Theorem Let $f : \mathbb{R} \to \mathbb{R}$ be a function with a γ-Lipschitz continuous first derivative and $x_0 < x_1 \in \mathbb{R}$. Let $f(x_0) = f(x_1) = 0$, while z is a real number $x_0 < z < x_1$ such that $f'(z) = 0$. Then

$$\left|f'(x_0) - f'(z)\right| \leq \gamma \frac{x_1 - x_0}{2}.$$

Proof. By definition of Lipschitz continuity, $\left|f'(x_0) - f'(z)\right| \leq \gamma(z - x_0)$. It must also be true that

$$\int_{x_0}^{z} f'(t)\ dt = -\int_{z}^{x_1} f'(t)\ dt. \tag{1}$$

Assume by way of contradiction that $\left|f'(x_0) - f'(z)\right| > \gamma(x_1 - x_0)/2$. Define $x_{1/2} = (x_0 + x_1)/2$. Knowing γt is an upper bound for the rate of change of $f'(t)$, it can be concluded that $z > x_{1/2}$ from

$$\left|\int_{x_0}^{z} f'(t)\right| > \int_{x_0}^{x_{1/2}} \gamma t\ dt. \tag{2}$$

Now it must be conversely true that $x_1 - z < x_{1/2} - x_0$ and hence

$$\left|\int_{z}^{x_1} f'(t)\right| < \int_{x_0}^{x_{1/2}} \gamma t\ dt. \tag{3}$$

However, (2) and (3) together contradict (1). Therefore

$$|f'(x_0) - f'(z)| \leq \gamma \frac{x_1 - x_0}{2}. \tag{4}$$

$\square$