ASTR3007 Assignment 4

Timothy Crundall - U5018130

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1 Galaxy Classification

1.1 Hubble's Galaxy Classification

E0 and E5 represent elliptical galaxies which show no spiral structure. They can appear perfectly round or cigar shaped based on our perspective room Earth.

SO denotes galaxes taht appear to have a spiral disc but no visible arms.

The elements after the fork refer to spiral galaxies with clearly visible arms. The further along the fork the sparser the arms become. The top branch features galaxies with gently curving arms stemming from the nucleus. The bottom branch features barred spiral galaxies - galaxies that have a prominent bar through the nucleus.

1.2 Classifying 10 Galaxies

The pictured galaxies are classified from left to right, top to bottom.

1.2.1 SBa

There is evidence of a bar feature which puts this galaxy on the bottom fork. The nucleus is quite large with respect to the rest of the galaxy which means it is an SBa.

1.2.2 Sb

It is difficult to see this galaxy clearly since the image is side on, however it is still possible to see spiral arms. The tightness and number of the arms suggests Sb.

1.2.3 Sa

This galaxy shows a spiral structure with thick, tight arms stemming from the nucleus with no sign of a bar.

1.2.4 Sc

There are several clearly defined spiral arms in this galaxy. There is no bar putting it on the top branch. The spiral arms are well defined. Since there are gaps between the arms this an Sc.

1.2.5 SBb

There is evidence of a bar feature putting the galaxy on the bottom branch. The nucleus is evident but not large, therefore SBb.

1.2.6 Im

There is no clear structure to this galaxy. It is an irregular.

1.2.7 E3

There is no sign of a spiral disc. So it is an elliptical galaxy. It is an E3 due to it slight elongated nature.

1.2.8 SBb

Evidence of bar puts it on the bottom branch of the fork. The

1.2.9 SBc

There is evidence of a bar. The arms have plenty of space between them which makes this galaxy an SBc.

1.2.10 S0

Seems to be spiral disc but no visible arms

1.3 Apparent Axis Ratios

2 Galaxy Properties

I am looking at Galaxy NGC5018.

2.1

NGC5018 is an elliptical galaxy. It falls under the category of E4.

The equatorial (J2000) coordinates of NGC5018 are:
R.A. 13h13m01.033s
DEC -19d31m05.49s
The Galactic Plane coordinates are:
RA 12h 51m 26.282s
Dec 27d07m42.0s

So the angular distance is 46.93861 degrees.

3 The Invisibles

4 Milky Way

5 Stellar Photometry

5.1

(insert plot here)

5.2

Best fit gives a temperature of 9800K. This corresponds to the border between B and A stars.

5.3

The strongest absorptions lines are in the blue end of the spectrum. These lines most probably are due to Hydrogen absorption, specifically the Balmer Series. The Balmer Series has absorptions at the following wavelengths in angstroms: 6563, 4861, 4340, 4102, 3970, 3646.

All of these wavelengths excepting 3646 correspond to an absorption line on the plot.

5.4

There is an absorption line which corresponds with H-α. H-α is 656.3nm, but on the plot it is measured to be 656.18 nm. The difference is 0.12nm.

So the radial velocity is:

$$\begin{aligned} v &= \frac{\Delta\lambda}{wavelength}c \\ &= \frac{0.12}{656.18}(2.997e5km/s) \\ &= 54.824km/s \end{aligned}$$

5.5

(instert plot here)

5.6

The filter curves are multiplied with the spectrum. The resulting spectrum is integrated (i.e. the array of y-values are summed since the units are arbitrary).

$$\begin{aligned} f_{blue} &= 1010.87 \\ f_{red} &= 1045.06 \end{aligned}$$

So the flux ratio is:

$$\frac{f_{blue}}{f_{red}} = 0.967$$

We can now find the corresponding $m_{blue} - m_{red}$ colour of that star:

$$\begin{aligned} -2.5log_{10}\frac{f_{blue}}{f_{red}} &= -2.5log_{10}(0.967) \\ &= 0.0361mag \end{aligned}$$

6

$$\begin{aligned} L[y_1] &= y_1'' + tY_1' + y \\ &= (t^2e^{-\frac{t^2}{2}} - e^{-\frac{t^2}{2}}) + t(-t * e^{-\frac{t^2}{2}}) + e^{-\frac{t^2}{2}} \\ &= 0 \end{aligned}$$

