Bayesian_HW1

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3.1

a

$$\Pr(Y_1 = y_1, \dots Y_{100} = y_{100}|\theta) = \prod_{i=1}^{100} \theta^{y_i}(1-\theta)^{100-y_i} = \theta^{\Sigma_{i=1}^{100} y_i}(1-\theta)^{100-\Sigma_{i=1}^{100} y_i}$$

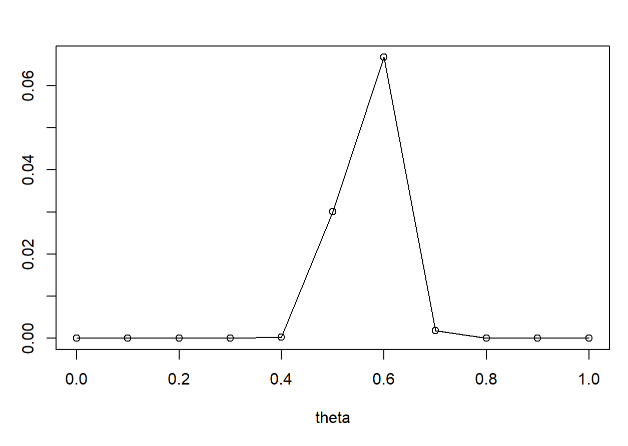
$$\Pr(\Sigma Y_i = y|\theta) = \binom{100}{y}\theta^y(1-\theta)^{100-y}$$

b

```r
prob <- function(n, y, theta){
  x = n-y
  return(factorial(n)/(factorial(y)*factorial(x))* theta**y * (1-theta)**(x))
}
theta = seq(0,1,0.1)
probs = prob(100,57,theta=theta)
data.frame(theta=theta, Relative_probability=round(probs, 4))
```

```
##    theta Relative_probability
## 1    0.0               0.0000
## 2    0.1               0.0000
## 3    0.2               0.0000
## 4    0.3               0.0000
## 5    0.4               0.0002
## 6    0.5               0.0301
## 7    0.6               0.0667
## 8    0.7               0.0019
## 9    0.8               0.0000
## 10   0.9               0.0000
## 11   1.0               0.0000
```

```r
plot(theta, probs, type = "o",ylab = "Relative probability")
```



c

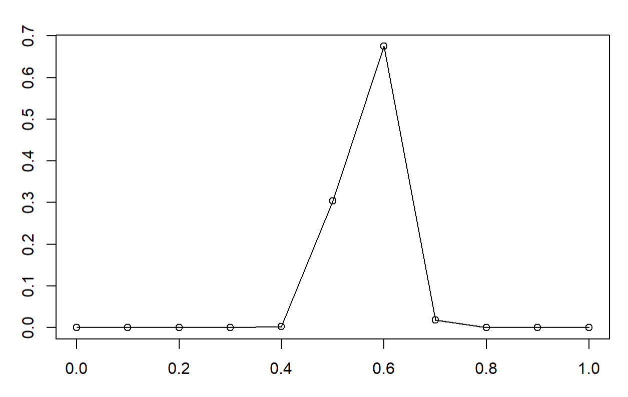
$$p(\theta|\Sigma_{i=1}^n Y_i = 57) = \frac{p(\Sigma_{i=1}^n Y_i = 57|\theta)p(\theta)}{p(\Sigma_{i=1}^n Y_i = 57)} \propto p(\Sigma_{i=1}^n Y_i = 57|\theta)$$

We can get the posterior distribution by the relative probabilities of question 3.1b

```r
new_probs = probs/sum(probs)
data.frame(theta=theta, probability=round(new_probs, 4))
```

```
##    theta probability
## 1    0.0      0.0000
## 2    0.1      0.0000
## 3    0.2      0.0000
## 4    0.3      0.0000
## 5    0.4      0.0023
## 6    0.5      0.3041
## 7    0.6      0.6749
## 8    0.7      0.0187
## 9    0.8      0.0000
## 10   0.9      0.0000
## 11   1.0      0.0000
```

