Vol. 45, No. 3 DUKE MATHEMATICAL JOURNAL© September 1978

ON THE DISTRIBUTION OF FEKETE POINTS

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1. Introduction

Let $E \subset R^n$, $n \geq 3$, be a compact set and N a given positive integer. A system of points $P_1, \cdots, P_N \in E$ which minimizes $\sum\sum_{i \neq j} |P_i - P_j|^{2-n}$ is called a system of Fekete points of E. (Notice that this represents a stable equilibrium distribution of N equal point charges on E). The purpose of this note is to find estimates of the distance d from a Fekete point P_i to its closest neighbour P_i^*. Using complex methods, Kövari and Pommerenke [1] found that if $E \subset R^2$ is a sufficiently smooth curve then $C_1N^{-1} \leq d \leq C_2N^{-1}$. In the case when $n \geq 3$ and E is a closed $C^{1,\alpha}$ surface, Sjögren [3] found the estimate $d \leq C\, N^{-\gamma}$, where $\gamma = \frac{1}{2}(n-1)^{-2}$. We can show the following estimate:

THEOREM. *Let $S \subset R^n$, $n \geq 3$, be a closed, compact $C^{1,\alpha}$-surface, where $0 < \alpha < 1$, that separates R^n into two components. Then there are positive numbers $C_i = C_i(S)$, $i = 1, 2$, such that if N is a positive integer and $P_1, \cdots, P_N$ is a system of Fekete points of S then*

$$C_1 r_N \leq |P_i - P_i^*| \leq C_2 r_N,\ 1 \leq i \leq N, \tag{1.1}$$

where $r_N = N^{-1/(n-1)}$.

2. The main result

We start by recalling that a $C^{1,\alpha}$-surface in R^n is a closed, bounded $(n-1)$-dimensional surface S such that S can be covered by finitely many open right circular cylinders whose bases have a positive distance to S and to each cylinder C there is an orthonormal coordinate system (x, y), $x \in R^{n-1}$, $y \in R$, such that the y-axis is parallel to the axis of symmetry of C and $C \cap S = C\,\{(x, y): y = \phi(x)\}$, where $\phi: R^{n-1} \to R$ is a C^1-function such that $|\nabla\phi(x) - \nabla\phi(z)| \leq M|x - z|^\alpha$, where ∇ denotes the gradient.

We shall from now on assume that $S \subset R^n$, $n \geq 3$, is a $C^{1,\alpha}$-surface for some α, $0 < \alpha < 1$, such that S separates R^n into two components D and D_∞ where D_∞ denotes the unbounded one. We denote by dS the surface measure element on S and by λ the equilibrium measure of S, i.e., the unique positive measure on S with total mass 1 minimizing

$$\iint |P - Q|^{2-n} d\lambda(P) d\lambda(Q).$$

Received October 15, 1977. Revision received March 14, 1978.

If μ is a measure we put $K\mu(P) = \int |P - Q|^{2-n} d\mu(Q)$. If V^{-1} denotes the capacity of S then $K\lambda = V$ in $\bar{D}$. If $v = V - K\lambda$ then v is positive and harmonic in D_∞ and it is a classical fact that grad v has a Hölder continuous extension to $\bar{D}_\infty$ and

$$-(\partial/\partial n)v \geq c > 0 \text{ on } S, \tag{2.1}$$

where n denotes the unit normal to S pointing into D_∞. For the methods of proving these facts see e.g., Widman [4].

Let now $P_1, \cdots, P_N$ be a system of Fekete points of S and denote by μ the mass distribution consisting of point masses N^{-1} at each P_i. Put

$$V_N = \inf_S K\mu.$$

Then it is known that there is a number $C = C(S) > 0$ such that

$$V_N \geq V - Cr_N. \tag{2.2}$$

For a proof of this fact we refer to Sjögren [3]. From the maximum principle we have

$$K\mu \geq V_N\omega \tag{2.3}$$

where $\omega = K\lambda/V$. From the smoothness of ω and (2.2) follows now the existence of a number $C = C(S) > 0$ such that if $d(P) \leq r_N$ then

$$K\mu(P) \geq V - Cr_N. \tag{2.4}$$

Here $d(P)$ denotes the distance from P to D.

