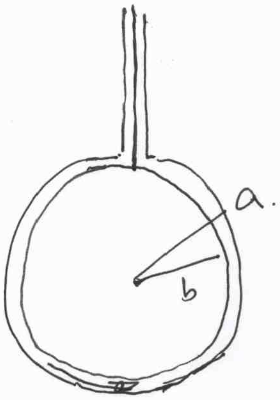
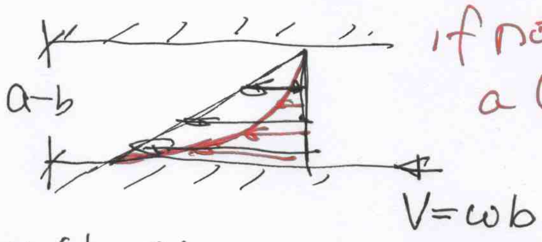
3.21. Viscous pump. Find power.



Width W

$\omega = 600\,rpm = 20\pi$ rad/s

if $a > 100(b-a)$, problem can be treated as flow ~~betwen~~ between two parallel plates



if not, can not assume a linear profile.

When $a > 100(b-a)$

$$\tau = \mu \frac{\omega b}{a-b}.$$

$$F = \tau A = \mu \frac{\omega b}{a-b} 2\pi b W$$

$$P = F b\omega = 2\pi \mu \frac{\omega^2 b^2}{a-b} W$$

When a linear profile is not valid.

NS equation on page 575, here $V_r = 0$, steady.

$$\begin{cases} \frac{1}{r}\frac{\partial}{\partial r}(\cancel{r V_r}) + \frac{1}{r}\frac{\partial V_\theta}{\partial \theta} = 0 \Rightarrow \frac{\partial V_\theta}{\partial \theta} = 0 \\ \cancel{\frac{\partial V_\theta}{\partial t}} + \left(\cancel{V_r}\frac{\partial V_\theta}{\partial r} + \cancel{\frac{1}{r}V_\theta\frac{\partial V_\theta}{\partial \theta}}\right) + \cancel{\frac{V_r V_\theta}{r}} = -\cancel{\frac{1}{\rho}\frac{\partial p}{r\partial \theta}} \cancel{+ g_\theta} \end{cases}$$

$$+ \nu\left(\nabla^2 V_\theta + \cancel{\frac{2}{r^2}\frac{\partial V_r}{\partial \theta}} - \frac{V_\theta}{r^2}\right)$$

$$\Rightarrow \nabla^2 V_\theta + \frac{\cancel{V_\theta}}{r^2}\cancel{\partial V_r} = 0$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V_\theta}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V_\theta}{\partial \theta^2} = 0$$

$$\boxed{r^2\frac{\partial^2 V_\theta}{\partial r^2} + r\frac{\partial V_\theta}{\partial r} - V_\theta = 0}$$

求解过程见刘锋同学作业.

解 $V_\theta = c_1 r + c_2 \frac{1}{r}$

B.C. $r = a$, $V_\theta = 0$,
$r = b$, $V_\theta = \omega b$.

$$\Rightarrow c_1 = \frac{\omega b^2}{b^2 - a^2}, \quad c_2 = -\frac{\omega a^2 b^2}{b - a^2}$$

$$\boxed{V_\theta = \frac{\omega b^2}{b^2 - a^2} r - \frac{\omega a^2 b^2}{r(b^2 - a^2)}}$$

$$\tau = \mu \frac{\partial v_\theta}{\partial r}\Big|_{r=b} = \mu \left[\frac{\omega(b^2)}{b^2-a^2} + \frac{\omega a^2 b^2}{b^2(b^2-a^2)} \right] = \mu\omega \frac{a^2+b^2}{b^2-a^2}$$

$$P = -\tau \cdot (2\pi b W)\, b\omega = 2\pi\mu \frac{\omega^2 b^2 (a^2+b^2)}{b^2-a^2} W$$

power for pump with unit width

$$\frac{P}{W} = 2\pi\mu \frac{\omega^2 b^2 (a^2+b^2)}{(b^2-a^2)}$$

$$= 2\times 3.14 \times \left(0.26\ \frac{N\cdot s}{m^2}\right) \frac{(0.1^2+0.09^2)\ m^2\ \ 0.09^2\ m^2\ (20\times 3.14)^2}{(0.1^2-0.09^2)\ m^2}$$

$$= 495\ \frac{W}{m}$$

Compare magnitude of τ

Linear profile.

$$\tau_L = \mu \frac{\omega b}{a-b}$$

non-linear profile

$$\tau_N = \mu \frac{\omega b}{a-b} \left(\frac{a^2+b^2}{ab+b^2} \right)$$

$$\cancel{\approx \mu \frac{\omega b}{a-b} \, 2b}$$

if $a \approx b$.
两者相等.

