Transverse Momentum Dependent Factorization Recipe Sheet Part 1: Semi-Inclusive Deeply Inelastic Scattering (SIDIS)

T. C. Rogers, N. Sato, B. Wang*

Department of Physics, Old Dominion University, Norfolk, VA 23529, USA and
Theory Center, Jefferson Lab, 12000 Jefferson Avenue, Newport News, VA 23606, USA

(Dated: August 4, 2016)

This work intended to be used both as a reference list of basic formulas for doing TMD factorization calculations and as a Rosetta stone for translating notational conventions throughout the existing TMD literature. The formulas here should be checked and verified frequently and updated as needed. This version deals with SIDIS.

CONTENTS

* trogers@odu.edu

I. COLOR CODING

- **Red color alerts to potential notation/terminology clashes.**

- **Blue color alerts to potential typos.**

- **Green color means general comments, stuff to add.**

A. Initials Glossary

- JCC : J.C. Collins's textbook [1] and related work.

- MOS : Meng-Olness-Soper [2] and related work.

- NSY : Nadolsky-Stump-Yuan [3] and related work. Very closely related to MOS . $\Longrightarrow$ **Need to fill in**

- PJM : Mulders and Tangerman [4] and related work. Overlaps heavily with Ref. [5]. See also Ref. [6].

- JMY : Ji-Ma-Yuan [7] and related work.

- KNV : Koike-Nagashima-Vogelsang [8] and related work. $\Longrightarrow$ **Need to fill in**

- BDR : Barone-Drago-Ratcliffe [9]. $\Longrightarrow$ **Need to fill in**

- CPGRSW : Collins-Prokudin-Gamberg-Rogers-Sato-Wang [10]. $\Longrightarrow$ **Need to fill in**

II. SEMI-INCLUSIVE DEEP INELASTIC SCATTERING (SIDIS): BASIC EXPRESSIONS

A. Kinematics

The proton has momentum P, the virtual photon has momentum q, the produced hadron has momentum P_h, and the incoming and scattered leptons have momenta l and l' respectively. Except when specified, it should be assumed that the frame is one where $P \cdot q \approx P^+ q^- = O(Q^2)$. The mass of the target proton is M and the mass of the produced hadron is M_B.

1. Lorentz Invariant Variables

The conventional kinematic variables are:

$$Q^2 = -q^2 = -(l - l')^2 \tag{1}$$

$$x_{\rm bj} = \frac{Q^2}{2P \cdot q} \tag{2}$$

$$x_{\rm n} = \frac{2x_{\rm bj}}{1 + \sqrt{1 + \frac{4x_{\rm bj}^2 Q^2}{M^2}}} \tag{3}$$

$$y = \frac{P \cdot q}{P \cdot l} \tag{4}$$

$$z_{\rm h} = \frac{P \cdot P_B}{P \cdot q} \tag{5}$$

$$s = (l + P)^2 \tag{6}$$

$$W = (q + P)^2 \ . \tag{7}$$

The inclusive deep inelastic limit is $Q/\Lambda_{\rm QCD} \to \infty$ with fixed $x_{\rm n}$ and $z_{\rm h}$. The kinematical variables obey

$$Q^2 = x_{\rm n} y (s - M^2 - m_l^2) \approx x_{\rm n} y s. \tag{8}$$

The target hadron has a spin vector S.

We will use Nachtmann everywhere $x_{\rm n}$ rather than the usual $x_{\rm bj}$.

2. Frames

There are at least four important reference frames.

- JCC photon frame:

There are two reference frames in JCC [1, Sec.13.15.1]. JCC uses P_A instead of P and p_B instead of P_B.

In the photon frame, the virtual photon and the initial proton both have zero transverse momentum, while the final state produce hadron acquires non-zero transverse momentum. It is thus analogous to the Collins-Soper frame for Drell-Yan scattering and is the frame more closely related to experimental observables. JCC defines

$$q_{{\rm JCC},\gamma} = \left(-x_{\rm n} P^+_{{\rm JCC},\gamma}, \frac{Q^2}{2x_{\rm n} P^+_{{\rm JCC},\gamma}}, \mathbf{0}_T \right) , \tag{9}$$

$$P_{{\rm JCC},\gamma} = \left(P^+_{{\rm JCC},\gamma}, \frac{M^2}{2P^+_{{\rm JCC},\gamma}}, \mathbf{0}_T \right) , \tag{10}$$

$$P_{B\,{\rm JCC},\gamma} = \left(\frac{P_B^2 + M_B^2}{2P^-_{B\,{\rm JCC},\gamma}}, P^-_{B\,{\rm JCC},\gamma}, \mathbf{P}_{BT,{\rm JCC},\gamma} \right) . \tag{11}$$

In the JCC photon frame

$$P^-_{B\,{\rm JCC},\gamma} \approx \frac{z_{\rm h} Q^2}{2x_{\rm n} P^+_{{\rm JCC},\gamma}} \tag{12}$$

up to mass corrections.

- JCC hadron frame:

In the JCC hadron frame, the incoming hadron and final state hadron are exactly back-to-back (zero relative transverse momentum) while the virtual photon generally has non-zero transverse momentum. The JCC hadron frame is especially useful for setting up factorization. (See [1, Sec.13.15.5].) The components of the four-momenta are:

$$q_{\mathrm{JCC},h} = \left(q^{+}_{\mathrm{JCC},h}, q^{-}_{\mathrm{JCC},h}, \boldsymbol{q}_{\mathrm{T}\,\mathrm{JCC},h} \right) , \tag{13}$$

$$P_{\mathrm{JCC},h} = \left(P^{+}_{h}, \frac{M^2}{2P^{+}_{h}}, \mathbf{0}_{T} \right) , \tag{14}$$

$$P_{B\,\mathrm{JCC},h} = \left(\frac{M_B^2}{2P_{B\,\mathrm{JCC},h}^{-}}, P_{B\,\mathrm{JCC},h}^{-}, \mathbf{0}_{\mathrm{T}} \right) . \tag{15}$$

From [1, Eq.(13.104)] and Eq. (12), and from the requirement that $\mathbf{P}_{BT,\mathrm{JCC},h}$ is obtained from $\mathbf{P}_{BT,\mathrm{JCC},\gamma}$ by Lorentz boosting to zero transverse momentum,

$$\boldsymbol{q}_{\mathrm{T}\,\mathrm{JCC},h} = -\frac{\mathbf{P}_{BT,\mathrm{JCC},\gamma}}{z_{\mathrm{h}}} . \tag{16}$$

(This is only valid for zero hadron masses.) Note that the transverse momenta on the left and right sides of Eq. (16) are in different reference frames; the left side is in the JCC hadron frame while the right side is in the JCC photon frame.

(In the Trento Conventions [11], the photon four momentum is labeled by k and Eq. (16) is expressed as $k_{\mathbf{T}} = -P_{hT}/z_{\mathrm{h}}$ where the h subscript labels the final state hadron.)

- MOS hadron frame:

There are two reference frames used by Meng-Olness-Soper(MOS) [2], the MOS hadron frame and the MOS HERA frame.

The MOS hadron frame in Ref. [2] is essentially the Breit frame. In light-cone coordinates

$$q_{\mathrm{MOS},h} = \left(-\frac{Q}{\sqrt{2}}, \frac{Q}{\sqrt{2}}, \mathbf{0} \right) \tag{17}$$

and

$$P_{\mathrm{MOS},h} = \left(\frac{Q}{x_{\mathrm{n}}\sqrt{2}}, 0, \mathbf{0} \right) , \tag{18}$$

where in the second equation masses are neglected. MOS use P_A instead of P for the hadron target momentum.

Note: The JCC hadron frame has zero transverse momentum for the produced hadron and non-zero transverse momentum for the virtual photon, which is *opposite* the situation in the MOS hadron frame. The MOS hadron frame corresponds to the JCC photon frame.

MOS define a Lorentz invariant four-vector ([2, Eq. (10)]) that measures the deviation from the back-to-back configuration:

$$q_t = q - \frac{P_B \cdot q}{P \cdot P_B} P - \frac{P \cdot q}{P \cdot P_B} P_B . \tag{19}$$

This definition removes the components of q along P and P_B. The Lorentz scalar

$$-\mathbf{q}_T^2 \equiv q_t^2 \tag{20}$$

is a measure of the deviation from a back-to-back configuration. (Note that in the actual MOS hadron frame, the two dimensional photon transverse momentum is zero.) From [2, Eq. (11)] and [1, Eq. (13.104)] one may verify that the JCC hadron frame q_{hT}^2 is the same as the MOS q_T^2 (assuming massless hadrons):

$$\mathbf{q}_T^2\big|_{\mathrm{MOS}} = \boldsymbol{q}_{\mathrm{T}\,\mathrm{JCC},h}^{2} \tag{21}$$

Restricting to the MOS hadron frame, MOS use [2, Eq. (11)] and [2, Eq. (13)] and ${P_B}^2 = 0$ to find [2, Eq. (12)], which in light-cone coordinates is

$$P_{B\,\mathrm{MOS},h} = z_{\mathrm{h}} \left(\frac{q_T^2}{Q\sqrt{2}}, \frac{Q}{\sqrt{2}}, |\mathbf{q}_T|, 0 \right) . \tag{22}$$

Hadron masses are neglected. **Note that in the MOS hadron frame, the transverse part of P_B is always in the x_{n} direction and is always positive.**

- MOS HERA Frame:

Come back...

- Other Authors:

Most other authors use variations of the JCC and MOS reference frames, with $\mathbf{P}_{BT}$ and $\mathbf{q}_T$ corresponding to the JCC $\mathbf{P}_{BT,\text{JCC},\gamma}$ and $\boldsymbol{q}_{\text{T}\,\text{JCC},h}$ respectively. Mulders and Tangerman [4, Eqs. (15-17)] give general expressions for four vector components that include the effects of hadron masses. The reference frames essentially correspond to JCC hadron and/or photon frames. References such as [5, 11, 12] specialize the photon frame to the target rest frame rather than the Breit frame. $\mathbf{P}_{BT,\text{JCC},\gamma}$ is invariant, however, with respect to boosts in the z direction.