We shall now prove the left hand side inequality of (1.1).

Suppose that $|P_i - P_i^*| \leq r_N/4$, otherwise there is nothing to prove. Put $h(P) = K\mu(P) - N^{-1}|P - P_i|^{2-n}$ and $f(P) = h(P) - N^{-1}|P - P_i^*|^{2-n}$. Since $P_1, \cdots, P_N$ is a system of Fekete points it follows that $h(P_i) = \inf_S h$. We note that $h = K\mu^*$, where μ^* is a positive measure with total mass less than 1. Hence we have from Fubini's theorem that

$$\int_S (h - V)d\lambda = \int_S K(\mu^* - \lambda)d\lambda = \int V\, d(\mu^* - \lambda) < 0,$$

which gives that

$$h(P_i) = \inf_S h < V. \tag{2.5}$$

Let $B = B(P_i, r_N)$ be the ball with radius r_N and center at P_i. If $P \in \partial B$ then it follows from the definition of f that $f(P) \geq K\mu(P) - (1 + (4/3)^{n-2})r_N$. Taking into account (2.4) we have

$$\inf_{\partial B} f \geq V - Cr_N.$$

Since f is superharmonic, it follows from the minimum principle that

$$f(P_i) \geq V - Cr_N. \tag{2.6}$$

We now find from (2.5) and (2.6) that

$$V \geq h(P_i) = f(P_i) + N^{-1}|P_i - P_i^*|^{2-n} \geq V - Cr_N + N^{-1}|P_i - P_i^*|^{2-n},$$

which yields that $|P_i - P_i^*| \geq cr_N$. Hence the left side inequality of (1.1) is proven.

In passing, let us note that since $h > f$, it follows from (2.5) and (2.6) that

$$V - Cr_N \leq h(P_i) \leq V. \tag{2.7}$$

We can now choose a constant $\alpha > 0$ independent of N such that $|P_i - P_i^*| \geq 10\alpha r_N$. Put $\rho_N = \alpha r_N$ and put

$$|\nabla_2 h| = \sum_{i,j} |\partial^2 h/\partial x_i \partial x_j|.$$

It's easily seen that

$$|\nabla_2 h(P)| \leq C \sum_{k \neq i} r_N^{n-1}|P - P_k|^{-n}.$$

If $P \in B(P_i, \rho_N)$ and $Q \in B(P_k, \rho_N)$, $k \neq i$ then

$$|P_i - Q| \leq |P_i - P| + |P - P_k| + |P_k - Q| \leq 2\rho_N + |P - P_k|.$$

Since $|P_i - Q| \geq |P_i - P_k| - |P_k - Q| \geq 9\rho_N$, it follows that $|P_i - Q| \leq 2|P - P_k|$. Hence

$$\sup\{|\nabla_2 h(P)|: P \in B(P_i, \rho_N)\} \leq C \sum_{k \neq i} r_N^{n-1} \inf\{|Q - P_i|^{-n}: Q \in B(P_k, \rho_N)\}.$$

Since

$$\int_{B(P_j,\rho_N) \cap S} dS \geq C\, r_N^{n-1},$$

it follows that

$$\max_{B(P_i,\rho_N)} |\nabla_2 h(P)| \leq C \int_{S - B(P_i,\rho_N)} |P_i - Q|^{-n} dS(Q) \leq C\, r_N^{-1}.$$

From this estimate it follows that if $\tilde{P}$ denotes the point such that $P + \tilde{P} = 2\, P_i$ and $P \in B(P_i, \rho_N)$ then

$$|h(P) + h(\tilde{P}) - 2h(P_i)| \leq C\, r_N^2 \sup\{|\nabla_2 h(P)|: P \in B(P_i, \rho_N)\} \leq Cr_N. \tag{2.8}$$

