Exponential Stability for Multi-Vehicle Coordination over Time-Varying Stochastic Networks

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***Abstract*— This paper presents a distributed control protocol for a fleet of multiple vehicles, connected over a specific class of stochastic communication networks, to solve time-critical coordination problems. Under the assumption that the underlying network topology is given by a balanced directed graph, sufficient conditions are given for exponential stability of the coordination dynamics with a guaranteed rate of convergence. The likelihood of achieving exponential stability is estimated based on the stochastic properties of the underlying network topology.**

I. Introduction

Recent advances in autonomous systems have accelerated the forthcoming leap of self-driving cars and urban air mobility solutions from research environments to our daily lives. This impending scenario opens a new paradigm where autonomous vehicles can benefit from collaboration and coordination to improve mission safety and efficacy. To this end, vehicles need to exchange information over a wireless communication network with a time-varying topology. There are three fundamental challenges regarding the distributed algorithms that must be tackled to ensure the safety of such cooperative frameworks:

1) understanding how the quality of service of the network affects the overall system safety and performance;
2) a gradual relaxation of the network assumptions until realistic necessary and sufficient criteria that guarantee the stability of the overall system are found; and
3) given a stochastic model of the network connections, estimating the likelihood that the necessary and sufficient conditions that guarantee stability will be met for a particular scenario.

Cooperative trajectory generation frameworks such as [1] and [2] design a set of trajectories $\boldsymbol{p}_{d,i}(t_d) : [0\ t_d^*] \to \mathbb{R}^{n_d}$, $n_d \in \{2,3\}$, $i \in \{1,\dots,n\}$ that map the desired mission time t_d to a position in space for a fleet of n cooperating vehicles, such that agents maintain a safety distance d_s among each other

$$\|\boldsymbol{p}_{d,i}(t_d) - \boldsymbol{p}_{d,j}(t_d)\| > d_s, \qquad \forall i \neq j \in \{1,\dots,n\}.$$

In such multi-agent scenarios coordination can be mission critical or even safety critical. For instance, some scenarios require agents to maintain a desired formation to accomplish a mission goal; in other cases, when multiple paths intersect, the trajectory generation algorithm in [1] will leverage the time dimension to ensure that vehicles do not coincide at the intersection point at the the same time. However, during the execution of these missions vehicles are hit with disturbances and partial failures that may put the vehicles behind or ahead the planned schedule. Not only can these events cause discoordination, but they can also render the initial plan non-feasible by forcing the vehicles to violate their dynamic constraints. To alleviate this problem, the authors in [3] introduce an additional degree of freedom into the system through a virtual target, which was later used in [4]–[6]. The virtual target, which defines the goal position along the path of the ith vehicle at time t, can slide along the path slightly faster or slower than planned in the trajectory generation phase to reduce the path-following error. For instance, if a vehicle precedes its virtual target, then the virtual target will speed up to catch up with the vehicle; and if the vehicle is falling behind (e.g. due to strong headwinds), then the virtual target will slow down. Thus, the virtual target artificially modifies the mission time for each vehicle and holds the potential to reduce the vehicle control effort, but can lead to vehicle discoordination, jeopardizing the success and safety of the mission. To overcome this problem, the vehicles in [4]–[6] exchange their artificially modified mission times, henceforth referred to as coordination states, over the network. In these synchronized path-following algorithms, a proportional-integral (PI) consensus protocol is used to force agreement on the coordination states of the fleet, hence ensuring that the vehicles are coordinated in real time, not just in the trajectory planning phase. The work in [6] proposes a PI consensus protocol where only a subset of agents have knowledge of the desired mission rate, and defines performance bounds under the passivity conditions defined in [7] for a bidirectional time-varying network without delays. The work in [8] analyzes the effects of communication delays of a PI consensus protocol for static bidirectional networks. The authors in [9] add absolute temporal constraints to the PI consensus algorithm proposed in [10], such as the time of arrival. The work in [11] relies on results from adaptive control and persistency of excitation to derive tighter performance bounds for time-varying bidirectional graphs.

The contributions of this paper are threefold. First, we work out the problem of time-critical coordination over a more general class of network topologies compared with the previous work. The distributed consensus protocol was initially studied in [12] for bidirectional time-varying graphs,

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subject to the persistency of excitation condition introduced in [7]. In this paper, we derive sufficient conditions for the global exponential stability of the consensus protocol in [12] for a balanced time-varying directed graph by leveraging tools from switched systems [13], [14]. Second, a guaranteed lower bound for the exponential rate of convergence under such conditions is provided. Finally, this paper evaluates the likelihood of the aforementioned exponential consensus being achieved when the switching of the network topology is modelled as a Poisson process and the time interval between switching instants is modelled as a Markov process.

The rest of the paper is organized as follows: important concepts and preliminary results are provided in Section II. The problem formulation is discussed in Section III. A distributed control protocol is presented and analyzed in Section IV. Simulation results are provided in Section V. Concluding remarks are included in Section VI.

II. Preliminaries

Notation, key notions, and some preliminary results that are leveraged through the paper are introduced in this section.

A. Notation

Let $rank(A)$ be the rank of a matrix A. Let 0_n, 1_n, and ρ_n denote the vectors in $\mathbb{R}^n$ whose elements are all 0, 1, and $\rho \in \mathbb{R}$, respectively. Henceforth, the identity matrix with dimension n will be represented by $\mathbb{I}_n$.

B. Graph Theory

Consider a digraph represented by $G = (V, E)$, where V is the set of vertices in G with cardinality n, and E is the set of directed edges in G. The Laplacian matrix of G is defined by $L = D - A$, where D is the degree matrix, and A is the adjacency matrix. From the definition of the Laplacian matrix, one has that $L1_n = 0_n$ for all digraphs.

Lemma 1: (Proposition 3.10. in [15]) The Laplacian matrix L of a digraph G has at least one eigenvalue at 0 and all the other eigenvalues lie in the open right-half complex plane by the Gershgorin disk theorem [16].

