Relative positioning with GPS

Undifferenced observation equations

The definition of the code pseudorange is given by the following equation:

$$P_B^s\left(\tilde{t}_B\right)=c\left(\tilde{t}_B-\tilde{t}^s\right)=c\Delta\tilde{t}_B^s \tag{1}$$

where

$P_B^s\left(\tilde{t}_B\right)$	pseudorange measured at time $\tilde{t}_B$ by receiver B to satellite s
$\tilde{t}_B$	nominal time of the signal reception measured by the clock of receiver B
$\tilde{t}^s$	nominal time of signal transmission measured by the clock of satellite s

The clock errors are defined by the following equations:

$$\begin{aligned} t_B &= \tilde{t}_B-\delta t_B \\ t^s &= \tilde{t}^s-\delta t^s \end{aligned} \tag{2}$$

By introducing (2) into Equation (1), we get:

$$P_B^s(\tilde{t}_B)=\rho_B^s(t_B)+c\,\delta t_B-c\,\delta t^s \tag{3}$$

The topocentric distance $\rho_B^s(t_B)$ is the true geometric distance travelled by the signal emitted by satellite s at time t^s and received by receiver B at the true time instance t_B. Since the true time t_B is unknown, the topocentric distance must be linearised around the known nominal receiver time $\tilde{t}_B$:

$$\rho_B^s(t_B)=\rho_B^s(\tilde{t}_B)-\dot{\rho}_B^s(\tilde{t}_B)\delta t_B \tag{4}$$

Equation (3) can now be rewritten as:

$$P_B^s(\tilde{t}_B)=\rho_B^s(\tilde{t}_B)+\left(c-\dot{\rho}_B^s(\tilde{t}_B)\right)\delta t_B-c\delta t^s \tag{5}$$

The term $\dot{\rho}_B^s(\tilde{t}_B)\delta t_B$ in equation (5) is often neglected if the equation is used for the code single point positioning. The reason is that the absolute value of the topocentric range rate $\dot{\rho}_B^s(\tilde{t}_B)$ is always less than 800 m/s. If the receiver clock correction is $\delta t_B=10\ \mu s$, then $\dot{\rho}_B^s(\tilde{t}_B)\delta t_B=8\ mm$, which is far below the code observation noise level. However, this correction is significant in case of relative positioning with phase pseudoranges.
The observation equation for carrier phases:

$$\lambda\varphi_B^s(\tilde{t}_B) = \rho_B^s(\tilde{t}_B) + \left(c - \dot{\rho}_B^s(\tilde{t}_B)\right)\delta t_B - c\delta t^s + \lambda N_B^s \tag{6}$$

The most convenient way of how to deal with $\dot{\rho}_B^s(\tilde{t}_B)\delta t_B$ correction in equation (6) is to use the receiver clock correction $\delta\hat{t}_B$ estimated in the navigation solution, i.e. single point positioning using code pseudoranges. The precision of this estimate is usually better than 1 μs, hence the precision of computed $\dot{\rho}_B^s(\tilde{t}_B)\delta t_B$ is better than 0.8 mm. Equation (6) can be rearranged, so that the known and measured quantities are at the left side:

$$\lambda\varphi_B^s(\tilde{t}_B) + \dot{\rho}_B^s(\tilde{t}_B)\delta\hat{t}_B + c\delta t^s = \rho_B^s(\tilde{t}_B) + c\delta t_B + \lambda N_B^s \tag{7}$$

Similarly for code pseudoranges:

$$P_B^s(\tilde{t}_B) + \dot{\rho}_B^s(\tilde{t}_B)\delta\hat{t}_B + c\delta t^s = \rho_B^s(\tilde{t}_B) + c\delta t_B \tag{8}$$

Both for the relative and single point positioning, it is necessary to compute the topocentric distance, which is given by the following equation:

$$\rho_B^s(\tilde{t}_B) = \sqrt{\left(X^s - x_B\right)^2 + \left(Y^s - y_B\right)^2 + \left(Z^s - z_B\right)^2} \tag{9}$$

where

X^s, Y^s, Z^s Cartesian coordinates of satellite s in WGS84 at time t^s

x_B, y_B, z_B Cartesian coordinates of receiver B expressed in non-rotating (inertial) reference frame, which coincides with WGS84 at time of signal transmission t^s.

