

Exercises 2. Matrices

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1. Suppose

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$$

(a) Check that $A^3 = 0$ where 0 is a 3×3 matrix with every entry equal to 0.

```
A <- matrix(c(1,1,3,5,2,6,-2,-1,-3), nrow = 3, byrow = TRUE)
A%%A%%A
```

```
##      [,1] [,2] [,3]
## [1,]    0    0    0
## [2,]    0    0    0
## [3,]    0    0    0
```

(b) Replace the third column of A by the sum of the second and third columns.

```
A[,3] <- A[,2]+A[,3]
A
```

```
##      [,1] [,2] [,3]
## [1,]    1    1    4
## [2,]    5    2    8
## [3,]   -2   -1   -4
```

2. Create the following matrix B with 15 rows

$$B = \begin{bmatrix} 10 & -10 & 10 \\ 10 & -10 & 10 \\ \dots & \dots & \dots \\ 10 & -10 & 10 \end{bmatrix}$$

Calculate the 3×3 matrix $B^T B$. (Look at the help for `crossprod`.)

```
B <- matrix(c(rep(10,15),rep(-10,15),rep(10,15)), nrow = 15)
crossprod(B)
```

```
##      [,1] [,2] [,3]
## [1,] 1500 -1500 1500
## [2,] -1500 1500 -1500
## [3,] 1500 -1500 1500
```

3. Create a 6 x 6 matrix matE with every element equal to 0.

Check what the functions `row` and `col` return when applied to `matE`. Hence create the 6×6 matrix:

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

```
matE <- matrix(rep(0,36), nrow = 6, byrow = TRUE)
matE[abs(row(matE)-col(matE))==1] <- 1
matE
```

```
##      [,1] [,2] [,3] [,4] [,5] [,6]
## [1,]    0    1    0    0    0    0
## [2,]    1    0    1    0    0    0
## [3,]    0    1    0    1    0    0
## [4,]    0    0    1    0    1    0
## [5,]    0    0    0    1    0    1
## [6,]    0    0    0    0    1    0
```

4. Look at the help for the function `outer()`. Hence, create the following patterned matrix:

$$\begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 & 7 \\ 4 & 5 & 6 & 7 & 8 \end{bmatrix}$$

```
a <- 0:4
A <- outer(a,a,"+")
A
```

```
##      [,1] [,2] [,3] [,4] [,5]
## [1,]    0    1    2    3    4
## [2,]    1    2    3    4    5
## [3,]    2    3    4    5    6
## [4,]    3    4    5    6    7
## [5,]    4    5    6    7    8
```

5. Create the following patterned matrices. In each case, your solution should make use of the special form of the matrix—this means that the solution should easily generalise to creating a larger matrix with the same structure and should not involve typing in all the entries in the matrix.

(a)

$$\begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 & 0 \\ 2 & 3 & 4 & 0 & 1 \\ 3 & 4 & 0 & 1 & 2 \\ 4 & 0 & 1 & 2 & 3 \end{bmatrix}$$

```
a <- 0:4
A <- outer(a,a,"+")%%5
A
```

```
##      [,1] [,2] [,3] [,4] [,5]
## [1,]    0    1    2    3    4
## [2,]    1    2    3    4    0
## [3,]    2    3    4    0    1
## [4,]    3    4    0    1    2
## [5,]    4    0    1    2    3
```

(b)

$$\begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 8 & 9 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 9 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{bmatrix}$$

```
b <- 0:9
B <- outer(b,b,"+")%%10
B
```

```
##      [,1] [,2] [,3] [,4] [,5] [,6] [,7] [,8] [,9] [,10]
## [1,]    0    1    2    3    4    5    6    7    8    9
## [2,]    1    2    3    4    5    6    7    8    9    0
## [3,]    2    3    4    5    6    7    8    9    0    1
## [4,]    3    4    5    6    7    8    9    0    1    2
## [5,]    4    5    6    7    8    9    0    1    2    3
## [6,]    5    6    7    8    9    0    1    2    3    4
## [7,]    6    7    8    9    0    1    2    3    4    5
## [8,]    7    8    9    0    1    2    3    4    5    6
## [9,]    8    9    0    1    2    3    4    5    6    7
## [10,]    9    0    1    2    3    4    5    6    7    8
```