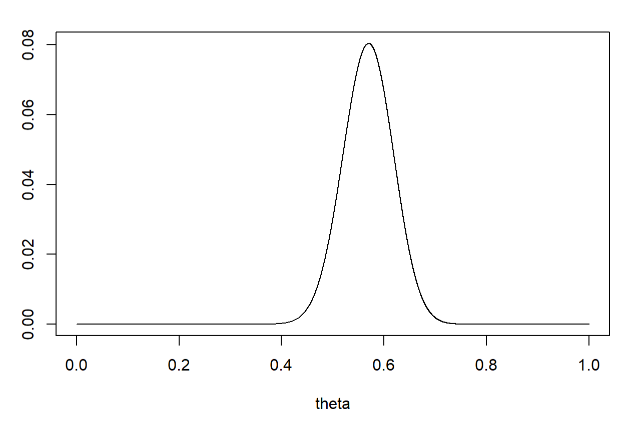
```r
plot(theta, new_probs, type = "o",ylab = "Probability")
```



d

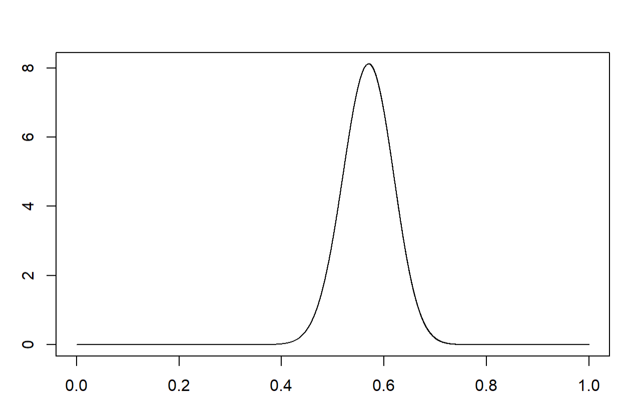
$$p(\theta) * \Pr(\Sigma_{i=1}^n Y_i = 57|\theta) = p(\theta)\binom{100}{57}\theta^{57}(1-\theta)^{100-57} = \binom{100}{57}\theta^{57}(1-\theta)^{100-57}$$

```r
theta=seq(0,1,1e-4)
probs = prob(n=100, y=57, theta=theta)
plot(theta, probs, type = "l", ylab = "Posterior density")
```



e

```r
betas = dbeta(seq(0,1,1e-4), shape1=58, shape2=44)
plot(theta, betas, type="l", ylab="Density")
```



Discussion: 1b and 1e is the posterior before normalization.

1c is the posterior after scaling

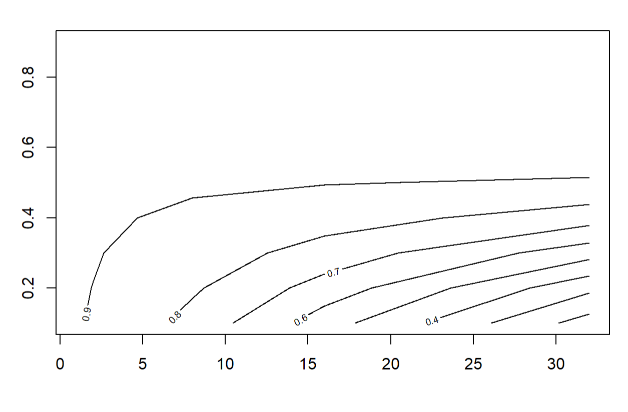
e is the posterior density given a uniform prior $Beta(1,1)$

