Solutions to Exercises of Chapter 5

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Problem 5.1 (Covariate balance in the CRE)

Consider a completely randomized experiment (CRE) with n units, among which n_1 are assigned to treatment ($Z_i = 1$) and n_0 to control ($Z_i = 0$). Let $X_i \in \{1, \dots, K\}$ be the stratum membership of unit i, where K is the number of strata. Let $n_{[k]} \equiv \sum_{i=1}^{n} \mathbf{1}(X_i = k)$ be the number of units in stratum k, $\pi_{[k]} \equiv n_{[k]}/n$ be the proportion of units in stratum k, and $n_{[k]z} \equiv \sum_{i=1}^{n} \mathbf{1}(X_i = k, Z_i = z)$ be the number of units in stratum k assigned to treatment z.

We are asked to show that

$$\mathrm{E}\left(\frac{n_{[k]1}}{n_1} - \frac{n_{[k]0}}{n_0}\right) = 0. \tag{5.2}$$

This can directly be shown by the following calculation:

$$\begin{aligned}
&\mathrm{E}\left(\frac{n_{[k]1}}{n_1} - \frac{n_{[k]0}}{n_0}\right) \\
&= \mathrm{E}\left(\frac{n_{[k]1}}{n_1}\right) - \mathrm{E}\left(\frac{n_{[k]0}}{n_0}\right) && \text{(Linearity of } \mathrm{E}(\cdot)) \\
&= \frac{1}{n_1}\mathrm{E}\left(\sum_{i=1}^{n} Z_i \mathbf{1}(X_i = k)\right) - \frac{1}{n_0}\mathrm{E}\left(\sum_{i=1}^{n}(1 - Z_i)\mathbf{1}(X_i = k)\right) && \text{(Definition of } n_{[k]z}) \\
&= \frac{1}{n_1}\sum_{i=1}^{n} \mathbf{1}(X_i = k)\,\mathrm{E}(Z_i) - \frac{1}{n_0}\sum_{i=1}^{n} \mathbf{1}(X_i = k)\,\mathrm{E}(1 - Z_i) && \text{(Linearity of } \mathrm{E}(\cdot)) \\
&= \frac{1}{n_1}\sum_{i=1}^{n} \mathbf{1}(X_i = k)\frac{n_1}{n} - \frac{1}{n_0}\sum_{i=1}^{n} \mathbf{1}(X_i = k)\frac{n_0}{n} && (\mathrm{E}(Z_i) = n_1/n \text{ in the CRE}) \\
&= \frac{1}{n}\sum_{i=1}^{n} \mathbf{1}(X_i = k) - \frac{1}{n}\sum_{i=1}^{n} \mathbf{1}(X_i = k) && \text{(Simplification)} \\
&= 0.
\end{aligned}$$

Problem 5.2 (Consequence of the constant propensity score)

Following the notation in Problem 5.1, let $e_{[k]} \equiv n_{[k]1}/n_{[k]}$ be the propensity score of stratum k. Define

$$\begin{aligned}
\hat{\tau} &\equiv \frac{1}{n_1}\sum_{i=1}^{n} Z_i Y_i - \frac{1}{n_0}\sum_{i=1}^{n}(1 - Z_i) Y_i, \\
\hat{\tau}_{\mathrm{S}} &\equiv \sum_{k=1}^{K} \pi_{[k]} \hat{\tau}_{[k]},
\end{aligned}$$

where

$$\hat{\tau}_{[k]} \equiv \frac{1}{n_{[k]1}} \sum_{i:X_i=k} Z_i Y_i - \frac{1}{n_{[k]0}} \sum_{i:X_i=k} (1 - Z_i) Y_i.$$

We are asked to show that when the propensity score is constant across strata,

$$\hat{\tau} = \hat{\tau}_{\mathrm{S}}. \tag{5.3}$$

Notice that if the propensity score is constant across strata, then $e_{[k]} = n_{[k]1}/n_{[k]} = n_1/n$ for all $k = 1, \ldots, K$. Therefore, we have

$$\begin{aligned}
&\hat{\tau}_{\mathrm{S}} \\
&\equiv \sum_{k=1}^{K} \pi_{[k]} \hat{\tau}_{[k]} && \text{(Definition of } \hat{\tau}_{\mathrm{S}}\text{)} \\
&= \sum_{k=1}^{K} \frac{n_{[k]}}{n} \left(\frac{1}{n_{[k]1}} \sum_{i:X_i=k} Z_i Y_i - \frac{1}{n_{[k]0}} \sum_{i:X_i=k} (1 - Z_i) Y_i \right) && \text{(Definition of } \pi_{[k]} \text{ and } \hat{\tau}_{[k]}\text{)} \\
&= \frac{1}{n} \sum_{k=1}^{K} \left(\frac{n_{[k]}}{n_{[k]1}} \sum_{i:X_i=k} Z_i Y_i - \frac{n_{[k]}}{n_{[k]0}} \sum_{i:X_i=k} (1 - Z_i) Y_i \right) && \text{(Factor out } 1/n\text{)} \\
&= \frac{1}{n} \sum_{k=1}^{K} \left(\frac{n}{n_1} \sum_{i:X_i=k} Z_i Y_i - \frac{n}{n_0} \sum_{i:X_i=k} (1 - Z_i) Y_i \right) && \text{(Use } e_{[k]} = n_1/n\text{)} \\
&= \frac{1}{n_1} \sum_{k=1}^{K} \sum_{i:X_i=k} Z_i Y_i - \frac{1}{n_0} \sum_{k=1}^{K} \sum_{i:X_i=k} (1 - Z_i) Y_i && \text{(Factor out } n/n_z\text{)} \\
&= \frac{1}{n_1} \sum_{i=1}^{n} Z_i Y_i - \frac{1}{n_0} \sum_{i=1}^{n} (1 - Z_i) Y_i && \text{(Combine sums)} \\
&\equiv \hat{\tau}, && \text{(Definition of } \hat{\tau}\text{)}
\end{aligned}$$

which completes the proof.

Problem 5.3 (Consquence of constant individual causal effects)

Problem 5.4 (Compare the CRE and SRE)

Problem 5.5 (From the CRE to the SRE)

Problem 5.6 (More FRTs for Section 5.2.2)

Acronyms

CRE completely randomized experiment. 1