(c)

$$\begin{bmatrix} 0 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \\ 1 & 0 & 8 & 7 & 6 & 5 & 4 & 3 & 2 \\ 2 & 1 & 0 & 8 & 7 & 6 & 5 & 4 & 3 \\ 3 & 2 & 1 & 0 & 8 & 7 & 6 & 5 & 4 \\ 4 & 3 & 2 & 1 & 0 & 8 & 7 & 6 & 5 \\ 5 & 4 & 3 & 2 & 1 & 0 & 8 & 7 & 6 \\ 6 & 5 & 4 & 3 & 2 & 1 & 0 & 8 & 7 \\ 7 & 6 & 5 & 4 & 3 & 2 & 1 & 0 & 8 \\ 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 & 0 \end{bmatrix}$$

```
c <- 0:8
C <- outer(c,c,"-")%%9
C
```

```
##      [,1] [,2] [,3] [,4] [,5] [,6] [,7] [,8] [,9]
## [1,]    0    8    7    6    5    4    3    2    1
## [2,]    1    0    8    7    6    5    4    3    2
## [3,]    2    1    0    8    7    6    5    4    3
## [4,]    3    2    1    0    8    7    6    5    4
## [5,]    4    3    2    1    0    8    7    6    5
## [6,]    5    4    3    2    1    0    8    7    6
## [7,]    6    5    4    3    2    1    0    8    7
```

```
## [8,] 7 6 5 4 3 2 1 0 8
## [9,] 8 7 6 5 4 3 2 1 0
```

6. Solve the following system of linear equations in five unknowns by considering an appropriate matrix equation $Ax = y$.

$$\begin{aligned}x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 &= 7 \\2x_1 + x_2 + 2x_3 + 3x_4 + 4x_5 &= -1 \\3x_1 + 2x_2 + x_3 + 2x_4 + 3x_5 &= -3 \\4x_1 + 3x_2 + 2x_3 + x_4 + 2x_5 &= 5 \\5x_1 + 4x_2 + 3x_3 + 2x_4 + x_5 &= 17\end{aligned}$$

Make use of the special form of the matrix A. The method used for the solution should easily generalize to a larger set of equations where the matrix A has the same structure; hence the solution should not involve typing in every number of A.

```
Y <- c(7,-1,-3,5,17)
A <- matrix(rep(0,25), nrow = 5, byrow = TRUE)
A <- abs(row(A)-col(A))+1
A
```

```
##      [,1] [,2] [,3] [,4] [,5]
## [1,] 1 2 3 4 5
## [2,] 2 1 2 3 4
## [3,] 3 2 1 2 3
## [4,] 4 3 2 1 2
## [5,] 5 4 3 2 1
```

```
X <- solve(A,Y)
X
```

```
## [1] -2 3 5 2 -4
```

7. Create a 6 x 10 matrix of random integers chosen from 1,2,...,10 by executing the following two lines of code:

```
set.seed(75)
aMat <- matrix(sample(10, size=60, replace=TRUE), nr=6)
aMat
```

```
##      [,1] [,2] [,3] [,4] [,5] [,6] [,7] [,8] [,9] [,10]
## [1,] 3 6 7 7 2 4 3 7 1 4
## [2,] 1 9 8 7 2 6 10 9 5 2
## [3,] 7 10 8 4 10 5 4 8 4 4
## [4,] 4 3 1 1 3 3 9 7 4 2
## [5,] 1 8 1 9 9 8 1 3 7 7
## [6,] 2 6 7 5 6 10 4 6 10 1
```

(a) Find the number of entries in each row which are greater than 4.

```
sum(aMat>4)
```

```
## [1] 32
```

(b) Which rows contain exactly two occurrences of the number seven?

```
which(rowSums(aMat==7)==2)
```

```
## [1] 5
```

(c) Find those pairs of columns whose total (over both columns) is greater than 75. The answer should be a matrix with two columns; so, for example, the row (1,2) in the output matrix means that the sum of columns 1 and 2 in the original matrix is greater than 75. Repeating a column is permitted; so, for example, the final output matrix could contain the rows (1,2), (2,1), and (2,2). What if repetitions are not permitted? Then only (1,2) from (1,2), (2,1) and (2,2) would be permitted.

```
sumtwocolumns <- colSums(aMat)
which(outer(sumtwocolumns,sumtwocolumns,"+")>75,arr.ind = TRUE)
```

```
##      row col
## [1,]    2  2
## [2,]    6  2
## [3,]    8  2
## [4,]    2  6
## [5,]    8  6
## [6,]    2  8
## [7,]    6  8
## [8,]    8  8
```

8. Calculate

$$(a) \sum_{i=1}^{20} \sum_{j=1}^5 \frac{i^4}{(3+j)}$$

```
sum((1:20)^4)*sum(1/(3+(1:5)))
```

```
## [1] 639215.3
```

$$(b) \sum_{i=1}^{20} \sum_{j=1}^5 \frac{i^4}{(3+ij)}$$

```
sum((1:20)^4/(3+outer(1:20,1:5,"*")))
```

```
## [1] 89912.02
```

$$(c) \sum_{i=1}^{10} \sum_{j=1}^i \frac{i^4}{(3+ij)}$$

```
i <- 1:10; j <- 1:10
ij <- function(i,j) {
  ifelse(i>=j,1/(i*j+3),0)
}
sum((matrix(i^4,nrow=1,byrow=FALSE))%*%(outer(i,j,ij)))
```

```
## [1] 6944.743
```
