Problem

True or false?

$$999! < 500^{999} \tag{1}$$

Solution

We will use Geometric Mean $\leq$ Arithmetic Mean, i.e. for non-negative x, and y,

$$\sqrt{x \times y} \leq \frac{x+y}{2} \tag{2}$$

with equality iff $x = y$.

Elementary proof:

$$\begin{aligned} & (a-b)^2 \geq 0, && \text{with equality iff } a = b && (3) \\ \therefore \quad & a^2 + b^2 \geq 2ab && && (4) \\ \therefore \quad & \frac{x+y}{2} \geq \sqrt{xy}, && \text{where } x = a^2, y = b^2 && (5) \end{aligned}$$

Now split each term in $n!$ into a $\sqrt{\cdot}$ pair, rearrange and regroup, before applying the GM≤AM inequality on each:

$$\begin{aligned} n! &= \sqrt{n \times 1} \quad \sqrt{(n-1) \times 2} \quad \dots \quad \sqrt{2 \times (n-1)} \; \sqrt{1 \times n} && (6) \\ &< \frac{n+1}{2} \quad \frac{n+1}{2} \quad \dots \quad \frac{n+1}{2} \quad \frac{n+1}{2} && (7) \\ &= \left(\frac{n+1}{2}\right)^n && (8) \end{aligned}$$

(the inequality $\leq$ has become strict $<$ because at least one of the term pairs are different).

Set $n = 999$ to answer the problem with the affirmative:

$$999! < \left(\frac{999+1}{2}\right)^{999} = 500^{999} \tag{9}$$

Discussion

How tight is this bound? Not very! It is made from a product of n terms, each larger than the term it replaces. Further, the terms “at the ends” consist of pairs of numbers which are most different, and from the proof for the inequality one can see that these have the weakest bound (or conversely, the bound is tightest when the two numbers are most similar, becoming exact when the two numbers are equal).

Is there are a different way to arrange the $\sqrt{\cdot}$ pairs to produce a tighter bound?

The above approach pairs numbers as follows:

$$\begin{array}{ccccccc} 1 & 2 & 3 & \dots & n-2 & n-1 & n \\ | & | & | & & | & | & | \\ n & n-1 & n-2 & \dots & 3 & 2 & 1 \end{array}$$

We can bring the pairs numerically closer to one another by “rotating” the bottom row:

$$
\begin{array}{cccccc|c}
1 & 2 & 3 & \dots & n-2 & n-1 & n \\
| & | & | & & | & | & | \\
n-1 & n-2 & n-3 & \dots & 2 & 1 & n
\end{array}
$$

Rotating by 3 steps matches as follows:

$$
\begin{array}{ccccc|ccc}
1 & 2 & 3 & \dots & n-3 & n-2 & n-1 & n \\
| & | & | & & | & | & | & | \\
n-3 & n-4 & n-5 & \dots & 1 & n & n-1 & n-2
\end{array}
$$

This makes

- $n-3$ pairs which individually sum to $n-2$, and
- 3 pairs which individually sum to $2n-2$.

The GM-AM inequality approach then produces

$$
\begin{aligned}
n! &= \sqrt{(n-3) \times 1}\ \sqrt{(n-4) \times 2} \dots \sqrt{1 \times (n-3)} && (10) \\
&\quad \times \sqrt{n \times (n-2)}\ \sqrt{(n-1) \times (n-1)}\ \sqrt{(n-2) \times n} && (11) \\
&\quad < \left(\frac{n-2}{2}\right)^{n-3} \times \left(\frac{2n-2}{2}\right)^{3} && (12)
\end{aligned}
$$

Generalising further with an arbitrary rotation by k steps,

$$
\begin{array}{ccccc|cccc}
1 & 2 & 3 & \dots & n-k & n-k+1 & \dots & n-1 & n \\
| & | & | & & | & | & & | & | \\
n-k & n-k-1 & n-k-2 & \dots & 1 & n & \dots & n-k+2 & n-k+1
\end{array}
$$

and again we have two set of pairs:

