POLYNOMIAL INTERPOLATION

The central problem in function approximation is to take a given function $f(x)$ and construct a simple function $s(x)$ which approximates $f(x)$ in some sense. The precise meaning of "simple" and "approximates" depends on the problem context and goals. Polynomial interpolation represents a first attempt to solve the approximation problem.

Let $f(x)$ be a function defined on $[a, b]$, and let x_0, x_1, …, x_n be $n+1$ distinct points in $[a, b]$. The Lagrange polynomials for x_0, …, x_n are defined by

$$L_k(x) = \prod_{\substack{i=0 \\ i \neq k}}^{n} \frac{(x - x_i)}{(x_k - x_i)}, \quad k = 0, 1, \ldots, n.$$

Note that $L_k(x_i) = \delta_{ik} = \begin{cases} 1, k = i \\ 0, k \neq i \end{cases}$. A polynomial $P(x)$ is said to *interpolate* f at x_0, …, x_n if $P(x_i) = f(x_i), \quad i = 0, 1, \ldots, n$.

Theorem. Let x_0, x_1, …, x_n be $n+1$ distinct points, and $f(x_0)$, …, $f(x_n)$ arbitrary values. Then $\exists$ a unique polynomial $P(x)$ of degree $\leq n$ such that $P(x_i) = f(x_i), \quad i = 0, 1, \ldots, n$.

Proof. The polynomial $P(x) = \sum_{k=0}^{n} f(x_k) L_k(x)$, called the Lagrange interpolating polynomial to $f(x)$ at x_0, …, x_n, has degree $\leq n$, and clearly satisfies $P(x_i) = f(x_i), \quad i = 0, 1, \ldots, n$. To prove the uniqueness, suppose $Q(x)$ is another polynomial of degree $\leq n$ also interpolating f at x_0, …, x_n. Then $P(x_i) - Q(x_i) = f(x_i) - f(x_i) = 0$ for $i = 0, 1, \ldots, n$. Thus $P - Q$ is a polynomial of degree $\leq n$ with $n+1$ distinct roots, which implies $P - Q \equiv 0$, i.e., $P = Q$.

Q. E. D.

The Hermite interpolating polynomial

$$H(x) = \sum_{k=0}^{n} f(x_k)\psi_k(x) + \sum_{k=0}^{n} f'(x_k)\Psi_k(x),$$

where

$$\psi_k(x) = (1 - 2L_k'(x_k)(x - x_k))L_k^2(x), \quad \Psi_k(x) = (x - x_k)L_k^2(x),$$

matches both f and f' at x_0, …, x_n, and is unique.

Disadvantages of Lagrange forms:

1) difficult to incorporate higher order derivative data or mixed data,
2) Lagrange form is expensive to evaluate,
3) new data cannot be easily incorporated.

Let $P(x)$ be the unique polynomial of degree $\leq n$ interpolating $f(x)$ at distinct points x_0, …, x_n. The Newton form of $P(x)$ is

$$\begin{aligned} P(x) &= \sum_{k=0}^{n} a_k \prod_{i=0}^{k-1} (x - x_i) \\ &= a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \cdots + a_n(x - x_0)(x - x_1)\cdots(x - x_{n-1}). \end{aligned}$$

Note that any polynomial can be expanded this way since 1, $(x - x_0)$, $(x - x_0)(x - x_1)$, ..., $(x - x_0)(x - x_1) \cdots (x - x_{n-1})$ are a basis for the vector space of polynomials of degree $\leq n$. The Newton form and Lagrange form of the interpolating polynomial are just different expansions of the same polynomial $P(x)$.

Definition. $a_k = f[x_0, \ldots, x_k]$ is called the kth divided difference of f at x_0, ..., x_k.

Properties of divided differences:

1) Let i_0, i_1, ..., i_k be any permutation of 0, 1, ..., k. Then $f[x_{i_0}, x_{i_1}, \ldots, x_{i_k}] = f[x_0, x_1, \ldots, x_k]$.
2) $f[x_0, x_1, \ldots, x_k] = \dfrac{f[x_1, \ldots, x_k] - f[x_0, \ldots, x_{k-1}]}{x_k - x_0}, \quad f[x_i] = f(x_i)$.
3) $f[x_0, x_1, \ldots, x_k]$ is a continuous function of x_0, ..., x_k.
4) $f[x_0, x_1, \ldots, x_k] = \dfrac{f^{(k)}(\xi)}{k!}$ for some point ξ, $\min(x_0, \ldots, x_k) < \xi < \max(x_0, \ldots, x_k)$.