Therefore y_1 is indeed a solution. We can now sue the method of reduction of order to show that a second solution is

$$y_2(t) = e^{-\frac{t^2}{2}} \int_0^t e^{\frac{s^2}{2}} ds$$

We look for y_2 of the form $y_2 = u(t)h(t)$ where $h(t) = y_1$. We find the first and second derivatives of y_2:

$$\begin{aligned} y_2' &= u'h + uh' \\ y_2'' &= u''h + 2u'h' + uh'' \end{aligned}$$

For the general homogeneous DE: $L[y] = y'' + p(t)y' + q(t) = 0$:

$$\begin{aligned} L[y_2] = (u''h + 2u'h' + uh'') + p(u'h + uh') + quh &= 0 \\ u''h + 2u'h' + uh'' + pu'h + puh' + quh &= 0 \\ u''h + 2u'h' + pu'h + u(h'' + ph' + qh) &= 0 \\ u''h + (2h' + ph)u' &= 0 \end{aligned}$$

Plugging in y_1 for h:

$$
\begin{aligned}
(e^{-\frac{t^2}{2}})u'' + (2(-te^{-\frac{t^2}{2}}) + t(e^{-\frac{t^2}{2}}) &= 0 \\
e^{-\frac{t^2}{2}}u'' - te^{-\frac{t^2}{2}}u' &= 0 \\
(e^{-\frac{t^2}{2}}u')' &= 0 \\
e^{-\frac{t^2}{2}}u' &= A \\
u' = Ae^{\frac{t^2}{2}} & \\
u = \int Ae^{\frac{t^2}{2}}\,dt &
\end{aligned}
$$

We can pick $A = 1$ and make the integral a definite integral:

$$u = \int_0^t e^{\frac{s^2}{2}}\,dt$$

Therefore

$$y_2 = y_1 * u = e^{-\frac{t^2}{2}} \int_0^t e^{\frac{s^2}{2}}\,dt$$

Part i

To show that $y_1(t)$ and $y_2(t)$ form a fundamental set of solutions on $-\infty < t < +\infty$ we need the Wronskian.

$$W[y_1, y_2](t) = \begin{vmatrix} y_1 & y_2 \\ \frac{dy_1}{dt} & \frac{dy_2}{dt} \end{vmatrix}$$

$$
\begin{aligned}
y_1 &= e^{-\frac{t^2}{2}} \\
\frac{dy_1}{dt} &= -te^{-\frac{t^2}{2}}
\end{aligned}
$$

$$
\begin{aligned}
y_2 &= e^{-\frac{t^2}{2}} \int_0^t e^{\frac{s^2}{2}}\,ds \\
\frac{dy_2}{dt} &= -te^{-\frac{t^2}{2}} \int_0^t e^{\frac{s^2}{2}}\,ds + e^{-\frac{t^2}{2}} * e^{\frac{t^2}{2}} \\
\frac{dy_2}{dt} &= -te^{-\frac{t^2}{2}} \int_0^t e^{\frac{s^2}{2}}\,ds + 1
\end{aligned}
$$

$$
\begin{aligned}
W &= y_1\frac{dy_2}{dt} - y_2\frac{dy_1}{dt} \\
&= (e^{-\frac{t^2}{2}})(-te^{-\frac{t^2}{2}}\int_0^t e^{\frac{s^2}{2}}ds + 1) - (e^{-\frac{t^2}{2}}\int_0^t e^{-\frac{s^2}{2}}ds)(-te^{-\frac{t^2}{2}} \\
&= -te^{-t^2}\int_0^t e^{\frac{s^2}{2}}ds + e^{-\frac{t^2}{2}} + te^{-t^2}\int_0^t e^{\frac{s^2}{2}}ds \\
&= e^{-\frac{t^2}{2}}
\end{aligned}
$$

Therefore $W \neq 0$ for all t. This implies that the solutions are linearly independent, and hence form a fundamental set on all t.

Part ii

Solve the initial value problem

$$
\begin{aligned}
&L[y] = 0 \\
&y(0) = 0, y'(0) = 1
\end{aligned}
$$

$$
\begin{aligned}
y(t) &= Ay_1 + By_2 \\
y(t) &= Ae^{-\frac{t^2}{2}} + Be^{-\frac{t^2}{2}}\int_0^t e^{\frac{s^2}{2}}ds \\
y(0) &= 0
\end{aligned}
$$

Therefore

$$
\begin{aligned}
(0) &= A*(1) + B*(1)*(0) \\
A &= 0
\end{aligned}
$$

$$
\begin{aligned}
y' &= By_2' \\
y' &= B(-te^{-\frac{t^2}{2}}\int_0^t e^{\frac{s^2}{2}}ds + 1) \\
y'(0) &= 1
\end{aligned}
$$

