INF587 Exercise sheet 2

Exercise 1. *Show that a normal operator/matrix is*

1. *Hermitian if and only if it has real eigenvalues,*
2. *Positive if and only if it has positive eigenvalues.*

Exercise 2. *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be* $\mathcal{P} \in \{$*Normal, Unitary, Hermitian, Projector, Positive*$\}$. *Show that* $\mathbf{A} \otimes \mathbf{B}$ *is* $\mathcal{P}$.

Exercise 3 (Exponential of Pauli matrices)**.**

1. *Compute*
$$\exp(\theta \mathbf{X})$$
2. *Let* $\mathbf{v} \in \mathbb{R}^3$ *with Euclidean norm* 1 *and* $\theta \in \mathbb{R}$. *Show that*
$$\exp(i\theta \mathbf{v} \cdot \sigma) = \cos(\theta)\mathbf{I}_2 + i\sin(\theta)\mathbf{v} \cdot \sigma$$
where $\mathbf{v} \cdot \sigma \stackrel{def}{=} \sum_{i=1}^{3} v_i \sigma_i = v_1\mathbf{X} + v_2\mathbf{Y} + v_3\mathbf{Z}$.

Hint: *compute* $(\mathbf{v} \cdot \sigma)^2$, *you can use that* $\mathbf{XY} + \mathbf{YX} = \mathbf{XZ} + \mathbf{ZX} = \mathbf{YZ} + \mathbf{ZY} = \mathbf{0}$

Exercise 4 (Some projective measurements for qubits)**.**

1. *Show that* $\mathbf{X}, \mathbf{Y}$ *and* $\mathbf{Z}$ *are Hermitian and give their spectral decomposition in an orthonormal basis (eigenvalues with associated unit eigenvectors)*
2. *Suppose that we have a qubit in the state* $|0\rangle$, *and we measure the observable* $\mathbf{X}$. *What is the average value of* $\mathbf{X}$? *What is the standard deviation for* $\mathbf{X}$?
3. *Show that the average value of the observable* $\mathbf{X} \otimes \mathbf{Z}$ *for a two qubits system measured in the state* $\frac{|00\rangle + |11\rangle}{\sqrt{2}}$ *is zero.*
4. *Show that* $\mathbf{v} \cdot \sigma$ *(see Exercise 3) has eigenvalues* ± 1 *and that the projectors onto the corresponding eigenspaces are given by* $\mathbf{P}_{\pm 1} = \frac{(\mathbf{I}_2 \pm \mathbf{v} \cdot \sigma)}{2}$.

5. *Calculate the probability of obtaining the result $+1$ for a measurement of $\mathbf{v} \cdot \sigma$ given that the state prior ot measurement is $|0\rangle$. What is the state of the system after the measurement if $+1$ is obtained?*

Exercise 5 (About the POVM formalism)**.**

1. *Prove that no quantum measurement are capable of distinguishing non-orthogonal states.*

$2^\star$. *Give a* POVM $(\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3)$ *that never makes error to distinguish the following quantum states:*

$$|\psi_1\rangle = |0\rangle \quad \textit{and} \quad |\psi_2\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} = |+\rangle$$

Exercise 6 (projective measurements versus quantum measurements)**.** *Our aim in this exercise is to show that projective measurements together with unitary dynamics are sufficient to implement a general measurement. The rough idea is "to increase the dimension" (sometimes called Naimark's dilatation trick).*

Let $(\mathbf{M}_m)_{m\in\mathcal{M}}$ be a quantum measurement that we want to perform on a state space Q. Notice that possible outcomes form a (finite) set $\mathcal{M}$.