The inertial coordinates are related to WGS84 coordinates by:

$$\begin{aligned} x_B &= X_B - \dot{\Omega}_e Y_B \Delta\tilde{t}_B^s \\ y_B &= Y_B + \dot{\Omega}_e X_B \Delta\tilde{t}_B^s \\ z_B &= Z_B \end{aligned} \tag{10}$$

where $\Delta\tilde{t}_B^s$ is the signal travel time, which can be computed with sufficient precision from Equation (1).

To be able to use the standard linear LSQ, equation (9) has to be linearised:

$$\rho_B^s(\tilde{t}_B) = \rho_{B0}^s(\tilde{t}_B) + \frac{\partial\rho_{B0}^s(\tilde{t}_B)}{\partial X_{B0}}\Delta X + \frac{\partial\rho_{B0}^s(\tilde{t}_B)}{\partial Y_{B0}}\Delta Y + \frac{\partial\rho_{B0}^s(\tilde{t}_B)}{\partial Z_{B0}}\Delta Z \tag{11}$$

$$
\begin{aligned}
\frac{\partial\rho_{B0}^{s}(t_B)}{\partial X_{B0}} &= a_X^s = -\frac{X^s - X_{B0}}{\rho_{B0}^s} \\
\frac{\partial\rho_{B0}^{s}(t_B)}{\partial Y_{B0}} &= a_Y^s = -\frac{Y^s - Y_{B0}}{\rho_{B0}^s} \\
\frac{\partial\rho_{B0}^{s}(t_k)}{\partial Z_{B0}} &= a_Z^s = -\frac{Z^s - Z_{B0}}{\rho_{B0}^s}
\end{aligned}
\tag{12}
$$

Please note, that in our computation, we need to linearise equation (3) only for receiver ROV, since the coordinates of receiver REF are considered as known.

Differenced observation equations

In the case of differential or relative positioning, at least two GPS receivers measure the pseudoranges to a set of common satellites at the same time (synchronously). Let two receivers on points A and B measure a satellite s and the point A is a known, reference point. The code and phase observation equations can be written as:

$$
\begin{aligned}
&\lambda\varphi_A^s(\tilde{t}_A) + \dot{\rho}_A^s(\tilde{t}_A)\delta\hat{t}_A + c\delta t^s = \lambda\varphi_A^s(t) + c\delta t^s = \rho_A^s(\tilde{t}_A) + c\delta t_A + \lambda N_A^s \\
&\lambda\varphi_B^p(\tilde{t}_B) + \dot{\rho}_B^s(\tilde{t}_B)\delta\hat{t}_B + c\delta t^s = \lambda\varphi_B^s(t) + c\delta t^s = \rho_B^s(\tilde{t}_B) + c\delta t_B + \lambda N_B^s \\
&P_A^s(\tilde{t}_A) + \dot{\rho}_A^s(\tilde{t}_A)\delta\hat{t}_A + c\delta t^s = P_A^s(t) + c\delta t^s = \rho_A^s(\tilde{t}_A) + c\delta t_A \\
&P_B^s(\tilde{t}_B) + \dot{\rho}_B^s(\tilde{t}_B)\delta\hat{t}_B + c\delta t^s = P_B^s(t) + c\delta t^s = \rho_B^s(\tilde{t}_B) + c\delta t_B
\end{aligned}
\tag{13}
$$

