INF587 Exercise sheet 3

Exercise 1 (Bloch sphere)**.**

1. *Why does any qubit can be written as*

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle$$

The number θ and φ define a point on the unit three-dimensional sphere as:

$$(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$$

It is shown in Figure 1: it is the Bloch sphere representation.

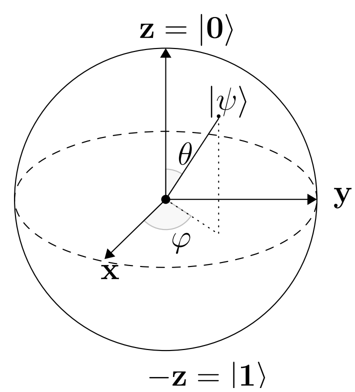


Figure 1: Bloch sphere.

2. *How are the Bloch representations of orthogonal qubits?*

This description has an important generalization to mixed states, the Bloch ball representation*, as follows.*

3. *Show that an arbitrary density matrix for a mixed state qubit may be written as*

$$\rho = \frac{\mathbf{I}_2 + \mathbf{v}\cdot\sigma}{2}$$

where $\mathbf{v} \in \mathbb{R}^3$ has Euclidean norm ≤ 1. Here $\mathbf{v}\cdot\sigma \stackrel{def}{=} \sum_{i=1}^{3} v_i\sigma_i = v_1\mathbf{X} + v_2\mathbf{Y} + v_3\mathbf{Z}$ where recall that

$$\mathbf{X} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{Y} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \textit{and} \quad \mathbf{Z} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The vector $\mathbf{v}$ *is known as the Bloch vector for the state* ρ *and it gives the representation of the state as a point in the unit ball.*

Hint: *treat first the case where* ρ *is a pure state.*

4. *Show that for pure states the description of the Bloch vector we have given coincides with that of question* 1.

5. *What is the Bloch vector representation for the state* $\rho \overset{def}{=} \mathbf{I}_2/2$*?*

6. *Show that a state* ρ *is pure if and only if* $\|\mathbf{v}\|_2 = 1$.

Solution:

1. Any qubit can be written as

$$|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$$

where $|\alpha|^2 + |\beta|^2 = 1$. Therefore $|\alpha| = \cos\frac{\theta}{2}$ and $|\beta| = \sin\frac{\theta}{2}$ for some θ. Using the definition of the complex exponential we have $\alpha = e^{i\varphi_1}\cos\frac{\theta}{2}$ and $\beta = e^{i\varphi_2}\sin\frac{\theta}{2}$. It leads to

$$|\psi\rangle = e^{i\varphi_1}\left(\cos\frac{\theta}{2}\,|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}\,|1\rangle\right)$$

where $\varphi \overset{\text{def}}{=} \varphi_2 - \varphi_1$. Here $e^{i\varphi_1}$ is a global phase that can be ignored (see Lecture 2, statistics of measurement, even after arbitrary unitary, will be the same with or without the global phase).

2. They will be opposite on the sphere, namely $\theta \longleftrightarrow \theta + \pi$.

3. Let us start by the case where $\rho = |\psi\rangle\langle\psi|$. We have:

$$\begin{aligned}
|\psi\rangle\langle\psi| &= \cos^2\frac{\theta}{2}\,|0\rangle\langle 0| + e^{i\varphi}\sin\frac{\theta}{2}\cos\frac{\theta}{2}\,|1\rangle\langle 0| + e^{-i\varphi}\sin\frac{\theta}{2}\cos\frac{\theta}{2}\,|0\rangle\langle 1| + \sin^2\frac{\theta}{2}\,|1\rangle\langle 1| \\
&= \begin{pmatrix} \cos^2\frac{\theta}{2} & e^{-i\varphi}\sin\frac{\theta}{2}\cos\frac{\theta}{2} \\ e^{i\varphi}\sin\frac{\theta}{2}\cos\frac{\theta}{2} & \sin^2\frac{\theta}{2} \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} 1+v_3 & v_1 - iv_2 \\ v_1 + iv_2 & 1 - v_3 \end{pmatrix}
\end{aligned}$$

where

$$1 + v_3 = 2\cos^2\frac{\theta}{2}, \quad v_1 = 2\cos\varphi\cos\frac{\theta}{2}\sin\frac{\theta}{2} \quad \text{and} \quad v_2 = 2\sin\varphi\cos\frac{\theta}{2}\sin\frac{\theta}{2} \tag{1}$$

