Ph21 Problem Set 1

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Problem One

We start with the step update function as:

$$x_{i+1} = x_i - f(x_i)\frac{x_i - x_{i-1}}{f(x_i) - f(x_{i-1})}$$

.

Or,

$$\epsilon_{i+1} = \epsilon_i - f(x_i)\frac{\epsilon_i - \epsilon_{i-1}}{f(x_i) - f(x_{i-1})}$$

.

Now, expanding each $f(x_i)$ in terms of the Taylor expansion

$$f(x+\epsilon) = f(x) + \epsilon f'(x) + \epsilon^2 \frac{f''(x)}{2} + ...$$

:

We obtain

$$\epsilon_{i+1} = \epsilon_i - \frac{(\epsilon_i - \epsilon_{i-1})\left(\frac{1}{2}\epsilon_i^2 f''(x) + \epsilon_i f'(x)\right)}{-\frac{1}{2}\epsilon_{i-1}^2 f''(x) + \frac{1}{2}\epsilon_i^2 f''(x) - \epsilon_{i-1} f'(x) + \epsilon_i f'(x)}$$

.

Simplifying, this becomes

$$\epsilon_{i+1} = \epsilon_i - \frac{\epsilon_i\left(\epsilon_i f''(x) + 2f'(x)\right)}{(\epsilon_{i-1} + \epsilon_i) f''(x) + 2f'(x)}$$

.

We can now expand the denominator assuming that $\epsilon_{i-1} + \epsilon_i$ is small:

$$\frac{\epsilon_i \left(\frac{\epsilon_i f''(x)}{2f'(x)} + 1\right)}{\frac{(\epsilon_{i-1}+\epsilon_i) f''(x)}{2f'(x)} + 1} \approx \epsilon_i \left(\frac{\epsilon_i f''(x)}{2f'(x)} + 1\right) \left(1 - \frac{(\epsilon_{i-1} + \epsilon_i) f''(x)}{2f'(x)}\right)$$

.

Now, expanding this out and subtracting from ϵ_i, we obtain

$$\epsilon_{i+1} = \frac{\epsilon_i^3 f''(x)^2}{4f'(x)^2} + \frac{\epsilon_{i-1}\epsilon_i^2 f''(x)^2}{4f'(x)^2} + \frac{\epsilon_{i-1}\epsilon_i f''(x)}{2f'(x)}$$

.

And ignoring terms of order ϵ^3 and higher, this becomes

$$\epsilon_{i+1} = \frac{\epsilon_{i-1}\epsilon_i f''(x)}{2f'(x)}$$

.

Now, we solve this recurrance relation assuming that $\epsilon_{i+1} = C\epsilon_i^r$, thus solving as

$$2Ca(n)^r = \frac{a(n)^{\frac{1}{r}+1} f''(x)}{Cf'(x)}$$

. Solving for r, this becomes $r = 1 + \frac{1}{r}$, or $r = \frac{1+\sqrt{5}}{2}$, the Golden ratio!