3.2

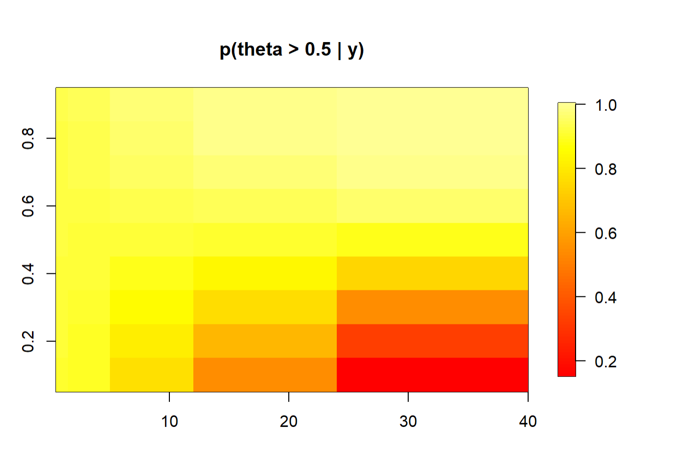
```r
theta0s = seq(0.1,0.9, 0.1)
n0s = c(1,2,8,16,32)
w = 0.5
n = 100
y = 57

prior_as <- n0s %*% t(theta0s)
prior_bs <- n0s %*% t((1-theta0s))
post_prob <- matrix(nrow = length(n0s), ncol = length(theta0s))
post_a <- prior_as + y
post_b <- prior_bs + n - y

for (i in 1:length(n0s)){
  for (j in 1:length(theta0s)){
    post_prob[i,j] <- 1-pbeta(w,post_a[i,j],post_b[i,j])
  }
}
contour(n0s,theta0s,post_prob)
```



```r
image.plot(n0s, theta0s,post_prob,xlab="n0",ylab="theta0",
           main="p(theta > 0.5 | y)",col=heat.colors(100)[1:90])
```



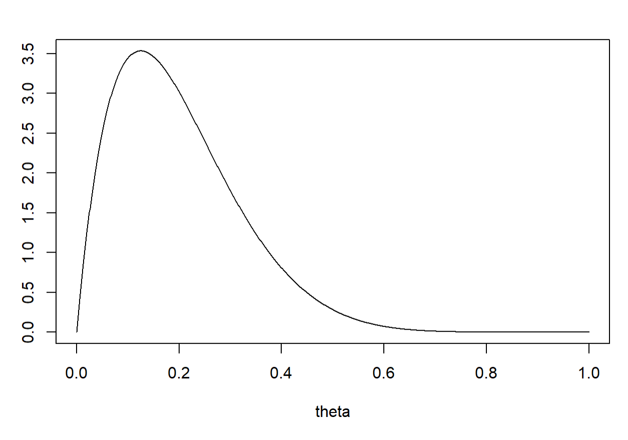
From the 2D contour plot, we could say that most priors result in posterior theta > 0.5 except those with very large n0 and very small theta0.