- Lab Frame (Target Rest Frame):

Certain calculations are simplified by working in coordinates where the target proton is at rest and the lepton has a large energy along the $+z$ direction. In the lab frame, neglecting the lepton mass,

$$l = (E_{\text{lab}}, 0, 0, E_{\text{lab}})\,; \qquad l' = (E'_{\text{lab}}, 0, E'_{\text{lab}} \sin\theta_{\text{lab}}, E'_{\text{lab}} \cos\theta_{\text{lab}})\,; \qquad P = (M, 0, 0, 0)\ . \tag{23}$$

Specializing to the lab frame gives for the kinematic variables,

$$y = 1 = \frac{E'_{\text{lab}}}{E_{\text{lab}}}; \qquad x_{\text{n}} = \frac{Q^2}{2M\nu}; \qquad Q^2 = 2E_{\text{lab}}E'_{\text{lab}}(1 - \cos\theta_{\text{lab}})\,. \tag{24}$$

Here we have defined the lab frame quantity

$$\nu = E - E' = Ey\,. \tag{25}$$

JMY [13] reference the MOS definition of the hadron frame rather than the JCC definition.

3. Summary

From here forward, we will use JCC notation for frames unless otherwise specified. So we will no longer include MOS, JCC, PJM, JMY, etc subscripts for frames. Transverse components of q will be relative to the JCC hadron frame, and transverse components of P_B will be relative to the JCC photon frame, unless otherwise specified. That is, from here forward

$$\mathbf{q}_T \equiv \mathbf{q}_{T,h} \equiv \mathbf{q}_{T,\text{JCC},h} \tag{26}$$
$$\mathbf{P}_{BT} \equiv \mathbf{P}_{BT,\gamma} \equiv \mathbf{P}_{BT,\text{JCC},\gamma}\,. \tag{27}$$

The angles ψ and ϕ will represent the azimuthal angles of the final state lepton and produced hadron respectively in a photon frame.

B. Hadronic and Leptonic Tensors

JCC's normalization convention for $L_{\mu\nu}$ and $W_{\mu\nu}$ [1, Eq. (2.15)] for *inclusive* DIS is

$$E' \frac{d\sigma}{d^3\mathbf{l}'} = \frac{2\,\alpha_{\text{em}}^2}{sQ^4}\, L_{\mu\nu} W^{\mu\nu}_{\text{JCC},\text{incl}}, \tag{28}$$

where the leptonic tensor is defined as,

$$L_{\mu\nu} = 2(l_\mu l'_\nu + l'_\mu l_\nu - g_{\mu\nu} l \cdot l'). \tag{29}$$

This is a factor of 2 larger than the conventions used by Diehl and Sapeta [12, Eq. (22)]. It is also a factor of 2 larger than the analogous tensor in [1, Eq.(12.7)] used for fragmentation in e^+e^- annihilation. MOS [2, Eq. (36)] use a leptonic tensor that differs from JCC by a factor of $1/(2\pi\alpha_{\text{em}})$:

$$L^{\text{JCC}}_{\mu\nu} = \frac{1}{2\pi\alpha_{\text{em}}} L^{\text{MOS}}_{\mu\nu}\ . \tag{30}$$

The convention in Eq. (29) appears to match most other authors. MOS actually write a general $L_{\mu\nu}$ that include all electroweak couplings. For simplicity, in this section we will focus attention on the photon.

The inclusive hadronic tensor following JCC's normalization conventions [1, Eq. (2.18)] is

$$W^{\mu\nu}_{\text{JCC}}(P,q) \equiv 4\pi^3 \sum_X \delta^{(4)}(P+q-P_X)\,\langle P,S|j^\mu(0)|X\rangle\langle X|j^\nu(0)|P,S\rangle. \tag{31}$$

Using

$$\langle P,S|j^\mu(z)|X\rangle = \langle P,S|j^\mu(0)|X\rangle\, e^{i(P-P_X)\cdot z}$$

one may write instead

$$W^{\mu\nu}_{\text{JCC}}(P,q,P_B) \equiv \frac{1}{4\pi}\sum_X \int d^4z\, e^{iq\cdot z}\langle P,S|j^\mu(z)|X\rangle\langle X|j^\nu(0)|P,S\rangle. \tag{32}$$

This differs by a factor of $1/2$ from Eq.(3.1.5) of BDR.

There are several common notational conventions for generalizing the hadronic tensor to the TMD SIDIS case:

- JCC convention:

The $4\pi^3$ from Eq. (31) is *not* included in the definition of the hadronic tensor [1, Eq. (13.111)]:

$$W^{\mu\nu}_{\text{JCC},\text{SIDIS}}(P,q,P_B) \equiv \sum_X \delta^{(4)}(P+q-P_B-P_X)\,\langle P,S|j^\mu(0)|P_B,X\rangle\langle P_B,X|j^\nu(0)|P,S\rangle, \tag{33}$$

Also,

$$W^{\mu\nu}_{\text{JCC},\text{SIDIS}}(P,q,P_B) \equiv \frac{1}{(2\pi)^4}\sum_X \int d^4z\, e^{iq\cdot z}\langle P,S|j^\mu(z)|P_B,X\rangle\langle P_B,X|j^\nu(0)|P,S\rangle. \tag{34}$$

Equation (28) in the TMD SIDIS case becomes,

$$4P_{B\gamma}^{\;0}E'_\gamma \frac{d\sigma}{d^3\mathbf{l}'_\gamma\, d^3\mathbf{P}_{B\gamma}} = \frac{2\,\alpha_{\text{em}}^2}{sQ^4}\, L_{\mu\nu}W^{\mu\nu}_{\text{JCC},\text{SIDIS}}\,. \tag{35}$$

The factor of $2(2\pi)^3$ from the extra phase space factor for P_B has been canceled by the $4\pi^3$ that was originally on the right-hand side of Eq. (31) leaving only a factor of $4P_{B\gamma}^{\;0}$. For the sake of definiteness, we will specialize four momentum components to the photon frame, despite the Lorentz invariance of the phase space.

- MOS convention:

MOS work with the energy flow (tailored to HERA experiments) rather than leaving the cross section differential in z_h. Energy flow is defined as

$$\int_0^1 dz_\text{h}\, z_\text{h}^2 \left(4P_{B\gamma}^{\;0}E'_\gamma \frac{d\sigma}{d^3\mathbf{l}'_\gamma\, d^3\mathbf{P}_{B\gamma}}\right) \propto \frac{2\,\alpha_{\text{em}}^2}{sQ^4}\int_0^1 dz_\text{h}\, z_\text{h}^2 \left(L^{\text{MOS}}{}_{\mu\nu}W^{\mu\nu}_{\text{MOS},\text{SIDIS}}\right)\,. \tag{36}$$

To match with other definitions of the hadron tensor, we have taken the integration $\int_0^1 dz_\text{h} z_\text{h}^2$ outside of the definition of $W^{\mu\nu}_{\text{MOS},\text{SIDIS}}$ rather than leaving it inside the definition as in [2, Eq. (38)].

- JMY convention:

There is a $1/(4z_\text{h})$ in JMY [7, Eq. (11)] relative to the JCC definition:

$$W^{\mu\nu}_{\text{JMY},\text{SIDIS}}(P,q,P_B) = \frac{1}{4z_\text{h}}W^{\mu\nu}_{\text{JCC},\text{SIDIS}}(P,q,P_B). \tag{37}$$

So,

$$P_B^0 E' \frac{d\sigma}{d^3\mathbf{l}'_\gamma\, d^3\mathbf{P}_{B\gamma}} = \frac{2z_\text{h}\,\alpha_{\text{em}}^2}{sQ^4}\, L_{\mu\nu}W^{\mu\nu}_{\text{JMY},\text{SIDIS}}. \tag{38}$$

- PJM Convention:

Finally, the PJM convention for the hadronic tensor is like the JCC definition but with an extra $1/(2M)$. These are the conventions used, for example, in Ref. [5]. (See, also, Mulders notes [6, Eq. (3.38)]). So,

$$4P_B^0 E' \frac{d\sigma}{d^3\mathbf{l}'_\gamma\, d^3\mathbf{P}_{B\gamma}} = \frac{2\,\alpha_{\text{em}}^2(2M)}{sQ^4}\, L_{\mu\nu}W^{\mu\nu}_{\text{PJM},\text{SIDIS}}. \tag{39}$$

1. Variable Changes

To bring the cross section into a form more common in phenomenological studies, we make the variable changes (see, e.g., [1, Eq.(A.15)]):

$$\frac{d^3\mathbf{P}_B}{P_B^0} \to \frac{d^2\mathbf{P}_{BT,\gamma}\,dP_{B\gamma}^-}{P_{B\gamma}^-} \to \frac{d^2\mathbf{P}_{BT,\gamma}\,dz_\mathrm{h}}{z_\mathrm{h}}. \tag{40}$$

Working in the lab frame, it is easy to perform a sequence of additional variable changes:

$$\frac{d^3\mathbf{l'}_\mathrm{lab}}{E'_\mathrm{lab}} \to E'_\mathrm{lab}dE'_\mathrm{lab}d\phi_\mathrm{lab}d(\cos\theta_\mathrm{lab}) \to \frac{sy}{2}dx_\mathrm{n}dyd\phi_\mathrm{lab} \to \frac{y}{2x_\mathrm{n}}dx_\mathrm{n}dQ^2d\phi_\mathrm{lab}\,. \tag{41}$$

Note that

$$dx_\mathrm{n}dQ^2 = x_\mathrm{n}sdx_\mathrm{n}dy = \frac{Q^2}{y}dx_\mathrm{n}dy\,. \tag{42}$$

Since in this section we only consider unpolarized, azimuthally independent cross sections, we integrate over ϕ_lab. Equation (35) becomes

$$\frac{d\sigma}{dx_\mathrm{n}dydz_\mathrm{h}d^2\mathbf{P}_{B\gamma}} = \frac{\pi\alpha_\mathrm{em}^2 y}{2Q^4 z_\mathrm{h}}\,L_{\mu\nu}W^{\mu\nu}_{\mathrm{JCC},\mathrm{SIDIS}} = \frac{2\pi\alpha_\mathrm{em}^2 y}{Q^4}\,L_{\mu\nu}W^{\mu\nu}_{\mathrm{JMY},\mathrm{SIDIS}} = \frac{\pi\alpha_\mathrm{em}^2 yM}{Q^4 z_\mathrm{h}}\,L_{\mu\nu}W^{\mu\nu}_{\mathrm{PJM},\mathrm{SIDIS}}. \tag{43}$$

