B-SPLINES (Carl deBoor)

Definition. Let $\xi = \left(\xi_i\right)_1^{\ell+1}$ be a strictly increasing sequence of points, $k > 0$, and $P_1, \ldots, P_\ell$ a sequence of ℓ polynomials each of order k (degree $< k$). The corresponding *piecewise polynomial* f of order k is defined by

use the function specified for the containing interval

$$f(x) = P_i(x), \qquad \xi_i < x < \xi_{i+1}; \quad i = 1, \ldots, \ell.$$

ξ_i are called the *breakpoints* of f. By convention,

use the edge function for points outside of range

$$f(x) = \begin{cases} P_1(x), & x \leq \xi_1, \\ P_\ell(x), & x \geq \xi_{\ell+1}, \end{cases} \qquad \text{and} \qquad f(\xi_i) = f(\xi_i+) \quad \text{right continuous.}$$

$\mathcal{P}_{k,\xi} = \{$piecewise polynomial functions of order k with breakpoint sequence $\left(\xi_i\right)_1^{\ell+1}\}$, and dim $\mathcal{P}_{k,\xi} = k\ell$.

Sum of

$$\underbrace{\begin{bmatrix} f1 \\ f2 \\ \cdots \end{bmatrix}}_{k} \ell$$

Let $\nu = \left(\nu_i\right)_2^{\ell}$ be a vector of nonnegative integers, related to the jump conditions

$$\text{jump}_{\xi_i} D^{j-1} f = 0 \text{ for } j = 1, \ldots, \nu_i \text{ and } i = 2, \ldots, \ell.$$

$\mathcal{P}_{k,\xi,\nu} = \{f \in \mathcal{P}_{k,\xi} \mid |f$ satisfies the above jump conditions$\}$ is a subspace of $\mathcal{P}_{k,\xi}$ with dimension

order breaks continuity

$$\sum_{i=1}^{\ell} k - \nu_i \qquad (\nu_1 = 0).$$

since these functions satisfy jump conditions, they have (necessarily) less information

$\boxed{\text{A basis for } \mathcal{P}_{k,\xi} \text{ is}}$

HUGE

$$\phi_{ij} = \begin{cases} (x - \xi_1)^j / j!, & i = 1, \\ (x - \xi_i)_+^j / j!, & i = 2, \ldots, \ell, \end{cases} \qquad j = 0, \ldots, k-1,$$

where

each function is right cont.

$(x-$ closest break point$)$ jump

$$(x - \xi_i)_+^j = \begin{cases} 0, & x < \xi_i, \\ (x - \xi_i)^j, & x \geq \xi_i. \end{cases}$$

doesn't match?

A basis for $\mathcal{P}_{k,\xi,\nu}$ is ϕ_{ij}, $j = \nu_i, \ldots, k-1$ and $i = 1, \ldots, \ell$. That these are bases follows from the fact that they have the right number of elements, and are independent since $\exists$ linear functionals λ_{ij} such that

$$\lambda_{ij}\phi_{rs} = \delta_{ir}\delta_{js}. \qquad [\lambda_{ij} f = \text{jump}_{\xi_i} D^j f.]$$

Definition. Let $t = (t_i)$ be a nondecreasing sequence (finite, infinite, or biinfinite). The *ith B-spline of order k for the knot sequence t* is denoted by $B_{i,k,t}$ and is defined by

$$B_{i,k,t}(x) = (t_{i+k} - t_i)\,(\tau - x)_+^{k-1}\,[t_i, \ldots, t_{i+k}], \qquad \text{all } x \in E.$$

(The divided difference is applied to $(\tau - x)_+^{k-1}$ considered as a function of τ.) If k and t are understood, write B_i instead of $B_{i,k,t}$.

Properties of B-splines:

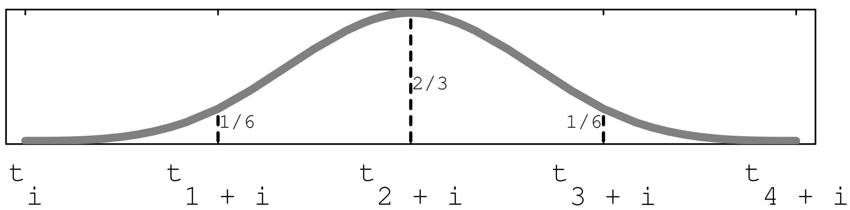
(i) $B_i(x) = 0$ for $x \notin [t_i, t_{i+k}]$.

