Journal of Computational and Applied Mathematics 393 (2021) 113494

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

The equivalence canonical form of five quaternion matrices with applications to imaging and Sylvester-type equations

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ARTICLE INFO

Article history:
Received 14 June 2020

MSC:
15A09
15A24
15B33
15B57

Keywords:
Quaternion matrix decomposition
Quaternion matrix equation
General solution
Solvability
Imaging

ABSTRACT

The equivalence canonical form of five quaternion matrices is investigated. Applications that are discussed include Sylvester-type quaternion matrix equations and color image encryption. A system of one-sided Sylvester-type quaternion matrix equations with six unknowns and five equations is considered by using this equivalence canonical form. Two different types of necessary and sufficient conditions for a solution to this system in terms of ranks and block matrices are presented. An expression of the general solution to the system is provided when it is solvable. Five color images can be encrypted simultaneously by using this equivalence canonical form. Some algorithms and examples are given to illustrate the main result.

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1. Introduction

In this paper, we study the equivalence canonical form of five quaternion matrices

$$\begin{array}{c c} & \begin{array}{ccc} q_1 & q_2 & q_3 \end{array} \\ \begin{array}{c} p_1 \\ p_2 \\ p_3 \end{array} & \begin{pmatrix} A_1 & A_2 & \\ & A_3 & A_4 \\ & & A_5 \end{pmatrix} \end{array}, \tag{1}$$

where A_1, A_2, A_3, A_4, and A_5 are given quaternion matrices. We will transform the five quaternion matrices in (1) to some quasi-diagonal matrices which have only zeros and identity matrices. Moreover, we make use of the equivalence canonical form that give some applications in Sylvester-type equations and color image encryption.

Quaternion was first introduced by W.R. Hamilton in 1843, which is an associative and noncommutative division algebra over the real number field. Nowadays quaternions and quaternion matrices play an important role in computer science, signal and color image processing, quantum mechanics, and so on (e.g., [1–5]). The decompositions of quaternion matrices can be used in image inpainting [6] and signal processing [7]. Theories and computational algorithms of quaternion matrix decompositions have been investigated by some authors (e.g., [8–10]).

Sylvester-type matrix equations is one of the main topics in matrix theory and control theory. Applications of Sylvester-type matrix equations include, for example, singular system control [11], robust control [12], output feedback control [13], descriptor systems control theory [14], the almost noninteracting control by measurement feedback problem [15].

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https://doi.org/10.1016/j.cam.2021.113494
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A large number of papers have given several methods for solving several kinds of Sylvester-type matrix equations over the fields or quaternion algebra (e.g., [5,15–29]). One-sided generalized Sylvester matrix equation over the complex number field

$$AX - YB = C \tag{2}$$

was first studied in 1952 [30]. A solvability condition to Eq. (2) in terms of generalized inverses was derived in [31]. An invariant proof of Roth's theorem was given in [32]. The consistency of Eq. (2) over Bezout domains was considered in [33].

The research on the one-sided Sylvester-type matrix equations with multiple variables and multiple equations is active in recent years. A necessary and sufficient condition for the existence of a simultaneous solution to a pair of generalized Sylvester equations of the form

$$\begin{cases} A_1X - YB_1 = C_1, \\ A_2X - YB_2 = C_2 \end{cases} \tag{3}$$

was presented in [15]. The solution to the system (3) was given in [34]. The system of Sylvester-type matrix equations with three variables over a field

$$\begin{cases} A_1X - YB_1 = C_1, \\ A_2Z - YB_2 = C_2 \end{cases} \tag{4}$$

was established in [35]. Some computable necessary and sufficient conditions for the system (4) were provided in [36]. The systems of Sylvester-type matrix equations with five variables and four equation over the quaternion algebra were considered in [9,37,38]

$$\begin{cases} A_1X_1 - X_2B_1 = C_1, \\ A_2X_3 - X_2B_2 = C_2, \\ A_3X_3 - X_4B_3 = C_3, \\ A_4X_5 - X_4B_4 = C_4, \end{cases} \quad \begin{cases} X_1A_1 - B_1X_2 = C_1, \\ X_2A_2 - B_2X_3 = C_2, \\ X_3A_3 - B_3X_4 = C_3, \\ X_4A_4 - B_4X_5 = C_4, \end{cases} \quad \begin{cases} A_1X_1 - X_2B_1 = C_1 \\ A_2X_3 - X_2B_2 = C_2 \\ A_3X_3 - X_4B_3 = C_3 \\ A_4X_4 - X_5B_4 = C_4. \end{cases} \tag{5}$$

Note that most of the research work related to one-sided Sylvester-type matrix equations builds upon the existing work for the number of equations is less than five. However, to our knowledge, there has been little information on the one-sided Sylvester-type quaternion matrix equation with more than four equations and more than five variables (even in the complex number field). Motivated by the mentioned above, we in this paper consider the following system of Sylvester-type quaternion matrix equation with six unknowns and five equations

$$\begin{cases} A_1X_1 - X_2B_1 = C_1, \\ A_2X_3 - X_2B_2 = C_2, \\ A_3X_3 - X_4B_3 = C_3, \\ A_4X_5 - X_4B_4 = C_4, \\ A_5X_5 - X_6B_5 = C_5, \end{cases} \tag{6}$$

where A_i, B_i, C_i are given quaternion matrices with appropriate sizes, $X_1, \ldots, X_6$ are unknowns ($i = 1, \ldots, 6$). Note that the coefficient matrices A_i and B_i can be arranged in the matrix array (1) and

$$\begin{array}{c c} & \begin{array}{ccc} n_1 & n_2 & n_3 \end{array} \\ \begin{array}{c} t_1 \\ t_2 \\ t_3 \end{array} & \begin{pmatrix} B_1 & B_2 & \\ & B_3 & B_4 \\ & & B_5 \end{pmatrix} \end{array}. \tag{7}$$

We make use of the equivalence canonical form of five quaternion matrices (1) and (7) that bring the system (6) to a canonical form. Therefore another contribution of this paper is to give some necessary and sufficient conditions for the existence of the general solution to the system (6) in terms of the ranks of the given matrices.

In the past decade, a variety of encryption schemes have been proposed to color image processing (e.g., [39–41]). However, to our knowledge, there has been little work on encrypting five color images simultaneously. Based on the equivalence canonical form of five quaternion matrices (1), we encrypt five color images simultaneously.

The main contribution of the paper is twofold. Firstly, we investigate the equivalence canonical form for five quaternion matrices. Secondly, we use the equivalence canonical form to Sylvester-type equations (6) and color image encryption.

The remainder of this paper is organized as follows. In Section 2, we give the equivalence canonical form for five quaternion matrices. In Section 3, we give two types of necessary and sufficient conditions for the system of quaternion matrix equations (6). In Section 4, we present its general solution to the system (6) and an example. In Section 5, we consider the application of the equivalence canonical form in the color image encryption.

Let $\mathbb{R}$ and $\mathbb{H}^{m\times n}$ stand, respectively, for the real number field and the space of all $m\times n$ matrices over the real quaternion algebra

$$\mathbb{H} = \left\{a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} \,\middle|\, \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1, a_0, a_1, a_2, a_3 \in \mathbb{R}\right\}.$$

The rank of $A \in \mathbb{H}^{m\times n}$ is defined as the (quaternion) dimension of

$$\mathrm{Ran}(A) := \{Ax : \ x \in \mathbb{H}^{m\times 1}\},$$

the range of A (Definition 3.2.3 in [42]). There exist invertible matrices P and Q such that

$$PAQ = \begin{pmatrix} I_{r(A)} & 0 \\ 0 & 0 \end{pmatrix}, \tag{8}$$

where $r(A)$ means the rank of the quaternion matrix A, I is the identity matrix and the zeros stand for zero matrices with appropriate sizes.

2. The equivalence canonical form of five quaternion matrices

In this section, we propose the equivalence canonical form of five quaternion matrices (1). De Moor, Van Dooren, and Zha derived the structure of generalization singular value decomposition (GSVD) for five complex matrices [43,44]. The following result gives the equivalence canonical form for five quaternion matrices.

Theorem 2.1. *Consider a set of five quaternion matrices: $A_1 \in \mathbb{H}^{p_1\times q_1}$, $A_2 \in \mathbb{H}^{p_1\times q_2}$, $A_3 \in \mathbb{H}^{p_2\times q_2}$, $A_4 \in \mathbb{H}^{p_2\times q_3}$, and $A_5 \in \mathbb{H}^{p_3\times q_3}$. Then there exist nonsingular matrices $P_j \in \mathbb{H}^{p_j\times p_j}$ and $Q_j \in \mathbb{H}^{q_j\times q_j}$, $(j = 1, 2, 3)$, such that*

$$P_1A_1Q_1 = S_{a_1}, \qquad P_1A_2Q_2 = S_{a_2}, \qquad P_2A_3Q_2 = S_{a_3}, \qquad P_2A_4Q_3 = S_{a_4}, \qquad P_3A_5Q_3 = S_{a_5},$$

where

$$\left(\begin{array}{ccc} S_{a_1} & S_{a_2} & \\ \hline & S_{a_3} & S_{a_4} \\ \hline & & S_{a_5} \end{array}\right)$$

$$= \left(\begin{array}{cccccccccccccccccccccccccccc}
I&0&0&0&0&0&I&0&0&0&0&0&0&0&0&0&0&0&&&&&&&&&& \\
0&I&0&0&0&0&0&I&0&0&0&0&0&0&0&0&0&0&&&&&&&&&& \\
0&0&I&0&0&0&0&0&I&0&0&0&0&0&0&0&0&0&&&&&&&&&& \\
0&0&0&I&0&0&0&0&0&I&0&0&0&0&0&0&0&0&&&&&&&&&& \\
0&0&0&0&I&0&0&0&0&0&0&0&0&0&0&0&0&0&&&&&&&&&& \\
0&0&0&0&0&0&0&0&0&0&I&0&0&0&0&0&0&0&&&&&&&&&& \\
0&0&0&0&0&0&0&0&0&0&0&I&0&0&0&0&0&0&&&&&&&&&& \\
0&0&0&0&0&0&0&0&0&0&0&0&I&0&0&0&0&0&&&&&&&&&& \\
0&0&0&0&0&0&0&0&0&0&0&0&0&I&0&0&0&0&&&&&&&&&& \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&&&&&&&&&& \\
\hline
&&&&&&I&0&0&0&0&0&0&0&0&0&0&0&I&0&0&0&0&0&0&0&0&0 \\
&&&&&&0&I&0&0&0&0&0&0&0&0&0&0&0&I&0&0&0&0&0&0&0&0 \\
&&&&&&0&0&I&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
&&&&&&0&0&0&0&I&0&0&0&0&0&0&0&0&0&I&0&0&0&0&0&0&0 \\
&&&&&&0&0&0&0&0&I&0&0&0&0&0&0&0&0&0&I&0&0&0&0&0&0 \\
&&&&&&0&0&0&0&0&0&I&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
&&&&&&0&0&0&0&0&0&0&0&I&0&0&0&0&0&0&0&I&0&0&0&0&0 \\
&&&&&&0&0&0&0&0&0&0&0&0&I&0&0&0&0&0&0&0&I&0&0&0&0 \\
&&&&&&0&0&0&0&0&0&0&0&0&0&I&0&0&0&0&0&0&0&0&0&0&0 \\
&&&&&&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&I&0&0&0 \\
&&&&&&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&I&0&0 \\
&&&&&&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
\hline
&&&&&&&&&&&&&&&&&&I&0&0&0&0&0&0&0&0&0 \\
&&&&&&&&&&&&&&&&&&0&0&I&0&0&0&0&0&0&0 \\
&&&&&&&&&&&&&&&&&&0&0&0&0&I&0&0&0&0&0 \\
&&&&&&&&&&&&&&&&&&0&0&0&0&0&0&I&0&0&0 \\
&&&&&&&&&&&&&&&&&&0&0&0&0&0&0&0&0&I&0 \\
&&&&&&&&&&&&&&&&&&0&0&0&0&0&0&0&0&0&0
\end{array}\right). \tag{9}$$

Proof. We will use the decomposition (8) and elementary row and column operations to transform the five matrices A_1, A_2, A_3, A_4, A_5 to quasi-diagonal matrices in turn. We form the matrix array:

$$\begin{array}{c} \\ p_1 \\ p_2 \\ p_3 \end{array} \begin{array}{c} \begin{array}{ccc} q_1 & q_2 & q_3 \end{array} \\ \begin{pmatrix} A_1 & A_2 & \\ & A_3 & A_4 \\ & & A_5 \end{pmatrix} \end{array}. \tag{10}$$

Any operation done on the rows of A_1 and A_3 is also done on those of A_2 and A_4 and conversely, so that both are multiplied from the left by the same nonsingular invertible matrix. The row operation on A_5 has no effect on A_1 or A_3. Likewise, the column operations on $\{A_2, A_3\}$ and $\{A_4, A_5\}$ are the same, respectively, but the columns of A_1 can be manipulated independently of $\{A_2, A_3\}$ and $\{A_4, A_5\}$.

Below is a sequence of displays of the matrix block that illustrates the transformations. The symbols $\ast$ and $\diamond$ are nonzero block matrices. We apply the decomposition (8) or operation to the objective matrix $\diamond$ in every step.

$$\begin{array}{cc} & \begin{matrix} q_1 & q_2 & q_3 \end{matrix} \\ \begin{matrix} p_1 \\ p_2 \\ p_3 \end{matrix} & \begin{pmatrix} A_1 & A_2 & \\ & A_3 & A_4 \\ & & A_5 \end{pmatrix} \end{array} \xrightarrow{S_1A_1M_1=\left(\begin{smallmatrix} I & 0\\ 0 & 0\end{smallmatrix}\right)} \left(\begin{array}{cccc} I & 0 & \ast & \\ 0 & 0 & \diamond & \\ \hline & & A_3 & A_4 \\ \hline & & & A_5 \end{array}\right) \xrightarrow{S_2\diamond M_2=\left(\begin{smallmatrix} I & 0\\ 0 & 0\end{smallmatrix}\right)}$$

$$\left(\begin{array}{ccccc} I & 0 & \diamond & \ast & \\ 0 & 0 & I & 0 & \\ 0 & 0 & 0 & 0 & \\ \hline & & \ast & \ast & A_4 \\ \hline & & & & A_5 \end{array}\right) \xrightarrow{\text{Elementary matrix operations}} \left(\begin{array}{ccccc} I & 0 & 0 & \diamond & \\ 0 & 0 & I & 0 & \\ 0 & 0 & 0 & 0 & \\ \hline & & \ast & \ast & A_4 \\ \hline & & & & A_5 \end{array}\right) \xrightarrow{S_3\diamond M_3=\left(\begin{smallmatrix} I & 0\\ 0 & 0\end{smallmatrix}\right)}$$

$$\left(\begin{array}{ccccccc} I & 0 & 0 & 0 & I & 0 & \\ 0 & I & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & I & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 0 & \\ \hline & & & \ast & \ast & \diamond & A_4 \\ \hline & & & & & & A_5 \end{array}\right) \xrightarrow{S_4\diamond M_4=\left(\begin{smallmatrix} I & 0\\ 0 & 0\end{smallmatrix}\right)} \left(\begin{array}{cccccccc} I & 0 & 0 & 0 & I & 0 & 0 & \\ 0 & I & 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & I & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\ \hline & & & \diamond & \diamond & I & 0 & \ast \\ & & & \ast & \ast & 0 & 0 & \ast \\ \hline & & & & & & & A_5 \end{array}\right)$$

$$\xrightarrow{\text{Elementary matrix operations}} \left(\begin{array}{cccccccc} I & 0 & 0 & 0 & I & 0 & 0 & \\ 0 & I & 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & I & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\ \hline & & & 0 & 0 & I & 0 & \ast \\ & & & \ast & \diamond & 0 & 0 & \ast \\ \hline & & & & & & & A_5 \end{array}\right) \xrightarrow{S_5\diamond M_5=\left(\begin{smallmatrix} I & 0\\ 0 & 0\end{smallmatrix}\right)}$$