Or,

$$\frac{d\sigma}{dx_\mathrm{n}dydz_\mathrm{h}dP_{B\gamma}^2} = \frac{\pi^2\alpha_\mathrm{em}^2 y}{2Q^4 z_\mathrm{h}}\,L_{\mu\nu}W^{\mu\nu}_{\mathrm{JCC},\mathrm{SIDIS}} = \frac{2\pi^2\alpha_\mathrm{em}^2 y}{Q^4}\,L_{\mu\nu}W^{\mu\nu}_{\mathrm{JMY},\mathrm{SIDIS}} = \frac{\pi^2\alpha_\mathrm{em}^2 yM}{Q^4 z_\mathrm{h}}\,L_{\mu\nu}W^{\mu\nu}_{\mathrm{PJM},\mathrm{SIDIS}}. \tag{44}$$

From Eq. (16),

$$\frac{d\sigma}{dx_\mathrm{n}dydz_\mathrm{h}dq_\mathrm{T}^2} = \frac{\pi^2\alpha_\mathrm{em}^2 z_\mathrm{h} y}{2Q^4}\,L_{\mu\nu}W^{\mu\nu}_{\mathrm{JCC},\mathrm{SIDIS}} = \frac{2\pi^2\alpha_\mathrm{em}^2 y z_\mathrm{h}^2}{Q^4}\,L_{\mu\nu}W^{\mu\nu}_{\mathrm{JMY},\mathrm{SIDIS}} = \frac{\pi^2\alpha_\mathrm{em}^2 y z_\mathrm{h} M}{Q^4}\,L_{\mu\nu}W^{\mu\nu}_{\mathrm{PJM},\mathrm{SIDIS}}. \tag{45}$$

Or from Eq. (42),

$$\frac{d\sigma}{dx_\mathrm{n}dQ^2dz_\mathrm{h}dq_\mathrm{T}^2} = \frac{\pi^2\alpha_\mathrm{em}^2 z_\mathrm{h}}{2x_\mathrm{n}^2 s^2 Q^2}\,L_{\mu\nu}W^{\mu\nu}_{\mathrm{JCC},\mathrm{SIDIS}} = \frac{2\pi^2\alpha_\mathrm{em}^2 z_\mathrm{h}^2}{x_\mathrm{n}^2 s^2 Q^2}\,L_{\mu\nu}W^{\mu\nu}_{\mathrm{JMY},\mathrm{SIDIS}} = \frac{\pi^2\alpha_\mathrm{em}^2 z_\mathrm{h} M}{x_\mathrm{n}^2 s^2 Q^2}\,L_{\mu\nu}W^{\mu\nu}_{\mathrm{PJM},\mathrm{SIDIS}}. \tag{46}$$

Compare Eq. (45) with [4, Eq. (4)]. Compare Eq. (43) with [7, Eq. (9)]. Note that $\mathbf{P}_{B\gamma}$ is the same as $\mathbf{P}_h$ of Ref. [5].

The totally inclusive cross section is,

$$\frac{d^2\sigma}{dx_\mathrm{n}\,dy} = \frac{Q^2}{y}\frac{d^2\sigma}{dx_\mathrm{n}\,dQ^2} = \frac{2\pi\alpha_\mathrm{em}^2 y}{Q^4}L_{\mu\nu}W^{\mu\nu}_{tot}. \tag{47}$$

C. Unpolarized Structure Functions

The normalizations of the TMD structure functions can be fixed by requiring that one obtains Eq. (47) after an integration of Eq. (43) over z and $\mathbf{P}_B$. Thus,

$$\begin{aligned}
&\frac{1}{4z_\mathrm{h}}\int z_\mathrm{h}dz_\mathrm{h}\,d^2\mathbf{P}_{BT,\gamma}W^{\mu\nu}_{\mathrm{JCC},\mathrm{SIDIS}} = \int z_\mathrm{h}dz_\mathrm{h}\,d^2\mathbf{P}_{BT,\gamma}W^{\mu\nu}_{\mathrm{JMY},SIDIS} = \frac{M}{2z_\mathrm{h}}\int z_\mathrm{h}dz_\mathrm{h}\,d^2\mathbf{P}_{BT,\gamma}W^{\mu\nu}_{\mathrm{PJM},\mathrm{SIDIS}}\\
&= \frac{1}{4}\int z_\mathrm{h}^2dz_\mathrm{h}\,d^2\boldsymbol{q}_\mathrm{T}W^{\mu\nu}_{\mathrm{JCC},\mathrm{SIDIS}} = \int z_\mathrm{h}^3dz_\mathrm{h}\,d^2\boldsymbol{q}_\mathrm{T}W^{\mu\nu}_{\mathrm{JMY},SIDIS} = \frac{M}{2}\int z_\mathrm{h}^2dz_\mathrm{h}\,d^2\boldsymbol{q}_\mathrm{T}W^{\mu\nu}_{\mathrm{PJM},\mathrm{SIDIS}} = W^{\mu\nu}_{tot}.
\end{aligned} \tag{48}$$

$W^{\mu\nu}_{tot}$ can be expressed in terms of the usual $F_1(x_\mathrm{n},Q^2)$ and $F_2(x_\mathrm{n},Q^2)$ by following the usual structure function decomposition (e.g., Ref [1] Eq.(2.20)). Therefore, we apply an analogous structure function decomposition to the unintegrated cross section:

$$\frac{1}{4z_{\rm h}} W^{\mu\nu}_{\rm JCC,SIDIS} = W^{\mu\nu}_{\rm JMY, \mathit{SIDIS}} = \frac{M}{2z_{\rm h}} W^{\mu\nu}_{\rm PJM,SIDIS}$$
$$= \left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right) F_1(x_{\rm n}, z_{\rm h}, \mathbf{P}_{B\gamma}, Q^2) + \frac{(P^\mu - q^\mu P\cdot q/q^2)(P^\nu - q^\nu P\cdot q/q^2)}{P\cdot q} F_2(x_{\rm n}, z_{\rm h}, \mathbf{P}_{B\gamma}, Q^2). \tag{49}$$

Then,

$$\frac{d\sigma}{dx_{\rm n} dy dz_{\rm h} d^2\mathbf{P}_{BT,\gamma}} = \frac{4\pi\alpha_{\rm em}^2}{x_{\rm n} y Q^2}\left[\left(1 - y - \frac{x_{\rm n}^2 y^2 M^2}{Q^2}\right) F_2(x_{\rm n}, z_{\rm h}, \mathbf{P}_{BT,\gamma}, Q^2) + y^2 x_{\rm n} F_1(x_{\rm n}, z_{\rm h}, \mathbf{P}_{BT,\gamma}, Q^2)\right]$$
$$= \frac{4\pi\alpha_{\rm em}^2}{x_{\rm n} y Q^2}\left[2x_{\rm n}\left(1 - y + \frac{y^2}{2} - \frac{x_{\rm n}^2 y^2 M^2}{Q^2}\right) F_T(x_{\rm n}, z_{\rm h}, \mathbf{P}_{BT,\gamma}, Q^2) + \left(1 - y - \frac{x_{\rm n}^2 y^2 M^2}{Q^2}\right) F_L(x_{\rm n}, z_{\rm h}, \mathbf{P}_{BT,\gamma}, Q^2)\right]. \tag{50}$$

In the last line we have extended the standard definitions, $F_T \equiv F_1$ and $F_L \equiv F_2 - 2x_{\rm n} F_1$, from the integrated case to the TMD case. This agrees with [5, Eq. (2.14)] in the limit that hadron masses are neglected.

Note that [5, Eq. (2.18)] defines F_T with a different normalization: $F_T = 2x_{\rm n} F_1$ so that $F_2 = F_L + F_T$; this needs to be taken into account to get agreement with Eq. (50).

From Eqs. (43,47,49,50),

$$\int z_{\rm h} dz_{\rm h}\, d^2\mathbf{P}_{BT,\gamma}\, F_T(x_{\rm n}, z_{\rm h}, \mathbf{P}_{BT,\gamma}, Q^2) = \int z_{\rm h}^3 dz_{\rm h}\, d^2\boldsymbol{q}_{\rm T}\, F_T(x_{\rm n}, z_{\rm h}, z_{\rm h}\boldsymbol{q}_{\rm T}, Q^2) = F_T(x_{\rm n}, Q^2) \tag{51}$$
$$\int z_{\rm h} dz_{\rm h}\, d^2\mathbf{P}_{BT,\gamma}\, F_L(x_{\rm n}, z_{\rm h}, \mathbf{P}_{BT,\gamma}, Q^2) = \int z_{\rm h}^3 dz_{\rm h}\, d^2\boldsymbol{q}_{\rm T}\, F_L(x_{\rm n}, z_{\rm h}, z_{\rm h}\boldsymbol{q}_{\rm T}, Q^2) = F_L(x_{\rm n}, Q^2). \tag{52}$$

$\Longrightarrow$ **Contribution from Bowen Wang**

NSY write the structure function [3, Eq.(31)] decomposition as a sum over A_n and V_n functions. For the azimuthally symmetric case,

$$\frac{d\sigma}{dx_{\rm n} dQ^2 dz dq_{\rm T}^2} = 2\pi \sum_{\alpha=1,2} V_{AB}^{(\alpha)} A_\alpha\,. \tag{53}$$

Here, from [3, Eq. (33)]

$$A_1 = 1 + \cosh^2\psi\,, \tag{54}$$
$$A_2 = -2\,, \tag{55}$$

where we have used the NSY notation

$$\cosh\psi = \frac{2}{y} - 1\,. \tag{56}$$

Using Eqs. (11), (12) in Ref [3] for l and l', and using Eqs. (49,50) for $W^{\mu\nu}_{\rm JCC,SIDIS}$, the cross section in Eq. (46) is written as

$$\frac{d\sigma}{dx_{\rm n} dQ^2 dz dq_{\rm T}^2} = \frac{\pi^2\alpha_{\rm em}^2 z}{2x_{\rm n}^2 s^2 Q^2} L_{\mu\nu} W^{\mu\nu}_{\rm JCC,SIDIS} = \frac{4\pi^2\alpha_{\rm em}^2 z^2}{x_{\rm n}^2 s^2}\left[F_1 + (1/4x_{\rm n})(\cosh^2\psi - 1)F_2\right] \tag{57}$$
$$= \frac{4\pi^2\alpha_{\rm em}^2 z^2}{x_{\rm n}^2 s^2}\left[\frac{1}{2}\left(\cosh^2\psi + 1\right) F_T + \frac{1}{4x_{\rm n}}(\cosh^2\psi - 1) F_L\right] \tag{58}$$
$$= \frac{4\pi^2\alpha_{\rm em}^2 z^2}{x_{\rm n}^2 s^2}\left[\frac{1}{4x_{\rm n}}\left(\cosh^2\psi + 1\right) F_2 - \frac{1}{2x_{\rm n}} F_L\right]. \tag{59}$$