Definition 1: (Definition 3.7. in [15]) A digraph G is a rooted out-branching graph if there is a vertex $v_i \in V$ (called a root) such that it can reach out to any other vertex through a directed path, and the digraph does not have a directed cycle.

Lemma 2: (Proposition 3.8. in [15]) A digraph G of order n has $rank(L) = n-1$ if and only if it contains a subgraph, which is a rooted out-branching graph. In this case, the null space of L is spanned by 1_n.

Definition 2: (Definition 3.14. in [15]) If the in-degree and out-degree are equal for all vertices in a digraph, that is, $1_n^\top L = 0_n^\top$ in addition to $L1_n = 0_n$, then the digraph is called balanced.

C. Coordination States

Let us assume that a desired path $\boldsymbol{p}_{d,i}(t_d) : [0 \ t_d^*] \to \mathbb{R}^{n_d}$, $n_d \in \{2, 3\}$ is given for each vehicle i involved in the cooperative effort. Here, $\boldsymbol{p}_{d,i}(\cdot)$ is parameterized by the desired mission time t_d, which is different from the actual time t that evolves as the mission unfolds. The mission is supposed to end at a prespecified time t_d^* called the mission duration time. The coordination state (or virtual time) $x_i(t)$ is the mission time t_d at the actual time t. The virtual target is a point $\boldsymbol{p}_{d,i}(x_i(t))$ on the path to be followed by the ith vehicle. The length $l_i(t)$ traveled by the virtual target is defined by $l_i(t) \triangleq \int_0^{x_i(t)} v_{d,i}(\tau) d\tau$, where $v_{d,i}(\tau) \triangleq \| \frac{d\boldsymbol{p}_{d,i}(\tau)}{d\tau} \|_2$ is the desired speed profile along the desired path.

Example. Consider a path $\boldsymbol{p}_{d,i}(t_d) : [0 \ 6] \to [t_d \ \ \sin t_d]^\top$ plotted in Figure 1 and a coordination state $x_i(t) = 0.8t$. At actual time $t = 5s$, the value of $x_i(t)$ is 4. In this setting, as can be seen in Figure 1, the virtual target is located at $\boldsymbol{p}_{d,i}(x_i(t)) = [4 \ -0.7568]^\top$ and $l_i(t) = 4.9666$.

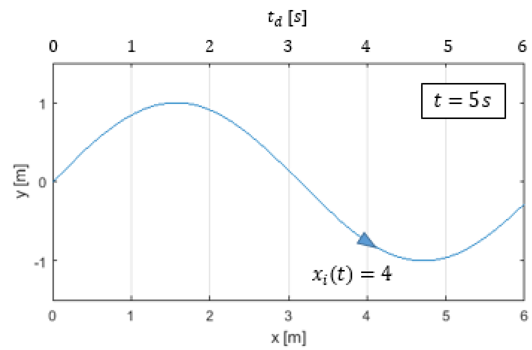


Fig. 1: The value of the coordination state $x_i(t)$ and the location of the virtual target (blue triangle) at actual time $t = 5s$.

D. Path Following

In this paper, we use the path following algorithm presented in [6], where nonlinear feedback control laws for the pitch and yaw rates of a vehicle are designed using the kinematic model of the vehicle and the virtual target. The path following control laws drive the errors in position and heading angle between the virtual target and the vehicle to 0 exponentially. To ensure the algorithm works, it requires that the speed of the virtual target be governed by

$$\dot{l}_i(t) = (v_i(t)\hat{\mathbf{w}}_\mathbf{i}(t) + k_l \mathbf{e}_\mathbf{i}(t)) \cdot \hat{\mathbf{t}}_\mathbf{i}(t), \tag{1}$$

where v_i is the speed of the ith vehicle; $\hat{\mathbf{w}}_\mathbf{i}$ is the unit vector in the direction of the velocity of it; k_l is a positive constant gain; $\mathbf{e}_\mathbf{i}$ is the error vector from the position of the virtual target to that of the ith vehicle; and $\hat{\mathbf{t}}_\mathbf{i}$ is the unit vector tangential to the desired path at the position of the virtual target.

III. Problem Formulation

Given a set of desired paths $\boldsymbol{p}_{d,i}(t_d)$ associated with the ith vehicle involved in a cooperative mission, our goal is to design a distributed control protocol for time-critical coordination so that 1) the vehicles are located at the relatively desired positions, 2) while progressing with the same desired mission rate. In this task, we use the path following control algorithm described in Section II-D.

Under this setting, our problem can be formulated as follows

$$x_i(t) - x_j(t) \xrightarrow{t\to\infty} 0, \quad \forall i,j \in \{1,\dots,n\}, \tag{2a}$$

$$\dot{x}_i(t) \xrightarrow{t\to\infty} \rho, \quad \forall i \in \{1,\dots,n\}, \tag{2b}$$

where $\rho > 0$ is the desired mission rate. The condition $x_i(t) - x_j(t) = 0$ for $\forall i,j \in \{1,\cdots,n\}$ and $\forall t \geq 0$ implies that the virtual targets to be followed by the vehicles are at the relatively desired positions at $\forall t \geq 0$ so that they arrive at their final destinations simultaneously. The additional condition $\dot{x}_i(t) = \rho$ for $\forall i \in \{1,\cdots,n\}$ and $\forall t \geq 0$ implies that they progress with the same mission rate ρ so that they eventually arrive at their final destinations in time $\frac{t_d^*}{\rho}$.

To achieve the above consensus, coordination states should be transmitted and received among the vehicles within the underlying communication network. It is assumed that if a vehicle receives $k(\geq 0)$ coordination states from its neighbors, it also transmits its coordination state to k other vehicles. It is also assumed that the network is stochastically time variant and not necessarily connected all the time. Putting all of these assumptions together, the network is well delineated by a stochastically time-varying balanced digraph $G(t) = (V, E(t))$. The fleet of n vehicles is divided into two groups: virtual leaders and followers. Virtual leaders have knowledge of the desired mission rate ρ in (2b), while followers do not.