Therefore

$$
\begin{aligned}
(1) &= B(-(0)*(1)*(0) + 1) \\
B &= 1
\end{aligned}
$$

Therefore solution to the IVP is given by:

$$
y = e^{-\frac{t^2}{2}}\int_0^t e^{\frac{s^2}{2}}ds
$$

Question 2

Part A

Considering the differential equation

$$y'' + p(x)y' + q(x)y = 0$$

i

We need to show that $y_1(x) = x^2$ can never be a solution if the functions p and q are continuous at $x = 0$.

We can use Abel's Theorem and proof by contradiction to show that y_1 cannot be a solution.

Let us assume $y_1 = x^2$ is a solution. Then since our DE is a second order homogeneous DE there must exist a second linearly independent solution y_2

Calculating the Wronskian for y_1 and our arbitrary y_2:

$$\begin{aligned} W[y_1, y_2](x) &= \begin{vmatrix} x^2 & y_2 \\ 2x & y_2' \end{vmatrix} \\ &= x^2 y_2' - 2xy_2 \\ &= x(xy_2' - 2y_2 \end{aligned}$$

It can be seen that at $x = 0$, $W = 0$. Since p and q are continuous, Abel's Theorem holds and it can be seen that $W(x) = 0$ for all x on I where $I : -h < x < h$ for some h.

Hence y_1 and y_2 are linearly dependent. Since y_2 is arbitrary, there is no second solution which is linearly independent of our first solution. This is a contradiction because a second order DE has two solutions which are linearly independent. Therefore y_1 is not a solution of our DE.

ii

We need to show that $y_1 = e^{2x}$ and $y_2 = x^2$ cannot both be solutions in the interval

$$I(\frac{1}{2} < x < 2)$$

if the functions p and q are continuous on I.

Using a similar approach as in part i, we assume y_1 and y_2 are both solutions, and we take the Wronskian of the two solutions.

$$\begin{aligned} W[y_1, y_2](x) &= \begin{vmatrix} e^{2x} & x^2 \\ 2e^{2x} & 2x \end{vmatrix} \\ &= e^{2x} 2x - 2x^2 e^{2x} \\ &= 2xe^{2x}(1 - x) \end{aligned}$$

Since p and q are continuous on I, the Wronskian must either be equal to 0 for all x in I, or not equal to 0 for all x in I.

However at $x = 1$, $W = 0$ and at $x = 3/2$, $W \neq 0$. We have a contradiction. Therefore our assumption must have been false. This implies y_1 and y_2 aren't both solutions on I.

Part B

We need to find a general solution to the differential equation

$$(D-1)^2(D+1)y = x^2e^x$$

The characteristic equation for the homogeneous case is

$$(r-1)^2(r+1) = 0$$

The roots are 1 repeated, -1. Therefore

$$y_c = C_1e^x + C_2xe^x + C_3e^{-x}$$

Using the Inverse Operator Method:

$$\begin{aligned}
y_p &= \frac{1}{(D-1)^2(D+1)}x^2e^x \\
&= e^x\frac{1}{D^2(D+2)}x^2 \\
&= e^x\frac{1}{D^2}\frac{1}{2}\frac{1}{1+\frac{D}{2}}x^2 \\
&= e^x\frac{1}{D^2}\frac{1}{2}[1 - \frac{D}{2} + \frac{D^2}{4} - \frac{D^3}{8} + \ldots]x^2 \\
&= e^x\frac{1}{2}[D^{-2} - \frac{1}{2}D^{-1} + \frac{1}{4} - \frac{1}{8}D + \ldots]x^2 \\
&= \frac{1}{2}e^x[\int\int x^2dxdx - \frac{1}{2}\int x^2dx + \frac{1}{4}x^2] \\
&= \frac{1}{2}e^x[\frac{x^4}{12} - \frac{x^3}{6} + \frac{1}{4}x^2] \\
&= \frac{1}{24}x^2e^x[x^2 - 2x + 3]
\end{aligned}$$

Hence

$$y = \frac{1}{24}x^2e^x[x^2 - 2x + 3] + C_1e^x + C_2xe^x + C_3e^{-x}$$

Question 3

Part A

We have the initial value problem

$$L[y] = y''' = f(x)$$

$$y(0) = y'(0) = y''(0) = 0$$

i

We need to show that $y_1(x) = 1$, $y_2(x) = x4$ and $y_3(x) = x^2$ form a fundamental set of solutions that span the null space of L. In order to satisfy this criterion the three equations must be solutions of the homogeneous equation,

$$y''' = 0$$

and be independent.