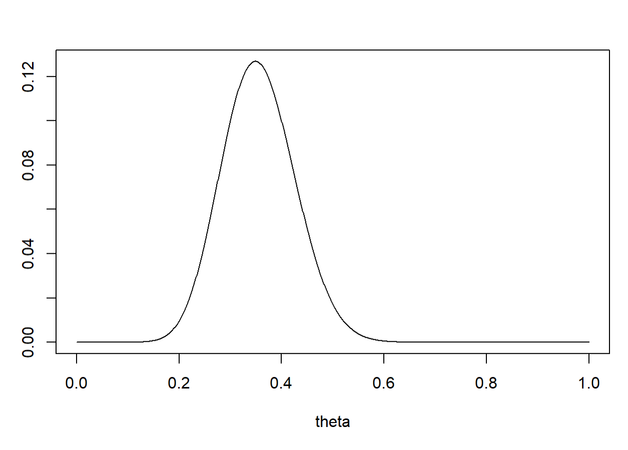
3.4

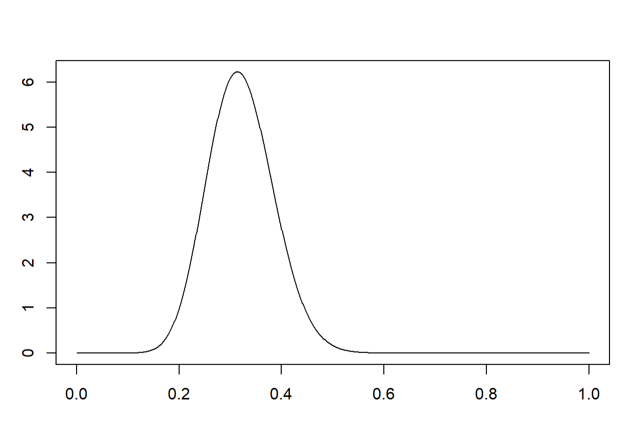
a

$$\begin{aligned}
&p(\theta) \sim Beta(2,8)\\
&p(y|\theta) \sim Binomial(43,\theta)\\
&p(\theta|y) \sim Beta(15+2, 43-15+8) \equiv Beta(17,36)\\
&E(\theta|y) = 17/(17+36) \approx 0.3207\\
&Mode = \arg\max f(\theta|y) = (17-1)/(17+36-2) \approx 0.3137\\
&Var(\theta|y) = [17/(17+36)\cdot(36/17+36)]/(17+36+1) \approx 0.0635\\
&95\%\text{Confidence Interval} : (0.2033, 0.4510)
\end{aligned}$$

```r
y = 15
n = 43
a = 2
b = 8
theta = seq(0,1,1e-3)
ploting <- function(theta,y,n,a,b){
  new_a = y + a
  new_b = n-y+b
  plot(theta, dbeta(theta,a,b), type="l", ylab="p(theta)")
  plot(theta, dbinom(y,n,theta), type="l", ylab="p(y|theta)")
  plot(theta, dbeta(theta,a+y,b+n-y), type="l", ylab="p(theta)")
  x1 <- data.frame(mean=new_a/(new_a+new_b),
           mode=(new_a-1)/(new_a+new_b-2),
           sd=sqrt((new_a/(new_a+new_b))*(new_b/(new_a+new_b))/(new_a+new_b+1)),
           Confidence_Interval=paste(round(qbeta(c(0.025, 0.975), new_a, new_b),4), collapse = ",")
           )
  rownames(x1) <- paste0("Beta(",a,",",b,")")
  return(x1)
}
ploting(theta=theta, y=y,n=n,a=a,b=b)
```







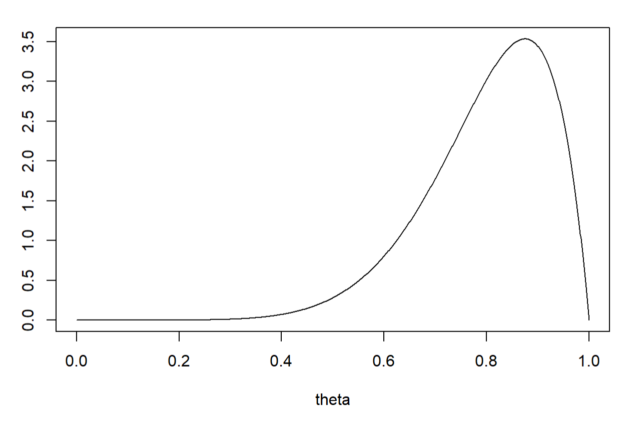

```
##                 mean      mode         sd Confidence_Interval
## Beta(2,8) 0.3207547 0.3137255 0.0635189        0.2033,0.451
```