IV. Distributed Consensus Protocol

To address the problem formulated in Section III, we design a distributed proportional-integral protocol in Section IV-A and then, present *deterministic* conditions on network switching that guarantee exponential convergence of (2a) and (2b) in Section IV-B. Then, Section IV-C extends the findings in Section IV-B by relating the *deterministic* conditions with *stochastic* conditions. Sometimes, we omit time dependency (t) for simplicity when it is obvious from the context that a variable is dependent on time t.

A. Proportional-Integral (PI) Protocol

From the definition of l_i, one has $\dot{l}_i = \dot{x}_i v_{d,i}(x_i)$. Replacing $\dot{l}_i$ with (1) yields

$$\dot{x}_i = \frac{(v_i \hat{\mathbf{w}}_\mathbf{i} + k_l \mathbf{e_i}) \cdot \hat{\mathbf{t}}_\mathbf{i}}{v_{d,i}(x_i)}.$$

Letting the ith vehicle fly with the speed $v_i = \frac{u_i v_{d,i}(x_i) - k_l \mathbf{e_i} \cdot \hat{\mathbf{t}}_\mathbf{i}}{\hat{\mathbf{w}}_\mathbf{i} \cdot \hat{\mathbf{t}}_\mathbf{i}}$ leads to the following dynamics of coordination states

$$\begin{aligned} \dot{x}_i &= u_i, \quad i \in V_l \triangleq \{1,\dots,n_l\}, \\ \dot{x}_i &= u_i + d_i, \quad i \in V_f \triangleq \{n_l+1,\dots,n\}, \end{aligned} \tag{3}$$

where $x_i \in \mathbb{R}$, $u_i \in \mathbb{R}$, and $d_i \in \mathbb{R}$ are the coordination state, its control input, and an unknown constant disturbance, respectively. Here, V_l and V_f are sets of virtual leaders and followers, respectively.

To achieve the time coordination (2), we adopt the following distributed proportional-integral protocol:

$$\begin{aligned} u_i &= -k_P \sum_{j\in\mathcal{N}_i(t)} (x_i - x_j) + \rho, \quad i \in V_l, \\ u_i &= -k_P \sum_{j\in\mathcal{N}_i(t)} (x_i - x_j) + \chi_i, \quad i \in V_f, \\ \dot{\chi}_i &= -k_I \sum_{j\in\mathcal{N}_i(t)} (x_i - x_j), \quad i \in V_f, \end{aligned} \tag{4}$$

where $k_P > 0$ and $k_I > 0$ are constant proportional gain and integral gain of the PI controller, respectively; $\chi_i \in \mathbb{R}$ is the integrator state; and $\mathcal{N}_i(t)$ denotes a set of incoming neighboring vertices of $i \in V$ at time t.

Note that the PI structure in (4) enables the followers to learn the desired mission rate ρ effectively compensating for the unknown constant disturbance d_i.

B. Collective Dynamics and Convergence Analysis

The network-wide system (3) with the protocol (4) has the closed-loop collective dynamics

$$\dot{x} = -k_P L(t) x + \begin{bmatrix} \rho_{n_l} \\ \chi + d \end{bmatrix}, \tag{5a}$$

$$\dot{\chi} = -k_I C^\top L(t) x, \tag{5b}$$

where $L(t)$ is the Laplacian matrix of $G(t)$, $C^\top \triangleq [0 \quad \mathbb{I}_{n-n_l}] \in \mathbb{R}^{(n-n_l)\times n}$, and

$$\begin{aligned} x &\triangleq [x_1 \ \dots \ x_n]^\top \in \mathbb{R}^n, \\ \chi &\triangleq [\chi_{n_l+1} \ \dots \ \chi_n]^\top \in \mathbb{R}^{n-n_l}, \\ d &\triangleq [d_{n_l+1} \ \dots \ d_n]^\top \in \mathbb{R}^{n-n_l}. \end{aligned}$$

A convergence analysis of the collective dynamics (5) can be done by reformulating the coordination problem into a stabilization problem. According to [12], $\zeta_1 \triangleq Qx \in \mathbb{R}^{n-1}$ represents errors between coordination states x_i and x_j, $\forall i,j \in \{1,\dots,n\}$ for any matrix $Q \in \mathbb{R}^{(n-1)\times n}$ that satisfies $Q1_n = 0$ and $QQ^\top = \mathbb{I}_{n-1}$. Also, let us define $\zeta_2 \triangleq \chi - \rho_{n-n_l} + d \in \mathbb{R}^{n-n_l}$. We call $\zeta \triangleq [\zeta_1^\top \ \zeta_2^\top]^\top \in \mathbb{R}^{2n-n_l-1}$ coordination error state.

Lemma 3: (Lemma 7 in [6]) For a matrix $Q \in \mathbb{R}^{(n-1)\times n}$ that satisfies $Q1_n = 0$ and $QQ^\top = \mathbb{I}_{n-1}$, the null space of Q is spanned by 1_n. Also, it holds that $\Pi \triangleq Q^\top Q = \mathbb{I}_n - \frac{1_n 1_n^\top}{n}$.

By Lemma 3, $\zeta_1 = 0_{n-1}$ leads to $x \in \text{span}\{1_n\}$. And with $\zeta_2 = 0_{n-n_l}$ and $x \in \text{span}\{1_n\}$, the equation (5a) boils down to $\dot{x} = \rho_n$. Therefore, $\zeta = [\zeta_1^\top \ \zeta_2^\top]^\top = 0_{2n-n_l-1}$ is equivalent to $x_i - x_j = 0$ and $\dot{x}_i = \rho$ for all $i,j \in \{1,\cdots,n\}$, which corresponds to our objective (2).

The coordination error state ζ is governed by the following dynamics

$$\dot{\zeta} = A_\zeta(t)\zeta, \quad \zeta(0) = \zeta_0, \tag{6}$$

where $A_\zeta(t) \in \mathbb{R}^{(2n-n_l-1)\times(2n-n_l-1)}$ is given by

$$A_\zeta(t) \triangleq \begin{bmatrix} -k_P \overline{L}(t) & QC \\ -k_I C^\top Q^\top \overline{L}(t) & 0 \end{bmatrix}, \tag{7}$$

and $\overline{L}(t) \triangleq QL(t)Q^\top$. Note that $A_\zeta(t)$ can be presented as in (7) because $G(t)$ is a balanced digraph (Definition 2).