$$\left(\begin{array}{cccccccccc} I & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & \\ 0 & I & 0 & 0 & 0 & 0 & I & 0 & 0 & \\ 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\ \hline & & & & 0 & 0 & 0 & I & 0 & \ast \\ & & & & \diamond & I & 0 & 0 & 0 & \ast \\ & & & & \ast & 0 & 0 & 0 & 0 & \ast \\ \hline & & & & & & & & & A_5 \end{array}\right) \xrightarrow{\text{Elementary matrix operations}} \left(\begin{array}{cccccccccc} I & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & \\ 0 & I & 0 & 0 & 0 & 0 & I & 0 & 0 & \\ 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\ \hline & & & & 0 & 0 & 0 & I & 0 & \ast \\ & & & & 0 & I & 0 & 0 & 0 & \ast \\ & & & & \diamond & 0 & 0 & 0 & 0 & \ast \\ \hline & & & & & & & & & A_5 \end{array}\right)$$

$$\xrightarrow{S_6\diamond M_6=\left(\begin{smallmatrix} I & 0\\ 0 & 0\end{smallmatrix}\right)} \left(\begin{array}{ccccccccccc} I & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & \\ 0 & I & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & \\ 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\ \hline & & & & 0 & 0 & 0 & 0 & I & 0 & \ast \\ & & & & 0 & 0 & I & 0 & 0 & 0 & \ast \\ & & & & I & 0 & 0 & 0 & 0 & 0 & \ast \\ & & & & 0 & 0 & 0 & 0 & 0 & 0 & \diamond \\ \hline & & & & & & & & & & A_5 \end{array}\right) \xrightarrow{S_7\diamond M_7=\left(\begin{smallmatrix} I & 0\\ 0 & 0\end{smallmatrix}\right)}$$

$$\left(\begin{array}{cccccccccccc} I & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & & \\ 0 & I & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & & \\ 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & \\ 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & & \\ 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & \\ \hline & & & & 0 & 0 & 0 & 0 & I & 0 & \diamond & \ast \\ & & & & 0 & 0 & I & 0 & 0 & 0 & \diamond & \ast \\ & & & & I & 0 & 0 & 0 & 0 & 0 & \diamond & \ast \\ & & & & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 \\ & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline & & & & & & & & & & \ast & \ast \end{array}\right) \xrightarrow{\text{Elementary matrix operations}} \left(\begin{array}{cccccccccccc} I & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & & \\ 0 & I & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & & \\ 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & \\ 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & & \\ 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & \\ \hline & & & & 0 & 0 & 0 & 0 & I & 0 & 0 & \ast \\ & & & & 0 & 0 & I & 0 & 0 & 0 & 0 & \ast \\ & & & & I & 0 & 0 & 0 & 0 & 0 & 0 & \diamond \\ & & & & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 \\ & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline & & & & & & & & & & \ast & \ast \end{array}\right)$$

$$\xrightarrow{S_8\diamond M_8=\left(\begin{smallmatrix} I & 0\\ 0 & 0\end{smallmatrix}\right)} \left(\begin{array}{cccccccccccccc} I & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & & & \\ 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & & & \\ 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & \\ 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & & & \\ 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & \\ \hline & & & & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & \diamond & \ast \\ & & & & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & \diamond & \ast \\ & & & & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 \\ & & & & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 \\ & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline & & & & & & & & & & & \ast & \ast & \ast \end{array}\right) \xrightarrow{\text{Elementary matrix operations}}$$

$$
\left(\begin{array}{lll}
I~0~0~0 & 0~0~0~I~0~0~0 & \\
0~I~0~0 & 0~0~0~0~I~0~0 & \\
0~0~I~0 & 0~0~0~0~0~0~0 & \\
0~0~0~0 & I~0~0~0~0~0~0 & \\
0~0~0~0 & 0~I~0~0~0~0~0 & \\
0~0~0~0 & 0~0~I~0~0~0~0 & \\
0~0~0~0 & 0~0~0~0~0~0~0 & \\
\hline
 & 0~0~0~0~0~I~0 & 0~0~* \\
 & 0~0~0~I~0~0~0 & 0~0~\diamond \\
 & I~0~0~0~0~0~0 & 0~I~0 \\
 & 0~I~0~0~0~0~0 & 0~0~0 \\
 & 0~0~0~0~0~0~0 & I~0~0 \\
 & 0~0~0~0~0~0~0 & 0~0~0 \\
\hline
 & & *~*~*
\end{array}\right)
\xrightarrow{S_9\diamond M_9=\begin{pmatrix} I & 0\\ 0 & 0\end{pmatrix}}
\left(\begin{array}{lll}
I~0~0~0~0 & 0~0~0~I~0~0~0~0 & \\
0~I~0~0~0 & 0~0~0~0~I~0~0~0 & \\
0~0~I~0~0 & 0~0~0~0~0~I~0~0 & \\
0~0~0~I~0 & 0~0~0~0~0~0~0~0 & \\
0~0~0~0~0 & I~0~0~0~0~0~0~0 & \\
0~0~0~0~0 & 0~I~0~0~0~0~0~0 & \\
0~0~0~0~0 & 0~0~I~0~0~0~0~0 & \\
0~0~0~0~0 & 0~0~0~0~0~0~0~0 & \\
\hline
 & 0~0~0~0~0~0~I~0 & 0~0~\diamond~* \\
 & 0~0~0~I~0~0~0~0 & 0~0~I~0 \\
 & 0~0~0~0~I~0~0~0 & 0~0~0~0 \\
 & I~0~0~0~0~0~0~0 & 0~I~0~0 \\
 & 0~I~0~0~0~0~0~0 & 0~0~0~0 \\
 & 0~0~0~0~0~0~0~0 & I~0~0~0 \\
 & 0~0~0~0~0~0~0~0 & 0~0~0~0 \\
\hline
 & & *~*~*~*
\end{array}\right)
$$

$$
\xrightarrow{\text{Elementary matrix operations}}
\left(\begin{array}{lll}
I~0~0~0~0 & 0~0~0~I~0~0~0~0 & \\
0~I~0~0~0 & 0~0~0~0~I~0~0~0 & \\
0~0~I~0~0 & 0~0~0~0~0~I~0~0 & \\
0~0~0~I~0 & 0~0~0~0~0~0~0~0 & \\
0~0~0~0~0 & I~0~0~0~0~0~0~0 & \\
0~0~0~0~0 & 0~I~0~0~0~0~0~0 & \\
0~0~0~0~0 & 0~0~I~0~0~0~0~0 & \\
0~0~0~0~0 & 0~0~0~0~0~0~0~0 & \\
\hline
 & 0~0~0~0~0~0~I~0 & 0~0~0~\diamond \\
 & 0~0~0~I~0~0~0~0 & 0~0~I~0 \\
 & 0~0~0~0~I~0~0~0 & 0~0~0~0 \\
 & I~0~0~0~0~0~0~0 & 0~I~0~0 \\
 & 0~I~0~0~0~0~0~0 & 0~0~0~0 \\
 & 0~0~0~0~0~0~0~0 & I~0~0~0 \\
 & 0~0~0~0~0~0~0~0 & 0~0~0~0 \\
\hline
 & & *~*~*~*
\end{array}\right)
\xrightarrow{S_{10}\diamond M_{10}=\begin{pmatrix} I & 0\\ 0 & 0\end{pmatrix}}
$$

$$
\left(\begin{array}{lll}
I~0~0~0~0 & 0~0~0~I~0~0~0~0~0 & \\
0~I~0~0~0 & 0~0~0~0~I~0~0~0~0 & \\
0~0~I~0~0 & 0~0~0~0~0~I~0~0~0 & \\
0~0~0~I~0 & 0~0~0~0~0~0~0~0~0 & \\
0~0~0~0~0 & I~0~0~0~0~0~0~0~0 & \\
0~0~0~0~0 & 0~I~0~0~0~0~0~0~0 & \\
0~0~0~0~0 & 0~0~I~0~0~0~0~0~0 & \\
0~0~0~0~0 & 0~0~0~0~0~0~0~0~0 & \\
\hline
 & 0~0~0~0~0~0~I~0~0 & 0~0~0~I~0 \\
 & 0~0~0~0~0~0~0~I~0 & 0~0~0~0~0 \\
 & 0~0~0~I~0~0~0~0~0 & 0~0~I~0~0 \\
 & 0~0~0~0~I~0~0~0~0 & 0~0~0~0~0 \\
 & I~0~0~0~0~0~0~0~0 & 0~I~0~0~0 \\
 & 0~I~0~0~0~0~0~0~0 & 0~0~0~0~0 \\
 & 0~0~0~0~0~0~0~0~0 & I~0~0~0~0 \\
 & 0~0~0~0~0~0~0~0~0 & 0~0~0~0~0 \\
\hline
 & & *~*~*~*~\diamond
\end{array}\right)
\xrightarrow{S_{11}\diamond M_{11}=\begin{pmatrix} I & 0\\ 0 & 0\end{pmatrix}}
\left(\begin{array}{lll}
I~0~0~0~0 & 0~0~0~I~0~0~0~0~0 & \\
0~I~0~0~0 & 0~0~0~0~I~0~0~0~0 & \\
0~0~I~0~0 & 0~0~0~0~0~I~0~0~0 & \\
0~0~0~I~0 & 0~0~0~0~0~0~0~0~0 & \\
0~0~0~0~0 & I~0~0~0~0~0~0~0~0 & \\
0~0~0~0~0 & 0~I~0~0~0~0~0~0~0 & \\
0~0~0~0~0 & 0~0~I~0~0~0~0~0~0 & \\
0~0~0~0~0 & 0~0~0~0~0~0~0~0~0 & \\
\hline
 & 0~0~0~0~0~0~I~0~0 & 0~0~0~I~0~0 \\
 & 0~0~0~0~0~0~0~I~0 & 0~0~0~0~0~0 \\
 & 0~0~0~I~0~0~0~0~0 & 0~0~I~0~0~0 \\
 & 0~0~0~0~I~0~0~0~0 & 0~0~0~0~0~0 \\
 & I~0~0~0~0~0~0~0~0 & 0~I~0~0~0~0 \\
 & 0~I~0~0~0~0~0~0~0 & 0~0~0~0~0~0 \\
 & 0~0~0~0~0~0~0~0~0 & I~0~0~0~0~0 \\
 & 0~0~0~0~0~0~0~0~0 & 0~0~0~0~0~0 \\
\hline
 & & \diamond~\diamond~\diamond~\diamond~I~0 \\
 & & *~*~*~*~0~0
\end{array}\right)
$$

$$
\xrightarrow{\text{Elementary matrix operations}}
\left(\begin{array}{lll}
I~0~0~0~0 & 0~0~0~I~0~0~0~0~0 & \\
0~I~0~0~0 & 0~0~0~0~I~0~0~0~0 & \\
0~0~I~0~0 & 0~0~0~0~0~I~0~0~0 & \\
0~0~0~I~0 & 0~0~0~0~0~0~0~0~0 & \\
0~0~0~0~0 & I~0~0~0~0~0~0~0~0 & \\
0~0~0~0~0 & 0~I~0~0~0~0~0~0~0 & \\
0~0~0~0~0 & 0~0~I~0~0~0~0~0~0 & \\
0~0~0~0~0 & 0~0~0~0~0~0~0~0~0 & \\
\hline
 & 0~0~0~0~0~0~I~0~0 & 0~0~0~I~0~0 \\
 & 0~0~0~0~0~0~0~I~0 & 0~0~0~0~0~0 \\
 & 0~0~0~I~0~0~0~0~0 & 0~0~I~0~0~0 \\
 & 0~0~0~0~I~0~0~0~0 & 0~0~0~0~0~0 \\
 & I~0~0~0~0~0~0~0~0 & 0~I~0~0~0~0 \\
 & 0~I~0~0~0~0~0~0~0 & 0~0~0~0~0~0 \\
 & 0~0~0~0~0~0~0~0~0 & I~0~0~0~0~0 \\
 & 0~0~0~0~0~0~0~0~0 & 0~0~0~0~0~0 \\
\hline
 & & 0~0~0~0~I~0 \\
 & & *~*~*~\diamond~0~0
\end{array}\right)
\xrightarrow{S_{12}\diamond M_{12}=\begin{pmatrix} I & 0\\ 0 & 0\end{pmatrix}}
$$

$$
\left(\begin{array}{lll}
I~0~0~0~0 & 0~0~0~I~0~0~0~0~0~0 & \\
0~I~0~0~0 & 0~0~0~0~I~0~0~0~0~0 & \\
0~0~I~0~0 & 0~0~0~0~0~I~0~0~0~0 & \\
0~0~0~I~0 & 0~0~0~0~0~0~0~0~0~0 & \\
0~0~0~0~0 & I~0~0~0~0~0~0~0~0~0 & \\
0~0~0~0~0 & 0~I~0~0~0~0~0~0~0~0 & \\
0~0~0~0~0 & 0~0~I~0~0~0~0~0~0~0 & \\
0~0~0~0~0 & 0~0~0~0~0~0~0~0~0~0 & \\
\hline
 & 0~0~0~0~0~0~I~0~0~0 & 0~0~0~I~0~0~0 \\
 & 0~0~0~0~0~0~0~I~0~0 & 0~0~0~0~I~0~0 \\
 & 0~0~0~0~0~0~0~0~I~0 & 0~0~0~0~0~0~0 \\
 & 0~0~0~I~0~0~0~0~0~0 & 0~0~I~0~0~0~0 \\
 & 0~0~0~0~I~0~0~0~0~0 & 0~0~0~0~0~0~0 \\
 & I~0~0~0~0~0~0~0~0~0 & 0~I~0~0~0~0~0 \\
 & 0~I~0~0~0~0~0~0~0~0 & 0~0~0~0~0~0~0 \\
 & 0~0~0~0~0~0~0~0~0~0 & I~0~0~0~0~0~0 \\
 & 0~0~0~0~0~0~0~0~0~0 & 0~0~0~0~0~0~0 \\
\hline
 & & 0~0~0~0~0~I~0 \\
 & & \diamond~\diamond~\diamond~I~0~0~0 \\
 & & *~*~*~0~0~0~0
\end{array}\right)
\xrightarrow{\text{Elementary matrix operations}}
$$

$$
\left(\begin{array}{ccc}
\begin{matrix} I&0&0&0&0\\ 0&I&0&0&0\\ 0&0&I&0&0\\ 0&0&0&I&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0 \end{matrix} &
\begin{matrix} 0&0&0&I&0&0&0&0&0&0\\ 0&0&0&0&I&0&0&0&0&0\\ 0&0&0&0&0&I&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ I&0&0&0&0&0&0&0&0&0\\ 0&I&0&0&0&0&0&0&0&0\\ 0&0&I&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0 \end{matrix} & \\
\hline
 & \begin{matrix} 0&0&0&0&0&0&I&0&0&0\\ 0&0&0&0&0&0&0&I&0&0\\ 0&0&0&0&0&0&0&0&I&0\\ 0&0&0&I&0&0&0&0&0&0\\ 0&0&0&0&I&0&0&0&0&0\\ I&0&0&0&0&0&0&0&0&0\\ 0&I&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0 \end{matrix} &
\begin{matrix} 0&0&0&I&0&0&0\\ 0&0&0&0&I&0&0\\ 0&0&0&0&0&0&0\\ 0&0&I&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&I&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ I&0&0&0&0&0&0\\ 0&0&0&0&0&0&0 \end{matrix} \\
\hline
 & & \begin{matrix} 0&0&0&0&0&I&0\\ 0&0&0&I&0&0&0\\ *&*&\diamond&0&0&0&0 \end{matrix}
\end{array}\right)
\xrightarrow{S_{13}\diamond M_{13}=\begin{pmatrix} I&0\\ 0&0 \end{pmatrix}}
$$