Since $h(Q) = K\mu(Q) - \alpha^{2-n} r_N$ for $Q \in \partial B(P_i, \rho_N)$ it follows from (2.7) and (2.8) that $|K\mu(P) + K\mu(\tilde{P}) - 2V| < C\, r_N$. From (2.4) follows that $K\mu(Q) - V \geq -C\, r_N$ for $Q \in \partial B(P_i, \rho_N)$ which yields that

$$|K\mu(P) - V| \leq C\, r_N \quad \text{for} \quad P \in \partial B(P_i, \rho_N). \tag{2.9}$$

LEMMA 1. *Let* $V_N = \inf_S K\mu$. *Then there are positive numbers* 1 *and* L *such that if* $r_N \leq t \leq 2r_N$ *then*

$$\{K\lambda > V - 1t\} \subset \{K\mu > V_N - t\} \subset \{K\lambda > V - Lt\}. \tag{2.10}$$

Proof. We have from (2.3) that $K\mu > V_N K\lambda V^{-1}$. Hence the left hand side inclusion of (2.10) holds whenever $0 < 1 < V_N^{-1}V$.

From (2.1) and the smoothness of $K\lambda$ follows the existence of a neighbourhood Ω of $\bar{D}$ such that if $P \in \Omega$ then

$$C_1 d(P) \leq V - K\lambda(P) \leq C_2 d(P). \tag{2.11}$$

From (2.9) and the maximum principle it follows that if $P \notin \bigcup_j B(P_j, \rho_N)$ then $K\mu(P) \leq V + Cr_N$. In particular we have that

$$\sup\{K\mu(P)\colon d(P) = \rho_N\} \leq V + Cr_N.$$

From (2.11) it follows that

$$\inf\{K\lambda(P)\colon d(P) = \rho_N\} \geq V - Cr_N.$$

The maximum principle therefore yields that if $d(P) > \rho_N$ then

$$K\mu(P) \leq (1 - Cr_N)^{-1} K\lambda(P).$$

Recalling the estimate $V_N \geq V - C\, r_N$ we see that if L is chosen large enough, and if $K\mu(P) > V_N - t$, $d(P) > \rho_N$ then $K\lambda(P) > V - Lt$. Also, it follows from (2.11) that if L is chosen large enough then $\{K\lambda > V - Lt\}$ contains the set $\{d(P) < \rho_N\}$ which yields the Lemma.

For each N we choose a number t^*, $r_N \leq t^* \leq 2r_N$ such that $\{K\mu > V_N - t^*\} = D^*$ has a smooth boundary. (The possibility of doing so follows from Sard's theorem).

LEMMA 2. *Fix a point* $P^* \in D$. *If* u *is non-negative and harmonic in a neighbourhood of* $\bar{D}^*$ *then there is a constant* C *only depending on* P^* *and* D *such that*

$$u(P^*) \leq C \int u \, d\mu.$$

Proof. Put $q = K\mu - V_N + t^*$. There is a positive number γ, only depending on n such that $\Delta q = -\gamma\mu$. Hence it follows from Green's formula that

$$\gamma \int u \, d\mu = \int_{\partial D^*} u(\partial/\partial n)\, q \, ds,$$

where n denotes the unit inward normal on ∂D^*. Define ϕ by $\phi = 0$ on ∂D^*, $\phi = t^*$ in $\bar{D}$ and ϕ is harmonic in $D^* - \bar{D}$. Then $\phi \leq q$ in $D^* - D$ which implies that

$$\gamma \int u \, d\mu \geq \int_{\partial D^*} u(\partial/\partial n)\, \phi dS.$$

For $P \in S$ let $n(P)$ denote the unit normal to S pointing into D_∞. We have on S $(\partial/\partial n)\phi < 0$. From the above we have

$$\gamma \int u \, d\mu \geq - \int_S u(\partial/\partial n)\phi \, dS. \tag{2.12}$$

From Lemma 1 follows the existence of a number $m_0 > 0$ such that

$$m_0(V - K\lambda) \leq t^* \text{ on } \partial D^*.$$

Hence $t^* - \phi \geq m_0(V - K\lambda)$ in $D^* - \bar{D}$, which together with (2.1) implies that $-(\partial/\partial n)\phi \geq m > 0$ on S. Hence the Lemma follows from (2.12).

