

# Transverse Momentum Dependent Factorization Recipe Sheet Part 1: Semi-Inclusive Deeply Inelastic Scattering (SIDIS)

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(Dated: February 12, 2017)

This work intended to be used both as a reference list of basic formulas for doing TMD factorization calculations and as a Rosetta stone for translating notational conventions throughout the existing TMD literature. The formulas here should be checked and verified frequently and updated as needed. This version deals with SIDIS.

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## I. GLOSSARIES

### A. Color Coding

- **Red color alerts to potential notation/terminology clashes.**
- **Blue color alerts to potential typos.**
- **Green color means general comments, stuff to add.**

### B. Initials Glossary

- **JCC**: J.C. Collins’s textbook [1] and related work.
- **MOS**: Meng-Olness-Soper [2] and related work.
- **NSY**: Nadolsky-Stump-Yuan [3] and related work. Very closely related to **MOS**.  $\Rightarrow$  **Need to fill in**
- **PJM**: Mulders and Tangerman [4] and related work. Overlaps heavily with Ref. [5]. See also Ref. [6].
- **JMY**: Ji-Ma-Yuan [7] and related work.
- **KNV**: Koike-Nagashima-Vogelsang [8] and related work.  $\Rightarrow$  **Need to fill in**
- **BDR**: Barone-Drago-Ratcliffe [9].  $\Rightarrow$  **Need to fill in**
- **CPGRSW**: Collins-Prokudin-Gamberg-Rogers-Sato-Wang [10].  $\Rightarrow$  **Need to fill in**

### C. Notation Glossary

- $\otimes$  = longitudinal momentum fraction convolution.
- $\circledast$  = transverse momentum convolution.
- $m$  = any small mass that does not grow with  $Q$ , e.g.  $m \in \{m_q, M, \Lambda_{\text{QCD}}, \dots\}$ . It could also be a small transverse momentum.
- “ $\approx$ ” means “dropping  $m/Q$  power-suppressed corrections.”

## II. BASICS OF SIDIS

The proton has momentum  $P$ , the virtual photon has momentum  $q$ , the produced hadron has momentum  $P_h$ , and the incoming and scattered leptons have momenta  $l$  and  $l'$  respectively. Except when specified, it should be assumed that the frame is one where  $P \cdot q \approx P^+ q^- = O(Q^2)$ . The mass of the target proton is  $M$  and the mass of the produced hadron is  $M_B$ .

### A. Lorentz Invariant Variables

The conventional kinematic variables are:

$$Q^2 = -q^2 = -(l - l')^2, \quad x_{\text{bj}} = \frac{Q^2}{2P \cdot q}, \quad x_n = \frac{2x_{\text{bj}}}{1 + \sqrt{1 + \frac{4x_{\text{bj}}^2 Q^2}{M^2}}}, \quad (1)$$

$$y = \frac{P \cdot q}{P \cdot l}, \quad z_h = \frac{P \cdot P_B}{P \cdot q}, \quad s = (l + P)^2, \quad (2)$$

$$W = (q + P)^2. \quad (3)$$

The inclusive deep inelastic limit is  $m/Q \rightarrow \infty$  with fixed  $x_n$  and  $z_h$ . Here  $m$  is any small mass scale like a quark mass,  $M_B$ ,  $M$ ,  $\Lambda_{\text{QCD}}$ , or a small transverse momentum. The kinematical variables obey

$$Q^2 = x_n y (s - M^2 - m_l^2) \approx x_n y s. \quad (4)$$

The target hadron has a spin vector  $S$ .

We will use Nachtmann everywhere  $x_n$  rather than the usual  $x_{\text{bj}}$ .

### B. Kinematical Map of SIDIS

SIDIS includes at least four distinct kinematical regions, depending on the momentum  $\mathbf{P}_B$  of the produced hadron. Each region, in principle, corresponds to a different factorization theorem and a different physical picture for the production mechanism.

## III. FRAGMENTATION REGION

### A. Frames

There are at least four important reference frames.

- JCC photon frame:

There are two reference frames in JCC [1, Sec.13.15.1]. JCC uses  $P_A$  instead of  $P$  and  $p_B$  instead of  $P_B$ .

In the photon frame, the virtual photon and the initial proton both have zero transverse momentum, while the final state produce hadron acquires non-zero transverse momentum. It is thus analogous to the Collins-Soper frame for Drell-Yan scattering and is the frame more closely related to experimental observables. JCC defines

$$q_{\text{JCC},\gamma} = \left( -x_n P_{\text{JCC},\gamma}^+, \frac{Q^2}{2x_n P_{\text{JCC},\gamma}^+}, \mathbf{0}_T \right), \quad (5)$$

$$P_{\text{JCC},\gamma} = \left( P_{\text{JCC},\gamma}^+, \frac{M^2}{2P_{\text{JCC},\gamma}^+}, \mathbf{0}_T \right), \quad (6)$$

$$P_{B\text{JCC},\gamma} = \left( \frac{P_B^2 + M_B^2}{2P_{B\text{JCC},\gamma}^-}, P_{B\text{JCC},\gamma}^-, \mathbf{P}_{BT,\text{JCC},\gamma} \right). \quad (7)$$

In the JCC photon frame

$$P_{B\text{JCC},\gamma}^- \approx \frac{z_h Q^2}{2x_n P_{\text{JCC},\gamma}^+} \quad (8)$$

up to mass corrections.

- JCC hadron frame:

In the **JCC** hadron frame, the incoming hadron and final state hadron are exactly back-to-back (zero relative transverse momentum) while the virtual photon generally has non-zero transverse momentum. The **JCC** hadron frame is especially useful for setting up factorization. (See [1, Sec.13.15.5].) The components of the four-momenta are:

$$q_{\text{JCC},h} = \left( q_{\text{JCC},h}^+, q_{\text{JCC},h}^-, \mathbf{q}_{\text{TJCC},h} \right), \quad (9)$$

$$P_{\text{JCC},h} = \left( P_h^+, \frac{M^2}{2P_h^+}, \mathbf{0}_T \right), \quad (10)$$

$$P_{B\text{JCC},h} = \left( \frac{M_B^2}{2P_{B\text{JCC},h}^-}, P_{B\text{JCC},h}^-, \mathbf{0}_T \right). \quad (11)$$

From [1, Eq.(13.104)] and Eq. (8), and from the requirement that  $\mathbf{P}_{BT,\text{JCC},h}$  is obtained from  $\mathbf{P}_{BT,\text{JCC},\gamma}$  by Lorentz boosting to zero transverse momentum,

$$\mathbf{q}_{\text{TJCC},h} = -\frac{\mathbf{P}_{BT,\text{JCC},\gamma}}{z_h}. \quad (12)$$

(This is only valid for zero hadron masses.) Note that the transverse momenta on the left and right sides of Eq. (12) are in different reference frames; the left side is in the **JCC** hadron frame while the right side is in the **JCC** photon frame.

**(In the Trento Conventions [11], the photon four momentum is labeled by  $k$  and Eq. (12) is expressed as  $k_T = -P_{hT}/z_h$  where the  $h$  subscript labels the final state hadron.)**

- **MOS** hadron frame:

There are two reference frames used by Meng-Olness-Soper(MOS) [2], the **MOS** hadron frame and the **MOS** HERA frame.

The **MOS** hadron frame in Ref. [2] is essentially the Breit frame. In light-cone coordinates

$$q_{\text{MOS},h} = \left( -\frac{Q}{\sqrt{2}}, \frac{Q}{\sqrt{2}}, \mathbf{0} \right) \quad (13)$$

and

$$P_{\text{MOS},h} = \left( \frac{Q}{x_n \sqrt{2}}, 0, \mathbf{0} \right), \quad (14)$$

where in the second equation masses are neglected. **MOS** use  $P_A$  instead of  $P$  for the hadron target momentum.

**Note: The JCC hadron frame has zero transverse momentum for the produced hadron and non-zero transverse momentum for the virtual photon, which is *opposite* the situation in the MOS hadron frame. The MOS hadron frame corresponds to the JCC photon frame.**

**MOS** define a Lorentz invariant four-vector ([2, Eq. (10)]) that measures the deviation from the back-to-back configuration:

$$q_t = q - \frac{P_B \cdot q}{P \cdot P_B} P - \frac{P \cdot q}{P \cdot P_B} P_B. \quad (15)$$

This definition removes the components of  $q$  along  $P$  and  $P_B$ . The Lorentz scalar

$$-q_t^2 \equiv q_T^2 \quad (16)$$

is a measure of the deviation from a back-to-back configuration. (Note that in the actual **MOS** hadron frame, the two dimensional photon transverse momentum is zero.) From [2, Eq. (11)] and [1, Eq. (13.104)] one may verify that the **JCC** hadron frame  $q_{hT}^2$  is the same as the **MOS**  $q_T^2$  (assuming massless hadrons):

$$q_T^2|_{\text{MOS}} = q_{\text{TJCC},h}^2 \quad (17)$$

Restricting to the **MOS** hadron frame, **MOS** use [2, Eq. (11)] and [2, Eq. (13)] and  $P_B^2 = 0$  to find [2, Eq. (12)], which in light-cone coordinates is

$$P_{B\text{MOS},h} = z_h \left( \frac{q_T^2}{Q\sqrt{2}}, \frac{Q}{\sqrt{2}}, |\mathbf{q}_T|, 0 \right). \quad (18)$$

Hadron masses are neglected. **Note that in the MOS hadron frame, the transverse part of  $P_B$  is always in the  $x_n$  direction and is always positive.**

- **MOS** HERA Frame:

Come back...

