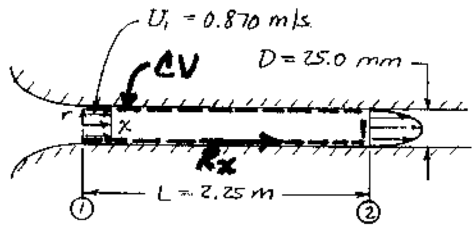
Problem 4.84

Given: Uniform flow into, fully developed flow from duct shown.

Air



$$\frac{u(r)}{U_c} = 1 - \left(\frac{r}{R}\right)^2 \quad \text{at } 2$$

$$p_1 - p_2 = 1.92 \text{ N/m}^2$$

Find: Total force exerted by tube on the flowing air.

Solution: Apply continuity and momentum to CV, CS shown.

Basic equations: $$0 = \cancel{\frac{\partial}{\partial t}}^{=0(1)} \int_{CV} \rho \, d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A}$$

$$F_{S_x} + \cancel{F_{B_x}}^{=0(4)} = \cancel{\frac{\partial}{\partial t}}^{=0(1)} \int_{CV} u \rho \, d\forall + \int_{CS} u \rho \vec{V} \cdot d\vec{A}$$

Assumptions: (1) Steady flow (3) Uniform flow at inlet
(2) Incompressible flow (4) $F_{B_x} = 0$

Then $$0 = \{-|\rho U_1 A_1|\} + \int_{2} \rho u \, dA = -\rho U_1 \pi R^2 + \int_0^R \rho U_c \left[1 - \left(\tfrac{r}{R}\right)^2\right] 2\pi r \, dr$$

$$0 = -\rho U_1 \pi R^2 + 2\rho \pi R^2 U_c \int_0^1 (1-\lambda^2)\lambda \, d\lambda \quad \text{or} \quad 0 = -U_1 + 2U_c \left[\frac{\lambda^2}{2} - \frac{\lambda^4}{4}\right]_0^1$$

Thus $0 = -U_1 + \frac{1}{2} U_c$ or $U_c = 2U_1$ $\quad (\lambda = r/R)$

From momentum $$R_x + p_1 A_1 - p_2 A_2 = u_1\{-|\rho U_1 A_1|\} + \int_{2} u_2 \{+\rho u_2 \, dA_2\}$$

$$u_1 = U_1 \qquad u_2 = U_c\left[1 - \left(\tfrac{r}{R}\right)^2\right]$$

So $$\int_{2} = \int_0^R U_c\left[1-\left(\tfrac{r}{R}\right)^2\right] \rho U_c \left[1-\left(\tfrac{r}{R}\right)^2\right] 2\pi r \, dr = 2\pi \rho U_c^2 R^2 \int_0^1 (1-\lambda^2)(1-\lambda^2)\lambda \, d\lambda$$

$$= 2\pi \rho U_c^2 R^2 \int_0^1 (1 - 2\lambda^2 + \lambda^4)\lambda \, d\lambda = 2\pi \rho U_c^2 R^2 \left[\frac{\lambda^2}{2} - \frac{\lambda^4}{2} + \frac{\lambda^6}{6}\right]_0^1 = \frac{1}{3}\pi \rho U_c^2 R^2$$

Substituting,

$$R_x + (p_1 - p_2)\pi R^2 = -\pi \rho U_1^2 R^2 + \frac{1}{3}\pi \rho U_c^2 R^2 = -\pi \rho U_1^2 R^2 + \frac{1}{3}\pi \rho (2U_1)^2 R^2$$

$$R_x = -(p_1 - p_2)\frac{\pi D^2}{4} + \frac{1}{3}\rho U_1^2 \frac{\pi D^2}{4}$$

$$= -1.92 \frac{\text{N}}{\text{m}^2} \times \frac{\pi}{4}(0.025)^2 \text{m}^2 + \frac{1}{3} \times 1.23 \frac{\text{kg}}{\text{m}^3} \times (0.870)^2 \frac{\text{m}^2}{\text{s}^2} \times \frac{\pi}{4}(0.025)^2 \text{m}^2 \times \frac{\text{N} \cdot \text{s}^2}{\text{kg} \cdot \text{m}}$$

$R_x = -7.90 \times 10^{-4}$ N (to left on CV, since < 0) $\qquad R_x$