如果相距较大. 则 τ 相差增大
a 比 b 大得越多
τ_N 比 τ_L 大的量也越多
这和示意图中标出的速度曲线是相符的

我们这个例子中相差 6%

流体力学　力学2班　刘犇　学号：21003026

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DATE　9

3-21：θ momentum：

$$\cancel{\frac{\partial v_\theta}{\partial t}} + (\underline{v}\cdot\nabla) v_\theta + \frac{v_r v_\theta}{r} = -\frac{1}{\rho r}\cancel{\frac{\partial p}{\partial \theta}} + \cancel{g_\theta} + \nu\left[\nabla^2 v_\theta + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r^2}\right]$$

$$\left(\cancel{v_r\frac{\partial}{\partial r}} + \frac{1}{r}v_\theta\frac{\partial}{\partial \theta}\right)v_\theta + \cancel{\frac{v_r v_\theta}{r}} = \nu\left\{\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right]v_\theta + \cancel{\frac{2}{r^2}\frac{\partial v_r}{\partial \theta}} - \frac{v_\theta}{r^2}\right\}$$

$$\therefore \frac{1}{r}v_\theta\cancel{\frac{\partial v_\theta}{\partial \theta}} = \nu\left\{\frac{1}{r}\frac{\partial v_\theta}{\partial r} + \frac{\partial^2 v_\theta}{\partial r^2} + \cancel{\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2}} - \frac{v_\theta}{r^2}\right\}$$

$$\therefore 0 = \frac{1}{r}\frac{\partial v_\theta}{\partial r} + \frac{\partial^2 v_\theta}{\partial r^2} - \frac{v_\theta}{r^2} = 0$$

即 $r^2\frac{\partial^2 v_\theta}{\partial r^2} + r\frac{\partial v_\theta}{\partial r} - v_\theta = 0$　①

设：$r = e^t$，$\therefore t = \ln r$

$$\therefore \frac{\partial v_\theta}{\partial r} = \frac{\partial v_\theta}{\partial t}\frac{\partial t}{\partial r} = \frac{\partial v_\theta}{\partial t}\frac{1}{r}$$ ✓

$$\frac{\partial^2 v_\theta}{\partial r^2} = \frac{\partial^2 v_\theta}{\partial t^2}\frac{1}{r^2} - \frac{\partial v_\theta}{\partial t}\frac{1}{r^2}$$

∴①式变为　$\frac{\partial^2 v_\theta}{\partial t^2} - \frac{\partial v_\theta}{\partial t} + \frac{\partial v_\theta}{\partial t} - v_\theta = 0$

即　$\frac{\partial^2 v_\theta}{\partial t^2} - v_\theta = 0$

设　$v_\theta = ce^{\lambda t}$ 代入上式，得

$$v_\theta = c_1 e^t + c_2 e^{-t} = c_1 r + c_2\frac{1}{r}$$ ✓

$$\because \begin{cases} v_\theta|_{r=b} = \omega b \\ v_\theta|_{r=a} = 0 \end{cases} \quad \therefore \begin{cases} c_1 b + \frac{c_2}{b} = \omega b \\ c_1 a + \frac{c_2}{a} = 0 \end{cases}$$

$$\therefore \begin{cases} c_1 = \frac{\omega b^2}{b^2-a^2} \\ c_2 = -\frac{\omega a^2 b^2}{b^2-a^2} \end{cases}$$

$$\therefore \begin{cases} v_\theta = \frac{\omega b^2}{b^2-a^2}r - \frac{\omega a^2 b^2}{(b^2-a^2)r} \\ v_r = 0 \end{cases}$$

∴内圆柱表面粘性力为：$\tau = \mu\frac{\partial v_\theta}{\partial r} = \frac{\omega b^2}{(b^2-a^2)} + \frac{\omega a^2 b^2}{r^2(b^2-a^2)}$ ✓

$$= \mu\frac{\omega(a^2+b^2)}{(b^2-a^2)}$$