- $n-k$ pairs which individually sum to $n-k+1$, and
- k pairs which individually sum to $2n-k+1$

The GM-AM inequality approach then produces

$$
n! < \left(\frac{n-k+1}{2}\right)^{n-k} \times \left(\frac{2n-k+1}{2}\right)^{k} \qquad (13)
$$

Splitting down the middle for an even n, i.e. $n = 2k$, then

$$
\begin{aligned}
n! &< \left(\frac{n-\frac{n}{2}+1}{2}\right)^{\frac{n}{2}} \times \left(\frac{2n-\frac{n}{2}+1}{2}\right)^{\frac{n}{2}} && (14) \\
&= \left(\frac{2n-n+2}{4}\right)^{\frac{n}{2}} \times \left(\frac{4n-n+2}{4}\right)^{\frac{n}{2}} && (15) \\
&= \left(\frac{(n+2)(3n+2)}{16}\right)^{\frac{n}{2}} && (16)
\end{aligned}
$$

For large n, this behaves like

$$
\left(\frac{3n^2}{16}\right)^{\frac{n}{2}} = \frac{n^n}{\left(\frac{4}{\sqrt{3}}\right)^n} \qquad (17)
$$

which compares favorably to the large n behavior of the original bound,

$$\left(\frac{n+1}{2}\right)^n \approx \frac{n^n}{2^n}, \qquad \text{for large } n \tag{18}$$

So our "large n" upper bound for $n!$ has reduced (improved) by a factor of

$$\left(\frac{n^n}{\left(\frac{4}{\sqrt{3}}\right)^n}\right) / \left(\frac{n^n}{2^n}\right) = \left(\frac{2\sqrt{3}}{4}\right)^n = \left(\frac{\sqrt{3}}{2}\right)^n \approx (0.87)^n \tag{19}$$

Another bound on $n!$

To better understand the tightness of the bounds achieved by the approach above, consider the proof of a different bound on $n!$,

$$e\left(\frac{n}{e}\right)^n \ \le \ n! \ \le \ ne\left(\frac{n}{e}\right)^n \tag{20}$$

Proof of lower bound

Start with a bound on the exponential function (see Appendix)

$$e^x \ge 1 + x \qquad\qquad \text{for } |x| < 1 \tag{21}$$

$$e^{\frac{1}{k}} \ge 1 + \frac{1}{k} \tag{22}$$

$$e^{\frac{1}{k}} \ge \frac{k+1}{k} \tag{23}$$

$$e \ge \left(\frac{k+1}{k}\right)^k \tag{24}$$

$$\prod_{k=1}^{n-1} e \ge \prod_{k=1}^{n-1} \left(\frac{k+1}{k}\right)^k \tag{25}$$

$$e^{n-1} \ge \left(\frac{2}{1}\right)^1 \left(\frac{3}{2}\right)^2 \left(\frac{4}{3}\right)^3 \ldots \left(\frac{n}{(n-1)^{n-1}}\right) \tag{26}$$

$$e^{n-1} \ge \left(\frac{1}{1}\right)\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)\ldots\left(\frac{n^{n-1}}{n-1}\right) \tag{27}$$

$$e^{n-1} \ge \frac{n^{n-1}}{(n-1)!} \tag{28}$$

$$e^{n-1} \ge \frac{n^n}{n!} \tag{29}$$

Rearranging gives the lower bound:

$$e\left(\frac{n}{e}\right)^n \le n! \tag{30}$$

Proof of upper bound

Proceed in a similar way as above but with a different substitution,

$$e^x \ge 1 + x \tag{31}$$

$$e^{-\frac{1}{k+1}} \geq 1 - \frac{1}{k+1} \tag{32}$$

$$e^{-\frac{1}{k+1}} \geq \frac{k}{k+1} \tag{33}$$

$$e \leq \left(\frac{k+1}{k}\right)^{k+1} \tag{34}$$

$$\prod_{k=1}^{n-1} e \leq \prod_{k=1}^{n-1} \left(\frac{k+1}{k}\right)^{k+1} \tag{35}$$

