Problem

Given $x^5 = 1$ and $x \neq 1$, find

$$a = \left(\frac{1}{x^2+x+1} + \frac{1}{x^2-x+1}\right)^5 \tag{1}$$

Solution

We will cross-multiply to create a common denominator. So start by observing that

$$\begin{aligned}
(x^2+x+1)(x^2-x+1) &= x^4+x^2+1 && && (2)\\
&= \frac{x^6-1}{x^2-1} && (\text{GP with factor } x^2) && (3)\\
&= \frac{x-1}{(x-1)(x+1)} && (x^5=1) && (4)\\
&= \frac{1}{x+1} && (x \neq 1) && (5)
\end{aligned}$$

Therefore

$$\begin{aligned}
a &= \left[\frac{2x^2+2}{\frac{1}{x+1}}\right]^5 && && (6)\\
&= 2^5(1+x+x^2+x^3)^5 && && (7)\\
&= 2^5\left(\frac{x^4-1}{x-1}\right)^5 && (\text{GP with factor } x) && (8)\\
&= 2^5\left(\frac{x^5-x}{x(x-1)}\right)^5 && && (9)\\
&= 2^5\left(\frac{1-x}{x(x-1)}\right)^5 && (x^5=1) && (10)\\
&= -2^5\left(\frac{1}{x}\right)^5 && (x \neq 1) && (11)\\
&= -2^5 && (x^5=1) && (12)\\
&= -32 && && (13)
\end{aligned}$$