Problem

True or false?

$$999! < 500^{999} \tag{1}$$

Solution

We will use Geometric Mean $\leq$ Arithmetic Mean, i.e. for non-negative x, and y,

$$\sqrt{x \times y} \leq \frac{x+y}{2} \tag{2}$$

with equality iff $x = y$.

Proof:

$$\begin{aligned}
& (a-b)^2 \geq 0, && \text{with equality iff } a = b && (3)\\
\therefore \quad & a^2 + b^2 \geq 2ab && && (4)\\
\therefore \quad & \frac{x+y}{2} \geq \sqrt{xy}, && \text{where } x = a^2, y = b^2 && (5)
\end{aligned}$$

Now split each term in $n!$ into a $\sqrt{\cdot}$ pair, rearrange and regroup, before applying the GM$\leq$AM inequality on each:

$$\begin{aligned}
n! &= \sqrt{n \times 1} \quad \sqrt{(n-1) \times 2} \quad \ldots \quad \sqrt{2 \times (n-1)} \quad \sqrt{1 \times n} && (6)\\
&< \frac{n+1}{2} \quad \frac{n+1}{2} \quad \ldots \quad \frac{n+1}{2} \quad \frac{n+1}{2} && (7)\\
&= \left(\frac{n+1}{2}\right)^n && (8)
\end{aligned}$$

(the inequality $\leq$ has become strict $<$ because at least one of the term pairs are different).

Set $n = 999$ to answer the problem with the affirmative:

$$999! < \left(\frac{999+1}{2}\right)^{999} = 500^{999} \tag{9}$$

Discussion

How tight is this bound? Not very! It is made from a product of n terms, each larger than the term it replaces. Further, the terms “at the ends” consist of pairs of numbers which are most different, and from the proof for the inequality one can see that these have the weakest bound (or conversely, the bound is tightest when the two numbers are most similar, becoming exact when the two numbers are equal).

Is there are a different way to arrange the $\sqrt{\cdot}$ pairs to produce a tighter bound?

The above approach pairs numbers as follows:

$$\begin{array}{ccccccc}
1 & 2 & 3 & \ldots & n-2 & n-1 & n \\
| & | & | & & | & | & | \\
n & n-1 & n-2 & \ldots & 3 & 2 & 1
\end{array}$$

We can bring the pairs numerically closer to one another by offsetting by 1 and wrapping around at the end:

$$
\begin{array}{cccccc|c}
1 & 2 & 3 & \dots & n-2 & n-1 & n \\
\vert & \vert & \vert & & \vert & \vert & \vert \\
n-1 & n-2 & n-3 & \dots & 2 & 1 & n
\end{array}
$$

Offsetting instead by 3:

$$
\begin{array}{ccccc|ccc}
1 & 2 & 3 & \dots & n-3 & n-2 & n-1 & n \\
\vert & \vert & \vert & & \vert & \vert & \vert & \vert \\
n-3 & n-4 & n-5 & \dots & 1 & n & n-1 & n-2
\end{array}
$$

There are
- $n-3$ pairs which individually sum to $n-2$, and
- 3 pairs which individually sum to $2n-2$.

The GM-AM inequality approach then produces

$$
\begin{aligned}
n! &= \sqrt{(n-3) \times 1}\ \sqrt{(n-4) \times 2} \dots \sqrt{1 \times (n-3)} && (10) \\
&\times \sqrt{n \times (n-2)}\ \sqrt{(n-1) \times (n-1)}\ \sqrt{(n-2) \times n} && (11) \\
&< \left(\frac{n-2}{2}\right)^{n-3} \times \left(\frac{2n-2}{2}\right)^{3} && (12)
\end{aligned}
$$

Generalising further with an arbitrary offset of k,

$$
\begin{array}{ccccc|cccc}
1 & 2 & 3 & \dots & n-k & n-k+1 & \dots & n-1 & n \\
\vert & \vert & \vert & & \vert & \vert & & \vert & \vert \\
n-k & n-k-1 & n-k-2 & \dots & 1 & n & \dots & n-k+2 & n-k+1
\end{array}
$$

and again we have two set of pairs:
- $n-k$ pairs which individually sum to $n-k+1$, and
- k pairs which individually sum to $2n-k+1$

The GM-AM inequality approach then produces

$$
n! < \left(\frac{n-k+1}{2}\right)^{n-k} \times \left(\frac{2n-k+1}{2}\right)^{k} \tag{13}
$$

Splitting down the middle for an even n, i.e. $n = 2k$, then

$$
\begin{aligned}
n! &< \left(\frac{n-\frac{n}{2}+1}{2}\right)^{\frac{n}{2}} \times \left(\frac{2n-\frac{n}{2}+1}{2}\right)^{\frac{n}{2}} && (14) \\
&= \left(\frac{2n-n+2}{4}\right)^{\frac{n}{2}} \times \left(\frac{4n-n+2}{4}\right)^{\frac{n}{2}} && (15) \\
&= \left(\frac{(n+2)(3n+2)}{16}\right)^{\frac{n}{2}} && (16)
\end{aligned}
$$

