Kalman filtering

Linearised position-velocity model

A dynamic process can be generally described by the following matrix differential equation

$$\dot{\mathbf{x}}(t) = \mathbf{F}(t)\mathbf{x}(t) + \mathbf{G}(t)\mathbf{u}(t), \tag{1}$$

where $\mathbf{x}$ is state vector, $\mathbf{F}$ is system dynamic matrix, $\mathbf{G}$ is distribution matrix, t is time and $\mathbf{u}$ is a vector forcing function, whose elements are white noise. The term $\mathbf{G}(t)\mathbf{u}(t)$ describes model's uncertainty. In the case of the position-velocity (PV) model we assume that a vehicle is moving with a constant velocity and that the velocity vector is changing randomly. The state vector contains position vector (coordinates) and velocity vector:

$$\mathbf{x} = \begin{bmatrix} e & n & v_e & v_n \end{bmatrix}^T = \begin{bmatrix} \mathbf{p}^T & \mathbf{v}^T \end{bmatrix}^T \tag{2}$$

The kinematic model:

$$\begin{aligned} \dot{\mathbf{p}} &= \mathbf{v} \\ \dot{\mathbf{v}} &= \mathbf{0} + \boldsymbol{\omega}_a \end{aligned} \tag{3}$$

The matrixes from equation (1) become:

$$\mathbf{F} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} \omega_{ae} \\ \omega_{an} \end{bmatrix} \tag{4}$$

The discrete solution of the differential equation (1) can be generally written as:

$$\mathbf{x}_k = \mathbf{T}_{k-1,k}\mathbf{x}_{k-1} + \mathbf{w}_{k-1,k} \tag{5}$$

where subscript k is a short notation of t_k, which is time of a discrete epoch k and

$$\mathbf{T}_{k-1,k} = e^{\mathbf{F}\cdot\Delta t} \approx \mathbf{I} + \mathbf{F}\Delta t + \frac{1}{2}\mathbf{F}^2\Delta t^2 + \frac{1}{3!}\mathbf{F}^3\Delta t^3 + \frac{1}{4!}\mathbf{F}^4\Delta t^4 + \dots \tag{6}$$

is the transition matrix between epochs k and k+1, where $\Delta t = t_k - t_{k-1}$ and

$$\mathbf{w}_k = \int_{k-1}^{k} \mathbf{T}_{k-1,\tau}\mathbf{G}_\tau\mathbf{u}_\tau d\tau \tag{7}$$

is the driven response at epoch k due to the presence of the white noise input during interval Δt. The covariance of $\mathbf{w}_k$, can be expressed as:

$$\begin{gathered}\mathbf{Q}_k = E\left[\mathbf{w}_{k-1,k}\mathbf{w}_{k-1,k}^T\right] = \\ E\left[\int_{k-1}^{k}\int_{k-1}^{k}\mathbf{T}(k-1,s)\mathbf{G}(s)\mathbf{u}(s)\mathbf{u}^T(t)\mathbf{G}^T(t)\mathbf{T}^T(k-1,t)\,dt\,ds\right] = \\ \int_{k-1}^{k}\int_{k-1}^{k}\mathbf{T}(k-1,s)\mathbf{G}(s)\mathbf{E}\left[\mathbf{u}(s)\mathbf{u}^T(t)\right]\mathbf{G}^T(t)\mathbf{T}^T(k-1,t)\,dt\,ds\end{gathered} \tag{8}$$

$$E\left[\mathbf{u}(s)\mathbf{u}^T(t)\right] = \mathbf{Q} = \begin{bmatrix} q_{ae} & 0 \\ 0 & q_{an} \end{bmatrix} \tag{9}$$

q_{ae} and q_{an} are power spectral density (PSD) of the random acceleration. The unit of q_{ae} and q_{an} is $\left[\left(\frac{m}{s^2\sqrt{Hz}}\right)^2\right]$. Using Equation (6), the solution of the integral (8) can be approximated by the following expansion:

$$\begin{aligned}\mathbf{Q}_k &= E\left[\mathbf{w}_{k-1,k}\mathbf{w}_{k-1,k}^T\right] \approx \mathbf{Q}_G\Delta t + \left(\mathbf{F}\mathbf{Q}_G + \mathbf{Q}_G\mathbf{F}^T\right)\frac{\Delta t^2}{2} + \\ &+\left[\mathbf{F}^2\mathbf{Q}_G + 2\mathbf{F}\mathbf{Q}_G\mathbf{F}^T + \mathbf{Q}_G\left(\mathbf{F}^T\right)^2\right]\frac{\Delta t^3}{6} + \\ &+\left[\mathbf{F}^3\mathbf{Q}_G + 3\mathbf{F}\mathbf{Q}_G\left(\mathbf{F}^T\right)^2 + 3\mathbf{F}^2\mathbf{Q}_G\mathbf{F}^T + \mathbf{Q}_G\left(\mathbf{F}^T\right)^3\right]\frac{\Delta t^4}{24} + \dots\end{aligned} \tag{10}$$

where

$$\mathbf{Q}_G = \mathbf{G}\mathbf{Q}\mathbf{G}^T \tag{11}$$

Since, in the case of our PV model $\mathbf{F}^n = \mathbf{0}, \quad n \geq 2$, the process noise covariance matrix will become exactly:

$$\mathbf{Q}_k = \mathbf{Q}_G\Delta t + \left(\mathbf{F}\mathbf{Q}_G + \mathbf{Q}_G\mathbf{F}^T\right)\frac{\Delta t^2}{2} + \mathbf{F}\mathbf{Q}_G\mathbf{F}^T\frac{\Delta t^3}{3} \tag{12}$$

Taking into account (4) we get:

$$\mathbf{Q}_k = \begin{bmatrix} \frac{q_e \Delta t^3}{3} & 0 & \frac{q_e \Delta t^2}{2} & 0 \\ 0 & \frac{q_n \Delta t^3}{3} & 0 & \frac{q_n \Delta t^2}{2} \\ \frac{q_e \Delta t^2}{2} & 0 & q_e \Delta t & 0 \\ 0 & \frac{q_n \Delta t^2}{2} & 0 & q_n \Delta t \end{bmatrix} \tag{13}$$

The transition matrix **T** can be computed by equation (6) as:

$$\mathbf{T}_{k-1,k} = \mathbf{I} + \mathbf{F}_k \Delta t = \begin{bmatrix} 1 & 0 & \Delta t & 0 \\ 0 & 1 & 0 & \Delta t \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{14}$$

Kalman filtering

The discrete Kalman filter algorithm has the following steps:

1. Initialisation:

$$\mathbf{x}_0, \quad \mathbf{Q}_{x0} = \mathrm{var}[\mathbf{x}_0] \tag{15}$$

2. Time propagation

$$\mathbf{x}_k^- = \mathbf{T}_{k-1,k}\mathbf{x}_{k-1}, \quad \mathbf{Q}_{x,k}^- = \mathbf{T}_{k-1,k}\mathbf{Q}_{x,k-1}\mathbf{T}_{k-1,k}^T + \mathbf{Q}_k \tag{16}$$

3. Gain calculation:

$$\mathbf{K}_k = \mathbf{Q}_{x,k}^- \mathbf{H}_k^T \left[\mathbf{R}_k + \mathbf{H}_k \mathbf{Q}_{x,k}^- \mathbf{H}_k^T \right]^{-1} \tag{17}$$

4. Measurement update

$$\mathbf{x}_k = \mathbf{x}_k^- + \mathbf{K}_k \left[\tilde{\mathbf{L}}_k - \mathbf{h}_k(\mathbf{x}_k^-) \right] \tag{18}$$

5. Covariance update

$$\mathbf{Q}_{x,k} = \left[\mathbf{I} - \mathbf{K}_k \mathbf{H}_k \right] \mathbf{Q}_{x,k}^- \tag{19}$$