Let M be an ancilla system *with dimension $\sharp\mathcal{M}$. Let $(|m\rangle)_{m\in\mathcal{M}}$ be an orthonormal basis of M.*

1. *Let $\mathbf{U}$ be the following operator on $Q \otimes M$ (not linear as not defined over the whole space):*

$$\mathbf{U} : |\psi\rangle\,|0\rangle \in Q \otimes M \mapsto \sum_m (\mathbf{M}_m \otimes \mathbf{I})\,|\psi\rangle\,|m\rangle$$

Show that:

$$\langle\varphi|\,\langle 0|\,\mathbf{U}^\dagger\mathbf{U}\,|\psi\rangle\,|0\rangle = \langle\varphi|\psi\rangle$$

2. *Show that $\mathbf{U}$ can be extended as a unitary operator on the space $Q \otimes M$.*

3. *Let $\mathbf{P}_m \overset{def}{=} \mathbf{I}_Q \otimes |m\rangle\langle m|$. Show that $(\mathbf{P}_m)_{m\in\mathcal{M}}$ is a projective measurement. In particular, given $\mathbf{U}\,|\psi\rangle\,|0\rangle$, what is the probability to outcome m? What becomes the $\mathbf{U}\,|\psi\rangle\,|0\rangle$ after measuring m?*

4. *Conclude.*

Exercise 7 (On Pauli matrices)**.**

1. *Let* $\mathbf{M} = \begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix}$. *Show that it exists* $\alpha, \beta \in \mathbb{C}$ *such that* $\mathbf{M} = \alpha\mathbf{X} + \beta\mathbf{Y}$.

2. *Let* $\mathbf{M}$ *be any* 2×2 *complex matrix. Show that it exists* $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ *such that* $\mathbf{M} = \alpha\mathbf{I}_2 + \beta\mathbf{X} + \gamma\mathbf{Y} + \delta\mathbf{Z}$.

3. *Compute* $\mathbf{XZ}, \mathbf{XY}$ *and* $\mathbf{YZ}$. *Let* $\mathbf{P}_1, \mathbf{P}_2 \in \{\mathbf{I}_2, \mathbf{X}, \mathbf{Y}, \mathbf{Z}\}$. *Show that* $\mathrm{tr}(\mathbf{P}_1\mathbf{P}_2) = 0$ *if* $\mathbf{P}_1 \neq \mathbf{P}_2$ *and* $\mathrm{tr}(\mathbf{P}_1\mathbf{P}_2) = 2$ *if* $\mathbf{P}_1 = \mathbf{P}_2$.

4. *Let* $\mathbf{U}$ *be any unitary matrix on* 1 *qubit. We can hence write* $\mathbf{U} = \alpha\mathbf{I} + \beta\mathbf{X} + \gamma\mathbf{Y} + \delta\mathbf{Z}$. *Show that*
$$|\alpha|^2 + |\beta|^2 + |\gamma|^2 + |\delta|^2 = 1.$$

Exercise 8 (Heisenberg uncertainty principle)**.** *Given two Hermitian operators* $\mathbf{A}, \mathbf{B}$ *we define*

$$[\mathbf{A}, \mathbf{B}] \stackrel{def}{=} \mathbf{AB} - \mathbf{BA} \text{ (commutator)} \quad \text{and} \quad \{\mathbf{A}, \mathbf{B}\} \stackrel{def}{=} \mathbf{AB} + \mathbf{BA} \text{ (anti-commutator)}$$

1. *Show that*
$$|\langle\psi|\,[\mathbf{A}, \mathbf{B}]\,|\psi\rangle|^2 + |\langle\psi|\,\{\mathbf{A}, \mathbf{B}\}\,|\psi\rangle|^2 = 4|\langle\psi|\,\mathbf{AB}\,|\psi\rangle|^2$$
Deduce that
$$|\langle\psi|\,[\mathbf{A}, \mathbf{B}]\,|\psi\rangle|^2 \leq 4\,\langle\psi|\,\mathbf{A}^2\,|\psi\rangle\,\langle\psi|\,\mathbf{B}^2\,|\psi\rangle$$

2. *Show that for two measurables* $\mathbf{C}$ *and* $\mathbf{D}$ *we have (Heisenberg uncertainty principle)*
$$\Delta(\mathbf{C})\,\Delta(\mathbf{D}) \geq \frac{|\langle\psi|\,[\mathbf{C}, \mathbf{D}]\,|\psi\rangle|}{2}$$
What is your interpretation of this inequation?

3. $\mathbf{X}$ *and* $\mathbf{Y}$ *are two measurables. What are their outcomes? What does the uncertainty principle tells with these measurables when measured for the quantum state* $|0\rangle$*?*