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Multi-leader coordination algorithm for networks with switching topology and quantized information[☆]

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ABSTRACT

This paper analyzes the stability and convergence properties of a proportional–integral protocol for coordination of a network of agents with dynamic information flow and quantized information exchange. In the setup adopted, each agent is only required to exchange its coordination state with its neighboring agents, and the desired reference rate is only available to a group of leaders. We show that the integral term of the protocol allows the agents to learn the reference rate, rather than have it available a priori, and also provides disturbance rejection capabilities. The paper addresses the case where the graph that captures the underlying network topology is not connected during some interval of time or even fails to be connected at all times.

1. Introduction

Worldwide, there has been growing interest in the use of autonomous vehicles to execute complex cooperative missions with limited involvement of human operators. A key enabling element for the successful execution of such missions is the availability of advanced algorithms for motion control of autonomous vehicles. In Xargay et al. (2013), for example, the authors address the development of strategies for cooperative missions in which a fleet of unmanned aerial vehicles is required to follow collision-free paths and arrive at their respective final destinations at the same time. To enforce the temporal constraints of the mission, the coordination algorithm relies on a distributed protocol (first introduced in Kaminer, Yakimenko, Pascoal, and Ghabcheloo (2006)) that has a proportional–integral (PI) structure in which each agent is only required to exchange its coordination state with its neighbors, and the constant *time*-reference rate is only available to a single leader. The integral term in the consensus algorithm allows the follower vehicles to learn the reference rate from the leader, and also provides disturbance rejection capabilities against steady winds.

A generalization of this PI protocol was proposed in Bai, Arcak, and Wen (2008), where the authors developed an adaptive algorithm to reconstruct a time-varying reference velocity that is available only to a single leader. The paper used a passivity framework to show that a network of nonlinear agents with fixed connected topology asymptotically achieves coordination. The work in Carli, Chiuso, Schenato, and Zampieri (2008) also used a (discrete-time) PI consensus protocol to synchronize networks of clocks with fixed connected topology. In this application, the integral part of the controller was critical to eliminate the different initial clock offsets. A PI estimation algorithm was also proposed in Freeman, Yang, and Lynch (2006) for dynamic average consensus in sensing networks. In particular, the paper analyzed the stability and convergence properties of the developed PI estimator, by deriving conditions on both constant and time-varying information flows that ensure stability of the estimator.

Motivated by the work in Xargay et al. (2013), the present paper modifies the PI protocol proposed in Kaminer et al. (2006) to include *multiple leaders*, and analyzes the convergence properties of the protocol for coordination of a network of agents with *dynamic information flow* and *quantized information exchange*, the latter being a topic that has received increased attention in recent years (Censi & Murray, 2009; Ceragioli, De Persis, & Frasca, 2011; De Persis, 2011; Kashyap, Başar, & Srikant, 2007; Nedić, Olshevsky, Ozdaglar, & Tsitsiklis, 2009; Yu, LaValle, & Liberzon, 2012). On one hand, the use of multiple leaders improves the robustness of the network to a single-point failure (Ren & Beard, 2008). On the other hand, the exchange of information over networks with finite-rate

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communication links motivates the interest in quantized consensus problems. The main contribution of this paper is twofold. First, we present lower bounds on the convergence rate of the collective dynamics as a function of the number of leaders and the *quality of service* (QoS) of the network, which in the context of this work represents a measure of the level of connectivity of the dynamic graph that captures the underlying network topology. In particular, we explicitly address the case where this graph is only connected in an integral sense, not necessarily pointwise in time. And second, we analyze the effect of quantization on the closed-loop collective dynamics. We note that some of the results in this paper were presented, without proofs, in Xargay, Choe, Hovakimyan, and Kaminer (2012).

The paper is organized as follows. Section 2 describes the problem formulation. Section 3 presents the (unquantized) protocol adopted in this paper and analyzes its convergence properties. In Section 4, we study the collective dynamics under quantization. Simulation results illustrating the theoretical findings are presented in Section 5, while Section 6 summarizes concluding remarks.

2. Problem formulation

Consider a network of *n integrator-agents*

$$\dot{x}_i = u_i, \quad i \in \mathcal{I}_\ell := \{1, \dots, n_\ell\}, \tag{1a}$$

$$\dot{x}_i = u_i + d_i, \quad i \in \mathcal{I}_f := \{n_\ell + 1, \dots, n\}, \tag{1b}$$

with *dynamic information flow* $\mathcal{G}(t) := (\mathcal{V}, \mathcal{E}(t))$. In the above formulation, $x_i(t) \in \mathbb{R}$ is the *coordination state* of the ith agent, $u_i(t) \in \mathbb{R}$ is its control input, and $d_i \in \mathbb{R}$ is an unknown constant disturbance.

The control objective is to design a *distributed protocol* that solves the *coordination problem*

$$x_i(t) - x_j(t) \xrightarrow{t\to\infty} 0, \quad \forall i, j \in \{1, \dots, n\}, \tag{2a}$$

$$\dot{x}_i(t) \xrightarrow{t\to\infty} \rho, \quad \forall i \in \{1, \dots, n\}, \tag{2b}$$

where ρ is the desired (constant) reference rate, with a guaranteed *rate of convergence*.

The network and the communications between agents satisfy the following assumptions:

Assumption 1. The ith agent can only exchange information with a set of neighboring agents $\mathcal{N}_i(t)$.

Assumption 2. Communications between two agents are bidirectional ($\mathcal{G}(t)$ is undirected) and the information is transmitted continuously with no delays.

Assumption 3. The connectivity of graph $\mathcal{G}(t)$ satisfies the persistency of excitation (PE)-like condition (Arcak, 2007)

$$\frac{1}{n}\frac{1}{T}\int_t^{t+T} \boldsymbol{Q}\boldsymbol{L}(\tau)\boldsymbol{Q}^\top d\tau \geq \mu \mathbb{I}_{\boldsymbol{n-1}}, \quad \forall t \geq 0, \tag{3}$$

where $\boldsymbol{L}(t) \in \mathbb{R}^{n\times n}$ is the piecewise-constant Laplacian of graph $\mathcal{G}(t)$, and $\boldsymbol{Q}$ is any $(n-1) \times n$ matrix satisfying $\boldsymbol{Q}\boldsymbol{1_n} = \boldsymbol{0}$ and $\boldsymbol{Q}\boldsymbol{Q}^\top = \mathbb{I}_{\boldsymbol{n-1}}$, with $\boldsymbol{1_n}$ being the vector in $\mathbb{R}^n$ whose components are all 1. Parameters $T > 0$ and $\mu \in (0, 1]$ characterize the QoS of the communications network, which in the context of this paper represents a measure of the level of connectivity of graph $\mathcal{G}(t)$.

Remark 4. The network dynamics (1) assume the presence of n_ℓ agents with disturbance-free dynamics. These agents can be *virtual agents*, implemented on n_ℓ nodes as part of the protocol proposed in the present paper. In Xargay et al. (2012), we analyze the effect of adding virtual agents to a given network, and provide a lower bound on the connectivity of the resulting extended network. The paper also shows that these agents with disturbance-free dynamics are critical to effectively solve the coordination problem (2).

Remark 5. Condition (3) requires the graph $\mathcal{G}(t)$ to be connected in an integral sense, not pointwise in time. In fact, the graph may be disconnected during some interval of time or may even fail to be connected at all times. A similar type of condition can be found in Lin, Francis, and Maggiore (2007).

3. Distributed consensus protocol

3.1. Proportional–integral protocol

To solve the coordination problem (2), we adopt the following distributed protocol:

$$\begin{aligned} u_i &= -k_P \sum_{j\in\mathcal{N}_i} (x_i - x_j) + \rho, \quad i \in \mathcal{I}_\ell, \\ u_i &= -k_P \sum_{j\in\mathcal{N}_i} (x_i - x_j) + \chi_i \\ \dot{\chi}_i &= -k_I \sum_{j\in\mathcal{N}_i} (x_i - x_j), \qquad i \in \mathcal{I}_f, \end{aligned} \tag{4}$$

where vehicles 1 through n_ℓ are elected as (virtual) leaders, and $k_P > 0$ and $k_I > 0$ are coordination control gains. This protocol has a PI structure in which each agent is only required to exchange its coordination state $x_i(t)$ with its neighbors, and the reference rate ρ is only available to the n_ℓ leaders. Also, note that the leaders adjust their dynamics according to information exchanged with their neighboring agents, rather than running as isolated agents.

Protocol (4) can be rewritten in compact form as

$$\begin{aligned} \boldsymbol{u} &= -k_P \boldsymbol{L}(t)\boldsymbol{x} + \begin{bmatrix} \rho \boldsymbol{1}_{\boldsymbol{n_\ell}} \\ \boldsymbol{\chi} \end{bmatrix}, \\ \dot{\boldsymbol{\chi}} &= -k_I \boldsymbol{C}^\top \boldsymbol{L}(t)\boldsymbol{x}, \end{aligned} \tag{5}$$

where $\boldsymbol{u}(t)$, $\boldsymbol{x}(t)$, and $\boldsymbol{\chi}(t)$ are defined as

$$\begin{aligned} \boldsymbol{u}(t) &:= [u_1(t), \dots, u_n(t)]^\top \in \mathbb{R}^n, \\ \boldsymbol{x}(t) &:= [x_1(t), \dots, x_n(t)]^\top \in \mathbb{R}^n, \\ \boldsymbol{\chi}(t) &:= [\chi_{n_\ell+1}(t), \dots, \chi_n(t)]^\top \in \mathbb{R}^{n-n_\ell}, \end{aligned}$$

and $\boldsymbol{C}^\top := [\boldsymbol{0} \quad \mathbb{I}_{\boldsymbol{n-n_\ell}}] \in \mathbb{R}^{(n-n_\ell)\times n}$.

3.2. Collective dynamics and convergence analysis

Protocol (4) leads to the *closed-loop collective dynamics*

$$\begin{aligned} \dot{\boldsymbol{x}} &= -k_P \boldsymbol{L}(t)\boldsymbol{x} + \begin{bmatrix} \rho \boldsymbol{1}_{\boldsymbol{n_\ell}} \\ \boldsymbol{\chi} + \boldsymbol{d} \end{bmatrix}, \\ \dot{\boldsymbol{\chi}} &= -k_I \boldsymbol{C}^\top \boldsymbol{L}(t)\boldsymbol{x}, \end{aligned} \tag{6}$$

where $\boldsymbol{d} := [d_{n_\ell+1}, \dots, d_n]^\top \in \mathbb{R}^{n-n_\ell}$ is the (constant) disturbance vector. Note that the solutions (in the sense of Carathéodory (Hájek, 1979)) of the collective dynamics above exist and are unique, since $\boldsymbol{L}(t)$ is piecewise constant in t.