$\mathbf{x}_0$ is initial state vector and $\mathbf{Q}_{x0}$ is its covariance matrix. The tilde symbol (~) denotes measured and minus in superscript predicted quantity. $\tilde{\mathbf{L}}_k$ is vector of observations with covariance matrix **R**.

$$\mathbf{R} = \begin{bmatrix} \sigma_e^2 & 0 & 0 \\ 0 & \sigma_n^2 & 0 \\ 0 & 0 & \sigma_v^2 \end{bmatrix} \tag{20}$$

The observation equations (generally non-linear) for epoch k can be written as:

$$\mathbf{L}_k = \mathbf{h}_k(\mathbf{x}_k) \tag{21}$$

H is the design matrix:

$$\mathbf{H}_k = \left.\frac{\partial \mathbf{h}}{\partial \mathbf{x}_k}\right|_{\mathbf{x}=\mathbf{x}_k^-} \tag{22}$$

If we measure position and absolute value of the velocity vector, then the observation equations are as follows:

$$\mathbf{L}_k = \mathbf{h}_k(\mathbf{x}_k) = \begin{bmatrix} e_k & n_k & \sqrt{v_{ek}^2 + v_{nk}^2} \end{bmatrix}^T \tag{23}$$

or introducing measured values and their random errors $\boldsymbol{\varepsilon}$:

$$\tilde{\mathbf{L}}_k + \boldsymbol{\varepsilon}_k = \begin{bmatrix} \tilde{e}_k \\ \tilde{n}_k \\ \tilde{v}_k \end{bmatrix} + \boldsymbol{\varepsilon}_k = \begin{bmatrix} e_k \\ n_k \\ \sqrt{v_{e,k}^2 + v_{n,k}^2} \end{bmatrix} \tag{24}$$

The last equation in (24)

$$v_k = \tilde{v}_k + \varepsilon_{v,k} = \sqrt{v_{e,k}^2 + v_{n,k}^2} \tag{25}$$

is not linear, therefore it is linearised around the predicted values $v_{e,k}^-$ and $v_{n,k}^-$

$$\begin{aligned} v_k &= v_k^- + \frac{\partial v_k^-}{\partial v_{e,k}^-}\Delta v_{e,k} + \frac{\partial v_k^-}{\partial v_{n,k}^-}\Delta v_{n,k} + \cdots \\ v_k^- &= \sqrt{(v_{e,k}^-)^2 + (v_{n,k}^-)^2} \end{aligned} \tag{26}$$

Design matrix (22) contains partial derivatives of $\mathbf{h}_k(\mathbf{x}_k)$ (Equation (21)) with respect to all variables in the state vector:

$$\mathbf{H}_k = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{v_{e,k}^-}{v_k^-} & \frac{v_{n,k}^-}{v_k^-} \end{bmatrix} \tag{27}$$

$$\mathbf{h}_k(\mathbf{x}_k^-) = \begin{bmatrix} e_k^- & n_k^- & \sqrt{(v_{ek}^-)^2 + (v_{nk}^-)^2} \end{bmatrix}^T \tag{28}$$

Smoothing

After the last observation we get an estimation of the state vector denoted as $\mathbf{x}_N$, and covariance matrix $\mathbf{Q}_{x,N}$. The smoothed estimation for the previous epochs is given by