$$
\left(\begin{array}{ccc}
\begin{matrix} I&0&0&0&0&0\\ 0&I&0&0&0&0\\ 0&0&I&0&0&0\\ 0&0&0&I&0&0\\ 0&0&0&0&I&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{matrix} &
\begin{matrix} 0&0&0&I&0&0&0&0&0&0&0\\ 0&0&0&0&I&0&0&0&0&0&0\\ 0&0&0&0&0&I&0&0&0&0&0\\ 0&0&0&0&0&0&I&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0\\ I&0&0&0&0&0&0&0&0&0&0\\ 0&I&0&0&0&0&0&0&0&0&0\\ 0&0&I&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0 \end{matrix} & \\
\hline
 & \begin{matrix} 0&0&0&0&0&0&0&I&0&0&0\\ 0&0&0&0&0&0&0&0&I&0&0\\ 0&0&0&0&0&0&0&0&0&I&0\\ 0&0&0&I&0&0&0&0&0&0&0\\ 0&0&0&0&I&0&0&0&0&0&0\\ 0&0&0&0&0&I&0&0&0&0&0\\ I&0&0&0&0&0&0&0&0&0&0\\ 0&I&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0 \end{matrix} &
\begin{matrix} 0&0&0&0&I&0&0&0\\ 0&0&0&0&0&I&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&I&0&0&0&0&0\\ 0&0&0&I&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&I&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ I&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{matrix} \\
\hline
 & & \begin{matrix} 0&0&0&0&0&0&I&0\\ 0&0&0&0&I&0&0&0\\ \diamond&\diamond&I&0&0&0&0&0\\ *&*&0&0&0&0&0&0 \end{matrix}
\end{array}\right)
\xrightarrow{\text{Elementary matrix operations}}
$$

$$
\left(\begin{array}{ccc}
\begin{matrix} I&0&0&0&0&0\\ 0&I&0&0&0&0\\ 0&0&I&0&0&0\\ 0&0&0&I&0&0\\ 0&0&0&0&I&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{matrix} &
\begin{matrix} 0&0&0&I&0&0&0&0&0&0&0\\ 0&0&0&0&I&0&0&0&0&0&0\\ 0&0&0&0&0&I&0&0&0&0&0\\ 0&0&0&0&0&0&I&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0\\ I&0&0&0&0&0&0&0&0&0&0\\ 0&I&0&0&0&0&0&0&0&0&0\\ 0&0&I&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0 \end{matrix} & \\
\hline
 & \begin{matrix} 0&0&0&0&0&0&0&I&0&0&0\\ 0&0&0&0&0&0&0&0&I&0&0\\ 0&0&0&0&0&0&0&0&0&I&0\\ 0&0&0&I&0&0&0&0&0&0&0\\ 0&0&0&0&I&0&0&0&0&0&0\\ 0&0&0&0&0&I&0&0&0&0&0\\ I&0&0&0&0&0&0&0&0&0&0\\ 0&I&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0 \end{matrix} &
\begin{matrix} 0&0&0&0&I&0&0&0\\ 0&0&0&0&0&I&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&I&0&0&0&0&0\\ 0&0&0&I&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&I&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ I&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{matrix} \\
\hline
 & & \begin{matrix} 0&0&0&0&0&0&I&0\\ 0&0&0&0&I&0&0&0\\ 0&0&I&0&0&0&0&0\\ *&\diamond&0&0&0&0&0&0 \end{matrix}
\end{array}\right)
\xrightarrow{S_{14}\diamond M_{14}=\begin{pmatrix} I&0\\ 0&0 \end{pmatrix}}
$$

$$
\left(\begin{array}{ccc}
\begin{matrix} I&0&0&0&0&0\\ 0&I&0&0&0&0\\ 0&0&I&0&0&0\\ 0&0&0&I&0&0\\ 0&0&0&0&I&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{matrix} &
\begin{matrix} 0&0&0&0&I&0&0&0&0&0&0&0\\ 0&0&0&0&0&I&0&0&0&0&0&0\\ 0&0&0&0&0&0&I&0&0&0&0&0\\ 0&0&0&0&0&0&0&I&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ I&0&0&0&0&0&0&0&0&0&0&0\\ 0&I&0&0&0&0&0&0&0&0&0&0\\ 0&0&I&0&0&0&0&0&0&0&0&0\\ 0&0&0&I&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0 \end{matrix} & \\
\hline
 & \begin{matrix} 0&0&0&0&0&0&0&0&I&0&0&0\\ 0&0&0&0&0&0&0&0&0&I&0&0\\ 0&0&0&0&0&0&0&0&0&0&I&0\\ 0&0&0&0&I&0&0&0&0&0&0&0\\ 0&0&0&0&0&I&0&0&0&0&0&0\\ 0&0&0&0&0&0&I&0&0&0&0&0\\ I&0&0&0&0&0&0&0&0&0&0&0\\ 0&I&0&0&0&0&0&0&0&0&0&0\\ 0&0&I&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0 \end{matrix} &
\begin{matrix} 0&0&0&0&0&I&0&0&0\\ 0&0&0&0&0&0&I&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&I&0&0&0&0&0\\ 0&0&0&0&I&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&I&0&0&0&0&0&0&0\\ 0&0&I&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ I&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0 \end{matrix} \\
\hline
 & & \begin{matrix} 0&0&0&0&0&0&0&I&0\\ 0&0&0&0&0&I&0&0&0\\ 0&0&0&I&0&0&0&0&0\\ \diamond&I&0&0&0&0&0&0&0\\ *&0&0&0&0&0&0&0&0 \end{matrix}
\end{array}\right)
\xrightarrow{\text{Elementary matrix operations}}
$$

$$
\left(\begin{array}{ccc}
\begin{matrix}
I & 0 & 0 & 0 & 0 & 0 \\
0 & I & 0 & 0 & 0 & 0 \\
0 & 0 & I & 0 & 0 & 0 \\
0 & 0 & 0 & I & 0 & 0 \\
0 & 0 & 0 & 0 & I & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{matrix} &
\begin{matrix}
0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{matrix} & \\
\hline
 &
\begin{matrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 \\
0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 \\
I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{matrix} &
\begin{matrix}
0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{matrix} \\
\hline
 & &
\begin{matrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 \\
0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 \\
0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 \\
0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\diamond & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{matrix}
\end{array}\right)
\xrightarrow{S_{15}\diamond M_{15}=\left(\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right)}
$$

$$
\left(\begin{array}{ccc}
\begin{matrix}
I & 0 & 0 & 0 & 0 & 0 \\
0 & I & 0 & 0 & 0 & 0 \\
0 & 0 & I & 0 & 0 & 0 \\
0 & 0 & 0 & I & 0 & 0 \\
0 & 0 & 0 & 0 & I & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{matrix} &
\begin{matrix}
0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{matrix} & \\
\hline
 &
\begin{matrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 \\
0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 \\
I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{matrix} &
\begin{matrix}
0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{matrix} \\
\hline
 & &
\begin{matrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{matrix}
\end{array}\right)
\xrightarrow{\text{Interchange some block columns and rows}}
$$

$$
\left(\begin{array}{ccc}
\begin{matrix}
I & 0 & 0 & 0 & 0 & 0 \\
0 & I & 0 & 0 & 0 & 0 \\
0 & 0 & I & 0 & 0 & 0 \\
0 & 0 & 0 & I & 0 & 0 \\
0 & 0 & 0 & 0 & I & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{matrix} &
\begin{matrix}
I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{matrix} & \\
\hline
 &
\begin{matrix}
I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{matrix} &
\begin{matrix}
I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{matrix} \\
\hline
 & &
\begin{matrix}
I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{matrix}
\end{array}\right). \qquad \square
$$

Remark 2.2. We can also give the expressions of all the identities in the equivalence canonical form.

3. Two different types of solvability conditions for the system (6)

In this section, we use the equivalence canonical form for five quaternion matrices (Theorem 2.1) to present two different types of solvability conditions for the system of quaternion matrix equations

$$
\begin{cases}
A_1X_1 - X_2B_1 = C_1, \\
A_2X_3 - X_2B_2 = C_2, \\
A_3X_3 - X_4B_3 = C_3, \\
A_4X_5 - X_4B_4 = C_4, \\
A_5X_5 - X_6B_5 = C_5.
\end{cases} \tag{11}
$$

The following theorem gives a necessary and sufficient condition for a solution to the system (11) in terms of ranks.

Theorem 3.1. *The system (11) is consistent if and only if the following 15 rank equalities hold*

$$r\begin{pmatrix} C_i & A_i \\ B_i & 0 \end{pmatrix} = r(A_i) + r(B_i), \quad (i = 1, \dots, 5), \tag{12}$$

$$r\begin{pmatrix} C_1 & C_2 & A_1 & A_2 \\ B_1 & B_2 & 0 & 0 \end{pmatrix} = r\begin{pmatrix} A_1 & A_2 \end{pmatrix} + r\begin{pmatrix} B_1 & B_2 \end{pmatrix}, \tag{13}$$

$$r\begin{pmatrix} C_2 & A_2 \\ C_3 & A_3 \\ B_2 & 0 \\ B_3 & 0 \end{pmatrix} = r\begin{pmatrix} A_2 \\ A_3 \end{pmatrix} + r\begin{pmatrix} B_2 \\ B_3 \end{pmatrix}, \tag{14}$$

$$r\begin{pmatrix} C_3 & C_4 & A_3 & A_4 \\ B_3 & B_4 & 0 & 0 \end{pmatrix} = r\begin{pmatrix} A_3 & A_4 \end{pmatrix} + r\begin{pmatrix} B_3 & B_4 \end{pmatrix}, \tag{15}$$

$$r\begin{pmatrix} C_4 & A_5 \\ C_4 & A_5 \\ B_4 & 0 \\ B_5 & 0 \end{pmatrix} = r\begin{pmatrix} A_4 \\ A_5 \end{pmatrix} + r\begin{pmatrix} B_4 \\ B_5 \end{pmatrix}, \tag{16}$$

$$r\begin{pmatrix} C_1 & C_2 & A_1 & A_2 \\ 0 & C_3 & 0 & A_3 \\ B_1 & B_2 & 0 & 0 \\ 0 & B_3 & 0 & 0 \end{pmatrix} = r\begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix} + r\begin{pmatrix} B_1 & B_2 \\ 0 & B_3 \end{pmatrix}, \tag{17}$$

$$r\begin{pmatrix} C_2 & 0 & A_2 & 0 \\ C_3 & C_4 & A_3 & A_4 \\ B_2 & 0 & 0 & 0 \\ B_3 & B_4 & 0 & 0 \end{pmatrix} = r\begin{pmatrix} A_2 & 0 \\ A_3 & A_4 \end{pmatrix} + r\begin{pmatrix} B_2 & 0 \\ B_3 & B_4 \end{pmatrix}, \tag{18}$$

$$r\begin{pmatrix} C_3 & C_4 & A_3 & A_4 \\ 0 & C_5 & 0 & A_5 \\ B_3 & B_4 & 0 & 0 \\ 0 & B_5 & 0 & 0 \end{pmatrix} = r\begin{pmatrix} A_3 & A_4 \\ 0 & A_5 \end{pmatrix} + r\begin{pmatrix} B_3 & B_4 \\ 0 & B_5 \end{pmatrix}, \tag{19}$$

$$r\begin{pmatrix} C_1 & C_2 & 0 & A_1 & A_2 & 0 \\ 0 & C_3 & C_4 & 0 & A_3 & A_4 \\ B_1 & B_2 & 0 & 0 & 0 & 0 \\ 0 & B_3 & B_4 & 0 & 0 & 0 \end{pmatrix} = r\begin{pmatrix} A_1 & A_2 & 0 \\ 0 & A_3 & A_4 \end{pmatrix} + r\begin{pmatrix} B_1 & B_2 & 0 \\ 0 & B_3 & B_4 \end{pmatrix}, \tag{20}$$

$$r\begin{pmatrix} C_2 & 0 & A_2 & 0 \\ C_3 & C_4 & A_3 & A_4 \\ 0 & C_5 & 0 & A_5 \\ B_2 & 0 & 0 & 0 \\ B_3 & B_4 & 0 & 0 \\ 0 & B_5 & 0 & 0 \end{pmatrix} = r\begin{pmatrix} A_2 & 0 \\ A_3 & A_4 \\ 0 & A_5 \end{pmatrix} + r\begin{pmatrix} B_2 & 0 \\ B_3 & B_4 \\ 0 & B_5 \end{pmatrix}, \tag{21}$$

$$r\begin{pmatrix} C_1 & C_2 & 0 & A_1 & A_2 & 0 \\ 0 & C_3 & C_4 & 0 & A_3 & A_4 \\ 0 & 0 & C_5 & 0 & 0 & A_5 \\ B_1 & B_2 & 0 & 0 & 0 & 0 \\ 0 & B_3 & B_4 & 0 & 0 & 0 \\ 0 & 0 & B_5 & 0 & 0 & 0 \end{pmatrix} = r\begin{pmatrix} A_1 & A_2 & 0 \\ 0 & A_3 & A_4 \\ 0 & 0 & A_5 \end{pmatrix} + r\begin{pmatrix} B_1 & B_2 & 0 \\ 0 & B_3 & B_4 \\ 0 & 0 & B_5 \end{pmatrix}. \tag{22}$$

Proof. Sufficiency: We divide it into three steps.

- Transform the system (11) into a simple form by using Theorem 2.1. See **Step 1** for more details.
- Give some necessary and sufficient conditions for the solvability of the simple system by using block matrices. See **Step 2** for more details.
- Prove that the block matrices solvability conditions are equivalent with the rank solvability conditions. See **Step 3** for more details.

Step 1. In this step, we transform the system (11) into a simple form. Observe that the coefficient matrices A_i and B_i can be arranged in the following matrix arrays

$$\begin{array}{c} \\ p_1 \\ p_2 \\ p_3 \end{array} \begin{array}{c} \begin{array}{ccc} q_1 & q_2 & q_3 \end{array} \\ \begin{pmatrix} A_1 & A_2 & \\ & A_3 & A_4 \\ & & A_5 \end{pmatrix} \end{array}, \tag{23}$$

$$\begin{array}{c c} & \begin{array}{ccc} n_1 & n_2 & n_3 \end{array} \\ \begin{array}{c} t_1 \\ t_2 \\ t_3 \end{array} & \begin{pmatrix} B_1 & B_2 & \\ & B_3 & B_4 \\ & & B_5 \end{pmatrix} \end{array}. \tag{24}$$

It follows from Theorem 2.1 that there exist nonsingular matrices

$$P_j \in \mathbb{H}^{p_j \times p_j}, \qquad Q_j \in \mathbb{H}^{q_j \times q_j}, \qquad T_j \in \mathbb{H}^{t_j \times t_j}, \qquad N_j \in \mathbb{H}^{n_j \times n_j}, \quad (j = 1, 2, 3),$$

such that

$$P_1A_1Q_1 = S_{a_1}, \qquad P_1A_2Q_2 = S_{a_2}, \qquad P_2A_3Q_2 = S_{a_3}, \qquad P_2A_4Q_3 = S_{a_4}, \qquad P_3A_5Q_3 = S_{a_5}, \tag{25}$$

$$T_1B_1N_1 = S_{b_1}, \qquad T_1B_2N_2 = S_{b_2}, \qquad T_2B_3N_2 = S_{b_3}, \qquad T_2B_4N_3 = S_{b_4}, \qquad T_3B_5N_3 = S_{b_5}, \tag{26}$$

where the forms of S_{a_i} and S_{b_i} are given in (9). S_{a_i} and S_{b_i} have the same forms except the dimensions of the identities and zeros. Then the system (11) becomes

$$\begin{cases} P_1^{-1}S_{a_1}Q_1^{-1}X_1 - X_2T_1^{-1}S_{b_1}N_1^{-1} = C_1, \\ P_1^{-1}S_{a_2}Q_2^{-1}X_3 - X_2T_1^{-1}S_{b_2}N_2^{-1} = C_2, \\ P_2^{-1}S_{a_3}Q_2^{-1}X_3 - X_4T_2^{-1}S_{b_3}N_2^{-1} = C_3, \\ P_2^{-1}S_{a_4}Q_3^{-1}X_5 - X_4T_2^{-1}S_{b_4}N_3^{-1} = C_4, \\ P_3^{-1}S_{a_5}Q_4^{-1}X_5 - X_6T_3^{-1}S_{b_5}N_3^{-1} = C_5, \end{cases}$$

i.e.,

$$\begin{cases} S_{a_1}(Q_1^{-1}X_1N_1) - (P_1X_2T_1^{-1})S_{b_1} = P_1C_1N_1, \\ S_{a_2}(Q_2^{-1}X_3N_2) - (P_1X_2T_1^{-1})S_{b_2} = P_1C_2N_2, \\ S_{a_3}(Q_2^{-1}X_3N_2) - (P_2X_4T_2^{-1})S_{b_3} = P_2C_3N_2, \\ S_{a_4}(Q_3^{-1}X_5N_3) - (P_2X_4T_2^{-1})S_{b_4} = P_2C_4N_3, \\ S_{a_5}(Q_3^{-1}X_5N_3) - (P_3X_6T_3^{-1})S_{b_5} = P_3C_5N_3. \end{cases}$$

