Bounding Linear Interpolant Error

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Lemma 1

Let $S \subset \mathbb{R}^d$ be open and convex, $f : \mathbb{R}^d \to \mathbb{R}$, and $\nabla f \in Lip_{(\gamma, \|\cdot\|_2)}(S)$, the set of γ-Lipschitz continuous functions in the 2-norm. Then for all $x, y \in S$

$$\left| f(y) - f(x) - \langle \nabla f(x), y - x \rangle \right| \leq \frac{\gamma \|y - x\|_2^2}{2}.$$

Proof

Consider the function $g(t) = f\big((1-t)x + ty\big), 0 \leq t \leq 1$, whose derivative[1] $g'(t) = \big\langle \nabla f\big((1-t)x + ty\big), y - x \big\rangle$ is the directional derivative of f in the direction $(y - x)$.

$$\begin{aligned}
&\left| f(y) - f(x) - \langle \nabla f(x), y - x \rangle \right| \\
&\qquad = \left| g(1) - g(0) - g'(0) \right| \\
&\qquad = \left| \int_0^1 g'(t) - g'(0)\, dt \right| \\
&\qquad \leq \int_0^1 \left| g'(t) - g'(0) \right| dt \\
&\qquad = \int_0^1 \left| \Big\langle \nabla f\big((1-t)x + ty\big) - \nabla f(x), y - x \Big\rangle \right| dt \\
&\qquad \leq \int_0^1 \left\| \nabla f\big((1-t)x + ty\big) - \nabla f(x) \right\|_2 \|y - x\|_2\, dt \\
&\qquad \leq \int_0^1 \big(\gamma\, \|y - x\|_2\big) \big(\|y - x\|_2\big) t\, dt \\
&\qquad = \frac{\gamma \|y - x\|_2^2}{2}.
\end{aligned}$$

□

Lemma 2

Let $x, y, v_i \in \mathbb{R}^d$, $c_i \in \mathbb{R}$, and $|\langle y - x, v_i \rangle| \le c_i$ for $i = 1, \ldots, d$. If $M = (v_1, \ldots, v_d)$ is nonsingular, then

$$\|y - x\|_2^2 \le \frac{1}{\sigma_d^2} \sum_{i=1}^{d} c_i^2,$$

where σ_d is the smallest singular value of M.

Proof

Using the facts that M and M^t have the same singular values, and $\|M^t w\|_2 \ge \sigma_d \|w\|_2$, gives

$$\begin{aligned} \|y - x\|_2^2 &\le \frac{\|M^t(y - x)\|_2^2}{\sigma_d^2} \\ &= \frac{1}{\sigma_d^2} \sum_{i=1}^{d} \langle y - x, v_i \rangle^2 \\ &\le \frac{1}{\sigma_d^2} \sum_{i=1}^{d} c_i^2. \end{aligned}$$

□

Lemma 3

Given f, γ, S as in *Lemma 1*, let $X = \{x_0, x_1, \ldots, x_d\} \subset S$ be the vertices of a d-simplex, and let $\hat{f}(x) = \langle c, x - x_0 \rangle + f(x_0)$, $c \in \mathbb{R}^d$ be the linear function interpolating f on X.

Let σ_d be the smallest singular value of the matrix $M = (x_1 - x_0, \ldots, x_d - x_0)$, and $k = \max_{1 \le j \le d} \|x_j - x_0\|_2$. Then

$$\left\|\nabla f(x_0) - c\right\|_2 \le \frac{\gamma k^2 \sqrt{d}}{\sigma_d}.$$

Proof

Consider $f(x) - \hat{f}(x)$ along the line segment $z(t) = (1 - t)x_0 + tx_j$, $0 \le t \le 1$. By Rolle's Theorem, for some $0 < \hat{t} < 1$, $\left\langle \nabla f\left(z(\hat{t}\,)\right) - c, x_j - x_0 \right\rangle = 0$. Now