From now on, we can omit the time indication (t), since all observables are corrected to the same time instance, i.e.:

$$
\begin{aligned}
P_A^s &= P_A^s(t_A) = P_A^s(\tilde{t}_A) + \dot{\rho}_A^s(\tilde{t}_A)\delta\hat{t}_A \\
\lambda\varphi_A^s &= \lambda\varphi_A^s(t_A) = \lambda\varphi_A^s(\tilde{t}_A) + \dot{\rho}_A^s(\tilde{t}_A)\delta\hat{t}_A
\end{aligned}
\tag{14}
$$

To reduce the atmospheric effects that are common to both stations, it is useful to form differenced equations. The single differences:

$$
\begin{aligned}
\lambda\varphi_{AB}^s &= \lambda\varphi_B^s - \lambda\varphi_A^s = \rho_{AB}^s + c\delta t_{AB} + \lambda N_{AB}^s \\
P_{AB}^s &= P_B^s - P_A^s = \rho_{AB}^s + c\delta t_{AB}
\end{aligned}
\tag{15}
$$

where

$$
\begin{aligned}
&\rho_{AB}^s = \rho_B^s - \rho_A^s = \rho_{B0}^s + a_{X,B}^s\Delta x + a_{Y,B}^s\Delta y + a_{Z,B}^s\Delta z - \rho_A^s = \\
&= \rho_{AB,0}^s + a_{X,B}^s\Delta x + a_{Y,B}^s\Delta y + a_{Z,B}^s\Delta z \\
&\delta t_{AB} = \delta t_B - \delta t_A \\
&N_{AB}^s = N_B^s - N_A^s
\end{aligned}
\tag{16}
$$

In the next step, it is possible to form double differences, which are differences between two single differences. The most important feature of the double differences is the cancellation of the receiver clock errors.

For each observed satellite a pair of single difference equation similar to (15) can be formed:

$$\begin{aligned}
\lambda\varphi_{AB}^{s} &= \rho_{AB}^{s} + c\delta t_{AB} + \lambda N_{AB}^{s}\\
\lambda\varphi_{AB}^{t} &= \rho_{AB}^{t} + c\delta t_{AB} + \lambda N_{AB}^{t}\\
P_{AB}^{s} &= \rho_{AB}^{s} + c\delta t_{AB}\\
P_{AB}^{t} &= \rho_{AB}^{t} + c\delta t_{AB}
\end{aligned} \tag{17}$$

By subtracting code and phase equations, respectively, we get the following double difference equations:

$$\begin{aligned}
\lambda\varphi_{AB}^{st} &= \lambda\varphi_{AB}^{t} - \lambda\varphi_{AB}^{s} = \rho_{AB}^{st} + \lambda N_{AB}^{st}\\
P_{AB}^{st} &= P_{AB}^{t} - P_{AB}^{s} = \rho_{AB}^{st}
\end{aligned} \tag{18}$$

where

$$\begin{aligned}
\rho_{AB}^{st} &= \rho_{AB}^{t} - \rho_{AB}^{s}\\
N_{AB}^{st} &= N_{AB}^{t} - N_{AB}^{s}
\end{aligned} \tag{19}$$

Taking into account linearised topocentric distance from Equation (11), the observation equations (18) can be written in linear form as:

$$\begin{aligned}
\lambda\varphi_{AB}^{st} - \rho_{AB,0}^{st} &= a_{XB}^{st}\Delta X + a_{YB}^{st}\Delta Y + a_{ZB}^{st}\Delta Z + \lambda N_{AB}^{st}\\
P_{AB}^{st} - \rho_{AB,0}^{st} &= a_{XB}^{st}\Delta X + a_{YB}^{st}\Delta Y + a_{ZB}^{st}\Delta Z
\end{aligned} \tag{20}$$