Problem 2

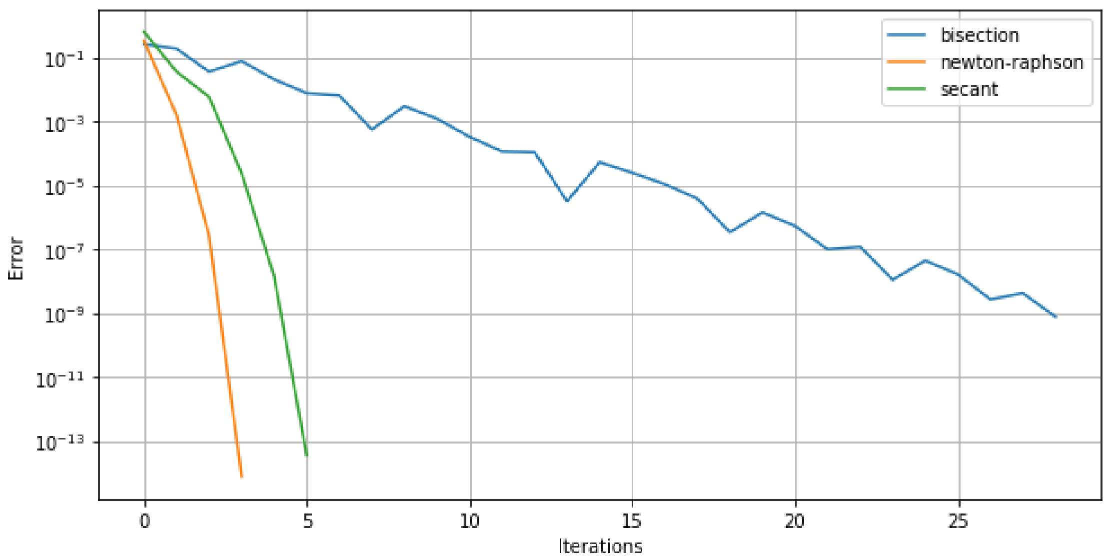
Root-Finding Implementations

```
import numpy as np
import matplotlib.pyplot as plt

def bisection(f, x1, x2, tolerance_needed=.01,
              prev_iters=[], save_iters=False):
    # signs should differ
    assert np.sign(f(x1)) != np.sign(f(x2))
    if (np.abs(x1 - x2) < tolerance_needed):
        if save_iters:
            return (x1, x2), prev_iters
        return np.mean([x1, x2])
    sign_f_x1 = np.sign(f(x1))
    # Find midpoint between x1 and x2
    x0 = (x1 + x2) / 2
    # check sign of f(midpoint)
    sign_f_x0 = np.sign(f(x0))
    # branch off depending on sign
    b1 = x0
```

```python
    if (sign_f_x0 == sign_f_x1):
        # bracket = [x0, x2]
        b2 = x2
    else:
        # bracket = [x1, x0]
        b2 = x1

    if save_iters:
        return bisection(
            f, b1, b2, tolerance_needed=tolerance_needed,
            prev_iters=prev_iters + [np.mean([b1, b2])], save_iters=True)
    return bisection(f, b1, b2, tolerance_needed=tolerance_needed)

def newton_raphson(f, fPrime, x1, tolerance_needed=.01, last_guess=np.inf,
                   prev_iters=[], save_iters=False):
    # we can get info on our precision by using |f(x_last) / f'(x_last)|
    if (np.abs(x1 - last_guess) < tolerance_needed):
        if save_iters:
            return x1, prev_iters
        return x1
    # step update function
    x2 = x1 - (f(x1) / fPrime(x1))
    if save_iters:
        return newton_raphson(
            f, fPrime, x2, tolerance_needed=tolerance_needed,
            last_guess=x1, prev_iters=prev_iters + [x1], save_iters=True)
    return newton_raphson(f, fPrime, x2, tolerance_needed=tolerance_needed,
                          last_guess=x1)

def secant(f, x2, x1, tolerance_needed=.01, prev_iters=[],
           save_iters=False):
    # estimate precision as with newton-raphson
    if (np.abs(x2 - x1) < tolerance_needed):
        if save_iters:
            return x2, prev_iters
        return x2
    # approximate derv with slope of line
    fPrime_approx = (f(x2) - f(x1)) / (x2 - x1)
    # step update function
    x3 = x2 - f(x2) * ((x2 - x1) / (f(x2) - f(x1)))
    if (save_iters):
        return secant(f, x3, x2, tolerance_needed=tolerance_needed,
                      prev_iters=prev_iters + [x2], save_iters=True)
    return secant(f, x3, x2, tolerance_needed=tolerance_needed)


def test_f(x):
    return np.sin(x) + .2

def test_fPrime(x):
    return np.cos(x)

# We'll test these each with a tolerance of 1e-8 and save our previous
# iterations to plot.
tol = 1e-8
bisecs = bisection(test_f, 4, .3, tolerance_needed=tol, save_iters=True)
raphs = newton_raphson(test_f, test_fPrime, 3,
                       tolerance_needed=tol, save_iters=True)
secants = secant(test_f, 4, 3, tolerance_needed=tol, save_iters=True)
```

Convergence Test

```python
# calc in Mathematica
actual_root = 3.342950574380124
plt.figure(figsize=(10, 5))
plt.plot(np.abs(np.array(bisecs[1]) - actual_root), label='bisection')
plt.plot(np.abs(np.array(raphs[1]) - actual_root), label='newton-raphson')
plt.plot(np.abs(np.array(secants[1]) - actual_root), label='secant')
plt.yscale('log')
plt.grid()
plt.legend()
plt.xlabel('Iterations')
plt.ylabel('Error')
plt.show()
```



Here we do see that the Newton-Raphson method slightly outperforms the secant method, and both hugely outperform the bisection method.

