Pure Mathematics for Engineers and Scientists

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Preface

The basic aim is to provide exposure to a solid foundation of mathematics based on proofs. The exit goals would be:

-Readers will be able to write valid proofs using the standard techniques of modern mathematics.

-Readers will be able to read and understand proofs at some reasonable level.

-Readers will have working knowledge of broad foundations of mathematics, and be able to confidently take dedicated courses in each area, or pursue independent studies.

From my experience, I certainly never got this stuff explicitly, but on retrospect, it certainly seems like it would help.

Topics to be covered:

Mathematical Foundations - Mathematical logic, basic notation, axiomatic set theory (Zermelo Frankel), Peano Axioms of Arithmetic, Godel theorems.

Methods of Proof: Induction, Contradiction, Direct Proofs. Examples of classic proofs - infinite primes, irrationality of square root of two, etc. Plenty of practice examples.

Real Analysis - Study of real numbers, including construction by Dedekind Cuts etc, point set topology, continuity, metric spaces (Basically a compressed version of Rudin or equivalent).

Modern Algebra: Groups, Rings, Fields, Transformations, Representation Theory, Modules, Galois Theory

Non-euclidean geometry and topology: Various spaces and properties. Invariants, knot theory, homotopy, homology.

Number theory:

Category theory:

Computational complexity theory: Finite Automata, lambda calculus, NP and other spaces.

Contents

Chapter 1

Mathematical Foundations

The objective of this chapter is to present some basic concepts that underlie much of the remaining material in this book.

1.1 Basic Notation

The reader may be familiar with some if not all of the following notation, but it is presented here for concreteness and review.

- $\forall$ is read as "for all"
- $\exists$ is read as "exists"
- $\subset$ is read as "subset of"
- $\in$ is read as "in"
- $\mathbb{Z}$ is the set of integers: $\ldots, -2, -1, 0, 1, 2, 3, \ldots$
- $\mathbb{N}$ is the set of natural numbers: $0, 1, 2, 3, \ldots$ (note: Sometimes these start with 0, and sometimes with 1. Usually doesn't matter to the argument in question, but the reader should be aware.)
- $\mathbb{Q}$ is the set of rational numbers. These are numbers of the form $\frac{p}{q}$ where p, q are integers, and $q \neq 0$
- $\mathbb{R}$ is the set of real numbers. These will later be defined precisely in terms of *Dedikind Cuts* but for now we'll just use the informal notion of all the points on a coordinate line. However, hopefully the reader can see this is not good enough for precise work.
- We use curly braces to indicate sets in general. For example, $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, 3 \ldots\}$ defines the set of integers using the curly braces and examples. It is hoped the reader is able to abstract that the dots to the left imply the list has

no smallest negative value and the dots to the right imply that there is no largest positive value.

- Set builder notation: There are other ways to specify the set within the braces. For example, we can indicate the property that each member of the set has: $S = \{p | p = 2m + 1, m \in \mathbb{Z}\}$. This is read as the set S consists of those values p of the form $2m+1$ where m is an integer. In other words, S the set of odd integers. The vertical bar in the braces is read as "such that" and the comma is read as "and."

Exercise
Write the definition of the set of rational numbers using set builder notation.
Solution
$\mathbb{Q} = \left\{ r | r = \frac{p}{q}, p, q \in \mathbb{Z}, q \neq 0 \right\}$

1.2 Mathematical Logic

1.3 Zermelo Frankel Axiomatic Set Theory

1.4 Peano Axioms of Arithmetic

1.5 Godel Theorems

Chapter 2

Methods of Proof

2.1 Induction

2.1.1 Weak Induction

2.1.2 Strong Induction

2.2 Proof by Contradiction

Proof by contradiction is a very useful method of proof which is used quite frequently. The usual pattern is one assumes the converse of what is being asserted, and then following a sequence of logical deductions arriving at a contradiction. Once this contradiction has been uncovered, then it implies the original assumption must have been false thus completing the proof. Here is an example to illustrate the points.

Prove that there is no rational number whose square is 12. (Note: This was originally an exercise in Rudin PMA 3rd Edition).

Proof:

Suppose that there was a rational number whose square was 12**.** This means that

$$\left(\frac{p}{q}\right)^2 = 12$$

for integers p, q. We assume that p and q are not both even. In other words, all common factors of 2 have been cancelled. This can always been done for a rational number.

Upon expansion we obtain:

$$p^2 = 12q^2 = 2\left(6q^2\right)$$

This implies that p^2 and p are both even. Thus q is odd. So we write $p = 2m$ for integer m and obtain:

$$(2m)^2 = 4m^2 = 12q^2$$

This implies that

$$m^2 = 3q^2$$

Since q is assumed to be odd, q^2 is odd, and $3q^2$ is also odd. Now, m^2 must be therefore be odd, and likewise for m. Therefore, for integers r, n we can rewrite the above as:

$$(2r+1)^2 = 3\left(2n+1\right)^2$$

Now, expanding both sides yields:

$$4r^2 + 4r + 1 = 3\left(4n^2 + 4n + 1\right)$$

$$4r^2 + 4r + 1 = 12n^2 + 12n + 3$$

$$4r^2 + 4r = 12n^2 + 12n + 2$$

$$4r^2 + 4r = 12n^2 + 12n + 2$$

The left side is clearly divisible by 4.

$$r^2 + r = 3n^2 + 3n + \frac{2}{4}$$

The left side is an integer, but the right side is not, which is a contradiction. Therefore our assumption about both p, q not even is false. However, since this can always be done for the rational number p/q it implies that there is no rational p/q such that $(p/q)^2 = 12$.

2.3 Direct Proof