

## Solutions to Exercises of Chapter 6

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### Problem 6.7 (Lin's estimator for covariate adjustment)

Consider two ordinary least squares (OLS) problems:

$$\min_{\gamma_1, \beta_1} \sum_{i=1}^n Z_i (Y_i - \gamma_1 - \beta_1^\top X_i)^2 \quad \text{and} \quad \min_{\gamma_0, \beta_0} \sum_{i=1}^n (1 - Z_i) (Y_i - \gamma_0 - \beta_0^\top X_i)^2.$$

The solutions are  $\hat{\gamma}_1, \hat{\beta}_1$  and  $\hat{\gamma}_0, \hat{\beta}_0$ . Define the estimator

$$\begin{aligned} \hat{\tau}(\hat{\beta}_1, \hat{\beta}_0) &= \frac{1}{n_1} \sum_{i=1}^n Z_i (Y_i - \hat{\beta}_1^\top X_i) - \frac{1}{n_0} \sum_{i=1}^n (1 - Z_i) (Y_i - \hat{\beta}_0^\top X_i) \\ &= \hat{\gamma}_1 - \hat{\gamma}_0. \end{aligned} \tag{6.4}$$

We are asked to show the Proposition 6.2:

The estimator  $\hat{\tau}(\hat{\beta}_1, \hat{\beta}_0)$  in Equation (6.4) equals the coefficient of  $Z_i$  in the OLS fit of  $Y_i$  on  $(1, Z_i, X_i, Z_i \times X_i)$ , which is Lin's estimator,  $\hat{\tau}_L$ .

Let's write the OLS problem of regressing  $Y_i$  on  $(1, Z_i, X_i, Z_i \times X_i)$ :

$$\min_{\alpha_0, \alpha_1, \alpha_2, \alpha_3} \sum_{i=1}^n (Y_i - \alpha_0 - \alpha_1 Z_i - \alpha_2^\top X_i - \alpha_3^\top (Z_i X_i))^2.$$

It is equivalent to the following problem:

$$\min_{\alpha_0, \alpha_1, \alpha_2, \alpha_3} \sum_{i=1}^n Z_i (Y_i - (\alpha_0 + \alpha_1) - (\alpha_2 + \alpha_3)^\top X_i)^2 + \sum_{i=1}^n (1 - Z_i) (Y_i - \alpha_0 - \alpha_2^\top X_i)^2.$$

Let  $\gamma_1 = \alpha_0 + \alpha_1$ ,  $\beta_1 = \alpha_2 + \alpha_3$ ,  $\gamma_0 = \alpha_0$ , and  $\beta_0 = \alpha_2$ . Then the above problem becomes

$$\min_{\gamma_1, \beta_1, \gamma_0, \beta_0} \sum_{i=1}^n Z_i (Y_i - \gamma_1 - \beta_1^\top X_i)^2 + \sum_{i=1}^n (1 - Z_i) (Y_i - \gamma_0 - \beta_0^\top X_i)^2.$$

This is exactly a reparametrization of the two separate OLS problems we started with. Therefore, the solutions are related by

$$\begin{aligned} \hat{\gamma}_1 &= \hat{\alpha}_0 + \hat{\alpha}_1, \\ \hat{\beta}_1 &= \hat{\alpha}_2 + \hat{\alpha}_3, \\ \hat{\gamma}_0 &= \hat{\alpha}_0, \\ \hat{\beta}_0 &= \hat{\alpha}_2, \end{aligned}$$

and we have

$$\hat{\tau}(\hat{\beta}_1, \hat{\beta}_0) = \hat{\gamma}_1 - \hat{\gamma}_0 = (\hat{\alpha}_0 + \hat{\alpha}_1) - \hat{\alpha}_0 = \hat{\alpha}_1 = \hat{\tau}_L.$$