(ii) $\displaystyle\sum_i B_i(x) = \sum_{i=r+1-k}^{s-1} B_i(x) = 1$ for all $t_r < x < t_s$.

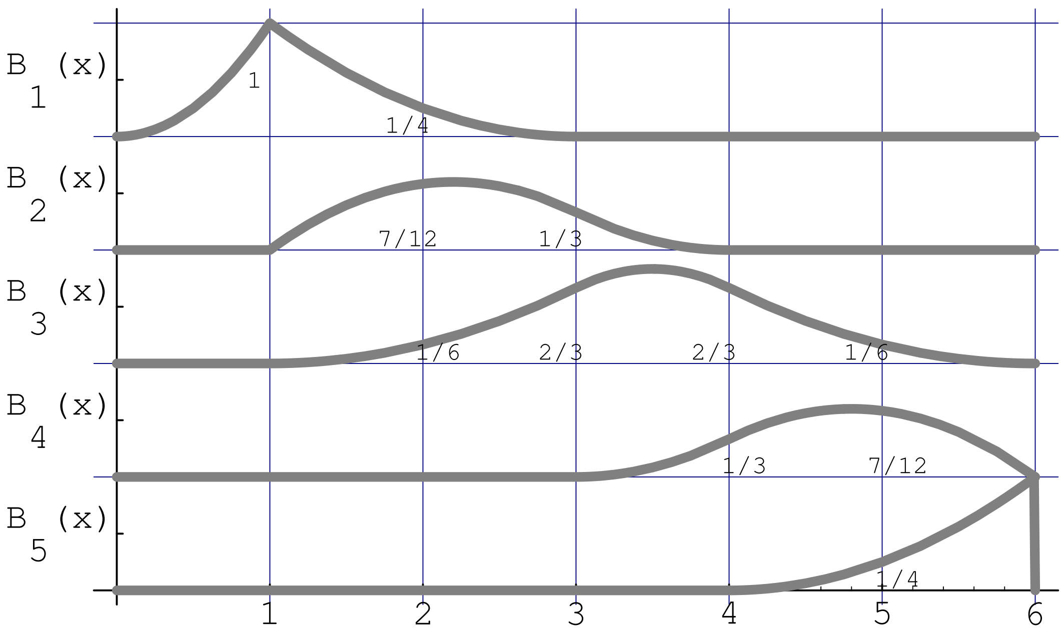
(iii) $B_i(x) > 0$ for $t_i < x < t_{i+k}$.

For t_i equally spaced a distance h apart, cubic ($k = 4$) B-splines are given by

$$B_i(x) = \frac{2}{3h^3}\left(\frac{1}{4}(x-t_i)_+^3 - (x-t_{i+1})_+^3 + \frac{3}{2}(x-t_{i+2})_+^3 \right.$$
$$\left. - (x-t_{i+3})_+^3 + \frac{1}{4}(x-t_{i+4})_+^3\right).$$



Cubic B-spline $B_i(x)$ for equally spaces knots t_j.



Parabolic B-splines $B_{i,3}(x)$ for the knot sequence $t = (0, 1, 1, 3, 4, 6, 6, 6)$.

Definition. A spline function of order k with knot sequence t is any linear combination of B-splines of order k for the knot sequence t. The collection of all such functions is denoted by $\mathcal{S}_{k,t}$.

Theorem (Curry, Schönberg). For a given strictly increasing sequence $\xi = \big(\xi_i\big)_1^{\ell+1}$, and a given nonnegative integer sequence $\nu = (\nu_i)_2^{\ell}$ with $\nu_i \le k$, all i, set

$$n = k + \sum_{i=2}^{\ell} k - \nu_i = k\ell - \sum_{i=2}^{\ell} \nu_i = \dim\, \mathcal{P}_{k,\xi,\nu}$$

and let $t = \big(t_i\big)_1^{n+k}$ be any nondecreasing sequence so that

(i) $t_1 \le t_2 \le \cdots \le t_k \le \xi_1$ and $\xi_{\ell+1} \le t_{n+1} \le t_{n+2} \le \cdots \le t_{n+k}$;

(ii) for $i = 2, \ldots, \ell$, the number ξ_i occurs exactly $k - \nu_i$ times in t.