Problem 3

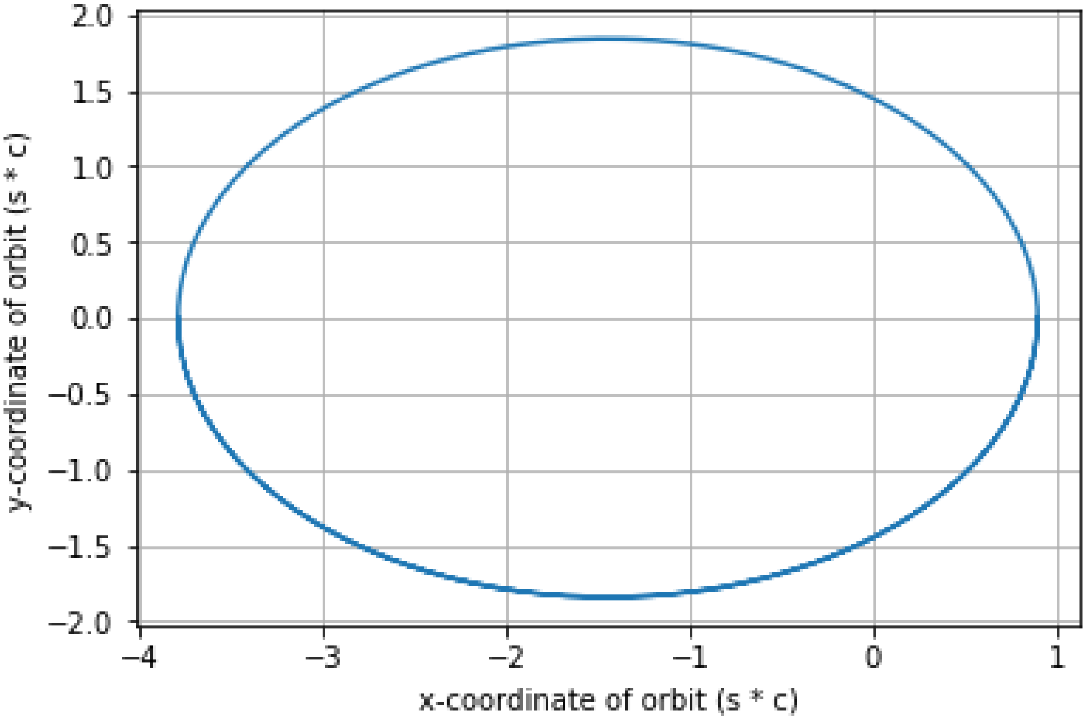
We start with $e = .617139$, $T = 27906.98161$, $a = 2.34186s \times c$. We wish to solve for the elliptical orbit of the system by finding ξ in terms of t and the equations for x, y in terms of ξ.

We start by solving the equation $\frac{T}{2\pi}(\xi - e\sin\xi) - t^* = 0$ and then plugging this value into the equations $x = a(\cos\xi - e), y = a\sqrt{1-e^2}\sin\xi)$.

```python
e_val = .617139
T_val = 27906.98161 # units of s
a_val = 2.34186 # units of s * c

def test_func(xi, tStar, e, T, a):
    return (T / (2 * np.pi)) * (xi - e * np.sin(xi)) - tStar
```

```python
def xi_func(tStar, e, T, a):
    def xi_one_var(xi):
        return (T / (2 * np.pi)) * (xi - e * np.sin(xi)) - tStar
    return xi_one_var

def solve_xi(tStar, e, T, a):
    xi_solve_func = xi_func(tStar, e, T, a)
    secant_xi_solve = secant(xi_solve_func, 0, 2 * np.pi, tolerance_needed=.0001)
    return secant_xi_solve

def get_x_y(xi, e, a):
    x = a * (np.cos(xi) - e)
    y = a * np.sqrt(1 - e ** 2) * np.sin(xi)
    return (x, y)

def solve_pos(tStar, e, T, a):
    xi = solve_xi(tStar, e, T, a)
    x, y = get_x_y(xi, e, a)
    return (x, y)

def solve_orbit(e, T, a, tStar_vals):
    # solve for orbit in terms of other times
    xy_vals = list(map(lambda tStar: solve_pos(tStar, e, T, a), tStar_vals))
    return np.array(xy_vals)
```

```python
tStar_vals = np.linspace(-T_val / 2, T_val, 500)
orbit_solve = solve_orbit(e_val, T_val, a_val, tStar_vals)
x_vals = orbit_solve[::, 0]
y_vals = orbit_solve[::, 1]

plt.plot(x_vals, y_vals)
plt.xlabel('x-coordinate of orbit (s * c)')
plt.ylabel('y-coordinate of orbit (s * c)')
plt.grid()
plt.show()
```



And here we see the orbit is an ellipse as expected!

Problem 4

We will obtain the velocities of the orbit through finite-difference formulas:

$$x'(t) \approx [x(t+\Delta t) - x(t)]/\Delta t, y'(t) \approx [y(t+\Delta t) - y(t)]/\Delta t$$

```
def get_velocity(pos_data, dt):
    # assume that dt is the same for each value in pos_data add array to index
    # + 1 (can cycically shift since last index and first index are connected
    # since orbit is periodic))
    velocity_data = (np.roll(pos_data, -1) - pos_data) / dt
    return velocity_data

def get_radial_vel(x_data, y_data, t_data, phi):
    dt = t_data[1] - t_data[0]
    x_vels = get_velocity(x_data, dt)
    y_vels = get_velocity(y_data, dt)
    # project {x'(t), y'(t)} onto unit vector (in terms of phi)
    r_vels = np.dot(np.array([x_vels, y_vels]).T,
                    np.array([np.cos(phi), np.sin(phi)]))
    return r_vels
```

After testing out various values of ϕ, the value of $\frac{-\pi}{2}$ gave the most qualitative agreement with Fig. 3 from the assignment:

```
r_vels = get_radial_vel(x_vals, y_vals, tStar_vals, -.5 * np.pi)
# Don't plot last value so that the last line doesn't go straight-up
# divide by the max to put in units of t/T
# convert to km / s
plt.plot((tStar_vals[:-1] / np.max(tStar_vals)), (r_vels * 3 * 10 ** 5)[:-1])
plt.xlabel('Phase (t / T)')
plt.ylabel('Radial Velocity (km / s)')
plt.grid()
plt.show()
```