Series convergence

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1 Series

The kernel matrix in Williams & Porter (2009) is $\mathrm{i} = \sqrt{-1}$ multiplied by the following:

$$G_{mn}^{(j)} = \sum_{\alpha\in\mathscr{S}_j} B_j^{(2)}(\alpha)F_m^{(j)}(\alpha)F_n^{(j)}(\alpha). \tag{1}$$

In practice we truncate this series so there are only N imaginary roots in $\mathscr{S}_j$, i.e.

$$G_{mn}^{(j)}(N) = \sum_{\alpha\in\mathscr{S}_j^{(N)}} B_j^{(2)}(\alpha)F_m^{(j)}(\alpha)F_n^{(j)}(\alpha), \tag{2}$$

where $\mathscr{S}_j^{(N)} \subset \mathscr{S}_j$.

$$\begin{aligned}
F_n^{(j)}(\alpha) &= \frac{2}{H_1}\int_{-H}^{-\sigma_1} \varphi^{(j)}(z,\alpha)\mathscr{C}_{2n}\left(\frac{z+H}{H_1}\right)dz \\
&= 2\mathrm{sech}(\gamma H_j)\int_0^1 \cos(\kappa t)\mathscr{C}_{2n}(t)dt \\
&= \frac{(-1)^n\Gamma(\beta)}{\cosh(\gamma H_j)}\left(\frac{\kappa}{2}\right)^{-\beta}(2n+\beta)J_{2n+\beta}(\kappa),
\end{aligned} \tag{3}$$

where $\kappa = -\mathrm{i}\gamma H_2$.

Now let $\tilde{\gamma}_{j,r} = \mathrm{i}n\pi/H_j$, and $\tilde{\kappa}_{j,r} = -\mathrm{i}\tilde{\gamma}_{j,r}H_2 = r\pi H_2/H_j = r\pi/\hat{H}_j$ ($r = 1, 2, \ldots$). Then

$$B_j^{(2)}(\tilde{\gamma}_{j,r}) = \frac{\mathrm{i}}{r\pi}, \tag{4a}$$

$$\begin{aligned}
J_{2n+\beta}(\tilde{\kappa}_{j,r}) &= \sqrt{\frac{2}{\pi\tilde{\kappa}_{j,r}}}\cos\left(\tilde{\kappa}_{j,r} - (2n+\beta)\frac{\pi}{2} - \frac{\pi}{4}\right) \\
&= (-1)^n r^{-1/2}\sqrt{\frac{2\hat{H}_j}{\pi^2}}\cos\left(\tilde{\kappa}_{j,r} - \beta'\right),
\end{aligned} \tag{4b}$$

$$\beta' = \frac{\pi}{2}\left(\beta + \frac{1}{2}\right). \tag{4c}$$

Thus

$$F_n^{(j)}(\tilde{\gamma}_{j,r}) = (-1)^r r^{-(2\beta+1)/2} f_{j,n}(\beta) \cos\left(\tilde{\kappa}_{j,r} - \beta'\right), \tag{5a}$$

$$f_{j,n}(\beta) = \Gamma(\beta)(2n+\beta) \left(\frac{\pi}{2\hat{H}_j}\right)^{-\beta} \sqrt{\frac{2\hat{H}_j}{\pi^2}}, \tag{5b}$$

and

$$\begin{aligned}
\tilde{G}_{mn}^{(j)}(N) &= \sum_{r=1}^{N} B_j^{(2)}(\tilde{\gamma}_{j,r}) F_m^{(j)}(\tilde{\gamma}_{j,r}) F_n^{(j)}(\tilde{\gamma}_{j,r}) \\
&= \frac{\mathrm{i}}{\pi} f_{j,n}(\beta) f_{j,m}(\beta) \sum_{r=1}^{N} r^{-s} \cos^2\left(\tilde{\kappa}_{j,r} - \beta'\right).
\end{aligned} \tag{6}$$

