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POSITIVITY OF CUBIC POLYNOMIALS ON INTERVALS AND POSITIVE SPLINE INTERPOLATION

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Abstract.

A criterion for the positivity of a cubic polynomial on a given interval is derived. By means of this result a necessary and sufficient condition is given under which cubic C^1-spline interpolants are nonnegative. Further, since such interpolants are not uniquely determined, for selecting one of them the geometric curvature is minimized. The arising optimization problem is solved numerically via dualization.

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1. Introduction.

This paper is mainly concerned with the positive interpolation by cubic C^1-splines. Let (x_0, y_0), $(x_1, y_1), \ldots, (x_n, y_n)$ be given points with nonnegative ordinates,

$$y_0 \geqq 0, \quad y_1 \geqq 0, \ldots, \quad y_n \geqq 0, \tag{1.1}$$

while the abscissas may define a partition Δ of an interval, say $[0, 1]$,

$$\Delta: x_0 = 0 < x_1 < \ldots < x_n = 1. \tag{1.2}$$

The problem of positive spline interpolation reads as follows: Are there spline interpolants s given on Δ which are nonnegative on $[0, 1]$, i.e.

$$s(x_i) = y_i \quad (i = 0, \ldots, n) \tag{1.3}$$

and

$$s(x) \geqq 0 \quad \text{for all } x \in [0, 1]? \tag{1.4}$$

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Besides convex and monotone interpolation this is a further type of shape preserving interpolation being also of practical interest.

In [7] the present authors dealt with the positive interpolation by quadratic splines. Now, cubic C^1-splines will be considered. It turns out that these splines make the problem of positive interpolation always solvable. This is a consequence of a necessary and sufficient criterion on the positivity of cubic spline interpolants which is presented here and which may be considered as a counterpart to the criteria of Neuman [4] and Fritsch/Carlson [3] on convexity and monotonicity, respectively.

For proving the criterion a general result on the positivity of cubic polynomials on a given interval is derived.

Since positive interpolants are not uniquely determined one of them is selected by minimizing the curvature. The arising optimization problem can be solved effectively via dualization since there is a dual program which is unconstrained. The used concept of dualization was elaborated by Burmeister/Heß/Schmidt [1] and Dietze/Schmidt [2] and was applied there with success in convex and in monotone spline interpolation.

2. Positivity of cubic polynomials on a given interval.

For a cubic polynomial p conditions will be derived under which

$$p(t) = at^3+bt^2+ct+d \geqq 0 \quad \text{for all } t \in [0, 1]. \tag{2.1}$$

By substituting $t = s/(1+s)$ it is easily seen that (2.1) is equivalent to

$$\varrho(s) = \alpha s^3+\beta s^2+\gamma s+\delta \geqq 0 \quad \text{for all } s \geqq 0, \tag{2.2}$$

if the coefficients are

$$\alpha = a+b+c+d, \quad \beta = b+2c+3d, \quad \gamma = c+3d, \quad \delta = d. \tag{2.3}$$

The necessary condition for positivity $p(0) \geqq 0$, $p(1) \geqq 0$ leads to

$$\alpha \geqq 0, \quad \delta \geqq 0. \tag{2.4}$$

Let us consider the case $\alpha > 0$, $\delta > 0$. Then, by means of the substitution

$$\beta = \alpha^{2/3}\delta^{1/3}\eta, \quad \gamma = \alpha^{1/3}\delta^{2/3}\xi, \quad \sigma = \alpha^{1/3}\delta^{-1/3}s \tag{2.5}$$

problem (2.2) is equivalently transformed into

$$\varphi(\sigma) = \sigma^3+\eta\sigma^2+\xi\sigma+1 \geqq 0 \quad \text{for all } \sigma \geqq 0. \tag{2.6}$$

By the substitution $\sigma = 1/\tau$ this problem is seen to be symmetric with respect to ξ and η.