Reading off the coefficients from Eqs. (54,55) gives,

$$F_2 = V_{AB}^{(1)} \frac{2x_{\rm n}^3 s^2}{\pi\alpha_{\rm em}^2 z^2} \tag{60}$$
$$F_L = V_{AB}^{(2)} \frac{2x_{\rm n}^3 s^2}{\pi\alpha_{\rm em}^2 z^2} \tag{61}$$
$$F_T = F_1 = \left(V_{AB}^{(1)} - V_{AB}^{(2)}\right)\frac{x_{\rm n}^2 s^2}{\pi\alpha_{\rm em}^2 z^2}\,. \tag{62}$$

III. TMD FACTORIZATION EXPRESSIONS

A. SIDIS: TMD

1. Large and small transverse momentum separation

We will mainly follow the notation for expressing the TMD-factorization formula of Collins [1, Eq. (13.116)]. In terms of a transverse momentum convolution integral involving TMD PDFs, the hadronic tensor is:

$$\begin{aligned}
W^{\mu\nu}_{\text{SIDIS},\text{JCC}} &= \overbrace{T^{\mu\nu}_{\text{SIDIS}}}^{\text{“W-term,” “L-term”}} \qquad + Y^{\mu\nu}_{\text{SIDIS}} \\
&= \sum_f |\mathcal{H}_f(Q/\mu, \alpha_s(\mu))^2|^{\mu\nu} \\
&\times \int d^2\mathbf{k}_{1T}\, d^2\mathbf{k}_{2T} \delta^{(2)}(\mathbf{k}_{1T} + \mathbf{q}_T - \mathbf{k}_{2T}) F^{[+]}_{f/p}(x_\text{n}, \mathbf{k}_{1T}; \zeta_\text{PDF}; \mu)\, D_{h/f}(z, z\mathbf{k}_{2T}; \zeta_\text{FF}; \mu) + Y^{\mu\nu}_{\text{SIDIS}} \\
&= \sum_f |\mathcal{H}_f(Q/\mu, \alpha_s(\mu))^2|^{\mu\nu} F^{[+]}_{f/p}(x_\text{n}, \mathbf{k}_{1T}; \zeta_\text{PDF}; \mu) \circledast D_{h/f}(z, z\mathbf{k}_{2T}; \zeta_\text{FF}; \mu) + Y^{\mu\nu}_{\text{SIDIS}}.
\end{aligned} \tag{63}$$

where the last line defines the shorthand "$\circledast$" for the transverse momentum convolution integral. The hadronic tensor (or cross section) separates into two terms: the first accurately approximates the low momentum region $q_\text{T} \ll O(Q)$ while the second is a correction for the the region of $q_\text{T} \sim Q$. The first term is conventionally called the "W-term" or the "L-term." It has also frequently been called the "resummed term," but this terminology will be disfavored in this document since the factorization into renormalized operators goes beyond resummation. Since W and L also denote the hadronic and leptonic tensors, we will use T to denote the low momentum term, as indicated in Eq. (63).

The second term is almost always called the "Y-term."

For this section, we will focus on the properties of the T-term. We will return to the Y-term in Sec. X. The T-term by itself is

$$\begin{aligned}
T^{\mu\nu}_{\text{SIDIS}} &= \sum_f |\mathcal{H}_f(Q/\mu, \alpha_s(\mu))^2|^{\mu\nu} \\
&\times \int d^2\mathbf{k}_{1T}\, d^2\mathbf{k}_{2T} \delta^{(2)}(\mathbf{k}_{1T} + \mathbf{q}_T - \mathbf{k}_{2T}) F^{[+]}_{f/p}(x_\text{n}, \mathbf{k}_{1T}; \zeta_\text{PDF}; \mu)\, D_{h/f}(z, z\mathbf{k}_{2T}; \zeta_\text{FF}; \mu) \\
&= \sum_f |\mathcal{H}_f(Q/\mu, \alpha_s(\mu))^2|^{\mu\nu} F^{[+]}_{f/p}(x_\text{n}, \mathbf{k}_{1T}; \zeta_\text{PDF}; \mu) \circledast D_{h/f}(z, z\mathbf{k}_{2T}; \zeta_\text{FF}; \mu)\,.
\end{aligned} \tag{64}$$

2. Hard Factor

The precise definition of the hard factor $|\mathcal{H}_f(Q/\mu, \alpha_s(\mu))^2|^{\mu\nu}$ will vary by a normalization factor depending on the convention for $W^{\mu\nu}$ (see Sec. II B). The $[+]$ on Eq. (63) is to indicate that the TMD PDF is defined with a future pointing Wilson line. Comparing with Collins [1] Eq. (13.116), the normalization for the hard part in that convention is:

$$\sum_f |\mathcal{H}_f(Q/\mu, \alpha_s(\mu))^2|^{\mu\nu}_{\text{JCC}} = \frac{z}{Q^2} \sum_f \text{Tr}\left[\hat{\not{k}}_{A,\gamma} H^\nu_f(Q/\mu, \alpha_s(\mu)) \hat{\not{k}}_{A,\gamma} H^\mu_f(Q/\mu, \alpha_s(\mu))^\dagger\right]. \tag{65}$$

The momenta $\hat{k}_{A,\gamma}$ and $\hat{k}_{B,\gamma}$ are the approximate parton momenta with the approximation defined in the photon frame. The components in the photon frame are

$$\hat{k}_{A,\gamma} = \left(x_\text{n} P^+, 0, \mathbf{0}_\text{T}\right)\,, \qquad \hat{k}_{B,\gamma} = \left(0, \frac{Q^2}{2x_\text{n} P^+}, \mathbf{0}_\text{T}\right)\,. \tag{66}$$

Note typo in Eq. [1, Eq.(13.115)]; the plus component of $\hat{k}_{A,\gamma}$ should be positive. $H_f^\nu(Q;\mu)$ can be written as scalar function times the tree level electromagnetic vertex:

$$H_f^\nu(Q/\mu,\alpha_s(\mu)) = -ie_f\gamma^\mu\,\Gamma(\mu/Q,\alpha_s(\mu))_{\rm SIDIS} \tag{67}$$

where $\Gamma(\mu/Q,\alpha_s(\mu))_{\rm SIDIS}$ is a hard vertex function equal to one at zeroth order. Using Eqs. (29,65,67):

$$\begin{aligned}&L_{\mu\nu}\sum_f|\mathcal{H}_f(Q/\mu,\alpha_s(\mu))^2|^{\mu\nu}_{\rm JCC} = 4zL_{\mu\nu}\sum_f|\mathcal{H}_f(Q/\mu,\alpha_s(\mu))^2|^{\mu\nu}_{\rm JMY} = 2M_pL_{\mu\nu}\sum_f|\mathcal{H}_f(Q/\mu,\alpha_s(\mu))^2|^{\mu\nu}_{\rm PJM}\\ &= \sum_f\frac{8Q^2ze_q^2}{y^2}\left(1-y+\frac{y^2}{2}\right)|\Gamma(\mu/Q,\alpha(\mu))_{\rm SIDIS}|^2.\end{aligned} \tag{68}$$

3. Small q_T Cross section in terms of TMD functions

Substituting Eq. (68) into Eq. (43) and keeping only the T-term from Eq. (64) gives

$$\begin{aligned}&\left.\frac{d\sigma}{dx_{\rm n}dydzd^2\mathbf{P}_{BT,\gamma}}\right|_{\rm unpol,\ T-part} = \frac{1}{z^2}\left.\frac{d\sigma}{dx_{\rm n}dydzd^2\boldsymbol{q}_{\rm T}}\right|_{\rm unpol,\ T-part}\\ &= \sum_f\frac{4\pi\alpha_{\rm em}^2e_f^2}{Q^2y}(1-y+y^2/2)|\Gamma(\mu/Q,\alpha_s(\mu))_{\rm SIDIS}|^2F^{[+]}_{f/p}(x_{\rm n},\mathbf{k}_{1T};\zeta_{\rm PDF},\mu)\circledast D_{h/f}(z,z\mathbf{k}_{2T};\zeta_{\rm FF};\mu)\\ &= \sum_f\frac{4\pi\alpha_{\rm em}^2e_f^2sx_{\rm n}}{Q^4}(1-y+y^2/2)|\Gamma(\mu/Q,\alpha_s(\mu))_{\rm SIDIS}|^2F^{[+]}_{f/p}(x_{\rm n},\mathbf{k}_{1T};\zeta_{\rm PDF};\mu)\circledast D_{h/f}(z,z\mathbf{k}_{2T};\zeta_{\rm FF};\mu)\,.\end{aligned} \tag{69}$$

In the last line, we have used $x_{\rm n}ys \approx Q^2$. If one integrates over the hadron's azimuthal angle, the right side gets an extra factor of 2π. This agrees with JCC [1] Eq. (12.91) and JMY [13] Eq. (50).

Comparing with Eq. (50), it is clear that the structure function contributions from the T-term TMD part (i.e., excluding the Y-term) are:

$$F_T(x_{\rm n},z,\mathbf{P}_{BT,\gamma},Q^2)^{\rm T-term} \approx \sum_f e_f^2|\Gamma(\mu/Q,\alpha(\mu))_{\rm SIDIS}|^2F^{[+]}_{f/p}(x_{\rm n},\mathbf{k}_{1T};\zeta_{\rm PDF};\mu)\circledast D_{h/f}(z,z\mathbf{k}_{2T};\zeta_{\rm FF};\mu) \tag{70}$$

and,

$$F_L(x_{\rm n},z,\mathbf{P}_{BT,\gamma},Q^2)^{\rm T-term}\approx 0. \tag{71}$$

The "≈" is to emphasize that terms suppressed by M_p^2/Q^2 have been dropped.