$$\begin{aligned} y_1' &= 0 \\ y_2'' &= 0 \\ y_3''' &= 0 \end{aligned}$$

To prove linear independence we can use the Wronskian. The theorem states that if $f_1, ...f_n$ are differentiable on an open interval I, and $W[f_1, ...f_n](x_0) \neq 0$ for some x_0, then $f_1, ...f_n$ are linearly independent on I.

$$\begin{aligned} W[y_1, y_2, y_3] &= \begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & 2x \\ 0 & 0 & 2 \end{vmatrix} \\ &= 2 \\ &\neq 0 \end{aligned}$$

Therefore y_1, y_2, y_3 are linearly independent and hence form a fundamental set of solutiosnt hat span the null space of L.

ii

We need to find a particular solution of the form

$$y_p(x) = v_1(x) + v_2(x)x + v_3(x)x^2$$

in order to show that

$$y_p(x) = \int_0^x \frac{1}{2}(x-t)^2 f(t)dt$$

is a particular solution that satisfies the initial conditions.

$$\begin{aligned} y_p(x) &= v_1(x) + v_2(x)x + v_3(x)x^2 \\ y_p'(x) &= v_1' + v_2'x + v_2 + v_3'x^2 + 2v_3x \end{aligned}$$

We impose the condition that $v_1' + v_2'x + v_3'x^2 = 0$

$$\begin{aligned} y_p'(x) &= v_2 + 2v_3x \\ y_p''(x) &= v_2' + 2v_3'x + 2v_3 \end{aligned}$$

We impose the condition that $v_2' + 2v_3'x = 0$

$$\begin{aligned} y_p''(x) &= 2v_3 \\ y_p'''(x) &= 2v_3' \end{aligned}$$

So we are left with three conditions for our $v(x)$ terms:

$$\begin{aligned} 2v_3 &= f(x) \\ v_2' + 2v_3'x &= 0 \\ v_1' + v_2'x + v_3'x^2 &= 0 \end{aligned}$$

These can now be solved simultaenously

$$\begin{aligned} v_3' &= \frac{1}{2}f(x) \\ v_2' &= -f(x)x \end{aligned}$$

$$\begin{aligned} v_1' + (-f(x)x)x + (\frac{1}{2}f(x))x^2 &= 0 \\ v_1' &= \frac{1}{2}f(x)x^2 \end{aligned}$$

Integrating each of the $v(x)$ terms and plugging it into our equation for y_p gives

$$\begin{aligned} y_p &= \int_0^x \frac{1}{2}f(t)t^2dt + \int_0^x -f(t)tdtx + \int_0^x \frac{1}{2}f(t)dtx^2 \\ &= \int_0^x \frac{1}{2}f(t)t^2 - f(t)tx + \frac{1}{2}f(t)x^2dt \\ &= \int_0^x \frac{1}{2}(x^2 - xt + t^2)f(t)dt \\ &= \int_0^x \frac{1}{2}(x-t)^2f(t)dt \end{aligned}$$

iii

We need to prove that

$$\int_0^x \int_0^t \int_0^s f(r)drdsdt = \int_0^x \frac{1}{2}(x-t)^2f(t)dt$$

We know that $y''' = f(x)$ Hence

$$\begin{aligned}
\text{LHS} &= \int_0^x \int_0^t \int_0^s y'''(r)drdsdt \\
&= \int_0^x \int_0^t y''(s)dsdt \\
&= \int_0^x y'(t)dt \\
&= y_p
\end{aligned}$$

We are left only with y_p since $y_c''' = 0$

$$\begin{aligned}
\text{RHS} &= \int_0^x \frac{1}{2}(x-t)^2 f(t)dt \\
&= y_p \\
&= \text{LHS}
\end{aligned}$$

Part B

We have a corollary that states $L[y] = p_0y'' + p_1y' + p_2y = r(x)$ is exact iff $p_2 = p_1' - p_0''$ and $L[y] = \frac{d}{dx}[p_0y' + (p_1 - p_0')y]$

Our differential equation is:

$$x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 2x$$

Let us check to see if it is exact:

$$\begin{aligned}
p_0 &= x \\
p_1 &= 2 \\
p_2 &= 0
\end{aligned}$$

$$p_2 = p_1' - p_0'$$

$$\text{LHS} = 0$$

$$\begin{aligned}
\text{RHS} &= (0) - (0) \\
&= 0 \\
&= \text{LHS}
\end{aligned}$$