The reader can refer to Appendix A for more details of the derivation of (6).

The time varying network is assumed to be piecewise constant in time. Then, the Laplacian matrix $L(t)$ of the balanced digraph $G(t)$ can be expressed as a switching Laplacian matrix $L_{\sigma(t)}$, where $\sigma(t):[0,\infty)\to\mathcal{I}_N\triangleq\{1,\ldots,N\}$ is a piecewise constant switching signal. Here, N is the number of topologies the network can take. A set $\{L_j : j\in\mathcal{I}_N\}$ composes a family of Laplacian matrices induced from the network topologies. If L_j is the Laplacian matrix of a balanced digraph containing a rooted out-branching graph, $-\overline{L}_j=-QL_jQ^\top$ is Hurwitz stable by Lemmas 1, 2, and 4. Otherwise, $-\overline{L}_j$ is marginally stable by Lemma 5.

Lemma 4: For the Laplacian matrix L of a digraph D, the spectrum of $\overline{L}\triangleq QLQ^\top\in\mathbb{R}^{(n-1)\times(n-1)}$ is the same as that of L without the eigenvalue 0 corresponding to the eigenvector 1_n.

Proof: Suppose that $Lv=\lambda v$. It follows that $QLv=\lambda Qv$. Then, by Lemma 3, the left hand side of the equation becomes $QLv=QL\left(\mathbb{I}_n-\frac{1_n1_n^\top}{n}\right)v=QLQ^\top Qv=\overline{L}Qv$. ■

Lemma 5: If L is the Laplacian matrix of a digraph not containing a rooted out-branching graph, $-\overline{L}$ is marginally stable.

Proof: Suppose that the algebraic multiplicity of the eigenvalue 0 of L is $k(\geq 2)$; then the geometric multiplicity of it is also equal to k (Corollary 4.2 in [17]). We can find a set of orthogonal vectors 1_n, v_1, ... , v_{k-1} that spans the null space of L. From the definition of $\overline{L}$ and the property $L\Pi=L$, $Lv_i=0$ $(1\leq i\leq k-1)$ implies $\overline{L}Qv_i=0$. It means that Qv_i $(1\leq i\leq k-1)$ are eigenvectors of $\overline{L}$ corresponding to the eigenvalue 0. Consider $\sum_{i=1}^{k-1}\alpha_iQv_i=0$. Premultiplication by $Q^\top$ yields $\sum_{i=1}^{k-1}\alpha_iQ^\top Qv_i=0$. Considering $Q^\top Q=\mathbb{I}_n-\frac{1_n1_n^\top}{n}$, it can be rewritten as $\sum_{i=1}^{k-1}\alpha_iv_i-\frac{1_n1_n^\top}{n}\sum_{i=1}^{k-1}\alpha_iv_i=\sum_{i=1}^{k-1}\alpha_iv_i=0$, where $\frac{\mathbf{1}_n\mathbf{1}_n^\top}{n}\sum_{i=1}^{k-1}\alpha_iv_i=0$ because 1_n, v_1, ... , v_{k-1} are orthogonal to one another. For the same reason, $\sum_{i=1}^{k-1}\alpha_iv_i=0$ implies $\alpha_1=\ldots=\alpha_{k-1}$. We have shown that the eigenvectors Qv_1, ..., Qv_{k-1} of $-\overline{L}$ corresponding to the eigenvalue 0 are linearly independent, from which and Lemmas 1 and 4, it follows that $-\overline{L}$ is marginally stable. ■

Since there exist both marginally stable and Hurwitz stable matrices in $\{-\overline{L}_j : j\in\mathcal{I}_N\}$, let us assume without loss of generality that $-\overline{L}_1,\ldots,-\overline{L}_r$ $(r<N)$ are marginally stable and the remaining matrices are Hurwitz stable. Then, we can always find a set of scalars a_j and $\lambda_j>0$ such that

$$\left\|e^{-\overline{L}_jt}\right\|\leq\begin{cases}e^{a_j}, & 1\leq j\leq r,\\ e^{a_j-\lambda_jt}, & r<j\leq N.\end{cases}\tag{8}$$

Now consider the following assumptions on the number of the network switches and the total length of time the balanced digraph $G(t)$ contains a rooted out-branching graph. Notice that switching systems theory has been widely used for convergence analysis of consensus problems over time-varying networks (e.g., [18], [19]). The following Assumptions 1 and 2 are from [13] and reformulated appropriately in accordance with our problem setting.

Assumption 1: Let $N_{\sigma(t)}(t,0)$ be the number of the network switches over the interval $[0,t)$. It is assumed that there exist two positive numbers N_0 and τ_a such that $N_{\sigma(t)}(t,0)\leq N_0+\frac{t}{\tau_a}$, where N_0 is the chatter bound and τ_a is the average dwell time that satisfies $\tau_a\geq\tau_a^*=\frac{a}{\lambda^*-\lambda}$. Here, $a\triangleq max_{1\leq j\leq N}\ a_i$ and λ and λ^* are any constants that satisfy $0<\lambda<\lambda^*<\lambda_c\triangleq min_{r+1\leq j\leq N}\ \lambda_i$.

Assumption 2: Let $T_c(t,0)$ (resp., $T_u(t,0)$) denote the total length of time when the balanced digraph $G(t)$ contains a rooted out-branching graph (resp., not contain a rooted out-branching graph) over the interval $[0,t)$. We assume the following inequality holds: $T_c(t,0)\geq T_u(t,0)\frac{\lambda^*}{\lambda_c-\lambda^*}$.

The implication of Assumption 1 is that the network does not switch too frequently. Assumption 2 implies that the network should be connected through a rooted-out branching graph at least for a certain length of time out of the interval $[0,t)$.