Then the sequence $B_1, \ldots, B_n$ of B-splines of order k for the knot sequence t is a basis for $\mathcal{P}_{k,\xi,\nu}$, considered as functions on $\big[t_k, t_{n+1}\big]$, i.e.,

$$\mathcal{S}_{k,t} = \mathcal{P}_{k,\xi,\nu} \qquad \text{on } \big[t_k, t_{n+1}\big].$$

NOTE: number of continuity conditions at ξ +
number of knots at ξ $= k$

Proof. From the definition of divided differences, for any sufficiently smooth function g $\exists$ constants d_i, …, d_{i+k} such that

$$g\big[t_i, \ldots, t_{i+k}\big] = \sum_{r=i}^{i+k} d_r g^{(j_r)}(t_r),$$

with $j_r = \max\{s \mid r - s \ge i,\ t_{r-s} = t_r\}$, $r = i$, …, $i + k$. Thus

$$B_i(x) = (t_{i+k} - t_i) \sum_{r=i}^{i+k} d_r \big(t_r - x\big)_+^{k-1-j_r} (k-1)!/(k-1-j_r)!,$$

which is clearly a piecewise polynomial function of order k with breakpoints at t_i, …, t_{i+k}, i.e., at some of the points ξ_2, …, ξ_ℓ (and possibly at some other points outside $(\xi_1, \xi_{\ell+1})$, but these don't matter). B_i has a jump in its s-th derivative at the breakpoint ξ_j only if for some $r \in [i, i+k]$, we have $\xi_j = t_r$ and $k-1-j_r = s$. Since j_r counts the number of t_m's equal to t_r and with $i \le m < r$, it follows that j_r must be less than $k - \nu_j$ which is the total number of t_m's equal to $\xi_j = t_r$ by construction of t. This says that always $s \ge \nu_j$, and so

$$\text{jump}_{\xi_j} D^m B_i = 0 \qquad \text{for } m = 0, \ldots, \nu_j - 1.$$

Therefore $B_i \in \mathcal{P}_{k,\xi,\nu}$, all i.

Since there are n B_i's and $\dim\, \mathcal{P}_{k,\xi,\nu} = n$, it suffices to show that the sequence $\big(B_i\big)_1^n$ is linearly independent. This follows from:

Lemma (deBoor, Fix, 1973). Let λ_i be the linear functional given by the rule

$$\lambda_i f = \sum_{r=0}^{k-1} (-1)^{k-1-r} \psi^{(k-1-r)}(\tau_i)\, D^r f(\tau_i),$$

all f, with $\psi(t) = \big(t_{i+1} - t\big) \cdots \big(t_{i+k-1} - t\big)/(k-1)!$, and τ_i some arbitrary point in the open interval (t_i, t_{i+k}). Then

$$\lambda_i\, B_j = \delta_{ij}, \qquad \text{all } j.$$

Q. E. D.

B-SPLINE INTERPOLATION.

Let $t = \big(t_i\big)_1^{n+k}$ be a nondecreasing knot sequence with $t_i < t_{i+k}$, all i, and $\big(B_i\big)_1^n$ the corresponding B-splines of order k. The span $\mathcal{S}_{k,t}$ of $B_1, \ldots, B_n$ is n-dimensional. Given a strictly increasing sequence $\tau = \big(\tau_i\big)_1^n$ and function g, the problem is to find $f \in \mathcal{S}_{k,t}$ such that $f(\tau_i) = g(\tau_i)$ $\forall i$. Or, find spline coefficients α_j such that

$$\sum_{j=1}^{n} \alpha_j B_j(\tau_i) = g(\tau_i), \qquad i = 1, \ldots, n.$$

Theorem (Schönberg, Whitney). The matrix $\big(B_j(\tau_i)\big)$ is invertible $\Leftrightarrow$

$$B_i(\tau_i) \neq 0, \qquad i = 1, \ldots, n,$$

i.e., $t_i < \tau_i < t_{i+k}$, all i.

Theorem (Karlin). The matrix $\big(B_j(\tau_i)\big)$ is totally positive (all minors ≥ 0).

Observation. $\big(B_j(\tau_i)\big)$ has bandwidth less than k if $t_i < \tau_i < t_{i+k}$, all i.