Our equation is exact and thus

$$
\begin{aligned}
L[y] = \frac{d}{dx}[xy' + y] &= 2x \\
xy' + y &= x^2 + C \\
(xy)' &= x^2 + C \\
xy &= \frac{1}{3}x^3 + Cx + D \\
y &= \frac{1}{3}x^2 + C + \frac{D}{x}
\end{aligned}
$$

Question 4

Considering the differential equation

$$L[y] = x(1 + 3x^2)y'' + 2y' - 6xy = 0$$

Part A

We have $y_1 = \frac{1}{x}$ as our first solution. Using the method of reduction of order we can find a second solution of the form $y_2 = u(x)y_1(x)$ When we plug y_2 into our DE we get

$$y_1 u'' + (2y_1' + py_1)u' = 0$$

Where $p(x) = \frac{2}{x(1+3x^2)}$ in this instance. Thus:

$$
\begin{aligned}
\frac{1}{x}u'' + (-\frac{2}{x^2} + \frac{2}{x(1+3x^2)}\frac{1}{x})u' &= 0 \\
u'' + (\frac{-2(1+3x^2)+2}{x(1+3x^2)}u' &= 0 \\
u'' + \frac{-6x^2}{x(1+3x^2)}u' &= 0 \\
u'' - \frac{6x}{1+3x^2}u' &= 0
\end{aligned}
$$

We now have a second order DE for u. We can find an integrating factor μ

$$
\begin{aligned}
\mu(x) &= e^{\int \frac{-6x}{1+3x^2}dx} \\
&= e^{\int \frac{-1}{s}ds} \text{where} s = 1 + 3x^2 \\
&= e^{-\log s} \\
&= e^{\log s^{-1}} \\
&= s^{-1} \\
&= \frac{1}{1+3x^2}
\end{aligned}
$$

Now that we have our integrating factor, the DE in u becomes:

$$
\begin{aligned}
(\frac{u'}{1+3x^2})' &= 0 \\
u' &= A(1+3x^2) \\
u &= A(x+x^3)+B
\end{aligned}
$$

thus

$$
\begin{aligned}
y_2 &= uy_1 \\
&= (A(x+x^3)+B)\frac{1}{x} \\
&= A(1+x^2)+\frac{B}{x} \\
&= (1+x^2)
\end{aligned}
$$

since A is an arbitrary constant and the term with B was our first solution.

Part B

It is given to us that the particular solution to a linear second order differential equation is

$$y_p(x)\int_{x_0}^{x}\frac{g(s)}{W(s)}[y_1(s)y_2(x)-y_2(s)y_1(x)]ds$$

where

$$G(x,s)=\frac{[y_1(s)y_2(x)-y_2(s)y_1(x)]}{W(s)}$$

and $g(s)$ is the forcing term when the DE is in standard form. We first want to find our Wronskian:

$$
\begin{aligned}
W &= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \\
&= \begin{vmatrix} \frac{1}{x} & 1+x^2 \\ \frac{-1}{x^2} & 2x \end{vmatrix} \\
&= 2+\frac{1}{x^2}+1 \\
&= \frac{1}{x^2}(3x^2+1)
\end{aligned}
$$

Thus our greens function is

$$
\begin{aligned}
G(x,s) &= \frac{\frac{1}{s}(1+x^2)-(1+s^2)\frac{1}{x}}{\frac{1}{s^2}(3s^2+1)} \\
&= \frac{1}{x}\frac{(sx+sx^3)-(s^2+s^4)}{3s^2+1}
\end{aligned}
$$

For $x > s$

Putting our DE into standard form:

$$\begin{aligned} L[y] = x(1+3x^2)y'' + 2y' - 6xy = g(x) \\ y'' + \frac{2}{x(1+3x^2}y' - \frac{6}{1+3x^2}y = \frac{g(x)}{x(1+3x^2)} \end{aligned}$$

So once in standard form our forcing term becomes

$$L[y] = \frac{g(x)}{x(1+3x^2)}$$

Plugging everything into our equation for y_p gives

$$\begin{aligned} y_p &= \int_1^x \frac{1}{x}\frac{(sx+sx^3)-(s^2+s^4)}{3s^2+1}\frac{g(s)}{s(1+3s^2)}ds \\ &= \frac{1}{x}\int_1^x \frac{(x+x^3)-(s+s^3)}{(3s^2+1)^2}g(s)ds \end{aligned}$$

