ODEs

Given a vector field $F: \mathbb{R}^n \to \mathbb{R}^n$, a system of <u>ordinary differential equations</u> is a vector equation of the form

$$(*) \qquad x'(t) = F(x(t)) \qquad (\text{often written } \dot{x} = F(x))$$

A function $\phi: \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is called the <u>flow</u> of the system if it satisfies

$$\frac{\partial}{\partial t}(\phi(x_0,t)) = F(\phi(x_0,t)), \quad \phi(x_0,0) = x_0$$

for all $x_0 \in \mathbb{R}^n$, $t \in \mathbb{R}$.

<u>Intuition</u> $\phi(x_0,t)$ describes a solution of the system $(*)$; in other words it describes the state of the system governed by $(*)$ at time t with the initial condition of being in state x_0 at time 0.

<u>Example</u> (1D linear, constant coefficients)

Let $a \in \mathbb{R}$ be a given constant and consider the equation

$$\begin{cases} \dot{x} = ax & (\text{so } F(x) = ax) \\ x(0) = x_0. \end{cases}$$

The flow is $\phi(x_0, t) = e^{at} x_0$ since

$$\frac{\partial}{\partial t}(\phi(x_0,t)) = \frac{\partial}{\partial t}(e^{at} x_0) = a e^{at} x_0 = a\phi(x_0,t),$$

$$\phi(x_0, 0) = e^{a(0)} x_0 = x_0.$$

<u>Example</u> (n-dimensional system, constant coefficients)

Let A be a given $n \times n$ matrix and consider the system $\begin{cases} \dot{x} = Ax \\ x(0) = x_0 \end{cases}$

The flow here is $\phi(x_0, t) = e^{tA} x_0$ where

$$e^{tA} = I + tA + \frac{t^2}{2} A^2 + \frac{t^3}{3!} A^3 + \cdots$$

Stochastic flow-kick system

Let $\phi(x,t)$ denote the flow of a given ODE system $\dot{x} = F(x)$. Let $(T_n)_{n\geq 0}$ be a given random sequence such that

$$0 = T_0 < T_1 < T_2 < T_3 < \dots$$

and let $(K_n)_{n\geq 1} \subseteq \mathbb{R}^n$ be an i.i.d. sequence. These are called the kick times and kicks respectively. Define for all $n \geq 1$,

$$\tau_n = T_n - T_{n-1}.$$

These are called the flow times. Given $x_0 \in \mathbb{R}^n$ let

$$X_t = \begin{cases} x_0 & t = 0 \\ \phi(X_{T_{n-1}}, t - T_{n-1}) & T_{n-1} < t < T_n \\ K_n + \phi(X_{T_{n-1}}, t) & t = T_n \end{cases}$$

for all $n \geq 1$, $t \in [0,\infty)$. This is called the stochastic flow-kick system.

Given $A \subseteq \mathbb{R}^n$, $x_0 \in \mathbb{R}^n$, the <u>first hitting time</u> of A is the random variable

$$T_{x_0}^{A} = \inf\{ t > 0 : X_t \in A \}.$$

Notice $P(T_{x_0}^{A} > t) = P(X_s \notin A \text{ for all } 0 \leq s \leq t)$.