## Problem

True or false?

$$999! < 500^{999} \tag{1}$$

## Solution

We will use Geometric Mean $\leq$ Arithmetic Mean, i.e. for non-negative $x$, and $y$,

$$\sqrt{x \times y} \leq \frac{x+y}{2} \tag{2}$$

with equality iff $x = y$.

Proof:

$$(a-b)^2 \geq 0, \qquad \text{with equality iff } a = b \tag{3}$$

$$\therefore \quad a^2 + b^2 \geq 2ab \tag{4}$$

$$\therefore \quad \frac{x+y}{2} \geq \sqrt{xy}, \qquad \text{where } x = a^2, y = b^2 \tag{5}$$

Now split each term in $n!$ into a $\sqrt{\cdot}$ pair, rearrange and regroup, before applying the GM$\leq$AM inequality on each:

$$n! = \sqrt{n \times 1} \quad \sqrt{(n-1) \times 2} \quad \ldots \quad \sqrt{2 \times (n-1)} \ \sqrt{1 \times n} \tag{6}$$

$$< \frac{n+1}{2} \quad \frac{n+1}{2} \quad \ldots \quad \frac{n+1}{2} \quad \frac{n+1}{2} \tag{7}$$

$$= \left(\frac{n+1}{2}\right)^n \tag{8}$$

(the inequality $\leq$ has become strict $<$ because at least one of the term pairs are different).

Set $n = 999$ to answer the problem with the affirmative:

$$999! < \left(\frac{999+1}{2}\right)^{999} = 500^{999} \tag{9}$$