Now, the minimizer of φ,

$$\sigma^* = \{-\eta+(\eta^2-3\xi)^{1/2}\}/3, \tag{2.7}$$

is of interest. Property (2.6) occurs only in one of the following three cases:

Case 1: σ^* is non-real, i.e.

$$\eta^2 < 3\xi; \tag{2.8}$$

Case 2: σ^* real, $\leqq 0$, i.e.

$$\xi \geqq 0,\ \eta \geqq 0,\ \eta^2 \geqq 3\xi; \tag{2.9}$$

Case 3: σ^* real, $\geqq 0$ and $\varphi(\sigma^*) \geqq 0$.

Inequality $\sigma^* \geqq 0$ occurs when

$$\xi \leqq 0, \tag{2.10}$$

or when

$$\xi \geqq 0, \quad \eta \leqq 0, \quad \eta^2 \geqq 3\xi, \tag{2.11}$$

while $\varphi(\sigma^*) \geqq 0$ means

$$2\eta^3-9\xi\eta+27-2(\eta^2-3\xi)^{3/2} \geqq 0.$$

This inequality is immediately seen to be equivalent to

$$f(\xi,\eta) = 4\xi^3+4\eta^3+27-18\xi\eta-\xi^2\eta^2 \geqq 0, \tag{2.12}$$

$$g(\xi,\eta) = 2\eta^3-9\xi\eta+27 \geqq 0. \tag{2.13}$$

For $\eta \leqq 0$ inequality (2.13) is satisfied if (2.12) holds. Indeed, because of

$$f(2\eta^2/9+3/\eta,\eta) = 4(27-\eta^3)^3/(9\eta)^3$$

there is no solution of $f(\xi,\eta) = 0$, $g(\xi,\eta) = 0$ with $\eta \leqq 0$, and moreover $f(0,\eta) = 0$, $g(0,\tilde{\eta}) = 0$ imply $0 > \eta > \tilde{\eta}$.

Further, there is no intersection point of $f(\xi,\eta) = 0$ with the parabola $\eta^2 = 3\xi$ if $\eta \leqq 0$. For $\xi \to +\infty$ the parabola $\eta^2 = 4\xi$ approximates $f(\xi,\eta) = 0$ asymptotically. Taking also into account the symmetry of problem (2.6) one gets

PROPOSITION 1: *The polynomial $\varphi = \varphi(\sigma)$ given by* (2.6) *is nonnegative for all* $\sigma \geqq 0$ *if and only if*

$$(\xi, \eta) \in D \cup E, \quad \text{where} \tag{2.14}$$

$$\begin{cases} D = \{(\xi, \eta) : \xi \geqq 0, \eta \geqq 0\}, \\ E = \{(\xi, \eta) : 4\xi^3 + 4\eta^3 + 27 - 18\xi\eta - \xi^2\eta^2 \geqq 0\}; \end{cases} \tag{2.15}$$

(see Figure 1.)

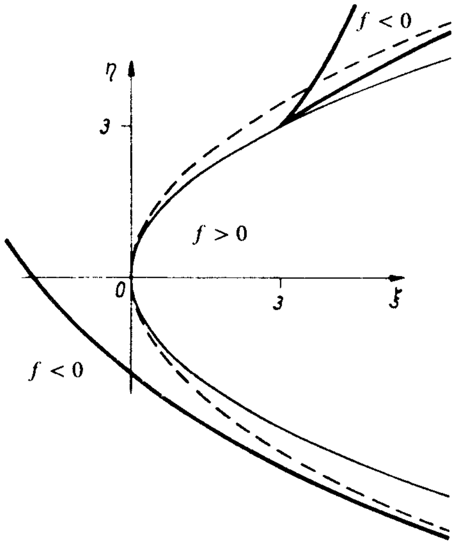

Fig. 1. Positivity region $D \cup E$. Boldfaced line: $f(\xi, \eta) = 0$; broken line: parabola $\eta^2 = 4\xi$; thin line: parabola $\eta^2 = 3\xi$.

In accordance with the convexity of problem (2.6) the set $D \cup E$ is convex.

The function $-f$ is known to be the discriminant of φ.