- **Other Authors**:

Most other authors use variations of the **JCC** and **MOS** reference frames, with  $\mathbf{P}_{BT}$  and  $\mathbf{q}_T$  corresponding to the **JCC**  $\mathbf{P}_{BT,\text{JCC},\gamma}$  and  $\mathbf{q}_{T,\text{JCC},h}$  respectively. Mulders and Tangerman [4, Eqs. (15-17)] give general expressions for four vector components that include the effects of hadron masses. The reference frames essentially correspond to **JCC** hadron and/or photon frames. References such as [5, 11, 12] specialize the photon frame to the target rest frame rather than the Breit frame.  $\mathbf{P}_{BT,\text{JCC},\gamma}$  is invariant, however, with respect to boosts in the  $z$  direction.

- **Lab Frame (Target Rest Frame)**:

Certain calculations are simplified by working in coordinates where the target proton is at rest and the lepton has a large energy along the  $+z$  direction. In the lab frame, neglecting the lepton mass,

$$l = (E_{\text{lab}}, 0, 0, E_{\text{lab}}); \quad l' = (E'_{\text{lab}}, 0, E'_{\text{lab}} \sin \theta_{\text{lab}}, E'_{\text{lab}} \cos \theta_{\text{lab}}); \quad P = (M, 0, 0, 0). \quad (19)$$

Specializing to the lab frame gives for the kinematic variables,

$$y = 1 = \frac{E'_{\text{lab}}}{E_{\text{lab}}}; \quad x_n = \frac{Q^2}{2M\nu}; \quad Q^2 = 2E_{\text{lab}}E'_{\text{lab}}(1 - \cos \theta_{\text{lab}}). \quad (20)$$

Here we have defined the lab frame quantity

$$\nu = E - E' = Ey. \quad (21)$$

**JMY [13] reference the MOS definition of the hadron frame rather than the JCC definition.**

### 1. Summary

From here forward, we will use **JCC** notation for frames unless otherwise specified. So we will no longer include **MOS**, **JCC**, **PJM**, **JMY**, etc subscripts for frames. Transverse components of  $q$  will be relative to the **JCC** hadron frame, and transverse components of  $P_B$  will be relative to the **JCC** photon frame, unless otherwise specified. That is, from here forward

$$\mathbf{q}_T \equiv \mathbf{q}_{T,h} \equiv \mathbf{q}_{T,\text{JCC},h} \quad (22)$$

$$\mathbf{P}_{BT} \equiv \mathbf{P}_{BT,\gamma} \equiv \mathbf{P}_{BT,\text{JCC},\gamma}. \quad (23)$$

The angles  $\psi$  and  $\phi$  will represent the azimuthal angles of the final state lepton and produced hadron respectively in a photon frame.

## B. Hadronic and Leptonic Tensors

**JCC**'s normalization convention for  $L_{\mu\nu}$  and  $W_{\mu\nu}$  [1, Eq. (2.15)] for *inclusive* DIS is

$$E' \frac{d\sigma}{d^3l'} = \frac{2\alpha_{\text{em}}^2}{sQ^4} L_{\mu\nu} W_{\text{JCC},\text{incl}}^{\mu\nu}, \quad (24)$$

where the leptonic tensor is defined as,

$$L_{\mu\nu} = 2(l_\mu l'_\nu + l'_\mu l_\nu - g_{\mu\nu} l \cdot l'). \quad (25)$$

**This is a factor of 2 larger than the conventions used by Diehl and Sapeta [12, Eq. (22)]. It is also a factor of 2 larger than the analogous tensor in [1, Eq.(12.7)] used for fragmentation in  $e^+e^-$  annihilation. MOS [2, Eq. (36)] use a leptonic tensor that differs from JCC by a factor of  $1/(2\pi\alpha_{\text{em}})$ :**

$$L_{\mu\nu}^{\text{JCC}} = \frac{1}{2\pi\alpha_{\text{em}}} L_{\mu\nu}^{\text{MOS}}. \quad (26)$$

The convention in Eq. (25) appears to match most other authors. **MOS** actually write a general  $L_{\mu\nu}$  that include all electroweak couplings. For simplicity, in this section we will focus attention on the photon.

The inclusive hadronic tensor following **JCC**'s normalization conventions [1, Eq. (2.18)] is

$$W_{\text{JCC}}^{\mu\nu}(P, q) \equiv 4\pi^3 \sum_X \delta^{(4)}(P + q - P_X) \langle P, S | j^\mu(0) | X \rangle \langle X | j^\nu(0) | P, S \rangle. \quad (27)$$

Using

$$\langle P, S | j^\mu(z) | X \rangle = \langle P, S | j^\mu(0) | X \rangle e^{i(P - P_X) \cdot z}$$

one may write instead

$$W_{\text{JCC}}^{\mu\nu}(P, q, P_B) \equiv \frac{1}{4\pi} \sum_X \int d^4z e^{iq \cdot z} \langle P, S | j^\mu(z) | X \rangle \langle X | j^\nu(0) | P, S \rangle. \quad (28)$$

**This differs by a factor of 1/2 from Eq.(3.1.5) of BDR.**

There are several common notational conventions for generalizing the hadronic tensor to the TMD SIDIS case:

- **JCC convention:**

The  $4\pi^3$  from Eq. (27) is *not* included in the definition of the hadronic tensor [1, Eq. (13.111)]:

$$W_{\text{JCC}, \text{SIDIS}}^{\mu\nu}(P, q, P_B) \equiv \sum_X \delta^{(4)}(P + q - P_B - P_X) \langle P, S | j^\mu(0) | P_B, X \rangle \langle P_B, X | j^\nu(0) | P, S \rangle, \quad (29)$$

Also,

$$W_{\text{JCC}, \text{SIDIS}}^{\mu\nu}(P, q, P_B) \equiv \frac{1}{(2\pi)^4} \sum_X \int d^4z e^{iq \cdot z} \langle P, S | j^\mu(z) | P_B, X \rangle \langle P_B, X | j^\nu(0) | P, S \rangle. \quad (30)$$

Equation (24) in the TMD SIDIS case becomes,

$$4P_{B\gamma}^0 E'_\gamma \frac{d\sigma}{d^3\mathbf{l}'_\gamma d^3\mathbf{P}_{B\gamma}} = \frac{2\alpha_{\text{em}}^2}{sQ^4} L_{\mu\nu} W_{\text{JCC}, \text{SIDIS}}^{\mu\nu}. \quad (31)$$

The factor of  $2(2\pi)^3$  from the extra phase space factor for  $P_B$  has been canceled by the  $4\pi^3$  that was originally on the right-hand side of Eq. (27) leaving only a factor of  $4P_{B\gamma}^0$ . For the sake of definiteness, we will specialize four momentum components to the photon frame, despite the Lorentz invariance of the phase space.

- **MOS convention:**

**MOS** work with the energy flow (tailored to HERA experiments) rather than leaving the cross section differential in  $z_h$ . Energy flow is defined as

$$\int_0^1 dz_h z_h^2 \left( 4P_{B\gamma}^0 E'_\gamma \frac{d\sigma}{d^3\mathbf{l}'_\gamma d^3\mathbf{P}_{B\gamma}} \right) \propto \frac{2\alpha_{\text{em}}^2}{sQ^4} \int_0^1 dz_h z_h^2 \left( L^{\text{MOS}}_{\mu\nu} W_{\text{MOS}, \text{SIDIS}}^{\mu\nu} \right). \quad (32)$$

**To match with other definitions of the hadron tensor, we have taken the integration  $\int_0^1 dz_h z_h^2$  outside of the definition of  $W_{\text{MOS}, \text{SIDIS}}^{\mu\nu}$  rather than leaving it inside the definition as in [2, Eq. (38)].**

- **JMY convention:**

There is a  $1/(4z_h)$  in **JMY** [7, Eq. (11)] relative to the **JCC** definition:

$$W_{\text{JMY}, \text{SIDIS}}^{\mu\nu}(P, q, P_B) = \frac{1}{4z_h} W_{\text{JCC}, \text{SIDIS}}^{\mu\nu}(P, q, P_B). \quad (33)$$

So,

$$P_B^0 E'_\gamma \frac{d\sigma}{d^3\mathbf{l}'_\gamma d^3\mathbf{P}_{B\gamma}} = \frac{2z_h \alpha_{\text{em}}^2}{sQ^4} L_{\mu\nu} W_{\text{JMY}, \text{SIDIS}}^{\mu\nu}. \quad (34)$$

- **PJM Convention:**

Finally, the **PJM** convention for the hadronic tensor is like the **JCC** definition but with an extra  $1/(2M)$ . These are the conventions used, for example, in Ref. [5]. (See, also, Mulders notes [6, Eq. (3.38)]). So,

$$4P_B^0 E'_\gamma \frac{d\sigma}{d^3\mathbf{l}'_\gamma d^3\mathbf{P}_{B\gamma}} = \frac{2\alpha_{\text{em}}^2(2M)}{sQ^4} L_{\mu\nu} W_{\text{PJM}, \text{SIDIS}}^{\mu\nu}. \quad (35)$$

### 1. Variable Changes

To bring the cross section into a form more common in phenomenological studies, we make the variable changes (see, e.g., [1, Eq.(A.15)]):

$$\frac{d^3\mathbf{P}_B}{P_B^0} \rightarrow \frac{d^2\mathbf{P}_{BT,\gamma} dP_{B\gamma}^-}{P_{B\gamma}^-} \rightarrow \frac{d^2\mathbf{P}_{BT,\gamma} dz_h}{z_h}. \quad (36)$$

Working in the lab frame, it is easy to perform a sequence of additional variable changes:

$$\frac{d^3\mathbf{l}'_{\text{lab}}}{E'_{\text{lab}}} \rightarrow E'_{\text{lab}} dE'_{\text{lab}} d\phi_{\text{lab}} d(\cos\theta_{\text{lab}}) \rightarrow \frac{sy}{2} dx_n dy d\phi_{\text{lab}} \rightarrow \frac{y}{2x_n} dx_n dQ^2 d\phi_{\text{lab}}. \quad (37)$$