Set

$$Y_1 = Q_1^{-1}X_1N_1, \qquad Y_2 = P_1X_2T_1^{-1}, \qquad Y_3 = Q_2^{-1}X_3N_2,$$

$$Y_4 = P_2X_4T_2^{-1}, \qquad Y_5 = Q_3^{-1}X_5N_3, \qquad Y_6 = P_3X_6T_3^{-1},$$

$$D_1 = P_1C_1N_1, \qquad D_2 = P_1C_2N_2, \qquad D_3 = P_2C_3N_2, \qquad D_4 = P_2C_4N_3, \qquad D_5 = P_3C_5N_3.$$

The system (11) is transformed to the following simple form

$$\begin{cases} S_{a_1}Y_1 - Y_2S_{b_1} = D_1, \\ S_{a_2}Y_3 - Y_2S_{b_2} = D_2, \\ S_{a_3}Y_3 - Y_4S_{b_3} = D_3, \\ S_{a_4}Y_5 - Y_4S_{b_4} = D_4, \\ S_{a_5}Y_5 - Y_6S_{b_5} = D_5. \end{cases} \tag{27}$$

Step 2. In this step, we solve the simple system (27). We give some necessary and sufficient conditions for the solvability of the system (27) by using block matrices. Let the matrices

$$Y_1 := \begin{pmatrix} Y_{11}^1 & \cdots & Y_{16}^1 \\ \vdots & \ddots & \vdots \\ Y_{61}^1 & \cdots & Y_{66}^1 \end{pmatrix}, \qquad Y_2 := \begin{pmatrix} Y_{11}^2 & \cdots & Y_{1,10}^2 \\ \vdots & \ddots & \vdots \\ Y_{10,1}^2 & \cdots & Y_{10,10}^2 \end{pmatrix}, \qquad Y_3 := \begin{pmatrix} Y_{11}^3 & \cdots & Y_{1,12}^3 \\ \vdots & \ddots & \vdots \\ Y_{12,1}^3 & \cdots & Y_{12,12}^3 \end{pmatrix},$$

$$Y_4 := \begin{pmatrix} Y_{11}^4 & \cdots & Y_{1,12}^4 \\ \vdots & \ddots & \vdots \\ Y_{12,1}^4 & \cdots & Y_{12,12}^4 \end{pmatrix}, \qquad Y_5 := \begin{pmatrix} Y_{11}^5 & \cdots & Y_{1,10}^5 \\ \vdots & \ddots & \vdots \\ Y_{10,1}^5 & \cdots & Y_{10,10}^5 \end{pmatrix}, \qquad Y_6 := \begin{pmatrix} Y_{11}^6 & \cdots & Y_{16}^6 \\ \vdots & \ddots & \vdots \\ Y_{61}^6 & \cdots & Y_{66}^6 \end{pmatrix},$$

and

$$D_1 := \left(D_{ij}^1\right)_{10\times 6}, \qquad D_2 := \left(D_{ij}^2\right)_{10\times 12}, \qquad D_3 := \left(D_{ij}^3\right)_{12\times 12}, \qquad D_4 := \left(D_{ij}^4\right)_{12\times 10}, \qquad D_5 := \left(D_{ij}^5\right)_{6\times 12} \tag{28}$$

be partitioned in accordance with S_{a_i}, S_{b_i} and (27). Substituting $Y_1, \ldots, Y_6$ into the system (27) yields

$$\begin{pmatrix} Y^1_{11}-Y^2_{11} & Y^1_{12}-Y^2_{12} & Y^1_{13}-Y^2_{13} & Y^1_{14}-Y^2_{14} & Y^1_{15}-Y^2_{15} & Y^1_{16} \\ Y^1_{21}-Y^2_{21} & Y^1_{22}-Y^2_{22} & Y^1_{23}-Y^2_{23} & Y^1_{24}-Y^2_{24} & Y^1_{25}-Y^2_{25} & Y^1_{26} \\ Y^1_{31}-Y^2_{31} & Y^1_{32}-Y^2_{32} & Y^1_{33}-Y^2_{33} & Y^1_{34}-Y^2_{34} & Y^1_{35}-Y^2_{35} & Y^1_{36} \\ Y^1_{41}-Y^2_{41} & Y^1_{42}-Y^2_{42} & Y^1_{43}-Y^2_{43} & Y^1_{44}-Y^2_{44} & Y^1_{45}-Y^2_{45} & Y^1_{46} \\ Y^1_{51}-Y^2_{51} & Y^1_{52}-Y^2_{52} & Y^1_{53}-Y^2_{53} & Y^1_{54}-Y^2_{54} & Y^1_{55}-Y^2_{55} & Y^1_{56} \\ -Y^2_{61} & -Y^2_{62} & -Y^2_{63} & -Y^2_{64} & -Y^2_{65} & 0 \\ -Y^2_{71} & -Y^2_{72} & -Y^2_{73} & -Y^2_{74} & -Y^2_{75} & 0 \\ -Y^2_{81} & -Y^2_{82} & -Y^2_{83} & -Y^2_{84} & -Y^2_{85} & 0 \\ -Y^2_{91} & -Y^2_{92} & -Y^2_{93} & -Y^2_{94} & -Y^2_{95} & 0 \\ -Y^2_{10,1} & -Y^2_{10,2} & -Y^2_{10,3} & -Y^2_{10,4} & -Y^2_{10,5} & 0 \end{pmatrix} = \left(D^1_{ij}\right)_{10\times 6}, \tag{29}$$

$$\begin{pmatrix} Y^3_{11}-Y^2_{11} & Y^3_{12}-Y^2_{12} & Y^3_{13}-Y^2_{13} & Y^3_{14}-Y^2_{14} & Y^3_{15}-Y^2_{16} & Y^3_{16}-Y^2_{17} & Y^3_{17}-Y^2_{18} & Y^3_{18}-Y^2_{19} & Y^3_{19} & Y^3_{1,10} & Y^3_{1,11} & Y^3_{1,12} \\ Y^3_{21}-Y^2_{21} & Y^3_{22}-Y^2_{22} & Y^3_{23}-Y^2_{23} & Y^3_{24}-Y^2_{24} & Y^3_{25}-Y^2_{26} & Y^3_{26}-Y^2_{27} & Y^3_{27}-Y^2_{28} & Y^3_{28}-Y^2_{29} & Y^3_{29} & Y^3_{2,10} & Y^3_{2,11} & Y^3_{2,12} \\ Y^3_{31}-Y^2_{31} & Y^3_{32}-Y^2_{32} & Y^3_{33}-Y^2_{33} & Y^3_{34}-Y^2_{34} & Y^3_{35}-Y^2_{36} & Y^3_{36}-Y^2_{37} & Y^3_{37}-Y^2_{38} & Y^3_{38}-Y^2_{39} & Y^3_{39} & Y^3_{3,10} & Y^3_{3,11} & Y^3_{3,12} \\ Y^3_{41}-Y^2_{41} & Y^3_{42}-Y^2_{42} & Y^3_{43}-Y^2_{43} & Y^3_{44}-Y^2_{44} & Y^3_{45}-Y^2_{46} & Y^3_{46}-Y^2_{47} & Y^3_{47}-Y^2_{48} & Y^3_{48}-Y^2_{49} & Y^3_{49} & Y^3_{4,10} & Y^3_{4,11} & Y^3_{4,12} \\ -Y^2_{51} & -Y^2_{52} & -Y^2_{53} & -Y^2_{54} & -Y^2_{56} & -Y^2_{57} & -Y^2_{58} & -Y^2_{59} & 0 & 0 & 0 & 0 \\ Y^3_{51}-Y^2_{61} & Y^3_{52}-Y^2_{62} & Y^3_{53}-Y^2_{63} & Y^3_{54}-Y^2_{64} & Y^3_{55}-Y^2_{66} & Y^3_{56}-Y^2_{67} & Y^3_{57}-Y^2_{68} & Y^3_{58}-Y^2_{69} & Y^3_{59} & Y^3_{5,10} & Y^3_{5,11} & Y^3_{5,12} \\ Y^3_{61}-Y^2_{71} & Y^3_{62}-Y^2_{72} & Y^3_{63}-Y^2_{73} & Y^3_{64}-Y^2_{74} & Y^3_{65}-Y^2_{76} & Y^3_{66}-Y^2_{77} & Y^3_{67}-Y^2_{78} & Y^3_{68}-Y^2_{79} & Y^3_{69} & Y^3_{6,10} & Y^3_{6,11} & Y^3_{6,12} \\ Y^3_{71}-Y^2_{81} & Y^3_{72}-Y^2_{82} & Y^3_{73}-Y^2_{83} & Y^3_{74}-Y^2_{84} & Y^3_{75}-Y^2_{86} & Y^3_{76}-Y^2_{87} & Y^3_{77}-Y^2_{88} & Y^3_{78}-Y^2_{89} & Y^3_{79} & Y^3_{7,10} & Y^3_{7,11} & Y^3_{7,12} \\ Y^3_{81}-Y^2_{91} & Y^3_{82}-Y^2_{92} & Y^3_{83}-Y^2_{93} & Y^3_{84}-Y^2_{94} & Y^3_{85}-Y^2_{96} & Y^3_{86}-Y^2_{97} & Y^3_{87}-Y^2_{98} & Y^3_{88}-Y^2_{99} & Y^3_{89} & Y^3_{8,10} & Y^3_{8,11} & Y^3_{8,12} \\ -Y^2_{10,1} & -Y^2_{10,2} & -Y^2_{10,3} & -Y^2_{10,4} & -Y^2_{10,6} & -Y^2_{10,7} & -Y^2_{10,8} & -Y^2_{10,9} & 0 & 0 & 0 & 0 \end{pmatrix} = \left(D^2_{ij}\right)_{10\times 12}, \tag{30}$$

$$\begin{pmatrix} Y^3_{11}-Y^4_{11} & Y^3_{12}-Y^4_{12} & Y^3_{13}-Y^4_{13} & Y^3_{14} & Y^3_{15}-Y^4_{14} & Y^3_{16}-Y^4_{15} & Y^3_{17}-Y^4_{16} & Y^3_{18} & Y^3_{19}-Y^4_{17} & Y^3_{1,10}-Y^4_{18} & Y^3_{1,11}-Y^4_{19} & Y^3_{1,12} \\ Y^3_{21}-Y^4_{21} & Y^3_{22}-Y^4_{22} & Y^3_{23}-Y^4_{23} & Y^3_{24} & Y^3_{25}-Y^4_{24} & Y^3_{26}-Y^4_{25} & Y^3_{27}-Y^4_{26} & Y^3_{28} & Y^3_{29}-Y^4_{27} & Y^3_{2,10}-Y^4_{28} & Y^3_{2,11}-Y^4_{29} & Y^3_{2,12} \\ Y^3_{31}-Y^4_{31} & Y^3_{32}-Y^4_{32} & Y^3_{33}-Y^4_{33} & Y^3_{34} & Y^3_{35}-Y^4_{34} & Y^3_{36}-Y^4_{35} & Y^3_{37}-Y^4_{36} & Y^3_{38} & Y^3_{39}-Y^4_{37} & Y^3_{3,10}-Y^4_{38} & Y^3_{3,11}-Y^4_{39} & Y^3_{3,12} \\ Y^3_{51}-Y^4_{41} & Y^3_{52}-Y^4_{42} & Y^3_{53}-Y^4_{43} & Y^3_{54} & Y^3_{55}-Y^4_{44} & Y^3_{56}-Y^4_{45} & Y^3_{57}-Y^4_{46} & Y^3_{58} & Y^3_{59}-Y^4_{47} & Y^3_{5,10}-Y^4_{48} & Y^3_{5,11}-Y^4_{49} & Y^3_{5,12} \\ Y^3_{61}-Y^4_{51} & Y^3_{62}-Y^4_{52} & Y^3_{63}-Y^4_{53} & Y^3_{64} & Y^3_{65}-Y^4_{54} & Y^3_{66}-Y^4_{55} & Y^3_{67}-Y^4_{56} & Y^3_{68} & Y^3_{69}-Y^4_{57} & Y^3_{6,10}-Y^4_{58} & Y^3_{6,11}-Y^4_{59} & Y^3_{6,12} \\ Y^3_{71}-Y^4_{61} & Y^3_{72}-Y^4_{62} & Y^3_{73}-Y^4_{63} & Y^3_{74} & Y^3_{75}-Y^4_{64} & Y^3_{76}-Y^4_{65} & Y^3_{77}-Y^4_{66} & Y^3_{78} & Y^3_{79}-Y^4_{67} & Y^3_{7,10}-Y^4_{68} & Y^3_{7,11}-Y^4_{69} & Y^3_{7,12} \\ Y^3_{91}-Y^4_{71} & Y^3_{92}-Y^4_{72} & Y^3_{93}-Y^4_{73} & Y^3_{94} & Y^3_{95}-Y^4_{74} & Y^3_{96}-Y^4_{75} & Y^3_{97}-Y^4_{76} & Y^3_{98} & Y^3_{99}-Y^4_{77} & Y^3_{9,10}-Y^4_{78} & Y^3_{9,11}-Y^4_{79} & Y^3_{9,12} \\ Y^3_{10,1}-Y^4_{81} & Y^3_{10,2}-Y^4_{82} & Y^3_{10,3}-Y^4_{83} & Y^3_{10,4} & Y^3_{10,5}-Y^4_{84} & Y^3_{10,6}-Y^4_{85} & Y^3_{10,7}-Y^4_{86} & Y^3_{10,8} & Y^3_{10,9}-Y^4_{87} & Y^3_{10,10}-Y^4_{88} & Y^3_{10,11}-Y^4_{89} & Y^3_{10,12} \\ Y^3_{11,1}-Y^4_{91} & Y^3_{11,2}-Y^4_{92} & Y^3_{11,3}-Y^4_{93} & Y^3_{11,4} & Y^3_{11,5}-Y^4_{94} & Y^3_{11,6}-Y^4_{95} & Y^3_{11,7}-Y^4_{96} & Y^3_{11,8} & Y^3_{11,9}-Y^4_{97} & Y^3_{11,10}-Y^4_{98} & Y^3_{11,11}-Y^4_{99} & Y^3_{11,12} \\ -Y^4_{10,1} & -Y^4_{10,2} & -Y^4_{10,3} & 0 & -Y^4_{10,4} & -Y^4_{10,5} & -Y^4_{10,6} & 0 & -Y^4_{10,7} & -Y^4_{10,8} & -Y^4_{10,9} & 0 \\ -Y^4_{11,1} & -Y^4_{11,2} & -Y^4_{11,3} & 0 & -Y^4_{11,4} & -Y^4_{11,5} & -Y^4_{11,6} & 0 & -Y^4_{11,7} & -Y^4_{11,8} & -Y^4_{11,9} & 0 \\ -Y^4_{12,1} & -Y^4_{12,2} & -Y^4_{12,3} & 0 & -Y^4_{12,4} & -Y^4_{12,5} & -Y^4_{12,6} & 0 & -Y^4_{12,7} & -Y^4_{12,8} & -Y^4_{12,9} & 0 \end{pmatrix}$$
$$= \left(D^3_{ij}\right)_{12\times 12}, \tag{31}$$