To analyze the convergence properties of protocol (5), we reformulate the coordination problem (2) into a stabilization problem. To this end, we define the *projection matrix* $\boldsymbol{\Pi} := \mathbb{I}_{\boldsymbol{n}} - (1/n)\boldsymbol{1_n}\boldsymbol{1_n}^\top \in \mathbb{R}^{n\times n}$, and note that this matrix satisfies $\boldsymbol{\Pi} = \boldsymbol{\Pi}^\top = \boldsymbol{\Pi}^2$, $\boldsymbol{Q}^\top\boldsymbol{Q} = \boldsymbol{\Pi}$, and $\boldsymbol{L}(t)\boldsymbol{\Pi} = \boldsymbol{\Pi}\boldsymbol{L}(t) = \boldsymbol{L}(t)$. Moreover, the spectrum of the matrix $\bar{\boldsymbol{L}}(t) := \boldsymbol{Q}\boldsymbol{L}(t)\boldsymbol{Q}^\top \in \mathbb{R}^{(n-1)\times(n-1)}$ is equal to the spectrum of $\boldsymbol{L}(t)$ without the eigenvalue $\lambda_1 = 0$ corresponding to the eigenvector $\mathbf{1_n}$. Finally, we define the *coordination error state* $\boldsymbol{\zeta}(t) := [\boldsymbol{\zeta_1}^\top(t), \boldsymbol{\zeta_2}^\top(t)]^\top$ as

$$\boldsymbol{\zeta_1}(t) := \boldsymbol{Q}\boldsymbol{x}(t) \ \in \mathbb{R}^{n-1}, \tag{7a}$$

$$\boldsymbol{\zeta_2}(t) := \boldsymbol{\chi}(t) - \rho \mathbf{1}_{\boldsymbol{n}-\boldsymbol{n}_\ell} + \boldsymbol{d} \ \in \mathbb{R}^{n-n_\ell}. \tag{7b}$$

Note that, by definition, $\boldsymbol{\zeta}(t) = \mathbf{0}$ is equivalent to $\boldsymbol{x}(t) \in \operatorname{span}\{\mathbf{1_n}\}$ and $\dot{\boldsymbol{x}}(t) = \rho\mathbf{1_n}$. With the above notation, the collective dynamics can be reformulated as (Xargay et al., 2012)

$$\dot{\boldsymbol{\zeta}} = \boldsymbol{A}_\zeta(t)\boldsymbol{\zeta}, \qquad \boldsymbol{A}_\zeta(t) := \begin{bmatrix} -k_P\bar{\boldsymbol{L}}(t) & \boldsymbol{Q}\,\boldsymbol{C} \\ -k_I\boldsymbol{C}^\top\boldsymbol{Q}^\top\bar{\boldsymbol{L}}(t) & \mathbf{0}\end{bmatrix}. \tag{8}$$

Next we show that, if Assumption 3 holds, then protocol (5) solves the consensus problem (2).

Theorem 6. *Assume that the information flow $\mathcal{G}(t)$ verifies the PE-like condition (3) for some parameters $\mu, T > 0$. Then, there exist control gains k_P and k_I such that the origin of the collective dynamics (8) is exponentially stable with guaranteed rate of convergence*

$$\bar{\lambda}_c := \frac{k_P n\mu}{(1+k_P nT)^2}\left(1+\beta_k\frac{n}{n_\ell}\right)^{-1}, \quad \beta_k \geq 2. \tag{9}$$

Furthermore, the following bounds hold:

$$|x_i - x_j| \leq \kappa_{x0}\|\boldsymbol{\zeta}(0)\|\mathrm{e}^{-\bar{\lambda}_c t}, \quad \forall i,j \in \{1,\ldots,n\}, \tag{10a}$$

$$|\dot{x}_i - \rho| \leq \kappa_{\dot{x}0}\|\boldsymbol{\zeta}(0)\|\mathrm{e}^{-\bar{\lambda}_c t}, \quad \forall\, i \in \{1,\ldots,n\}, \tag{10b}$$

for some positive constants $\kappa_{x0}, \kappa_{\dot{x}0} \in (0,\infty)$.

Proof. The proof is given in Appendix A.

Remark 7. The proof of Theorem 6 is constructive and explicitly specifies a particular choice for the control gains k_P and k_I that ensures exponential stability of the collective dynamics; see Eq. (A.7) in Appendix A. According to the proof of the theorem, the admissible control gains lie in the following intervals:

$$k_P > 0, \quad \frac{k_P n\mu}{(1+k_P nT)^2}\,\frac{2\frac{n}{n_\ell}}{1+2\frac{n}{n_\ell}} \leq \frac{k_I}{k_P} < \frac{k_P n\mu}{(1+k_P nT)^2}.$$

Remark 8. Theorem 6 indicates that the guaranteed rate of convergence of the collective dynamics is limited by the QoS of the network. According to the theorem, for a given QoS, the maximal guaranteed convergence rate $\bar{\lambda}_c^*$ is achieved by setting $k_P = \frac{1}{Tn}$, which results in

$$\bar{\lambda}_c^* := \frac{\mu}{4T}\left(1+\beta_k\frac{n}{n_\ell}\right)^{-1}, \quad \beta_k \geq 2.$$

Note that $\bar{\lambda}_c^*$ scales with the ratio (n_ℓ/n). Note also that, as T goes to zero (and the graph becomes connected pointwise in time), the convergence rate can be set arbitrarily high by increasing the gains k_P and k_I.

Remark 9. The formulation above assumes the presence of constant disturbances in the agents' dynamics. In the case of slowly-varying bounded disturbances $d_i(t)$, protocol (4) will lead to *ultimate boundedness* of the coordination error state, rather than exponential stability.

4. Convergence under quantization

In this section we analyze the effect of quantization on the stability and convergence properties of the collective dynamics. For the sake of simplicity, we consider only the case of *uniform quantization* with step size Δ.

4.1. Protocol and collective dynamics

When only quantized information from other agents is available, protocol (4) becomes

$$\begin{aligned} \boldsymbol{u} &= -k_P\left(\boldsymbol{D}(t)\boldsymbol{x} - \boldsymbol{A}(t)\,\mathbf{q}(\boldsymbol{x})\right) + \begin{bmatrix}\rho\mathbf{1}_{\boldsymbol{n}_\ell} \\ \boldsymbol{\chi}\end{bmatrix}, \\ \dot{\boldsymbol{\chi}} &= -k_I\boldsymbol{C}^\top\left(\boldsymbol{D}(t)\boldsymbol{x} - \boldsymbol{A}(t)\,\mathbf{q}(\boldsymbol{x})\right), \end{aligned} \tag{11}$$

where the time-varying matrices $\boldsymbol{D}(t)$ and $\boldsymbol{A}(t)$ are, respectively, the *degree* and *adjacency matrices* of $\boldsymbol{L}(t)$, while $\mathbf{q}(\boldsymbol{x}(t)) := [\mathrm{q}_\Delta(x_1(t)), \ldots, \mathrm{q}_\Delta(x_n(t))]^\top \in \mathbb{Z}^n\Delta$ is the quantized coordination state, with $\mathrm{q}_\Delta(\cdot) : \mathbb{R} \to \mathbb{Z}\Delta$ being defined as

$$\mathrm{q}_\Delta(\xi) := \operatorname{sgn}(\xi)\Delta\left\lfloor \frac{|\xi|}{\Delta} + \frac{1}{2}\right\rfloor, \quad \xi \in \mathbb{R}.$$

In the definition above, $\lfloor\cdot\rfloor$ denotes the *floor function*. Protocol (11) assumes that only information exchanged over the network is subject to quantization and, therefore, each agent uses its own unquantized state.

The closed-loop collective dynamics can be written as

$$\begin{aligned} \dot{\boldsymbol{x}} &= -k_P\left(\boldsymbol{D}(t)\boldsymbol{x} - \boldsymbol{A}(t)\,\mathbf{q}(\boldsymbol{x})\right) + \begin{bmatrix}\rho\mathbf{1}_{\boldsymbol{n}_\ell} \\ \boldsymbol{\chi}+\boldsymbol{d}\end{bmatrix}, \\ \dot{\boldsymbol{\chi}} &= -k_I\boldsymbol{C}^\top\left(\boldsymbol{D}(t)\boldsymbol{x} - \boldsymbol{A}(t)\,\mathbf{q}(\boldsymbol{x})\right). \end{aligned} \tag{12}$$

In terms of the coordination error state $\boldsymbol{\zeta}(t)$, the collective dynamics can be expressed as:

$$\dot{\boldsymbol{\zeta}} = \boldsymbol{A}_\zeta(t)\boldsymbol{\zeta} + \boldsymbol{f}_{\mathbf{q}}, \tag{13}$$

where $\boldsymbol{A}_\zeta(t)$ was defined in (8), and $\boldsymbol{f}_{\mathbf{q}}(t)$ is given by $\boldsymbol{f}_{\mathbf{q}}(t) := \begin{bmatrix} k_P\,\boldsymbol{Q}\boldsymbol{A}(t)\,\boldsymbol{e_x}(t) \\ k_I\,\boldsymbol{C}^\top\boldsymbol{A}(t)\,\boldsymbol{e_x}(t)\end{bmatrix}$, with $\boldsymbol{e_x}(t) := \mathbf{q}(\boldsymbol{x}(t)) - \boldsymbol{x}(t)$ being the *quantization error vector*.

Note that, in this case, the right-hand side of the collective error dynamics (13) is discontinuous not only due to the switching network topology, but also due to the presence of quantized states. As proven in Ceragioli et al. (2011), Carathéodory solutions might not exist for quantized consensus problems, implying that a weaker concept of solution has to be considered. Similar to Ceragioli et al. (2011), we will consider solutions *in the sense of Krasovskii* (Hájek, 1979), which we define next.

Definition 10 (*Krasovskii Solution (Hájek, 1979)*)**.** Let $\boldsymbol{\xi} : J \to \mathbb{R}^n$ (J an interval in $\mathbb{R}$) be absolutely continuous on each compact subinterval of J. Then, $\boldsymbol{\xi}$ is called a *Krasovskii solution* of the vector differential equation $\dot{\boldsymbol{\xi}} = \boldsymbol{f}(t,\boldsymbol{\xi})$ if

$$\dot{\boldsymbol{\xi}}(t) \in \mathrm{K}\left(\boldsymbol{f}(t,\boldsymbol{\xi}(t))\right) \quad \text{almost everywhere in } J,$$

where the operator $\mathrm{K}(\cdot)$ is defined as

$$\mathrm{K}\left(\boldsymbol{f}(t,\boldsymbol{\xi})\right) := \bigcap_{\epsilon>0}\overline{\mathrm{co}}\,\boldsymbol{f}(t,\boldsymbol{\xi}+\epsilon\mathcal{B}),$$

with $\mathcal{B}$ being the open unit ball in $\mathbb{R}^n$.

To show that Krasovskii solutions of (13) exist (at least) locally, we note that, during continuous evolution of the system between "quantization jumps", the quantized collective error dynamics (13) are linear, with the quantized state $\mathbf{q}(\boldsymbol{x}(t))$ acting as a bounded exogenous input. This implies that the solutions $\boldsymbol{x}(t)$ are locally bounded (no *finite escape time* occurs). Then, local existence of Krasovskii solutions is guaranteed by the fact that the right-hand side of (13) is measurable and locally bounded (Hájek, 1979). At this point, however, we cannot claim that Krasovskii solutions to (13) are complete; for this, we will need to prove that solutions are bounded.

4.2. (Krasovskii) equilibria

In this section we analyze the existence of equilibria for the quantized collective error dynamics (13). In particular, we show that (i) unlike the unquantized case, $\boldsymbol{\zeta}_{\mathbf{eq}} = \mathbf{0}$ is not an equilibrium point; and (ii) other (undesirable) equilibria might exist, depending on the step size of the quantizers. The former follows easily from the error dynamics (13) and the definition of the coordination error state $\boldsymbol{\zeta}(t)$. The proof of the latter result is, instead, more involved, and we only show it for the case of static and connected topologies.