Note that

$$dx_n dQ^2 = x_n s dx_n dy = \frac{Q^2}{y} dx_n dy. \quad (38)$$

Since in this section we only consider unpolarized, azimuthally independent cross sections, we integrate over  $\phi_{\text{lab}}$ . Equation (31) becomes

$$\frac{d\sigma}{dx_n dy dz_h d^2\mathbf{P}_{B\gamma}} = \frac{\pi\alpha_{\text{em}}^2 y}{2Q^4 z_h} L_{\mu\nu} W_{\text{JCC},\text{SIDIS}}^{\mu\nu} = \frac{2\pi\alpha_{\text{em}}^2 y}{Q^4} L_{\mu\nu} W_{\text{JMY},\text{SIDIS}}^{\mu\nu} = \frac{\pi\alpha_{\text{em}}^2 y M}{Q^4 z_h} L_{\mu\nu} W_{\text{PJM},\text{SIDIS}}^{\mu\nu}. \quad (39)$$

Or,

$$\frac{d\sigma}{dx_n dy dz_h dP_{B\gamma}^2} = \frac{\pi^2\alpha_{\text{em}}^2 y}{2Q^4 z_h} L_{\mu\nu} W_{\text{JCC},\text{SIDIS}}^{\mu\nu} = \frac{2\pi^2\alpha_{\text{em}}^2 y}{Q^4} L_{\mu\nu} W_{\text{JMY},\text{SIDIS}}^{\mu\nu} = \frac{\pi^2\alpha_{\text{em}}^2 y M}{Q^4 z_h} L_{\mu\nu} W_{\text{PJM},\text{SIDIS}}^{\mu\nu}. \quad (40)$$

From Eq. (12),

$$\frac{d\sigma}{dx_n dy dz_h dq_T^2} = \frac{\pi^2\alpha_{\text{em}}^2 z_h y}{2Q^4} L_{\mu\nu} W_{\text{JCC},\text{SIDIS}}^{\mu\nu} = \frac{2\pi^2\alpha_{\text{em}}^2 y z_h^2}{Q^4} L_{\mu\nu} W_{\text{JMY},\text{SIDIS}}^{\mu\nu} = \frac{\pi^2\alpha_{\text{em}}^2 y z_h M}{Q^4} L_{\mu\nu} W_{\text{PJM},\text{SIDIS}}^{\mu\nu}. \quad (41)$$

Or from Eq. (38),

$$\frac{d\sigma}{dx_n dQ^2 dz_h dq_T^2} = \frac{\pi^2\alpha_{\text{em}}^2 z_h}{2x_n^2 s^2 Q^2} L_{\mu\nu} W_{\text{JCC},\text{SIDIS}}^{\mu\nu} = \frac{2\pi^2\alpha_{\text{em}}^2 z_h^2}{x_n^2 s^2 Q^2} L_{\mu\nu} W_{\text{JMY},\text{SIDIS}}^{\mu\nu} = \frac{\pi^2\alpha_{\text{em}}^2 z_h M}{x_n^2 s^2 Q^2} L_{\mu\nu} W_{\text{PJM},\text{SIDIS}}^{\mu\nu}. \quad (42)$$

Compare Eq. (41) with [4, Eq. (4)]. Compare Eq. (39) with [7, Eq. (9)]. Note that  $\mathbf{P}_{B\gamma}$  is the same as  $\mathbf{P}_h$  of Ref. [5]. The totally inclusive cross section is,

$$\frac{d^2\sigma}{dx_n dy} = \frac{Q^2}{y} \frac{d^2\sigma}{dx_n dQ^2} = \frac{2\pi\alpha_{\text{em}}^2 y}{Q^4} L_{\mu\nu} W_{\text{tot}}^{\mu\nu}. \quad (43)$$

### C. Unpolarized Structure Functions

The normalizations of the TMD structure functions can be fixed by requiring that one obtains Eq. (43) after an integration of Eq. (39) over  $z$  and  $\mathbf{P}_B$ . Thus,

$$\begin{aligned} \frac{1}{4z_h} \int z_h dz_h d^2\mathbf{P}_{BT,\gamma} W_{\text{JCC},\text{SIDIS}}^{\mu\nu} &= \int z_h dz_h d^2\mathbf{P}_{BT,\gamma} W_{\text{JMY},\text{SIDIS}}^{\mu\nu} = \frac{M}{2z_h} \int z_h dz_h d^2\mathbf{P}_{BT,\gamma} W_{\text{PJM},\text{SIDIS}}^{\mu\nu} \\ &= \frac{1}{4} \int z_h^2 dz_h d^2\mathbf{q}_T W_{\text{JCC},\text{SIDIS}}^{\mu\nu} = \int z_h^3 dz_h d^2\mathbf{q}_T W_{\text{JMY},\text{SIDIS}}^{\mu\nu} = \frac{M}{2} \int z_h^2 dz_h d^2\mathbf{q}_T W_{\text{PJM},\text{SIDIS}}^{\mu\nu} = W_{\text{tot}}^{\mu\nu}. \end{aligned} \quad (44)$$

$W_{\text{tot}}^{\mu\nu}$  can be expressed in terms of the usual  $F_1(x_n, Q^2)$  and  $F_2(x_n, Q^2)$  by following the usual structure function decomposition (e.g., Ref [1] Eq.(2.20)). Therefore, we apply an analogous structure function decomposition to the

unintegrated cross section:

$$\begin{aligned} \frac{1}{4z_h} W_{\text{JCC}, \text{SIDIS}}^{\mu\nu} &= W_{\text{JMY}, \text{SIDIS}}^{\mu\nu} = \frac{M}{2z_h} W_{\text{PJM}, \text{SIDIS}}^{\mu\nu} \\ &= \left( -g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2} \right) F_1(x_n, z_h, \mathbf{P}_{B\gamma}, Q^2) + \frac{(P^\mu - q^\mu P \cdot q/q^2)(P^\nu - q^\nu P \cdot q/q^2)}{P \cdot q} F_2(x_n, z_h, \mathbf{P}_{B\gamma}, Q^2). \end{aligned} \quad (45)$$

Then,

$$\begin{aligned} \frac{d\sigma}{dx_n dy dz_h d^2\mathbf{P}_{BT,\gamma}} &= \frac{4\pi\alpha_{\text{em}}^2}{x_n y Q^2} \left[ \left( 1 - y - \frac{x_n^2 y^2 M^2}{Q^2} \right) F_2(x_n, z_h, \mathbf{P}_{BT,\gamma}, Q^2) + y^2 x_n F_1(x_n, z_h, \mathbf{P}_{BT,\gamma}, Q^2) \right] \\ &= \frac{4\pi\alpha_{\text{em}}^2}{x_n y Q^2} \left[ 2x_n \left( 1 - y + \frac{y^2}{2} - \frac{x_n^2 y^2 M^2}{Q^2} \right) F_T(x_n, z_h, \mathbf{P}_{BT,\gamma}, Q^2) + \left( 1 - y - \frac{x_n^2 y^2 M^2}{Q^2} \right) F_L(x_n, z_h, \mathbf{P}_{BT,\gamma}, Q^2) \right]. \end{aligned} \quad (46)$$

In the last line we have extended the standard definitions,  $F_T \equiv F_1$  and  $F_L \equiv F_2 - 2x_n F_1$ , from the integrated case to the TMD case. This agrees with [5, Eq. (2.14)] in the limit that hadron masses are neglected.

**Note that [5, Eq. (2.18)] defines  $F_T$  with a different normalization:  $F_T = 2x_n F_1$  so that  $F_2 = F_L + F_T$ ; this needs to be taken into account to get agreement with Eq. (46).**

From Eqs. (39,43,45,46),

$$\int z_h dz_h d^2\mathbf{P}_{BT,\gamma} F_T(x_n, z_h, \mathbf{P}_{BT,\gamma}, Q^2) = \int z_h^3 dz_h d^2\mathbf{q}_T F_T(x_n, z_h, z_h \mathbf{q}_T, Q^2) = F_T(x_n, Q^2) \quad (47)$$

$$\int z_h dz_h d^2\mathbf{P}_{BT,\gamma} F_L(x_n, z_h, \mathbf{P}_{BT,\gamma}, Q^2) = \int z_h^3 dz_h d^2\mathbf{q}_T F_L(x_n, z_h, z_h \mathbf{q}_T, Q^2) = F_L(x_n, Q^2). \quad (48)$$

**⇒ Contribution from Bowen Wang**

**NSY** write the structure function [3, Eq.(31)] decomposition as a sum over  $A_n$  and  $V_n$  functions. For the azimuthally symmetric case,

$$\frac{d\sigma}{dx_n dQ^2 dz dq_T^2} = 2\pi \sum_{\alpha=1,2} V_{AB}^{(\alpha)} A_\alpha. \quad (49)$$

Here, from [3, Eq. (33)]

$$A_1 = 1 + \cosh^2 \psi, \quad (50)$$

$$A_2 = -2, \quad (51)$$

where we have used the **NSY** notation

$$\cosh \psi = \frac{2}{y} - 1. \quad (52)$$

Using Eqs. (11), (12) in Ref [3] for  $l$  and  $l'$ , and using Eqs. (45,46) for  $W_{\text{JCC}, \text{SIDIS}}^{\mu\nu}$ , the cross section in Eq. (42) is written as

$$\frac{d\sigma}{dx_n dQ^2 dz dq_T^2} = \frac{\pi^2 \alpha_{\text{em}}^2 z}{2x_n^2 s^2 Q^2} L_{\mu\nu} W_{\text{JCC}, \text{SIDIS}}^{\mu\nu} = \frac{4\pi^2 \alpha_{\text{em}}^2 z^2}{x_n^2 s^2} [F_1 + (1/4x_n)(\cosh^2 \psi - 1)F_2] \quad (53)$$

$$= \frac{4\pi^2 \alpha_{\text{em}}^2 z^2}{x_n^2 s^2} \left[ \frac{1}{2} (\cosh^2 \psi + 1) F_T + \frac{1}{4x_n} (\cosh^2 \psi - 1) F_L \right] \quad (54)$$

$$= \frac{4\pi^2 \alpha_{\text{em}}^2 z^2}{x_n^2 s^2} \left[ \frac{1}{4x_n} (\cosh^2 \psi + 1) F_2 - \frac{1}{2x_n} F_L \right]. \quad (55)$$