4. Coordinate space

In it convenient to work with the convolution $F^{[+]}_{f/p}\circledast D_{h/f}$ in transverse coordinate space. The Fourier transforms of the TMD functions are:

$$F^{[+]}_{f/p}(x_{\rm n},\mathbf{k}_{2T}-\boldsymbol{q}_{\rm T};\zeta_{\rm PDF};\mu) = \frac{1}{(2\pi)^2}\int d^2\boldsymbol{b}_{\rm T}e^{i\boldsymbol{b}_{\rm T}\cdot(\mathbf{k}_{2T}-\boldsymbol{q}_{\rm T})}\tilde{F}^{[+]}_{f/p}(x_{\rm n},\boldsymbol{b}_{\rm T};\zeta_{\rm PDF};\mu) \tag{72}$$

$$D_{h/f}(z,z\mathbf{k}_{2T};\zeta_{\rm FF};\mu) = \frac{1}{(2\pi)^2}\int d^2\boldsymbol{b}_{\rm T}e^{-i\boldsymbol{b}_{\rm T}\cdot\mathbf{k}_{2T}}\tilde{D}_{h/f}(z,\boldsymbol{b}_{\rm T};\zeta_{\rm FF};\mu)\,. \tag{73}$$

So the convolution in Eq. (63) is

$$
\begin{aligned}
&F^{[+]}_{f/p}(x_{\rm n}, \mathbf{k}_{1T}; \zeta_{\rm PDF}; \mu) \circledast D_{h/f}(z, z\mathbf{k}_{2T}; \zeta_{\rm FF}; \mu) \\
&\quad = \int d^2\mathbf{k}_{2T} F^{[+]}_{f/p}(x_{\rm n}, \mathbf{k}_{2T} - \boldsymbol{q}_{\rm T}; \zeta_{\rm PDF}; \mu)\, D_{h/f}(z, z\mathbf{k}_{2T}; \zeta_{\rm FF}; \mu) \\
&\quad = \frac{1}{(2\pi)^4} \int d^2\mathbf{k}_{2T} \int d^2\boldsymbol{b}_{\rm T} \int d^2\boldsymbol{b}'_{\rm T} e^{-i\boldsymbol{b}_{\rm T}\cdot\boldsymbol{q}_{\rm T}} e^{i\mathbf{k}_{2T}\cdot(\boldsymbol{b}'_{\rm T}-\boldsymbol{b}_{\rm T})} \tilde{F}^{[+]}_{f/p}(x_{\rm n}, \boldsymbol{b}'_{\rm T}; \zeta_{\rm PDF}; \mu) \tilde{D}_{h/f}(z, \boldsymbol{b}_{\rm T}; \zeta_{\rm FF}; \mu) \\
&\quad = \int \frac{d^2\boldsymbol{b}_{\rm T}}{(2\pi)^2} e^{-i\boldsymbol{b}_{\rm T}\cdot\boldsymbol{q}_{\rm T}} \tilde{F}^{[+]}_{f/p}(x_{\rm n}, \boldsymbol{b}_{\rm T}; \zeta_{\rm PDF}; \mu) \tilde{D}_{h/f}(z, \boldsymbol{b}_{\rm T}; \zeta_{\rm FF}; \mu)
\end{aligned}
\tag{74}
$$

So, for example, Eq. (69) can be written:

$$
\begin{aligned}
&\left.\frac{1}{z^2}\frac{d\sigma}{dx_{\rm n}\,dy\,dz\,d^2\boldsymbol{q}_{\rm T}}\right|_{\text{unpol, T}-\text{part}} \\
&= \sum_f \frac{4\pi\alpha_{\rm em}^2 e_f^2}{Q^2 y}(1 - y + y^2/2)|\Gamma(\mu/Q, \alpha_s(\mu))_{\rm SIDIS}|^2 \int \frac{d^2\boldsymbol{b}_{\rm T}}{(2\pi)^2} e^{-i\boldsymbol{b}_{\rm T}\cdot\boldsymbol{q}_{\rm T}} \tilde{F}^{[+]}_{f/p}(x_{\rm n}, \boldsymbol{b}_{\rm T}; \zeta_{\rm PDF}; \mu) \tilde{D}_{h/f}(z, \boldsymbol{b}_{\rm T}; \zeta_{\rm FF}; \mu)\,.
\end{aligned}
\tag{75}
$$

In situations with azimuthal symmetry,

$$
\begin{aligned}
&\left.\frac{d\sigma}{dx_{\rm n}\,dy\,dz\,dq_{\rm T}}\right|_{\text{unpol, T}-\text{part}} \\
&= \sum_f \frac{4\pi z^2\alpha_{\rm em}^2 e_f^2 q_{\rm T}}{Q^2 y}(1 - y + y^2/2)|\Gamma(\mu/Q, \alpha_s(\mu))_{\rm SIDIS}|^2 \int db_{\rm T}\, b_{\rm T} J_0(q_{\rm T} b_{\rm T}) \tilde{F}^{[+]}_{f/p}(x_{\rm n}, \boldsymbol{b}_{\rm T}; \zeta_{\rm PDF}; \mu) \tilde{D}_{h/f}(z, \boldsymbol{b}_{\rm T}; \zeta_{\rm FF}; \mu)\,.
\end{aligned}
\tag{76}
$$

Note that the sign in the Fourier transform exponential $e^{-i\boldsymbol{b}_{\rm T}\cdot\boldsymbol{q}_{\rm T}}$ in Eq. (74) differs from that of JCC, Eq. (13.116). This will not matter for azimuthally symmetric cross sections, but may cause differences in azimuthal asymmetries.

Many authors use the notation:

$$
\mathcal{C}\left[F^{[+]}_{f/p}\, D_{h/f}\right] \equiv F^{[+]}_{f/p}(x_{\rm n}, \mathbf{k}_{1T}; \zeta_{\rm PDF}; \mu) \circledast D_{h/f}(z, z\mathbf{k}_{2T}; \zeta_{\rm FF}; \mu) \tag{77}
$$

for the convolution in Eq. (74).

5. Conventions for Factors of z

A common notation is to change variables in Eq. (77) so that z does not multiply $\mathbf{k}_{2T}$ in $D_{h/f}(z, z\mathbf{k}_{2T}; \zeta_{\rm FF}; \mu)$. One defines $z\mathbf{k}_{2T} = \underline{k}$. Then a change of variables gives

$$\begin{aligned}\mathcal{C}\left[F^{[+]}_{f/p}\, D_{h/f}\right] &= \sum_f \int d^2\mathbf{k}_{1T}\, d^2\mathbf{k}_{2T} \delta^{(2)}(\mathbf{k}_{1T} + \mathbf{q}_T - \mathbf{k}_{2T}) F^{[+]}_{f/p}(x, \mathbf{k}_{1T}; \zeta_{\rm PDF}; \mu)\, D_{h/f}(z, z\mathbf{k}_{2T}; \zeta_{\rm FF}; \mu) \\ &= \sum_f \int d^2\mathbf{k}_{1T}\, d^2\mathbf{k}_{2T} \delta^{(2)}(\mathbf{k}_{1T} + \mathbf{P}_{BT,\gamma}/z - \underline{\mathbf{k}}/z) F^{[+]}_{f/p}(x, \mathbf{k}_{1T}; \zeta_{\rm PDF}; \mu)\, D_{h/f}(z, \underline{\mathbf{k}}; \zeta_{\rm FF}; \mu) \\ &= \sum_f \int d^2\mathbf{k}_{1T}\, d^2\underline{\mathbf{k}}\, \delta^{(2)}(z\mathbf{k}_{1T} + \mathbf{P}_{BT,\gamma} - \underline{\mathbf{k}}) F^{[+]}_{f/p}(x, \mathbf{k}_{1T}; \zeta_{\rm PDF}; \mu)\, D_{h/f}(z, \underline{\mathbf{k}}; \zeta_{\rm FF}; \mu)\,.\end{aligned} \tag{78}$$

Then, $\underline{k}$ is the transverse momentum of the hadronizing parton relative to its parent jet.

IV. TMD FUNCTIONS

A. Further Notation and Conventions

It will be useful to have a specific scheme for cutting off the behavior of certain perturbatively calculated expressions at large-$\boldsymbol{b}_{\rm T}$. For this, many authors use the "b-star" method by defining:

$$\boldsymbol{b}_*(\boldsymbol{b}_T) \to \begin{cases} \boldsymbol{b}_T & b_T \ll b_{\max} \\ \boldsymbol{b}_{\max} & b_T \gg b_{\max}\,. \end{cases} \tag{79}$$

where $\boldsymbol{b}_{\max} = b_{\max} \frac{\boldsymbol{b}_{\rm T}}{\|\boldsymbol{b}_{\rm T}\|}$.

The standard $\overline{\rm MS}$ renormalization group scale is μ, and one commonly uses scales

$$\mu_Q \equiv C_2 Q \tag{80}$$
$$\mu_b \equiv C_1/b_{\rm T} \tag{81}$$
$$\mu_{b_*} \equiv C_1/b_*\,, \tag{82}$$

where C_1 and C_2 are arbitrary constants that are ultimately to be chosen to optimize perturbative convergence.

B. TMD Parton Distributions

The definition of a TMD PDF in coordinate space is:

The evolution equations are:

The most general and basic way to write the solution is evolve from some reference scales $\mu \to \mu_0$, $\zeta_{\rm PDF} \to Q_0^2$ to some arbitrary μ and $\zeta_{\rm PDF}$.

$$\begin{aligned}&\tilde{F}_{f/P}(x, \mathbf{b}_T; \zeta_{\rm PDF}, \mu) \\ &\quad = \tilde{F}_{f/P}(x, \mathbf{b}_T; Q_0^2, \mu_0) \\ &\quad \times \exp\left\{ \ln\frac{\sqrt{\zeta_{\rm PDF}}}{Q_0} \tilde{K}(b_*; \mu_{b_*}) + \int_{\mu_0}^{\mu} \frac{d\mu'}{\mu'} \left[\gamma_F(\alpha_s(\mu'); 1) - \ln\frac{\sqrt{\zeta_{\rm PDF}}}{\mu'} \gamma_K(g(\mu'))\right] \right. \\ &\quad \left. + \int_{\mu_0}^{\mu_{b_*}} \frac{d\mu'}{\mu'} \ln\frac{\sqrt{\zeta_{\rm PDF}}}{Q_0} \gamma_K(\alpha_s(\mu')) \right\} \\ &\quad \times \exp\left\{ -g_K(b_T) \ln\frac{\sqrt{\zeta_{\rm PDF}}}{Q_0} \right\}.\end{aligned} \tag{83}$$

Ultimately, one typically sets $\mu = Q$ and $\zeta_{\rm PDF} = Q^2$. The scales μ_0 and Q_0 should be large enough to admit perturbative calculations of the anomalous dimensions.