To this effect, we start by noting that $\dot{\boldsymbol{\zeta}}(t) \equiv \mathbf{0}$ is equivalent to $\dot{\boldsymbol{x}}(t) \in \operatorname{span}\{\mathbf{1_n}\}$ and $\dot{\boldsymbol{\chi}}(t) \equiv \mathbf{0}$ holding simultaneously. Hence, $\boldsymbol{\zeta}_{\mathbf{eq}} := [\boldsymbol{\zeta}_{\mathbf{1eq}}^\top, \boldsymbol{\zeta}_{\mathbf{2eq}}^\top]^\top$ is an equilibrium of (13) if the following inclusions hold for all $t \geq 0$:

$$\gamma(t)\mathbf{1_n} \in \mathrm{K}\left(-k_P\left(\boldsymbol{D}\boldsymbol{x}_{\mathbf{eq}}(t) - \boldsymbol{A}\,\mathbf{q}(\boldsymbol{x}_{\mathbf{eq}}(t))\right) + \begin{bmatrix}\rho\mathbf{1}_{\boldsymbol{n_\ell}}\\ \boldsymbol{\chi}_{\mathbf{eq}} + \boldsymbol{d}\end{bmatrix}\right),$$

$$\mathbf{0} \in \mathrm{K}\left(-k_I\boldsymbol{C}^\top\left(\boldsymbol{D}\boldsymbol{x}_{\mathbf{eq}}(t) - \boldsymbol{A}\,\mathbf{q}(\boldsymbol{x}_{\mathbf{eq}}(t))\right)\right),$$

where $\gamma(t) \in \mathbb{R}$ is an arbitrary signal; $\boldsymbol{x}_{\mathbf{eq}}(t)$ is a continuous coordination-state trajectory satisfying $\boldsymbol{\zeta}_{\mathbf{1eq}} = \boldsymbol{Q}\boldsymbol{x}_{\mathbf{eq}}(t)$; while $\boldsymbol{\chi}_{\mathbf{eq}} = \boldsymbol{\zeta}_{\mathbf{2eq}} + \rho\mathbf{1}_{\boldsymbol{n}-\boldsymbol{n_\ell}} - \boldsymbol{d}$. The second inclusion above and continuity of $\boldsymbol{x}_{\mathbf{eq}}(t)$, along with the fact that the network is assumed to be static and connected, preclude the existence of equilibria involving time-varying coordination-state trajectories, i.e. $\gamma(t) \equiv 0$ (or equivalently $\dot{\boldsymbol{x}}_{\mathbf{eq}}(t) \equiv 0$). Then, the set of (Krasovskii) equilibria of (13) can be defined as:

$$\Theta := \left\{(\boldsymbol{\zeta}_{\mathbf{1eq}}, \boldsymbol{\zeta}_{\mathbf{2eq}}) \in \mathbb{R}^{n-1}\times\mathbb{R}^{n-n_\ell} \;:\; \boldsymbol{\zeta}_{\mathbf{1eq}} = \boldsymbol{Q}\boldsymbol{x}_{\mathbf{eq}},\right.$$
$$\left.\mathbf{0} \in \mathrm{K}\left(\begin{bmatrix}-k_P\left(\boldsymbol{D}\boldsymbol{x}_{\mathbf{eq}} - \boldsymbol{A}\,\mathbf{q}(\boldsymbol{x}_{\mathbf{eq}})\right) + \rho\mathbf{1_n} + \begin{bmatrix}\mathbf{0}\\ \boldsymbol{\zeta}_{\mathbf{2eq}}\end{bmatrix}\\ -k_I\boldsymbol{C}^\top\left(\boldsymbol{D}\boldsymbol{x}_{\mathbf{eq}} - \boldsymbol{A}\,\mathbf{q}(\boldsymbol{x}_{\mathbf{eq}})\right)\end{bmatrix}\right)\right\}.$$

Next we show that, under sufficiently fine quantization, the set Θ is empty. We also prove the existence of (undesirable) equilibria under coarse quantization.

Lemma 11. *Assume that the network topology is static and connected. If the step size of the quantizers satisfies*

$$\Delta < \frac{2n_\ell}{(3n-2n_\ell)(n-1)}\,\frac{|\rho|}{k_P}, \tag{14}$$

then the set of equilibria Θ is empty.

Proof. The proof is given in Appendix B.

The next corollary follows from the proof of Lemma 11.

Corollary 12. *Assume that the network topology is static and connected. If the following inequality holds:*

$$\left\|\frac{\rho}{k_P}\boldsymbol{D}^{-1}\begin{bmatrix}\mathbf{1}_{\boldsymbol{n_\ell}}\\ \mathbf{0}\end{bmatrix}\right\|_\infty < \frac{\Delta}{2}, \tag{15}$$

then, for any $k \in \mathbb{Z}$, the point

$$(\hat{\boldsymbol{x}}, \hat{\boldsymbol{\chi}}) = \left(k\Delta\mathbf{1_n} + \frac{\rho}{k_P}\boldsymbol{D}^{-1}\begin{bmatrix}\mathbf{1}_{\boldsymbol{n_\ell}}\\ \mathbf{0}\end{bmatrix}, -\boldsymbol{d}\right) \tag{16}$$

is a "zero-speed" equilibrium point of system (12).

Lemma 13. *Consider the collective dynamics (12), and assume that the network topology is static and connected. Further, assume that the step size of the quantizers satisfies inequality (15). Then, the "zero-speed" equilibrium points defined in (16) are locally asymptotically stable.*

Proof. The proof is given in Appendix C.

Remark 14. The equilibrium points characterized by (16) correspond to solutions in which the agents have "zero speed" (i.e. $\dot{\boldsymbol{x}}(t) \equiv \mathbf{0}$) and, therefore, do not satisfy the control objective (2). Unfortunately, as shown in Lemma 13, these equilibrium points are asymptotically stable, which implies that the quantizers are to be designed to preclude the existence of such equilibria. The bound in (14) should thus be understood as a design constraint for the quantizers that prevents the existence of such undesirable equilibria when using protocol (11).

4.3. Convergence analysis

Next we show that, if the connectivity of graph $\mathcal{G}(t)$ verifies the PE-like condition (3), then protocol (11) solves the coordination problem (2) in a *practical sense* and, in addition, the coordination error vector degrades gracefully with the value of the quantizer step size.

Theorem 15. *Consider the collective error dynamics (13) and suppose that the information flow satisfies the PE-like condition (3) for some parameters $\mu, T > 0$. Then, there exist control gains k_P and k_I ensuring that there is a finite time $T_b \geq 0$ such that the solution of the quantized collective error dynamics (13) satisfies*

$$\|\boldsymbol{\zeta}(t)\| \leq \kappa_{\zeta 0}\|\boldsymbol{\zeta}(0)\|\mathrm{e}^{-\lambda_c t}, \quad \forall\, 0 \leq t < T_b, \tag{17a}$$

$$\|\boldsymbol{\zeta}(t)\| \leq \kappa_{\zeta 1}\Delta \quad \forall\, t \geq T_b, \tag{17b}$$

for some positive constants $\kappa_{\zeta 0}, \kappa_{\zeta 1} \in (0, \infty)$, and with $\lambda_c := \bar{\lambda}_c(1-\theta_\lambda)$, where $\bar{\lambda}_c$ was defined in (9) and θ_λ is a positive constant verifying $0 < \theta_\lambda < 1$.

Proof. The proof is given in Appendix D.

4.4. Coordination with fully quantized information

Next we propose a modification of protocol (11) that retains $\boldsymbol{\zeta}_{\mathbf{eq}} = \mathbf{0}$ as an equilibrium point of the quantized collective error dynamics. We also show that, for the case of connected network topologies and sufficiently fine quantization, $\boldsymbol{\zeta}_{\mathbf{eq}} = \mathbf{0}$ is the only equilibrium point.

To this effect, consider the following protocol:

$$\begin{aligned}\boldsymbol{u} &= -k_P\boldsymbol{L}(t)\,\mathbf{q}(\boldsymbol{x}) + \begin{bmatrix}\rho\mathbf{1}_{\boldsymbol{n_\ell}}\\ \boldsymbol{\chi}\end{bmatrix},\\ \dot{\boldsymbol{\chi}} &= -k_I\boldsymbol{C}^\top\boldsymbol{L}(t)\,\mathbf{q}(\boldsymbol{x}),\end{aligned} \tag{18}$$

which, unlike protocol (11), uses only quantized information. The collective dynamics can now be written as

$$\begin{aligned}\dot{\boldsymbol{x}} &= -k_P\boldsymbol{L}(t)\,\mathbf{q}(\boldsymbol{x}) + \begin{bmatrix}\rho\mathbf{1}_{\boldsymbol{n_\ell}}\\ \boldsymbol{\chi} + \boldsymbol{d}\end{bmatrix},\\ \dot{\boldsymbol{\chi}} &= -k_I\boldsymbol{C}^\top\boldsymbol{L}(t)\,\mathbf{q}(\boldsymbol{x}),\end{aligned} \tag{19}$$

leading to the quantized collective error dynamics

$$\dot{\boldsymbol{\zeta}} = \boldsymbol{A}_\zeta(t)\boldsymbol{\zeta} + \boldsymbol{f}'_{\mathbf{q}}, \tag{20}$$

where $\boldsymbol{A}_\zeta(t)$ was defined in (8) and $\boldsymbol{f}'_{\mathbf{q}}(t)$ is given by $\boldsymbol{f}'_{\mathbf{q}}(t) := \begin{bmatrix}k_P\,\boldsymbol{Q}\boldsymbol{L}(t)\,\boldsymbol{e_x}(t)\\ k_I\,\boldsymbol{C}^\top\boldsymbol{L}(t)\,\boldsymbol{e_x}(t)\end{bmatrix}$. In this case, it can be shown that the set of (Krasovskii) equilibria of (20) is characterized by:

$$\Theta' := \left\{(\boldsymbol{\zeta}_{\mathbf{1eq}}, \boldsymbol{\zeta}_{\mathbf{2eq}}) \in \mathbb{R}^{n-1}\times\mathbb{R}^{n-n_\ell} \;:\; \boldsymbol{\zeta}_{\mathbf{1eq}} = \boldsymbol{Q}\boldsymbol{x}_{\mathbf{eq}}(t),\right.$$
$$\left.\begin{bmatrix}\gamma(t)\mathbf{1_n}\\ \mathbf{0}\end{bmatrix} \in \mathrm{K}\left(\begin{bmatrix}-k_P\boldsymbol{L}(t)\,\mathbf{q}(\boldsymbol{x}_{\mathbf{eq}}(t)) + \rho\mathbf{1_n} + \begin{bmatrix}\mathbf{0}\\ \boldsymbol{\zeta}_{\mathbf{2eq}}\end{bmatrix}\\ -k_I\boldsymbol{C}^\top\boldsymbol{L}(t)\,\mathbf{q}(\boldsymbol{x}_{\mathbf{eq}}(t))\end{bmatrix}\right)\right\},$$

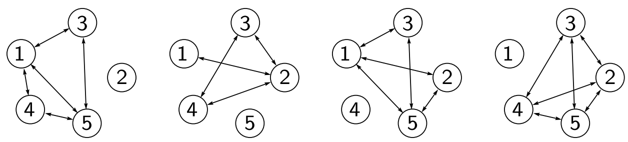


Fig. 1. Network topologies.

where $\gamma(t) \in \mathbb{R}$ is an arbitrary signal. Next, we provide some insights into the set of equilibria Θ' defined above. Moreover, similar to Theorem 15, we show that the coordination error vector degrades gracefully with the value of the quantizer step size.