Reading off the coefficients from Eqs. (50,51) gives,

$$F_2 = V_{AB}^{(1)} \frac{2x_n^3 s^2}{\pi \alpha_{\text{em}}^2 z^2} \quad (56)$$

$$F_L = V_{AB}^{(2)} \frac{2x_n^3 s^2}{\pi \alpha_{\text{em}}^2 z^2} \quad (57)$$

$$F_T = F_1 = \left( V_{AB}^{(1)} - V_{AB}^{(2)} \right) \frac{x_n^2 s^2}{\pi \alpha_{\text{em}}^2 z^2}. \quad (58)$$



## IV. TMD FACTORIZATION EXPRESSIONS

### A. SIDIS: TMD

#### 1. Large and small transverse momentum separation

We will mainly follow the notation for expressing the TMD-factorization formula of Collins [1, Eq. (13.116)]. In terms of a transverse momentum convolution integral involving TMD PDFs, the hadronic tensor is:

$$\begin{aligned}
W_{\text{SIDIS}, \text{JCC}}^{\mu\nu} &= \overbrace{T_{\text{SIDIS}}^{\mu\nu}}^{\text{“W-term,” “L-term”}} + Y_{\text{SIDIS}}^{\mu\nu} \\
&= \sum_f |\mathcal{H}_f(Q/\mu, \alpha_s(\mu))^2|^{\mu\nu} \\
&\times \int d^2\mathbf{k}_{1T} d^2\mathbf{k}_{2T} \delta^{(2)}(\mathbf{k}_{1T} + \mathbf{q}_T - \mathbf{k}_{2T}) F_{f/p}^{[+]}(x_n, \mathbf{k}_{1T}; \zeta_{\text{PDF}}; \mu) D_{h/f}(z, z\mathbf{k}_{2T}; \zeta_{\text{FF}}; \mu) + Y_{\text{SIDIS}}^{\mu\nu} \\
&= \sum_f |\mathcal{H}_f(Q/\mu, \alpha_s(\mu))^2|^{\mu\nu} F_{f/p}^{[+]}(x_n, \mathbf{k}_{1T}; \zeta_{\text{PDF}}; \mu) \circledast D_{h/f}(z, z\mathbf{k}_{2T}; \zeta_{\text{FF}}; \mu) + Y_{\text{SIDIS}}^{\mu\nu}. \tag{59}
\end{aligned}$$

where the last line defines the shorthand “ $\circledast$ ” for the transverse momentum convolution integral. The hadronic tensor (or cross section) separates into two terms: the first accurately approximates the low momentum region  $q_T \ll O(Q)$  while the second is a correction for the the region of  $q_T \sim Q$ . The first term is conventionally called the “W-term” or the “L-term.” It has also frequently been called the “resummed term,” but this terminology will be disfavored in this document since the factorization into renormalized operators goes beyond resummation. Since  $W$  and  $L$  also denote the hadronic and leptonic tensors, we will use  $T$  to denote the low momentum term, as indicated in Eq. (59).

The second term is almost always called the “Y-term.”

For this section, we will focus on the properties of the T-term. We will return to the Y-term in Sec. XI. The T-term by itself is

$$\begin{aligned}
T_{\text{SIDIS}}^{\mu\nu} &= \sum_f |\mathcal{H}_f(Q/\mu, \alpha_s(\mu))^2|^{\mu\nu} \\
&\times \int d^2\mathbf{k}_{1T} d^2\mathbf{k}_{2T} \delta^{(2)}(\mathbf{k}_{1T} + \mathbf{q}_T - \mathbf{k}_{2T}) F_{f/p}^{[+]}(x_n, \mathbf{k}_{1T}; \zeta_{\text{PDF}}; \mu) D_{h/f}(z, z\mathbf{k}_{2T}; \zeta_{\text{FF}}; \mu) \\
&= \sum_f |\mathcal{H}_f(Q/\mu, \alpha_s(\mu))^2|^{\mu\nu} F_{f/p}^{[+]}(x_n, \mathbf{k}_{1T}; \zeta_{\text{PDF}}; \mu) \circledast D_{h/f}(z, z\mathbf{k}_{2T}; \zeta_{\text{FF}}; \mu). \tag{60}
\end{aligned}$$

#### 2. Hard Factor

The precise definition of the hard factor  $|\mathcal{H}_f(Q/\mu, \alpha_s(\mu))^2|^{\mu\nu}$  will vary by a normalization factor depending on the convention for  $W^{\mu\nu}$  (see Sec. III B). The  $[+]$  on Eq. (59) is to indicate that the TMD PDF is defined with a future pointing Wilson line. Comparing with Collins [1] Eq. (13.116), the normalization for the hard part in that convention is:

$$\sum_f |\mathcal{H}_f(Q/\mu, \alpha_s(\mu))^2|^{\mu\nu}_{\text{JCC}} = \frac{z}{Q^2} \sum_f \text{Tr} \left[ \hat{k}_{A,\gamma} H_f^\nu(Q/\mu, \alpha_s(\mu)) \hat{k}_{A,\gamma} H_f^\mu(Q/\mu, \alpha_s(\mu))^\dagger \right]. \tag{61}$$

The momenta  $\hat{k}_{A,\gamma}$  and  $\hat{k}_{B,\gamma}$  are the approximate parton momenta with the approximation defined in the photon frame. The components in the photon frame are

$$\hat{k}_{A,\gamma} = (x_n P^+, 0, \mathbf{0}_T), \quad \hat{k}_{B,\gamma} = \left( 0, \frac{Q^2}{2x_n P^+}, \mathbf{0}_T \right). \tag{62}$$

**Note typo in Eq. [1, Eq.(13.115)]; the plus component of  $\hat{k}_{A,\gamma}$  should be positive.**  $H_f^\nu(Q; \mu)$  can be written as scalar function times the tree level electromagnetic vertex:

$$H_f^\nu(Q/\mu, \alpha_s(\mu)) = -ie_f \gamma^\mu \Gamma(\mu/Q, \alpha_s(\mu))_{\text{SIDIS}} \quad (63)$$

where  $\Gamma(\mu/Q, \alpha_s(\mu))_{\text{SIDIS}}$  is a hard vertex function equal to one at zeroth order. Using Eqs. (25,61,63):

$$\begin{aligned} L_{\mu\nu} \sum_f |\mathcal{H}_f(Q/\mu, \alpha_s(\mu))^2|_{\text{JCC}}^{\mu\nu} &= 4z L_{\mu\nu} \sum_f |\mathcal{H}_f(Q/\mu, \alpha_s(\mu))^2|_{\text{JMY}}^{\mu\nu} = 2M_p L_{\mu\nu} \sum_f |\mathcal{H}_f(Q/\mu, \alpha_s(\mu))^2|_{\text{PJM}}^{\mu\nu} \\ &= \sum_f \frac{8Q^2 z e_q^2}{y^2} \left(1 - y + \frac{y^2}{2}\right) |\Gamma(\mu/Q, \alpha_s(\mu))_{\text{SIDIS}}|^2. \end{aligned} \quad (64)$$

### 3. Small $q_T$ Cross section in terms of TMD functions

Substituting Eq. (64) into Eq. (39) and keeping only the  $T$ -term from Eq. (60) gives

$$\begin{aligned} \left. \frac{d\sigma}{dx_n dy dz d^2\mathbf{P}_{BT,\gamma}} \right|_{\text{unpol, T-part}} &= \frac{1}{z^2} \left. \frac{d\sigma}{dx_n dy dz d^2\mathbf{q}_T} \right|_{\text{unpol, T-part}} \\ &= \sum_f \frac{4\pi\alpha_{\text{em}}^2 e_f^2}{Q^2 y} (1 - y + y^2/2) |\Gamma(\mu/Q, \alpha_s(\mu))_{\text{SIDIS}}|^2 F_{f/p}^{[+]}(x_n, \mathbf{k}_{1T}; \zeta_{\text{PDF}}, \mu) \otimes D_{h/f}(z, z\mathbf{k}_{2T}; \zeta_{\text{FF}}, \mu) \\ &= \sum_f \frac{4\pi\alpha_{\text{em}}^2 e_f^2 s x_n}{Q^4} (1 - y + y^2/2) |\Gamma(\mu/Q, \alpha_s(\mu))_{\text{SIDIS}}|^2 F_{f/p}^{[+]}(x_n, \mathbf{k}_{1T}; \zeta_{\text{PDF}}, \mu) \otimes D_{h/f}(z, z\mathbf{k}_{2T}; \zeta_{\text{FF}}, \mu). \end{aligned} \quad (65)$$

In the last line, we have used  $x_n y s \approx Q^2$ . If one integrates over the hadron's azimuthal angle, the right side gets an extra factor of  $2\pi$ . This agrees with **JCC** [1] Eq. (12.91) and **JMY** [13] Eq. (50).