$$\begin{pmatrix} Y^5_{11}-Y^4_{11} & Y^5_{12}-Y^4_{12} & Y^5_{13}-Y^4_{14} & Y^5_{14}-Y^4_{15} & Y^5_{15}-Y^4_{17} & Y^5_{16}-Y^4_{18} & Y^5_{17}-Y^4_{1,10} & Y^5_{18}-Y^4_{1,11} & Y^5_{19} & Y^5_{1,10} \\ Y^5_{21}-Y^4_{21} & Y^5_{22}-Y^4_{22} & Y^5_{23}-Y^4_{24} & Y^5_{24}-Y^4_{25} & Y^5_{25}-Y^4_{27} & Y^5_{26}-Y^4_{28} & Y^5_{27}-Y^4_{2,10} & Y^5_{28}-Y^4_{2,11} & Y^5_{29} & Y^5_{2,10} \\ -Y^4_{31} & -Y^4_{32} & -Y^4_{34} & -Y^4_{35} & -Y^4_{37} & -Y^4_{38} & -Y^4_{3,10} & -Y^4_{3,11} & 0 & 0 \\ Y^5_{31}-Y^4_{41} & Y^5_{32}-Y^4_{42} & Y^5_{33}-Y^4_{44} & Y^5_{34}-Y^4_{45} & Y^5_{35}-Y^4_{47} & Y^5_{36}-Y^4_{48} & Y^5_{37}-Y^4_{4,10} & Y^5_{38}-Y^4_{4,11} & Y^5_{39} & Y^5_{3,10} \\ Y^5_{41}-Y^4_{51} & Y^5_{42}-Y^4_{52} & Y^5_{43}-Y^4_{54} & Y^5_{44}-Y^4_{55} & Y^5_{45}-Y^4_{57} & Y^5_{46}-Y^4_{58} & Y^5_{47}-Y^4_{5,10} & Y^5_{48}-Y^4_{5,11} & Y^5_{49} & Y^5_{4,10} \\ -Y^4_{61} & -Y^4_{62} & -Y^4_{64} & -Y^4_{65} & -Y^4_{67} & -Y^4_{68} & -Y^4_{6,10} & -Y^4_{6,11} & 0 & 0 \\ Y^5_{51}-Y^4_{71} & Y^5_{52}-Y^4_{72} & Y^5_{53}-Y^4_{74} & Y^5_{54}-Y^4_{75} & Y^5_{55}-Y^4_{77} & Y^5_{56}-Y^4_{78} & Y^5_{57}-Y^4_{7,10} & Y^5_{58}-Y^4_{7,11} & Y^5_{59} & Y^5_{5,10} \\ Y^5_{61}-Y^4_{81} & Y^5_{62}-Y^4_{82} & Y^5_{63}-Y^4_{84} & Y^5_{64}-Y^4_{85} & Y^5_{65}-Y^4_{87} & Y^5_{66}-Y^4_{88} & Y^5_{67}-Y^4_{8,10} & Y^5_{68}-Y^4_{8,11} & Y^5_{69} & Y^5_{6,10} \\ -Y^4_{91} & -Y^4_{92} & -Y^4_{94} & -Y^4_{95} & -Y^4_{97} & -Y^4_{98} & -Y^4_{9,10} & -Y^4_{9,11} & 0 & 0 \\ Y^5_{71}-Y^4_{10,1} & Y^5_{72}-Y^4_{10,2} & Y^5_{73}-Y^4_{10,4} & Y^5_{74}-Y^4_{10,5} & Y^5_{75}-Y^4_{10,7} & Y^5_{76}-Y^4_{10,8} & Y^5_{77}-Y^4_{10,10} & Y^5_{78}-Y^4_{10,11} & Y^5_{79} & Y^5_{7,10} \\ Y^5_{81}-Y^4_{11,1} & Y^5_{82}-Y^4_{11,2} & Y^5_{83}-Y^4_{11,4} & Y^5_{84}-Y^4_{11,5} & Y^5_{85}-Y^4_{11,7} & Y^5_{86}-Y^4_{11,8} & Y^5_{87}-Y^4_{11,10} & Y^5_{88}-Y^4_{11,11} & Y^5_{89} & Y^5_{8,10} \\ -Y^4_{12,1} & -Y^4_{12,2} & -Y^4_{12,4} & -Y^4_{12,5} & -Y^4_{12,7} & -Y^4_{12,8} & -Y^4_{12,10} & -Y^4_{12,11} & 0 & 0 \end{pmatrix} = \left(D^4_{ij}\right)_{12\times 10}, \tag{32}$$

$$\begin{pmatrix} Y^5_{11}-Y^6_{11} & Y^5_{12} & Y^5_{13}-Y^6_{12} & Y^5_{14} & Y^5_{15}-Y^6_{13} & Y^5_{16} & Y^5_{17}-Y^6_{14} & Y^5_{18} & Y^5_{19}-Y^6_{15} & Y^5_{1,10} \\ Y^5_{31}-Y^6_{21} & Y^5_{32} & Y^5_{33}-Y^6_{22} & Y^5_{34} & Y^5_{35}-Y^6_{23} & Y^5_{36} & Y^5_{37}-Y^6_{24} & Y^5_{38} & Y^5_{39}-Y^6_{25} & Y^5_{3,10} \\ Y^5_{51}-Y^6_{31} & Y^5_{52} & Y^5_{53}-Y^6_{32} & Y^5_{54} & Y^5_{55}-Y^6_{33} & Y^5_{56} & Y^5_{57}-Y^6_{34} & Y^5_{58} & Y^5_{59}-Y^6_{35} & Y^5_{5,10} \\ Y^5_{71}-Y^6_{41} & Y^5_{72} & Y^5_{73}-Y^6_{42} & Y^5_{74} & Y^5_{75}-Y^6_{43} & Y^5_{76} & Y^5_{77}-Y^6_{44} & Y^5_{78} & Y^5_{79}-Y^6_{45} & Y^5_{7,10} \\ Y^5_{91}-Y^6_{51} & Y^5_{92} & Y^5_{93}-Y^6_{52} & Y^5_{94} & Y^5_{95}-Y^6_{53} & Y^5_{96} & Y^5_{97}-Y^6_{54} & Y^5_{98} & Y^5_{99}-Y^6_{55} & Y^5_{9,10} \\ -Y^6_{61} & 0 & -Y^6_{62} & 0 & -Y^6_{63} & 0 & -Y^6_{64} & 0 & -Y^6_{65} & 0 \end{pmatrix} = \left(D^5_{ij}\right)_{6\times 10}. \tag{33}$$

Hence, the system (27) is consistent if and only if

$$D^1_{66} = 0, \quad D^1_{76} = 0, \quad D^1_{86} = 0, \quad D^1_{96} = 0, \quad D^1_{10,6} = 0, \tag{34}$$

$$D^2_{59} = 0, \quad D^2_{5,10} = 0, \quad D^2_{5,11} = 0, \quad D^2_{5,12} = 0, \quad D^2_{10,9} = 0, \quad D^2_{10,10} = 0, \quad D^2_{10,11} = 0, \quad D^2_{10,12} = 0, \tag{35}$$

$$\begin{gathered} D^3_{10,4} = 0, \quad D^3_{11,4} = 0, \quad D^3_{12,4} = 0, \quad D^3_{10,8} = 0, \quad D^3_{11,8} = 0, \quad D^3_{12,8} = 0, \quad D^3_{10,12} = 0, \\ D^3_{11,12} = 0, \quad D^3_{12,12} = 0, \end{gathered} \tag{36}$$

$$D^4_{39} = 0, \quad D^4_{3,10} = 0, \quad D^4_{69} = 0, \quad D^4_{6,10} = 0, \quad D^4_{99} = 0, \quad D^4_{9,10} = 0, \quad D^4_{12,9} = 0, \quad D^4_{12,10} = 0, \tag{37}$$

$$D^5_{62} = 0, \quad D^5_{64} = 0, \quad D^5_{66} = 0, \quad D^5_{68} = 0, \quad D^5_{6,10} = 0, \tag{38}$$

$$D^1_{10,1} = D^2_{10,1}, \quad D^1_{10,2} = D^2_{10,2}, \quad D^1_{10,3} = D^2_{10,3}, \quad D^1_{10,4} = D^2_{10,4}, \tag{39}$$

$$D^2_{1,12} = D^3_{1,12}, \quad D^2_{2,12} = D^3_{2,12}, \quad D^2_{3,12} = D^3_{3,12}, \quad D^2_{6,12} = D^3_{4,12}, \quad D^2_{7,12} = D^3_{5,12}, \quad D^2_{8,12} = D^3_{6,12}, \tag{40}$$

$$D^3_{12,1} = D^4_{12,1}, \quad D^3_{12,2} = D^4_{12,2}, \quad D^3_{12,5} = D^4_{12,3}, \quad D^3_{12,6} = D^4_{12,4}, \quad D^3_{12,9} = D^4_{12,5}, \quad D^3_{12,10} = D^4_{12,6}, \tag{41}$$

$$D^4_{1,10} = D^5_{1,10}, \quad D^4_{4,10} = D^5_{2,10}, \quad D^4_{7,10} = D^5_{3,10}, \quad D^4_{10,10} = D^5_{4,10}, \tag{42}$$

$$D^2_{64} = D^3_{44} + D^1_{64}, \quad D^2_{74} = D^3_{54} + D^1_{74}, \quad D^2_{84} = D^3_{64} + D^1_{84}, \tag{43}$$

$$D^3_{39} = D^2_{39} + D^4_{35}, \quad D^3_{3,10} = D^2_{3,10} + D^4_{36}, \quad D^3_{69} = D^2_{89} + D^4_{65}, \quad D^3_{6,10} = D^2_{8,10} + D^4_{66}, \tag{44}$$

$$D^4_{10,2} = D^3_{10,2} + D^5_{42}, \quad D^4_{10,4} = D^3_{10,6} + D^5_{44}, \quad D^4_{10,6} = D^3_{10,10} + D^5_{46}, \tag{45}$$

$$D^1_{81} + D^3_{61} = D^2_{81} + D^4_{61}, \quad D^1_{82} + D^3_{62} = D^2_{82} + D^4_{62}, \tag{46}$$

$$D^2_{1,10} + D^4_{16} = D^3_{1,10} + D^5_{16}, \quad D^2_{6,10} + D^4_{46} = D^3_{4,10} + D^5_{26}, \tag{47}$$

$$D^1_{62} + D^3_{42} + D^5_{22} = D^2_{62} + D^4_{42}. \tag{48}$$

Step 3. In this step, we show that (12)–(22)$\Longrightarrow$(34)–(48).

- Note that

$$r\begin{pmatrix} C_1 & A_1 \\ B_1 & 0 \end{pmatrix} = r(A_1) + r(B_1) \Longrightarrow r\begin{pmatrix} P_1C_1N_1 & P_1A_1Q_1 \\ T_1B_1N_1 & 0 \end{pmatrix} = r(P_1A_1Q_1) + r(T_1B_1N_1)$$
$$\Longrightarrow r\begin{pmatrix} D_1 & S_{a_1} \\ S_{b_1} & 0 \end{pmatrix} = r(S_{a_1}) + r(S_{b_1}) \Longrightarrow (34).$$

  Similarly, it can be found that

$$r\begin{pmatrix} C_2 & A_2 \\ B_2 & 0 \end{pmatrix} = r(A_2) + r(B_2) \Longrightarrow (35),$$
$$r\begin{pmatrix} C_3 & A_3 \\ B_3 & 0 \end{pmatrix} = r(A_3) + r(B_3) \Longrightarrow (36),$$
$$r\begin{pmatrix} C_4 & A_4 \\ B_4 & 0 \end{pmatrix} = r(A_4) + r(B_4) \Longrightarrow (37),$$
$$r\begin{pmatrix} C_5 & A_5 \\ B_5 & 0 \end{pmatrix} = r(A_5) + r(B_5) \Longrightarrow (38).$$

- Note that

$$r\begin{pmatrix} C_1 & C_2 & A_1 & A_2 \\ B_1 & B_2 & 0 & 0 \end{pmatrix} = r\begin{pmatrix} A_1 & A_2 \end{pmatrix} + r\begin{pmatrix} B_1 & B_2 \end{pmatrix} \Longrightarrow$$

$$r\begin{pmatrix} P_1C_1N_1 & P_1C_2N_2 & P_1A_1Q_1 & P_1A_2Q_2 \\ T_1B_1N_1 & T_1B_2N_2 & 0 & 0 \end{pmatrix} = r\begin{pmatrix} P_1A_1Q_1 & P_1A_2Q_2 \end{pmatrix} + r\begin{pmatrix} T_1B_1N_1 & T_1B_2N_2 \end{pmatrix}$$

$$\Longrightarrow r\begin{pmatrix} D_1 & D_2 & S_{a_1} & S_{a_2} \\ S_{b_1} & S_{b_2} & 0 & 0 \end{pmatrix} = r\begin{pmatrix} S_{a_1} & S_{a_2} \end{pmatrix} + r\begin{pmatrix} S_{b_1} & S_{b_2} \end{pmatrix} \Longrightarrow (39).$$

Similarly, it can be found that

$$r\begin{pmatrix} C_2 & A_2 \\ C_3 & A_3 \\ B_2 & 0 \\ B_3 & 0 \end{pmatrix} = r\begin{pmatrix} A_2 \\ A_3 \end{pmatrix} + r\begin{pmatrix} B_2 \\ B_3 \end{pmatrix} \Longrightarrow (40),$$

$$r\begin{pmatrix} C_3 & C_4 & A_3 & A_4 \\ B_3 & B_4 & 0 & 0 \end{pmatrix} = r\begin{pmatrix} A_3 & A_4 \end{pmatrix} + r\begin{pmatrix} B_3 & B_4 \end{pmatrix} \Longrightarrow (41),$$

$$r\begin{pmatrix} C_4 & A_5 \\ C_4 & A_5 \\ B_4 & 0 \\ B_5 & 0 \end{pmatrix} = r\begin{pmatrix} A_4 \\ A_5 \end{pmatrix} + r\begin{pmatrix} B_4 \\ B_5 \end{pmatrix} \Longrightarrow (42).$$

- Note that

$$r\begin{pmatrix} C_1 & C_2 & A_1 & A_2 \\ 0 & C_3 & 0 & A_3 \\ B_1 & B_2 & 0 & 0 \\ 0 & B_3 & 0 & 0 \end{pmatrix} = r\begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix} + r\begin{pmatrix} B_1 & B_2 \\ 0 & B_3 \end{pmatrix} \Longrightarrow$$

$$r\begin{pmatrix} P_1C_1N_1 & P_1C_2N_2 & P_1A_1Q_1 & P_1A_2Q_2 \\ 0 & P_2C_3N_2 & 0 & P_2A_3Q_2 \\ T_1B_1N_1 & T_1B_2N_2 & 0 & 0 \\ 0 & T_2B_3N_2 & 0 & 0 \end{pmatrix} = r\begin{pmatrix} P_1A_1Q_1 & P_1A_2Q_2 \\ 0 & P_2A_3Q_2 \end{pmatrix} + r\begin{pmatrix} T_1B_1N_1 & T_1B_2N_2 \\ 0 & T_2B_3N_2 \end{pmatrix}$$

$$r\begin{pmatrix} D_1 & D_2 & S_{a_1} & S_{a_2} \\ 0 & D_3 & 0 & S_{a_3} \\ S_{b_1} & S_{b_2} & 0 & 0 \\ 0 & S_{b_3} & 0 & 0 \end{pmatrix} = r\begin{pmatrix} S_{a_1} & S_{a_2} \\ 0 & S_{a_3} \end{pmatrix} + r\begin{pmatrix} S_{b_1} & S_{b_2} \\ 0 & S_{b_3} \end{pmatrix} \Longrightarrow (43).$$