Lemma 16. *Consider the quantized collective error dynamics (20). We have that:*

(i) $\boldsymbol{\zeta}_{\mathbf{eq}} = \mathbf{0}$ *is an equilibrium point, independently of the quantizer resolution and the information flow;*
(ii) *for the case of connected (undirected) network topologies, if the step size of the quantizers satisfies*

$$\Delta < \frac{1}{2(n-n_\ell)} \frac{|\rho|}{k_P}, \tag{21}$$

then $\boldsymbol{\zeta}_{\mathbf{eq}} = \mathbf{0}$ *is the only equilibrium point.*

Proof. The proof is given in Appendix E.

Remark 17. The bound in (21) should be understood as a design constraint for the quantizers that prevents the existence of equilibria other than $\boldsymbol{\zeta}_{\mathbf{eq}} = \mathbf{0}$. Note also that the bound in (21) is less restrictive than the bound in (14).

Theorem 18. *Consider the collective error dynamics (20) and suppose that the information flow satisfies the PE-like condition (3) for some parameters* $\mu, T > 0$. *Then, there exist control gains* k_P *and* k_I *ensuring that there is a finite time* $T_b \geq 0$ *such that the solution of the quantized collective error dynamics (20) satisfies*

$$\|\boldsymbol{\zeta}(t)\| \leq \kappa'_{\zeta 0} \|\boldsymbol{\zeta}(0)\| e^{-\lambda_c t}, \quad \forall\, 0 \leq t < T_b, \tag{22a}$$

$$\|\boldsymbol{\zeta}(t)\| \leq \kappa'_{\zeta 1} \Delta \quad \forall\, t \geq T_b, \tag{22b}$$

for some positive constants $\kappa'_{\zeta 0}, \kappa'_{\zeta 1} \in (0, \infty)$, *and with* $\lambda_c := \bar{\lambda}_c(1-\theta_\lambda)$, *where* $\bar{\lambda}_c$ *was defined in (9) and* θ_λ *is again a positive constant verifying* $0 < \theta_\lambda < 1$.

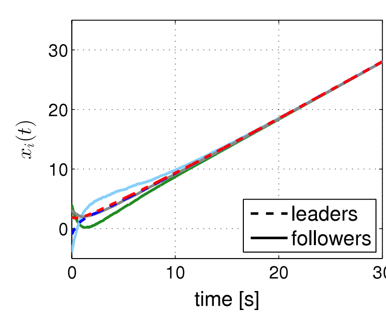
Proof. The proof of this result is similar to the proof of Theorem 15, and is therefore omitted.

Remark 19. While protocol (18) overcomes some of the shortcomings of protocol (11), it has been observed that the use of fully quantized information can induce chattering, a phenomenon that has not been noticed with protocol (11). To prevent chattering from occurring, one could resort to the *hysteretic quantizers* proposed in Ceragioli et al. (2011).

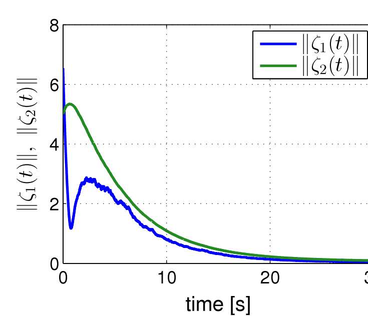
5. Simulation results

We now present simulation results that illustrate the theoretical findings of the paper. To this end, we consider a network of five integrator-agents, two of which with disturbance-free dynamics (i.e. $n = 5$ and $n_\ell = 2$). At a given time t, the information flow of the network is characterized by one of the graphs in Fig. 1; note that all four graphs are *not* connected. The control objective is to design a distributed PI protocol that solves the coordination problem (2) with $\rho = 1$ (in a practical sense). In all of the simulations, the initial coordination-state vector $\boldsymbol{x_0}$ and the disturbance vector $\boldsymbol{d}$ are given by $\boldsymbol{x_0} = [-1,\ 2,\ 4,\ -4,\ 3]^\top$ and $\boldsymbol{d} = [-3,\ 4,\ 1]^\top$.

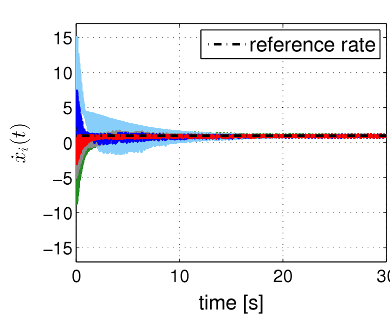
To solve the consensus problem, we implement the protocol with *partially quantized feedback* (protocol (11)) and the protocol with *fully quantized feedback* (protocol (18)). The PI gains of both protocols are set to $k_P = 0.60$ and $k_I = 0.10$, with initial integrator state $\boldsymbol{\chi_0} = \mathbf{0}$. Figs. 2 and 3 present the computed evolution of the closed-loop collective dynamics for the two protocols with quantizer step size $\Delta = 0.15$ (note that this step size verifies inequalities (14) and (21)). In particular, the figures show the time evolution of the coordination states, their time-derivative, the integrator states, and the 2-norms of the consensus error states $\boldsymbol{\zeta_1}(t)$



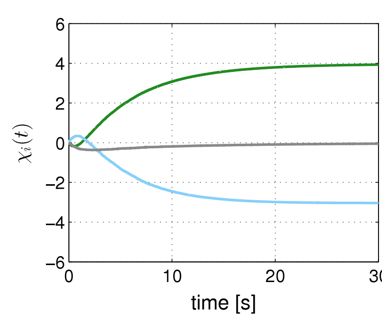
(a) Coordination states.



(b) Coordination error states.

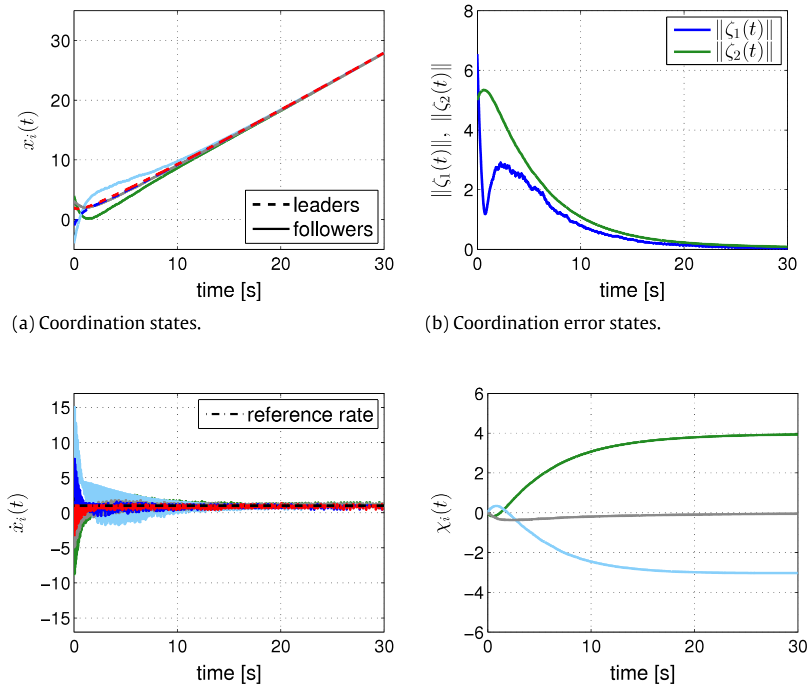


(c) Coordination-state rates.



(d) Integrator states.

Fig. 2. Quantized closed-loop collective dynamics with *partially quantized feedback* ($\Delta = 0.15$).



(a) Coordination states.

(b) Coordination error states.

(c) Coordination-state rates.

(d) Integrator states.

Fig. 3. Quantized closed-loop collective dynamics with *fully quantized feedback* ($\Delta = 0.15$).

and $\boldsymbol{\zeta}_2(t)$. Additionally, Fig. 4 shows an estimate of the QoS of the network, computed as

$$\hat{\mu}(t) := \lambda_{\min}\left(\frac{1}{n}\,\frac{1}{T}\int_{t-T}^{t} \boldsymbol{Q}\boldsymbol{L}(\tau)\boldsymbol{Q}^{\top}\,\mathrm{d}\tau\right), \quad t \geq T,$$

with $T = 0.5$ s. The results demonstrate that both PI protocols allow the followers to 'learn' the reference rate command ρ and reach (practical) agreement with the leaders, while effectively compensating for the (constant) disturbances present in the network. In fact, for this resolution of the quantizers, the two protocols lead to similar results, with comparable levels of performance.

Next, we illustrate the behavior of the two protocols in the presence of coarse quantization. To this end, we consider the same simulation scenario as in Figs. 2–4, but change the quantizer step size to $\Delta = 3.0$, which does not verify inequalities (14) or (21). The computed responses of the collective dynamics for the two protocols are shown in Figs. 5 and 6, respectively. While the protocol with fully quantized feedback achieves the desired agreement (in a practical sense), the protocol with partially quantized feedback is not able to coordinate the agents and, in fact, the solution converges to a neighborhood[2] of one of the "zero-speed" equilibrium points analyzed in Corollary 12 and Lemma 13. We note that, for the case of the fully quantized protocol, the solution converges to a neighborhood of the desired equilibrium $\boldsymbol{\zeta}_{\mathbf{eq}} = \mathbf{0}$ even if the network topology is not connected at any time; this can be explained form the fact that the network satisfies the PE-like condition (3), which implies that the topology is uniformly jointly connected.

6. Conclusions

We analyzed the stability and convergence properties of a distributed, multi-leader, proportional–integral protocol for coordination of a network of agents subject to constant disturbances.

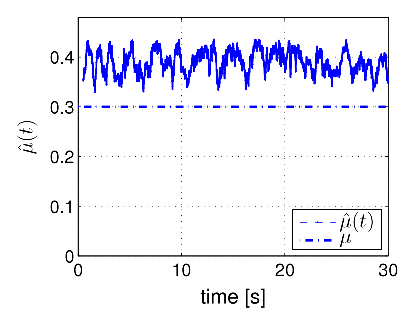


Fig. 4. Estimate of the QoS of the network.

In particular, we provided conditions under which the agents reach (in a practical sense) the desired agreement in the presence of switching network topologies and quantized information exchange. We also derived lower bounds on the convergence rate of the network dynamics as a function of the number of leaders and the quality of service of the network. We addressed explicitly the situation where the graph that captures the information flow is disconnected during some interval of time or even fails to be connected at all times. Further work is needed to investigate the convergence properties of the proposed fully quantized protocol. Future work will also analyze stability of the protocols under time-varying *directed* graphs.

Acknowledgments

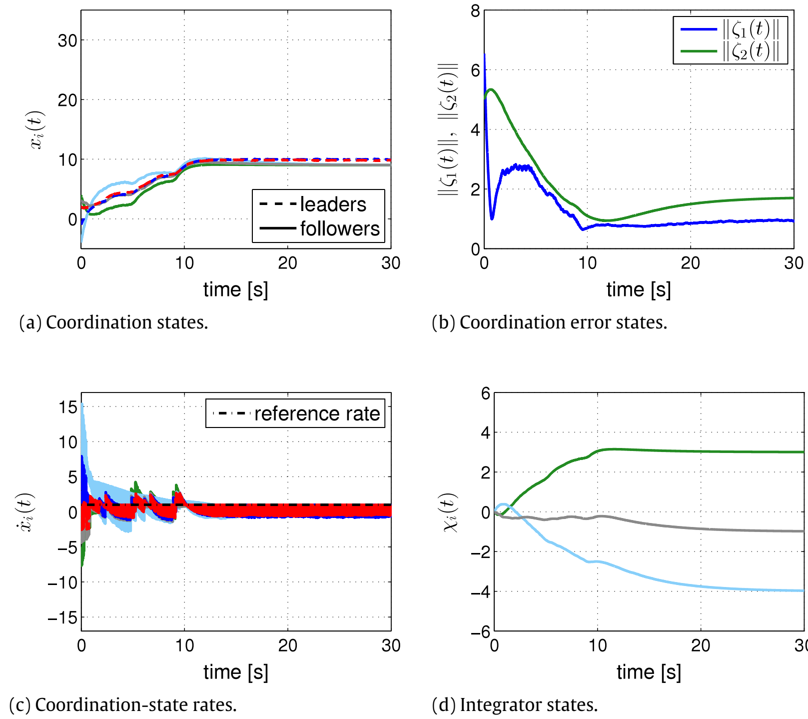
The authors would like to thank the anonymous reviewers for their valuable comments and suggestions.