Now, by means of substitution (2.5) this statement is easily reformulated for the polynomial ϱ. Though in (2.5) it is assumed that $\alpha > 0$, $\delta > 0$ the following proposition even holds for $\alpha \geqq 0$, $\delta \geqq 0$, E.g. for $\alpha = 0$, $\delta > 0$ the polynomial ϱ, now quadratic, satisfies (2.2) only for $\beta \geqq 0$, $\gamma \geqq 0$ or for $4\beta\delta \geqq \gamma^2$ which is in accordance with

PROPOSITION 2: *The polynomial $\varrho = \varrho(s)$ defined by* (2.2) *is nonnegative for all* $s \geqq 0$ *if and only if*

$$(\alpha, \beta, \gamma, \delta) \in A \cup B, \quad \text{where} \tag{2.16}$$

$$\begin{cases} A = \{(\alpha, \beta, \gamma, \delta) : \alpha \geqq 0, \beta \geqq 0, \gamma \geqq 0, \delta \geqq 0\}, \\ B = \{(\alpha, \beta, \gamma, \delta) : \alpha \geqq 0,\ \delta \geqq 0,\ 4\alpha\gamma^3 + 4\delta\beta^3 + 27\alpha^2\delta^2 - 18\alpha\beta\gamma\delta - \beta^2\gamma^2 \geqq 0\}. \end{cases} \tag{2.17}$$

Indeed, the set $A \cup B$ is rather complicated. Thus there is an interest in more simple subsets. As easily seen the set of points $(\alpha, \beta, \gamma, \delta)$ with $\alpha \geqq 0$, $\delta \geqq 0$ and

$$\beta \geqq \alpha - 2\sqrt{(\alpha\delta)}, \quad \gamma \geqq \delta - 2\sqrt{(\alpha\delta)} \tag{2.18}$$

belongs to $A \cup B$.

3. Positivity of quadratic polynomials on a given interval.

Along the lines of the previous section positivity conditions for quadratic polynomials are easily derived. By using the substitution $t = s/(1+s)$ the condition

$$p(t) = at^2 + bt + c \geqq 0 \text{ for all } t \in [0, 1] \tag{3.1}$$

is equivalently transformed into

$$\varrho(s) = \beta s^2 + \gamma s + \delta \geqq 0 \quad \text{for all } s \geqq 0, \quad \text{where} \tag{3.2}$$

$$\beta = a + b + c, \quad \gamma = b + 2c, \quad \delta = c. \tag{3.3}$$

Now, proposition 2 says that (3.1) holds exactly for $\beta \geqq 0$, $\gamma \geqq 0$, $\delta \geqq 0$, or for $\delta \geqq 0$, $4\beta\delta \geqq \gamma^2$, i.e. for

$$\beta \geqq 0, \quad \delta \geqq 0, \quad \gamma \geqq -2\sqrt{(\beta\delta)}. \tag{3.4}$$

PROPOSITION 3: *The polynomial $\varrho = \varrho(s)$ introduced by* (3.2) *is nonnegative for all $s \geqq 0$ if and only if* (3.4) *is valid.*

4. Positive interpolation with cubic C^1-splines.

A cubic spline s defined on the grid Δ may be given as follows:

$$s(x) = y_{i-1} + m_{i-1}h_i t + (3\tau_i - 2m_{i-1} - m_i)h_i t^2 + (m_{i-1} + m_i - 2\tau_i)h_i t^3, \quad t \in [0, 1], \tag{4.1}$$

with $x = x_{i-1} + h_i t$, $i = 1, \dots, n$. Here τ_i denotes the slope

$$\tau_i = (y_i - y_{i-1})/h_i, \quad h_i = x_i - x_{i-1}. \tag{4.2}$$

Indeed, this spline s is continuously differentiable. Moreover, s satisfies the interpolation condition (1.3), and one gets

$$s'(x_i) = m_i, \quad i = 0, \ldots, n. \tag{4.3}$$

The quantities $m_0, m_1, \ldots, m_n$ now are used in order to satisfy the positivity condition (1.4) while the numbers $y_0 \geqq 0,\ y_1 \geqq 0, \ldots, y_n \geqq 0$ are given. For applying the results of sections 2, identify the right hand side of (4.1) with the polynomial p of (2.1). Then, via (2.3) it follows

$$\alpha = y_i, \quad \beta = 3y_i - m_i h_i, \quad \gamma = 3y_{i-1} + m_{i-1} h_i, \quad \delta = y_{i-1}. \tag{4.4}$$

Now, proposition 2 implies the following statement for positive interpolation.