It defines $\mathbf{v} \in \mathbb{R}^3$ such that
$$\rho = \frac{\mathbf{I}_2 + \mathbf{v} \cdot \sigma}{2}$$
Let us show now that $\mathbf{v}$ has Euclidean norm ≤ 1. We have the following computation
$$\begin{aligned}
\|\mathbf{v}\|^2 &= \left(2\cos^2\frac{\theta}{2} - 1\right)^2 + \left(2\cos\varphi\cos\frac{\theta}{2}\sin\frac{\theta}{2}\right)^2 + \left(2\sin\varphi\cos\frac{\theta}{2}\sin\frac{\theta}{2}\right)^2 \\
&= \left(2\cos^2\frac{\theta}{2} - 1\right)^2 + 4\cos^2\frac{\theta}{2}\sin^2\frac{\theta}{2} \\
&= 4\cos^2\frac{\theta}{2}\left(\cos^2\frac{\theta}{2} - 1 + \sin^2\frac{\theta}{2}\right) + 1 \\
&= 1
\end{aligned}$$
In particular $\|\mathbf{v}\|$ has norm one when ρ is a pure state.

To consider the general case, let us suppose that $\rho = \sum_k p_k |k\rangle\langle k|$ with $\sum_k p_k = 1$ and the p_k's be positive. Let us write
$$|k\rangle\langle k| = \frac{\mathbf{I}_2 + \mathbf{v}^{(k)} \cdot \sigma}{2}$$
with $\|\mathbf{v}^{(k)}\| = 1$. Using that the p_k's sum to one:
$$\rho = \frac{\mathbf{I}_2 + \left(\sum_k \mathbf{v}^{(k)}\right) \cdot \sigma}{2}$$
We now have a linear combination of unit vectors $\sum_k \mathbf{v}^{(k)}$ with $\sum_k p_k = 1$. This sum will correspond to some other vectors $\mathbf{v}$ such that $\|\mathbf{v}\| \leq 1$.

4. We have seen in 3 that for pure states we have $\|\mathbf{v}\| = 1$ with coordinates given in Equation (1). Using trigonometric formulas concludes the question.

5. It is the center.

6. We have shown in Question 3 that ρ is a pure state implies that $\|\mathbf{v}\| = 1$. Furthermore, recall that when $\|\mathbf{v}\| = 1$ we have $(\mathbf{v} \cdot \sigma)^2 = \mathbf{I}_2$ (see Exercise Session 2). Therefore in that case
$$\rho^2 = \frac{\mathbf{I}_2 + \mathbf{v} \cdot \sigma}{2}$$
We deduce that $\operatorname{tr}(\rho^2) = 1$ when $\|\mathbf{v}\| = 1$ which shows that it is a pure state (see the lecture).

□

Exercise 2 (von Neumann entropy). *The von Neumann entropy of a quantum system, expressed as density operator ρ is (with the convention $0 \log 0 = 0$)*

$$S(\rho) \stackrel{def}{=} -\operatorname{tr}(\rho \log \rho)$$

1. *Why is $S(\rho)$ well defined? Give the expression of $S(\rho)$ according to the eigenvalues of ρ,*
2. *Give the entropy of the states $\rho_0 \stackrel{def}{=} |0\rangle\langle 0|$ and $\rho_1 \stackrel{def}{=} \frac{|0\rangle\langle 0| + |1\rangle\langle 1|}{2}$,*
3. *Prove that the von Neumann entropy of pure states is 0,*
4. *Give the entropy of the probabilistic mixture of $|+\rangle$ with prob. $\frac{1}{2}$ and $|-\rangle$ with prob. $\frac{1}{2}$,*
5. *Prove that the von Neumann entropy of ρ is zero if and only ρ is a pure state. What happens if ρ is a mixed quantum state?*

Solution:

1. As ρ is density operator, it is in particular positive. Therefore ρ is diagonal in some orthonormal basis $(|i\rangle)_i$ with positive eigenvalues $(\lambda_i)_i$, namely
$$\rho = \sum_i \lambda_i |i\rangle\langle i|$$
As the λ_i's are positive, then $\rho \log \rho$ is well defined (via the function $x \mapsto x \log x$) as $\sum_i \lambda_i \log \lambda_i |i\rangle\langle i|$ and we have (using that $(|i\rangle)_i$ is an orthonormal basis)
$$-\operatorname{tr}(\rho \log \rho) = -\operatorname{tr}\left(\sum_i \lambda_i \log(\lambda_i) |i\rangle\langle i|\right) = -\sum_i \lambda_i \log(\lambda_i)$$
2. The entropy of $\rho_0 = |0\rangle\langle 0|$ is $S(\rho_0) = -1 \log(1) = 0$. The operator $\rho_1 = \frac{|0\rangle\langle 0| + |1\rangle\langle 1|}{2}$ is diagonal with eigenvalues $1/2$. Therefore $S(\rho_1) = -\frac{1}{2} \log\left(\frac{1}{2}\right) - \frac{1}{2} \log\left(\frac{1}{2}\right) = \log(2)$.
3. Notice that any pure state can be extended into an orthonormal basis (Gram-Schmidt process). Therefore, the representation of any density operator which is a pure state is in this basis a matrix with only zeros on the diagonal except in the first position for which it is one. Since $\log 1 = 0$, the entropy of pure states is 0.

4. The operator $\rho = \frac{1}{2}|+\rangle\langle+| + \frac{1}{2}|-\rangle\langle-|$ has, $(|+\rangle, |-\rangle)$ is an orthonormal basis (Hadamard basis), which eigenvalues 1/2. Therefore, $S(\rho) = \log(2)$.

5. If ρ is a pure state, then its entropy is zero as shown in Question 3. Conversely, suppose that $S(\rho) = 0$. As shown in Question 1,

$$S(\rho) = -\sum_i \lambda_i \log \lambda_i$$

where the λ_i's are the eigenvalues of ρ. Now $\mathrm{tr}(\rho) = 1$, therefore $\sum_i \lambda_i = 1$. Furthermore, the λ_i's are ≥ 0 as ρ is a positive operator. We deduce that

$$\forall i, \quad \lambda_i \in [0, 1]$$

Cases where all the $\lambda_i' s$ are zero or all in $[0, 1)$ are impossible as $\sum_i \lambda_i = 1$ and $S(\rho) = 0$. Therefore, as least one $\lambda_{i_0} = 1$, but using once again that $\mathrm{tr}(\rho) = 1$ and the λ_i's are ≥ 0, shows that all the λ_i's are zero except $\lambda_{i_0} = 1$. It concludes the exercise.

$\square$

Exercise 3. *Suppose a composite of systems A and B is in the state $|a\rangle|b\rangle$, where $|a\rangle$ is a pure state of A, and $|b\rangle$ is a pure state of B. Show that the reduced density operator of system A alone is a pure state.*

Solution: Let $\rho \overset{\text{def}}{=} |a\rangle|b\rangle\langle a|\langle b|$. By definition of the partial trace, if one traces out over the system B,

$$\rho_A = \mathrm{tr}_B(\rho) = |a\rangle\langle a| \,\mathrm{tr}(|b\rangle\langle b|) = (\langle b|b\rangle)\,|a\rangle\langle a| = |a\rangle\langle a|$$

where we used that $|b\rangle$ is a pure quantum state and therefore has norm 1. $\square$

Exercise 4. *For each state $|\psi_{AB}\rangle$, give the reduced density matrices*

$$\rho_A = \mathrm{tr}_B(|\psi_{AB}\rangle\langle\psi_{AB}|) \quad \textit{and} \quad \rho_B = \mathrm{tr}_A(|\psi_{AB}\rangle\langle\psi_{AB}|).$$

You can write your answers in Dirac's "ket,bra" notation or in matrix form. Compute also $S(\rho_A)$ in each case (see Exercise 2). You can use $\log_2(3) \approx 1.585$.

1. $|\psi_{AB}\rangle = \frac{1}{\sqrt{2}}(|0\rangle|-\rangle + |1\rangle|+\rangle)$.

2. $|\psi_{AB}\rangle = \frac{1}{2}(|00\rangle - |01\rangle - |10\rangle + |11\rangle)$.

3. $|\psi_{AB}\rangle = \sqrt{\frac{3}{8}}\,|00\rangle + \sqrt{\frac{3}{8}}\,|01\rangle - \sqrt{\frac{1}{8}}\,|10\rangle + \sqrt{\frac{1}{8}}\,|11\rangle$.