$$e^{n-1} \leq \left(\frac{2}{1}\right)^1 \left(\frac{3}{2}\right)^2 \left(\frac{4}{3}\right)^3 \dots \left(\frac{n}{n-1}\right)^n \tag{36}$$

$$e^{n-1} \leq \left(\frac{1}{1}\right) \left(\frac{1}{2}\right) \left(\frac{1}{3}\right) \dots \left(\frac{n^n}{n-1}\right) \tag{37}$$

$$e^{n-1} \leq \frac{n^n}{(n-1)!} \tag{38}$$

$$e^{n-1} \leq \frac{n^{n+1}}{n!} \tag{39}$$

Rearranging gives the upper bound:

$$n! \leq ne\left(\frac{n}{e}\right)^n \tag{40}$$

Note that both of these bounds are reasonably tight because the substitution into the inequality keeps x small, and hence $1 + x \approx e^x$.

As an aside, it is worth observing that from the middle steps (Equation 24 and Equation 34) of these lower and upper bounds, we find

$$\left(\frac{k+1}{k}\right)^k = \quad \left(1 + \frac{1}{k}\right)^k \leq e \leq \left(1 + \frac{1}{k}\right)\left(1 + \frac{1}{k}\right)^k \quad = \left(\frac{k+1}{k}\right)^{k+1} \tag{41}$$

from which we have the limit definition of $e = \lim_{k\to\infty} \left(1 + \frac{1}{k}\right)^k$.

The approach of using the $1 + x \leq e^x$ inequality with a repeated product, converting the exponential argument into addition, is reminiscent of a lovely proof of the generalised AM-GM inequality.

Another proof of AM-GM inequality

Let

$$p_i \geq 0 \tag{42}$$

$$a_i > 0 \tag{43}$$

$$\sum_i^n p_i = 1 \tag{44}$$

$$A = \sum_i^n p_i a_i \tag{45}$$

$$G = \prod_i^n a_i^{p_i} \tag{46}$$

From $x + 1 \leq e^x$ for $x > -1$, we have $x \leq e^{x-1}$ for $x > 0$. Let $x = a_i / A$,

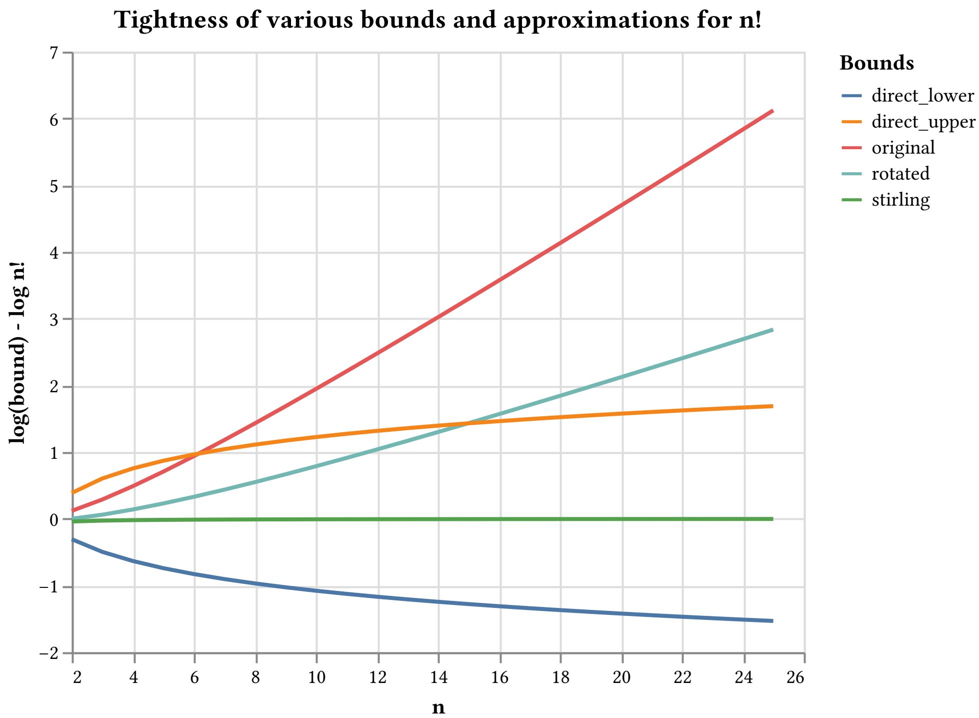
$$\frac{a_i}{A} \leq e^{\frac{a_i}{A}-1} \tag{47}$$