Comparing with Eq. (46), it is clear that the structure function contributions from the  $T$ -term TMD part (i.e., excluding the  $Y$ -term) are:

$$F_T(x_n, z, \mathbf{P}_{BT,\gamma}, Q^2)^{\text{T-term}} \approx \sum_f e_f^2 |\Gamma(\mu/Q, \alpha_s(\mu))_{\text{SIDIS}}|^2 F_{f/p}^{[+]}(x_n, \mathbf{k}_{1T}; \zeta_{\text{PDF}}, \mu) \otimes D_{h/f}(z, z\mathbf{k}_{2T}; \zeta_{\text{FF}}, \mu) \quad (66)$$

and,

$$F_L(x_n, z, \mathbf{P}_{BT,\gamma}, Q^2)^{\text{T-term}} \approx 0. \quad (67)$$

The “ $\approx$ ” is to emphasize that terms suppressed by  $M_p^2/Q^2$  have been dropped.

### 4. Coordinate space

In it convenient to work with the convolution  $F_{f/p}^{[+]} \otimes D_{h/f}$  in transverse coordinate space. The Fourier transforms of the TMD functions are:

$$F_{f/p}^{[+]}(x_n, \mathbf{k}_{2T} - \mathbf{q}_T; \zeta_{\text{PDF}}, \mu) = \frac{1}{(2\pi)^2} \int d^2\mathbf{b}_T e^{i\mathbf{b}_T \cdot (\mathbf{k}_{2T} - \mathbf{q}_T)} \tilde{F}_{f/p}^{[+]}(x_n, \mathbf{b}_T; \zeta_{\text{PDF}}, \mu) \quad (68)$$

$$D_{h/f}(z, z\mathbf{k}_{2T}; \zeta_{\text{FF}}, \mu) = \frac{1}{(2\pi)^2} \int d^2\mathbf{b}_T e^{-i\mathbf{b}_T \cdot \mathbf{k}_{2T}} \tilde{D}_{h/f}(z, \mathbf{b}_T; \zeta_{\text{FF}}, \mu). \quad (69)$$

So the convolution in Eq. (59) is

$$\begin{aligned}
& F_{f/p}^{[+]}(x_n, \mathbf{k}_{1T}; \zeta_{\text{PDF}}; \mu) \otimes D_{h/f}(z, z\mathbf{k}_{2T}; \zeta_{\text{FF}}; \mu) \\
&= \int d^2\mathbf{k}_{2T} F_{f/p}^{[+]}(x_n, \mathbf{k}_{2T} - \mathbf{q}_T; \zeta_{\text{PDF}}; \mu) D_{h/f}(z, z\mathbf{k}_{2T}; \zeta_{\text{FF}}; \mu) \\
&= \frac{1}{(2\pi)^4} \int d^2\mathbf{k}_{2T} \int d^2\mathbf{b}_T \int d^2\mathbf{b}'_T e^{-i\mathbf{b}_T \cdot \mathbf{q}_T} e^{i\mathbf{k}_{2T} \cdot (\mathbf{b}'_T - \mathbf{b}_T)} \tilde{F}_{f/p}^{[+]}(x_n, \mathbf{b}'_T; \zeta_{\text{PDF}}; \mu) \tilde{D}_{h/f}(z, \mathbf{b}_T; \zeta_{\text{FF}}; \mu) \\
&= \int \frac{d^2\mathbf{b}_T}{(2\pi)^2} e^{-i\mathbf{b}_T \cdot \mathbf{q}_T} \tilde{F}_{f/p}^{[+]}(x_n, \mathbf{b}_T; \zeta_{\text{PDF}}; \mu) \tilde{D}_{h/f}(z, \mathbf{b}_T; \zeta_{\text{FF}}; \mu)
\end{aligned} \tag{70}$$

So, for example, Eq. (65) can be written:

$$\begin{aligned}
& \frac{1}{z^2} \frac{d\sigma}{dx_n dy dz d^2\mathbf{q}_T} \Big|_{\text{unpol, T-part}} \\
&= \sum_f \frac{4\pi\alpha_{\text{em}}^2 e_f^2}{Q^2 y} (1 - y + y^2/2) |\Gamma(\mu/Q, \alpha_s(\mu))_{\text{SIDIS}}|^2 \int \frac{d^2\mathbf{b}_T}{(2\pi)^2} e^{-i\mathbf{b}_T \cdot \mathbf{q}_T} \tilde{F}_{f/p}^{[+]}(x_n, \mathbf{b}_T; \zeta_{\text{PDF}}; \mu) \tilde{D}_{h/f}(z, \mathbf{b}_T; \zeta_{\text{FF}}; \mu).
\end{aligned} \tag{71}$$

In situations with azimuthal symmetry,

$$\begin{aligned}
& \frac{d\sigma}{dx_n dy dz dq_T} \Big|_{\text{unpol, T-part}} \\
&= \sum_f \frac{4\pi z^2 \alpha_{\text{em}}^2 e_f^2 q_T}{Q^2 y} (1 - y + y^2/2) |\Gamma(\mu/Q, \alpha_s(\mu))_{\text{SIDIS}}|^2 \int db_T b_T J_0(q_T b_T) \tilde{F}_{f/p}^{[+]}(x_n, \mathbf{b}_T; \zeta_{\text{PDF}}; \mu) \tilde{D}_{h/f}(z, \mathbf{b}_T; \zeta_{\text{FF}}; \mu).
\end{aligned} \tag{72}$$

**Note that the sign in the Fourier transform exponential  $e^{-i\mathbf{b}_T \cdot \mathbf{q}_T}$  in Eq. (70) differs from that of JCC, Eq. (13.116). This will not matter for azimuthally symmetric cross sections, but may cause differences in azimuthal asymmetries.**

Many authors use the notation:

$$\mathcal{C} \left[ F_{f/p}^{[+]} D_{h/f} \right] \equiv F_{f/p}^{[+]}(x_n, \mathbf{k}_{1T}; \zeta_{\text{PDF}}; \mu) \otimes D_{h/f}(z, z\mathbf{k}_{2T}; \zeta_{\text{FF}}; \mu) \tag{73}$$

for the convolution in Eq. (70).

### 5. Conventions for Factors of $z$

A common notation is to change variables in Eq. (73) so that  $z$  does not multiply  $\mathbf{k}_{2T}$  in  $D_{h/f}(z, z\mathbf{k}_{2T}; \zeta_{\text{FF}}; \mu)$ . One defines  $z\mathbf{k}_{2T} = \underline{\mathbf{k}}$ . Then a change of variables gives

$$\begin{aligned}
\mathcal{C} \left[ F_{f/p}^{(+)} D_{h/f} \right] &= \sum_f \int d^2\mathbf{k}_{1T} d^2\mathbf{k}_{2T} \delta^{(2)}(\mathbf{k}_{1T} + \mathbf{q}_T - \mathbf{k}_{2T}) F_{f/p}^{(+)}(x, \mathbf{k}_{1T}; \zeta_{\text{PDF}}; \mu) D_{h/f}(z, z\mathbf{k}_{2T}; \zeta_{\text{FF}}; \mu) \\
&= \sum_f \int d^2\mathbf{k}_{1T} d^2\mathbf{k}_{2T} \delta^{(2)}(\mathbf{k}_{1T} + \mathbf{P}_{BT,\gamma}/z - \underline{\mathbf{k}}/z) F_{f/p}^{(+)}(x, \mathbf{k}_{1T}; \zeta_{\text{PDF}}; \mu) D_{h/f}(z, \underline{\mathbf{k}}; \zeta_{\text{FF}}; \mu) \\
&= \sum_f \int d^2\mathbf{k}_{1T} d^2\underline{\mathbf{k}} \delta^{(2)}(z\mathbf{k}_{1T} + \mathbf{P}_{BT,\gamma} - \underline{\mathbf{k}}) F_{f/p}^{(+)}(x, \mathbf{k}_{1T}; \zeta_{\text{PDF}}; \mu) D_{h/f}(z, \underline{\mathbf{k}}; \zeta_{\text{FF}}; \mu).
\end{aligned} \tag{74}$$

Then,  $\underline{\mathbf{k}}$  is the transverse momentum of the hadronizing parton relative to its parent jet.

## V. TMD FUNCTIONS

### A. Further Notation and Conventions

It will be useful to have a specific scheme for cutting off the behavior of certain perturbatively calculated expressions at large- $\mathbf{b}_T$ . For this, many authors use the “b-star” method by defining:

$$\mathbf{b}_*(\mathbf{b}_T) \rightarrow \begin{cases} \mathbf{b}_T & b_T \ll b_{\text{max}} \\ \mathbf{b}_{\text{max}} & b_T \gg b_{\text{max}} \end{cases} \tag{75}$$

where  $b_{\text{max}} = b_{\text{max}} \frac{b_T}{\|\mathbf{b}_T\|}$ .

The standard  $\overline{\text{MS}}$  renormalization group scale is  $\mu$ , and one commonly uses scales

$$\mu_Q \equiv C_2 Q \tag{76}$$

$$\mu_b \equiv C_1/b_T \tag{77}$$

$$\mu_{b_*} \equiv C_1/b_*, \tag{78}$$

where  $C_1$  and  $C_2$  are arbitrary constants that are ultimately to be chosen to optimize perturbative convergence.