Appendix A. Proof of Theorem 6

To prove that the origin of the collective error dynamics (8) is globally uniformly exponentially stable (GUES) under the connectivity condition (3), we first consider the system

$$\dot{\phi} = -k_P\bar{\boldsymbol{L}}(t)\phi, \quad \phi(t) \in \mathbb{R}^{n-1}. \tag{A.1}$$

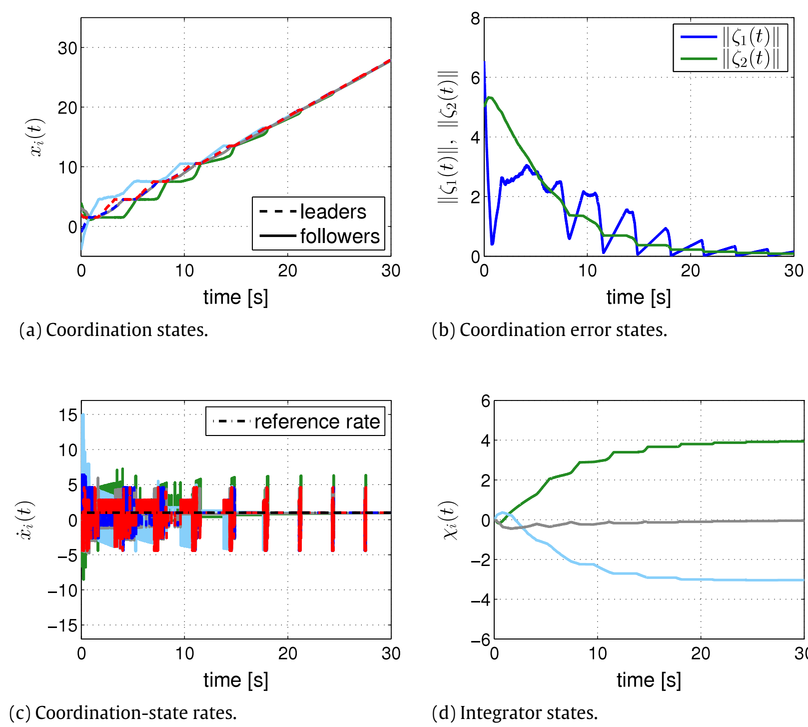
[2] Note that the results in Section 4.2 are derived for network topologies that are static and connected; instead, the simulations presented here consider a switching information flow.



(a) Coordination states. (b) Coordination error states.

(c) Coordination-state rates. (d) Integrator states.

Fig. 5. Quantized closed-loop collective dynamics with *partially quantized feedback* ($\Delta = 3.0$).



(a) Coordination states. (b) Coordination error states.

(c) Coordination-state rates. (d) Integrator states.

Fig. 6. Quantized closed-loop collective dynamics with *fully quantized feedback* ($\Delta = 3.0$).

Letting $\mathbf{\Lambda}(t)$ be the time-varying *incidence matrix*, $\mathbf{L}(t) = \mathbf{\Lambda}(t)\mathbf{\Lambda}^\top(t)$, we can rewrite the system above as $\dot{\phi} = -k_P(\mathbf{Q}\mathbf{\Lambda}(t))(\mathbf{Q}\mathbf{\Lambda}(t))^\top\phi$. Then, since $\mathbf{Q}\mathbf{\Lambda}(t)$ is piecewise constant in time and $\|\mathbf{Q}\mathbf{\Lambda}(t)\|^2 \leq n$, one can prove that system (A.1) is GUES and the following bound holds:

$$\|\phi(t)\| \leq \kappa_\phi \|\phi(0)\| \,\mathrm{e}^{-\gamma_c t},$$

with $\kappa_\phi = 1$ and $\gamma_c \geq \bar{\gamma}_c := (k_P n\mu)/(1 + k_P nT)^2$. This result can be proven along the same lines as Lemma 5 in Loría and Panteley (2002) or Lemma 3 in Panteley and Loría (2000). Since $\dot{\bar{\mathbf{L}}}(t)$ is continuous for almost all $t \geq 0$ and uniformly bounded, and system (A.1) is GUES, then Lemma 1 in Panteley and Loría (2000) and a similar argument as in Theorem 4.12 in Khalil (2002) imply that, for

$$\dot{V} = \boldsymbol{z}^\top \begin{bmatrix} \dot{\boldsymbol{P}}_\mathbf{0}(t) - k_P\left(\bar{\boldsymbol{L}}(t)\boldsymbol{P}_\mathbf{0}(t) + \boldsymbol{P}_\mathbf{0}(t)\bar{\boldsymbol{L}}(t)\right) + \frac{k_I}{k_P}\left(\boldsymbol{QCC}^\top\boldsymbol{Q}^\top\boldsymbol{P}_\mathbf{0}(t) + \boldsymbol{P}_\mathbf{0}(t)\boldsymbol{QCC}^\top\boldsymbol{Q}^\top\right) & \left(\boldsymbol{P}_\mathbf{0}(t) - \frac{k_P}{k_I}\mathbb{I}_{\boldsymbol{n-1}}\right)\boldsymbol{QC} \\ \boldsymbol{C}^\top\boldsymbol{Q}^\top\left(\boldsymbol{P}_\mathbf{0}(t) - \frac{k_P}{k_I}\mathbb{I}_{\boldsymbol{n-1}}\right) & -2\frac{k_P^2}{k_I^2}\mathbb{I}_{\boldsymbol{n-n_\ell}} \end{bmatrix}\boldsymbol{z} \tag{A.6}$$

Box I.

$$\begin{bmatrix} -\bar{c}_3\mathbb{I}_{\boldsymbol{n-1}} + \frac{k_I}{k_P}\left(\boldsymbol{QCC}^\top\boldsymbol{Q}^\top\boldsymbol{P}_\mathbf{0}(t) + \boldsymbol{P}_\mathbf{0}(t)\boldsymbol{QCC}^\top\boldsymbol{Q}^\top\right) & \left(\boldsymbol{P}_\mathbf{0}(t) - \frac{k_P}{k_I}\mathbb{I}_{\boldsymbol{n-1}}\right)\boldsymbol{QC} \\ \boldsymbol{C}^\top\boldsymbol{Q}^\top\left(\boldsymbol{P}_\mathbf{0}(t) - \frac{k_P}{k_I}\mathbb{I}_{\boldsymbol{n-1}}\right) & -2\frac{k_P^2}{k_I^2}\mathbb{I}_{\boldsymbol{n-n_\ell}} \end{bmatrix} \le -2\bar{\lambda}_c \begin{bmatrix} \bar{c}_2\mathbb{I}_{\boldsymbol{n-1}} & \mathbf{0} \\ \mathbf{0} & \frac{k_P^3}{k_I^3}\left(\boldsymbol{C}^\top\boldsymbol{Q}^\top\boldsymbol{QC}\right)^{-1} \end{bmatrix} \tag{A.8}$$

Box II.

any constants $\bar{c}_3$ and $\bar{c}_4$ satisfying $0 < \bar{c}_3 \le \bar{c}_4$, there exists a continuous, piecewise-differentiable matrix $\boldsymbol{P}_\mathbf{0}(t) = \boldsymbol{P}_\mathbf{0}^\top(t)$ such that

$$\bar{c}_1\mathbb{I}_{\boldsymbol{n-1}} := \frac{\bar{c}_3}{2k_P n}\mathbb{I}_{\boldsymbol{n-1}} \le \boldsymbol{P}_\mathbf{0}(t) \le \frac{\bar{c}_4}{2\bar{\gamma}_c}\mathbb{I}_{\boldsymbol{n-1}} =: \bar{c}_2\mathbb{I}_{\boldsymbol{n-1}},$$

$$\dot{\boldsymbol{P}}_\mathbf{0}(t) - k_P\bar{\boldsymbol{L}}(t)\boldsymbol{P}_\mathbf{0}(t) - k_P\boldsymbol{P}_\mathbf{0}(t)\bar{\boldsymbol{L}}(t) \le -\bar{c}_3\mathbb{I}_{\boldsymbol{n-1}}. \tag{A.2}$$

Next, we apply the change of variables

$$\boldsymbol{z}(t) := \boldsymbol{S}_\zeta\boldsymbol{\zeta}(t), \quad \boldsymbol{S}_\zeta := \begin{bmatrix} \mathbb{I}_{\boldsymbol{n-1}} & 0 \\ -\frac{k_I}{k_P}\boldsymbol{C}^\top\boldsymbol{Q}^\top & \mathbb{I}_{\boldsymbol{n-n_\ell}} \end{bmatrix}, \tag{A.3}$$

to the original collective error dynamics (8), which leads to

$$\dot{\boldsymbol{z}} = \begin{bmatrix} -k_P\bar{\boldsymbol{L}}(t) + \frac{k_I}{k_P}\boldsymbol{QCC}^\top\boldsymbol{Q}^\top & \boldsymbol{QC} \\ -\frac{k_I^2}{k_P^2}\boldsymbol{C}^\top\boldsymbol{Q}^\top\boldsymbol{QCC}^\top\boldsymbol{Q}^\top & -\frac{k_I}{k_P}\boldsymbol{C}^\top\boldsymbol{Q}^\top\boldsymbol{QC} \end{bmatrix}\boldsymbol{z}. \tag{A.4}$$

Consider now the Lyapunov function candidate

$$V(t,\boldsymbol{z}) := \boldsymbol{z}^\top\boldsymbol{P}(t)\boldsymbol{z},$$

$$\boldsymbol{P}(t) := \begin{bmatrix} \boldsymbol{P}_\mathbf{0}(t) & \mathbf{0} \\ \mathbf{0} & \frac{k_P^3}{k_I^3}\left(\boldsymbol{C}^\top\boldsymbol{Q}^\top\boldsymbol{QC}\right)^{-1} \end{bmatrix}. \tag{A.5}$$

The time derivative of V along the trajectories of system (A.4) is given in Box I. Now, for any $\beta_k \ge 2$, define

$$\bar{\lambda}_c = \bar{\gamma}_c\left(1 + \beta_k\frac{n}{n_\ell}\right)^{-1}.$$

Then, letting

$$k_P > 0, \quad k_I = k_P\bar{\lambda}_c\frac{n}{n_\ell}\beta_k, \quad \bar{c}_3 = \bar{c}_4 = \frac{\bar{\gamma}_c}{\bar{\lambda}_c}\frac{2n_\ell}{\beta_k n}, \tag{A.7}$$

and noting that $\|\boldsymbol{QC}\| = 1$ and $\lambda_{\min}(\boldsymbol{C}^\top\boldsymbol{Q}^\top\boldsymbol{QC}) = \frac{n_\ell}{n}$, one can use inequalities (A.2) and Schur complements to prove that inequality (A.8) in Box II holds for all $t \ge 0$.