or, in matrix form

$$\mathbf{L} = \mathbf{A}\mathbf{x} \tag{21}$$

$$\begin{aligned}
a_{XB}^{st} &= a_{XB}^{t} - a_{XB}^{s}\\
a_{YB}^{st} &= a_{YB}^{t} - a_{YB}^{s}\\
a_{ZB}^{st} &= a_{ZB}^{t} - a_{ZB}^{s}
\end{aligned} \tag{22}$$

$$\mathbf{x} = \begin{bmatrix} \Delta X & \Delta Y & \Delta Z & N_{AB}^{1,2} & \cdots & N_{AB}^{1,6} \end{bmatrix}^{T} \tag{23}$$

$$\mathbf{L} = \begin{bmatrix} \lambda\varphi_{AB}^{1,2} - \rho_{AB,0}^{1,2}\\ \vdots \\ \lambda\varphi_{AB}^{1,6} - \rho_{AB,0}^{1,6}\\ P_{AB,1}^{1,2} - \rho_{AB,0}^{1,2}\\ \vdots \\ P_{AB,1}^{1,6} - \rho_{AB,0}^{1,6} \end{bmatrix} \tag{24}$$

$$
\mathbf{A}=\begin{bmatrix}
a_{XB}^{1,2} & a_{YB}^{1,2} & a_{ZB}^{1,2} & \lambda_1 & 0 & 0 & 0 & 0\\
a_{XB}^{1,3} & a_{YB}^{1,3} & a_{ZB}^{1,3} & 0 & \lambda_1 & 0 & 0 & 0\\
a_{XB}^{1,4} & a_{YB}^{1,4} & a_{ZB}^{1,4} & 0 & 0 & \lambda_1 & 0 & 0\\
a_{XB}^{1,5} & a_{YB}^{1,5} & a_{ZB}^{1,5} & 0 & 0 & 0 & \lambda_1 & 0\\
a_{XB}^{1,6} & a_{YB}^{1,6} & a_{ZB}^{1,6} & 0 & 0 & 0 & 0 & \lambda_1\\
a_{XB}^{1,2} & a_{YB}^{1,2} & a_{ZB}^{1,2} & 0 & 0 & 0 & 0 & 0\\
a_{XB}^{1,3} & a_{YB}^{1,3} & a_{ZB}^{1,3} & 0 & 0 & 0 & 0 & 0\\
a_{XB}^{1,4} & a_{YB}^{1,4} & a_{ZB}^{1,4} & 0 & 0 & 0 & 0 & 0\\
a_{XB}^{1,5} & a_{YB}^{1,5} & a_{ZB}^{1,5} & 0 & 0 & 0 & 0 & 0\\
a_{XB}^{1,6} & a_{YB}^{1,6} & a_{ZB}^{1,6} & 0 & 0 & 0 & 0 & 0
\end{bmatrix} \tag{25}
$$

The LSQ solution of equation (21) is:

$$
\mathbf{X}=\left(\mathbf{A}^T\mathbf{PA}\right)^{-1}\mathbf{A}^T\mathbf{PL} \tag{26}
$$

where **P** is weight matrix of double difference observations **L**:

$$
\mathbf{P}=\begin{bmatrix}\mathbf{P}_\Phi & \mathbf{0}\\ \mathbf{0} & \mathbf{P}_P\end{bmatrix} \tag{27}
$$

$$
\mathbf{P}_i=\frac{1}{2\sigma_i^2\left(n_{dd}+1\right)}\begin{bmatrix}
n_{dd} & -1 & -1 & \cdots\\
-1 & n_{dd} & -1 & \cdots\\
-1 & & \ddots & \\
\vdots & \cdots & & n_{dd}
\end{bmatrix}, i=\Phi,P \tag{28}
$$

where, n_{dd} is number of double difference observations, σ_Φ and σ_P is standard deviation of undifferenced phase, resp. code pseudorange. For most of the receivers σ_Φ = 2 mm, σ_P = 0.3 m. Note: Equation (28) is derived by application of law of error propagation to single and double difference equations.