Algorithm for Constructing Piecewise Quintic Monotone Interpolating Splines

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When provided data that has no assigned first and second derivative values, the derivative data is filled by a linear fit of neighboring data points. End points are set to be the slope between the end and its nearest neighbor.

The method finding a maximal value on a line is the Golden Section search. This will be referred to in pseudo code as `line_search(`a`,`b`)` where a and b are n-tuples for integer n.

After assigning function values and derivative values, an interpolating function is constructed by solving for the unique weights of a set of quintic B-splines in a linear system.

1 Verifying Monotonicity of a Quintic Polynomial

Let f be a quintic polynomial over a closed interval $[x_0, x_1] \subset \mathbb{R}$. Now f is uniquely defined by the evaluation tuples $\big(x_0,\ f(x_0),\ f'(x_0),\ f''(x_0)\big)$ and $\big(x_1,\ f(x_1),\ f'(x_1),\ f''(x_1)\big)$. Assume without loss of generality that $f(x_0) < f(x_1)$, where the case of monotonic decreasing f uses the negated the function values. This algorithm will determine whether or not f is monotone increasing on the interval $[x_0, x_1]$.

Algorithm 1a: `is_monotone`

0: `if` $(f'(x_0) = 0$ `or` $f'(x_1) = 0)$`; return is_monotone_simplified`
1: `if` $(f'(x_0) < 0$ `or` $f'(x_1) < 0)$`; return FALSE`

> *This can be seen clearly from the fact that f is analytic and there will exist some $0 < \epsilon < x_1 - x_0$ such that either $f'(x_0 + \epsilon)$ or $f'(x_1 - \epsilon)$ will be negative.*

2: $A = f'(x_0)\frac{x_1 - x_0}{f(x_1) - f(x_0)}$
3: $B = f'(x_1)\frac{x_1 - x_0}{f(x_1) - f(x_0)}$

> *The variables A and B correspond directly to the theoretical foundation for positive quartic polynomials laid in [2].*

8: $\gamma_0 = 4\frac{f'(x_0)}{f'(x_1)}(B/A)^{3/4}$
9: $\gamma_1 = \frac{x_1-x_0}{f'(x_1)}(B/A)^{3/4}$
4: $\alpha_0 = 4(B/A)^{1/4}$
5: $\alpha_1 = -\frac{x_1-x_0}{f'(x_1)}(B/A)^{1/4}$
6: $\beta_0 = 30 - \frac{12\left(f'(x_0)+f'(x_1)\right)(x_1-x_0)}{\left(f(x_1)-f(x_0)\right)\sqrt{A}\sqrt{B}}$
7: $\beta_1 = \frac{-3(x_1-x_0)^2}{2\left(f(x_1)-f(x_0)\right)\sqrt{A}\sqrt{B}}$

The γ, α, and β terms with subscripts 0 and 1 are algebraic reductions of the original variables from [2] that give the computation of each corresponding variable the form $v = v_0 + v_1 c$, where c is a term involving only the second derivative values.

11: $\gamma = \gamma_0 + \gamma_1 f''(x_0)$
10: $\alpha = \alpha_0 + \alpha_1 f''(x_1)$
12: $\beta = \beta_0 + \beta_1\left(f''(x_0) - f''(x_1)\right)$
13: `if` $(\beta \leq 6)$`; return` $\alpha > -(\beta+2)/2$
14: `else; return` $\gamma > -2\sqrt{\beta-2}$

1.1 Verifying Monotonicity of a Simplified Quintic

Consider the same initial conditions outlined in Section 1.

Algorithm 1b: `is_monotone_simplified`

- Compute

2 Enforcing Monotonicity of a Quintic Polynomial

Algorithm 2a: `make_monotone`

- Compute

3 Enforcing Monotonicity of a Simplified Quintic

Algorithm 2b: `make_monotone_simplified`

- Compute

This follows the simplified conditions outlined in proposition 2 of [1].

4 Constructing a Piecewise Quintic Monotone Spline

Let $f : \mathbb{R} \to \mathbb{R}$ be a function in $\mathcal{C}^2$. Proceed given evaluation tuples $\big(x_i,\, f(x_i),\, f'(x_i),\, f''(x_i)\big)$ for $i = 0, \ldots, N$ such that $x_i < x_{i+1}$ and (without loss of generality) $f(x_i) \leq f(x_{i+1})$ for $i = 1, \ldots, N-1$.

Algorithm 3: `monotone_spline`

- Compute

References

[1] Schmidt, J.W., Hess, W.: Positivity of cubic polynomials on intervals and positive spline interpolation. BIT Numerical Mathematics **28**(2), 340–352 (1988)

[2] Ulrich, G., Watson, L.T.: Positivity conditions for quartic polynomials. SIAM Journal on Scientific Computing **15**(3), 528–544 (1994)