using the box functions as a weighted sum.

Notes

The ordering that has been used when adding points to the box mesh thus far is:

- Add the x_i with the median valued y_i.
- Compute the approximation surface error at all remaining x_i, add the x_i to the surface with the most (relative) error.
- Repeat previous step until small enough maximum error is obtained.

It is important to note that this constant recalculation of error causes an increase in the runtime by a factor of n.