Solutions to Exercises of Chapter 12

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Problem 12.1 (A sanity check)

Problem 12.2 (An alternative form of the doubly robust estimator for τ)

Motivated by

$$\mu_1 = \mathrm{E}\left\{\frac{Z\{Y-\mu_1(X,\beta_1)\}}{e(X)}\right\} + \mathrm{E}\{\mu_1(X,\beta_1)\},$$

we have an alternative form of the doubly robust estimator for $\mu_1 = \mathrm{E}\{Y(1)\}$ as

$$\tilde{\mu}_1^{\mathrm{dr2}} = \frac{\mathrm{E}\left\{\frac{Z\{Y-\mu_1(X,\beta_1)\}}{e(X,\alpha)}\right\}}{\mathrm{E}\left\{\frac{Z}{e(X,\alpha)}\right\}} + \mathrm{E}\{\mu_1(X,\beta_1)\}.$$

We are asked to show that

$$\tilde{\mu}_1^{\mathrm{dr2}} = \mu_1$$

if either $e(X,\alpha) = e(X)$ or $\mu_1(X,\beta_1) = \mu_1(X)$.

Let

$$L = \mathrm{E}\left\{\frac{Z}{e(X,\alpha)}\right\}.$$

Note that

$$\begin{aligned}
&L\tilde{\mu}_1^{\mathrm{dr2}} - L\mu_1 \\
&= \mathrm{E}\left\{\frac{Z\{Y-\mu_1(X,\beta_1)\}}{e(X,\alpha)}\right\} + L\,\mathrm{E}\{\mu_1(X,\beta_1)\} - L\mu_1 && \text{(Definition)} \\
&= \mathrm{E}\left\{\frac{Z\{Y-\mu_1(X,\beta_1)\}}{e(X,\alpha)} - L\{Y(1)-\mu_1(X,\beta_1)\}\right\} && (\mu_1 = \mathrm{E}[Y(1)]) \\
&= \mathrm{E}\left\{\frac{Z\{Y(1)-\mu_1(X,\beta_1)\}}{e(X,\alpha)} - L\{Y(1)-\mu_1(X,\beta_1)\}\right\} && \text{(Consistency)} \\
&= \mathrm{E}\left\{\frac{Z - Le(X,\alpha)}{e(X,\alpha)}\{Y(1)-\mu_1(X,\beta_1)\}\right\} && \text{(Combine terms)} \\
&= \mathrm{E}\left\{\mathrm{E}\left\{\frac{Z - Le(X,\alpha)}{e(X,\alpha)}\{Y(1)-\mu_1(X,\beta_1)\} \,\middle|\, X\right\}\right\} && \text{(Iterated expectation)} \\
&= \mathrm{E}\left\{\frac{\mathrm{E}\{Z - Le(X,\alpha) \mid X\}}{e(X,\alpha)}\,\mathrm{E}\{Y(1)-\mu_1(X,\beta_1) \mid X\}\right\} && \text{(Ignorability)} \\
&= \mathrm{E}\left\{\left\{\frac{e(X)}{e(X,\alpha)} - \frac{Le(X,\alpha)}{e(X,\alpha)}\right\}\{\mu_1(X)-\mu_1(X,\beta_1)\}\right\}. && \text{(Linearity of expectation)}
\end{aligned}$$

$$= \mathrm{E}\left\{\left\{\frac{e(X)}{e(X,\alpha)} - L\right\}\{\mu_1(X) - \mu_1(X,\beta_1)\}\right\}. \qquad \text{(Simplification)}$$

Therefore, if $e(X,\alpha) = e(X)$, then

$$L = \mathrm{E}\left\{\frac{Z}{e(X,\alpha)}\right\} = \mathrm{E}\left\{\frac{Z}{e(X)}\right\} = 1,$$

and thus the last line becomes

$$\mathrm{E}\left\{\{1-1\}\{\mu_1(X) - \mu_1(X,\beta_1)\}\right\} = 0.$$

On the other hand, if $\mu_1(X,\beta_1) = \mu_1(X)$, then the last line becomes

$$\mathrm{E}\left\{\left\{\frac{e(X)}{e(X,\alpha)} - L\right\}\{0\}\right\} = 0.$$

In both cases, we have shown that $L\tilde{\mu}_1^{\mathrm{dr2}} - L\mu_1 = 0$, which implies that $\tilde{\mu}_1^{\mathrm{dr2}} = \mu_1$. Similarly, we can define

$$\tilde{\mu}_0^{\mathrm{dr2}} = \frac{\mathrm{E}\left\{\frac{(1-Z)\{Y-\mu_0(X,\beta_0)\}}{1-e(X,\alpha)}\right\}}{\mathrm{E}\left\{\frac{1-Z}{1-e(X,\alpha)}\right\}} + \mathrm{E}\left\{\mu_0(X,\beta_0)\right\}$$

and show that

$$\tilde{\mu}_0^{\mathrm{dr2}} = \mu_0$$

if either $e(X,\alpha) = e(X)$ or $\mu_0(X,\beta_0) = \mu_0(X)$.

The sample version of $\tilde{\mu}_1^{\mathrm{dr2}}$ and $\tilde{\mu}_0^{\mathrm{dr2}}$ give us an alternative form of the doubly robust estimator for τ as

$$\begin{aligned}
&\hat{\tau}^{\mathrm{dr2}} \\
&= \frac{\frac{1}{n}\sum_{i=1}^{n}\frac{Z_i\left\{Y_i-\mu_1(X_i,\hat{\beta}_1)\right\}}{e(X_i,\hat{\alpha})}}{\frac{1}{n}\sum_{i=1}^{n}\frac{Z_i}{e(X_i,\hat{\alpha})}} + \frac{1}{n}\sum_{i=1}^{n}\mu_1(X_i,\hat{\beta}_1) - \frac{\frac{1}{n}\sum_{i=1}^{n}\frac{(1-Z_i)\left\{Y_i-\mu_0(X_i,\hat{\beta}_0)\right\}}{1-e(X_i,\hat{\alpha})}}{\frac{1}{n}\sum_{i=1}^{n}\frac{1-Z_i}{1-e(X_i,\hat{\alpha})}} - \frac{1}{n}\sum_{i=1}^{n}\mu_0(X_i,\hat{\beta}_0).
\end{aligned}$$