### B. TMD Parton Distributions

The definition of a TMD PDF in coordinate space is:

The evolution equations are:

The most general and basic way to write the solution is evolve from some reference scales  $\mu \rightarrow \mu_0$ ,  $\zeta_{\text{PDF}} \rightarrow Q_0^2$  to some arbitrary  $\mu$  and  $\zeta_{\text{PDF}}$ .

$$\begin{aligned}
&\tilde{F}_{f/P}(x, \mathbf{b}_T; \zeta_{\text{PDF}}, \mu) \\
&= \tilde{F}_{f/P}(x, \mathbf{b}_T; Q_0^2, \mu_0) \\
&\times \exp \left\{ \ln \frac{\sqrt{\zeta_{\text{PDF}}}}{Q_0} \tilde{K}(b_*; \mu_{b_*}) + \int_{\mu_0}^{\mu} \frac{d\mu'}{\mu'} \left[ \gamma_F(\alpha_s(\mu'); 1) - \ln \frac{\sqrt{\zeta_{\text{PDF}}}}{\mu'} \gamma_K(g(\mu')) \right] \right. \\
&\quad \left. + \int_{\mu_0}^{\mu_{b_*}} \frac{d\mu'}{\mu'} \ln \frac{\sqrt{\zeta_{\text{PDF}}}}{Q_0} \gamma_K(\alpha_s(\mu')) \right\} \\
&\times \exp \left\{ -g_K(b_T) \ln \frac{\sqrt{\zeta_{\text{PDF}}}}{Q_0} \right\}.
\end{aligned} \tag{79}$$

Ultimately, one typically sets  $\mu = Q$  and  $\zeta_{\text{PDF}} = Q^2$ . The scales  $\mu_0$  and  $Q_0$  should be large enough to admit perturbative calculations of the anomalous dimensions.

More commonly, one evolves relative to  $\mu_b$  (from Eq. (77)) in order to permit  $C_1/b_T$  to be used as a hard scale for the application of an operator product expansion in the limit of small  $b_T$ . Then, the small  $b_T$  region is expressible in terms of collinear factorization with collinear PDFs:

$$\begin{aligned}
\tilde{F}_{f/P}(x, \mathbf{b}_T; \zeta_{\text{PDF}}, \mu) &= \overbrace{\sum_j \int_x^1 \frac{d\hat{x}}{\hat{x}} \tilde{C}_{f/j}(x/\hat{x}, b_*; \mu_{b_*}^2, \mu_{b_*}, \alpha_s(\mu_{b_*})) f_{j/P}(\hat{x}, \mu_{b_*})}^{\text{AA}} \\
&\times \overbrace{\exp \left\{ \ln \frac{\sqrt{\zeta_{\text{PDF}}}}{\mu_{b_*}} \tilde{K}(b_*; \mu_{b_*}) + \int_{\mu_{b_*}}^{\mu} \frac{d\mu'}{\mu'} \left[ \gamma_F(\alpha_s(\mu'); 1) - \ln \frac{\sqrt{\zeta_{\text{PDF}}}}{\mu'} \gamma_K(g(\mu')) \right] \right\}}^{\text{BB}} \\
&\times \overbrace{\exp \left\{ -g_{f/P}(x, b_T) - g_K(b_T) \ln \frac{\sqrt{\zeta_{\text{PDF}}}}{Q_0} \right\}}^{\text{CC}}. \tag{80}
\end{aligned}$$

### C. TMD Fragmentation Functions

The definitions of a TMD FFs are:

The evolutions equations are:

## VI. HARD FACTORS

The hard factors for SIDIS and DY at one loop in the  $\overline{\text{MS}}$  renormalization scheme are:

$$|\Gamma(Q; \mu/Q, \alpha_s(\mu))_{\text{SIDIS}}|^2 = 1 + 4C_F \left( \frac{3}{2} \ln(Q^2/\mu^2) - \frac{1}{2} \ln^2(Q^2/\mu^2) - 4 \right) \left( \frac{\alpha_s(\mu)}{4\pi} \right) + \mathcal{O} \left( \left( \frac{\alpha_s(\mu)}{4\pi} \right)^2 \right) \tag{81}$$

$$|\Gamma(Q; \mu/Q, \alpha_s(\mu))_{\text{DY}}|^2 = 1 + 4C_F \left( \frac{3}{2} \ln(Q^2/\mu^2) - \frac{1}{2} \ln^2(Q^2/\mu^2) - 4 + \frac{\pi^2}{2} \right) \left( \frac{\alpha_s(\mu)}{4\pi} \right) + \mathcal{O} \left( \left( \frac{\alpha_s(\mu)}{4\pi} \right)^2 \right). \tag{82}$$

## VII. $\overline{\text{MS}}$ COLLINS-SOPER KERNEL

The CS kernel  $K(b_T; \mu, \alpha_s(\mu))$  is known at least to order  $\alpha_s(\mu)$ :

$$\tilde{K}(b_T; \mu, \alpha_s(\mu)) = -4C_F [\ln(\mu^2 b_T^2) - \ln 4 + 2\gamma_E] \left( \frac{\alpha_s(\mu)}{4\pi} \right) + \mathcal{O}(\alpha_s(\mu)^2). \tag{83}$$

### VIII. CS KERNEL $\overline{\text{MS}}$ ANOMALOUS DIMENSION

The  $\overline{\text{MS}}$  anomalous dimension for  $K(b_T; \mu)$  has been calculated to three loops by Moch, Vermaseren and Vogt (MVV) [14, Eqs.(3.8,3.9)]:

$$\begin{aligned}
\gamma_K(\alpha_s(\mu)) = & 8C_F \left( \frac{\alpha_s(\mu)}{4\pi} \right) \\
& + \left[ 16C_F C_A \left( \frac{67}{18} - \zeta_2 \right) + 16C_F n_f \left( -\frac{5}{9} \right) \right] \left( \frac{\alpha_s(\mu)}{4\pi} \right)^2 \\
& + \left[ 32C_F C_A^2 \left( \frac{254}{24} - \frac{67}{9}\zeta_2 + \frac{11}{6}\zeta_3 + \frac{11}{5}\zeta_2^2 \right) + 32C_F^2 n_f \left( -\frac{55}{24} + 2\zeta_3 \right) \right. \\
& \quad \left. + 32C_F C_A n_f \left( -\frac{209}{108} + \frac{10}{9}\zeta_2 - \frac{7}{3}\zeta_3 \right) + 32C_F n_f^2 \left( -\frac{1}{27} \right) \right] \left( \frac{\alpha_s(\mu)}{4\pi} \right)^3 \\
& + \mathcal{O} \left( \left( \frac{\alpha_s(\mu)}{4\pi} \right)^4 \right).
\end{aligned} \tag{84}$$

**Note that MVV use the notation  $A$  for the anomalous dimension. It is related to  $\gamma_K$  by  $\gamma_K = 2A$ . The MVV  $A$  should not be used for the  $A$  function commonly used in the CSS formalism.**

Note the evolution equations (2.3,2.4) in MVV and the definition of the expansion in Eq. (2.5).

### IX. TMD PDFS AND FFS, $\overline{\text{MS}}$ ANOMALOUS DIMENSIONS

The anomalous dimensions for the TMD PDFs and FFS,  $\gamma_F$  and  $\gamma_D$  respectively, are known at least to order  $\alpha_s(\mu)$ . To order  $\alpha_s(\mu)$  they are the same for the TMD PDF and the TMD fragmentation function:

$$\gamma_{\text{PDF}}(\mu, \zeta_{\text{PDF}}/\mu^2, \alpha_s(\mu)) = 4C_F \left( \frac{3}{2} - \ln \left( \frac{\zeta_{\text{PDF}}}{\mu^2} \right) \right) \left( \frac{\alpha_s(\mu)}{4\pi} \right) + \mathcal{O} \left( \left( \frac{\alpha_s(\mu)}{4\pi} \right)^2 \right), \tag{85}$$

$$\gamma_{\text{FF}}(\mu, \zeta_{\text{FF}}/\mu^2, \alpha_s(\mu)) = 4C_F \left( \frac{3}{2} - \ln \left( \frac{\zeta_{\text{FF}}}{\mu^2} \right) \right) \left( \frac{\alpha_s(\mu)}{4\pi} \right) + \mathcal{O} \left( \left( \frac{\alpha_s(\mu)}{4\pi} \right)^2 \right). \tag{86}$$

### X. TMD PDFS AND FFS: $\overline{\text{MS}}$ WILSON COEFFICIENTS FOR SMALL- $b_T$ OPERATOR PRODUCT EXPANSION

In the region of small  $b_T$ , the TMD functions may be expanded perturbatively by exploiting the appearance of a hard scale  $1/b_T$ . The expansions for the unpolarized case are:

$$\tilde{F}_{f/P}(x_n, b_T; \zeta_{\text{PDF}}, \mu, \alpha_s(\mu)) = \sum_j \int_{x_n}^1 \frac{d\hat{x}}{\hat{x}} \tilde{C}_{f/j}^{\text{PDF}}(x_n/\hat{x}, b_T; \zeta_{\text{PDF}}, \mu, \alpha_s(\mu)) f_{j/P}(\hat{x}; \mu) + \mathcal{O}((\Lambda_{\text{QCD}} b_T)^a), \tag{87}$$

$$\tilde{D}_{H/f}(z, b_T; \zeta_{\text{FF}}, \mu, \alpha_s(\mu)) = \sum_j \int_z^1 \frac{d\hat{z}}{\hat{z}^{3-2\epsilon}} \tilde{C}_{j/f}^{\text{FF}}(z/\hat{z}, b_T; \zeta_{\text{FF}}, \mu, \alpha_s(\mu)) d_{h/j}(\hat{z}; \mu) + \mathcal{O}((\Lambda_{\text{QCD}} b_T)^a), \tag{88}$$

where  $a > 0$ .

