

## SHAPE PRESERVING SPLINE INTERPOLATION

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### 1. INTRODUCTION

The spline-under-tension, developed by Schweikert<sup>14</sup> and Cline<sup>2</sup>, introduces a parameter which gives some control on the shape of the spline curve. The tension spline involves the use of hyperbolic functions and can be considered within a general setting proposed by Pruess<sup>13</sup>, where other alternatives are discussed. In particular a rational spline due to Späth<sup>15</sup> is considered.

For parametric representations, Nielson<sup>11</sup> describes a polynomial alternative to the spline-under-tension. Here use is made of the additional freedom given by relaxing  $C^2$  parametric continuity to that of 'geometric' or 'visual'  $C^2$  continuity. This idea has more recently been taken up by Barsky and Beatty<sup>1</sup>.

The specific problem of shape preserving interpolation has been considered by a number of authors. Fritsch and Carlson<sup>6</sup> and Fritsch and Butland<sup>5</sup> discuss the interpolation of monotonic data using  $C^1$  piecewise cubic polynomials. McAllister, Passow and Roulier<sup>9</sup> and Passow and Roulier<sup>12</sup> consider the problem of interpolating monotonic and convex data. They make use of piecewise polynomial Bernstein-Bezier representations and introduce additional knots into their schemes. In particular McAllister and Roulier<sup>10</sup> describe an algorithm for quadratic spline interpolation.

In this paper we will discuss a rational spline solution to the problem of shape preserving interpolation based on references 3, 4, 7 and 8. The rational spline is represented in terms of first derivative values at the knots and provides an alternative to the spline-under-tension. The idea of making the shape control parameters dependent on the first derivative unknowns is then explored. We are then able to preserve the monotonic or convex shape of the interpolation data automatically through the solution of the resulting non-linear consistency equations of the spline.

## 2. A RATIONAL ALTERNATIVE TO THE SPLINE-UNDER-TENSION

Let  $(x_i, f_i), i = 1, \dots, n$ , be a given set of real data, where  $x_1 < x_2 < \dots < x_n$  and let  $d_i, i = 1, \dots, n$ , denote first derivative values defined at the knots  $x_i, i = 1, \dots, n$ . A function  $s \in C^1[x, x_n]$  such that

$$(2.1) \quad s(x_i) = f_i \quad \text{and} \quad s^{(1)}(x_i) = d_i, \quad i = 1, \dots, n$$

is piecewise defined for  $x \in [x_i, x_{i+1}]$ ,  $i = 1, \dots, n-1$  by  $s(x) = s_i(x; r_i)$ , where

$$(2.2) \quad s_i(x; r_i) = \frac{(1-\theta)^2(1-\theta+r_i\theta)f_i + (1-\theta)^2\theta h_i d_i - \theta^2(1-\theta)h_i d_{i+1} + \theta^2(\theta+r_i(1-\theta))f_{i+1}}{1 + (r_i-3)\theta(1-\theta)}$$

and

$$(2.3) \quad \theta = (x-x_i)/h_i, \quad h_i = x_{i+1} - x_i, \quad r_i > -1.$$

The parameters  $r_i$  will be used to control the shape of the curve  $s$ . The case  $r_i = 3$  is that of piecewise cubic Hermite interpolation.

The  $C^2$  spline constraints

$$(2.4) \quad s^{(2)}(x_{i-}) = s^{(2)}(x_{i+}), \quad i = 2, \dots, n-1,$$

give the consistency equations

$$(2.5) \quad h_i d_{i-1} + [h_i(r_{i-1}-1) + h_{i-1}(r_i-1)]d_i + h_{i-1}d_{i+1} \\ = h_i r_{i-1} \Delta_{i-1} + h_{i-1} r_i \Delta_i, \quad i = 2, \dots, n-1,$$

where

$$(2.6) \quad \Delta_i = (f_{i+1} - f_i)/h_i, \quad i = 1, \dots, n-1$$

and we assume that  $d_1$  and  $d_n$  are given as end conditions. The case  $r_i = 3$  corresponds to that of cubic spline interpolation.

Assume that

$$(2.7) \quad r_i \geq r > 2, \quad i = 1, \dots, n-1.$$

Then the linear system (2.5) is strictly diagonally dominant and hence has a unique solution. The solution is also bounded with respect to the  $r_i$  since

$$(2.8) \quad \max |d_i| \leq [r/(r-2)] \max |\Delta_i| .$$

The rational cubic (2.2) can be written as

$$(2.9) \quad s_i(x; r_i) = \ell_i(x) + e_i(x; r_i)$$

where  $\ell_i$  is the linear interpolant

$$(2.10) \quad \ell_i(x) = (1-\theta)f_i + \theta f_{i+1}$$

and

$$(2.11) \quad e_i(x; r_i) = \frac{\theta(1-\theta)[(2\theta-1)(f_{i+1}-f_i) + (1-\theta)h_i d_i - \theta h_i d_{i+1}]}{1 + (r_i - 3)\theta(1-\theta)}$$

Thus, as the parameters  $r_i$  are increased, it can be shown that the rational spline  $s$  converges uniformly to a piecewise defined linear interpolant.

An identical argument applies to the rational spline representation of parametric curves.