More commonly, one evolves relative to μ_b (from Eq. (81)) in order to permit $C_1/b_{\rm T}$ to be used as a hard scale for the application of an operator product expansion in the limit of small $b_{\rm T}$. Then, the small $b_{\rm T}$ region is expressible in terms of collinear factorization with collinear PDFs:

$$\begin{aligned}
\tilde{F}_{f/P}(x, \mathbf{b}_T; \zeta_{\rm PDF}, \mu) = &\overbrace{\sum_j \int_x^1 \frac{d\hat{x}}{\hat{x}} \tilde{C}_{f/j}(x/\hat{x}, b_*; \mu_{b_*}^2, \mu_{b_*}, \alpha_s(\mu_{b_*})) f_{j/P}(\hat{x}, \mu_{b_*})}^{\rm AA} \\
&\times \overbrace{\exp\left\{ \ln \frac{\sqrt{\zeta_{\rm PDF}}}{\mu_{b_*}} \tilde{K}(b_*; \mu_{b_*}) + \int_{\mu_{b_*}}^{\mu} \frac{d\mu'}{\mu'} \left[\gamma_F(\alpha_s(\mu'); 1) - \ln \frac{\sqrt{\zeta_{\rm PDF}}}{\mu'} \gamma_K(g(\mu')) \right] \right\}}^{\rm BB} \\
&\times \overbrace{\exp\left\{ -g_{f/P}(x, b_T) - g_K(b_T) \ln \frac{\sqrt{\zeta_{\rm PDF}}}{Q_0} \right\}}^{\rm CC} .
\end{aligned} \tag{84}$$

C. TMD Fragmentation Functions

The definitions of a TMD FFs are:

The evolutions equations are:

V. HARD FACTORS

The hard factors for SIDIS and DY at one loop in the $\overline{\rm MS}$ renormalization scheme are:

$$|\Gamma(Q; \mu/Q, \alpha_s(\mu))_{\rm SIDIS}|^2 = 1 + 4C_F \left(\frac{3}{2} \ln \left(Q^2/\mu^2 \right) - \frac{1}{2} \ln^2 \left(Q^2/\mu^2 \right) - 4 \right) \left(\frac{\alpha_s(\mu)}{4\pi} \right) + \mathcal{O}\left(\left(\frac{\alpha_s(\mu)}{4\pi} \right)^2 \right) \tag{85}$$

$$|\Gamma(Q; \mu/Q, \alpha_s(\mu))_{\rm DY}|^2 = 1 + 4C_F \left(\frac{3}{2} \ln \left(Q^2/\mu^2 \right) - \frac{1}{2} \ln^2 \left(Q^2/\mu^2 \right) - 4 + \frac{\pi^2}{2} \right) \left(\frac{\alpha_s(\mu)}{4\pi} \right) + \mathcal{O}\left(\left(\frac{\alpha_s(\mu)}{4\pi} \right)^2 \right) . \tag{86}$$

VI. $\overline{\rm MS}$ COLLINS-SOPER KERNEL

The CS kernel $K(b_T; \mu, \alpha_s(\mu))$ is known at least to order $\alpha_s(\mu)$:

$$\tilde{K}(b_T; \mu, \alpha_s(\mu)) = -4C_F \left[\ln(\mu^2 b_T^2) - \ln 4 + 2\gamma_{\rm E} \right] \left(\frac{\alpha_s(\mu)}{4\pi} \right) + \mathcal{O}(\alpha_s(\mu)^2). \tag{87}$$

VII. CS KERNEL $\overline{\text{MS}}$ ANOMALOUS DIMENSION

The $\overline{\text{MS}}$ anomalous dimension for $K(b_T;\mu)$ has been calculated to three loops by Moch, Vermaseren and Vogt (MVV) [14, Eqs.(3.8,3.9)]:

$$
\begin{aligned}
&\gamma_K(\alpha_s(\mu)) = \\
&\qquad 8C_F\left(\frac{\alpha_s(\mu)}{4\pi}\right) \\
&\qquad + \left[16C_FC_A\left(\frac{67}{18}-\zeta_2\right)+16C_Fn_f\left(-\frac{5}{9}\right)\right]\left(\frac{\alpha_s(\mu)}{4\pi}\right)^2 \\
&\qquad + \left[32C_FC_A^2\left(\frac{254}{24}-\frac{67}{9}\zeta_2+\frac{11}{6}\zeta_3+\frac{11}{5}\zeta_2^2\right)+32C_F^2n_f\left(-\frac{55}{24}+2\zeta_3\right)\right. \\
&\qquad\quad \left. + 32C_FC_An_f\left(-\frac{209}{108}+\frac{10}{9}\zeta_2-\frac{7}{3}\zeta_3\right)+32C_Fn_f^2\left(-\frac{1}{27}\right)\right]\left(\frac{\alpha_s(\mu)}{4\pi}\right)^3 \\
&\qquad + \mathcal{O}\left(\left(\frac{\alpha_s(\mu)}{4\pi}\right)^4\right).
\end{aligned}
\tag{88}
$$

Note that MVV use the notation A for the anomalous dimension. It is related to γ_K by $\gamma_K = 2A$. The MVV A should not be used for the A function commonly used in the CSS formalism.

Note the evolution equations (2.3,2.4) in MVV and the definition of the expansion in Eq. (2.5).

VIII. TMD PDFS AND FFS, $\overline{\text{MS}}$ ANOMALOUS DIMENSIONS

The anomalous dimensions for the TMD PDFs and FFs, $\gamma_{\rm F}$ and $\gamma_{\rm D}$ respectively, are known at least to order $\alpha_s(\mu)$. To order $\alpha_s(\mu)$ they are the same for the TMD PDF and the TMD fragmentation function:

$$
\gamma_{\rm PDF}(\mu,\zeta_{\rm PDF}/\mu^2,\alpha_s(\mu)) = 4C_{\rm F}\left(\frac{3}{2}-\ln\left(\frac{\zeta_{\rm PDF}}{\mu^2}\right)\right)\left(\frac{\alpha_s(\mu)}{4\pi}\right)+\mathcal{O}\left(\left(\frac{\alpha_s(\mu)}{4\pi}\right)^2\right), \tag{89}
$$

$$
\gamma_{\rm FF}(\mu,\zeta_{\rm FF}/\mu^2,\alpha_s(\mu)) = 4C_{\rm F}\left(\frac{3}{2}-\ln\left(\frac{\zeta_{\rm FF}}{\mu^2}\right)\right)\left(\frac{\alpha_s(\mu)}{4\pi}\right)+\mathcal{O}\left(\left(\frac{\alpha_s(\mu)}{4\pi}\right)^2\right). \tag{90}
$$

IX. TMD PDFS AND FFS: $\overline{\text{MS}}$ WILSON COEFFICIENTS FOR SMALL-b_T OPERATOR PRODUCT EXPANSION

In the region of small b_T, the TMD functions may be expanded perturbatively by exploiting the appearance of a hard scale $1/b_T$. The expansions for the unpolarized case are:

$$
\tilde{F}_{f/P}(x_{\rm n},b_T;\zeta_{\rm PDF},\mu,\alpha_s(\mu)) = \sum_j\int_{x_{\rm n}}^1\frac{d\hat{x}}{\hat{x}}\,\tilde{C}^{\rm PDF}_{f/j}(x_{\rm n}/\hat{x},b_T;\zeta_{\rm PDF},\mu,\alpha_s(\mu))\,f_{j/P}(\hat{x};\mu)+\mathcal{O}((\Lambda_{\rm QCD}b_T)^a), \tag{91}
$$

$$
\tilde{D}_{H/f}(z,b_T;\zeta_{\rm FF},\mu,\alpha_s(\mu)) = \sum_j\int_z^1\frac{d\hat{z}}{\hat{z}^{3-2\epsilon}}\,\tilde{C}^{\rm FF}_{j/f}(z/\hat{z},b_T;\zeta_{\rm FF},\mu,\alpha_s(\mu))\,d_{h/j}(\hat{z};\mu)+\mathcal{O}((\Lambda_{\rm QCD}b_T)^a)\,, \tag{92}
$$

where $a > 0$.

The Wilson coefficients in Eqs. (91, 92) to order $\alpha_s(\mu)$ are

- TMD PDF

1. *Quark flavor f inside quark flavor j:*

$$
\begin{aligned}
&\tilde{C}^{\rm PDF}_{f/j}(x_{\rm n}, \boldsymbol{b}_{\rm T}; \zeta_{\rm PDF}, \mu, \alpha_s(\mu)) = \\
&\qquad \delta_{fj}\delta(1-x_{\rm n}) \\
&\qquad + \delta_{fj} 2C_{\rm F}\left\{ 2\left[\ln\left(\frac{2}{\mu b_{\rm T}}\right) - \gamma_{\rm E}\right]\left[\left(\frac{2}{1-x_{\rm n}}\right)_+ - 1 - x_{\rm n}\right] + 1 - x_{\rm n} \right. \\
&\qquad \left. +\delta(1-x_{\rm n})\left[-\frac{1}{2}\left[\ln\left(b_{\rm T}^2\mu^2\right) - 2(\ln 2 - \gamma_{\rm E})\right]^2 - \left[\ln(b_{\rm T}^2\mu^2) - 2(\ln 2 - \gamma_{\rm E})\right]\ln\left(\frac{\zeta_{\rm PDF}}{\mu^2}\right)\right]\right\}\left(\frac{\alpha_s(\mu)}{4\pi}\right) \\
&\qquad + \mathcal{O}\left(\left(\frac{\alpha_s(\mu)}{4\pi}\right)^2\right) .
\end{aligned} \tag{93}
$$

2. *Quark flavor f inside gluon:*

$$
\begin{aligned}
&\tilde{C}^{\rm PDF}_{f/g}(x, \mathbf{b}_T; \mu; \zeta_{\rm PDF}/\mu^2, \alpha_s(\mu)) = \\
&\qquad 2T_{\rm F}\left(2\left[1-2x(1-x)\right]\left[\ln\left(\frac{2}{b_{\rm T}\mu}\right) - \gamma_{\rm E}\right] + 2x(1-x)\right)\left(\frac{\alpha_s(\mu)}{4\pi}\right) \\
&\qquad + \mathcal{O}\left(\left(\frac{\alpha_s(\mu)}{4\pi}\right)^2\right)
\end{aligned} \tag{94}
$$