The Wilson coefficients in Eqs. (87, 88) to order  $\alpha_s(\mu)$  are

- TMD PDF

1. *Quark flavor  $f$  inside quark flavor  $j$ :*

$$\begin{aligned}
\tilde{C}_{f/j}^{\text{PDF}}(x_{\text{n}}, \mathbf{b}_{\text{T}}; \zeta_{\text{PDF}}, \mu, \alpha_s(\mu)) = \\
\delta_{fj} \delta(1 - x_{\text{n}}) \\
+ \delta_{fj} 2C_{\text{F}} \left\{ 2 \left[ \ln \left( \frac{2}{\mu b_{\text{T}}} \right) - \gamma_{\text{E}} \right] \left[ \left( \frac{2}{1 - x_{\text{n}}} \right)_+ - 1 - x_{\text{n}} \right] + 1 - x_{\text{n}} \right. \\
+ \delta(1 - x_{\text{n}}) \left[ -\frac{1}{2} \left[ \ln(b_{\text{T}}^2 \mu^2) - 2(\ln 2 - \gamma_{\text{E}}) \right]^2 - \left[ \ln(b_{\text{T}}^2 \mu^2) - 2(\ln 2 - \gamma_{\text{E}}) \right] \ln \left( \frac{\zeta_{\text{PDF}}}{\mu^2} \right) \right] \left. \right\} \left( \frac{\alpha_s(\mu)}{4\pi} \right) \\
+ \mathcal{O} \left( \left( \frac{\alpha_s(\mu)}{4\pi} \right)^2 \right). \tag{89}
\end{aligned}$$

2. *Quark flavor  $f$  inside gluon:*

$$\begin{aligned}
\tilde{C}_{f/g}^{\text{PDF}}(x, \mathbf{b}_{\text{T}}; \mu; \zeta_{\text{PDF}}/\mu^2, \alpha_s(\mu)) = \\
2T_{\text{F}} \left( 2[1 - 2x(1 - x)] \left[ \ln \left( \frac{2}{b_{\text{T}} \mu} \right) - \gamma_{\text{E}} \right] + 2x(1 - x) \right) \left( \frac{\alpha_s(\mu)}{4\pi} \right) \\
+ \mathcal{O} \left( \left( \frac{\alpha_s(\mu)}{4\pi} \right)^2 \right) \tag{90}
\end{aligned}$$

• TMD Fragmentation Function

1. *Quark flavor  $f$  hadronizes to quark flavor  $j$ :*

$$\begin{aligned}
\tilde{C}_{j/f}^{\text{FF}}(z, \mathbf{b}_{\text{T}}; \zeta_{\text{FF}}, \mu, \alpha_s(\mu)) = \\
\delta_{jf} \delta(1 - z) \\
\delta_{jf} 2C_{\text{F}} \left\{ 2 \left[ \ln \left( \frac{2z}{\mu b_{\text{T}}} \right) - \gamma_{\text{E}} \right] \left[ \left( \frac{2}{1 - z} \right)_+ + \frac{1}{z^2} + \frac{1}{z} \right] + \frac{1}{z^2} - \frac{1}{z} + \right. \\
+ \delta(1 - z) \left[ -\frac{1}{2} \left[ \ln(b_{\text{T}}^2 \mu^2) - 2(\ln 2 - \gamma_{\text{E}}) \right]^2 - \left[ \ln(b_{\text{T}}^2 \mu^2) - 2(\ln 2 - \gamma_{\text{E}}) \right] \ln \left( \frac{\zeta_{\text{FF}}}{\mu^2} \right) \right] \left. \right\} \left( \frac{\alpha_s(\mu)}{4\pi} \right) \\
+ \mathcal{O} \left( \left( \frac{\alpha_s(\mu)}{4\pi} \right)^2 \right). \tag{91}
\end{aligned}$$

2. *Quark flavor  $f$  hadronizes into gluon:*

$$\begin{aligned}
\tilde{C}_{g/j'}^{\text{FF}}(z, \mathbf{b}_T; \zeta_{\text{FF}}, \mu, \alpha_s(\mu)) = \\
\frac{2C_F}{z^3} \left( 2 [1 + (1-z)^2] \left[ \ln \left( \frac{2z}{b_T \mu} \right) - \gamma_E \right] + z^2 \right) \left( \frac{\alpha_s(\mu)}{4\pi} \right) \\
+ \mathcal{O} \left( \left( \frac{\alpha_s(\mu)}{4\pi} \right)^2 \right)
\end{aligned} \tag{92}$$

**Note an overall factor of  $1/z^2$  in the fragmentation function Wilson coefficients relative to MOS, Ref. [15, Eq. (45)]. We suspect that this is due to an additional factor of  $z^2$  in the definition of the hadronic tensor in Ref. [2, Eq. (38)].**

One often wishes to evaluate these functions after evolution to the specific scales in Eqs. (76)-(78) and using the b-star prescription. So, for example,  $\tilde{C}_{f/j}^{\text{PDF}}(x, \mathbf{b}_T; \zeta_{\text{PDF}}, \mu, \alpha_s(\mu)) \rightarrow \tilde{C}_{f/j}^{\text{PDF}}(x, \mathbf{b}_*; \mu_{b_*}^2, \mu_{b_*}, \alpha_s(\mu_{b_*}))$ . The PDFs then become

- TMD PDF

1. *Quark flavor  $f$  inside quark flavor  $j$ :*

$$\begin{aligned}
\tilde{C}_{f/j}^{\text{PDF}}(x, \mathbf{b}_*; \mu_{b_*}^2, \mu_{b_*}, \alpha_s(\mu_{b_*})) = \\
\delta_{fj} \delta(1-x) \\
+ \delta_{fj} 2C_F \left\{ 2 \left[ \ln \left( \frac{2}{C_1} \right) - \gamma_E \right] \left[ \frac{1}{C_F} P_{fj}(x) - \frac{3}{2} \delta(1-x) \right] + 1 - x \right. \\
\left. + \delta(1-x) \left[ -\frac{1}{2} [\ln(C_1^2) - 2(\ln 2 - \gamma_E)]^2 - [\ln(C_1^2) - 2(\ln 2 - \gamma_E)] \ln(1) \right] \right\} \left( \frac{\alpha_s(\mu_{b_*})}{4\pi} \right) \\
+ \mathcal{O} \left( \left( \frac{\alpha_s(\mu_{b_*})}{4\pi} \right)^2 \right).
\end{aligned} \tag{93}$$

We have left the  $\ln(1) = 0$  explicit as a reminder of the need to evolve  $\zeta$  and  $\mu$  to the same scale.

## XI. Y-TERMS

### A. Notation and Terminology

- JCC notation:

The Y-terms are fixed by the requirements for factorization in the  $T$ -term region in the transverse momentum  $\ll C_1 Q$ , combined with the requirements for factorization in the region of  $\mathbf{q}_T \sim Q$ .

The basic logic is discussed in [1, pgs.(513-514)], which is based on earlier CSS work [16].

My starting point is the definition of the  $L$  in Collins [1, 13.71]:

$$\overbrace{L(q_T; Q)}^{\text{“W-term,” “resummed term”}} = T(q_T, Q)_{\text{from Eq. (59)}}^{\mu\nu} \equiv T_{\text{TMD}} W(q_T; Q)^{\mu\nu}. \tag{94}$$

The  $W^{\mu\nu}$  is the exact hadronic tensor for the process under consideration, and  $T_{\text{TMD}}$  is the low transverse momentum “approximator.” It is an instruction to replace  $W^{\mu\nu}$  by an expression,  $W^{\mu\nu}$ , that is a good approximation to  $W^{\mu\nu}$



for small  $q_T$ . **To avoid confusing the  $L$ -term (or  $W$ -term) with the notation for leptonic and hadronic tensors, we will use  $T^{\mu\nu}$  to denote the result of applying the  $T_{\text{TMD}}$  approximator to  $W^{\mu\nu}$ .**

There is another approximator valid for the case that  $q_T \sim Q$ . It is denoted in [1] by  $T_{\text{coll}}$ , and it acts on the hadronic tensor by replacing it an approximated expression that is good for  $q_T \sim Q$ .

The  $Y$  term is then (see the definition in Collins [1, 13.73]):

$$\begin{aligned} Y(q_T, Q)^{\mu\nu} &\equiv T_{\text{coll}}(W(q_T, Q)^{\mu\nu} - T(q_T, Q)^{\mu\nu}) \\ &= T_{\text{coll}}W(q_T; Q)^{\mu\nu} - T_{\text{coll}}T_{\text{TMD}}W(q_T; Q)^{\mu\nu}. \end{aligned} \quad (95)$$

• Other Standard Terminology:

In other literature, the first term on the second line of Eq. (95) is often called the “fixed order” contribution, while the second term is the “asymptotic” contribution. We will denote them by  $FO$  and  $ASY$ .

$$FO(q_T, Q)^{\mu\nu} = T_{\text{coll}}W(q_T, Q)^{\mu\nu} \quad (96)$$

$$ASY(q_T, Q)^{\mu\nu} = T_{\text{coll}}T_{\text{TMD}}W(q_T, Q)^{\mu\nu}. \quad (97)$$

So,

$$Y(q_T, Q)^{\mu\nu} \equiv FO(q_T, Q)^{\mu\nu} - ASY(q_T, Q)^{\mu\nu}. \quad (98)$$

## B. Unpolarized Semi-Inclusive Deep Inelastic Scattering

For unpolarized semi-inclusive deep inelastic scattering, we will mainly quote the results of **NSY**.