Solution:

1. $\rho_A = \frac{1}{2}\left(|0\rangle\langle 0| + |1\rangle\langle 1|\right)$. $\rho_B = \frac{1}{2}\left(|-\rangle\langle -| + |+\rangle\langle +|\right) = \frac{1}{2}\left(|0\rangle\langle 0| + |1\rangle\langle 1|\right)$. Then, $S(\rho_A) = 1$.

2. $|\psi_{AB}\rangle = |-\rangle\,|-\rangle$ so $\rho_A = \rho_B = |-\rangle\langle -|$. Then, $S(\rho_A) = 0$.

3. We can rewrite $|\psi_{AB}\rangle = \sqrt{\frac{3}{4}}\left(|0\rangle\,|+\rangle\right) - \sqrt{\frac{1}{4}}\left(|1\rangle\,|-\rangle\right)$. So $\rho_A = \frac{3}{4}\,|0\rangle\langle 0| + \frac{1}{4}\,|1\rangle\langle 1|$ and $\rho_B = \frac{3}{4}\,|-\rangle\langle -| + \frac{1}{4}\,|+\rangle\langle +|$.

$$S(\rho_A) = \frac{3}{4}\log_2(4/3) + \frac{1}{4}\log_2(4) = \frac{3}{4}\left(2 - \log_2(3)\right) + \frac{1}{4}2 = 2 - \frac{3\log_2(3)}{4} \approx 0.811.$$

□

Exercise 5 (Schmidt decomposition)**.** *Find the Schmidt decomposition and give the Schmidt number of the following two qubits:*

$$|\psi_1\rangle \stackrel{def}{=} \frac{|00\rangle + |11\rangle}{\sqrt{2}}, \quad |\psi_2\rangle \stackrel{def}{=} \frac{|00\rangle + |01\rangle + |10\rangle + |11\rangle}{2}$$

$$|\psi_3\rangle \stackrel{def}{=} \frac{|00\rangle + |01\rangle + |10\rangle - |11\rangle}{2}$$

Solution: The state $|\psi_1\rangle$ is already written in Schmidt form

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}\sum_b |b\rangle\,|b\rangle$$

Its Schmidt number is 2. Concerning $|\psi_2\rangle$ notice that

$$|\psi_2\rangle = \frac{|0\rangle\left(|0\rangle + |0\rangle\right) + |1\rangle\left(|0\rangle + |1\rangle\right)}{2} = |+\rangle\,|+\rangle$$

Therefore its Schmidt number is 1. For $|\psi_3\rangle$ we have

$$|\psi_3\rangle = \frac{|0\rangle\left(|0\rangle + |1\rangle\right) + |1\rangle\left(|0\rangle - |1\rangle\right)}{2} = |0\rangle\,|+\rangle + |1\rangle\,|-\rangle$$

We deduce that its Schmidt number is 2.

□

Exercise 6. *Prove that a state $|\psi\rangle$ of a composite system $A \otimes B$ is a product state if and only if it has Schmidt number* 1. *Prove that $|\psi\rangle$ is a product state if and only if ρ^A (and thus ρ^B) are pure states where $\rho = |\psi\rangle\langle\psi|$ and $\rho_A = \mathrm{tr}_B(\rho)$ and $\rho_B = \mathrm{tr}_A(\rho)$.*

Deduce the theorem of the lecture: a pure state $|\psi\rangle \in A \otimes B$ is entangled if and only if its Schmidt's number is > 1 if and only if ρ_A and ρ_B (defined as above) are mixed states.

Solution: If $|\psi\rangle$ has Schmidt number one then by definition it is a product state. Conversely, suppose that $|\psi\rangle$ is a product state. By definition $|\psi\rangle = |a\rangle |b\rangle$ where $|a\rangle$ and $|b\rangle$ are quantum state. In particular $\{|a\rangle\}$ and $\{|b\rangle\}$ are orthogonal families. It shows that $|\psi\rangle = |a\rangle |b\rangle$ is the Schmidt decomposition of $|\psi\rangle$ and its Schmidt number is one.

Recall the following result from the lecture:

$|\psi\rangle = \sum_{i=1}^{d} \lambda_i |i_A\rangle |i_B\rangle$ *be the Schmidt decomposition, (in particular $\lambda_i > 0$)*

then ρ_A and ρ_B have the same eigenvalues, the λ_i's and possibly 0.