- <u>TMD Fragmentation Function</u>

1. *Quark flavor f hadronizes to quark flavor j:*

$$
\begin{aligned}
&\tilde{C}^{\rm FF}_{j/f}(z, \boldsymbol{b}_{\rm T}; \zeta_{\rm FF}, \mu, \alpha_s(\mu)) = \\
&\qquad \delta_{jf}\delta(1-z) \\
&\qquad \delta_{jf} 2C_{\rm F}\left\{ 2\left[\ln\left(\frac{2z}{\mu b_{\rm T}}\right) - \gamma_{\rm E}\right]\left[\left(\frac{2}{1-z}\right)_+ + \frac{1}{z^2} + \frac{1}{z}\right] + \frac{1}{z^2} - \frac{1}{z} + \right. \\
&\qquad \left. +\delta(1-z)\left[-\frac{1}{2}\left[\ln\left(b_{\rm T}^2\mu^2\right) - 2(\ln 2 - \gamma_{\rm E})\right]^2 - \left[\ln(b_{\rm T}^2\mu^2) - 2(\ln 2 - \gamma_{\rm E})\right]\ln\left(\frac{\zeta_{\rm FF}}{\mu^2}\right)\right]\right\}\left(\frac{\alpha_s(\mu)}{4\pi}\right) \\
&\qquad + \mathcal{O}\left(\left(\frac{\alpha_s(\mu)}{4\pi}\right)^2\right) .
\end{aligned} \tag{95}
$$

2. *Quark flavor f hadronizes into gluon:*

$$\tilde{C}^{\rm FF}_{g/j'}(z, \boldsymbol{b}_{\rm T}; \zeta_{\rm FF}, \mu, \alpha_s(\mu)) =$$

$$\frac{2C_{\rm F}}{z^3}\left(2\left[1+(1-z)^2\right]\left[\ln\left(\frac{2z}{b_{\rm T}\mu}\right)-\gamma_{\rm E}\right]+z^2\right)\left(\frac{\alpha_s(\mu)}{4\pi}\right)$$

$$+\,\mathcal{O}\left(\left(\frac{\alpha_s(\mu)}{4\pi}\right)^2\right) \,. \tag{96}$$

Note an overall factor of $1/z^2$ in the fragmentation function Wilson coefficients relative to MOS, Ref. [15, Eq. (45)]. We suspect that this is due to an additional factor of z^2 in the definition of the hadronic tensor in Ref. [2, Eq. (38)].

One often wishes to evaluate these functions after evolution to the specific scales in Eqs. (80)-(82) and using the b-star prescription. So, for example, $\tilde{C}^{\rm PDF}_{f/j}(x, \boldsymbol{b}_{\rm T}; \zeta_{\rm PDF}, \mu, \alpha_s(\mu)) \to \tilde{C}^{\rm PDF}_{f/j}(x, \boldsymbol{b}_*; \mu_{b_*}^2, \mu_{b_*}, \alpha_s(\mu_{b_*}))$. The PDFs then become

- TMD PDF

  1. *Quark flavor f inside quark flavor j:*

$$\tilde{C}^{\rm PDF}_{f/j}(x, \boldsymbol{b}_*; \mu_{b_*}^2, \mu_{b_*}, \alpha_s(\mu_{b_*})) =$$

$$\delta_{fj}\delta(1-x)$$

$$+\,\delta_{fj}2C_{\rm F}\left\{2\left[\ln\left(\frac{2}{C_1}\right)-\gamma_{\rm E}\right]\left[\frac{1}{C_F}P_{fj}(x)-\frac{3}{2}\delta(1-x)\right]+1-x\right.$$
$$\left.+\delta(1-x)\left[-\frac{1}{2}\left[\ln\left(C_1^2\right)-2(\ln 2-\gamma_{\rm E})\right]^2-\left[\ln(C_1^2)-2(\ln 2-\gamma_{\rm E})\right]\ln\left(1\right)\right]\right\}\left(\frac{\alpha_s(\mu_{b_*})}{4\pi}\right)$$

$$+\,\mathcal{O}\left(\left(\frac{\alpha_s(\mu_{b_*})}{4\pi}\right)^2\right) \,. \tag{97}$$

We have left the $\ln(1) = 0$ explicit as a reminder of the need to evolve ζ and μ to the same scale.

X. Y-TERMS

A. Notation and Terminology

- JCC notation:

The Y-terms are fixed by the requirements for factorization in the T-term region in the transverse momentum $\ll C_1 Q$, combined with the requirements for factorization in the region of $\boldsymbol{q}_{\rm T} \sim Q$.

The basic logic is discussed in [1, pgs.(513-514)], which is based on earlier CSS work [16].

My starting point is the definition of the L in Collins [1, 13.71]:

$$\overbrace{L(q_{\rm T}; Q)}^{\text{“W-term,” “resummed term”}} = T(q_{\rm T}, Q)^{\mu\nu}_{\text{from Eq. (63)}} \equiv T_{\rm TMD} W(q_{\rm T}; Q)^{\mu\nu} \,. \tag{98}$$

The $W^{\mu\nu}$ is the exact hadronic tensor for the process under consideration, and $T_{\rm TMD}$ is the low transverse momentum "approximator." It is an instruction to replace $W^{\mu\nu}$ by an expression, $W^{\mu\nu}$, that is a good approximation to $W^{\mu\nu}$

for small q_T. **To avoid confusing the L-term (or W-term) with the notation for leptonic and hadronic tensors, we will use $T^{\mu\nu}$ to denote the result of applying the $T_{\rm TMD}$ approximator to $W^{\mu\nu}$.**

There is another approximator valid for the case that $q_T \sim Q$. It is denoted in [1] by $T_{\rm coll}$, and it acts on the hadronic tensor by replacing it an approximated expression that is good for $q_T \sim Q$.

The Y term is then (see the definition in Collins [1, 13.73]):

$$Y(q_T, Q)^{\mu\nu} \equiv T_{\rm coll}\left(W(q_T, Q)^{\mu\nu} - T(q_T, Q)^{\mu\nu}\right)$$
$$= T_{\rm coll} W(q_T; Q)^{\mu\nu} - T_{\rm coll} T_{\rm TMD} W(q_T; Q)^{\mu\nu} \,. \tag{99}$$

- Other Standard Terminology:

In other literature, the first term on the second line of Eq. (99) is often called the "fixed order" contribution, while the second term is the "asymptotic" contribution. We will denote them by FO and ASY.

$$FO(q_T, Q)^{\mu\nu} = T_{\rm coll} W(q_T, Q)^{\mu\nu} \tag{100}$$
$$ASY(q_T, Q)^{\mu\nu} = T_{\rm coll} T_{\rm TMD} W(q_T, Q)^{\mu\nu} \,. \tag{101}$$

So,

$$Y(q_T, Q)^{\mu\nu} \equiv FO(q_T, Q)^{\mu\nu} - ASY(q_T, Q)^{\mu\nu} \,. \tag{102}$$

B. Unpolarized Semi-Inclusive Deep Inelastic Scattering

For unpolarized semi-inclusive deep inelastic scattering, we will mainly quote the results of NSY .

1. NSY *Fixed order term*

The fixed order contribution to the cross section is expressed compactly by Ref. [3, Eq. (89)].

$$\left.\frac{d\sigma}{dx_{\rm n} dQ^2 dz dq_T^2}\right|_{FO} = \int_{x_{\rm n}+w}^{1} \frac{d\xi_a}{\xi_a - x_{\rm n}} M_{AB}(\xi_a, \xi_b, \hat{x}, \hat{z}, q_T) + \int_{z+w}^{1} \frac{d\xi_b}{\xi_b - z} M(\xi_a, \xi_b, \hat{x}, \hat{z}, q_T) \,, \tag{103}$$

with

$$M_{AB}(\xi_a, \xi_b, \hat{x}, \hat{z}, q_T) = \frac{e^4}{64\pi s^2 Q^4 x_{\rm n}^2} \frac{\alpha_s}{\pi} \hat{x}\hat{z} \sum_{a,b,j} e_j^2 d_{B/b}(\xi_b; \mu) f_{a/A}(\xi_a; \mu) \sum_{\alpha=1,2} \Xi_{ab}^{(\alpha)}(\hat{x}, \hat{z}, q_T, Q) A_{(\alpha)}$$
$$= \frac{\alpha_{e.m.}^2 \alpha_s}{4 s^2 Q^4 x_{\rm n}^2} \hat{x}\hat{z} \sum_{a,b,j} e_j^2 d_{B/b}(\xi_b; \mu) f_{a/A}(\xi_a; \mu) \sum_{\alpha=1,2} \Xi_{ab}^{(\alpha)}(\hat{x}, \hat{z}, q_T, Q) A_{(\alpha)} \,. \tag{104}$$

The $A_{(\alpha)}$ are the same as Eqs. (54,55). In NSY , the capital A and B subscripts of M label the species of target and produced hadron. The f and d functions are collinear PDFs and fragmentation functions, respectively. Lower case a and b label the flavor and species of the initial target parton and the final state hadronizing parton. The label j is for the flavor of the quark that couples to the photon. The variables $\hat{x}$ and $\hat{z}$ are:

$$\hat{x} \equiv \frac{x_{\rm n}}{\xi_a} \,, \qquad \hat{z} = \frac{z}{\xi_b} \,. \tag{105}$$

Also,

$$w \equiv \frac{q_T}{Q} \sqrt{x_{\rm n} z} \,. \tag{106}$$

From B. Wang: Note the extra $1/(2Q^2)$ in Eq. (104), relative to NSY [3, Eq. (85)]. This is needed to get units and matching with asymptotic term. (We suspect typo.) We have modified notation in the following ways: we use lower case f and d to denote the collinear pdf and fragmentation functions in order to distinguish them from the corresponding TMD functions. We have taken only the azimuthally

symmetric parts, so we have integrated over ϕ, and we have included a factor of 2π in the definition of M_{AB}. We use s rather than S_{eA} for the center-of-mass energy squared to match with earlier notation. We have substituted the explicit values for σ_0 and F_l from Ref. [Eqs.(37,38)][3],

$$\sigma_0 \equiv \frac{Q^2}{4\pi s x_{\rm n}^2}\left(\frac{e^2}{2}\right), \qquad F_l \equiv \frac{e^2}{2}\frac{1}{Q^2} \tag{107}$$

to get the overall normalization in Eq. (104). Finally, we use Ξ in place of the f used by NSY for the factors multiplying $A^{(\alpha)}$ to avoid confusing it with the collinear PDFs.

