Exercises 2. Matrices

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1. Suppose

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$$

(a) Check that $A^3 = 0$ where 0 is a 3×3 matrix with every entry equal to 0.

```
A <- matrix(c(1,1,3,5,2,6,-2,-1,-3), nrow = 3, byrow = TRUE)
A%*%A%*%A
```

```
##      [,1] [,2] [,3]
## [1,]    0    0    0
## [2,]    0    0    0
## [3,]    0    0    0
```

(b) Replace the third column of A by the sum of the second and third columns.

```
A[,3] <- A[,2]+A[,3]
A
```

```
##      [,1] [,2] [,3]
## [1,]    1    1    4
## [2,]    5    2    8
## [3,]   -2   -1   -4
```

---

2. Create the following matrix B with 15 rows

$$B = \begin{bmatrix} 10 & -10 & 10 \\ 10 & -10 & 10 \\ \dots & \dots & \dots \\ 10 & -10 & 10 \end{bmatrix}$$

Calculate the 3x3 matrix $B^T B$. (Look at the help for `crossprod`.)

```
B <- matrix(c(rep(10,15),rep(-10,15),rep(10,15)), nrow = 15)
crossprod(B)
```

```
##       [,1]  [,2]  [,3]
## [1,]  1500 -1500  1500
## [2,] -1500  1500 -1500
## [3,]  1500 -1500  1500
```

---

3. Create a 6 x 6 matrix matE with every element equal to 0.

Check what the functions `row` and `col` return when applied to `matE`. Hence create the 6×6 matrix:

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

```r
matE <- matrix(rep(0,36), nrow = 6, byrow = TRUE)
matE[abs(row(matE)-col(matE))==1] <- 1
matE
```

```
##      [,1] [,2] [,3] [,4] [,5] [,6]
## [1,]    0    1    0    0    0    0
## [2,]    1    0    1    0    0    0
## [3,]    0    1    0    1    0    0
## [4,]    0    0    1    0    1    0
## [5,]    0    0    0    1    0    1
## [6,]    0    0    0    0    1    0
```

---

4. Look at the help for the function outer(). Hence, create the following patterned matrix:

$$\begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 & 7 \\ 4 & 5 & 6 & 7 & 8 \end{bmatrix}$$

```r
a <- 0:4
A <- outer(a,a,"+")
A
```

```
##      [,1] [,2] [,3] [,4] [,5]
## [1,]    0    1    2    3    4
## [2,]    1    2    3    4    5
## [3,]    2    3    4    5    6
## [4,]    3    4    5    6    7
## [5,]    4    5    6    7    8
```

---

5. Create the following patterned matrices. In each case, your solution should make use of the special form of the matrix—this means that the solution should easily generalise to creating a larger matrix with the same structure and should not involve typing in all the entries in the matrix.

(a)

$$\begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 & 0 \\ 2 & 3 & 4 & 0 & 1 \\ 3 & 4 & 0 & 1 & 2 \\ 4 & 0 & 1 & 2 & 3 \end{bmatrix}$$

```
a <- 0:4
A <- outer(a,a,"+")%%5
A
```

```
##      [,1] [,2] [,3] [,4] [,5]
## [1,]    0    1    2    3    4
## [2,]    1    2    3    4    0
## [3,]    2    3    4    0    1
## [4,]    3    4    0    1    2
## [5,]    4    0    1    2    3
```

(b)

$$\begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 8 & 9 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 9 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{bmatrix}$$

```
b <- 0:9
B <- outer(b,b,"+")%%10
B
```

```
##       [,1] [,2] [,3] [,4] [,5] [,6] [,7] [,8] [,9] [,10]
##  [1,]    0    1    2    3    4    5    6    7    8     9
##  [2,]    1    2    3    4    5    6    7    8    9     0
##  [3,]    2    3    4    5    6    7    8    9    0     1
##  [4,]    3    4    5    6    7    8    9    0    1     2
##  [5,]    4    5    6    7    8    9    0    1    2     3
##  [6,]    5    6    7    8    9    0    1    2    3     4
##  [7,]    6    7    8    9    0    1    2    3    4     5
##  [8,]    7    8    9    0    1    2    3    4    5     6
##  [9,]    8    9    0    1    2    3    4    5    6     7
## [10,]    9    0    1    2    3    4    5    6    7     8
```

