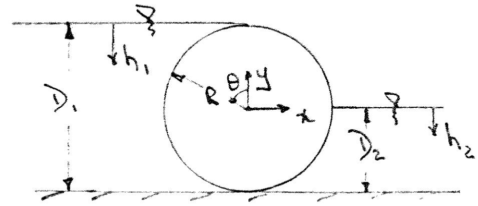
Problem 3.69

Given: Cylindrical weir of radius, $R = 1.5\,m$ and length, $L = 6\,m$ as shown. Liquid is water.
$D_1 = 3\,m \quad D_2 = 1.5\,m$



Find: Magnitude and direction of resultant force of water on the weir.

Solution

Basic equations: $\vec{F}_R = -\int P\,d\vec{A} \qquad \dfrac{dP}{dh} = \rho g$

Assumptions: (1) static fluid
(2) ρ = constant
(3) h is measured positive down from free surface

$F_{R_x} = \int dF_x = \vec{F}_R\cdot\hat{i} = \int d\vec{F}\cdot\hat{i} = -\int P\,d\vec{A}\cdot\hat{i} = -\int P\,dA\cos(90+\theta) = \int P\,dA\sin\theta$

$F_{R_y} = \int dF_y = \vec{F}_R\cdot\hat{j} = \int d\vec{F}\cdot\hat{j} = -\int P\,d\vec{A}\cdot\hat{j} = -\int P\,dA\cos\theta$

Since $dA = LR\,d\theta$,

$F_{R_x} = \int_0^{3\pi/2} PLR\sin\theta\,d\theta$ and $F_{R_y} = -\int_0^{3\pi/2} PLR\cos\theta\,d\theta$

We can obtain an expression for P as a function of h,

$\dfrac{dP}{dh} = \rho g \qquad dP = \rho g\,dh$ and $P - P_0 = \int_{P_0}^{P} dP = \int_0^h \rho g\,dh = \rho g h$

Since atmospheric pressure acts over the first quadrant of the cylinder and both free surfaces, the appropriate expression for P is $P = \rho g h$.

For

$0 \le \theta \le \pi$, $h_1 = R - R\cos\theta = R(1-\cos\theta)$ and hence $P_1 = \rho g R(1-\cos\theta)$

$\pi \le \theta \le \frac{3\pi}{2}$, $h_2 = -R\cos\theta$ and hence $P_2 = -\rho g R\cos\theta$

$$F_{R_x} = \int_0^{3\pi/2} PLR\sin\theta\,d\theta = \int_0^{\pi} \rho g R(1-\cos\theta)LR\sin\theta\,d\theta + \int_{\pi}^{3\pi/2}(-\rho g R\cos\theta)LR\sin\theta\,d\theta$$

$$= \rho g R^2 L\int_0^{\pi}(1-\cos\theta)\sin\theta\,d\theta - \rho g R^2 L\int_{\pi}^{3\pi/2}\cos\theta\sin\theta\,d\theta$$

$$= \rho g R^2 L\left[-\cos\theta - \tfrac{1}{2}\sin^2\theta\right]_0^{\pi} - \rho g R^2 L\left[\tfrac{1}{2}\sin^2\theta\right]_{\pi}^{3\pi/2} = \rho g R^2 L\left[2 - \tfrac{1}{2}\right] = \tfrac{3}{2}\rho g R^2 L$$

$$F_{R_x} = \frac{3}{2}\times 999\,\frac{kg}{m^3}\times 9.81\,\frac{m}{s^2}\times(1.5)^2\,m^2\times 6\,m\times\frac{N\cdot s^2}{kg\cdot m} = 198\ kN$$

$$F_{R_y} = -\int_0^{3\pi/2} PLR\cos\theta = -\int_0^{\pi}\rho g R(1-\cos\theta)LR\cos\theta\,d\theta - \int_{\pi}^{3\pi/2}(-\rho g R\cos\theta)LR\cos\theta\,d\theta$$

$$= -\rho g R^2 L\int_0^{\pi}(1-\cos\theta)\cos\theta\,d\theta + \rho g R^2 L\int_{\pi}^{3\pi/2}\cos^2\theta\,d\theta$$

$$= -\rho g R^2 L\left[\sin\theta - \frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_0^{\pi} + \rho g R^2 L\left[\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_{\pi}^{3\pi/2} = \rho g R^2 L\left[\frac{\pi}{2} + \frac{3\pi}{4} - \frac{\pi}{2}\right] = \frac{3\pi}{4}\rho g R^2 L$$

$$F_{R_y} = \frac{3\pi}{4}\times 999\,\frac{kg}{m^3}\times 9.81\,\frac{m}{s^2}\times(1.5)^2\,m^2\times 6\,m\times\frac{N\cdot s^2}{kg\cdot m} = 312\ kN$$

$\vec{F}_R = \hat{i}F_{R_x} + \hat{j}F_{R_y} = 198\hat{i} + 312\hat{j}\ kN$

$F_R = \sqrt{F_{R_x}^2 + F_{R_y}^2} = \left[(198)^2 + (312)^2\right]^{1/2}\ kN = 370\ kN$ ⟵ F_R

Since all elements of force $d\vec{F}$ are normal to the surface, the direction α,

$\alpha = \tan^{-1} F_{R_y}/F_{R_x} = \tan^{-1} 312/198 = 57.6^\circ$ ⟵ α