Similarly, it can be found that

$$r\begin{pmatrix} C_2 & 0 & A_2 & 0 \\ C_3 & C_4 & A_3 & A_4 \\ B_2 & 0 & 0 & 0 \\ B_3 & B_4 & 0 & 0 \end{pmatrix} = r\begin{pmatrix} A_2 & 0 \\ A_3 & A_4 \end{pmatrix} + r\begin{pmatrix} B_2 & 0 \\ B_3 & B_4 \end{pmatrix} \Longrightarrow (44),$$

$$r\begin{pmatrix} C_3 & C_4 & A_3 & A_4 \\ 0 & C_5 & 0 & A_5 \\ B_3 & B_4 & 0 & 0 \\ 0 & B_5 & 0 & 0 \end{pmatrix} = r\begin{pmatrix} A_3 & A_4 \\ 0 & A_5 \end{pmatrix} + r\begin{pmatrix} B_3 & B_4 \\ 0 & B_5 \end{pmatrix} \Longrightarrow (45),$$

$$r\begin{pmatrix} C_1 & C_2 & 0 & A_1 & A_2 & 0 \\ 0 & C_3 & C_4 & 0 & A_3 & A_4 \\ B_1 & B_2 & 0 & 0 & 0 & 0 \\ 0 & B_3 & B_4 & 0 & 0 & 0 \end{pmatrix} = r\begin{pmatrix} A_1 & A_2 & 0 \\ 0 & A_3 & A_4 \end{pmatrix} + r\begin{pmatrix} B_1 & B_2 & 0 \\ 0 & B_3 & B_4 \end{pmatrix} \Longrightarrow (46),$$

$$
r\begin{pmatrix} C_2 & 0 & A_2 & 0 \\ C_3 & C_4 & A_3 & A_4 \\ 0 & C_5 & 0 & A_5 \\ B_2 & 0 & 0 & 0 \\ B_3 & B_4 & 0 & 0 \\ 0 & B_5 & 0 & 0 \end{pmatrix} = r\begin{pmatrix} A_2 & 0 \\ A_3 & A_4 \\ 0 & A_5 \end{pmatrix} + r\begin{pmatrix} B_2 & 0 \\ B_3 & B_4 \\ 0 & B_5 \end{pmatrix} \Longrightarrow (47),
$$

$$
r\begin{pmatrix} C_1 & C_2 & 0 & A_1 & A_2 & 0 \\ 0 & C_3 & C_4 & 0 & A_3 & A_4 \\ 0 & 0 & C_5 & 0 & 0 & A_5 \\ B_1 & B_2 & 0 & 0 & 0 & 0 \\ 0 & B_3 & B_4 & 0 & 0 & 0 \\ 0 & 0 & B_5 & 0 & 0 & 0 \end{pmatrix} = r\begin{pmatrix} A_1 & A_2 & 0 \\ 0 & A_3 & A_4 \\ 0 & 0 & A_5 \end{pmatrix} + r\begin{pmatrix} B_1 & B_2 & 0 \\ 0 & B_3 & B_4 \\ 0 & 0 & B_5 \end{pmatrix} \Longrightarrow (48).
$$

For the other direction, assume that $(X_1^0, X_2^0, X_3^0, X_4^0, X_5^0, X_6^0)$ is a solution to the system (11), then clearly

$$
\begin{cases} A_1X_1^0 - X_2^0B_1 = C_1, \\ A_2X_3^0 - X_2^0B_2 = C_2, \\ A_3X_3^0 - X_4^0B_3 = C_3, \\ A_4X_5^0 - X_4^0B_4 = C_4, \\ A_5X_5^0 - X_6^0B_5 = C_5. \end{cases} \tag{49}
$$

We will make use of (49) and elementary matrix operations to prove the rank equalities (12)–(22). We only prove the most complicated rank equality (22). Note that

$$
\begin{aligned}
& r\begin{pmatrix} C_1 & C_2 & 0 & A_1 & A_2 & 0 \\ 0 & C_3 & C_4 & 0 & A_3 & A_4 \\ 0 & 0 & C_5 & 0 & 0 & A_5 \\ B_1 & B_2 & 0 & 0 & 0 & 0 \\ 0 & B_3 & B_4 & 0 & 0 & 0 \\ 0 & 0 & B_5 & 0 & 0 & 0 \end{pmatrix} \\
&= r\begin{pmatrix} A_1X_1^0 - X_2^0B_1 & A_2X_3^0 - X_2^0B_2 & 0 & A_1 & A_2 & 0 \\ 0 & A_3X_3^0 - X_4^0B_3 & A_4X_5^0 - X_4^0B_4 & 0 & A_3 & A_4 \\ 0 & 0 & A_5X_5^0 - X_6^0B_5 & 0 & 0 & A_5 \\ B_1 & B_2 & 0 & 0 & 0 & 0 \\ 0 & B_3 & B_4 & 0 & 0 & 0 \\ 0 & 0 & B_5 & 0 & 0 & 0 \end{pmatrix} \\
&= r\left[\begin{pmatrix} I & 0 & 0 & X_2^0 & 0 & 0 \\ 0 & I & 0 & 0 & X_4^0 & 0 \\ 0 & 0 & I & 0 & 0 & X_6^0 \\ 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & I \end{pmatrix} \begin{pmatrix} A_1X_1^0-X_2^0B_1 & A_2X_3^0-X_2^0B_2 & 0 & A_1 & A_2 & 0 \\ 0 & A_3X_3^0-X_4^0B_3 & A_4X_5^0-X_4^0B_4 & 0 & A_3 & A_4 \\ 0 & 0 & A_5X_5^0-X_6^0B_5 & 0 & 0 & A_5 \\ B_1 & B_2 & 0 & 0 & 0 & 0 \\ 0 & B_3 & B_4 & 0 & 0 & 0 \\ 0 & 0 & B_5 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} I & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 \\ -X_1^0 & 0 & 0 & I & 0 & 0 \\ 0 & -X_3^0 & 0 & 0 & I & 0 \\ 0 & 0 & -X_5^0 & 0 & 0 & I \end{pmatrix}\right] \\
&= r\begin{pmatrix} 0 & 0 & 0 & A_1 & A_2 & 0 \\ 0 & 0 & 0 & 0 & A_3 & A_4 \\ 0 & 0 & 0 & 0 & 0 & A_5 \\ B_1 & B_2 & 0 & 0 & 0 & 0 \\ 0 & B_3 & B_4 & 0 & 0 & 0 \\ 0 & 0 & B_5 & 0 & 0 & 0 \end{pmatrix} \\
&= r\begin{pmatrix} A_1 & A_2 & 0 \\ 0 & A_3 & A_4 \\ 0 & 0 & A_5 \end{pmatrix} + r\begin{pmatrix} B_1 & B_2 & 0 \\ 0 & B_3 & B_4 \\ 0 & 0 & B_5 \end{pmatrix}.
\end{aligned}
$$

The proof of (12)–(21) is similar to that of (22). □

Remark 3.2. He [45] used Moore–Penrose inverse to consider the generalization of the system (11), i.e., $A_iX_iB_i+C_iX_{i+1}D_i = E_i,\ i = \overline{1,k}$. He presented some necessary and sufficient conditions for the existence of a solution to the above mentioned system in terms of ranks. We in this paper use different method to consider the system (11).

Another necessary and sufficient condition for a solution to the system (11) follows from the proof of Theorem 3.1.

Theorem 3.3. *The system (11) is consistent if and only if the equalities (34)–(48) hold, where the blocks D^k_{uv} are defined in (28).*

Remark 3.4. The necessary and sufficient conditions in Theorem 3.1 are more computational.

4. General solution to the system (6)

In this section, we give the general solution to the system (11).

Theorem 4.1. *Assume that the system (11) is consistent. Then, the general solution to the system (11) can be expressed as*

$$X_1 = Q_1 Y_1 N_1^{-1}, \qquad X_2 = P_1^{-1} Y_2 T_1, \qquad X_3 = Q_2 Y_3 N_2^{-1}, \tag{50}$$

$$X_4 = P_2^{-1} Y_4 T_2, \qquad X_5 = Q_3 Y_5 N_3^{-1}, \qquad X_6 = P_3^{-1} Y_6 T_3, \tag{51}$$

where

$$Y_1 = \begin{pmatrix} D^1_{11}-D^2_{11}+D^3_{11}-D^4_{11}+D^5_{11}+Y^6_{11} & D^1_{12}-D^2_{12}+D^3_{12}-D^4_{12}+D^5_{12} & D^1_{13}-D^2_{13}+D^3_{13}+Y^4_{13} & D^1_{14}-D^2_{14}+D^3_{14} & D^1_{15}+Y^2_{15} & D^1_{16} \\ D^1_{21}-D^2_{21}+D^3_{21}-D^4_{21}+Y^5_{21} & D^1_{22}-D^2_{22}+D^3_{22}-D^4_{22}+Y^5_{22} & D^1_{23}-D^2_{23}+D^3_{23}+Y^4_{23} & D^1_{24}-D^2_{24}+D^3_{24} & D^1_{25}+Y^2_{25} & D^1_{26} \\ D^1_{31}-D^2_{31}+D^3_{31}-D^4_{31} & D^1_{32}-D^2_{32}+D^3_{32}-D^4_{32} & D^1_{33}-D^2_{33}+D^3_{33}+Y^4_{33} & D^1_{34}-D^2_{34}+D^3_{34} & D^1_{35}+Y^2_{35} & D^1_{36} \\ D^1_{41}-D^2_{41}+Y^3_{41} & D^1_{42}-D^2_{42}+Y^3_{42} & D^1_{43}-D^2_{43}+Y^3_{43} & D^1_{44}-D^2_{44}+Y^3_{44} & D^1_{45}+Y^2_{45} & D^1_{46} \\ D^1_{51}-D^2_{51} & D^1_{52}-D^2_{52} & D^1_{53}-D^2_{53} & D^1_{54}-D^2_{54} & D^1_{55}+Y^2_{55} & D^1_{56} \\ Y^1_{61} & Y^1_{62} & Y^1_{63} & Y^1_{64} & Y^1_{65} & Y^1_{66} \end{pmatrix},$$

$$Y_2 = \begin{pmatrix} -D^2_{11}+D^3_{11}-D^4_{11}+D^5_{11}+Y^6_{11} & -D^2_{12}+D^3_{12}-D^4_{12}+D^5_{12} & -D^2_{13}+D^3_{13}+Y^4_{13} & -D^2_{14}+D^3_{14} & Y^2_{15} & -D^2_{15}+D^3_{15}-D^4_{13}+D^5_{13}+Y^6_{12} & -D^2_{16}+D^3_{16}-D^4_{14}+D^5_{14} & -D^2_{17}+D^3_{17}+Y^4_{16} & -D^2_{18}+D^3_{18} & Y^2_{1,10} \\ -D^2_{21}+D^3_{21}-D^4_{21}+Y^5_{21} & -D^2_{22}+D^3_{22}-D^4_{22}+Y^5_{22} & -D^2_{23}+D^3_{23}+Y^4_{23} & -D^2_{24}+D^3_{24} & Y^2_{25} & -D^2_{25}+D^3_{25}-D^4_{23}+Y^5_{23} & -D^2_{26}+D^3_{26}-D^4_{24}+Y^5_{24} & -D^2_{27}+D^3_{27}+Y^4_{26} & -D^2_{28}+D^3_{28} & Y^2_{2,10} \\ -D^2_{31}+D^3_{31}-D^4_{31} & -D^2_{32}+D^3_{32}-D^4_{32} & -D^2_{33}+D^3_{33}+Y^4_{33} & -D^2_{34}+D^3_{35}-D^4_{33} & Y^2_{35} & -D^2_{35}+D^3_{35}-D^4_{33} & -D^2_{36}+D^3_{36}-D^4_{34} & -D^2_{37}+D^3_{37}+Y^4_{36} & -D^2_{38}+D^3_{38} & Y^2_{3,10} \\ -D^2_{41}+Y^3_{41} & -D^2_{42}+Y^3_{42} & -D^2_{43}+Y^3_{43} & -D^2_{44}+Y^3_{44} & Y^2_{45} & -D^2_{45}+Y^3_{45} & -D^2_{46}+Y^3_{46} & -D^2_{47}+Y^3_{47} & -D^2_{48}+Y^3_{48} & Y^2_{4,10} \\ -D^2_{51} & -D^2_{52} & -D^2_{53} & -D^2_{54} & Y^2_{55} & -D^2_{55} & -D^2_{56} & -D^2_{57} & -D^2_{58} & Y^2_{5,10} \\ -D^1_{61} & -D^1_{62} & -D^1_{63} & -D^1_{64} & -D^1_{65} & -D^2_{65}+D^3_{45}-D^4_{43}+D^5_{23}+Y^6_{22} & -D^2_{66}+D^3_{46}-D^4_{44}+D^5_{24} & -D^2_{67}+D^3_{47}+Y^4_{46} & -D^2_{68}+D^3_{48} & Y^2_{6,10} \\ -D^1_{71} & -D^1_{72} & -D^1_{73} & -D^1_{74} & -D^1_{75} & -D^2_{75}+D^3_{55}-D^4_{53}+Y^5_{43} & -D^2_{76}+D^3_{56}-D^4_{54}+Y^5_{44} & -D^2_{77}+D^3_{57}+Y^4_{56} & -D^2_{78}+D^3_{58} & Y^2_{7,10} \\ -D^1_{81} & -D^1_{82} & -D^1_{83} & -D^1_{84} & -D^1_{85} & -D^2_{85}+D^3_{65}-D^4_{63} & -D^2_{86}+D^3_{66}-D^4_{64} & -D^2_{87}+D^3_{67}+Y^4_{66} & -D^2_{88}+D^3_{68} & Y^2_{8,10} \\ -D^1_{91} & -D^1_{92} & -D^1_{93} & -D^1_{94} & -D^1_{95} & -D^2_{95}+Y^3_{85} & -D^2_{96}+Y^3_{86} & -D^2_{97}+Y^3_{87} & -D^2_{98}+Y^3_{88} & Y^2_{9,10} \\ -D^1_{10,1} & -D^1_{10,2} & -D^1_{10,3} & -D^1_{10,4} & -D^1_{10,5} & -D^2_{10,5} & -D^2_{10,6} & -D^2_{10,7} & -D^2_{10,8} & Y^2_{10,10} \end{pmatrix},$$