### 1. **NSY** Fixed order term

The fixed order contribution to the cross section is expressed compactly by Ref. [3, Eq. (89)].

$$\left. \frac{d\sigma}{dx_n dQ^2 dz dq_T^2} \right|_{FO} = \int_{x_n+w}^1 \frac{d\xi_a}{\xi_a - x_n} M_{AB}(\xi_a, \xi_b, \hat{x}, \hat{z}, q_T) + \int_{z+w}^1 \frac{d\xi_b}{\xi_b - z} M(\xi_a, \xi_b, \hat{x}, \hat{z}, q_T), \quad (99)$$

with

$$\begin{aligned} M_{AB}(\xi_a, \xi_b, \hat{x}, \hat{z}, q_T) &= \frac{e^4}{64\pi s^2 Q^4 x_n^2} \frac{\alpha_s}{\pi} \hat{x} \hat{z} \sum_{a,b,j} e_j^2 d_{B/b}(\xi_b; \mu) f_{a/A}(\xi_a; \mu) \sum_{\alpha=1,2} \Xi_{ab}^{(\alpha)}(\hat{x}, \hat{z}, q_T, Q) A_{(\alpha)} \\ &= \frac{\alpha_{e.m.}^2 \alpha_s}{4s^2 Q^4 x_n^2} \hat{x} \hat{z} \sum_{a,b,j} e_j^2 d_{B/b}(\xi_b; \mu) f_{a/A}(\xi_a; \mu) \sum_{\alpha=1,2} \Xi_{ab}^{(\alpha)}(\hat{x}, \hat{z}, q_T, Q) A_{(\alpha)}. \end{aligned} \quad (100)$$

The  $A_{(\alpha)}$  are the same as Eqs. (50,51). In **NSY**, the capital  $A$  and  $B$  subscripts of  $M$  label the species of target and produced hadron. The  $f$  and  $d$  functions are collinear PDFs and fragmentation functions, respectively. Lower case  $a$  and  $b$  label the flavor and species of the initial target parton and the final state hadronizing parton. The label  $j$  is for the flavor of the quark that couples to the photon. The variables  $\hat{x}$  and  $\hat{z}$  are:

$$\hat{x} \equiv \frac{x_n}{\xi_a}, \quad \hat{z} = \frac{z}{\xi_b}. \quad (101)$$

Also,

$$w \equiv \frac{q_T}{Q} \sqrt{x_n z}. \quad (102)$$

**From B. Wang: Note the extra  $1/(2Q^2)$  in Eq. (100), relative to **NSY** [3, Eq. (85)]. This is needed to get units and matching with asymptotic term. (We suspect typo.) We have modified notation in the following ways: we use lower case  $f$  and  $d$  to denote the collinear pdf and fragmentation functions in order to distinguish them from the corresponding TMD functions. We have taken only the azimuthally**

symmetric parts, so we have integrated over  $\phi$ , and we have included a factor of  $2\pi$  in the definition of  $M_{AB}$ . We use  $s$  rather than  $S_{eA}$  for the center-of-mass energy squared to match with earlier notation. We have substituted the explicit values for  $\sigma_0$  and  $F_l$  from Ref. [Eqs.(37,38)][3],

$$\sigma_0 \equiv \frac{Q^2}{4\pi s x_n^2} \left( \frac{e^2}{2} \right), \quad F_l \equiv \frac{e^2}{2} \frac{1}{Q^2} \quad (103)$$

to get the overall normalization in Eq. (100). Finally, we use  $\Xi$  in place of the  $f$  used by NSY for the factors multiplying  $A^{(\alpha)}$  to avoid confusing it with the collinear PDFs.

The expressions from  $\Xi$  are from Ref. [3, Eqs.(B1-B4)] Specializing to just the azimuthally independent structure functions,

$$\begin{aligned} & \sum_{\alpha=1,2} \Xi_{jk}^{(\alpha)}(\hat{x}, \hat{z}, q_T, Q) A_{(\alpha)} \\ &= 2\delta_{jk} C_F \hat{x} \hat{z} \left\{ \left[ \frac{1}{q_T^2} \left( \frac{Q^4}{\hat{x}^2 \hat{z}^2} + (Q^2 - q_T^2)^2 \right) + 6Q^2 \right] A_1 + 4Q^2 A_2 \right\}, \end{aligned} \quad \text{quark - to - quark} \quad (104)$$

$$\begin{aligned} & \sum_{\alpha=1,2} \Xi_{jg}^{(\alpha)}(\hat{x}, \hat{z}, q_T, Q) A_{(\alpha)} \\ &= \hat{x}(1 - \hat{x}) \left\{ \left[ \frac{Q^4}{q_T^2} \left( \frac{1}{\hat{x}^2 \hat{z}^2} - \frac{2}{\hat{x} \hat{z}} + 2 \right) + 2Q^2 \left( 5 - \frac{1}{\hat{x}} - \frac{1}{\hat{z}} \right) \right] A_1 + 8Q^2 A_2 \right\}, \end{aligned} \quad \text{quark - to - gluon} \quad (105)$$

$$\begin{aligned} & \sum_{\alpha=1,2} \Xi_{gj}^{(\alpha)}(\hat{x}, \hat{z}, q_T, Q) A_{(\alpha)} \\ &= 2C_F \hat{x}(1 - \hat{z}) \left\{ \left[ \frac{1}{\tilde{q}_T^2} \left( \frac{Q^4}{\hat{x}^2 (1 - \hat{z})^2} + (Q^2 - \tilde{q}_T^2)^2 \right) + 6Q^2 \right] A_1 + 4Q^2 A_2 \right\}. \end{aligned} \quad \text{gluon - to - quark} \quad (106)$$

Here,  $j$  and  $k$  indices represent quark flavors whereas  $g$  labels a gluon. In Eq. (106) we have used the NSY notation:

$$\tilde{q}_T \equiv \frac{\hat{z} q_T}{1 - \hat{z}}. \quad (107)$$

## 2. NSY Asymptotic term

The asymptotic term is given in NSY [3, Eq. (36)]:

$$\begin{aligned} \frac{d\sigma}{dx_n dQ^2 dz dq_T^2} \Big|_{ASY} &= \frac{\alpha_{e.m.}^2 \alpha_s(\mu_Q)}{2s^2 x_n^2} \frac{A_1}{q_T^2} \sum_j e_j^2 \left[ d_{B/j}(z; \mu_Q) \{ (P_{qq} \otimes f_{j/A})(x_n; \mu_Q) + (P_{qg} \otimes f_{g/A})(x_n; \mu_Q) \} \right. \\ &\quad + \{ (d_{B/j} \otimes P_{qq})(z; \mu_Q) + (d_{B/g} \otimes P_{gq})(z; \mu) \} f_{j/A}(x_n; \mu_Q) \\ &\quad \left. + 2C_F d_{B/j}(z; \mu_Q) f_{j/A}(x_n; \mu_Q) \left\{ \ln \left( \frac{\mu_Q^2}{q_T^2} \right) - \frac{3}{2} - \ln C_2^2 \right\} \right], \end{aligned} \quad (108)$$

with the longitudinal momentum fraction convolution integral defined by

$$(f \otimes g)(x_n; \mu_Q) \equiv \int_{x_n}^1 \frac{d\xi}{\xi} f(x_n/\xi; \mu_Q) g(\xi; \mu_Q), \quad (109)$$

and the splitting functions are

$$P_{qq}(x) = C_F \frac{1+x^2}{(1-x)_+} + \frac{3}{2} C_F \delta(1-x), \quad (110)$$

$$P_{qg}(x) = \frac{1}{2} (1 - 2x + 2x^2), \quad (111)$$

$$P_{gq}(x) = C_F \frac{1 + (1-x)^2}{x}. \quad (112)$$

In the above, any renormalization scale is valid of course, but we have specifically used  $\mu = \mu_Q = C_2 Q$  in anticipation of later results.

### 3. CPGRSW Asymptotic term

The modified asymptotic term from Ref. [10] first makes the following replacements in Eq. (108):

$$\frac{1}{q_T^2} \rightarrow \frac{C_2 b_0}{q_T \mu_Q C_5} K_1 \left( \frac{C_2 q_T b_0}{C_5 \mu_Q} \right) \quad (113)$$

$$\begin{aligned} \frac{1}{q_T^2} \ln \left( \frac{\mu_Q^2}{q_T^2} \right) &\rightarrow \frac{C_2 b_0}{q_T \mu_Q C_5} \left[ K_1 \left( \frac{C_2 q_T b_0}{C_5 \mu_Q} \right) \ln \left( \frac{C_2 \mu_Q}{C_5 q_T} \right) + \right. \\ &\quad \left. + K_1^{(1)} \left( \frac{C_2 q_T b_0}{C_5 \mu_Q} \right) \right]. \end{aligned} \quad (114)$$

The mathematical identities can be found in Ref. [17, Eqs. (B.10)-(B.13)]. Next, the result is multiplied by a function  $\Xi(q_T/Q, \eta)$  which is unity for small  $q_T/Q$  and vanishes for large  $q_T/Q$  with  $\eta$  being a parameter to determine the exact transition point. Then Eq. (108) becomes

$$\begin{aligned} \Xi \left( \frac{q_T}{Q}, \eta \right) &\frac{\alpha_s(\mu_Q)}{2\pi s x_A} \frac{C_2 b_0}{q_T \mu_Q C_5} \sigma_0 \sum_{q, \bar{q}} e_q^2 \left[ 2f_q(x_A, \mu_Q) D_q(z_B, \mu_Q) \left( C_F \left[ K_1 \left( \frac{C_2 q_T b_0}{C_5 \mu_Q} \right) \ln \left( \frac{C_2 \mu_Q}{C_5 q_T} \right) + \right. \right. \right. \\ &\quad \left. \left. + K_1^{(1)} \left( \frac{C_2 q_T b_0}{C_5 \mu_Q} \right) \right] - \left( \frac{3}{2} + \ln(C_2^2) \right) C_F K_1 \left( \frac{C_2 q_T b_0}{C_5 \mu_Q} \right) \right) \right. \\ &\quad \left. + K_1 \left( \frac{C_2 q_T b_0}{C_5 \mu_Q} \right) \left( f_q(x_A, \mu_Q) \otimes P_{qq}^{(0)} + f_g(x_A, \mu_Q) \otimes P_{qg}^{(0)} \right) D_q(z_B, \mu_Q) \right. \\ &\quad \left. + K_1 \left( \frac{C_2 q_T b_0}{C_5 \mu_Q} \right) f_q(x_A, \mu_Q) \left( D_q(z_B, \mu_Q) \otimes P_{qq}^{(0)} + D_g(z_B, \mu_Q) \otimes P_{gq}^{(0)} \right) \right]. \end{aligned} \quad (115)$$

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