Part C

We are given an IVP:

$$L[y] = x(1+3x^2)y'' + 2y' - 6xy = (1+3x^2)^2$$

$$y(1) = 2, \quad y'(1) = 4$$

Since we have non-homogeneous initial conditions we need to include the solutions to the homogeneous equation.

$$\begin{aligned} y &= A_1y_1 + A_2y_2 + y_p \\ y &= A_1\frac{1}{x} + A_2(1+x^2) + y_p \\ y(1) &= A_1 + 2A_2 + (0) = 2 \end{aligned}$$

$$\begin{aligned} y' &= -A_1\frac{1}{x^2} + A_2(2x) + (0) \\ y'(1) &= -A_1 + 2A_2 = 4 \end{aligned}$$

$$\begin{aligned} A_1 &= -1 \\ A_2 &= \frac{3}{2} \end{aligned}$$

We can now find y_p with our Green's function

$$
\begin{aligned}
y_p &= \frac{1}{x}\int_1^x \frac{x+x^3-s-s^3}{(3s^2+1)^2}(1+3s^2)^2 ds \\
&= \frac{1}{x}\int_1^x x+x^3-s-s^3 ds \\
&= \frac{1}{x}[(x+x^3)s-\frac{s^2}{2}-\frac{s^4}{4}]_1^x \\
&= \frac{1}{x}[((x+x^3)(x)-\frac{(x)^2}{2}-\frac{(x)^4}{4})-((x+x^3)(1)-\frac{(1)^2}{2}-\frac{(1)^4}{4})] \\
&= \frac{1}{x}[(\frac{x^2}{2}+\frac{3x^4}{4}-x-x^3+\frac{3}{4}] \\
&= \frac{x}{2}+\frac{3x^4}{3}-1-x^2+\frac{3}{4x}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
y &= y_c+y_p \\
&= (-\frac{1}{x}+\frac{3}{2}(1+x^2))+(\frac{x}{2}+\frac{3x^4}{4}-1-x^2+\frac{3}{4x}) \\
&= \frac{3x^3}{4}+\frac{x^2}{2}+\frac{x}{2}+\frac{1}{2}-\frac{1}{4x}
\end{aligned}
$$

Question 5

Part A

We need to evaluate the surface integral

$$\iint_S \vec{F}\wedge d\vec{S}$$

where

$$\vec{F} = \cos u\cos v\vec{i}+\cos u\sin v\vec{j}-\sin u\vec{k}$$

and S is the octant of the sphere

$$\vec{r}(u,v) = \rho\sin u\cos v\vec{i}+\rho\sin u\sin v\vec{j}+\rho\cos u\vec{k}$$

where:

$$0\leq u\leq\frac{\pi}{2}, 0\leq v\leq\frac{\pi}{2}$$

First we need $d\vec{S}$

$$d\vec{S} = (\frac{d\vec{r}}{du}\wedge\frac{d\vec{r}}{dv})dudv$$

$$\begin{aligned}
\vec{r}_u &= \rho \cos u \cos v \vec{i} + \rho \cos u \sin v \vec{j} - \rho \sin u \vec{k} \\
\vec{r}_v &= -\rho \sin u \sin v \vec{i} + \rho \sin u \cos v \vec{j}
\end{aligned}$$

$$\begin{aligned}
\vec{r}_u \wedge \vec{r}_v &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \rho \cos u \cos v & \rho \cos u \sin v & -\rho \sin u \\ -\rho \sin u \sin v & \rho \sin u \cos v & 0 \end{vmatrix} \\
&= \rho^2 \sin^2 u \cos v \vec{i} + \rho^2 \sin^2 u \sin v \vec{j} + \rho^2 \sin u \cos u \vec{k}
\end{aligned}$$

hence

$$d\vec{S} = (\rho^2 \sin^2 u \cos v \vec{i} + \rho^2 \sin^2 u \sin v \vec{j} + \rho^2 \sin u \cos u \vec{k}) dudv$$