Divided Difference Table

$$
\begin{array}{llllll}
x_0 & f(x_0) & & & & \\
 & & f[x_0, x_1] & & & \\
x_1 & f(x_1) & & f[x_0, x_1, x_2] & & \\
 & & f[x_1, x_2] & & f[x_0, x_1, x_2, x_3] & \\
x_2 & f(x_2) & & f[x_1, x_2, x_3] & & f[x_0, x_1, x_2, x_3, x_4] \\
 & & f[x_2, x_3] & & f[x_1, x_2, x_3, x_4] & \\
x_3 & f(x_3) & & f[x_2, x_3, x_4] & & \\
 & & f[x_3, x_4] & & & \\
x_4 & f(x_4) & & & &
\end{array}
$$

Evaluation algorithm for $P(x) = \sum_{k=0}^{n} a_k \prod_{i=0}^{k-1} (x - x_i)$ at $x = z$:

$b_n := a_n$;

for $k := n - 1$ step -1 until 0 do

$b_k := b_{k+1} * (z - x_k) + a_k$;

Then $b_0 = P(z)$, and $P(x) = b_0 + b_1(x - z) + b_2(x - z)(x - x_0) + \cdots + b_n(x - z)(x - x_0) \cdots (x - x_{n-2})$.

Theorem. Let $f \in C^{n+1}[a, b]$, $x_0, x_1, \ldots, x_n$ distinct points in $[a, b]$, and $P(x)$ be the unique polynomial of degree $\leq n$ interpolating f at $x_0, \ldots, x_n$. Then for any $x \in [a, b]$,

$$f(x) - P(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{i=0}^{n} (x - x_i),$$

where $\min(x, x_0, \ldots, x_n) < \xi < \max(x, x_0, \ldots, x_n)$.

Proof. If $x = x_i$ for some i, the result holds trivially. So assume $x \neq x_i$ for any i, and define

$$A(t) = f(t) - P(t) - \frac{f(x) - P(x)}{\prod_{i=0}^{n}(x - x_i)} \prod_{i=0}^{n}(t - x_i).$$

$A(t)$ has $n+2$ distinct zeros, x, x_0, …, x_n. By Rolle's Theorem, $A'(t)$ has at least $n+1$ distinct zeros separating the zeros of $A(t)$. Applying Rolle's Theorem repeatedly yields that $A^{(n+1)}(t)$ has at least one zero, say ξ. Then

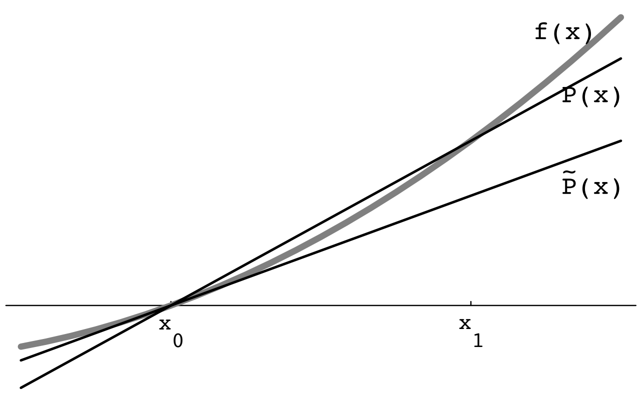
$$0 = A^{(n+1)}(\xi) = f^{(n+1)}(\xi) - \frac{f(x) - P(x)}{\prod_{i=0}^{n}(x - x_i)}(n+1)!,$$

which is the theorem. Q. E. D.

OSCULATORY INTERPOLATION

Question: What is the meaning of $P(x)$ as $x_1 \to x_0$?

