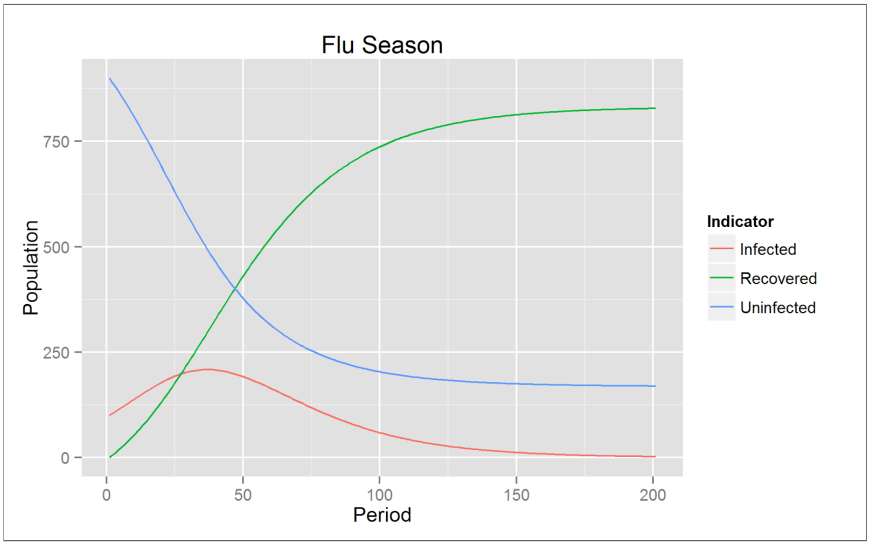
LECTURE 7: SIMPLE DYNAMIC MODELS

RETURN TO BIG PICTURE - MODEL TYPES

Conceptual…….Mathematical
Stochastic…….Deterministic
Lumped…….Spatially Distributed: *SPACE*
Static…….Dynamic : *TIME*
Abstract…….Physically/Process Based

DYNAMIC MODELS

Example of a time varying (dynamic) model
Modeling disease through time
source



SPATIAL MODELS THAT INTERACT - SIMILAR TO DYNAMIC

- Key idea is dependency from one point in time/space to the next!
- This implies a differential/difference equation to describe the process/transfer function

Spatially distributed - model is applied to different “patches” in space

- spatial units are independent

SPATIAL MODELS THAT INTERACT - SIMILAR TO DYNAMIC

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DYNAMIC MODELING

Dynamic modeling is common in environmental problems solving

- Similar issues: what happens at one place, depends on neighbors;
- what happens at one time; depends on previous time -

Space - two way; Time is usually one-way
Dynamic system modeling - quickly becomes complex (Engineering degrees spend a lot of time on this; there are books, entire journals etc on this topic)

WHY YOU SHOULD CARE

Many environmental problems and questions can be related to

- Diffusion
- Population

Both often require dynamics models; and both often require thinking about dynamics in space and in time

VOCAB

Some useful terminology

- **stocks** - variables that evolve over time
- **flows** - transfers between variables or from the system
- **parameters** - values that controls the relationship between stocks and flows
- **sink** - something that absorbs flows
- **source** - something that generates flows

MORE TERMINOLOGY

- **System state**: value of all variables need to describe the "entity that evolves through time" at a particular point in time
  - usually think of these as stores (soil moisture, bank account balance, number of individuals in a particular age class)
- **State-space**: description of the entity may require multiple variables - for a watershed this could be soil moisture, water currently in dam and water stored in trees, and for each "grid" in a watershed)
- **State-space trajectories**: how the system state evolves through time
- **Initial conditions**: values to describe the system state at the beginning

EXAMPLES - NOTE THE FEEDBACK LOOPS (TIME DEPENDENCY)

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HUMAN ENGINEERING EXAMPLE

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Feekback is a key feature of dynamic models {.scrollable}

- dynamic systems often have *feedbak* loops
  - positive
  - negative
- feedbacks often lead to non-linear responses
  - think “runaway” growth