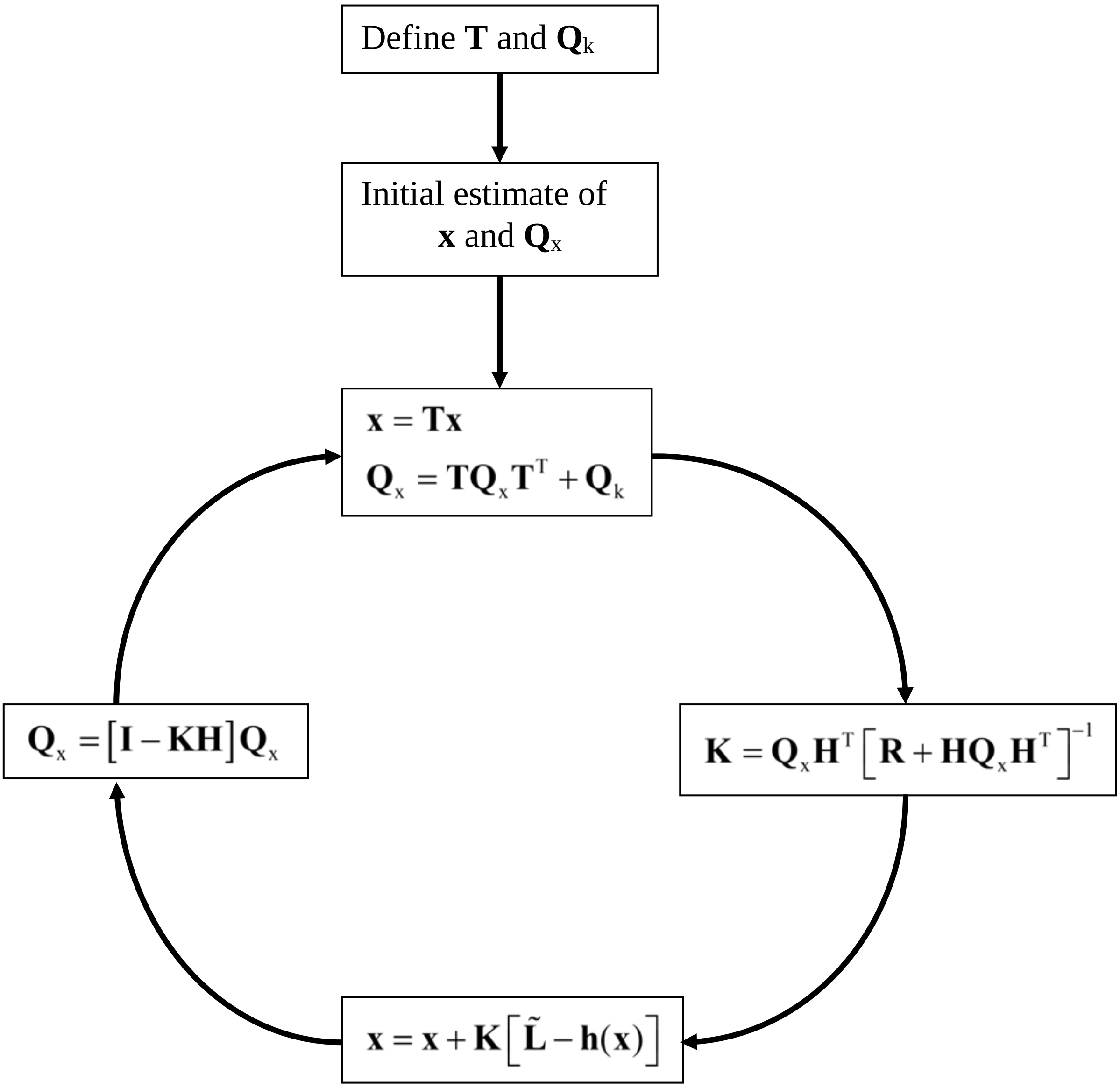
$$\hat{\mathbf{x}}_k = \mathbf{x}_k + \mathbf{D}_k \left[\hat{\mathbf{x}}_{k+1} - \mathbf{x}_{k+1}^- \right] \tag{29}$$

$$\mathbf{D}_k = \mathbf{Q}_{x,k} \mathbf{T}_{k-1,k}^T \left(\mathbf{Q}_{x,k+1}^- \right)^{-1} \tag{30}$$

and the covariance matrix of the smoothed state vector:

$$\hat{\mathbf{Q}}_{x,k} = \mathbf{Q}_{x,k} + \mathbf{D}_k \left[\hat{\mathbf{Q}}_{x,k+1} - \mathbf{Q}_{x,k+1}^- \right] \mathbf{D}_k^T \tag{31}$$

Kalman filter loop



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