THEOREM 4: *The cubic C^1-spline s defined by* (4.1) *is nonnegative on the interval* $[0, 1]$ *if and only if*

$$(m_{i-1}, m_i) \in W_i, \quad i = 1, \ldots, n, \quad \textit{where} \tag{4.5}$$

$$\begin{aligned} W_i = {} & \{(x, y) : h_i x \geqq -3y_{i-1}, h_i y \leqq 3y_i\} \cup \\ & \cup \{(x, y) : 36 y_{i-1} y_i (x^2 + xy + y^2 - 3\tau_i(x+y) + 3\tau_i^2) + \\ & + 3(y_i x - y_{i-1} y)(2h_i xy - 3y_i x + 3y_{i-1} y) + 4h_i(y_i x^3 - y_{i-1} y^3) - \\ & - h_i^2 x^2 y^2 \geqq 0\}. \end{aligned} \tag{4.6}$$

Since, e.g., $m_0 = m_1 = \ldots = m_n = 0$ satisfies (4.5) the problem of positive interpolation with cubic C^1-splines is always solvable. However, the solution is not unique.

If relation (2.18) is taken into account one is led to the following condition for the positivity of s which is only sufficient but simpler than (4.5):

$$(m_{i-1}, m_i) \in S_i, \qquad i = 1, \ldots, n, \qquad \text{where, with} \tag{4.7}$$

$$s_i = -2(y_{i-1} + \sqrt{(y_{i-1} y_i)})/h_i, \quad 2\tau_i - s_i = 2(y_i + \sqrt{(y_{i-1} y_i)})/h_i, \tag{4.8}$$

$$S_i = \{(x, y) : x \geqq s_i, y \leqq 2\tau_i - s_i\}. \tag{4.9}$$

Note that $S_i \subset W_i$ holds.

5. Determination of positive interpolants with minimal curvature.

As seen in the previous section there exists an infinite number of nonnegative cubic C^1-spline interpolants s. For selecting one of them here according to a

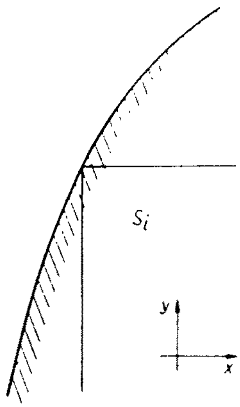

Fig. 2. Positivity region W_i and subset $S_i(y_i = 4,\ y_{i-1} = 1,\ h_i = 1)$.

recommmendation made in [1] the curvature

$$f_2(s) = \sum_{i=1}^{n} w_i \int_{x_{i-1}}^{x_i} s''(x)^2 dx \tag{5.1}$$

is used as objective function. The weights w_i may be, e.g., $w_i = 1$ or $w_i = 1/(1+\tau_i^2)^3$. The latter choice coincides with the geometric curvature if $s'(x)$ for $x \in [x_{i-1}, x_i]$ is approximated by τ_i. For the spline (4.1)

$$f_2(s) = \sum_{i=1}^{n} F_i(m_{i-1}, m_i) \quad \text{with} \tag{5.2}$$

$$F_i(x, y) = (4w_i/h_i)\{(x-\tau_i)^2 + (x-\tau_i)(y-\tau_i) + (y-\tau_i)^2\} \tag{5.3}$$

follows. Thus, one is led to a programming problem of the type:

$$\sum_{i=1}^{n} F_i(m_{i-1}, m_i) \to \min \text{ under the constraints} \tag{5.4}$$

$$(m_{i-1}, m_i) \in V_i, \quad i = 1, \ldots, n.$$

For solving this problem in [1], [2] it is proposed to consider the dual program

$$-\sum_{i=1}^{n} H_i^*(p_{i-1}, -p_i) \to \max \quad \text{with } p_0 = p_n = 0. \tag{5.5}$$

Here H_i^* denotes the Fenchel conjugate to F_i and V_i,

$$H_i^*(\xi, \eta) = \sup\{\xi x + \eta y - F_i(x, y) \colon (x, y) \in V_i\}. \tag{5.6}$$

In contrast to (5.4), the dual program (5.5) is unconstrained. Therefore, (5.5) can be treated numerically easier than (5.4). E.g., Newton's method applies to (5.5).