The expressions from Ξ are from Ref. [3, Eqs.(B1-B4)] Specializing to just the azimuthally independent structure functions,

$$\sum_{\alpha=1,2}\Xi_{jk}^{(\alpha)}(\hat{x},\hat{z},q_{\rm T},Q)A_{(\alpha)}$$
$$= 2\delta_{jk}C_F\hat{x}\hat{z}\left\{\left[\frac{1}{q_{\rm T}^2}\left(\frac{Q^4}{\hat{x}^2\hat{z}^2}+(Q^2-q_{\rm T}^2)^2\right)+6Q^2\right]A_1+4Q^2A_2\right\}, \qquad \text{quark}-\text{to}-\text{quark} \tag{108}$$

$$\sum_{\alpha=1,2}\Xi_{jg}^{(\alpha)}(\hat{x},\hat{z},q_{\rm T},Q)A_{(\alpha)}$$
$$\hat{x}(1-\hat{x})\left\{\left[\frac{Q^4}{q_{\rm T}^2}\left(\frac{1}{\hat{x}^2\hat{z}^2}-\frac{2}{\hat{x}\hat{z}}+2\right)+2Q^2\left(5-\frac{1}{\hat{x}}-\frac{1}{\hat{z}}\right)\right]A_1+8Q^2A_2\right\}, \qquad \text{quark}-\text{to}-\text{gluon} \tag{109}$$

$$\sum_{\alpha=1,2}\Xi_{gj}^{(\alpha)}(\hat{x},\hat{z},q_{\rm T},Q)A_{(\alpha)}$$
$$= 2C_F\hat{x}(1-\hat{z})\left\{\left[\frac{1}{\tilde{q}_{\rm T}^2}\left(\frac{Q^4}{\hat{x}^2(1-\hat{z})^2}+(Q^2-\tilde{q}_{\rm T}^2)^2\right)+6Q^2\right]A_1+4Q^2A_2\right\}. \qquad \text{gluon}-\text{to}-\text{quark} \tag{110}$$

Here, j and k indices represent quark flavors whereas g labels a gluon. In Eq. (110) we have used the NSY notation:

$$\tilde{q}_{\rm T} \equiv \frac{\hat{z}q_{\rm T}}{1-\hat{z}}. \tag{111}$$

2. NSY *Asymptotic term*

The asymptotic term is given in NSY [3, Eq. (36)]:

$$\left.\frac{d\sigma}{dx_{\rm n}dQ^2dzdq_{\rm T}^2}\right|_{ASY} = \frac{\alpha_{e.m.}^2\alpha_s(\mu_Q)}{2s^2x_{\rm n}^2}\frac{A_1}{q_{\rm T}^2}\sum_j e_j^2\Big[d_{B/j}(z;\mu_Q)\left\{\left(P_{qq}\otimes f_{j/A}\right)(x_{\rm n};\mu_Q)+\left(P_{qg}\otimes f_{g/A}\right)(x_{\rm n};\mu_Q)\right\}$$
$$+\left\{\left(d_{B/j}\otimes P_{qq}\right)(z;\mu_Q)+\left(d_{B/g}\otimes P_{gq}\right)(z;\mu)\right\}f_{j/A}(x_{\rm n};\mu_Q)$$
$$+2C_Fd_{B/j}(z;\mu_Q)f_{j/A}(x_{\rm n};\mu_Q)\left\{\ln\left(\frac{\mu_Q^2}{q_{\rm T}^2}\right)-\frac{3}{2}-\ln C_2^2\right\}\Big], \tag{112}$$

with the longitudinal momentum fraction convolution integral defined by

$$\left(f\otimes g\right)(x_{\rm n};\mu_Q) \equiv \int_{x_{\rm n}}^1\frac{d\xi}{\xi}f(x_{\rm n}/\xi;\mu_Q)g(\xi;\mu_Q), \tag{113}$$

and the splitting functions are

$$P_{qq}(x) = C_F\frac{1+x^2}{(1-x)_+}+\frac{3}{2}C_F\delta(1-x), \tag{114}$$
$$P_{qg}(x) = \frac{1}{2}\left(1-2x+2x^2\right), \tag{115}$$
$$P_{gq}(x) = C_F\frac{1+(1-x)^2}{x}. \tag{116}$$

In the above, any renormalization scale is valid of course, but we have specifically used $\mu=\mu_Q=C_2Q$ in anticipation of later results.

3. CPGRSW *Asymptotic term*

The modified asymptotic term from Ref. [10] first makes the following replacements in Eq. (112):

$$\frac{1}{q_{\mathrm{T}}^2} \to \frac{C_2 b_0}{q_{\mathrm{T}} \mu_Q C_5} K_1\left(\frac{C_2 q_{\mathrm{T}} b_0}{C_5 \mu_Q}\right) \tag{117}$$

$$\frac{1}{q_{\mathrm{T}}^2} \ln\left(\frac{\mu_Q^2}{q_{\mathrm{T}}^2}\right) \to \frac{C_2 b_0}{q_{\mathrm{T}} \mu_Q C_5}\left[K_1\left(\frac{C_2 q_{\mathrm{T}} b_0}{C_5 \mu_Q}\right) \ln\left(\frac{C_2 \mu_Q}{C_5 q_{\mathrm{T}}}\right) + \right.$$
$$\left. + K_1^{(1)}\left(\frac{C_2 q_{\mathrm{T}} b_0}{C_5 \mu_Q}\right)\right]. \tag{118}$$

The mathematical identities can be found in Ref. [17, Eqs. (B.10)-(B.13)]. Next, the result is multiplied by a function $\Xi(q_{\mathrm{T}}/Q, \eta)$ which is unity for small q_{T}/Q and vanishes for large q_{T}/Q with η being a parameter to determine the exact transition point. Then Eq. (112) becomes

$$\Xi\left(\frac{q_{\mathrm{T}}}{Q}, \eta\right) \frac{\alpha_s(\mu_Q)}{2\pi s x_A} \frac{C_2 b_0}{q_{\mathrm{T}} \mu_Q C_5} \sigma_0 \sum_{q,\bar{q}} e_q^2 \Big[2 f_q(x_A, \mu_Q) D_q(z_B, \mu_Q) \Big(C_F \Big[K_1\left(\frac{C_2 q_{\mathrm{T}} b_0}{C_5 \mu_Q}\right) \ln\left(\frac{C_2 \mu_Q}{C_5 q_{\mathrm{T}}}\right) +$$
$$+ K_1^{(1)}\left(\frac{C_2 q_{\mathrm{T}} b_0}{C_5 \mu_Q}\right)\Big] - \left(\frac{3}{2} + \ln(C_2^2)\right) C_F K_1\left(\frac{C_2 q_{\mathrm{T}} b_0}{C_5 \mu_Q}\right)\Big)$$
$$+ K_1\left(\frac{C_2 q_{\mathrm{T}} b_0}{C_5 \mu_Q}\right)\left(f_q(x_A, \mu_Q) \otimes P_{qq}^{(0)} + f_g(x_A, \mu_Q) \otimes P_{qg}^{(0)}\right) D_q(z_B, \mu_Q)$$
$$+ K_1\left(\frac{C_2 q_{\mathrm{T}} b_0}{C_5 \mu_Q}\right) f_q(x_A, \mu_Q)\left(D_q(z_B, \mu_Q) \otimes P_{qq}^{(0)} + D_g(z_B, \mu_Q) \otimes P_{gq}^{(0)}\right)\Big]. \tag{119}$$

---

[1] J. C. Collins, *Foundations of Perturbative QCD* (Cambridge University Press, Cambridge, 2011).
[2] R. Meng, F. I. Olness, and D. E. Soper, Nucl. Phys. **B371**, 79 (1992).
[3] P. Nadolsky, D. R. Stump, and C. P. Yuan, Phys. Rev. **D61**, 014003 (1999), arXiv:hep-ph/9906280 [hep-ph].
[4] P. J. Mulders and R. D. Tangerman, Nucl. Phys. **B461**, 197 (1996), arXiv:hep-ph/9510301 [hep-ph].
[5] A. Bacchetta *et al.*, JHEP **02**, 093 (2007), arXiv:hep-ph/0611265.
[6] P. J. Mulders, "Unpublished collected notes: http://www.nat.vu.nl/ mulders/correlations-0new.pdf," .
[7] X.-D. Ji, J.-P. Ma, and F. Yuan, Phys. Lett. **B597**, 299 (2004), arXiv:hep-ph/0405085 [hep-ph].
[8] Y. Koike, J. Nagashima, and W. Vogelsang, Nucl. Phys. **B744**, 59 (2006), arXiv:hep-ph/0602188 [hep-ph].
[9] V. Barone, A. Drago, and P. G. Ratcliffe, Phys. Rept. **359**, 1 (2002), hep-ph/0104283.
[10] *To be published*.
[11] A. Bacchetta, U. D'Alesio, M. Diehl, and C. A. Miller, Phys. Rev. **D70**, 117504 (2004), hep-ph/0410050.
[12] M. Diehl and S. Sapeta, Eur. Phys. J. **C41**, 515 (2005), arXiv:hep-ph/0503023.
[13] X.-D. Ji, J.-P. Ma, and F. Yuan, Phys. Rev. **D71**, 034005 (2005), arXiv:hep-ph/0404183.
[14] S. Moch, J. Vermaseren, and A. Vogt, JHEP **0508**, 049 (2005), arXiv:hep-ph/0507039 [hep-ph].
[15] R. Meng, F. I. Olness, and D. E. Soper, Phys. Rev. **D54**, 1919 (1996), arXiv:hep-ph/9511311.
[16] J. C. Collins, D. E. Soper, and G. Sterman, Nucl. Phys. **B250**, 199 (1985).
[17] G. Bozzi, S. Catani, D. de Florian, and M. Grazzini, Nucl. Phys. **B737**, 73 (2006), arXiv:hep-ph/0508068 [hep-ph].