Theorem 1: Assume that the underlying communication network represented by a time-varying balanced digraph $G(t)$ satisfies Assumptions 1 and 2. Then, for any $k_\beta\geq 2$, there exist control gains k_P and k_I such that the origin of the coordination error dynamics (6) is globally uniformly exponentially stable (GUES), satisfying

$$\|\zeta(t)\|\leq\kappa_0\|\zeta(0)\|e^{-\bar{\lambda}_ct},\tag{9}$$

for some positive constant κ_0 and with a guaranteed rate of convergence $\bar{\lambda}_c\triangleq\frac{k_P\lambda}{1+k_\beta\frac{n}{n_l}}$. Furthermore, the coordination state x_i and its rate $\dot{x}_i$ satisfy

$$|x_i(t)-x_j(t)|\leq 2\left(1-\frac{1}{n}\right)^{\frac{1}{2}}\|\zeta(t)\|\tag{10}$$

$$|\dot{x}_i(t)-\rho|\leq(k_Pn+1)\|\zeta(t)\|\tag{11}$$

for all $i,j\in\{1,\ldots,n\}$.

Proof: To prove GUES, we need to construct a Lyapunov function candidate for the dynamics (6). As a first step towards it, we consider the system

$$\dot{\phi}(t)=-k_P\overline{L}(t)\phi(t)=-k_P\overline{L}_{\sigma(t)}\phi(t),\quad\phi(t)\in\mathbb{R}^{n-1}.\tag{12}$$

Let $t_1,t_2,\ldots$ be the time instants where switching occurs and p_m denote the value of $\sigma(t)$ on $[t_{m-1},t_m)$. Then, for any t satisfying $0=t_0<t_1<\ldots<t_k\leq t<t_{k+1}$,

$$\phi(t)=e^{-k_P\overline{L}_{p_{k+1}}(t-t_k)}e^{-k_P\overline{L}_{p_k}(t_k-t_{k-1})}\cdots e^{-k_P\overline{L}_{p_1}(t_1-0)}\phi(0).$$

From the inequality (8), $\|\phi(t)\|$ can be estimated by putting together the terms with marginally stable matrices and Hurwitz stable matrices respectively

$$\begin{aligned}\|\phi(t)\|&\leq\left(\prod_{m=1}^{k+1}e^{a_{p_m}k_P}\right)e^{-k_P\lambda_cT_c(t,0)}\|\phi(0)\|\\&\leq e^{ak_P(1+k)}e^{-k_P\lambda_cT_c(t,0)}\|\phi(0)\|\\&\leq e^{ak_P+ak_PN_{\sigma(t)}(t,0)-k_P\lambda_cT_c(t,0)}\|\phi(0)\|.\end{aligned}$$

Note that the exponent satisfies

$$\begin{aligned} &ak_P + ak_P N_{\sigma(t)}(t,0) - k_P\lambda_c T_c(t,0) \\ &\le ak_P + ak_P\left(N_0 + \frac{t}{\tau_a}\right) - k_P\lambda^*[T_u(t,0)+T_c(t,0)] \\ &\le ak_P + ak_P N_0 + k_P(\lambda^*-\lambda)t - k_P\lambda^* t \\ &= ak_P(N_0+1) - k_P\lambda t, \end{aligned}$$

where Assumptions 1 and 2 were used. Now, it follows that

$$\|\phi(t)\| \le k_\phi e^{-\lambda_\phi t}\|\phi(0)\|, \quad \forall t \ge 0,$$

where $k_\phi = e^{ak_P(N_0+1)}$ and $\lambda_\phi = k_P\lambda$. Thus, the system (12) is globally uniformly exponentially stable (GUES) with the convergence rate λ_ϕ. Since $\overline{L}(t)$ is continuous for almost all $t \ge 0$ and uniformly bounded ($\exists M > 0$ such that $\left\|\overline{L}(t)\right\|^2 \le max_{1\le j\le N}\left\|\overline{L}_j\right\|^2 = M$), and the system (12) is GUES, Theorem 4.12 in [20] implies that, for any constants c_3 and c_4 satisfying $0 < c_3 \le c_4$, there exists a continuous, piecewise differentiable positive definite matrix $P(t) = P^\top(t)$ such that

$$c_1\mathbb{I}_{n-1} \triangleq \frac{c_3}{2Mk_P}\mathbb{I}_{n-1} \le P(t) \le \frac{k_\phi c_4}{2\lambda_\phi}\mathbb{I}_{n-1} \triangleq c_2\mathbb{I}_{n-1} \tag{13}$$

$$\dot{P}(t) - k_P\overline{L}(t)^\top P(t) - k_P P(t)\overline{L}(t) \le -c_3\mathbb{I}_{n-1}. \tag{14}$$

Now, we are ready to construct a Lyapunov function candidate for the error dynamics (6) using the matrix $P(t)$ along with the following change of variables for the coordination error state $\zeta(t)$ in (6)

$$z(t) = S_\zeta\zeta(t) \triangleq \begin{bmatrix} \mathbb{I}_{n-1} & 0 \\ -\frac{k_I}{k_P}C^\top Q^\top & \mathbb{I}_{n-n_l} \end{bmatrix}\zeta(t). \tag{15}$$

Let us consider the Lyapunov function candidate

$$V(t) \triangleq z^\top P_c(t) z = z^\top \begin{bmatrix} P(t) & 0 \\ 0 & \frac{k_P^3}{k_I^3}(C^\top Q^\top QC)^{-1} \end{bmatrix} z.$$

The time derivative of V along the trajectories of the dynamics (6) is

$$\dot{V}(t) = z^\top \begin{bmatrix} P_{11}(t) & P_{12}(t) \\ P_{21}(t) & -2\frac{k_P^2}{k_I^2}\mathbb{I}_{n-1} \end{bmatrix} z,$$

where

$$\begin{aligned} P_{11}(t) &= \dot{P}(t) - k_P\overline{L}(t)^\top P(t) - k_P P(t)\overline{L}(t) + \frac{k_I}{k_P}QCC^\top Q^\top P(t) \\ &\quad + \frac{k_I}{k_P}P(t)QCC^\top Q^\top, \\ P_{12}(t) &= P_{21}^\top(t) = P(t)QC - \frac{k_P}{k_I}QC. \end{aligned}$$