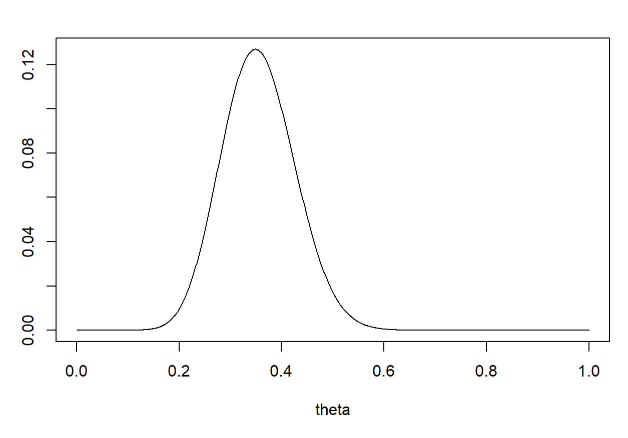
b

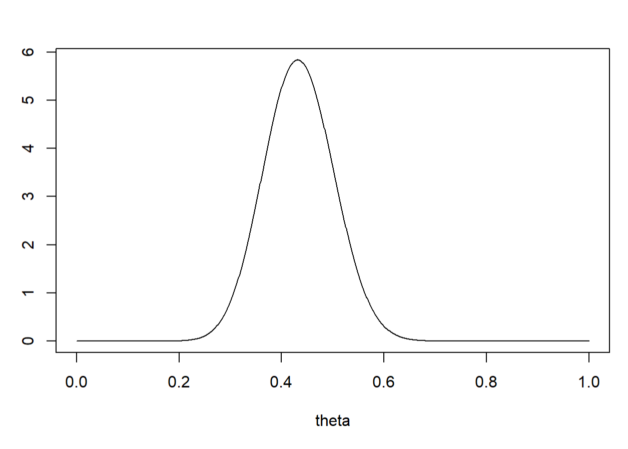
Since they're the same but a & b switched, I used the exact same function created from q3.4a for $p(\theta|y) \sim Beta(23,30)$

$$\begin{aligned}
&Mean \approx 0.4340\\
&Mode \approx 0.4314\\
&sd \approx 0.0674\\
&95\% CI : (0.3047, 0.5680)
\end{aligned}$$

```r
ploting(theta=theta, y=y,n=n,a=b,b=a)
```





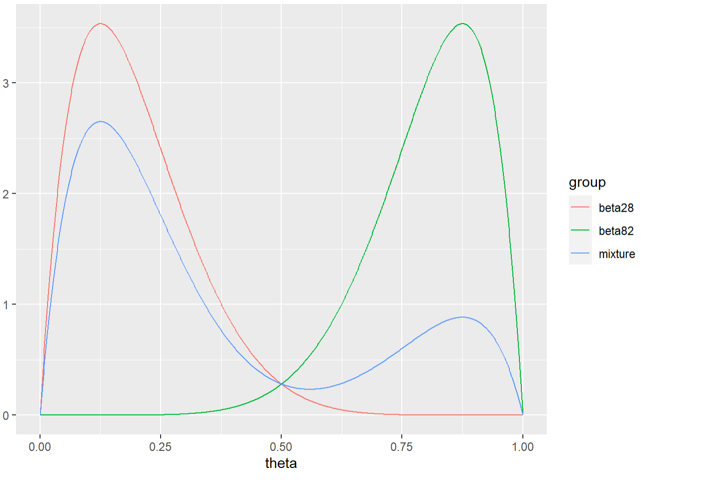



```
##                 mean      mode         sd Confidence_Interval
## Beta(8,2) 0.4339623 0.4313725 0.06744532        0.3047,0.568
```

c

The mixture shows a split prior opinions at around 0.125 and around 0.875.

```r
mixture = 0.75*dbeta(theta,2,8) + 0.25*dbeta(theta,8,2)
df <- data.frame(theta=theta, beta28=dbeta(theta,2,8),
           beta82=dbeta(theta,8,2), mixture=mixture) %>%
  pivot_longer(cols=c("beta28", "beta82", "mixture"), names_to="group", values_to = "density")
ggplot(df) +
  geom_line(aes(x=theta, y=density,col=group))
```



