Solutions to Exercises of Chapter 3

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Problem 3.1 (Exactness of p_{FRT})

Problem 3.2 (Monte Carlo error of $\hat{p}_{\text{FRT}}$)

Problem 3.3 (A finite-sample valid Monte Carlo approximation of p_{FRT})

Problem 3.4 (Fisher's exact test)

Problem 3.5 (More details for lady tasting tea)

Problem 3.6 (Covariate-adjusted FRT)

Problem 3.8 (An algebraic detail)

We are asked to show that

$$(n-1)s^2 = \sum_{Z_i=1}\left(Y_i - \hat{\bar{Y}}(1)\right)^2 + \sum_{Z_i=0}\left(Y_i - \hat{\bar{Y}}(0)\right)^2 + \frac{n_1 n_0}{n}\hat{\tau}^2. \tag{3.7}$$

First, notice that

$$\bar{Y} \equiv \frac{1}{n}\sum_{i=1}^{n} Y_i = \frac{1}{n}\left(\sum_{Z_i=1} Y_i + \sum_{Z_i=0} Y_i\right) = \frac{1}{n}\left(n_1\hat{\bar{Y}}(1) + n_0\hat{\bar{Y}}(0)\right),$$

which implies

$$\bar{Y} = \hat{\bar{Y}}(1) - \frac{n_0}{n}\underbrace{\left(\hat{\bar{Y}}(1) - \hat{\bar{Y}}(0)\right)}_{\hat{\tau}} = \hat{\bar{Y}}(0) + \frac{n_1}{n}\underbrace{\left(\hat{\bar{Y}}(1) - \hat{\bar{Y}}(0)\right)}_{\hat{\tau}}. \tag{1}$$

Therefore, we have

$$\begin{aligned}
&(n-1)s^2 \\
\equiv& \sum_{i=1}^{n}\left(Y_i - \bar{Y}\right)^2 && \text{(Definition)} \\
=& \sum_{Z_i=1}\left(Y_i - \bar{Y}\right)^2 + \sum_{Z_i=0}\left(Y_i - \bar{Y}\right)^2 && \text{(Partition by } Z_i\text{)} \\
=& \sum_{Z_i=1}\left(Y_i - \hat{\bar{Y}}(1) + \hat{\bar{Y}}(1) - \bar{Y}\right)^2 + \sum_{Z_i=0}\left(Y_i - \hat{\bar{Y}}(0) + \hat{\bar{Y}}(0) - \bar{Y}\right)^2 && \text{(Add and subtract)} \\
=& \sum_{Z_i=1}\left(Y_i - \hat{\bar{Y}}(1) + \frac{n_0}{n}\hat{\tau}\right)^2 + \sum_{Z_i=0}\left(Y_i - \hat{\bar{Y}}(0) - \frac{n_1}{n}\hat{\tau}\right)^2 && \text{(Use Equation (1))}
\end{aligned}$$

$$= \sum_{Z_i=1} \left(Y_i - \hat{\hat{Y}}(1)\right)^2 + \frac{n_0^2 n_1}{n^2}\hat{\tau}^2 + \sum_{Z_i=0} \left(Y_i - \hat{\hat{Y}}(0)\right)^2 + \frac{n_1^2 n_0}{n^2}\hat{\tau}^2 \qquad (\sum_{Z_i=z}(Y_i - \hat{\hat{Y}}(z)) = 0)$$

$$= \sum_{Z_i=1} \left(Y_i - \hat{\hat{Y}}(1)\right)^2 + \sum_{Z_i=0} \left(Y_i - \hat{\hat{Y}}(0)\right)^2 + \frac{n_1 n_0}{n}\hat{\tau}^2, \qquad \text{(Combine terms)}$$

which completes the proof.