Furthermore, as shown in Exercise 2, Question 5, any density operator is a pure state if and only if it has only 1 as non-zero eigenvalue. Therefore we have the following chain of equivalence: $|\psi\rangle$ is a pure state if and only if its Schmidt number is one if and only if ρ_A and ρ_B have one as non-zero eigenvalue if and only if ρ_A and ρ_B are pure states.

We easily deduce the theorem of the lecture session from the above result and the fact that a Schmidt number is ≥ 1 and that $|\psi\rangle$ that is not a product state is by definition entangled. $\square$

Exercise 7 (Bell states)**.** *The four Bell states are defined as*

$$|\beta_{00}\rangle \overset{def}{=} \frac{|00\rangle + |11\rangle}{\sqrt{2}}, \quad |\beta_{01}\rangle \overset{def}{=} \frac{|01\rangle + |10\rangle}{\sqrt{2}}$$

$$|\beta_{10}\rangle \overset{def}{=} \frac{|00\rangle - |11\rangle}{\sqrt{2}} \quad \textit{and} \quad |\beta_{11}\rangle \overset{def}{=} \frac{|01\rangle - |10\rangle}{\sqrt{2}}$$

For each of the four Bell states, find the reduced density operator for each qubit.

Solution: Let us make the computation for $|\beta_{00}\rangle$. First we have:

$$|\beta_{00}\rangle\langle\beta_{00}| = \frac{|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|}{2}$$

Tracing out the second value gives the following computation:

$$\begin{aligned}
\mathrm{tr}_2\left(|\beta_{00}\rangle\langle\beta_{00}|\right) &= \frac{|0\rangle\langle 0|\,\mathrm{tr}\left(|0\rangle\langle 0|\right) + |0\rangle\langle 1|\,\mathrm{tr}\left(|0\rangle\langle 1|\right) + |1\rangle\langle 0|\,\mathrm{tr}\left(|1\rangle\langle 0|\right) + |1\rangle\langle 1|\,\mathrm{tr}\left(|1\rangle\langle 1|\right)}{2} \\
&= \frac{|0\rangle\langle 0|\left(\langle 0|0\rangle\right) + |0\rangle\langle 1|\left(\langle 0|1\rangle\right) + |1\rangle\langle 0|\left(\langle 1|0\rangle\right) + |1\rangle\langle 1|\left(\langle 1|1\rangle\right)}{2} \\
&= \frac{|0\rangle\langle 0| + |1\rangle\langle 1|}{2} \\
&= \frac{\mathbf{I}_2}{2}
\end{aligned}$$

The same kind of computation also gives

$$\mathrm{tr}_1\left(|\beta_{00}\rangle\langle\beta_{00}|\right) = \frac{\mathbf{I}_2}{2}$$

We can also show that reduced density operators of all Bell states are $\frac{\mathbf{I}_2}{2}$. $\square$

Exercise 8. *Suppose $\{p_i, |\psi_i\rangle\}$ is an ensemble of states generating a density matrix $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$ for a quantum system A. Let R be a quantum system with orthonormal basis $(|i\rangle)$.*

1. *Show that $\sum_i \sqrt{p_i}\,|\psi_i\rangle\,|i\rangle$ is purification of ρ.*
2. *Suppose we measure R in the basis $(|i\rangle)$, obtaining the outcome i. With what probability do we obtain the result i, and what is the corresponding state of system A?*
3. *Let $|AR\rangle$ be any purification of ρ to the system $A \otimes R$. Show that there exists an orthonormal basis $(|i\rangle)$ in which R can be measured such that the corresponding post-measurement state for system A is $|\psi_i\rangle$ with probability p_i.*

Solution:

1. Let,

$$|\varphi\rangle \stackrel{\text{def}}{=} \sum_i \sqrt{p_i}\,|\psi_i\rangle\,|i\rangle$$

Therefore,

$$|\varphi\rangle\langle\varphi| = \sum_{i,j} \sqrt{p_i p_j}\, |\psi_i\rangle\langle\psi_j| \otimes |i\rangle\langle j|$$

By definition of the partial trace:

$$\mathrm{tr}_R\left(|\varphi\rangle\langle\varphi|\right) = \sum_{i,j} \mathrm{tr}_R\left(\sqrt{p_i p_j}\, |\psi_i\rangle\langle\psi_j| \otimes |i\rangle\langle j|\right) = \sum_{i,j} \sqrt{p_i p_j}\, |\psi_i\rangle\langle\psi_j|\, \langle i|j\rangle = \rho$$

where in the last equality we used that $(|i\rangle)_i$ is an orthonormal basis.