Both programs (5.4) and (5.5) are solvable. Further, if a solution $(p_0, p_1, \ldots, p_n)$ of (5.5) is known then the solution $(m_0, m_1, \ldots, m_n)$ of program (5.4) is explicitly given by means of the partial derivatives of H_i^*,

$$\text{(5.7)} \qquad \begin{cases} m_{i-1} = \partial_1 H_i^*(p_{i-1}, -p_i) \\ m_i = \partial_2 H_i^*(p_{i-1}, -p_i) \end{cases} \quad i = 1, \ldots, n.$$

Under assumptions which are now satisfied these duality statements, including the return-formula (5.7), are verified in [2] by applying Fenchel's theory.

It seems difficult to compute the Fenchel conjugates H_i^* explicitly for $V_i = W_i$ with W_i from (4.6). However, for sets V_i which are given by affine linear inequalities this is done straightforwardly. Thus, if one is willing to accept an approximately optimal spline, then e.g. the set $V_i = S_i$ defined by (4.9) may be used.

PROPOSITION 5: *The Fenchel conjugate H_i^* for F_i and S_i, with $\sigma_i = 12\tau_i w_i/h_i$ and $\varrho_i = (\tau_i - s_i)\sigma_i/\tau_i$, is equal to*

$$\text{(5.8)} \quad \frac{12w_i}{h_i} H_i^*(\xi, \eta)$$

$$\begin{aligned}
&= \sigma_i(\xi+\eta) + \xi^2 - \xi\eta + \eta^2 && \textit{for } \eta \leqq 2\xi + \varrho_i, \quad 2\eta \leqq \xi + \varrho_i \\
&= -\frac{\varrho_i^2}{4} + (\sigma_i + \varrho_i)\eta + \left(\sigma_i - \frac{\varrho_i}{2}\right)\xi + \tfrac{3}{4}\xi^2 && \textit{for } 2\eta \geqq \xi + \varrho_i, \quad \xi \geqq -\frac{\varrho_i}{3} \\
&= -\frac{\varrho_i^2}{4} + (\sigma_i - \varrho_i)\xi + \left(\sigma_i + \frac{\varrho_i}{2}\right)\eta + \tfrac{3}{4}\eta^2 && \textit{for } \eta \geqq 2\xi + \varrho_i, \quad \eta \leqq \frac{\varrho_i}{3} \\
&= -\frac{\varrho_i^2}{3} + (\sigma_i - \varrho_i)\xi + (\sigma_i + \varrho_i)\eta && \textit{for } \xi \leqq -\frac{\varrho_i}{3}, \quad \eta \geqq \frac{\varrho_i}{3}.
\end{aligned}$$

PROOF. By means of the Lagrangian to (5.6)

$$\Phi(x, y, \lambda, \mu) = -\xi x - \eta y + F_i(x, y) + \lambda(s_i - x) + \mu(y - 2\tau_i + s_i)$$

the Kuhn-Tucker conditions, here necessary and sufficient, read

$$\begin{aligned}
&\Phi_x = 0, && \Phi_y = 0 \\
&\Phi_\lambda \leqq 0, && \lambda \geqq 0, && \lambda\Phi_\lambda = 0 \\
&\Phi_\mu \leqq 0, && \mu \geqq 0, && \mu\Phi_\mu = 0.
\end{aligned}$$