DISCRETE VS CONTINUOUS MODELS

- do we break time/space in to “chunks” *discrete*
- or think of time/space as a stream. *continuous*

Discrete might be modeling something that occurs once every period; water withdrawls, tax revenues
Continuous would be things like pollution, water that are always changing in time
Discrete models for dynamic processes use *difference* equations Continuous models for dynamic processes use *differential* equations
Often it depends on how you look at the system -

DYNAMIC SYSTEM

Initial conditions

- to model how something evolves over space and time
  - you need to know where it starts from
  - setting initial conditions is usually part of dynamic modellinng

Boundary Value Problems

- especially for spatial dynamic models sometimes initial conditions are provided values at “end points”

STABILITY

Dynamic systems may lead to *unstable* or *stable* or *cyclic* states over time

- **stable**…output converges to a particular over time, or for *cyclic* a repeated patterns of values
- **unstable** …grows to infinity
- *chaotic* ..highly sensitive to initial conditions
- for the same dynamic systems whether you are stable can change with parameters and initial conditions

PARTNER EXERCIES

Discuss examples of dynamic systems * spatial interaction * temporal interaction * both

- Positive feedback
- Negative feedback

Think of a model where you might have a multi-dimensional system state

- what variables would define the state-space
- what would control the evolution of the state-space
- what would be the initial conditions

SIMPLE EXAMPLE

Exponential Growth - Simple Dynamic System

- rate of growth(change) = r * population(density)
- differential equation
- $dP / dt = rP$

EXPONENTIAL GROWTH - SIMPLE DYNAMIC SYSTEM

- differential equation

  $dP / dt = rP$

- an analytic solution exists so we can write Population as a function of time by integrating both sides
- $P = PO * exp(rt)$

  This is a regular input-output function - that gives population after some time t

SIMPLE IMPLEMENTATION

```
1 source(here("R/exppop.R"))
2 exppop
```

```
function (T, P0, r)
{
    P <- P0 * exp(r * T)
    return(P)
}
```

```
1 # lets look at how this models population over time
2 years <- seq(from = 1, to = 100, by = 2)
3 Ptime <- years %>% map_dbl(~ exppop(P0 = 20, r = 0.01, T = .x
4
5                             # keep track of what times we ran
6 Ptime <- data.frame(P = Ptime, years = years)
7 # plot
8 ggplot(Ptime, aes(years, P)) +
9   geom_point() +
10   labs(x = "years", y = "Rabbit Population")
```

INTEGRATION, OR SOLVING DIFFERENTIAL EQUATIONS

-We want the value of the dependent variation (population) over a range of values for independent variable (time)

- We know how dependent variable is changing (that's the differential equation)

$dP/dt = rP$

- For each P we can approximate the next P after a small time period
- $P_{t+1} = P + dP/dt * Timestep$
- But as P changes dP/dt changes so we have to keep time step small (really small if possible)

IMPLEMENTING DYNAMIC MODELS IN R

Dynamic models always involves derivatives (equations that express how things change from time step to time step or place to place)
Implement population growth as a derivative - a model of population change

```
# note that we include time here but we don't use it; we will
source(here("R/dexppop.R"))

dexppop
```

```
function (time, P, r)
{
    dexpop = r * P
    return(list(dexpop))
}
```

```
# see how it works
dexppop(P = 20, r = 0.01)
```

```
[[1]]
[1] 0.2
```

```
# what is this?

# notices this is the same as
dexppop(t = 100, P = 20, r = 0.01)
```

```
[[1]]
[1] 0.2
```

```
# lets look at this for a range of initial populations
pops <- seq(from = 1, to = 100)
tmp <- pops %>% map(~ dexppop(time = 0, r = 0.01, P = .x))
pchange <- unlist(tmp)
```

PLOTTING THE DERIVATIVE (RATE OF CHANGE)

```
pdyn <- data.frame(pops, pchange)
ggplot(pdyn, aes(pops, pchange)) +
  geom_point(col = "green", size = 1.5)
```