$$\prod_i^n \left(\frac{a_i}{A}\right)^{p_i} \leq \exp\left(\sum_i^n p_k \frac{a_k}{A} - \sum_i^n p_i\right) = \exp(1-1) = 1 \tag{48}$$

$$G = \prod_i^n a_i^{p_i} \leq \sum_i^n p_i a_i = A \tag{49}$$

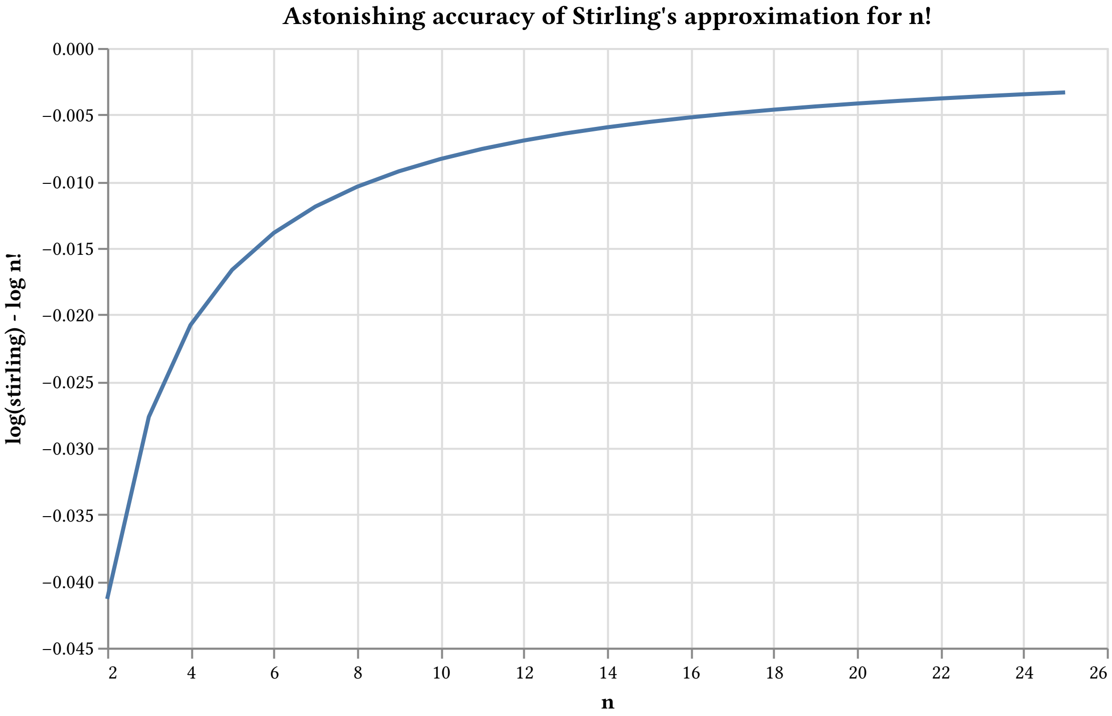
as required.

