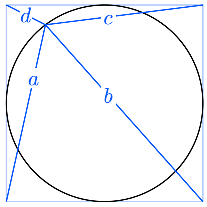
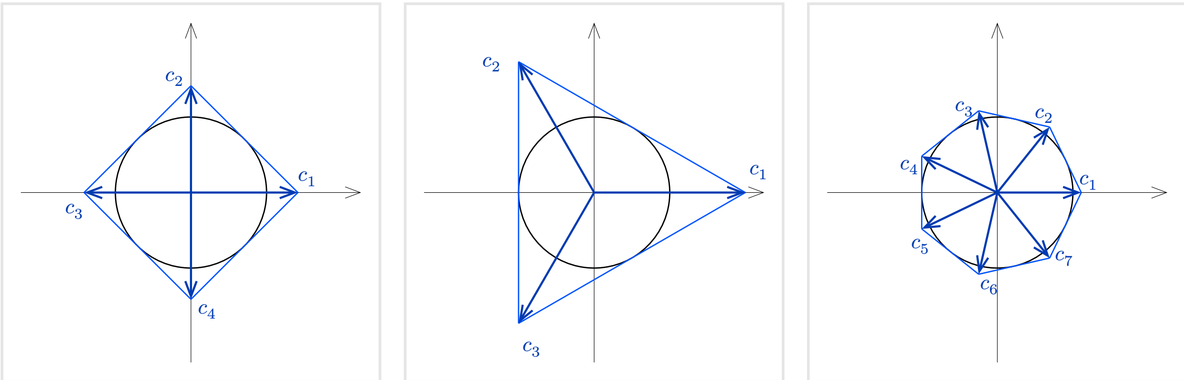
Problem

Find: $a^2 + b^2 + c^2 + d^2$ in the following



Solution

Centre the circle at the origin and rotate by 45°. Also, generalise to any number of corners, and let those corners be the vectors, c_k. For example,



These corner vectors all have the same length, and by symmetry sum to 0:

$$|c_k| = c \tag{1}$$

$$\sum_{k=1}^{n} c_k = 0 \tag{2}$$

Let r be any radius vector of the circle, and define x_k such that

$$r + x_k = c_k \tag{3}$$

For the case of $n = 4$, these x_k correspond to the a, b, c, d of the question. Hence, define the sum

$$S_n = \sum_{k=1}^{n} x_k \cdot x_k \tag{4}$$

$$= \sum_{k=1}^{n} (c_k - r) \cdot (c_k - r) \tag{5}$$

$$= \sum_{k=1}^{n} c_k \cdot c_k - 2r \cdot \sum_{k=1}^{n} c_k + \sum_{k=1}^{n} r \cdot r \tag{6}$$

The first term is nc^2 because of Equation 1, and the middle term is 0 because of Equation 2. Hence

$$S_n = n(c^2 + r^2) \tag{7}$$

For a unit circle with $r = 1$, simple geometry means $c = \cos\left(\frac{\pi}{n}\right)^{-1}$. Hence

$$S_n = n\left(\frac{1}{\cos\left(\frac{\pi}{n}\right)^2} + 1\right) \tag{8}$$

In the case of the original question, $n = 4$ which makes the answer $\boxed{S_4 = 12.}$

(Note that elsewhere on the internet the problem is described as having a unit *square*. This makes $r = 0.5$ and so $a^2 + b^2 + c^2 + d^2 = 3$.)

Evaluating for unit circle and various number of sides:

n	S_n	S_n/n
3	15.000	5.0000
4	12.000	3.0000
5	12.639	2.5279
6	14.000	2.3333
7	15.623	2.2319
8	17.373	2.1716
9	19.192	2.1325
10	21.056	2.1056
11	22.948	2.0862
12	24.862	2.0718
13	26.790	2.0608
14	28.729	2.0521
15	30.678	2.0452
16	32.633	2.0396
17	34.594	2.0349
18	36.560	2.0311
19	38.529	2.0278
20	40.502	2.0251

Some observations

The only integer values of S_n are when $n = 3, 4, 6$.

S_n is smallest when $n = 4$.

Normalised S_n (by dividing by the number of corners, n) looks like it converges to 2. Equation 8 confirms this:

$$\lim_{n\to\infty} \frac{S_n}{n} = \lim_{n\to\infty} \left(\frac{1}{\cos\left(\frac{\pi}{n}\right)^2} + 1 \right) = 1 + 1 = 2 \tag{9}$$

So in some sense, the average distance squared from a point on a circle to every other point is 2.