$$\begin{aligned}
&\left|\left\langle \nabla f(x_0) - c, x_j - x_0 \right\rangle\right| \\
&\qquad = \left|\left\langle \nabla f(x_0) - \nabla f\left(z(\hat{t}\,)\right) + \nabla f\left(z(\hat{t}\,)\right) - c, x_j - x_0 \right\rangle\right| \\
&\qquad = \left|\left\langle \nabla f(x_0) - \nabla f\left(z(\hat{t}\,)\right), x_j - x_0 \right\rangle\right| \\
&\qquad \le \left\|\nabla f(x_0) - \nabla f\left(z(\hat{t}\,)\right)\right\|_2 \|x_j - x_0\|_2 \\
&\qquad \le \gamma \|x_0 - z(\hat{t}\,)\|_2 \, \|x_j - x_0\|_2 \\
&\qquad \le \gamma \|x_j - x_0\|_2^2 \le \gamma k^2,
\end{aligned}$$

for all $1 \le j \le d$. Using *Lemma 2*, we have

$$\begin{aligned}
&\left\|\nabla f(x_i) - c\right\|_2^2 \le d\left(\frac{\gamma k^2}{\sigma_d}\right)^2 \\
\implies &\left\|\nabla f(x_i) - c\right\|_2 \le \frac{\gamma k^2 \sqrt{d}}{\sigma_d}.
\end{aligned}$$

□

Theorem

Under the assumptions of *Lemma 1* and *Lemma 3*, for $z \in S$,

$$\left| f(z) - \hat{f}(z) \right| \leq \frac{\gamma \|x_0 - z\|_2^2}{2} + \frac{\gamma k^2 \sqrt{d}}{\sigma_d} \|x_0 - z\|_2.$$

Proof

Consider the error bound construction in *Lemma 1* and let $\nabla f(x_0) - c = v$, $v \in \mathbb{R}^d$, where *Lemma 3* shows $\|v\|_2 \leq \frac{\gamma k^2 \sqrt{d}}{\sigma_d}$.

$$\begin{aligned}
\left| f(z) - \hat{f}(z) \right| &= \left| f(z) - f(x_0) - \langle c, z - x_0 \rangle \right| \\
&= \left| f(z) - f(x_0) - \langle \nabla f(x_0) - v, z - x_0 \rangle \right| \\
&= \left| f(z) - f(x_0) - \langle \nabla f(x_0), z - x_0 \rangle + \langle v, z - x_0 \rangle \right| \\
&\leq \left| f(z) - f(x_0) - \langle \nabla f(x_0), z - x_0 \rangle \right| + \left| \langle v, z - x_0 \rangle \right| \\
&\leq \left| f(z) - f(x_0) - \langle \nabla f(x_0), z - x_0 \rangle \right| + \|v\|_2 \, \|z - x_0\|_2 \\
&\leq \left| f(z) - f(x_0) - \langle \nabla f(x_0), z - x_0 \rangle \right| + \tfrac{\gamma k^2 \sqrt{d}}{\sigma_d} \|z - x_0\|_2 \\
&\;\;\vdots \\
&= \frac{\gamma \|z - x_0\|_2^2}{2} + \frac{\gamma k^2 \sqrt{d}}{\sigma_d} \|z - x_0\|_2.
\end{aligned}$$

□

Footnotes

1. $$\begin{aligned} g'(t) &= \lim_{\Delta t \to 0} \frac{g(t+\Delta t) - g(t)}{\Delta t} \\ &= \lim_{\Delta t \to 0} \frac{f\Big(\big[1-(t+\Delta t)\big]x + (t+\Delta t)y\Big) - f\Big((1-t)x + ty\Big)}{\Delta t} \\ &= \lim_{\Delta t \to 0} \frac{f\Big(z + \Delta t(y-x)\Big) - f(z)}{\Delta t} \qquad \text{for } z = (1-t)x + ty \\ &= \lim_{\|\Delta v\| \to 0} \frac{f\Big(z + \langle \Delta v, y-x \rangle (y-x)\Big) - f(z)}{\langle \Delta v, y-x \rangle} \qquad \text{for } \Delta v \in \mathbb{R}^d \\ &= \Big\langle \nabla f(z), y-x \Big\rangle \end{aligned}$$