To complete the proof of the Theorem we next observe the following. It is known that if Ω is a $C^{1,\alpha}$-domain and if G denotes the Green function of Ω and if ∇ denotes the gradient then

$$|\nabla_Q G(P, Q)| \leq M(\Omega)\delta(P)|P - Q|^{-n}, \tag{2.13}$$

where $\delta(P)$ denotes the distance from P to $\partial\Omega$. Also the constant $M(\Omega)$ depends on certain geometrical facts of Ω, see e.g., Widman [4, Theorem 2.3]. If we now recall that $\nabla K\lambda$ is $C^{1,\alpha}$ up to $\bar{D}_\infty$ (see Widman [4, Theorem (2.4) and (2.1)] we can now make the estimates of the Green functions of the domain $\{K\lambda > V - \epsilon\} = D_\epsilon$ uniform in ϵ, $0 < \epsilon < \epsilon_0$. In particular we have from (2.13) that

$$|\nabla_Q G_\epsilon(P, Q)| \leq C\, \delta_\epsilon(P)|P - Q|^{-n}, \; 0 < \epsilon < \epsilon_0, \tag{2.14}$$

where G_ϵ denotes the Green function of D_ϵ and δ_ϵ denotes the distance to ∂D_ϵ. In the same way, it follows that if $P^* \in D$ is fixed, then

$$(\partial/\partial n_Q)G_\epsilon(P^*, Q) \geq C > 0, \; 0 < \epsilon < \epsilon_0, \; Q \in \partial D_\epsilon, \tag{2.15}$$

where n_Q denotes the unit inward normal to ∂D_ϵ, c.f. Widman [4, Theorem 2.5]. The constants C which appear in (2.14) and (2.15) are independent of ϵ.

To prove the Theorem, it now remains to prove that the numbers $\eta > 0$, for which $B(Q, \eta r_N)$, $Q \in S$, does not meet $\{P_1, \cdots, P_N\}$ are uniformly bounded from above.

From Lemma 1 follows the existence of a number $\beta > 0$ such that $\Omega = \{K\lambda > V - \beta r_N\} \supset \overline{D^*}$. We shall assume that β has been chosen so large that

$$\bigcup_j B(P_j, 2\rho_N) \subset \Omega. \tag{2.16}$$

Suppose now that $Q \in S$ and $B(Q, \eta r_N)$ does not meet $\bigcup_j B(P_j, \rho_N)$ for some $\eta \geq 1$. Let $Z \in \partial\Omega$ be a point such that $|Q - Z| = \operatorname{dist}(Q, \partial\Omega)$. Define $u(P) = (\partial/\partial n_Z)G(P, Z)$, where G is the Green function of Ω. From (2.16) and Harnack's inequality follows now that

$$u(P_i) \leq C \inf\{u(P) : P \in B(P_i, \rho_N)\}.$$

Since the surface measure of $S \cap B(P_i, \rho_N)$ is larger than $Cr_N^{n-1} = CN^{-1}$ and the balls $B(P_i, \rho_N)$ are pairwise disjoint, it follows

$$\int_{S - B(Q, \eta r_N)} u \, d\mu \leq C \int u \, dS.$$

From the definition of Ω and (2.11) follows that $|Q - Z| \leq M\, r_N$, where M is independent of N. Suppose now that $\eta \geq 2M$. Then $|P - Q| \leq 2|P - Z|$ if $P \in S - B(Q, \eta r_N)$. From (2.14) we now have

$$\int u \, d\mu \leq C\, r_N \int_{S - B(Q, \eta r_N)} |P - Q|^{-n} dS(P) \leq C\eta^{-1}.$$

From Lemma 2 follows that

$$u(P^*) \leq C\eta^{-1}$$

which taken together with (2.15) shows that η is uniformly bounded from above, which yields the remaining part of the Theorem.

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