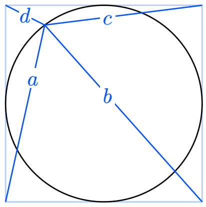
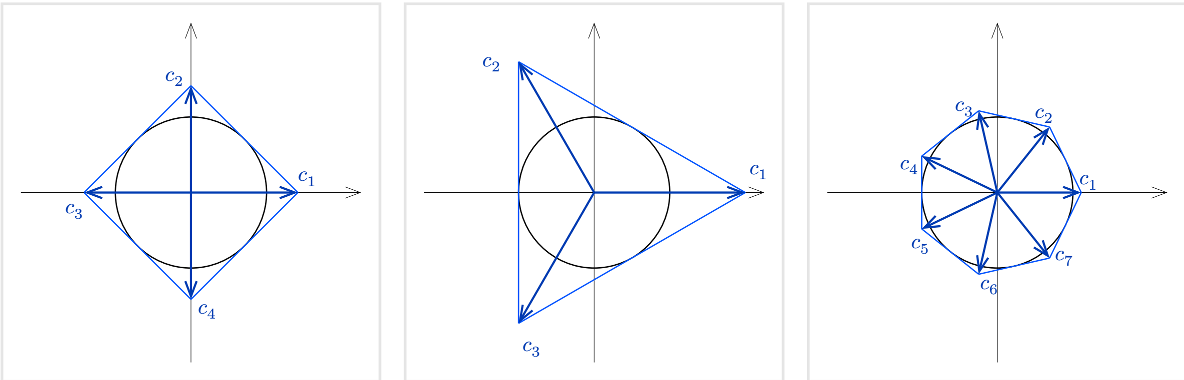
Problem

Find: $a^2 + b^2 + c^2 + d^2$ in the following



Solution

Centre the circle at the origin and rotate by 45°. Also, generalise to any number of corners, and let those corners be the vectors, c_k. For example,



These corner vectors all have the same length, and by symmetry sum to 0:

$$|c_k| = c \tag{1}$$

$$\sum_{k=1}^{n} c_k = 0 \tag{2}$$

Let r be any radius vector of the circle, and define x_k such that

$$r + x_k = c_k \tag{3}$$

For the case of $n = 4$, these x_k correspond to the a, b, c, d of the question. Hence, define the sum

$$S_n = \sum_{k=1}^{n} x_k \cdot x_k \tag{4}$$

$$= \sum_{k=1}^{n} (c_k - r) \cdot (c_k - r) \tag{5}$$

$$= \sum_{k=1}^{n} c_k \cdot c_k - 2r \cdot \sum_{k=1}^{n} c_k + \sum_{k=1}^{n} r \cdot r \tag{6}$$

The first term is nc^2 because of Equation 1, and the middle term is 0 because of Equation 2. Hence

$$S_n = n(c^2 + r^2) \tag{7}$$

For a unit circle with $r = 1$, simple geometry means $c = \cos(\frac{\pi}{n})^{-1}$. Hence

$$S_n = n\left(\frac{1}{\cos(\frac{\pi}{n})^2} + 1\right) \tag{8}$$

In the case of the original question, $n = 4$ which makes the answer $\boxed{S_4 = 12.}$

(Note that elsewhere on the internet the problem is described as having a unit *square*. This makes $r = 0.5$ and so $a^2 + b^2 + c^2 + d^2 = 3$.)

Evaluating for unit circle and various number of sides:

n	S_n
3	15
4	12
5	12.639
6	14
7	15.623
8	17.373
9	19.192
10	21.056
11	22.948
12	24.862
13	26.79
14	28.729
15	30.678
16	32.633
17	34.594
18	36.56
19	38.529
20	40.502

It is interesting to observe that the only integer solutions are for $n = 3, 4, 6$. Also that this quantity is smallest for $n = 4$.