$$
Y_3 = \begin{pmatrix}
D^3_{11}-D^4_{11}+D^5_{11}+Y^6_{11} & D^3_{12}-D^4_{12}+D^5_{12} & D^3_{13}+Y^4_{13} & D^3_{14} & D^3_{15}-D^4_{13}+D^5_{13}+Y^6_{12} & D^3_{16}-D^4_{14}+D^5_{14} & D^3_{17}+Y^4_{16} & D^3_{18} & D^2_{19} & D^2_{1,10} & D^2_{1,11} & D^2_{1,12} \\
D^3_{21}-D^4_{21}+Y^5_{21} & D^3_{22}-D^4_{22}+Y^5_{22} & D^3_{23}+Y^4_{23} & D^3_{24} & D^3_{25}-D^4_{23}+Y^5_{23} & D^3_{26}-D^4_{24}+Y^5_{24} & D^3_{27}+Y^4_{26} & D^3_{28} & D^2_{29} & D^2_{2,10} & D^2_{2,11} & D^2_{2,12} \\
D^3_{31}-D^4_{31} & D^3_{32}-D^4_{32} & D^3_{33}+Y^4_{33} & D^3_{34} & D^3_{35}-D^4_{33} & D^3_{36}-D^4_{34} & D^3_{37}+Y^4_{36} & D^3_{38} & D^2_{39} & D^2_{3,10} & D^2_{3,11} & D^2_{3,12} \\
Y^3_{41} & Y^3_{42} & Y^3_{43} & Y^3_{44} & Y^3_{45} & Y^3_{46} & Y^3_{47} & Y^3_{48} & D^2_{49} & D^2_{4,10} & D^2_{4,11} & D^2_{4,12} \\
-D^1_{61}+D^2_{61} & -D^1_{62}+D^2_{62} & -D^1_{63}+D^2_{63} & -D^1_{64}+D^2_{64} & D^3_{45}-D^4_{43}+D^5_{23}+Y^6_{22} & D^3_{46}-D^4_{44}+D^5_{24} & D^3_{47}+Y^4_{46} & D^3_{48} & D^2_{69} & D^2_{6,10} & D^2_{6,11} & D^2_{6,12} \\
-D^1_{71}+D^2_{71} & -D^1_{72}+D^2_{72} & -D^1_{73}+D^2_{73} & -D^1_{74}+D^2_{74} & D^3_{55}-D^4_{53}+Y^5_{43} & D^3_{56}-D^4_{54}+Y^5_{44} & D^3_{57}+Y^4_{56} & D^3_{58} & D^2_{79} & D^2_{7,10} & D^2_{7,11} & D^2_{7,12} \\
-D^1_{81}+D^2_{81} & -D^1_{82}+D^2_{82} & -D^1_{83}+D^2_{83} & -D^1_{84}+D^2_{84} & D^3_{65}-D^4_{63} & D^3_{66}-D^4_{64} & D^3_{67}+Y^4_{66} & D^3_{68} & D^2_{89} & D^2_{8,10} & D^2_{8,11} & D^2_{8,12} \\
-D^1_{91}+D^2_{91} & -D^1_{92}+D^2_{92} & -D^1_{93}+D^2_{93} & -D^1_{94}+D^2_{94} & Y^3_{85} & Y^3_{86} & Y^3_{87} & Y^3_{88} & D^2_{99} & D^2_{9,10} & D^2_{9,11} & D^2_{9,12} \\
D^3_{71}-D^4_{71}+D^5_{31}+Y^6_{31} & D^3_{72}-D^4_{72}+D^5_{32} & D^3_{73}+Y^4_{73} & D^3_{74} & D^3_{75}-D^4_{73}+D^5_{33}+Y^6_{32} & D^3_{76}-D^4_{74}+D^5_{34} & D^3_{77}+Y^4_{76} & D^3_{78} & D^3_{79}-D^4_{75}+D^5_{35}+Y^6_{33} & D^3_{7,10}-D^4_{76}+D^5_{36} & D^3_{7,11}+Y^4_{79} & D^3_{7,12} \\
D^3_{81}-D^4_{81}+Y^5_{61} & D^3_{82}-D^4_{82}+Y^5_{62} & D^3_{83}+Y^4_{83} & D^3_{8,4} & D^3_{85}-D^4_{83}+Y^5_{63} & D^3_{86}-D^4_{84}+Y^5_{64} & D^3_{87}+Y^4_{86} & D^3_{88} & D^3_{89}-D^4_{85}+Y^5_{65} & D^3_{8,10}-D^4_{86}+Y^5_{66} & D^3_{8,11}+Y^4_{89} & D^3_{8,12} \\
D^3_{91}-D^4_{91} & D^3_{92}-D^4_{92} & D^3_{93}+Y^4_{93} & D^3_{9,4} & D^3_{95}-D^4_{93} & D^3_{96}-D^4_{94} & D^3_{97}+Y^4_{96} & D^3_{98} & D^3_{99}-D^4_{95} & D^3_{9,10}-D^4_{96} & D^3_{9,11}+Y^4_{99} & D^3_{9,12} \\
Y^3_{12,1} & Y^3_{12,2} & Y^3_{12,3} & Y^3_{12,4} & Y^3_{12,5} & Y^3_{12,6} & Y^3_{12,7} & Y^3_{12,8} & Y^3_{12,9} & Y^3_{12,10} & Y^3_{12,11} & Y^3_{12,12}
\end{pmatrix},
$$

$$
Y_4 = \begin{pmatrix}
-D^4_{11}+D^5_{11}+Y^6_{11} & -D^4_{12}+D^5_{12} & Y^4_{13} & -D^4_{13}+D^5_{13}+Y^6_{12} & -D^4_{14}+D^5_{14} & Y^4_{16} & D^2_{19}-D^3_{19} & D^2_{1,10}-D^3_{1,10} & D^2_{1,11}-D^3_{1,11} & -D^4_{17}+D^5_{17}+Y^6_{14} & -D^4_{18}+D^5_{18} & Y^4_{1,12} \\
-D^4_{21}+Y^5_{21} & -D^4_{22}+Y^5_{22} & Y^4_{23} & -D^4_{23}+Y^5_{23} & -D^4_{24}+Y^5_{24} & Y^4_{26} & D^2_{29}-D^3_{29} & D^2_{2,10}-D^3_{2,10} & D^2_{2,11}-D^3_{2,11} & -D^4_{27}+Y^5_{27} & -D^4_{28}+Y^5_{28} & Y^4_{2,12} \\
-D^4_{31} & -D^4_{32} & Y^4_{33} & -D^4_{33} & -D^4_{34} & Y^4_{36} & D^2_{39}-D^3_{39} & D^2_{3,10}-D^3_{3,10} & D^2_{3,11}-D^3_{3,11} & -D^4_{37} & -D^4_{38} & Y^4_{3,12} \\
-D^1_{61}+D^2_{61}-D^3_{41} & -D^1_{62}+D^2_{62}-D^3_{42} & -D^1_{63}+D^2_{63}-D^3_{43} & -D^4_{43}+D^5_{23}+Y^6_{22} & -D^4_{44}+D^5_{24} & Y^4_{46} & D^2_{69}-D^3_{49} & D^2_{6,10}-D^3_{4,10} & D^2_{6,11}-D^3_{4,11} & -D^4_{47}+D^5_{27}+Y^6_{24} & -D^4_{48}+D^5_{28} & Y^4_{4,12} \\
-D^1_{71}+D^2_{71}-D^3_{51} & -D^1_{72}+D^2_{72}-D^3_{52} & -D^1_{73}+D^2_{73}-D^3_{53} & -D^4_{53}+Y^5_{43} & -D^4_{54}+Y^5_{44} & Y^4_{56} & D^2_{79}-D^3_{59} & D^2_{7,10}-D^3_{5,10} & D^2_{7,11}-D^3_{5,11} & -D^4_{57}+Y^5_{47} & -D^4_{58}+Y^5_{48} & Y^4_{5,12} \\
-D^1_{81}+D^2_{81}-D^3_{61} & -D^1_{82}+D^2_{82}-D^3_{62} & -D^1_{83}+D^2_{83}-D^3_{63} & -D^4_{63} & -D^4_{64} & Y^4_{66} & D^2_{89}-D^3_{69} & D^2_{8,10}-D^3_{6,10} & D^2_{8,11}-D^3_{6,11} & -D^4_{67} & -D^4_{68} & Y^4_{6,12} \\
-D^4_{71}+D^5_{31}+Y^6_{31} & -D^4_{72}+D^5_{32} & Y^4_{73} & -D^4_{73}+D^5_{33}+Y^6_{32} & -D^4_{74}+D^5_{34} & Y^4_{76} & -D^4_{75}+D^5_{35}+Y^6_{33} & -D^4_{76}+D^5_{36} & Y^4_{79} & -D^4_{77}+D^5_{37}+Y^6_{34} & -D^4_{78}+D^5_{38} & Y^4_{7,12} \\
-D^4_{81}+Y^5_{61} & -D^4_{82}+Y^5_{62} & Y^4_{83} & -D^4_{83}+Y^5_{63} & -D^4_{84}+Y^5_{64} & Y^4_{86} & -D^4_{85}+Y^5_{65} & -D^4_{86}+Y^5_{66} & Y^4_{89} & -D^4_{87}+Y^5_{67} & -D^4_{88}+Y^5_{68} & Y^4_{8,12} \\
-D^4_{91} & -D^4_{92} & Y^4_{93} & -D^4_{93} & -D^4_{94} & Y^4_{96} & -D^4_{95} & -D^4_{96} & Y^4_{99} & -D^4_{97} & -D^4_{98} & Y^4_{9,12} \\
-D^3_{10,1} & -D^3_{10,2} & -D^3_{10,3} & -D^3_{10,5} & -D^3_{10,6} & -D^3_{10,7} & -D^3_{10,9} & -D^3_{10,10} & -D^3_{10,11} & -D^4_{10,7}+D^5_{47}+Y^6_{44} & -D^4_{10,8}+D^5_{48} & Y^4_{10,12} \\
-D^3_{11,1} & -D^3_{11,2} & -D^3_{11,3} & -D^3_{11,5} & -D^3_{11,6} & -D^3_{11,7} & -D^3_{11,9} & -D^3_{11,10} & -D^3_{11,11} & -D^4_{11,7}+Y^5_{87} & -D^4_{11,8}+Y^5_{88} & Y^4_{11,12} \\
-D^3_{12,1} & -D^3_{12,2} & -D^3_{12,3} & -D^3_{12,5} & -D^3_{12,6} & -D^3_{12,7} & -D^3_{12,9} & -D^3_{12,10} & -D^3_{12,11} & -D^4_{12,7} & -D^4_{12,8} & Y^4_{12,12}
\end{pmatrix},
$$

$$Y_5=\begin{pmatrix}
D^5_{11}+Y^6_{11} & D^5_{12} & D^5_{13}+Y^6_{12} & D^5_{14} & D^2_{19}-D^3_{19}+D^4_{15} & D^2_{1,10}-D^3_{1,10}+D^4_{16} & D^5_{17}+Y^6_{14} & D^5_{18} & D^4_{19} & D^4_{1,10}\\
Y^5_{21} & Y^5_{22} & Y^5_{23} & Y^5_{24} & D^2_{29}-D^3_{29}+D^4_{25} & D^2_{2,10}-D^3_{2,10}+D^4_{26} & Y^5_{27} & Y^5_{28} & D^4_{29} & D^4_{2,10}\\
-D^1_{61}+D^2_{61}-D^3_{41}+D^4_{41} & -D^1_{62}+D^2_{62}-D^3_{42}+D^4_{42} & D^5_{23}+Y^6_{22} & D^5_{24} & D^2_{69}-D^3_{49}+D^4_{45} & D^2_{6,10}-D^3_{4,10}+D^4_{46} & D^5_{27}+Y^6_{24} & D^5_{28} & D^4_{49} & D^4_{4,10}\\
-D^1_{71}+D^2_{71}-D^3_{51}+D^4_{51} & -D^1_{72}+D^2_{72}-D^3_{52}+D^4_{52} & Y^5_{43} & Y^5_{44} & D^2_{79}-D^3_{59}+D^4_{55} & D^2_{7,10}-D^3_{5,10}+D^4_{56} & Y^5_{47} & Y^5_{48} & D^4_{59} & D^4_{5,10}\\
D^5_{31}+Y^6_{31} & D^5_{32} & D^5_{33}+Y^6_{32} & D^5_{34} & D^5_{35}+Y^6_{33} & D^5_{36} & D^5_{37}+Y^6_{34} & D^5_{38} & D^4_{79} & D^4_{7,10}\\
Y^5_{61} & Y^5_{62} & Y^5_{63} & Y^5_{64} & Y^5_{65} & Y^5_{66} & Y^5_{67} & Y^5_{68} & D^4_{89} & D^4_{8,10}\\
-D^3_{10,1}+D^4_{10,1} & -D^3_{10,2}+D^4_{10,2} & -D^3_{10,5}+D^4_{10,3} & -D^3_{10,6}+D^4_{10,4} & -D^3_{10,9}+D^4_{10,5} & -D^3_{10,10}+D^4_{10,6} & D^5_{47}+Y^6_{44} & D^5_{48} & D^4_{10,9} & D^4_{10,10}\\
-D^3_{11,1}+D^4_{11,1} & -D^3_{11,2}+D^4_{11,2} & -D^3_{11,5}+D^4_{11,3} & -D^3_{11,6}+D^4_{11,4} & -D^3_{11,9}+D^4_{11,5} & -D^3_{11,10}+D^4_{11,6} & Y^5_{87} & Y^5_{88} & D^4_{11,9} & D^4_{11,10}\\
D^5_{51}+Y^6_{51} & D^5_{52} & D^5_{53}+Y^6_{52} & D^5_{54} & D^5_{55}+Y^6_{53} & D^5_{56} & D^5_{57}+Y^6_{54} & D^5_{58} & D^5_{59}+Y^6_{55} & D^5_{5,10}\\
Y^5_{10,1} & Y^5_{10,2} & Y^5_{10,3} & Y^5_{10,4} & Y^5_{10,5} & Y^5_{10,6} & Y^5_{10,7} & Y^5_{10,8} & Y^5_{10,9} & Y^5_{10,10}
\end{pmatrix},$$

$$Y_6=\begin{pmatrix}
Y^6_{11} & Y^6_{12} & D^2_{19}-D^3_{19}+D^4_{15}-D^5_{15} & Y^6_{14} & D^4_{19}-D^5_{19} & Y^6_{16}\\
-D^1_{61}+D^2_{61}-D^3_{41}+D^4_{41}-D^5_{21} & Y^6_{22} & D^2_{69}-D^3_{49}+D^4_{45}-D^5_{25} & Y^6_{24} & D^4_{49}-D^5_{29} & Y^6_{26}\\
Y^6_{31} & Y^6_{32} & Y^6_{33} & Y^6_{34} & D^4_{79}-D^5_{39} & Y^6_{36}\\
-D^3_{10,1}+D^4_{10,1}-D^5_{41} & -D^3_{10,5}+D^4_{10,3}-D^5_{43} & -D^3_{10,9}+D^4_{10,5}-D^5_{45} & Y^6_{44} & D^4_{10,9}-D^5_{49} & Y^6_{46}\\
Y^6_{51} & Y^6_{52} & Y^6_{53} & Y^6_{54} & Y^6_{55} & Y^6_{56}\\
-D^6_{61} & -D^6_{63} & -D^6_{65} & -D^6_{67} & -D^6_{69} & Y^6_{66}
\end{pmatrix},$$

where $P_1, P_2, P_3, Q_1, Q_2, Q_3, N_1, N_2, N_3, T_1, T_2, T_3$ are defined in (25) and (26), and the remaining $Y^1_{j_1k_1}, Y^2_{j_2k_2}, Y^3_{j_3k_3}, Y^4_{j_4k_4}$ are arbitrary matrices over an arbitrary division ring with appropriate sizes.

We present an example to illustrate Theorem 3.1.