where $s = 2(\beta+1)$. Therefore

$$\begin{aligned}
\tilde{G}_{mn}^{(j)} &= \lim_{N\to\infty} \tilde{G}_{mn}^{(j)}(N) \\
&= \frac{\mathrm{i}}{2\pi} f_{j,n}(\beta) f_{j,m}(\beta) \sum_{r=1}^{\infty} r^{-s} \left(1 + \mathrm{Re}\left(\mathrm{e}^{-2\mathrm{i}\beta'} \mathrm{e}^{2\mathrm{i}\pi r/\hat{H}_j}\right)\right) \\
&= \frac{\mathrm{i}}{2\pi} f_{j,n}(\beta) f_{j,m}(\beta) \left(\zeta(s) + \mathrm{Re}\left(\mathrm{e}^{-2\mathrm{i}\beta'} \mathrm{Li}_s\left(\mathrm{e}^{2\mathrm{i}\pi/\hat{H}_j}\right)\right)\right),
\end{aligned} \tag{7}$$

where

$$\mathrm{Li}_s(z) = \sum_{r=1}^{\infty} \frac{z^r}{r^s} \quad \text{for } |z| \le 1 \tag{8}$$

is the polylogarithm function (Wood, 1992, also see the next section). Note that $\mathrm{Li}_s(1) = \zeta(s)$, so since $\hat{H}_2 = 1$,

$$\begin{aligned}
\tilde{G}_{mn}^{(2)} &= \frac{\mathrm{i}}{2\pi} f_{2,n}(\beta) f_{2,m}(\beta) \zeta(s) \left(1 + \cos(2\beta')\right), \\
&= \frac{\mathrm{i}}{\pi} f_{2,n}(\beta) f_{2,m}(\beta) \zeta(s) \cos^2(\beta').
\end{aligned} \tag{9}$$

Finally we approximate

$$G_{mn}^{(j)} \approx G_{mn}^{(j)}(N) - \tilde{G}_{mn}^{(j)}(N) + \tilde{G}_{mn}^{(j)}, \tag{10}$$

which gives the summand in the series (6) decays like $O(r^{-(s+1)})$. This means it can be truncated at a much lower value of N.

2 Calculation of the polylogarithm function

Li_s has the Bose-Einstein integral representation (Wood, 1992)

$$\mathrm{Li}_s(z) = \frac{z}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1}\mathrm{e}^{-t}}{1 - z\mathrm{e}^{-t}} \mathrm{d}t \quad \text{for } z \in \mathbb{C}\backslash[1,\infty), \tag{11}$$

where Li_s has a branch cut. It converges reasonably well using Gauss-Laguerre quadrature unless z gets too close to this branch cut. By taking the Mellin transform of this integral, Wood (1992) derives an expansion in $w = \log z$ which works extremely well near $z = 1$ ($|w| \leq 2\pi$):

$$\mathrm{Li}_s(\mathrm{e}^w) = \sum_{k=0}^{\infty} \zeta(s-k)\frac{w^k}{k!} + \Gamma(1-s)(-w)^{s-1} \quad (s \neq 1, 2, \ldots), \tag{12}$$

or

$$\mathrm{Li}_s(\mathrm{e}^w) = \sum_{k=0,\, k\neq s-1}^{\infty} \zeta(s-k)\frac{w^k}{k!} + f_s(w)\frac{w^{s-1}}{(s-1)!} \quad (s = 1, 2, \ldots) \tag{13}$$

where

$$f_s(w) = \sum_{k=1}^{s-1} \frac{1}{k} - \log(-w).$$

Near $z = 1$ ($|w| = |\log(-z)| < \pi$), the alternative series works well:

$$\mathrm{Li}_s(\mathrm{e}^{-w}) = -\sum_{k=0,\, k\neq s-1}^{\infty} \eta(s-k)\frac{w^k}{k!}, \tag{14}$$

where $\eta(s) = (1-2^{1-s})\zeta(s)$.

References

WILLIAMS, T. D. & PORTER, R. 2009 The effect of submergence on the scattering by the interface between two semi-infinite sheets. *Journal of Fluids and Structures* **25**, 777–793.

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