Problem

Given $x^5 = 1$ and $x \neq 1$, find

$$a = \left(\frac{1}{x^2 + x + 1} + \frac{1}{x^2 - x + 1}\right)^5 \tag{1}$$

Solution

Define $f_n(x) = 1 + x + x^2 + \ldots + x^n$.

The denominators in the question are for $f_2(x)$ and $f_2(-x)$, i.e. $n = 2$.

We proceed by considering arbitrary n, with $x^n = 1, x \neq 1$.

Noting that $f_n(x)$ is a geometric series and $x^{n+1} = x$,

$$f_n(x) = \frac{1 - x^{n+1}}{1 - x} \qquad = \frac{1 - x}{1 - x} = 1 \tag{2}$$

$$f_{2k}(-x) = \frac{1 - (-x)^{2k+1}}{1 - (-x)} = \frac{1 + x}{1 + x} = 1 \tag{3}$$

$$f_{2k+1}(-x) = \frac{1 - (-x)^{2k+2}}{1 - (-x)} = \frac{1 - x}{1 + x} \tag{4}$$

We will be cross-multiplying to make a common denominator:

$$\frac{1}{f_n(x)} + \frac{1}{f_n(-x)} = \frac{f_n(x) + f_n(-x)}{f_n(x) \times f_n(-x)} \tag{5}$$

Separating into even and odd n,

Case 1: $n = 2k$

$$\frac{1}{f_{2k}(x)} + \frac{1}{f_{2k}(-x)} = \frac{1 + 1}{1 \times 1} = 2 \tag{6}$$

Case 2: $n = 2k + 1$

$$\frac{1}{f_{2k+1}(x)} + \frac{1}{f_{2k+1}(-x)} = \frac{1 + \frac{1-x}{1+x}}{1 \times \frac{1-x}{1+x}} = \frac{2}{1 - x} \tag{7}$$

In the original question, we have $n = 2$, and hence

$$a = 2^5 = 32 \tag{8}$$

Observations

The 5^{th} power of the parentheses in the definition of a, and the 5^{th} root of x in the $x^5 = 1$ criteria are completely different "5′s".

This is really a question about rotational symmetry of roots of unity.