Example 4.2. Given the quaternion matrices:

$$A_1=\begin{pmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 2\mathbf{j} & 0 & 1\\ 0 & 0 & \mathbf{k}\end{pmatrix},\quad B_1=\begin{pmatrix}\mathbf{k} & 0 & 1\\ 2 & \mathbf{i} & \mathbf{j}\\ 0 & 0 & 2\end{pmatrix},\quad C_1=\begin{pmatrix}3\mathbf{k} & -1+2\mathbf{j}+\mathbf{k} & 4-2\mathbf{i}+\mathbf{j}\\ \mathbf{i}-2\mathbf{k} & 1-\mathbf{j}-2\mathbf{k} & 5\mathbf{j}\\ 0 & \mathbf{k} & \mathbf{j}\end{pmatrix},$$

$$A_2=\begin{pmatrix}3 & 0 & 1\\ \mathbf{i} & 0 & \mathbf{k}\\ \mathbf{i}+\mathbf{j} & 0 & \mathbf{k}\end{pmatrix},\quad B_2=\begin{pmatrix}2+\mathbf{k} & 1 & i\\ 0 & \mathbf{j} & 0\\ 1 & 0 & 0\end{pmatrix},\quad C_2=\begin{pmatrix}6-\mathbf{i}+3\mathbf{j}-3\mathbf{k} & 3 & 5+3\mathbf{i}\\ -2\mathbf{i}-2\mathbf{j}+\mathbf{k} & \mathbf{i}-\mathbf{j} & 2\mathbf{i}\\ -1-2\mathbf{i}+2\mathbf{j}+\mathbf{k} & 0 & 2\mathbf{i}+2\mathbf{j}-\mathbf{k}\end{pmatrix},$$

$$A_3=\begin{pmatrix}\mathbf{i}-\mathbf{k} & \mathbf{j} & -1\\ 0 & 1 & 2\\ 0 & 0 & 1\end{pmatrix},\quad B_3=\begin{pmatrix}2\mathbf{i}+\mathbf{j} & 1-\mathbf{k} & 0\\ 0 & -1 & 2\\ -1+\mathbf{i} & 0 & 0\end{pmatrix},\quad C_3=\begin{pmatrix}-2+2\mathbf{i}+\mathbf{k} & -1+\mathbf{i}+\mathbf{j}-\mathbf{k} & 1-3\mathbf{k}\\ \mathbf{j}+\mathbf{k} & 2 & -4+\mathbf{i}\\ 3-3\mathbf{i} & 0 & -1\end{pmatrix},$$

$$A_4=\begin{pmatrix}-2 & 0 & \mathbf{i}\\ \mathbf{j} & \mathbf{k} & 1\\ 0 & 1 & \mathbf{j}\end{pmatrix},\quad B_4=\begin{pmatrix}\mathbf{k} & \mathbf{i} & 0\\ 0 & \mathbf{j} & 1\\ 2 & \mathbf{j} & 3\mathbf{k}\end{pmatrix},\quad C_4=\begin{pmatrix}1-2\mathbf{i}-2\mathbf{j}+\mathbf{k} & 1-2\mathbf{k} & \mathbf{i}-2\mathbf{k}\\ -3+\mathbf{j} & \mathbf{j} & 1-3\mathbf{j}\\ -1 & \mathbf{i} & -6+\mathbf{j}\end{pmatrix},$$

$$A_5=\begin{pmatrix}0 & 3\mathbf{i} & 0\\ 1 & \mathbf{j} & 0\\ 2 & 3 & \mathbf{k}\end{pmatrix},\quad B_5=\begin{pmatrix}-1+\mathbf{j} & -2 & 1\\ 0 & \mathbf{k} & \mathbf{i}\\ -2+3\mathbf{k} & 0 & 1\end{pmatrix},\quad C_5=\begin{pmatrix}-1+\mathbf{j} & -5-\mathbf{i} & 1+\mathbf{k}\\ 5-\mathbf{j}+2\mathbf{k} & 4+\mathbf{j} & -1-\mathbf{k}\\ 2\mathbf{j}+\mathbf{k} & 3\mathbf{i}+4\mathbf{k} & 0\end{pmatrix}.$$

We consider the system (6). The rank of a quaternion matrix A can be calculated by using the following property (see [46])

$$r(A)=\frac{1}{2}r\begin{bmatrix}A_1 & A_2\\ -A_2 & A_1\end{bmatrix},$$

where $A = A_1 + A_2\mathbf{j}$, A_1 and A_2 are complex matrices, $\overline{A_1}$ means the conjugate of the matrix A_1. Direct computations yield

$$r\begin{pmatrix} C_i & A_i \\ B_i & 0 \end{pmatrix} = r(A_i) + r(B_i) = \begin{cases} 6, & \text{if } i = 1 \\ 5, & \text{if } i = 2 \\ 6, & \text{if } i = 3 \\ 6, & \text{if } i = 4 \\ 6, & \text{if } i = 5 \end{cases},$$

$$r\begin{pmatrix} C_1 & C_2 & A_1 & A_2 \\ B_1 & B_2 & 0 & 0 \end{pmatrix} = r\begin{pmatrix} A_1 & A_2 \end{pmatrix} + r\begin{pmatrix} B_1 & B_2 \end{pmatrix} = 6,$$

$$r\begin{pmatrix} C_2 & A_2 \\ C_3 & A_3 \\ B_2 & 0 \\ B_3 & 0 \end{pmatrix} = r\begin{pmatrix} A_2 \\ A_3 \end{pmatrix} + r\begin{pmatrix} B_2 \\ B_3 \end{pmatrix} = 6,$$

$$r\begin{pmatrix} C_3 & C_4 & A_3 & A_4 \\ B_3 & B_4 & 0 & 0 \end{pmatrix} = r\begin{pmatrix} A_3 & A_4 \end{pmatrix} + r\begin{pmatrix} B_3 & B_4 \end{pmatrix} = 6,$$

$$r\begin{pmatrix} C_4 & A_5 \\ C_4 & A_5 \\ B_4 & 0 \\ B_5 & 0 \end{pmatrix} = r\begin{pmatrix} A_4 \\ A_5 \end{pmatrix} + r\begin{pmatrix} B_4 \\ B_5 \end{pmatrix} = 6,$$

$$r\begin{pmatrix} C_1 & C_2 & A_1 & A_2 \\ 0 & C_3 & 0 & A_3 \\ B_1 & B_2 & 0 & 0 \\ 0 & B_3 & 0 & 0 \end{pmatrix} = r\begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix} + r\begin{pmatrix} B_1 & B_2 \\ 0 & B_3 \end{pmatrix} = 12,$$

$$r\begin{pmatrix} C_2 & 0 & A_2 & 0 \\ C_3 & C_4 & A_3 & A_4 \\ B_2 & 0 & 0 & 0 \\ B_3 & B_4 & 0 & 0 \end{pmatrix} = r\begin{pmatrix} A_2 & 0 \\ A_3 & A_4 \end{pmatrix} + r\begin{pmatrix} B_2 & 0 \\ B_3 & B_4 \end{pmatrix} = 11,$$

$$r\begin{pmatrix} C_3 & C_4 & A_3 & A_4 \\ 0 & C_5 & 0 & A_5 \\ B_3 & B_4 & 0 & 0 \\ 0 & B_5 & 0 & 0 \end{pmatrix} = r\begin{pmatrix} A_3 & A_4 \\ 0 & A_5 \end{pmatrix} + r\begin{pmatrix} B_3 & B_4 \\ 0 & B_5 \end{pmatrix} = 12,$$

$$r\begin{pmatrix} C_1 & C_2 & 0 & A_1 & A_2 & 0 \\ 0 & C_3 & C_4 & 0 & A_3 & A_4 \\ B_1 & B_2 & 0 & 0 & 0 & 0 \\ 0 & B_3 & B_4 & 0 & 0 & 0 \end{pmatrix} = r\begin{pmatrix} A_1 & A_2 & 0 \\ 0 & A_3 & A_4 \end{pmatrix} + r\begin{pmatrix} B_1 & B_2 & 0 \\ 0 & B_3 & B_4 \end{pmatrix} = 12,$$

$$r\begin{pmatrix} C_2 & 0 & A_2 & 0 \\ C_3 & C_4 & A_3 & A_4 \\ 0 & C_5 & 0 & A_5 \\ B_2 & 0 & 0 & 0 \\ B_3 & B_4 & 0 & 0 \\ 0 & B_5 & 0 & 0 \end{pmatrix} = r\begin{pmatrix} A_2 & 0 \\ A_3 & A_4 \\ 0 & A_5 \end{pmatrix} + r\begin{pmatrix} B_2 & 0 \\ B_3 & B_4 \\ 0 & B_5 \end{pmatrix} = 12,$$

$$r\begin{pmatrix} C_1 & C_2 & 0 & A_1 & A_2 & 0 \\ 0 & C_3 & C_4 & 0 & A_3 & A_4 \\ 0 & 0 & C_5 & 0 & 0 & A_5 \\ B_1 & B_2 & 0 & 0 & 0 & 0 \\ 0 & B_3 & B_4 & 0 & 0 & 0 \\ 0 & 0 & B_5 & 0 & 0 & 0 \end{pmatrix} = r\begin{pmatrix} A_1 & A_2 & 0 \\ 0 & A_3 & A_4 \\ 0 & 0 & A_5 \end{pmatrix} + r\begin{pmatrix} B_1 & B_2 & 0 \\ 0 & B_3 & B_4 \\ 0 & 0 & B_5 \end{pmatrix} = 18.$$

All the rank equalities in (12)–(22) hold. Hence, the system (6) is consistent. In addition, $X_1, \dots, X_6$ with the following structures satisfy the system (6)

$$X_1 = \begin{pmatrix} \mathbf{k} & \mathbf{i} & 0 \\ 1 & 2 & -\mathbf{j} \\ 0 & 1 & \mathbf{i} \end{pmatrix}, \quad X_2 = \begin{pmatrix} -3 & 0 & \mathbf{i} \\ \mathbf{j} & \mathbf{k} & \mathbf{i}+2\mathbf{j} \\ 0 & 0 & 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} \mathbf{j}-2\mathbf{k} & 0 & 2 \\ 0 & 1 & \mathbf{i} \\ 0 & 0 & -1 \end{pmatrix},$$

$$X_4 = \begin{pmatrix} \mathbf{k} & \mathbf{i} & 0 \\ 0 & 1 & \mathbf{j} \\ 0 & 0 & 3 \end{pmatrix}, \quad X_5 = \begin{pmatrix} \mathbf{j} & \mathbf{k} & 0 \\ 0 & \mathbf{i} & 0 \\ \mathbf{j} & 0 & 1 \end{pmatrix}, \quad X_6 = \begin{pmatrix} -1 & \mathbf{j} & 0 \\ 2 & \mathbf{i} & \mathbf{k} \\ \mathbf{k} & 0 & 0 \end{pmatrix}.$$

5. An application from color image encryption

In this section, we apply the equivalence canonical form of five quaternion matrices to the color image encryption. Note that a color image can be represented by a quaternion matrix. Based on Theorem 2.1, we can encrypt five color images simultaneously.

Let $A_1 \in \mathbb{H}^{p_1\times q_1}$, $A_2 \in \mathbb{H}^{p_1\times q_2}$, $A_3 \in \mathbb{H}^{p_2\times q_2}$, $A_4 \in \mathbb{H}^{p_2\times q_3}$ and $A_5 \in \mathbb{H}^{p_3\times q_3}$ be five original color images. Decompose the five color images at the same time and get the corresponding encrypted images $S_{a_1}, S_{a_2}, S_{a_3}, S_{a_4}$, and S_{a_5}. At the same time, six keys need to be saved to decrypt. When the five quaternion matrices to be processed are all full rank, and $p_i = q_i = n, i = 1, 2, 3, 4, 5$. We have $S_{a_1} = S_{a_2} = S_{a_3} = S_{a_4} = S_{a_5} = I_n$, where I_n is a nth order identity matrix.

Now we can give a frame diagram for the encryption and decryption of five color images in this case. (See Fig. 1.)

In this framework, five color images are encrypted into a binary image at the same time. We only need to save six keys to complete the decryption process. In order to increase the speed of the encryption and the decryption process, we use the real-preservation structure [2] to design the algorithm based on Theorem 2.1, which not only avoids complex quaternion operations, but also greatly reduces the space complexity. The detailed algorithm is as follows.

Algorithm 1

Input: Original color images $A_i^{n\times n}$, $i = 1, 2, 3, 4, 5$.
Output: Encrypted image $S_{a_i} = I_n$, $i = 1, 2, 3, 4, 5$, six keys

1. $T = \begin{pmatrix} A_1 & A_2 & \\ & A_3 & A_4 \\ & & A_5 \end{pmatrix}$, $S_1A_1M_1 = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$.
2. Let $P_1 = \begin{pmatrix} S_1 & \\ & I_{2n} \end{pmatrix}$, $Q_1 = \begin{pmatrix} M_1 & \\ & I_{2n} \end{pmatrix}$, and $T_1 = P_1AQ_1$
3. $X_1 = T_1(1:n, n+1:2n)$, $S_2X_1M_2 = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$
4. $P_2 = \begin{pmatrix} S_2 & \\ & I_{2n} \end{pmatrix}$, $Q_2 = \begin{pmatrix} I_n & & \\ & M_2 & \\ & & I_n \end{pmatrix}$, $IP_2 = \begin{pmatrix} inv(S_2) & \\ & I_{2n} \end{pmatrix}$.
5. $T_3 = T_2IP_2$, $P = P_2P_1$, $Q = Q_1Q_2IP_2$
$X_2 = T_3(1+n:2n, n+1:2n)$, $S_3X_2M_3 = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$
6. $P_3 = \begin{pmatrix} I_n & & \\ & S_3 & \\ & & I_n \end{pmatrix}$, $Q_3 = \begin{pmatrix} I_n & & \\ & M_3 & \\ & & I_n \end{pmatrix}$, $T_4 = P_3T_3Q_3$
7. $LQ_3 = \begin{pmatrix} inv(M_3) & \\ & I_{2n} \end{pmatrix}$ $RQ_3 = \begin{pmatrix} M_2 & \\ & I_{2n} \end{pmatrix}$, $T_4 = LQ_3T_4RQ_3$
8. $P = LQ_3P_3P$, $Q = QQ_3RQ_3$
9. $X_3 = T_4(1:n, n+1:2n)$, $S_4X_3M_4 = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$
10. $P_4 = \begin{pmatrix} I_n & & \\ & S_4 & \\ & & I_n \end{pmatrix}$, $Q_4 = \begin{pmatrix} I_{2n} & \\ & M_4 \end{pmatrix}$, $T_5 = P_4T_4Q_4$
11. $LP_4 = \begin{pmatrix} I_n & & \\ & inv(S_4) & \\ & & I_n \end{pmatrix}$ $RQ_4 = \begin{pmatrix} I_{2n} & \\ & S_4 \end{pmatrix}$, $T_5 = LP_4T_5RQ_4$
12. $P = LP_4P_4P$, $Q = QQ_4RQ_4$
13. $X_4 = T_5(1+2n:3n, 1+2n:3n)$, $S_5X_4M_5 = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$
14. $P_5 = \begin{pmatrix} I_{2n} & \\ & S_5 \end{pmatrix}$, $Q_5 = \begin{pmatrix} I_n & & \\ & M_5 & \\ & & I_n \end{pmatrix}$, $T_6 = P_5T_5Q_5$
15. $LP_5 = \begin{pmatrix} I_{2n} & \\ & M_5 \end{pmatrix}$, $RQ_5 = \begin{pmatrix} I_{2n} & \\ & inv(M_5) \end{pmatrix}$ $T_6 = LP_5T_6RQ_5$
16. $P = LP_5P_5P$, $Q = QQ_5RQ_5$
17. Get six keys from P, Q.

Remark 5.1. The operations are performed on the real structure-preserving, and the algorithm is only about $O(n^3)$ real flops. $S_1A_1M_1 = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$ is calculated by the structure-preserving quaternion SVD algorithm [47].

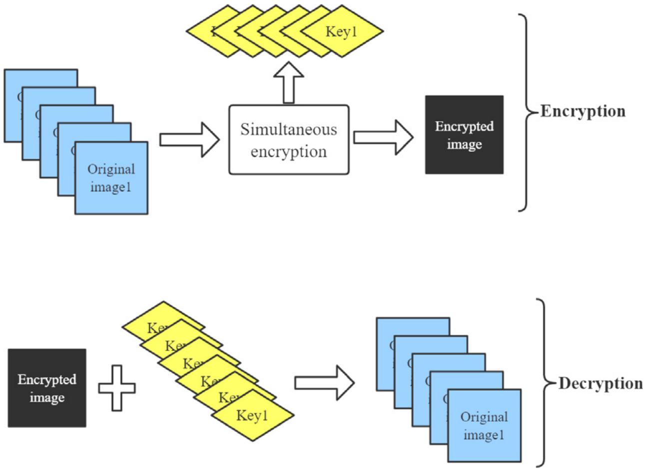

Fig. 1. Schematic diagram for the encryption and decryption.

Fig. 2. The result of simultaneous encryption.

Now we give an example of the encryption. In this experiment, we use the color images baboon, yellowlily, trees, onion and flower with the same size 128×128 pixels as original color images, the result of simultaneous encryption and decryption is as follows.

In Fig. 2, a1, a2, a3, a4 and a5 are five original color images, and b1 is the encrypted image. a1, a2, a3, a4 and a5 are decrypted images. It is easy to see visually that the effect of encryption and decryption is obvious. In particular, we can obtain five completely different pictures by only using one encrypted image and six keys, and the CPU time of the encryption and decryption process is 51.1335s. It shows that the algorithm has a good computing speed.

6. Conclusion

We have established the equivalence canonical form of five quaternion matrices (1). The equivalence canonical form has only identity matrices and zeros. We have used the equivalence canonical form to Sylvester-type quaternion matrix equations and color image encryption. We have provided two types of necessary and sufficient conditions for the existence of a solution to the system (6) in terms of ranks and block matrices. We have also presented the general solution to the system (6) in terms of block matrices. Based on the equivalence canonical form of five quaternion matrices (1), we have encrypted five color images simultaneously. Moreover, some algorithms and examples have been provided.

Acknowledgment

This research was supported by the grants from the National Natural Science Foundation of China (11801354).

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