

Series convergence

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Abstract

Reference: Williams & Porter (2009)

1 Series

$$G_{mn}^{(j)} = \sum_{\alpha \in \mathcal{S}_j} B_j^{(2)}(\alpha) F_m^{(j)}(\alpha) F_n^{(j)}(\alpha). \quad (1)$$

In practice we truncate this series so there are only N imaginary roots in \mathcal{S}_j , i.e.

$$G_{mn}^{(j)}(N) = \sum_{\alpha \in \mathcal{S}_j^{(N)}} B_j^{(2)}(\alpha) F_m^{(j)}(\alpha) F_n^{(j)}(\alpha), \quad (2)$$

where $\mathcal{S}_j^{(N)} \subset \mathcal{S}_j$.

$$\begin{aligned} F_n^{(j)}(\alpha) &= \frac{2}{H_1} \int_{-H}^{-\sigma_1} \varphi^{(j)}(z, \alpha) \mathcal{C}_{2n} \left(\frac{z+H}{H_1} \right) dz \\ &= 2 \operatorname{sech}(\gamma H_j) \int_0^1 \cos(\kappa t) \mathcal{C}_{2n}(t) dt \\ &= \frac{(-1)^n \Gamma(\beta)}{\cosh(\gamma H_j)} \left(\frac{\kappa}{2} \right)^{-\beta} (2n + \beta) J_{2n+\beta}(\kappa), \end{aligned} \quad (3)$$

where $\kappa = -i\gamma H_2$.

Now let $\tilde{\gamma}_{j,r} = in\pi/H_j$, and $\tilde{\kappa}_{j,r} = -i\tilde{\gamma}_{j,r}H_2 = r\pi H_2/H_j = r\pi/\hat{H}_j$ ($r = 1, 2, \dots$). Then

$$B_j^{(2)}(\tilde{\gamma}_{j,r}) = \frac{i}{r\pi}, \quad (4a)$$

$$\begin{aligned} J_{2n+\beta}(\tilde{\kappa}_{j,r}) &= \sqrt{\frac{2}{\pi \tilde{\kappa}_{j,r}}} \cos \left(\tilde{\kappa}_{j,r} - (2n + \beta) \frac{\pi}{2} - \frac{\pi}{4} \right) \\ &= (-1)^n r^{-1/2} \sqrt{\frac{2\hat{H}_j}{\pi^2}} \cos(\tilde{\kappa}_{j,r} - \beta'), \end{aligned} \quad (4b)$$

$$\beta' = \frac{\pi}{2} \left(\beta + \frac{1}{2} \right). \quad (4c)$$

Thus

$$F_n^{(j)}(\tilde{\gamma}_{j,r}) = (-1)^r r^{-(2\beta+1)/2} f_{j,n}(\beta) \cos(\tilde{\kappa}_{j,r} - \beta'), \quad (5a)$$

$$f_{j,n}(\beta) = \Gamma(\beta)(2n + \beta) \left(\frac{\pi}{2\hat{H}_j} \right)^{-\beta} \sqrt{\frac{2\hat{H}_j}{\pi^2}}, \quad (5b)$$

and

$$\begin{aligned} \tilde{G}_{mn}^{(j)}(N) &= \sum_{r=1}^N B_j^{(2)}(\tilde{\gamma}_{j,r}) F_m^{(j)}(\tilde{\gamma}_{j,r}) F_n^{(j)}(\tilde{\gamma}_{j,r}) \\ &= \frac{i}{\pi} f_{j,n}(\beta) f_{j,m}(\beta) \sum_{r=1}^N r^{-s} \cos^2(\tilde{\kappa}_{j,r} - \beta'). \end{aligned} \quad (6)$$

where $s = 2(\beta + 1)$. Therefore

$$\begin{aligned} \tilde{G}_{mn}^{(j)} &= \lim_{N \rightarrow \infty} \tilde{G}_{mn}^{(j)}(N) \\ &= \frac{i}{2\pi} f_{j,n}(\beta) f_{j,m}(\beta) \sum_{r=1}^{\infty} r^{-s} \left(1 + \operatorname{Re} \left(e^{-2i\beta'} e^{2i\pi r / \hat{H}_j} \right) \right) \\ &= \frac{i}{2\pi} f_{j,n}(\beta) f_{j,m}(\beta) \left(\zeta(s) + \operatorname{Re} \left(e^{-2i\beta'} \operatorname{Li}_s \left(e^{2i\pi / \hat{H}_j} \right) \right) \right), \end{aligned} \quad (7)$$

where

$$\begin{aligned} \operatorname{Li}_s(z) &= \sum_{r=1}^{\infty} \frac{z^r}{r^s} \\ &= \frac{z}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1} e^{-t}}{1 - z e^{-t}} dt \end{aligned} \quad (8)$$

is the polylogarithm function (??). This integral can be done very efficiently with Gauss-Laguerre quadrature. Also note that $\operatorname{Li}_s(1) = \zeta(s)$, which also implies that for $j = 2$, when $\hat{H}_2 = 1$,

$$\begin{aligned} \tilde{G}_{mn}^{(2)} &= \frac{i}{2\pi} f_{2,n}(\beta) f_{2,m}(\beta) \zeta(s) (1 + \cos(2\beta')), \\ &= \frac{i}{\pi} f_{2,n}(\beta) f_{2,m}(\beta) \zeta(s) \cos^2(\beta') \end{aligned} \quad (9)$$

Finally we approximate

$$G_{mn}^{(j)} \approx G_{mn}^{(j)}(N) - \tilde{G}_{mn}^{(j)}(N) + \tilde{G}_{mn}^{(j)}, \quad (10)$$

which gives the summand in the series (6) decays like $O(r^{-(s+1)})$. This means it can be truncated at a much lower value of N .

References

WILLIAMS, T. D. & PORTER, R. 2009 The effect of submergence on the scattering by the interface between two semi-infinite sheets. *Journal of Fluids and Structures* **25**, 777–793.