Now, for any $k_\beta \ge 2$, define $\bar{\lambda}_c \triangleq \frac{k_P\lambda}{1+k_\beta\frac{n}{n_l}}$. Then, let us set

$$k_P > 0, \quad k_I = k_P\bar{\lambda}_c\frac{n}{n_l}k_\beta, \quad c_3 = c_4 = \frac{\lambda_\phi}{k_\phi\bar{\lambda}_c}\frac{2n_l}{k_\beta n}. \tag{16}$$

Considering $\|QC\| = 1$ and $\lambda_{min}(C^\top Q^\top QC) = \frac{n_l}{n}$, it can be shown that

$$\dot{V}(t) \le -2\bar{\lambda}_c z^\top \begin{bmatrix} P(t) & 0 \\ 0 & \frac{k_P^3}{k_I^3}(C^\top Q^\top QC)^{-1} \end{bmatrix} z$$

using inequalities (13) and (14) and Schur complements. By the comparison lemma (see [20], Lemma 3.4)

$$V(t) \le V(0)e^{-2\bar{\lambda}_c t}.$$

And since

$$min\left\{c_1, \frac{k_P^3}{k_I^3}\right\}\|z(t)\|^2 \le V(t) \le max\left\{c_2, \frac{k_P^3}{k_I^3}\frac{n}{n_l}\right\}\|z(t)\|^2,$$

we find that

$$\|z(t)\| \le \left(max\left\{c_2, \frac{k_P^3}{k_I^3}\frac{n}{n_l}\right\}/min\left\{c_1, \frac{k_P^3}{k_I^3}\right\}\right)^{\frac{1}{2}}\|z(0)\|e^{-\bar{\lambda}_c t}.$$

The similarity transformation from (15) yields

$$\|\zeta(t)\| \le \kappa_0\|\zeta(0)\|e^{-\bar{\lambda}_c t}, \tag{17}$$

where $\kappa_0 = \|S_\zeta^{-1}\|\left(max\left\{c_2, \frac{k_P^3}{k_I^3}\frac{n}{n_l}\right\}/min\left\{c_1, \frac{k_P^3}{k_I^3}\right\}\right)^{\frac{1}{2}}\|S_\zeta\|$. Consequently, the origin of the coordination error dynamics (6) is globally uniformly exponentially stable with a guaranteed rate of convergence $\bar{\lambda}_c$. The reader is referred to the proof of Lemma 3 in [6] for derivation of (10), (11) from (17). ∎

Note that the control gains k_P and k_I in the distributed protocol (4) are set to (16). Considering the integral control gain in (4) is set by $k_I = k_P\bar{\lambda}_c\frac{n}{n_l}k_\beta$ and the guaranteed rate of convergence in (9) is $\bar{\lambda}_c = k_P\lambda/(1+k_\beta\frac{n}{n_l})$, a large value of k_P makes the followers learn the desired mission rate ρ very fast and quickly achieve the consensus (2). In other words, the fleet of n vehicles will find their desired relative positions very soon after the mission unfolds and the mission end time will get closer to $\frac{t_d^*}{\rho}$.

As compared to the previous results on time-critical coordination [12], where it was assumed that the communication network is represented by a bidirectional graph with a symmetric Laplacian matrix, Theorem 1 extends the framework to balanced directed graphs that do not have symmetric Laplacian matrices.

C. Stochastic Convergence Analysis

In this section, we analyze the exponential consensus derived in Theorem 1 in the presence of a stochastic time-varying network.

To discuss switching between graphs containing rooted out-branching graphs and graphs not containing rooted out-branching graphs, we introduce the following definition.

Definition 3: At time t, the network $G(t)$ is said to be in state c, if $G(t)$ contains a rooted out-branching graph. In the other case, where $G(t)$ does not contain a rooted out-branching graph, it is said to be in state u.

With the above definition, switching in the network can be categorized into 4 cases: 1) c to c, where $G(t)$ switches from a digraph containing a rooted out-branching graph to another digraph containing one or to itself; 2) c to u, where $G(t)$ switches from a digraph containing a rooted out-branching graph to a digraph not containing one; and likewise, 3) u to u and 4) u to c can be defined.

It is assumed that the switching meets the following two stochastic assumptions; one describes when switching occurs and the other one does how a change in the topology of the network occurs.

Assumption 3: The number of topology switching over the time interval $[0,t)$, $N_{\sigma(t)}(t,0)$, follows a Poisson distribution with rate γt for $\forall t \geq 0$.

Assumption 4: At each switching instant, the network $G(t)$ changes its topology according to a Markov chain with the transition matrix $P = \begin{bmatrix} p_{cc} & p_{cu} \\ p_{uc} & p_{uu} \end{bmatrix}$, where $p_{uu} \neq 1$ and p_{ab} denotes transition probability from state a to b.

Theorem 2: Assume that the underlying communication network represented by a time-varying balanced digraph $G(t)$ satisfies Assumptions 3 and 4. Then, the probability $\mathbb{P}$ that (10) and (11) hold is lower bounded by

$$\mathbb{P} \geq \left(1 - \frac{p_{cu}}{1-p_{uu}+p_{cu}} \frac{\gamma t+1}{\gamma t}(1+\Lambda)\right) \sum_{k=0}^{\lfloor N_0+\frac{t}{\tau_a^*}\rfloor} \frac{(\gamma t)^k e^{-\gamma t}}{k!},$$

where $\Lambda = \frac{\lambda^*}{\lambda_c - \lambda^*}$ and $\lfloor N_0 + \frac{t}{\tau_a^*} \rfloor$ is the the greatest integer less than or equal to $N_0 + \frac{t}{\tau_a^*}$.

Proof: We prove the statement by calculating a lower bound for probability that both Assumptions 1 and 2 are satisfied under Assumptions 3 and 4.