For large n, this behaves like

$$
\left(\frac{3n^2}{16}\right)^{\frac{n}{2}} = \frac{n^n}{\left(\frac{4}{\sqrt{3}}\right)^n} \tag{17}
$$

which compares favorably to the large n behavior of the original bound,

$$\left(\frac{n+1}{2}\right)^n \approx \frac{n^n}{2^n}, \qquad \text{for large } n \tag{18}$$

So our "large n" upper bound for $n!$ has reduced (improved) by a factor of

$$\left(\frac{n^n}{\left(\frac{4}{\sqrt{3}}\right)^n}\right) / \left(\frac{n^n}{2^n}\right) = \left(\frac{2\sqrt{3}}{4}\right)^n = \left(\frac{\sqrt{3}}{2}\right)^n \approx (0.87)^n \tag{19}$$

Could this be improved further? What is the smallest α such that

$$n! \lesssim \alpha^n n^n \qquad \text{for large } n \tag{20}$$

Restating without the $\lesssim$, what is the smallest α such that

$$n! < \alpha^n \left(n^m + a_{m-1} n^{m-1} + \ldots + a_1 n + a_0\right)^{\frac{n}{m}} \tag{21}$$

$$= \alpha^n \left(P_m(n)\right)^{\frac{n}{m}} \tag{22}$$

for some set of coefficients $a_0 \ldots a_{m-1}$ making the polynomial P_m of degree m. Note the coefficient of n^m in P_m is by definition equal to 1.

The solution to the original problem found an $\alpha_1 = 0.5$ with $P_1(n) = n + 1$.

Our mid-point pairing approach reduced this to $\alpha_2 = \sqrt{3}/4 \approx 0.43$ with $P_2(n) = n^2 + \frac{8}{3}n + \frac{4}{3}$.

Note we have not shown that these are the smallest α for a given degree polynomial.

The largest m is when $m = n$, so $P_n(n) = n^n + \ldots$ and is raised to the power of 1, i.e.,

$$n! < \alpha_n^n \left(n^n + a_{n-1} n^{n-1} + \ldots + a_0\right) \tag{23}$$

To be continued...

Another bound on $n!$

To help understand the tightness of the bounds achieved by the approach above, consider a different bound:

$$\frac{n^n}{e^{n-1}} \leq n! \leq \frac{n^{n+1}}{e^{n-1}} \tag{24}$$

To prove the lower bound, start with a bound made by truncating the taylor series for $\exp(\cdot)$,

$$e^x \geq 1 + x \tag{25}$$

$$e^{\frac{1}{k}} \geq 1 + \frac{1}{k}, \qquad \text{with } x = 1/k \tag{26}$$

$$e^{\frac{1}{k}} \geq \frac{k+1}{k} \tag{27}$$

$$e \geq \left(\frac{k+1}{k}\right)^k \tag{28}$$

$$\prod_{k=1}^{n-1} e \geq \prod_{k=1}^{n-1} \left(\frac{k+1}{k}\right)^k \tag{29}$$

$$e^{n-1} \geq \left(\frac{2}{1}\right)^1 \left(\frac{3}{2}\right)^2 \left(\frac{4}{3}\right)^3 \ldots \left(\frac{n}{(n-1)^{n-1}}\right) \tag{30}$$

$$e^{n-1} \geq \left(\frac{1}{1}\right) \left(\frac{1}{2}\right) \left(\frac{1}{3}\right) \ldots \left(\frac{n^{n-1}}{n-1}\right) \tag{31}$$

$$e^{n-1} \geq \frac{n^{n-1}}{(n-1)!} \tag{32}$$

$$e^{n-1} \geq \frac{n^n}{n!} \tag{33}$$

Rearranging gives the lower bound:

$$\frac{n^n}{e^{n-1}} \leq n! \tag{34}$$

To prove the upper bound, proceed in a similar way but with a different substitution,

$$e^x \geq 1 + x \tag{35}$$

$$e^{-\frac{1}{k+1}} \geq 1 - \frac{1}{k+1}, \qquad \text{with } x = -1/(k+1) \tag{36}$$

$$e^{-\frac{1}{k+1}} \geq \frac{k}{k+1} \tag{37}$$

$$e \leq \left(\frac{k+1}{k}\right)^{k+1} \tag{38}$$

$$\prod_{k=1}^{n-1} e \leq \prod_{k=1}^{n-1} \left(\frac{k+1}{k}\right)^{k+1} \tag{39}$$

$$e^{n-1} \leq \left(\frac{2}{1}\right)^1 \left(\frac{3}{2}\right)^2 \left(\frac{4}{3}\right)^3 \ldots \left(\frac{n}{n-1}\right)^n \tag{40}$$

$$e^{n-1} \leq \left(\frac{1}{1}\right) \left(\frac{1}{2}\right) \left(\frac{1}{3}\right) \ldots \left(\frac{n^n}{n-1}\right) \tag{41}$$

$$e^{n-1} \leq \frac{n^n}{(n-1)!} \tag{42}$$

$$e^{n-1} \leq \frac{n^{n+1}}{n!} \tag{43}$$

Rearranging gives the upper bound:

$$n! \leq \frac{n^{n+1}}{e^{n-1}} \tag{44}$$

Note that both of these bounds are reasonably tight because the substitution into the inequality keeps x small, and hence $1 + x \approx e^x$.