Answer: $\tilde{P}(x) = \lim_{x_1 \to x_0} P(x)$ matches *both* f and f' at x_0.



$$P(x) = f[x_0] + f[x_0, x_1](x - x_0)$$
$$\tilde{P}(x) = f(x_0) + f'(x_0)(x - x_0)$$

Since $\tilde{P}(x) = f(x_0) + f'(x_0)(x - x_0) = \lim_{x_1 \to x_0} P(x) = \lim_{x_1 \to x_0} (f[x_0] + f[x_0, x_1](x - x_0))$ it is reasonable to define $f[x_0, x_0] = f'(x_0)$, and in general $f[\underbrace{x_0, \ldots, x_0}_{k+1 \text{ times}}] = \dfrac{f^{(k)}(x_0)}{k!}$.

Definition. For $x_0 \le x_1 \le x_2 \le \cdots \le x_n$, distinct or not, define

$$f[x_0, x_1, \ldots, x_n] = \begin{cases} \dfrac{f[x_1, \ldots, x_n] - f[x_0, \ldots, x_{n-1}]}{x_n - x_0}, & x_n \neq x_0, \\ \dfrac{f^{(n)}(x_0)}{n!}, & x_n = x_0. \end{cases}$$

To construct a polynomial $P(x)$ such that

$$P^{(j)}(x_k) = f^{(j)}(x_k), \qquad \text{for } j = 0, 1, \ldots, r_k - 1, \quad k = 0, 1, \ldots, n,$$

simply compute the Newton form of the polynomial interpolating f at the points:

$$\underbrace{x_0, \ldots, x_0}_{r_0 \text{ times}}, \underbrace{x_1, \ldots, x_1}_{r_1 \text{ times}}, \underbrace{x_2, \ldots, x_2}_{r_2 \text{ times}}, \ldots, \underbrace{x_n, \ldots, x_n}_{r_n \text{ times}}.$$

For a rigorous proof see Isaacson and Keller (1966).

Weierstrass Approximation Theorem. Let f be a continuous function on a closed, bounded interval [a,b]. Then $\forall \epsilon > 0 \; \exists$ a polynomial P such that $\max\limits_{a \le x \le b} |f(x) - P(x)| < \epsilon$.

Proof (convolution with an "approximate identity"). Assume that $[a, b] = [0, 1]$ and $f(0) = f(1) = 0$ (possible by adding a linear function to f). Extend f to the whole line by $f(x) = 0$ $\forall x \notin [0, 1]$. Consider the kernal $Q_n(t) = c_n(1 - x^2)^n$, $\int_{-1}^{1} Q_n(x)dx = 1$. Define

$$P_n(x) = \int_{-1}^{1} Q_n(t) f(x+t)dt = \int_{-x}^{1-x} Q_n(t) f(x+t)dt = \int_{0}^{1} Q_n(t-x) f(t)dt,$$

(note that $P_n(x) \approx f(x)$ since Q_n has all its weight at 0) which is a polynomial in x. Let $|f(t)| \le M$ on $[0, 1]$ and δ be such that $|f(x) - f(y)| < \epsilon$ for $|x - y| < 2\delta$. Observe that $\forall \epsilon, \delta > 0 \; \exists N : Q_N(t) \le \epsilon$ for $|t| \ge \delta$. Then

$$\begin{aligned} |P_N(x) - f(x)| = \left| \int_{-1}^{1} Q_N(t)[f(x+t) - f(x)]dt \right| &\le \left| \int_{-1}^{-\delta} \right| + \left| \int_{-\delta}^{\delta} \right| + \left| \int_{\delta}^{1} \right| \\ &\le 2M\epsilon + \epsilon + 2M\epsilon = (4M+1)\epsilon. \end{aligned}$$

Q. E. D.

Theorem (Weierstrass). Let f be continuous on $[-\pi, \pi]$, $f(-\pi) = f(\pi)$. Then $\forall \epsilon > 0 \; \exists$ a trigonometric polynomial $T_n(x) = A_0 + \sum\limits_{k=1}^{n} (A_k \cos kx + B_k \sin kx)$ such that $||T_n - f||_\infty < \epsilon$.

Definition. Let s_i be the ith partial sum of a sequence a_i. Then the nth Cesáro mean is

$$\sigma_n = \frac{s_0 + s_1 + \cdots + s_n}{n+1}.$$

Fejér's Theorem. If f is continuous and periodic with period 2π on $(-\infty, \infty)$, then the Cesáro means of the Fourier series of f coverage uniformly to f on $(-\infty, \infty)$.