### Problem 6.8 (Predictive and projective estimators)

We can predict  $Y_i(z)$  based on  $X_i$  using the data from the treatment group ( $Z_i = 1$ ) and control group ( $Z_i = 0$ ) separately,

$$\begin{aligned}\hat{\mu}_1(X_i) &= \hat{\gamma}_1 + \hat{\beta}_1^\top X_i, \\ \hat{\mu}_0(X_i) &= \hat{\gamma}_0 + \hat{\beta}_0^\top X_i,\end{aligned}$$

where  $(\hat{\gamma}_1, \hat{\beta}_1)$  and  $(\hat{\gamma}_0, \hat{\beta}_0)$  are the solutions to the two OLS problems in Problem 6.7.

If we use the predictions to impute the missing potential outcomes, then we can estimate the average treatment effect (ATE) by

$$\hat{\tau}_{\text{pred}} = \frac{1}{n} \left\{ \sum_{Z_i=1} Y_i + \sum_{Z_i=0} \hat{\mu}_1(X_i) - \sum_{Z_i=1} \hat{\mu}_0(X_i) - \sum_{Z_i=0} Y_i \right\}, \quad (6.7)$$

which is called the *predictive estimator*.

Instead of predicting the missing potential outcomes, if we predict all potential outcomes even if they are observed, then we have the *projective estimator*:

$$\hat{\tau}_{\text{proj}} = \frac{1}{n} \sum_{i=1}^n \{\hat{\mu}_1(X_i) - \hat{\mu}_0(X_i)\}. \quad (6.9)$$

We are asked to show that  $\hat{\tau}_{\text{pred}}$  equals Lin's estimator  $\hat{\tau}_L$ . Furthermore,  $\hat{\tau}_{\text{proj}}$  also agrees with  $\hat{\tau}_L$ .

We first show  $\hat{\tau}_{\text{proj}} = \hat{\tau}_L$ . By substituting the definitions of  $\hat{\mu}_1(X_i)$  and  $\hat{\mu}_0(X_i)$ ,

$$\begin{aligned}\hat{\tau}_{\text{proj}} &= \frac{1}{n} \sum_{i=1}^n \{(\hat{\gamma}_1 - \hat{\gamma}_0) + (\hat{\beta}_1 - \hat{\beta}_0)^\top X_i\} \\ &= (\hat{\gamma}_1 - \hat{\gamma}_0) + (\hat{\beta}_1 - \hat{\beta}_0)^\top \bar{X} \\ &= \hat{\tau}_L + (\hat{\beta}_1 - \hat{\beta}_0)^\top \bar{X} \\ &= \hat{\tau}_L,\end{aligned}$$

where the last equality holds because of the centering of  $X_i$  (i.e.,  $\bar{X} = 0$ ).

By normal equations of the two OLS problems, we have

$$\sum_{Z_i=z} Y_i = \sum_{Z_i=z} (\hat{\gamma}_z + \hat{\beta}_z^\top X_i) = \sum_{Z_i=z} \hat{\mu}_z(X_i).$$

Plug in the above equation into Equation (6.7), we have

$$\begin{aligned}\hat{\tau}_{\text{pred}} &= \frac{1}{n} \left\{ \sum_{Z_i=1} \hat{\mu}_1(X_i) + \sum_{Z_i=0} \hat{\mu}_1(X_i) - \sum_{Z_i=1} \hat{\mu}_0(X_i) - \sum_{Z_i=0} \hat{\mu}_0(X_i) \right\} \\ &= \frac{1}{n} \sum_{i=1}^n \{\hat{\mu}_1(X_i) - \hat{\mu}_0(X_i)\} \\ &= \hat{\tau}_{\text{proj}} \\ &= \hat{\tau}_L.\end{aligned}$$

**Problem 6.12 (More on the difference-in-difference estimator in the CRE)**

**Acronyms**

ATE    average treatment effect. [2](#)  
OLS    ordinary least squares. [1](#), [2](#)