Given the Poisson distribution, the probability that the Assumption 1 holds can be calculated by

$$\begin{aligned} P(\tau_a \geq \tau_a^*) &= P\left(N_{\sigma(t)}(t,0) \leq N_0 + \frac{t}{\tau_a^*}\right) \\ &= \sum_{k=0}^{\lfloor N_0+\frac{t}{\tau_a^*}\rfloor} \frac{(\gamma t)^k e^{-\gamma t}}{k!}. \end{aligned} \tag{18}$$

Now we proceed to find a lower bound for probability of the Assumption 2 being satisfied:

$$P(T_c(t,0) \geq T_u(t,0)\Lambda) = P\left(T_u(t,0) \leq \frac{t}{1+\Lambda}\right). \tag{19}$$

Here, note that $T_u(t,0)$ can be described as $T_u(t,0) = I_0 T_0 + \sum_{i=1}^{N_{\sigma(t)}(t,0)} I_i T_i$, where T_i is the length of the ith interval and I_i is an indicator function, which is equal to 1 when the balanced digraph does not contain a rooted out-branching graph during the ith interval, otherwise 0. Under an assumption that the random variables I_i, T_i, and $N_{\sigma(t)}(t,0)$ are independent from one another, its expected value can be calculated as

$$\begin{aligned} \mathbb{E}[T_u(t,0)] &= \mathbb{E}\left[I_0 T_0 + \sum_{i=1}^{N_{\sigma(t)}(t,0)} I_i T_i\right] \\ &= \mathbb{E}[I_0]\mathbb{E}[T_0] + \mathbb{E}[N_{\sigma(t)}(t,0)]\mathbb{E}[I_i]\mathbb{E}[T_i] \\ &= P(I_0 = 1)\mathbb{E}[T_0] + \mathbb{E}[N_{\sigma(t)}(t,0)]P(I_i = 1)\mathbb{E}[T_i] \\ &= P(I_i = 1)(\mathbb{E}[N_{\sigma(t)}(t,0)] + 1)\mathbb{E}[T_i] \\ &= p_u \frac{\gamma t + 1}{\gamma}, \end{aligned}$$

where $p_u = \frac{p_{cu}}{1-p_{uu}+p_{cu}} < 1$ (because $p_{uu} \neq 1$) is the stationary probability that the network is in state u.

Application of the Markov inequality to the random variable $T_u(t,0)$ leads to

$$P\left(T_u(t,0) \geq \frac{t}{1+\Lambda}\right) \leq \frac{\mathbb{E}[T_u(t,0)]}{\frac{t}{1+\Lambda}}.$$

Then, from its relation to (19), one has a lower bound for probability that Assumption 2 holds:

$$P(T_c(t,0) \geq T_u(t,0)\Lambda) \geq 1 - \frac{\mathbb{E}[T_u(t,0)]}{\frac{t}{1+\Lambda}}. \tag{20}$$

Thus, the probability $\mathbb{P}$ of Assumptions 1 and 2 being satisfied can be estimated by multiplying the right hand sides of (18) and (20):

$$\begin{aligned} \mathbb{P} &\geq \left(1 - \frac{\frac{p_{cu}}{1-p_{uu}+p_{cu}} \frac{\gamma t+1}{\gamma}}{\frac{t}{1+\Lambda}}\right) \sum_{k=0}^{\lfloor N_0+\frac{t}{\tau_a^*}\rfloor} \frac{(\gamma t)^k e^{-\gamma t}}{k!} \\ &= \left(1 - \frac{p_{cu}}{1-p_{uu}+p_{cu}} \frac{\gamma t+1}{\gamma t}(1+\Lambda)\right) \sum_{k=0}^{\lfloor N_0+\frac{t}{\tau_a^*}\rfloor} \frac{(\gamma t)^k e^{-\gamma t}}{k!}. \end{aligned} \tag{21}$$

Since the probability that Assumptions 1 and 2 are met is greater than the right hand side of (21), the probability that (10) and (11) hold is greater than that as well. ■

Notice that Theorem 2 holds with any set of parameters N_0, λ^*, λ, and t. One may increase the probability by adjusting these parameters, but it will affect the transient performance by changing the values of κ_0 and $\bar{\lambda}_c$ in (9).

V. Simulation

In this section, we demonstrate the performance of the protocol designed in (4) with the path following controller in Section II-D over the communication network represented by a stochastically time-varying balanced digraph. For a fleet of 3 vehicles, we elect vehicle 1 as a virtual leader and vehicles 2 and 3 as followers. They are tasked to follow concentric arcs and simultaneously arrive at their final destinations in 23 seconds. The communication network switches among all the possible balanced digraphs of 3 nodes given in Figure 2.

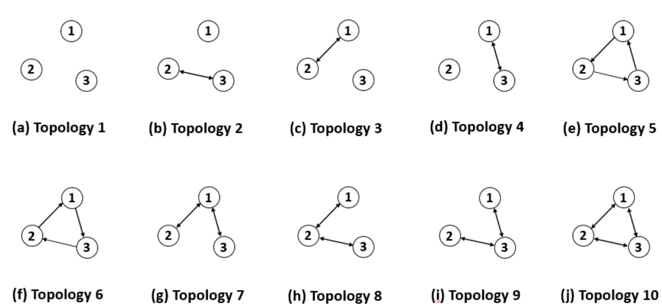
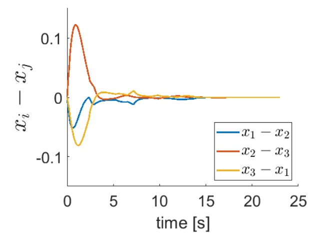


Fig. 2: Network topologies. The time-varying network assumes one of these graphs at any time t. The coordination information is transmitted in the direction of arrows.

