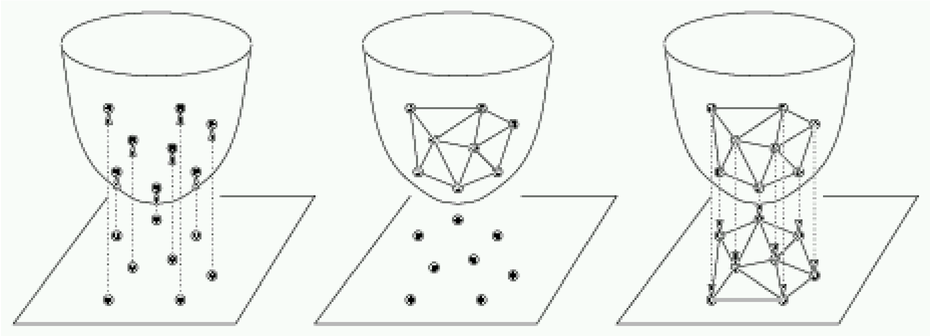

Figure 93: Delaunay triangulations and convex hull.

The question is, why does this work? To see why, we need to establish the connection between the triangles of the Delaunay triangulation and the faces of the convex hull of transformed points. In particular, recall that

Delaunay condition: Three points p, q, r, in S form a Delaunay triangle if and only if the circumcircle of these points contains no other point of S.

Convex hull condition: Three points p_0, q_0, r_0 in S_0 form a face of the convex hull of S_0 if and only if the plane passing through p_0, q_0, and r_0 has all the points of S_0 lying to one side.

Clearly, the connection we need to establish is between the emptiness of circumcircles in the plane and the emptiness of halfspaces in 3 space. We will prove the following claim.

Lemma: Consider 4 distinct points p, q, r, s in the plane, and let p_0, q_0, r_0 s_0 be their respective projections onto the paraboloid, $z = x^2 + y^2$. The point s lies within the circumcircle of p, q, r if and only if s_0 lies on the lower side of the plane passing through p_0, q_0, r_0.

To prove the lemma, first consider an arbitrary (nonvertical) plane in 3 space, which we assume is tangent to the paraboloid above some point (a, b) in the plane. To determine the equation of this tangent plane, we take derivatives of the equation $z = x^2 + y^2$ with respect to x and y, giving:

$dz/dx = 2x$

$dz/dy = 2y$

At the point $(a, b, a^2 + b^2)$ these evaluate to 2a and 2b. It follows that the plane passing through these point has the form

$z = 2ax + 2by + k$:

To solve for k we know that the plane passes through $(a, b, a^2 + b^2)$ so we solve giving

$a^2 + b^2 = 2 a^2 + 2 b^2 + k$;

Implying that $k = -(a^2 + b^2)$. Thus the plane equation is $z = 2ax + 2by - (a^2 + b^2)$:

If we shift the plane upwards by some positive amount r^2 we get the plane

$z = 2ax + 2by - (a^2 + b^2) + r^2$:

How does this plane intersect the paraboloid? Since the paraboloid is defined by $z = x^2 + y^2$

we can eliminate z giving

$x^2 + y^2 = 2ax + 2by - (a^2 + b^2) + r^2$.

which after some simple rearrangements is equal to

$(x - a)^2 + (y - b)^2 = r^2$

This is just a circle. Thus, we have shown that the intersection of a plane with the paraboloid produces a space curve (which turns out to be an ellipse), which when projected back onto the (x, y) coordinate plane is a circle centered at (a, b).

Thus, we conclude that the intersection of an arbitrary lower halfspace with the paraboloid, when projected onto the (x, y) plane is the interior of a circle. Going back to the lemma, when we project the points p; q; r onto the paraboloid, the projected points p_0, q_0 and r_0 define a plane. Since p_0, q_0, and r_0, lie at the intersection of the plane and paraboloid, the original points p, q, r lie on the projected circle. Thus this circle is the (unique) circumcircle passing through these p, q, and r. Thus, the point s lies within this circumcircle, if and only if its projection s_0 onto the paraboloid lies within the lower halfspace of the plane passing through

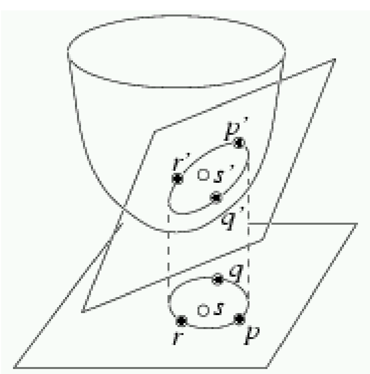

Figure 94: Planes and circles.

An applet that show simultaneously the delaunay triangulation and the 3D convex hull can be found here
It may be necessary to try a collection of patterns or sets of points (click on the left coordinate system near the bottom of the page).