For the choice of parameters (A.7), inequality (A.8) in Box II leads to

$$\dot{V} \le -2\bar{\lambda}_c\boldsymbol{z}^\top\begin{bmatrix} \boldsymbol{P}_\mathbf{0}(t) & \mathbf{0} \\ \mathbf{0} & \frac{k_P^3}{k_I^3}\left(\boldsymbol{C}^\top\boldsymbol{Q}^\top\boldsymbol{QC}\right)^{-1} \end{bmatrix}\boldsymbol{z} = -2\bar{\lambda}_c V.$$

Application of the comparison lemma (Khalil, 2002, Lemma 3.4) yields

$$V(t) \le V(0)\mathrm{e}^{-2\bar{\lambda}_c t}.$$

Then, the following inequalities:

$$\min\left\{\bar{c}_1, \frac{k_P^3}{k_I^3}\right\}\|\boldsymbol{z}(t)\|^2 \le V(t) \le \max\left\{\bar{c}_2, \frac{k_P^3}{k_I^3}\frac{n}{n_\ell}\right\}\|\boldsymbol{z}(t)\|^2,$$

along with the similarity transformation in (A.3), imply that

$$\|\boldsymbol{\zeta}(t)\| \le \kappa_{\zeta 0}\|\boldsymbol{\zeta}(0)\|\mathrm{e}^{-\bar{\lambda}_c t}, \tag{A.9}$$

where $\kappa_{\zeta 0}$ is given by

$$\kappa_{\zeta 0} := \|\boldsymbol{S}_\zeta^{-1}\|\left(\frac{\max\left\{\bar{c}_2, \frac{k_P^3}{k_I^3}\frac{n}{n_\ell}\right\}}{\min\left\{\bar{c}_1, \frac{k_P^3}{k_I^3}\right\}}\right)^{\frac{1}{2}}\|\boldsymbol{S}_\zeta\|. \tag{A.10}$$

To prove the bounds in (10), we note that from the dynamics (6), the definition of $\boldsymbol{\zeta}_\mathbf{1}(t)$ and $\boldsymbol{\zeta}_\mathbf{2}(t)$ in (7), and the properties of matrices $\boldsymbol{Q}$ and $\boldsymbol{\Pi}$, we have that

$$|x_i(t) - x_j(t)| \le 2\left(1 - \frac{1}{n}\right)^{\frac{1}{2}}\|\boldsymbol{\zeta}(t)\|, \tag{A.11a}$$

$$|\dot{x}_i(t) - \rho| \le (k_P n + 1)\|\boldsymbol{\zeta}(t)\|. \tag{A.11b}$$

Inequalities (A.9) and (A.11) lead to the bounds in (10) with

$$\kappa_{x0} = 2\left(1 - \frac{1}{n}\right)^{\frac{1}{2}}\kappa_{\zeta 0}, \qquad \kappa_{\dot{x}0} = (k_P n + 1)\kappa_{\zeta 0}. \quad \square \tag{A.12}$$

Appendix B. Proof of Lemma 11

Let $\hat{\boldsymbol{x}} \in \mathbb{R}^n$, and note that $\mathbf{q}(\hat{\boldsymbol{x}}) = \boldsymbol{k}\Delta$, for some $\boldsymbol{k} \in \mathbb{Z}^n$. Also, for any $\boldsymbol{v} \in \mathrm{K}(\mathbf{q}(\hat{\boldsymbol{x}}))$, we have that

$$v_i \begin{cases} = k_i\Delta, & \hat{x}_i \ne k_i\Delta - \frac{\Delta}{2} \\ \in [(k_i - 1)\Delta, k_i\Delta], & \hat{x}_i = k_i\Delta - \frac{\Delta}{2}, \end{cases}$$

where $v_i \in \mathbb{R}$, $\hat{x}_i \in \mathbb{R}$, $k_i \in \mathbb{Z}$ are the ith components of $\boldsymbol{v}$, $\hat{\boldsymbol{x}}$, and $\boldsymbol{k}$, respectively. Note that $|v_i - \hat{x}_i| \le \frac{\Delta}{2}$.

To prove the result of the lemma, it is enough to show that, if (14) holds, then there exists no 4-tuple $(\hat{\boldsymbol{x}}, \boldsymbol{v}_\mathbf{1}, \boldsymbol{v}_\mathbf{2}, \boldsymbol{\chi})$, $\hat{\boldsymbol{x}} \in \mathbb{R}^n$, $\boldsymbol{v}_\mathbf{1}, \boldsymbol{v}_\mathbf{2} \in \mathrm{K}(\mathbf{q}(\hat{\boldsymbol{x}}))$, and $\boldsymbol{\chi} \in \mathbb{R}^{n-n_\ell}$, such that the following equality holds:

$$\mathbf{0} = \begin{bmatrix} -k_P\left(\boldsymbol{D}\hat{\boldsymbol{x}} - \boldsymbol{A}\boldsymbol{v}_\mathbf{1}\right) + \begin{bmatrix} \rho\mathbf{1}_{\boldsymbol{n_\ell}} \\ \boldsymbol{\chi} + \boldsymbol{d} \end{bmatrix} \\ -k_I\boldsymbol{C}^\top\left(\boldsymbol{D}\hat{\boldsymbol{x}} - \boldsymbol{A}\boldsymbol{v}_\mathbf{2}\right) \end{bmatrix}. \tag{B.1}$$

To this end, we first consider the first n rows of equality (B.1) and multiply them on the left by $\boldsymbol{C}^\top$ to obtain

$$-k_P\boldsymbol{C}^\top(\boldsymbol{D}\hat{\boldsymbol{x}} - \boldsymbol{A}\boldsymbol{v}_\mathbf{1}) + \boldsymbol{\chi} + \boldsymbol{d} = \mathbf{0}.$$

Then, noting that the last $(n-n_\ell)$ rows of (B.1) imply that $\boldsymbol{C}^\top(\boldsymbol{D}\hat{\boldsymbol{x}}-\boldsymbol{A}\boldsymbol{v}_2)=\mathbf{0}$, it follows that equality (B.1) can be satisfied only if $\chi = k_P\boldsymbol{C}^\top\boldsymbol{A}(\boldsymbol{v}_2-\boldsymbol{v}_1)-\boldsymbol{d}$. This result implies that existence of a 4-tuple $(\hat{\boldsymbol{x}},\boldsymbol{v}_1,\boldsymbol{v}_2,\chi)$ satisfying (B.1) is equivalent to existence of a triple $(\hat{\boldsymbol{x}},\boldsymbol{v}_1,\boldsymbol{v}_2)$, $\hat{\boldsymbol{x}}\in\mathbb{R}^n$ and $\boldsymbol{v}_1,\boldsymbol{v}_2\in \mathrm{K}(\mathbf{q}(\hat{\boldsymbol{x}}))$, such that the following equality holds:

$$\boldsymbol{L}\boldsymbol{v}_1-\begin{bmatrix}\frac{\rho}{k_P}\mathbf{1}_{n_\ell}\\ \mathbf{0}\end{bmatrix}=\boldsymbol{D}(\boldsymbol{v}_1-\hat{\boldsymbol{x}})+\begin{bmatrix}\mathbf{0}\\ \boldsymbol{C}^\top\boldsymbol{A}(\boldsymbol{v}_2-\boldsymbol{v}_1)\end{bmatrix}. \tag{B.2}$$

The existence of vectors $\hat{\boldsymbol{x}}$, $\boldsymbol{v}_1$, and $\boldsymbol{v}_2$ such that equality (B.2) holds depends on the quantizer precision. For instance, if $\|\frac{\rho}{k_P}\boldsymbol{D}^{-1}\begin{bmatrix}\mathbf{1}_{n_\ell}\\ \mathbf{0}\end{bmatrix}\|_\infty<\frac{\Delta}{2}$, then the vectors

$$\hat{\boldsymbol{x}}=k\Delta\mathbf{1}_n+\frac{\rho}{k_P}\boldsymbol{D}^{-1}\begin{bmatrix}\mathbf{1}_{n_\ell}\\ \mathbf{0}\end{bmatrix},\qquad \boldsymbol{v}_1=\boldsymbol{v}_2=k\Delta\mathbf{1}_n, \tag{B.3}$$

verify (B.2) for any $k\in\mathbb{Z}$. On the contrary, if the bound in (14) holds, then there exist no vectors $\hat{\boldsymbol{x}}$, $\boldsymbol{v}_1$, and $\boldsymbol{v}_2$ such that (B.2) holds. To see this, consider the scalar equality

$$\frac{\rho}{k_P}n_\ell=\mathbf{1}_n^\top\boldsymbol{D}\left(\hat{\boldsymbol{x}}-\boldsymbol{v}_1\right)+\mathbf{1}_{n-n_\ell}^\top\boldsymbol{C}^\top\boldsymbol{A}(\boldsymbol{v}_1-\boldsymbol{v}_2), \tag{B.4}$$

which has been obtained from (B.2) by multiplying on the left by $\mathbf{1}_n^\top$. The right-hand side of this equality can be bounded as

$$|\mathbf{1}_n^\top\boldsymbol{D}\left(\hat{\boldsymbol{x}}-\boldsymbol{v}_1\right)+\mathbf{1}_{n-n_\ell}^\top\boldsymbol{C}^\top\boldsymbol{A}(\boldsymbol{v}_1-\boldsymbol{v}_2)|\le(3n-2n_\ell)(n-1)\frac{\Delta}{2}.$$

If the bound in (14) holds, then we have

$$|\mathbf{1}_n^\top\boldsymbol{D}\left(\hat{\boldsymbol{x}}-\boldsymbol{v}_1\right)+\mathbf{1}_{n-n_\ell}^\top\boldsymbol{C}^\top\boldsymbol{A}(\boldsymbol{v}_1-\boldsymbol{v}_2)|<\frac{|\rho|}{k_P}n_\ell,$$

which implies that no vectors $\hat{\boldsymbol{x}}$, $\boldsymbol{v}_1$, and $\boldsymbol{v}_2$ satisfy equality (B.4), and thus equality (B.2). In turn, this implies that there is no 4-tuple $(\hat{\boldsymbol{x}},\boldsymbol{v}_1,\boldsymbol{v}_2,\chi)$ such that equality (B.1) holds, and therefore the set Θ is empty. □

Appendix C. Proof of Lemma 13

Let $\boldsymbol{\eta}(t):=[\boldsymbol{\eta}_1^\top(t),\boldsymbol{\eta}_2^\top(t)]^\top$ be defined as

$$\boldsymbol{\eta}_1(t):=\boldsymbol{x}(t)-\hat{\boldsymbol{x}},\qquad \boldsymbol{\eta}_2(t):=\chi(t)-\hat{\chi},$$

where $\hat{\boldsymbol{x}}$ and $\hat{\chi}$ characterize the "zero-speed" equilibrium points introduced in (16). Since, by assumption, we have that $\|\frac{\rho}{k_P}\boldsymbol{D}^{-1}\begin{bmatrix}\mathbf{1}_{n_\ell}\\ \mathbf{0}\end{bmatrix}\|_\infty<\frac{\Delta}{2}$, it follows that $\mathbf{q}(\hat{\boldsymbol{x}})=k\Delta\mathbf{1}_n$, $k\in\mathbb{Z}$. Then, the closed-loop collective dynamics (12) can be rewritten in terms of the states $\boldsymbol{\eta}_1(t)$ and $\boldsymbol{\eta}_2(t)$ as

$$\dot{\boldsymbol{\eta}}_1=-k_P\left(\boldsymbol{D}\boldsymbol{\eta}_1-\boldsymbol{A}\,\mathbf{q}(\boldsymbol{\eta}_1+\hat{\boldsymbol{x}})\right)+\boldsymbol{C}\boldsymbol{\eta}_2-k_P\boldsymbol{A}k\Delta\mathbf{1}_n,$$
$$\dot{\boldsymbol{\eta}}_2=-k_I\boldsymbol{C}^\top\left(\boldsymbol{D}\boldsymbol{\eta}_1-\boldsymbol{A}\,\mathbf{q}(\boldsymbol{\eta}_1+\hat{\boldsymbol{x}})\right)-k_I\boldsymbol{C}^\top\boldsymbol{A}k\Delta\mathbf{1}_n.$$