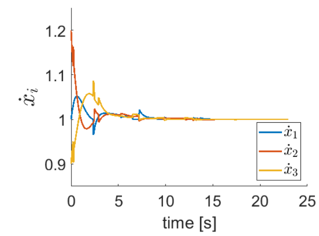
In this mission, the desired mission rate ρ is set to 1. The initial condition for x and χ are given by $x_0 = [0\ 0\ 0]^\top$ and $\chi_0 = [1\ 1]^\top$. The disturbance vector is $d = [0.2\ \ {-0.1}]^\top$. The coordination control gains are set to $k_P = 1$ and $k_I = 0.8571$. The rate parameter of the Poisson distribution is $\gamma = 1$ and the transition matrix is given by $P = \begin{bmatrix} 0.4 & 0.6 \\ 0.4 & 0.6 \end{bmatrix}$. At every switching instant, the network assumes one of the topologies in Figure 2 with probability 0.1.

Figures 3a and Figure 3b show that the control objective (2) was achieved: the state error $x_i - x_j$ converges to a neighborhood of 0 in Figure 3a and the rate of change of the state $\dot{x}_i$ gets close to the desired mission rate $\rho = 1$ in Figure 3b. The error $x_i - x_j$ and the rate $\dot{x}_i$ lie within the envelopes $|x_i - x_j| \leq 0.2720e^{-0.1429t}$ and $|\dot{x}_i - 1| \leq 0.6664e^{-0.1429t}$ given by Theorem 1, respectively. The probability of this event taking place is estimated to be 0.8274 by Theorem 2. Figure 3c shows that χ_2 and χ_3 converge to desired values and have the followers learn the desired mission rate $\rho = 1$, compensating for the disturbance $d = [0.2 \; -0.1]^\top$. Finally, Figure 3d shows the evolution of the communication network.

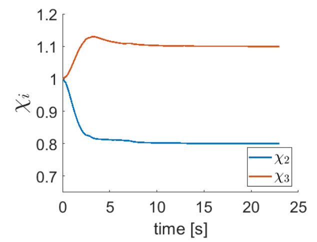
Simulation results for time-critical coordination and path following of the 3 vehicles are given in Figure 4. The vehicles whose travelled trajectories are denoted by solid curves follow their virtual targets (small circles) moving along the desired paths (dotted curves). As can be seen in Figure 4, as time goes by, initial offsets of the vehicles in position and heading angle converge to 0 by the path following algorithm. They arrive at their final destinations simultaneously at $t = 22.90s$ by the proposed distributed control protocol (4).



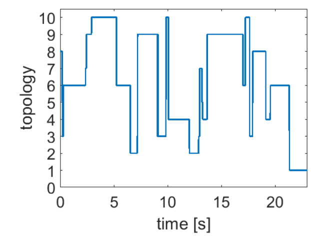
(a) Errors in coordination states.



(b) Coordination-state rates

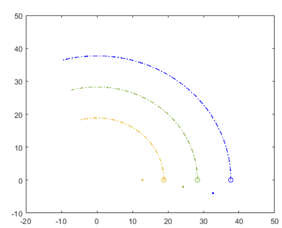


(c) Integrator states.

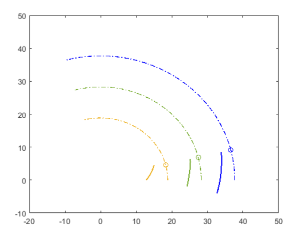


(d) Evolution of topology.

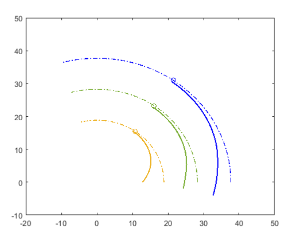
Fig. 3: Consensus in coordination states and coordination-state rates over a stochastic network.



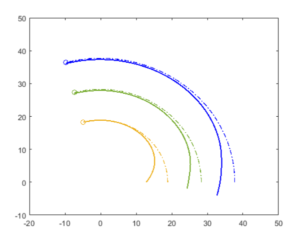
(a) $t = 0s$



(b) $t = 3s$



(c) $t = 12s$



(d) $t = 22.90s$

Fig. 4: Path following and time-critical coordination of 3 vehicles.

VI. Conclusion

In this paper, we studied time-critical coordination for a fleet of multiple vehicles over a time-varying network represented by a stochastically switching balanced digraph. It was shown that the consensus is achieved exponentially with a guaranteed convergence rate under the conditions on the connectivity and switching of the network. Finally, we estimated how likely the exponential consensus is achieved by calculating the probability of the conditions being met. The simulation results on the fleet of 3 vehicles demonstrated the performance of the proposed time-critical coordination strategy.

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APPENDIX

A. Closed-loop Coordination Error Dynamics

From the definition of ζ_1 and ζ_2, it follows that

$$\begin{aligned}\zeta_1 &= -k_P QL(t)x + Q\begin{bmatrix}\rho_{n-n_l}\\ \chi + d\end{bmatrix}\\ &= -k_P QL(t)x + Qx_n + Q\begin{bmatrix}0\\ \zeta_2\end{bmatrix}\\ &= -k_P QL(t)x + 0 + QC\zeta_2.\end{aligned}$$

Considering the matrix $\Pi \triangleq Q^\top Q = \mathbb{I}_n - \frac{1_n 1_n^\top}{n}$ in Lemma 3 satisfies $L(t)\Pi = L(t)$ for all digraphs, and $\Pi L(t) = L(t)$ for all balanced digraphs by Definition 2, one has

$$\begin{aligned}\dot{\zeta}_1 &= -k_P QL(t)\Pi x + QC\zeta_2\\ &= -k_P QL(t)Q^\top Qx + QC\zeta_2\\ &= -k_P \bar{L}(t)Qx + QC\zeta_2\\ &= -k_P \bar{L}(t)\zeta_1 + QC\zeta_2.\end{aligned} \tag{22}$$

Similarly, it follows that

$$\begin{aligned}\dot{\zeta}_2 &= -k_I C^\top L(t)x\\ &= -k_I C^\top \Pi L(t)\Pi x\\ &= -k_I C^\top Q^\top QL(t)Q^\top Qx\\ &= -k_I C^\top Q^\top \bar{L}(t)Qx\\ &= -k_I C^\top Q^\top \bar{L}(t)\zeta_1.\end{aligned} \tag{23}$$

Equations (22) and (23) lead to the dynamics (6).