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Lecture 19 Notes

These notes correspond to Section 3.3 in the text.

Osculatory Interpolation

Suppose that the interpolation points are perturbed so that two neighboring points x_i and x_{i+1} , $0 \leq i < n$, approach each other. What happens to the interpolating polynomial? In the limit, as $x_{i+1} \rightarrow x_i$, the interpolating polynomial $p_n(x)$ not only satisfies $p_n(x_i) = y_i$, but also the condition

$$p'_n(x_i) = \lim_{x_{i+1} \rightarrow x_i} \frac{y_{i+1} - y_i}{x_{i+1} - x_i}.$$

It follows that in order to ensure uniqueness, the data must specify the value of the derivative of the interpolating polynomial at x_i .

In general, the inclusion of an interpolation point x_i k times within the set x_0, \dots, x_n must be accompanied by specification of $p_n^{(j)}(x_i)$, $j = 0, \dots, k-1$, in order to ensure a unique solution. These values are used in place of divided differences of identical interpolation points in Newton interpolation.

Interpolation with repeated interpolation points is called *osculatory interpolation*, since it can be viewed as the limit of distinct interpolation points approaching one another, and the term “osculatory” is based on the Latin word for “kiss”.

Hermite Interpolation

In the case where each of the interpolation points x_0, x_1, \dots, x_n is repeated exactly once, the interpolating polynomial for a differentiable function $f(x)$ is called the *Hermite polynomial* of $f(x)$, and is denoted by $H_{2n+1}(x)$, since this polynomial must have degree $2n+1$ in order to satisfy the $2n+2$ constraints

$$H_{2n+1}(x_i) = f(x_i), \quad H'_{2n+1}(x_i) = f'(x_i), \quad i = 0, 1, \dots, n.$$

The Hermite polynomial can be described using Lagrange polynomials and their derivatives, but this representation is not practical because of the difficulty of differentiating and evaluating these polynomials. Instead, one can construct the Hermite polynomial using a divided-difference table, as discussed previously, in which each entry corresponding to two identical interpolation points is filled with the value of $f'(x)$ at the common point. Then, the Hermite polynomial can be represented using the Newton divided-difference formula.

Example We will use Hermite interpolation to construct the third-degree polynomial $p_3(x)$ that fits the data

| i | x_i | $f(x_i)$ | $f'(x_i)$ |
|-----|-------|----------|-----------|
| 0,1 | 0 | 0 | 1 |
| 2,3 | 1 | 0 | 1 |

In other words, we must have $p_3(0) = 0$, $p'_3(0) = 1$, $p_3(1) = 0$, and $p'_3(1) = 1$. To include the values of $f'(x)$ at the two distinct interpolation points, we repeat each point once, so that the number of interpolation points, including repetitions, is equal to the number of constraints described by the data.

First, we construct the *divided-difference table* from this data. The divided differences in the table are computed as follows:

$$\begin{aligned}
f[x_0] &= f(x_0) \\
&= 0 \\
f[x_1] &= f(x_1) \\
&= 0 \\
f[x_2] &= f(x_2) \\
&= 0 \\
f[x_3] &= f(x_3) \\
&= 0 \\
f[x_0, x_1] &= \frac{f[x_1] - f[x_0]}{x_1 - x_0} \\
&= f'(x_0) \\
&= 1 \\
f[x_1, x_2] &= \frac{f[x_2] - f[x_1]}{x_2 - x_1} \\
&= \frac{0 - 0}{1 - 0} \\
&= 0 \\
f[x_2, x_3] &= \frac{f[x_3] - f[x_2]}{x_3 - x_2} \\
&= f'(x_2) \\
&= 1 \\
f[x_0, x_1, x_2] &= \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} \\
&= \frac{0 - 1}{1 - 0}
\end{aligned}$$

$$\begin{aligned}
&= -1 \\
f[x_1, x_2, x_3] &= \frac{f[x_2, x_3] - f[x_1, x_2]}{x_3 - x_1} \\
&= \frac{1 - 0}{1 - 0} \\
&= 1 \\
f[x_0, x_1, x_2, x_3] &= \frac{f[x_1, x_2, x_3] - f[x_0, x_1, x_2]}{x_3 - x_0} \\
&= \frac{1 - (-1)}{1 - 0} \\
&= 2
\end{aligned}$$

Note that the values of the derivative are used whenever a divided difference of the form $f[x_i, x_{i+1}]$ is to be computed, where $x_i = x_{i+1}$. This makes sense because

$$\lim_{x_{i+1} \rightarrow x_i} f[x_i, x_{i+1}] = \lim_{x_{i+1} \rightarrow x_i} \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i} = f'(x_i).$$

The resulting divided-difference table is

| | | | | |
|-----------|--------------|-------------------|-------------------------|-----------------------------|
| $x_0 = 0$ | $f[x_0] = 0$ | | | |
| | | $f[x_0, x_1] = 1$ | | |
| $x_1 = 0$ | $f[x_1] = 0$ | | $f[x_0, x_1, x_2] = -1$ | |
| | | $f[x_1, x_2] = 0$ | | $f[x_0, x_1, x_2, x_3] = 2$ |
| $x_2 = 1$ | $f[x_2] = 0$ | | $f[x_1, x_2, x_3] = 1$ | |
| | | $f[x_2, x_3] = 1$ | | |
| $x_3 = 1$ | $f[x_3] = 0$ | | | |

It follows that the interpolating polynomial $p_3(x)$ can be obtained using the *Newton divided-difference formula* as follows:

$$\begin{aligned}
p_3(x) &= \sum_{j=0}^3 f[x_0, \dots, x_j] \prod_{i=0}^{j-1} (x - x_i) \\
&= f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) + \\
&\quad f[x_0, x_1, x_2, x_3](x - x_0)(x - x_1)(x - x_2) \\
&= 0 + (x - 0) + (-1)(x - 0)(x - 0) + 2(x - 0)(x - 0)(x - 1) \\
&= x - x^2 + 2x^2(x - 1).
\end{aligned}$$

We see that Hermite interpolation, using divided differences, produces an interpolating polynomial that is in the *Newton form*, with centers $x_0 = 0$, $x_1 = 0$, and $x_2 = 1$. \square