ADDITIVE COMBINATORICS COURSE
PROBLEMS 3

(1) (a) Let $(G,+)$ be an abelian group and let $A, B \subset G$ be finite sets. Let
$$E(A,B) = \#\{(a,a',b,b') \in A^2 \times B^2 : a+b = a'+b'\}.$$
Prove that $E(A,B) \geqslant |A|^2|B|^2/|A+B|$.
(b) Let $A = [N] \cup \{2^N, \ldots, 2^{2N}\}$. Show that $E(A,A) \gg |A|^3$ but $|A+A| \gg |A|^2$.

(2) Let $A \subset G$ be a finite subset of an abelian group $(G,+)$. Assume that $|E(A,A)| \geqslant |A|/K$ for some $K \geqslant 1$. Combine the Ruzsa triangle inequality with the Balog–Szemerédi–Gowers theorem to deduce that there exists a subset $A' \subset A$ such that $|A'| \gg K^{-O(1)}|A|$ and $|A'+A'| \ll K^{O(1)}|A'|$.

(3) Let $\varepsilon > 0$ and let A, B be finite sets. Let $H = (A,B,E)$ be a bipartite graph with edge set E between A and B, such that $|E| \geqslant |A||B|/K$ for some $K \geqslant 1$.
(a) The neighbourhood of $b \in B$ is defined to be $A' = A'(b) = \{a \in A : \{a,b\} \in E\}$. Show that
$$\mathbb{E}_{b \in A}|A'| \geqslant \frac{|A|}{K}.$$
(b) Let S be the set of $(a_1', a_2') \in A' \times A'$ such that a_1' and a_2' are connected by fewer than $\varepsilon|B|/(2K^2)$ paths of length 2. Show that
$$\mathbb{E}_{b \in A}|S| < \frac{\varepsilon|A|^2}{2K^2}.$$
(c) Deduce that there must exist $b \in B$ such that
$$|A'|^2 - \frac{|S|}{\varepsilon} \geqslant \frac{|A|^2}{2K^2}.$$
(d) Conclude that $|S| \leqslant \varepsilon|A'|^2$; i.e. all but at most $\varepsilon|A'|^2$ pairs of vertices $a_1', a_2' \in A'$ are joined by at least $\frac{\varepsilon}{2K^2}|B|$ paths of length 2.