Suppose  $f_i = f(x_i)$  and  $f_{i+1} = f(x_{i+1})$ , where  $f \in C^4[x_i, x_{i+1}]$ . Then an error bound for the rational cubic on  $[x_i, x_{i+1}]$  is given by

$$(2.12) \quad |f(x) - s_i(x; r_i)| \leq \frac{h_i}{4c_i} \max \left\{ |f_i^{(1)} - d_i|, |f_{i+1}^{(1)} - d_{i+1}| \right\}$$

$$+ \frac{1}{384c_i} \left[ h_i^4 \| f^{(4)} \| + |r_i - 3| \left\{ \frac{1}{4} h_i^4 \| f^{(4)} \| + 4h_i^3 \| f^{(3)} \| + 12h_i^2 \| f^{(2)} \| \right\} \right] ,$$

where

$$c_i = \min\{1, (1+r_i)/4\} \quad \text{and} \quad \|f\| = \max_{x \in [x_i, x_{i+1}]} |f(x)| .$$

This result will influence the choice of the parameters  $r_i$  when the interpolation data are monotonic or convex. In particular we wish to choose  $r_i$  such that  $r_i - 3 = O(h_i^2)$ , whilst maintaining monotonicity or convexity of the interpolant.

### 3. THE INTERPOLATION OF MONOTONIC DATA

Suppose that

$$(3.1) \quad f_1 < f_2 < \dots < f_n \quad (\text{monotonic increasing data})$$

and assume the derivative values  $d_i$  satisfy the necessary monotonicity conditions

$$(3.2) \quad d_i > 0 , \quad i = 2, \dots, n-1 ,$$

where  $d_1 > 0$  and  $d_n > 0$  are given. Then

$$(3.3) \quad r_i = 1 + (d_i + d_{i+1})/\Delta_i$$

ensures that  $s_i^{(1)}(x; r_i) > 0$  on  $[x_i, x_{i+1}]$  and hence the rational cubic is monotonic increasing.

Substituting (3.3) into the  $C^2$  consistency equations (2.5) gives the non-linear system

$$(3.4) \quad d_i[-c_i + a_{i-1}d_{i-1} + (a_{i-1} + a_i)d_i + a_i d_{i+1}] = b_i , \quad i = 2, \dots, n-1 ,$$

where

$$(3.5) \quad a_i = 1/(h_i \Delta_i) , \quad b_i = \Delta_{i-1}/h_{i-1} + \Delta_i/h_i , \quad c_i = 1/h_{i-1} + 1/h_i .$$

This system has a unique solution satisfying the monotonicity conditions (3.2). The "Gauss-Seidel" iteration

$$(3.6) \quad d_i^{(k+1)} = G_i(d_{i-1}^{(k+1)}, d_{i+1}^{(k)}) , \quad i = 2, \dots, n-1 ,$$

where

$$(3.7) \quad G_i(\xi, \eta) = \frac{1}{2(a_{i-1} + a_i)} \{ \xi + [\xi^2 + 4(a_{i-1} + a_i)b_i]^{\frac{1}{2}} \} ,$$

$$(3.8) \quad \xi = c_i - a_{i-1}\xi - a_i\eta ,$$

provides a robust algorithm for solving (3.4), being globally convergent to the required positive solution. The method also provides an  $O(h^4)$  accurate solution with reference to the error bound (2.12).

#### 4. THE INTERPOLATION OF CONVEX DATA

Let

$$(4.1) \quad \Delta_1 < \Delta_2 < \dots < \Delta_{n-1} \quad (\text{convex data}) ,$$

and assume the derivative values satisfy the necessary convexity conditions

$$(4.2) \quad \Delta_{i-1} < d_i < \Delta_i , \quad i = 2, \dots, n-1 ,$$

where  $d_1 < \Delta_1$  and  $d_n > \Delta_{n-1}$  are given. Then

$$(4.3) \quad r_i = 1 + (d_{i+1} - \Delta_i)/(\Delta_i - d_i) + (\Delta_i - d_i)/(d_{i+1} - \Delta_i)$$

ensures that  $s_i^{(2)}(x; r_i) > 0$  on  $[x_i, x_{i+1}]$  and hence the rational cubic is convex.

The  $C^2$  consistency equations are now

$$(4.4) \quad \left( \frac{\Delta_i - d_i}{d_i - \Delta_{i-1}} \right)^2 = \frac{h_i}{h_{i-1}} \left( \frac{d_{i+1} - \Delta_i}{\Delta_{i-1} - d_{i-1}} \right) , \quad i = 2, \dots, n-1 ,$$

giving a non-linear system with a unique solution satisfying the convexity conditions (4.2). This solution can be found using a "Gauss-Seidel" iteration as in (3.6), where we now take

$$(4.5) \quad G_i(\xi, \eta) = \frac{h_{i-1}^{\frac{1}{2}} (\Delta_{i-1} - \xi)^{\frac{1}{2}} \Delta_i + h_i^{\frac{1}{2}} (\eta - \Delta_i)^{\frac{1}{2}} \Delta_{i-1}}{h_{i-1}^{\frac{1}{2}} (\Delta_{i-1} - \xi)^{\frac{1}{2}} + h_i^{\frac{1}{2}} (\eta - \Delta_i)^{\frac{1}{2}}}$$

and choose initial values  $d_i^{(0)}$ ,  $i = 2, \dots, n-1$ , for the iteration such that the residuals alternate. Finally, the convex spline method like the monotonic spline is  $O(h^4)$  accurate.

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