

**ADDITIVE COMBINATORICS COURSE**  
**PROBLEMS 3**

- (1) (a) Let  $(G, +)$  be an abelian group and let  $A, B \subset G$  be finite sets. Let

$$E(A, B) = \#\{(a, a', b, b') \in A^2 \times B^2 : a + b = a' + b'\}.$$

Prove that  $E(A, B) \geq |A|^2|B|^2/|A+B|$ .

- (b) Let  $A = [N] \cup \{2^N, \dots, 2^{2N}\}$ . Show that  $E(A, A) \gg |A|^3$  but  $|A+A| \gg |A|^2$ .
- (2) Let  $A \subset G$  be a finite subset of an abelian group  $(G, +)$ . Assume that

$$E(A, A) \geq \frac{|A|^3}{K},$$

for some  $K \geq 1$ . Combine the Ruzsa triangle inequality with the Balog–Szemerédi–Gowers theorem to deduce that there exists a subset  $A' \subset A$  such that  $|A'| \gg K^{-O(1)}|A|$  and  $|A' + A'| \ll K^{O(1)}|A'|$ .

- (3) Let  $\varepsilon > 0$  and let  $A, B$  be finite sets. Let  $H = (A, B, E)$  be a bipartite graph with edge set  $E$  between  $A$  and  $B$ , such that  $|E| \geq |A||B|/K$  for some  $K \geq 1$ .

- (a) The neighbourhood of  $b \in B$  is defined to be  $A' = A'(b) = \{a \in A : \{a, b\} \in E\}$ . Show that

$$\mathbb{E}_{b \in B} |A'| \geq \frac{|A|}{K}.$$

- (b) Let  $S$  be the set of  $(a'_1, a'_2) \in A' \times A'$  such that  $a'_1$  and  $a'_2$  are connected by fewer than  $\varepsilon|B|/(2K^2)$  paths of length 2. Show that

$$\mathbb{E}_{b \in B} |S| < \frac{\varepsilon|A|^2}{2K^2}.$$

- (c) Deduce that there must exist  $b \in B$  such that

$$|A'|^2 - \frac{|S|}{\varepsilon} \geq \frac{|A|^2}{2K^2}.$$

- (d) Conclude that  $|S| \leq \varepsilon|A'|^2$ ; i.e. all but at most  $\varepsilon|A'|^2$  pairs of vertices  $a'_1, a'_2 \in A'$  are joined by at least  $\frac{\varepsilon}{2K^2}|B|$  paths of length 2.