(c)

$$\begin{bmatrix} 0 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \\ 1 & 0 & 8 & 7 & 6 & 5 & 4 & 3 & 2 \\ 2 & 1 & 0 & 8 & 7 & 6 & 5 & 4 & 3 \\ 3 & 2 & 1 & 0 & 8 & 7 & 6 & 5 & 4 \\ 4 & 3 & 2 & 1 & 0 & 8 & 7 & 6 & 5 \\ 5 & 4 & 3 & 2 & 1 & 0 & 8 & 7 & 6 \\ 6 & 5 & 4 & 3 & 2 & 1 & 0 & 8 & 7 \\ 7 & 6 & 5 & 4 & 3 & 2 & 1 & 0 & 8 \\ 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 & 0 \end{bmatrix}$$

```
c <- 0:8
C <- outer(c,c,"-")%%9
C
```

```
##      [,1] [,2] [,3] [,4] [,5] [,6] [,7] [,8] [,9]
##  [1,]    0    8    7    6    5    4    3    2    1
##  [2,]    1    0    8    7    6    5    4    3    2
##  [3,]    2    1    0    8    7    6    5    4    3
##  [4,]    3    2    1    0    8    7    6    5    4
##  [5,]    4    3    2    1    0    8    7    6    5
##  [6,]    5    4    3    2    1    0    8    7    6
##  [7,]    6    5    4    3    2    1    0    8    7
```

```
## [8,]    7    6    5    4    3    2    1    0    8
## [9,]    8    7    6    5    4    3    2    1    0
```

---

6. Solve the following system of linear equations in five unknowns by considering an appropriate matrix equation Ax = y.

$$x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 = 7$$
$$2x_1 + x_2 + 2x_3 + 3x_4 + 4x_5 = -1$$
$$3x_1 + 2x_2 + x_3 + 2x_4 + 3x_5 = -3$$
$$4x_1 + 3x_2 + 2x_3 + x_4 + 2x_5 = 5$$
$$5x_1 + 4x_2 + 3x_3 + 2x_4 + x_5 = 17$$

Make use of the special form of the matrix A. The method used for the solution should easily generalize to a larger set of equations where the matrix A has the same structure; hence the solution should not involve typing in every number of A.

```r
Y <- c(7,-1,-3,5,17)
A <- matrix(rep(0,25), nrow = 5, byrow = TRUE)
A <- abs(row(A)-col(A))+1
A
```

```
##      [,1] [,2] [,3] [,4] [,5]
## [1,]    1    2    3    4    5
## [2,]    2    1    2    3    4
## [3,]    3    2    1    2    3
## [4,]    4    3    2    1    2
## [5,]    5    4    3    2    1
```

```r
X <- solve(A,Y)
X
```

```
## [1] -2  3  5  2 -4
```

---

7. Create a 6 x 10 matrix of random integers chosen from 1,2,. . . ,10 by executing the following two lines of code:

```r
set.seed(75)
aMat <- matrix(sample(10, size=60, replace=TRUE), nr=6)
aMat
```

```
##      [,1] [,2] [,3] [,4] [,5] [,6] [,7] [,8] [,9] [,10]
## [1,]    3    6    7    7    2    4    3    7    1    4
## [2,]    1    9    8    7    2    6   10    9    5    2
## [3,]    7   10    8    4   10    5    4    8    4    4
## [4,]    4    3    1    1    3    3    9    7    4    2
## [5,]    1    8    1    9    9    8    1    3    7    7
## [6,]    2    6    7    5    6   10    4    6   10    1
```

(a) Find the number of entries in each row which are greater than 4.

```r
sum(aMat>4)
```

4

```
## [1] 32
```

(b) Which rows contain exactly two occurrences of the number seven?

```
which(rowSums(aMat==7)==2)
```

```
## [1] 5
```

(c) Find those pairs of columns whose total (over both columns) is greater than 75. The answer should be a matrix with two columns; so, for example, the row (1,2) in the output matrix means that the sum of columns 1 and 2 in the original matrix is greater than 75. Repeating a column is permitted; so, for example, the final output matrix could contain the rows (1,2), (2,1), and (2,2). What if repetitions are not permitted? Then only (1,2) from (1,2),(2,1) and (2,2) would be permitted.

```
sumtwocolumns <- colSums(aMat)
which(outer(sumtwocolumns,sumtwocolumns,"+")>75,arr.ind = TRUE)
```

```
##       row col
## [1,]   2   2
## [2,]   6   2
## [3,]   8   2
## [4,]   2   6
## [5,]   8   6
## [6,]   2   8
## [7,]   6   8
## [8,]   8   8
```

---

8. Calculate

(a) $\sum_{i=1}^{20} \sum_{j=1}^{5} \frac{i^4}{(3+j)}$

```
sum((1:20)^4)*sum(1/(3+(1:5)))
```

```
## [1] 639215.3
```

(b) $\sum_{i=1}^{20} \sum_{j=1}^{5} \frac{i^4}{(3+ij)}$

```
sum((1:20)^4/(3+outer(1:20,1:5,"*")))
```

```
## [1] 89912.02
```

(c) $\sum_{i=1}^{10} \sum_{j=1}^{i} \frac{i^4}{(3+ij)}$

```
sum((1:10)^4/(3+outer(1:10,1:10,"*")))
```

```
## [1] 7485.33
```