- why is this a straight line?
- how many new individuals are born at each population level

  try this - add a carrying capacity (dP/dt = 0 if P > carryingcapacity) `

INTEGRATION

What if we wanted to look at population in 20 years given an initial condition
Two options - First (and easiest)

- explicit solution to differential equation is known; e.g. you can integrate both sides of the equation! Not always possible but lets look at a case where it is possible
- you can get the answer exactly and quickly this way

EXPLICIT SOLUTION IS AVAILABLE

Even with carrying capacity there is an analytical solution, depending on how you think about carrying capacity

- if you think of carrying capacity as a *hard* limit on the population growth rate, then you can still use the same equation, and just add a limit

```
source(here("R/exppopK.R"))

exppopK
```

```
function (T, P0, r, K)
{
    P <- P0 * exp(r * T)
    if (P > K) {
        P <- K
    }
    return(P)
}
```

```
# gives population after any time given an initial population

# 20 rabbits, growth rate of 0.01 how many in 30 years
exppopK(T = 30, P0 = 20, r = 0.01, K = 1000)
```

```
[1] 26.99718
```

```
# if we want to see how population evolves over time - genera

initialrabbits <- 20
years <- seq(from = 1, to = 100, by = 2)
Ptime <- years %>% map_dbl(~ exppopK(P0 = initialrabbits, r =

# keep track of what times we ran
Ptime <- data.frame(P = Ptime, years = years)

ggplot(Ptime, aes(years, P)) +
  geom_point() +
  labs(x = "years", y = "Rabbit Population")
```

```r
# try generating results for maximum and minimum possible r v


max_r <- 0.1
min_r <- 0.01
K <- 1000

tmp <- years %>% map_dbl(~ exppopK(r = max_r, P0 = initialrab
Ptime$Pmaxr <- tmp
tmp <- years %>% map_dbl(~ exppopK(r = min_r, P0 = initialrab
Ptime$Pminr <- tmp

head(Ptime)
```

```
         P years    Pmaxr    Pminr
1 20.20100     1 22.10342 20.20100
2 20.60909     3 26.99718 20.60909
3 21.02542     5 32.97443 21.02542
4 21.45016     7 40.27505 21.45016
5 21.88349     9 49.19206 21.88349
6 22.32556    11 60.08332 22.32556
```

```r
Ptimep <- Ptime %>% gather(key = "r", value = "P", -years)
ggplot(Ptimep, aes(years, P, col = r)) +
  geom_point() +
  labs(x = "years", y = "Rabbit Population")
```

```r
# notice how population becomes unstable for high growth rate
```

INTEGRATION

What if we wanted to look at population in 20 years given an initial condition
Two options

- explicit solution to differential equation is known; e.g. you can integrate both sides of the equation! Not always possible but lets look at a case where it is possible
- must be solved by iteration; this is what we do when we can't integrate both sides

What if no analytical solutions is available

EXAMPLE

Continue looking at our rabbit population but we can't solve it..

- iterate through time
- calculate rate of change at initial conditions (e.g value of differential equation for initial conditions)
- add that value to initial conditions, to create state at time t + 1
- re-calculate rate of change (e.g value of differential equation for state at time t+1)
- keep doing this

Key questions is how much of a "jump" in t do we do
If you do this you are "kind of" turning the differential equation into a difference equation

SOLVING BY THINKING OF PROBLEM AS A DIFFERENCE EQUATIONS

Population models can be discrete (rather than continuous)
So we could implement them as difference equations and iterate

```
source(here("R/discrete_logistic_popK.R"))
# notice how a for loop is used to iterate

# how many rabbits after 50 years given a growth of 0.1
# starting with 1 rabbit - but a carrying capcity of 500

discrete_logistic_pop
```

```
function (P0, r, K, T = 10)
{
    pop <- P0
    for (i in 1:T) {
        pop <- pop + r * pop
        pop <- ifelse(pop > K, K, pop)
    }
    return(pop)
}
```

```
discrete_logistic_pop(P0 = 1, r = 0.05, K = 200, T = 50)
```

```
[1] 11.4674
```

```
# save results
discrete_result <- discrete_logistic_pop(P0 = 1, r = 0.05, K
# lets also keep the parameters for use later
P0 <- 1
r <- 0.05
K <- 200
T <- 50
```