2. By definition of the measure in the orthonormal basis $(|i\rangle)_i$ (the operators of measurement are the $|i\rangle\langle i|$'s) we obtain the outcome i with probability p_i and the resulting quantum state is $|\psi_i\rangle\,|i\rangle$. In particular, the corresponding state of system A is $|\psi_i\rangle$.

3. By definition

$$\mathrm{tr}_R\left(|AR\rangle\langle AR|\right) = \rho$$

Let us consider the Schmidt decomposition of $|AR\rangle$:

$$|AR\rangle = \sum_{i=1}^{d} \lambda_i\, |a_i\rangle\, |b_i\rangle$$

By (possibly) extending the orthogonal family $(|a_i\rangle)$ and $(|b_i\rangle)$ into orthonormal basis and choosing 0 as associated λ_i we can write

$$|AR\rangle = \sum_{i} \lambda_i\, |a_i\rangle\, |b_i\rangle$$

Using that $(|b_i\rangle)_i$ is by definition an orthogonal family we obtain

$$\rho = \mathrm{tr}_R\left(|AR\rangle\langle AR|\right) = \sum_{i} \lambda_i^2\, |a_i\rangle\langle a_i|$$

In particular,

$$\sum_{i} p_i\, |\psi_i\rangle\langle\psi_i| = \sum_{i=1} \lambda_i^2\, |a_i\rangle\langle a_i|$$

From the lecture, there exists a unitary $\mathbf{U} = (u_{i,j})_{i,j}$ be such that

$$\lambda_i\, |a_i\rangle = \sum_{j} u_{i,j} \sqrt{p_j}\, |\psi_j\rangle$$

Therefore,

$$|AR\rangle = \sum_i \lambda_i |a_i\rangle |b_i\rangle = \sum_j \sqrt{p_j} |\psi_j\rangle \left(\sum_i u_{i,j} |b_i\rangle \right)$$

Let us define

$$\forall j, \quad |b'_j\rangle \stackrel{\text{def}}{=} \sum_i u_{i,j} |b_i\rangle$$

Recall that the $|b_i\rangle$'s are an orthonormal basis and $\mathbf{U}$ is a unitary. We deduce that $(|b_j\rangle')_j$ form an orthogonal basis. Let us perform the measure of $|AR\rangle$ in this basis. But

$$|AR\rangle = \sum_j \sqrt{p_j} |\psi_j\rangle |b'_j\rangle$$

which shows that we obtain the outcome j with probability p_j while the resulting corresponding state of system A is $|\psi_j\rangle$.

$\square$

Exercise 9 (Bell inequality)**.**

Imagine that both Alice and Bob have a particle. Alice can perform two measurements on her particle, for instance Alice can measure its speed or its position. Alice doesn't know in advance which measurement she will choose to perform. Rather, when she receives the particle she decides randomly which measurement to perform. The situation is the same for Bob, with two other possible measurements.

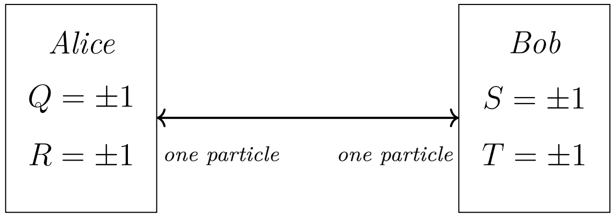
For the sake of simplicity, we suppose that measurements of Alice (resp. *Bob*) P_Q *and* P_R (resp. P_S *and* P_T) *have one of two outcomes* $Q, R \in \{-1, 1\}$ (resp. $S, T \in \{-1, 1\}$).

The timing of the experiment is so that Alice and Bob do their measurements at the same time. *Therefore, Alice's measurement cannot disturb Bob's measurement (and vice versa), since any physical influence cannot propagate faster than light.*

Alice and Bob perform many time this experiment and then meat together to use their common data to estimate the value of

$$QS + RS + RT - QT$$

Furthermore, we suppose that the two particles are prepared each time in the same way.



1. *Show that*
$$QS + RS + RT - QT = \pm 2.$$
Hint: *you may use* $QS + RS + RT - QT = (Q + R)S + (R - Q)T$.