d

i

```r
gamma(10)/(gamma(2)*gamma(8))/4
```

```
## [1] 18
```

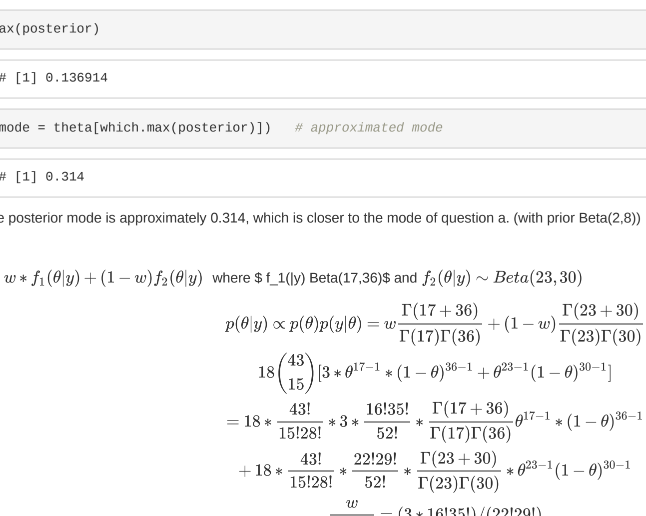
$$\begin{aligned}
p(\theta) * p(y|\theta) &= 18[3\theta(1-\theta)^7 + \theta^7(1-\theta)] * [\binom{43}{15}\theta^{15}(1-\theta)^{28}]\\
&= 18\binom{43}{15}[3\theta^{16}(1-\theta)^{35} + \theta^{22}(1-\theta)^{29}]\\
&= 18\binom{43}{15}\theta^{16}(1-\theta)^{29}[3(1-\theta)^6 + \theta^6]
\end{aligned}$$

ii

Since $p(\theta|y) \propto p(\theta) * p(y|\theta)$, the posterior distribution should be a mixture of the posteriors in above question 3.4a and 3.4b.

iii

```r
new_dens <- function(theta){
  18*(factorial(43)/(factorial(15)*factorial(43-15)))*(3*theta**16*(1-theta)**35
                                                       + theta**22*(1-theta)**29)
}
posterior <- new_dens(theta=theta)
plot(theta, posterior,type = "l", ylab="Posterior")
```




```r
max(posterior)
```

```
## [1] 0.136914
```

```r
(mode = theta[which.max(posterior)])   # approximated mode
```

```
## [1] 0.314
```

The posterior mode is approximately 0.314, which is closer to the mode of question a. (with prior Beta(2,8))

e

Let $w * f_1(\theta|y) + (1-w)f_2(\theta|y)$ where $ f_1(|y) Beta(17,36)$ and $f_2(\theta|y) \sim Beta(23,30)$

$$\begin{aligned}
p(\theta|y) \propto p(\theta)p(y|\theta) &= w\frac{\Gamma(17+36)}{\Gamma(17)\Gamma(36)} + (1-w)\frac{\Gamma(23+30)}{\Gamma(23)\Gamma(30)}\\
&18\binom{43}{15}[3*\theta^{17-1}*(1-\theta)^{36-1} + \theta^{23-1}(1-\theta)^{30-1}]\\
&= 18*\frac{43!}{15!28!}*3*\frac{16!35!}{52!}*\frac{\Gamma(17+36)}{\Gamma(17)\Gamma(36)}\theta^{17-1}*(1-\theta)^{36-1}\\
&+ 18*\frac{43!}{15!28!}*\frac{22!29!}{52!}*\frac{\Gamma(23+30)}{\Gamma(23)\Gamma(30)}*\theta^{23-1}(1-\theta)^{30-1}\\
\frac{w}{1-w} &= (3*16!35!)/(22!29!)\\
w &\approx 0.984
\end{aligned}$$

```r
# w/(1-w) =
x = 30*31*32*33*34*35/(22*21*20*19*6*17)
w = x/(1+x)
w
```

```
## [1] 0.9849087
```

3.9