The system $\Phi_x = 0$, $\Phi_y = 0$ has the solution

$$\bar{x} = \tau_i + (h_i/12w_i)(2\xi - \eta + 2\lambda + \mu)$$
$$\bar{y} = \tau_i + (\mathrm{h}_i/12w_i)(-\xi + 2\eta - \lambda - 2\mu).$$

The maximizers $(\bar{x}, \bar{y})$ of

$$d_i(x, y) = \xi x + \eta y - F_i(x, y)$$

which differ in the four cases $\lambda = \mu = 0$, $\lambda = \Phi_\mu = 0$, $\Phi_\lambda = \mu = 0$ and $\Phi_\lambda = \Phi_\mu = 0$, are easily determined, and then the optimal values $H_i^*(\xi, \eta) = d_i(\bar{x}, \bar{y})$ follow as given in (5.8).

For the set $V_i = W_i$ given by (4.6) we will indicate how to compute the first and second derivatives of the Fenchel conjugate H_i^* for fixed (ξ, η). Indeed, when applying Newton's method to (5.5) these derivatives are necessary.

First, compute the unconstrained maximizer of (5.6),

$$\begin{cases} \bar{x} = \bar{x}(\xi, \eta) = \tau_i + (h_i/12w_i)(2\xi - \eta), \\ \bar{y} = \bar{y}(\xi, \eta) = \tau_i + (h_i/12w_i)(-\xi + 2\eta), \end{cases} \tag{5.9}$$

and test if

$$(\bar{x}, \bar{y}) \in W_i. \tag{5.10}$$

If this relation holds the explicit formula

$$H_i^*(\xi, \eta) = \tau_i(\xi + \eta) + (h_i/12w_i)(\xi^2 - \xi\eta + \eta^2) \tag{5.11}$$

follows; compare with (5.8), first row. In the opposite case, i.e. for

$$(\bar{x}, \bar{y}) \notin W_i \tag{5.12}$$

the constraint

$$\begin{aligned} f_i(x, y) = {} & 36y_{i-1}y_i(x^2 + xy + y^2 - 3\tau_i(x+y) + 3\tau_i^2) + \\ & + 3(y_i x - y_{i-1} y)(2h_i xy - 3y_i x + 3y_{i-1} y) + \\ & + 4h_i(y_i x^3 - y_{i-1} y^3) - h_i^2 x^2 y^2 \geqq 0 \end{aligned}$$

becomes active. Now, with the Lagrange factor λ, the Kuhn-Tucker conditions to (5.6) read

$$\begin{cases} \xi - \partial_1 F_i(x, y) + \lambda \partial_1 f_i(x, y) = 0 \\ \eta - \partial_2 F_i(x, y) + \lambda \partial_2 f_i(x, y) 0 \\ f_i(x, y) = 0. \end{cases} \tag{5.13}$$

After eliminating λ this is a system of two nonlinear equations in x and y which may be solved numerically, e.g., by Newton's method. The solution

$$\tilde{x} = \tilde{x}(\xi, \eta), \quad \tilde{y} = \tilde{y}(\xi, \eta) \tag{5.14}$$

is the maximizer of (5.6). Moreover, the general theory yields

$$\partial_1 H_i^*(\xi, \eta) = \tilde{x}(\xi, \eta), \quad \partial_2 H_i^*(\xi, \eta) = \tilde{y}(\xi, \eta); \tag{5.15}$$

see [2]. Finally, differentiating the modified system (5.13) with respect to ξ and η, respectively, we get two systems of two linear equations with the unknowns

$$\partial_1^2 H_i^*(\xi, \eta) = \partial_1 \tilde{x}(\xi, \eta), \quad \partial_1 \partial_2 H_i^*(\xi, \eta) = \partial_1 \tilde{y}(\xi, \eta) \tag{5.16}$$

$$\partial_2 \partial_1 H_i^*(\xi, \eta) = \partial_2 \tilde{x}(\xi, \eta), \quad \partial_2^2 H_i^*(\xi, \eta) = \partial_2 \tilde{y}(\xi, \eta), \tag{5.17}$$

respectively. Thus, in the case (5.10) as well as in the case (5.12), the derivatives of H_i^* are available for fixed (ξ, η).