In a sufficiently small neighborhood of the origin $(\boldsymbol{\eta}_1,\boldsymbol{\eta}_2)=(\mathbf{0},\mathbf{0})$, we have that $\mathbf{q}(\boldsymbol{\eta}_1(t)+\hat{\boldsymbol{x}})=k\Delta\mathbf{1}_n$, and therefore the nonlinear dynamics above evolve according to the following linear equation:

$$\dot{\boldsymbol{\eta}}=\boldsymbol{A}_\eta\,\boldsymbol{\eta},\qquad \boldsymbol{A}_\eta:=\begin{bmatrix}-k_P\boldsymbol{D} & \boldsymbol{C}\\ -k_I\boldsymbol{C}^\top\boldsymbol{D} & \mathbf{0}\end{bmatrix}.$$

The characteristic polynomial of $\boldsymbol{A}_\eta$ is given by

$$p_{\boldsymbol{A}_\eta}(\lambda)=\prod_{i=1}^{n_\ell}(\lambda+k_P\delta_i)\prod_{i=n_\ell+1}^{n}(\lambda^2+k_P\delta_i\lambda+k_I\delta_i),$$

where δ_i is the ith diagonal element of the degree matrix $\boldsymbol{D}$. Since the communications graph is assumed to be connected, it follows that $1\le\delta_i\le n-1$, which implies that all of the eigenvalues of $\boldsymbol{A}_\eta$ have negative real part. □

Appendix D. Proof of Theorem 15

Let the function $\boldsymbol{\zeta}(t)=\begin{bmatrix}\boldsymbol{Q}\boldsymbol{x}(t)\\ \chi(t)-\rho\mathbf{1}_{n-n_\ell}+\boldsymbol{d}\end{bmatrix}$, $t\in I_t\subset\mathbb{R}$ be a Krasovskii solution of (13) on I_t, that is, $\boldsymbol{\zeta}(t)$ is absolutely continuous and satisfies the differential inclusion (Hájek, 1979)

$$\dot{\boldsymbol{\zeta}}-\boldsymbol{A}_\zeta(t)\boldsymbol{\zeta}\in\mathrm{K}\left(\boldsymbol{f}_{\boldsymbol{q}}(t)\right),$$

for almost every $t\in I_t$. Then, letting $\boldsymbol{z}(t):=\boldsymbol{S}_\zeta\boldsymbol{\zeta}(t)$, where $\boldsymbol{S}_\zeta$ was defined in (A.3), we have

$$\dot{\boldsymbol{z}}-\boldsymbol{S}_\zeta\boldsymbol{A}_\zeta(t)\boldsymbol{S}_\zeta^{-1}\boldsymbol{z}\in\boldsymbol{S}_\zeta\,\mathrm{K}\left(\boldsymbol{f}_{\boldsymbol{q}}(t)\right),\quad \text{almost everywhere in } I_t.$$

Consider now the same Lyapunov function candidate (A.5) as in the proof of Theorem 6. Then, letting $\boldsymbol{v}_1(t),\boldsymbol{v}_2(t),\in\mathrm{K}(\mathbf{q}(\boldsymbol{x}(t)))$ and following the same steps as in the proof of Theorem 6, we have that, for the choice of parameters in (A.7), the following inequality holds:

$$\dot V\le-2\bar\lambda_c V+2\boldsymbol{z}^\top\begin{bmatrix}k_P\boldsymbol{P}_0(t)\boldsymbol{Q}\boldsymbol{A}(t)\\ -\frac{k_P^3}{k_I^2}(\boldsymbol{C}^\top\boldsymbol{Q}^\top\boldsymbol{Q}\boldsymbol{C})^{-1}\boldsymbol{C}^\top\boldsymbol{Q}^\top\boldsymbol{Q}\boldsymbol{A}(t)\end{bmatrix}(\boldsymbol{v}_1-\boldsymbol{x})$$
$$+2\boldsymbol{z}^\top\begin{bmatrix}\mathbf{0}\\ \frac{k_P^3}{k_I^2}(\boldsymbol{C}^\top\boldsymbol{Q}^\top\boldsymbol{Q}\boldsymbol{C})^{-1}\boldsymbol{C}^\top\boldsymbol{A}(t)\end{bmatrix}(\boldsymbol{v}_2-\boldsymbol{x}).$$

Noting that $\|\boldsymbol{v}_i(t)-\boldsymbol{x}(t)\|\le\sqrt{n}\frac{\Delta}{2}$, $i=1,2$, and also that

$$\left\|\begin{bmatrix}k_P\boldsymbol{P}_0(t)\boldsymbol{Q}\boldsymbol{A}(t)\\ -\frac{k_P^3}{k_I^2}(\boldsymbol{C}^\top\boldsymbol{Q}^\top\boldsymbol{Q}\boldsymbol{C})^{-1}\boldsymbol{C}^\top\boldsymbol{Q}^\top\boldsymbol{Q}\boldsymbol{A}(t)\end{bmatrix}\right\|$$
$$\le\sqrt{2}k_P(n-1)\max\left\{\bar c_2,\frac{k_P^2}{k_I^2}\frac{n}{n_\ell}\right\}=:\sigma_{B1},$$

$$\left\|\begin{bmatrix}\mathbf{0}\\ \frac{k_P^3}{k_I^2}(\boldsymbol{C}^\top\boldsymbol{Q}^\top\boldsymbol{Q}\boldsymbol{C})^{-1}\boldsymbol{C}^\top\boldsymbol{A}(t)\end{bmatrix}\right\|\le k_P(n-1)\frac{k_P^2}{k_I^2}\frac{n}{n_\ell}=:\sigma_{B2},$$

it follows that

$$\dot V\le-2\bar\lambda_c V+\sqrt{n}\Delta(\sigma_{B1}+\sigma_{B2})\|\boldsymbol{z}\|.$$

We can now rewrite the above inequality as

$$\dot V\le-2\bar\lambda_c(1-\theta_\lambda)V-2\bar\lambda_c\theta_\lambda V+\sqrt{n}\Delta(\sigma_{B1}+\sigma_{B2})\|\boldsymbol{z}\|,$$

where $0<\theta_\lambda<1$. Then, for all $\boldsymbol{z}(t)$ satisfying

$$-2\bar\lambda_c\theta_\lambda V+\sqrt{n}\Delta(\sigma_{B1}+\sigma_{B2})\|\boldsymbol{z}\|\le0, \tag{D.1}$$

we have $\dot V(t)\le-2\bar\lambda_c(1-\theta_\lambda)V(t)$. Inequality (D.1) holds outside the bounded set D_Δ defined as

$$D_\Delta:=\left\{\boldsymbol{z}\in\mathbb{R}^{2n-n_\ell-1}\;:\;\|\boldsymbol{z}\|\le\frac{\sqrt{n}\Delta(\sigma_{B1}+\sigma_{B2})}{2\bar\lambda_c\theta_\lambda\min\left\{\bar c_1,(k_P^3/k_I^3)\right\}}\right\}.$$

The set D_Δ is in the interior of the compact set Ω_Δ given by

$$\Omega_\Delta:=\left\{\boldsymbol{z}\in\mathbb{R}^{2n-n_\ell-1}\;:\;V(t,\boldsymbol{z})\le\frac{n(\sigma_{B1}+\sigma_{B2})^2\max\left\{\bar c_2,(k_P^3/k_I^3)(n/n_\ell)\right\}}{4\bar\lambda_c^2\theta_\lambda^2\left(\min\left\{\bar c_1,(k_P^3/k_I^3)\right\}\right)^2}\Delta^2=:\kappa_V^2\,\Delta^2\right\}.$$

Then, using a proof similar to that of Theorem 4.18 in Khalil (2002), it can be shown that there is a time $T_b\ge0$ such that

$$\|\boldsymbol{\zeta}(t)\|\le\kappa_{\zeta0}\|\boldsymbol{\zeta}(0)\|\mathrm{e}^{-\bar\lambda_c(1-\theta_\lambda)t},\quad\forall\,0\le t<T_b,$$
$$\|\boldsymbol{\zeta}(t)\|\le\kappa_{\zeta1}\Delta,\quad\forall\,t\ge T_b,$$

where $\kappa_{\zeta0}$ was defined in (A.10) and $\kappa_{\zeta1}$ is given by

$$\kappa_{\zeta1}:=\|\boldsymbol{S}_\zeta^{-1}\|\left(\min\left\{\bar c_1,\frac{k_P^3}{k_I^3}\right\}\right)^{-\frac12}\kappa_V.\quad\square$$

Appendix E. Proof of Lemma 16

(i) To prove that $\boldsymbol{\zeta}_{\mathbf{eq}} = \mathbf{0}$ is an equilibrium point of the collective error dynamics (20), it is enough to notice that, for any admissible information flow, the following inclusion holds for the pair $(\boldsymbol{x}_{\mathbf{eq}}(t), \boldsymbol{\chi}_{\mathbf{eq}}) = ((x_0 + \rho t)\mathbf{1}_{\boldsymbol{n}}, \rho \mathbf{1}_{\boldsymbol{n}-\boldsymbol{n}_\ell} - \boldsymbol{d}), x_0 \in \mathbb{R}$:

$$\begin{bmatrix} \rho \mathbf{1}_{\boldsymbol{n}} \\ \mathbf{0} \end{bmatrix} \in \mathrm{K}\left(\begin{bmatrix} -k_P \boldsymbol{L}(t)\, \mathbf{q}(\boldsymbol{x}_{\mathbf{eq}}(t)) + \begin{bmatrix} \rho \mathbf{1}_{\boldsymbol{n}_\ell} \\ \boldsymbol{\chi}_{\mathbf{eq}} + \boldsymbol{d} \end{bmatrix} \\ -k_I \boldsymbol{C}^\top \boldsymbol{L}(t)\, \mathbf{q}(\boldsymbol{x}_{\mathbf{eq}}(t)) \end{bmatrix}\right).$$

(ii) To prove this second result, we show that, if the bound in (21) holds, then there exists no 4-tuple $(\gamma(t), \boldsymbol{v}_1(t), \boldsymbol{v}_2(t), \boldsymbol{\chi})$, with $\gamma(t) \in \mathbb{R}$, $\boldsymbol{v}_1(t), \boldsymbol{v}_2(t) \in \mathrm{K}(\mathbf{q}(\boldsymbol{x}(t)))$, and $\boldsymbol{\chi} \in \mathbb{R}^{n-n_\ell}$, other than $(\rho, \mathbf{q}((x_0+\rho t)\mathbf{1}_{\boldsymbol{n}}), \mathbf{q}((x_0+\rho t)\mathbf{1}_{\boldsymbol{n}}), \rho \mathbf{1}_{\boldsymbol{n}-\boldsymbol{n}_\ell} - \boldsymbol{d})$, $x_0 \in \mathbb{R}$, such that the following equality holds:

$$\begin{bmatrix} \gamma(t)\mathbf{1}_{\boldsymbol{n}} \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} -k_P \boldsymbol{L}(t)\boldsymbol{v}_1(t) + \begin{bmatrix} \rho \mathbf{1}_{\boldsymbol{n}_\ell} \\ \boldsymbol{\chi} + \boldsymbol{d} \end{bmatrix} \\ -k_I \boldsymbol{C}^\top \boldsymbol{L}(t)\boldsymbol{v}_2(t) \end{bmatrix}. \tag{E.1}$$

