# Possible collision model

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**Abstract**

Collisions reference: **?**

## 1 Series

$$G_{mn}^{(j)} = \sum_{\alpha\in\mathscr{S}_j} B_j^{(2)}(\alpha)F_m^{(j)}(\alpha)F_n^{(j)}(\alpha). \tag{1}$$

In practice we truncate this series so there are only $N$ imaginary roots in $\mathscr{S}_j$, i.e.

$$G_{mn}^{(j)}(N) = \sum_{\alpha\in\mathscr{S}_j^{(N)}} B_j^{(2)}(\alpha)F_m^{(j)}(\alpha)F_n^{(j)}(\alpha), \tag{2}$$

where $\mathscr{S}_j^{(N)} \subset \mathscr{S}_j$.

$$\begin{aligned}
F_n^{(j)}(\alpha) &= \frac{2}{H_1}\int_{-H}^{-\sigma_1} \varphi^{(j)}(z,\alpha)\mathscr{C}_{2n}\left(\frac{z+H}{H_1}\right)dz \\
&= 2\mathrm{sech}(\gamma H_j)\int_0^1 \cos(\kappa t)\mathscr{C}_{2n}(t)dt \\
&= \frac{(-1)^n\Gamma(\beta)}{\cosh(\gamma H_j)}\left(\frac{\kappa}{2}\right)^{-\beta}(2n+\beta)J_{2n+\beta}(\kappa),
\end{aligned} \tag{3}$$

where $\kappa = -\mathrm{i}\gamma H_2$.

Now let $\tilde{\gamma}_{j,r} = \mathrm{i}n\pi/H_j$, and $\tilde{\kappa}_{j,r} = -\mathrm{i}\tilde{\gamma}_{j,r}H_2 = r\pi H_2/H_j = r\pi/\hat{H}_j$ ($r = 1,2,\ldots$). Then

$$B_j^{(2)}(\tilde{\gamma}_{j,r}) = \frac{\mathrm{i}}{r\pi}, \tag{4a}$$

$$\begin{aligned}
J_{2n+\beta}(\tilde{\kappa}_{j,r}) &= \sqrt{\frac{2}{\pi\tilde{\kappa}_{j,r}}}\cos\left(\tilde{\kappa}_{j,r} - (2n+\beta)\frac{\pi}{2} - \frac{\pi}{4}\right) \\
&= (-1)^n r^{-1/2}\sqrt{\frac{2}{\pi^2\hat{H}_j}}\cos\left(\tilde{\kappa}_{j,r} - \beta'\right),
\end{aligned} \tag{4b}$$

$$\beta' = \frac{\pi}{2}\left(\beta + \frac{1}{2}\right). \tag{4c}$$

Thus

$$F_n^{(j)}(\tilde{\gamma}_{j,r}) = (-1)^r r^{-(2\beta+1)/2} f_{j,n}(\beta)\cos\left(\tilde{\kappa}_{j,r} - \beta'\right), \tag{5a}$$

$$f_{j,n}(\beta) = \Gamma(\beta)(2n+\beta)\left(\frac{\pi}{2\hat{H}_j}\right)^{-\beta}\sqrt{\frac{2}{\pi^2\hat{H}_j}}, \tag{5b}$$

and

$$\begin{aligned}
\tilde{G}_{mn}^{(j)}(N) &= \sum_{r=1}^{N} B_j^{(2)}(\tilde{\gamma}_{j,r})F_m^{(j)}(\tilde{\gamma}_{j,r})F_n^{(j)}(\tilde{\gamma}_{j,r}) \\
&= \frac{\mathrm{i}}{\pi} f_{j,n}(\beta) f_{j,m}(\beta)\sum_{r=1}^{N} r^{-s}\cos^2\left(\tilde{\kappa}_{j,r} - \beta'\right).
\end{aligned} \tag{6}$$

where $s = 2(\beta+1)$. Therefore

$$\begin{aligned}
\tilde{G}_{mn}^{(j)} &= \lim_{N\to\infty}\tilde{G}_{mn}^{(j)}(N) \\
&= \frac{\mathrm{i}}{2\pi} f_{j,n}(\beta) f_{j,m}(\beta)\sum_{r=1}^{\infty} r^{-s}\left(1 + \mathrm{Re}\left(\mathrm{e}^{-2\mathrm{i}\beta'}\mathrm{e}^{2\mathrm{i}\pi r/\hat{H}_j}\right)\right) \\
&= \frac{\mathrm{i}}{2\pi} f_{j,n}(\beta) f_{j,m}(\beta)\left(\zeta(s) + \mathrm{Re}\left(\mathrm{e}^{-2\mathrm{i}\beta'}\mathrm{Li}_s\left(\mathrm{e}^{2\mathrm{i}\pi/\hat{H}_j}\right)\right)\right),
\end{aligned} \tag{7}$$

where

$$\begin{aligned}
\mathrm{Li}_s(z) &= \sum_{r=1}^{\infty}\frac{z^r}{r^s} \\
&= \frac{z}{\Gamma(s)}\int_0^{\infty}\frac{t^{s-1}\mathrm{e}^{-t}}{1 - z\mathrm{e}^{-t}}\mathrm{d}t
\end{aligned} \tag{8}$$

is the polylogarithm function (??). This integral can be done very efficiently with Gauss-Laguerre quadrature. Also note that $\mathrm{Li}_s(1) = \zeta(s)$, which also implies that for $j = 2$, when $\hat{H}_2 = 1$,

$$\begin{aligned}
\tilde{G}_{mn}^{(2)} &= \frac{\mathrm{i}}{2\pi} f_{2,n}(\beta) f_{2,m}(\beta)\zeta(s)\left(1 + \cos(2\mathrm{i}\beta')\right), \\
&= \frac{\mathrm{i}}{\pi} f_{2,n}(\beta) f_{2,m}(\beta)\zeta(s)\cos^2(\beta')
\end{aligned} \tag{9}$$

Finally we approximate

$$G_{mn}^{(j)} \approx G_{mn}^{(j)}(N) - \tilde{G}_{mn}^{(j)}(N) + \tilde{G}_{mn}^{(j)}, \tag{10}$$

which gives the summand in the series (6) decays like $O(r^{-(s+1)})$. This means it can be truncated at a much lower value of $N$.