In order to demonstrate positive interpolation with cubic C^1-spline let the data set

x_i	0	1	2	3
y_i	2	0.07	4	7

be given. The positive interpolant with $m_0 = \ldots = m_n = 0$ is compared with the suboptimal ($V_i = S_i$) as well as with the optimal ($V_i = W_i$) spline, where in both cases we assume $w_i = 1/(1+\tau_i^2)^3$; see Figures 3–6. In this and other tested examples it turned out that at least suboptimal splines should be preferred. Indeed, the proposed optimization procedure generates more pleasant positive interpolants.

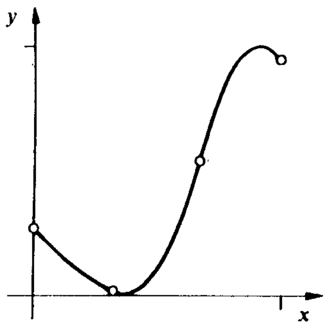

Fig. 3. Quadratic interpolant [7].

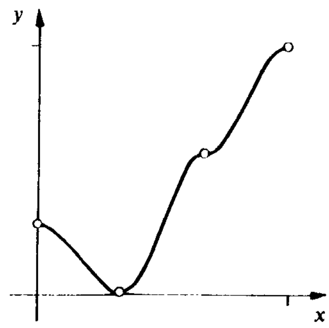

Fig. 4. Cubic interpolant with $m_0 = \ldots = m_4 = 0$.

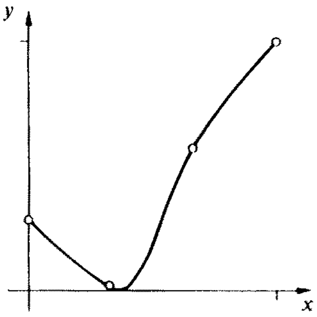

Fig. 5. Suboptimal cubic interpolant ($V_i = S_i$, $w_i = 1/(1+\tau_i^2)^3$; mean curvature = 0.0414).

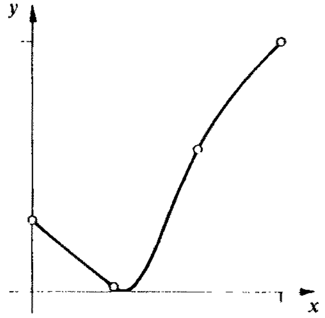

Fig. 6. Optimal cubic interpolant ($V_i = W_i$, $w_i = 1/(1+\tau_i^2)^3$; mean curvature = 0.0325).

6. Appendix I.

Here some known results will be shown to be immediate consequences of propositions 2 or 3.

6.1. Monotonicity of cubic C^1-splines.

The widely used criterion of Fritsch/Carlson [3] for the monotonicity of the spline (4.1) can be derived as follows. When starting from

$$s'(x) = m_{i-1} + 2(3\tau_i - 2m_{i-1} - m_i)t + 3(m_{i-1} + m_i - 2\tau_i)t^2 \geqq 0 \text{ for } t \in [0, 1]$$

the coefficients of the polynomial (3.2) are

$$\beta = m_{i-1}, \quad \gamma = 2(3\tau_i - m_{i-1} - m_i), \quad \delta = m_i,$$

and proposition 3 yields the condition

$$m_{i-1} \geqq 0, \quad m_i \geqq 0, \quad m_{i-1} - \sqrt{(m_{i-1}m_i)} + m_i \leqq 3\tau_i, \quad i = 1, \ldots, n. \tag{6.1}$$

This is the monotonicity criterion given in [3], though in an equivalent formulation.