To this end, we analyze separately the following cases:

(1) *$\gamma(t) \equiv 0$ and $x_i \neq k_i\Delta - \Delta/2$ for all $i \in \{1, \ldots, n\}$*: In this case, the existence of an equilibrium point for the collective error dynamics (20) is equivalent to the existence of a pair $(\boldsymbol{v}, \boldsymbol{\chi})$, with $\boldsymbol{v} \in \mathrm{K}(\mathbf{q}(\boldsymbol{x}))$ and $\boldsymbol{\chi} \in \mathbb{R}^{n-n_\ell}$, such that the following equality is satisfied:

$$\mathbf{0} = \begin{bmatrix} -k_P \boldsymbol{L}(t)\boldsymbol{v} + \begin{bmatrix} \rho \mathbf{1}_{\boldsymbol{n}_\ell} \\ \boldsymbol{\chi} + \boldsymbol{d} \end{bmatrix} \\ -k_I \boldsymbol{C}^\top \boldsymbol{L}(t)\boldsymbol{v} \end{bmatrix}. \tag{E.2}$$

Following similar derivations as in the proof of Lemma 11, it can be shown that existence of a pair $(\boldsymbol{v}, \boldsymbol{\chi})$ satisfying (E.2) is equivalent to existence of a vector $\boldsymbol{v}$, $\boldsymbol{v} \in \mathrm{K}(\mathbf{q}(\boldsymbol{x}))$, such that the following equality holds:

$$\boldsymbol{L}(t)\boldsymbol{v} - \begin{bmatrix} \dfrac{\rho}{k_P}\mathbf{1}_{\boldsymbol{n}_\ell} \\ \mathbf{0} \end{bmatrix} = \mathbf{0}. \tag{E.3}$$

Since $\boldsymbol{L}(t)\mathbf{1}_{\boldsymbol{n}} = \mathbf{0}$, $\boldsymbol{L}(t) = \boldsymbol{L}^\top(t)$, and $\boldsymbol{L}(t) \geq \mathbf{0}$, it follows that the vector $\begin{bmatrix} \mathbf{1}_{\boldsymbol{n}_\ell} \\ \mathbf{0} \end{bmatrix} \in \mathbb{R}^n$ is not in the column space of any admissible $\boldsymbol{L}(t)$. Hence, equality (E.3) does not hold for any vector $\boldsymbol{v} \in \mathbb{R}^n$.

(2) *$\gamma(t) \equiv 0$ and $x_i = k_i\Delta - \Delta/2$ for (at least) one $i \in \{1, \ldots, n\}$*: In this case, the existence of an equilibrium point for the collective error dynamics (20) is equivalent to the existence of a triple $(\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{\chi})$, with $\boldsymbol{v}_1, \boldsymbol{v}_2 \in \mathrm{K}(\mathbf{q}(\boldsymbol{x}))$ and $\boldsymbol{\chi} \in \mathbb{R}^{n-n_\ell}$, such that the following equality holds:

$$\mathbf{0} = \begin{bmatrix} -k_P \boldsymbol{L}(t)\boldsymbol{v}_1 + \begin{bmatrix} \rho \mathbf{1}_{\boldsymbol{n}_\ell} \\ \boldsymbol{\chi} + \boldsymbol{d} \end{bmatrix} \\ -k_I \boldsymbol{C}^\top \boldsymbol{L}(t)\boldsymbol{v}_2 \end{bmatrix}. \tag{E.4}$$

We first consider the first n rows of equality (E.4) and multiply them on the left by $\boldsymbol{C}^\top$ to obtain $-k_P \boldsymbol{C}^\top \boldsymbol{L}(t)\boldsymbol{v}_1 + \boldsymbol{\chi} + \boldsymbol{d} = \mathbf{0}$. Then, noting that the last $(n - n_\ell)$ rows of (E.4) imply that $\boldsymbol{C}^\top \boldsymbol{L}(t)\boldsymbol{v}_2 = \mathbf{0}$, it follows that equality (E.4) can be satisfied only if the following equality holds:

$$\boldsymbol{C}^\top \boldsymbol{L}(t)\,(\boldsymbol{v}_1 - \boldsymbol{v}_2) - \frac{1}{k_P}(\boldsymbol{\chi} + \boldsymbol{d}) = \mathbf{0}. \tag{E.5}$$

We now multiply (E.5) on the left by $\mathbf{1}^\top_{\boldsymbol{n}-\boldsymbol{n}_\ell}$ to obtain

$$\mathbf{1}^\top_{\boldsymbol{n}-\boldsymbol{n}_\ell} \boldsymbol{C}^\top \boldsymbol{L}(t)\,(\boldsymbol{v}_1 - \boldsymbol{v}_2) - \frac{1}{k_P}\mathbf{1}^\top_{\boldsymbol{n}-\boldsymbol{n}_\ell}(\boldsymbol{\chi} + \boldsymbol{d}) = \mathbf{0} \tag{E.6}$$

and, noting that from equality (E.4) it follows that $\rho n_\ell + \mathbf{1}^\top_{\boldsymbol{n}-\boldsymbol{n}_\ell}(\boldsymbol{\chi} + \boldsymbol{d}) = \mathbf{0}$ (which has been obtained by multiplying the first n rows of (E.4) on the left by $\mathbf{1}^\top_{\boldsymbol{n}}$), we can rewrite equality (E.6) as

$$\frac{\rho}{k_P}\, n_\ell = \mathbf{1}^\top_{\boldsymbol{n}-\boldsymbol{n}_\ell} \boldsymbol{C}^\top \boldsymbol{L}(t)\,(\boldsymbol{v}_2 - \boldsymbol{v}_1). \tag{E.7}$$

The right-hand side of this equality can be bounded as

$$|\mathbf{1}^\top_{\boldsymbol{n}-\boldsymbol{n}_\ell} \boldsymbol{C}^\top \boldsymbol{L}(t)\,(\boldsymbol{v}_2 - \boldsymbol{v}_1)| \leq 2n_\ell(n - n_\ell)\Delta.$$

If the bound in (21) holds, then we have

$$|\mathbf{1}^\top_{\boldsymbol{n}-\boldsymbol{n}_\ell} \boldsymbol{C}^\top \boldsymbol{L}(t)\,(\boldsymbol{v}_2 - \boldsymbol{v}_1)| < \frac{|\rho|}{k_P}\, n_\ell,$$

which implies that no vectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ satisfy (E.7). In turn, this implies that, if the bound in (21) is satisfied, then there is no triple $(\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{\chi})$ such that (E.4) holds.

(3) *$\gamma(t) \not\equiv 0$*: We start by noting that, in this case, the existence of an equilibrium point for the collective error dynamics (20) requires that, at any time t' between "quantization jumps", there exist a triple $(\gamma(t'), \boldsymbol{v}(t'), \boldsymbol{\chi})$, with $\gamma(t') \in \mathbb{R}$, $\boldsymbol{v}(t') \in \mathrm{K}(\mathbf{q}(\boldsymbol{x}(t')))$, and $\boldsymbol{\chi} \in \mathbb{R}^{n-n_\ell}$, such that the following equality holds:

$$\begin{bmatrix} \gamma(t')\mathbf{1}_{\boldsymbol{n}} \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} -k_P \boldsymbol{L}(t')\boldsymbol{v}(t') + \begin{bmatrix} \rho \mathbf{1}_{\boldsymbol{n}_\ell} \\ \boldsymbol{\chi} + \boldsymbol{d} \end{bmatrix} \\ -k_I \boldsymbol{C}^\top \boldsymbol{L}(t')\boldsymbol{v}(t') \end{bmatrix}. \tag{E.8}$$

Following again similar derivations as in the proof of Lemma 11, it can be shown that existence of a triple $(\gamma(t'), \boldsymbol{v}(t'), \boldsymbol{\chi})$ satisfying equality (E.8) is equivalent to the existence of a pair $(\gamma(t'), \boldsymbol{v}(t'))$, with $\gamma(t') \in \mathbb{R}$ and $\boldsymbol{v}(t') \in \mathrm{K}(\mathbf{q}(\boldsymbol{x}(t')))$, such that the following equality is satisfied:

$$\boldsymbol{L}(t')\boldsymbol{v}(t') - \begin{bmatrix} \dfrac{\gamma(t') - \rho}{k_P}\mathbf{1}_{\boldsymbol{n}_\ell} \\ \mathbf{0} \end{bmatrix} = \mathbf{0}. \tag{E.9}$$

Since $\boldsymbol{L}(t)\mathbf{1}_{\boldsymbol{n}} = \mathbf{0}$, $\boldsymbol{L}(t) = \boldsymbol{L}^\top(t)$, and $\boldsymbol{L}(t) \geq \mathbf{0}$, it follows that the vector $\begin{bmatrix} \mathbf{1}_{\boldsymbol{n}_\ell} \\ \mathbf{0} \end{bmatrix} \in \mathbb{R}^n$ is not in the column space of any admissible $\boldsymbol{L}(t)$. Hence, equality (E.9) can hold only if $\gamma(t') = \rho$. Moreover, if the network topology is connected at all times, then the null space of $\boldsymbol{L}(t)$ is equal to the span of $\mathbf{1}_{\boldsymbol{n}}$ for all $t \geq 0$, which implies that, in this case, equality (E.9) can hold only if $\boldsymbol{x}(t') \in \mathrm{span}\{\mathbf{1}_{\boldsymbol{n}}\}$. This implies that, between "quantization jumps", $\boldsymbol{x}(t)$ is required to evolve continuously according to $\boldsymbol{x}(t) \in \mathrm{span}\{\mathbf{1}_{\boldsymbol{n}}\}$ and $\dot{\boldsymbol{x}}(t) = \rho \mathbf{1}_{\boldsymbol{n}}$. From (E.8), it further follows that $\boldsymbol{\chi} = \rho \mathbf{1}_{\boldsymbol{n}-\boldsymbol{n}_\ell} - \boldsymbol{d}$ is required for $\boldsymbol{x}(t) \in \mathrm{span}\{\mathbf{1}_{\boldsymbol{n}}\}$ and $\dot{\boldsymbol{x}}(t) = \rho \mathbf{1}_{\boldsymbol{n}}$ to hold simultaneously. Finally, because the term $\boldsymbol{L}(t)\boldsymbol{v}(t)$ is bounded at the "quantization jumps", $\boldsymbol{x}(t)$ is continuous for all $t \geq 0$, implying that (E.8) only holds if $\gamma(t) = \rho$ for almost every $t \geq 0$, $\boldsymbol{x}(t) = (x_0 + \rho t)\mathbf{1}_{\boldsymbol{n}}$ for some $x_0 \in \mathbb{R}$ and all $t \geq 0$, and $\boldsymbol{\chi} = \rho \mathbf{1}_{\boldsymbol{n}-\boldsymbol{n}_\ell} + \boldsymbol{d}$.

We can thus conclude that, if the information flow $\mathcal{G}(t)$ is connected at all times and the step size of the quantizers is bounded as in (21), then $\boldsymbol{\zeta}_{\mathbf{eq}} = \mathbf{0}$ is the only equilibrium point of the collective error dynamics (20). □

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