$$\begin{aligned}
\vec{F} \wedge d\vec{S} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos u \cos v & \cos u \sin v & -\sin u \\ \rho^2 \sin^2 u \cos v & \rho^2 \sin^2 u \sin v & \rho^2 \sin u \cos u \end{vmatrix} dudv \\
&= ((\rho^2 \sin u \cos^2 u \sin v + \rho^2 \sin^3 u \sin v)\vec{i} + (-\rho^2 \sin^3 u \cos v - \rho^2 \sin u \cos^2 u \cos v)\vec{j} \\
&\quad (\rho^2 \sin^2 u \sin v \cos u \cos v - \rho^2 \sin^2 u \sin v \cos u \cos v)\vec{k}) dudv \\
&= (\rho^2 \sin u \sin v (\cos^2 u + \sin^2 u)\vec{i} - \rho^2 \sin u \cos v (\sin^2 u + \cos^2 u)\vec{j}) dudv \\
&= (\rho^2 \sin u \sin v \vec{i} - \rho^2 \sin u \cos v \vec{j}) dudv \\
&= \rho^2 \sin u [\sin v \vec{i} - \cos v \vec{j}] dudv
\end{aligned}$$

Now we just need to integrate over u and v over $[0, 2\pi]$

$$\begin{aligned}
\iint_S \vec{F} \wedge d\vec{S} &= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \rho^2 \sin u [\sin v \vec{i} - \cos v \vec{j}] dudv \\
&= \rho^2 \int_0^{\frac{\pi}{2}} \sin u du \int_0^{\frac{\pi}{2}} \sin v \vec{i} - \cos v \vec{j} dv \\
&= \rho^2 [-\cos u]_0^{\frac{\pi}{2}} [-\cos v - \sin v]_0^{\frac{\pi}{2}} \\
&= \rho^2 [-(0-1)][(-1\vec{j}) - (-\vec{i})] \\
&= \rho^2 (\vec{i} - \vec{j})
\end{aligned}$$

Part B

i

We need to calculate the outward flux of the vector field

$$\vec{F} = y\vec{i} + z\vec{k}$$

across the boundary of the solid cone

$$0 \leq z \leq 1 - \sqrt{x^2 + y^2}$$

Where flux $\vec{F}$ across S is given by

$$\iint_S \vec{F} \cdot d\vec{S}$$

We will treat the cone as a sum of the curved surface and flat base, both of which are orientable surfaces. The base shall be labelled S_1 while the curved surface shall be labelled S_2. In order to calculate the flux we need a parametrisation for the S_1

From the bounds it can be seen that the base is a circle on the x-y plane centred at the origin of radius 1. In polar coordinate this can be parametrised like so

$$\vec{r}(\theta, \rho) = \rho \cos\theta \vec{i} + \rho \sin\theta \vec{j}$$

Now to find $d\vec{S_1}$

$$d\vec{S_1} = (\frac{\delta \vec{r}}{\delta \theta} \wedge \frac{\delta \vec{r}}{\delta \rho}) d\theta d\rho$$

$$\begin{aligned}
\frac{\delta \vec{r}}{\delta \theta} &= -\rho \sin\theta \vec{i} + \rho \cos\theta \vec{j} \\
\frac{\delta \vec{r}}{\delta \rho} &= \cos\theta \vec{i} + \sin\theta \vec{j}
\end{aligned}$$

$$\begin{aligned}
d\vec{S_1} &= (\frac{\delta \vec{r}}{\delta \theta} \wedge \frac{\delta \vec{r}}{\delta \rho}) d\theta d\rho \\
&= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\rho \sin\theta & \rho \cos\theta & 0 \\ \cos\theta & \sin\theta & 0 \end{vmatrix} d\theta d\rho \\
&= (-\rho \sin^2\theta - \rho \cos^2\theta) \vec{k} d\theta d\rho \\
&= -\rho \vec{k} d\theta d\rho
\end{aligned}$$

And so the integral becomes

$$\begin{aligned}
\iint_{S_1} \vec{F} \cdot d\vec{S_1} &= \iint_{S_1} (y\vec{i} + z\vec{k}) \cdot (-\rho \vec{k}) d\theta d\rho \\
&= \iint_{S_1} -\rho z d\theta d\rho \\
&= 0
\end{aligned}$$

Since $z = 0$ for all values of θ and ρ. Therefore the flux through the base is 0.