6.2. Positivity of rational quadratic C^1-splines.

Let s be a rational quadratic spline on the grid Δ,

$$s(x) = y_{i-1} + \tau_i h_i t + (m_{i-1} - \tau_i) h_i t(1-t)/(1 + r_i t), \quad t \in [0, 1] \tag{6.2}$$

with $x = x_{i-1} + h_i t$ and parameters $r_1 \geqq 0, \ldots, \; r_n \geqq 0$. The positivity criterion for (6.2) given in [7] follows from proposition 3 by applying it to $(1 + r_i t)s(x)$. Indeed, the coefficients of polynomial (3.2) now are

$$\beta = (1 + r_i)y_i, \quad \gamma = m_{i-1}h_i + (2 + r_i)y_{i-1}, \quad \delta = y_{i-1}.$$

Thus, in view of (3.4), the spline (6.2) is nonnegative on [0, 1] if and only if

$$m_{i-1} \geqq -\{(2+r_i)y_{i-1}+2\sqrt{[(1+r_i)y_{i-1}y_i]}\}/h_i, \quad i = \lambda, \ldots, n. \tag{6.3}$$

These inequalities together with the condition of differentiability

$$m_{i-1}+(1+r_i)m_i = (2+r_i)\tau_i, \quad i = 1, \ldots, n-1,$$

can be satisfied for sufficiently large $r_1, \ldots, r_n$, if $y_1 > 0, \ldots, y_n > 0$; see [7].

6.3. Convexity of rational cubic C^1-splines.

Rational cubic C^1-splines s on the grid Δ may be defined by

$$s(x) = y_{i-1}+\tau_i h_i t+h_i t(1-t)\frac{(\tau_i-m_i)t+(m_{i-1}-\tau_i)(1-t)}{1+r_i t(1-t)}, \quad t \in [0, 1] \tag{6.4}$$

with $x = x_{i-1}+h_i t$, where $r_1 \geqq 0, \ldots, r_n \geqq 0$ are parameters. Using the relation

$$(1+r_i t(1-t))^3 h_i s''(x) = \varepsilon(t)(\tau_i-m_i)+\varepsilon(1-t)(m_{i-1}-\tau_i)$$

with $\varepsilon(t) = 2-6t-2r_i t^3$ the convexity of s reduces to the positivity of the right-hand side. The coefficients of the corresponding polynomial (2.2) are

$$\begin{aligned} \alpha &= 2\{m_{i-1}-\tau_i-(2+r_i)(\tau_i-m_i)\}, \\ \beta &= -6(m_{i-1}-\tau_i), \quad \gamma = -6(\tau_i-m_i), \\ \delta &= 2\{\tau_i-m_i-(2+r_i)(m_{i-1}-\tau_i)\}. \end{aligned}$$

Since $\alpha \geqq 0$, $\delta \geqq 0$ implies $\beta \geqq 0$, $\gamma \geqq 0$ it follows that $A \cup B = A$ for the sets A, B defined by (2.17). Thus, proposition 2 asserts that the spline (6.4) is convex on [0, 1] if and only if $\alpha \geqq 0$, $\delta \geqq 0$, i.e.

$$\begin{cases} (2+r_i)m_{i-1}+m_i \leqq (3+r_i)\tau_i \\ m_{i-1}+(2+r_i)m_i \geqq (3+r_i)\tau_i \end{cases} \quad i = \lambda, \ldots, n. \tag{6.5}$$

A first proof for (6.5) is given in [6]. Further, it is shown there that this condition holds for sufficiently large $r_1, \ldots, r_n$ if $\tau_1 < \tau_2 < \ldots < \tau_n$.

7. Appendix II.

Due to a suggestion of the referee proposition 1 can be verified also as follows. First notice that, in view of the Cardano formula, φ has three different real roots if and only if $f(\xi, \eta) < 0$, i.e. $(\xi, \eta) \notin E$.

Now let $(\xi, \eta) \in D \cup E$. For $(\xi, \eta) \in D$ (2.6) obviously holds. In the case $(\xi, \eta) \in E$ the polynomial φ has only one real root which, in view of $\varphi(0) = 1$ and $\varphi(-\infty) = -\infty$, is negative. Thus also in this case (2.6) follows.

On the other hand, (2.6) implies $(\xi, \eta) \in D \cup E$. Indeed, if φ has one real root then, with regard to the Cardano formula, $(\xi, \eta) \in E$ is valid. If φ has three different real roots they are all negative, and the Vieta formulas lead to $(\xi, \eta) \in D$.

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