COMPARE DISCRETE AND ANALYTIC RESULTS

Save the results from both to compare

```
source(here("R/exppop.R"))

exppopK(P0 = P0, r = r, K = K, T = T)
```

```
[1] 12.18249
```

```
analytic_result <- exppopK(P0 = P0, r = r, K = K, T = T)

analytic_result
```

```
[1] 12.18249
```

```
discrete_result
```

```
[1] 11.4674
```

```
# why are they different
# look at trajectories

growth_result <- data.frame(time = seq(from = 1, to = 100))

growth_result$Panalytic <- growth_result$time %>% map_dbl(~ e

growth_result$Pdiscrete <- growth_result$time %>% map_dbl(~ d
```

GRAPH

```
tmp <- growth_result %>% gather(key = "Ptype", value = "P", -
ggplot(tmp, aes(time, P, col = Ptype)) +
  geom_point()
```

```
# try running them for longer time periods to see what happen
# change the value of r, K , P0 - see how it effects the resu
```

SOLVING USING NUMERIC INTEGRATION

Using a solver….when you can't do the integration by hand :)
Solvers integrate by iteration but doing so in a way that more closely *approximates* analytic integration
Use mathematical tricks to deal with the fact that the rate of change keeps changing :)
There are different types of *solvers*, some work better than others depending on the form of the derivative

NUMERICAL INTEGRATION WITH ODE

Implement the differential equation as a function that

- returns the derivative (as a list)
- inputs time, the variable(s) and a parameter list

(it needs time even though you don't use it)
My convention: name derivative functions starting with *d* to remind myself that they are computing a derivative

ODE

Only works for Ordinary Differential Equations - single independent variable (in our case time)
Partial differential equations - more than 1 independent variable (e.g x and y if changing in space)
R has a solver called *ODE* for solving ordinary differential equations from package **desolve**

ODE REQUIRES

- initial conditions
- values of independent where we want values of dependent variable (e.g times where we want population)
- the derivative as a function
- a list that contains all parameter values (or if you have only one parameter then you can use a single value)

ODE EXAMPLE

```
source(here("R/dexppop.R"))

dexppop
```

```
function (time, P, r)
{
    dexpop = r * P
    return(list(dexpop))
}
```

```
library(deSolve)
initialrabbits <- 20
years <- seq(from = 1, to = 100, by = 2)

# run the solver
Ptime <- ode(y = initialrabbits, times = years, func = dexppo
head(Ptime)
```

```
     time        1
[1,]    1 20.00000
[2,]    3 20.40404
[3,]    5 20.81623
[4,]    7 21.23675
[5,]    9 21.66576
[6,]   11 22.10344
```

```
colnames(Ptime) <- c("year", "P")

# notice that there are additional pieces of information year
attributes(Ptime)
```

```
$dim
[1] 50  2

$dimnames
$dimnames[[1]]
NULL

$dimnames[[2]]
[1] "year" "P"
```

```
$istate
 [1]   2  52 105  NA   6   6   0  36  21  NA  NA  NA  NA   0
1   1  NA  NA  NA
[20]  NA  NA
```

```
# this also means you need to extract just the data frame for
ggplot(as.data.frame(Ptime), aes(year, P)) +
  geom_point() +
  labs(y = "Population", "years")
```

```
# this also works (of course function can be by order)
Ptime <- ode(initialrabbits, years, dexppop, 0.03)
colnames(Ptime) <- c("year", "P")
ggplot(as.data.frame(Ptime), aes(year, P)) +
  geom_point() +
  labs(y = "Population", "years")
```

HOMEWORK

Try to modify *dexppop* so that it includes carrying capacity and compare with what we did for our analytic and then discrete ways of doing this