We must now calculate the flux through the curved surface S_2.

$$\iint_S \vec{F} \cdot d\vec{S_2}$$

The parametrisation of S_2 is given by

$$\vec{r}(\theta, h) = (1-h)\cos\theta\vec{i} + (1-h)\sin\theta\vec{j} + h\vec{k}$$

where h is the distance above the x-y plane.
Similarly:

$$d\vec{S_2} = (\frac{\delta r}{\delta\theta} \wedge \frac{\delta r}{\delta h})d\theta dh$$

$$\frac{\delta r}{\delta\theta} = -(1-h)\sin\theta\vec{i} + (1-h)\cos\theta\vec{j}$$
$$\frac{\delta r}{\delta h} = -\cos\theta\vec{i} - \sin\theta\vec{j}$$

Calculating the cross product:

$$d\vec{S_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -(1-h)\sin\theta & (1-h)\cos\theta & 0 \\ -\cos\theta & -\sin\theta & 0 \end{vmatrix} d\theta dh$$
$$= (1-h)\cos\theta\vec{i} + (1-h)\sin\theta\vec{j} + ((1-h)\sin^2\theta + (1-h)\cos^2\theta)\vec{k})d\theta dh$$
$$= (1-h)[\cos\theta\vec{i} + \sin\theta\vec{j} + \vec{k}]d\theta dh$$

We need to convert $\vec{F}$ into our 'cone' coordinates:

$$\vec{F} = y\vec{i} + z\vec{k} = (1-h)\sin\theta\vec{i} + h\vec{k}$$

Now finding the dot product:

$$\vec{F} \cdot d\vec{S_2} = (1-h)^2\sin\theta\cos\theta\vec{i} + h(1-h)\vec{k}$$
$$= (1-2h+h^2\sin\theta\cos\theta\vec{i} + (h-h^2)\vec{k}$$

The integral thus becomes:

$$\begin{aligned}
\iint_S \vec{F}\cdot d\vec{S_2} &= \int_0^1\int_0^{2\pi}(1-2h+h^2)\sin\theta\cos\theta\vec{i}+(h-h^2)\vec{k}d\theta dh \\
&= \int_0^1 1-2h+h^2dh\int_0^{2\pi}\sin\theta\cos\theta d\theta\vec{i}+\int_0^1 h-h^2dh\int_0^{2\pi}d\theta\vec{k} \\
&= \int_0^1 1-2h+h^2dh\int_0^{2\pi}\frac{1}{2}\sin 2\theta d\theta\vec{i}+2\pi[\frac{h^2}{2}-\frac{h^3}{3}]_0^1\vec{k} \\
&= \int_0^1 1-2h+h^2dh[-\frac{1}{2}\cos 2\theta]_0^{2\pi}\vec{i}+2\pi(\frac{1}{2}-\frac{1}{3})\vec{k} \\
&= \int_0^1 1-2h+h^2dh(0)\vec{i}+\frac{\pi}{3}\vec{k} \\
&= \frac{\pi}{3}\vec{k}
\end{aligned}$$

ii

We need to calculate the volume integral

$$\iiint_V \mathrm{div}\vec{F}dV$$

where V is the volume of the cone. The parametrisation of the cone is given by

$$\vec{r}(\theta,\rho,h)=\rho\cos\vec{i}+\rho\sin\theta\vec{j}+h\vec{k}$$

where $0\le h\le 1$ and is the height above the x-y plane, $0\le r\le(1-h)$ and r is the distance from the origin along the x-y plane, and $0\le\theta\le 2\pi$ and is the angle from the x-axis on the x-y plane.

We need $\mathrm{div}\vec{F}$ and dV

$$\begin{aligned}
\mathrm{div}\vec{F} &= \vec{k} \\
dV &= \left|\frac{\delta\vec{r}}{\delta\theta}\right|\left|\frac{\delta\vec{r}}{\delta\rho}\right|\left|\frac{\delta\vec{r}}{\delta h}\right|
\end{aligned}$$

$$\begin{aligned}
\left|\frac{\delta\vec{r}}{\delta\theta}\right| &= \rho \\
\left|\frac{\delta\vec{r}}{\delta\rho}\right| &= 1 \\
\left|\frac{\delta\vec{r}}{\delta h}\right| &= 1
\end{aligned}$$

Thus

$$dV=\rho d\theta d\rho dh$$

And the integral becomes

$$\begin{aligned}
\iiint_V \mathrm{div}\vec{F}dV &= \int_0^1 \int_0^{(1-h)} \int_0^{2\pi} \rho d\theta d\rho dh\vec{k} \\
&= \int_0^1 [\frac{1}{2}\rho^2]_0^{(1-h)} 2\pi dh\vec{k} \\
&= \pi \int_0^1 (1-h)^2 dh\vec{k} \\
&= \pi \int_0^1 1 - 2h + h^2 dh\vec{k} \\
&= \pi[h - h^2 + \frac{h^3}{3}]_0^1\vec{k} \\
&= \pi[(1 - 1 + \frac{1}{3}) - (0)]\vec{k} \\
&= \frac{\pi}{3}\vec{k}
\end{aligned}$$