Bringing this back to the original problem, an upper bound on $n!$ can be found using a “AM-GM product inequality approach”, where the AM-GM inequality can be proven using an “exponential product inequality approach. Alternatively, a direct application of the “exponential product inequality approach” yields a tighter bound, as can be seen in the following plot.



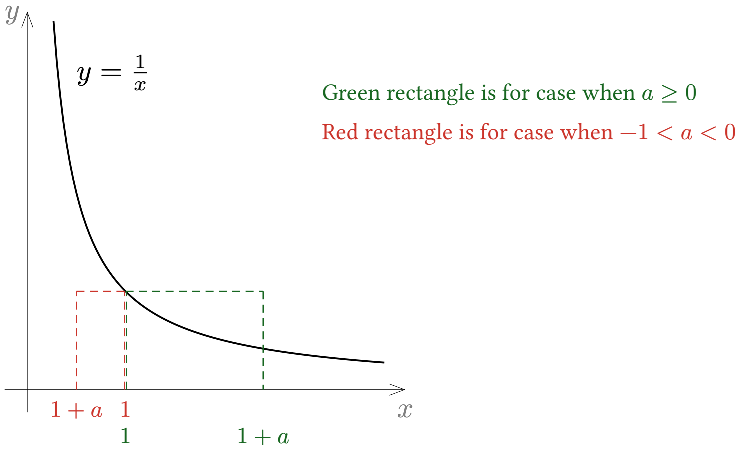
The y-axis is the log of the bound (or approximation) minus $\ln(n!)$. This makes it the logarithm of the ratio of the bound with the true value. Hence the best bound (or approximation) will be a horizontal line through $y = 0$.

The “original” bound is that of the original problem. The “rotated” is the modified version, which certainly lowers the upper bound, but still rises as a power too fast. The “direct_lower” and “direct_upper” are much better, but still not nearly as good as Stirling’s approximation, as shown in the following plot which only includes Stirling’s approximation (for a larger range of n):


Astonishing accuracy of Stirling's approximation for n!
log(stirling) - log n!
0.000
−0.005
−0.010
−0.015
−0.020
−0.025
−0.030
−0.035
−0.040
−0.045
2
4
6
8
10
12
14
16
18
20
22
24
26
n

Appendix - ln and exp bounds

Bounds on ln and exp can be proven by starting with the monotonically decreasing function $1/x$.



For $a \geq 0$,

$$\int_1^{1+a} \frac{1}{x} \, \mathrm{d}x \leq 1 \times ((1+a) - 1) \tag{50}$$

$$\ln(1+a) \leq a \tag{51}$$

For $-1 < a < 0$,

$$\int_{1+a}^{1} \frac{1}{x} \, \mathrm{d}x \geq 1 \times (1 - (1+a)) \tag{52}$$

$$-\ln(1+a) \geq -a \tag{53}$$

$$\ln(1+a) \leq a \tag{54}$$

Hence

$$\ln(1+x) \leq x \qquad \text{for all } x > -1, \text{with equality when } x = 0 \tag{55}$$

Since $\exp(\cdot)$ is monotonically increasing,

$$1 + x \leq e^x \qquad \text{for all } x > -1, \text{with equality when } x = 0 \tag{56}$$

An upper bound on $\exp(\cdot)$ can also be found by considering the Taylor series expansion:

$$e^x = \sum_{j=0}^{\infty} \frac{x^j}{j!} \quad = 1 + x + x^2 \sum_{j=2}^{\infty} \frac{x^{j-2}}{j!} \tag{57}$$

$$< 1 + x + x^2 \sum_{j=2}^{\infty} \frac{x^{j-2}}{(j-2)!} \tag{58}$$

$$= 1 + x + x^2 \sum_{j=0}^{\infty} \frac{x^j}{j!} \tag{59}$$

$$= 1 + x + x^2 e \tag{60}$$

Hence

$$e^x < 1 + x + ex^2 \tag{61}$$