2. *Deduce that*
$$\mathbb{E}(QS) + \mathbb{E}(RS) + \mathbb{E}(RT) - \mathbb{E}(QT) \leq 2.$$

The quantum experiment. Let $|\psi\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}$. *Alice has the first qubit while Bob has the second one. Define:*

$$Q: \quad \textit{meas. according to } \mathbf{Z}, \quad R: \quad \textit{meas. according to } \mathbf{X}$$

$$S: \quad \textit{meas. according to } \frac{-\mathbf{X}-\mathbf{Z}}{\sqrt{2}} \quad \textit{and} \quad T: \quad \textit{meas. according to } \frac{\mathbf{Z}-\mathbf{X}}{\sqrt{2}}$$

3. *Show that*
$$\langle QS \rangle + \langle RS \rangle + \langle RT \rangle - \langle QT \rangle = 2\sqrt{2}.$$

4. *Is it surprising?*

Solution:

1. We have the identity
$$QS + RS + RT - QT = (Q + R)S + (R - Q)T$$
Because $R, Q = \pm 1$ it follows that either $(Q + R)S = 0$ or $(R - Q)T = 0$. In either case, it is easy to see from the above equation that
$$QS + RS + RT - QT = \pm 2.$$

2. Let $p(q,r,s,t)$ be the probability that *before the measurements are performed*, the system is in a state $Q=q$, $R=r$, $S=s$ and $T=t$. We have the following computation by definition of the expectation

$$\begin{aligned}\mathbb{E}(QS+RS+RT-QT) &= \sum_{qrst} p(q,r,s,t)(qs+rs+rt-qt) \\ &\leq \sum_{qrst} p(q,r,s,t) \\ &= 2.\end{aligned}$$

Now by linearity of the expectation,

$$\mathbb{E}\left(QS+RS+RT-QT\right) = \mathbb{E}\left(QS\right)+\mathbb{E}\left(RS\right)+\mathbb{E}\left(RT\right)-\mathbb{E}\left(QT\right)$$

3. By definition,

$$\begin{aligned}\langle QS\rangle &= \left(\frac{\langle 01| - \langle 10|}{\sqrt{2}}\right)\left(\mathbf{Z}\otimes\left(\frac{-\mathbf{X}-\mathbf{Z}}{\sqrt{2}}\right)\right)\left(\frac{|01\rangle - |10\rangle}{\sqrt{2}}\right) \\ &= \frac{1}{2\sqrt{2}}\ \left(\langle 01| - \langle 10|\right)\left(-\mathbf{Z}\otimes\mathbf{X}-\mathbf{Z}\otimes\mathbf{Z}\right)\left(|01\rangle - |10\rangle\right) \\ &= \frac{1}{2\sqrt{2}}\left(\left(\langle 01| - \langle 10|\right)\left(-\mathbf{Z}\,|0\rangle\otimes\mathbf{X}\,|1\rangle + \mathbf{Z}\,|1\rangle\otimes\mathbf{X}\,|0\rangle - \mathbf{Z}\,|0\rangle\otimes\mathbf{Z}\,|1\rangle + \mathbf{Z}\,|1\rangle\otimes\mathbf{Z}\,|0\rangle\right)\right) \\ &= \frac{1}{2\sqrt{2}}\left(\left(\langle 01| - \langle 10|\right)\left(-|00\rangle - |11\rangle + |01\rangle - |10\rangle\right)\right) \\ &= \frac{1}{\sqrt{2}}\end{aligned}$$

The same kind of computations shows that

$$\langle QS\rangle = \langle RS\rangle = \langle RT\rangle = \frac{1}{\sqrt{2}} \quad \text{and} \quad \langle QT\rangle = -\frac{1}{\sqrt{2}}$$

which show the claimed equality.

4. Yes! Quantum mechanics predicts that the sum of above averages yields $2\sqrt{2}$ while the Bell inequality shows that it cannot the bigger than $2 < 2\sqrt{2}$.

What does this mean? It means that one or more of the assumptions that went into the derivation of the Bell inequality must be incorrect.

It turns out that there are two assumptions made in the proof of the Bell inequality:

(a) The assumption that the physical properties P_Q, P_R, P_S, P_T have definite values Q, R, S, T which exist independent of observation. This si sometimes known as the assumption of *realism*.

(b) The assumption that Alice performing her measurement does not influence the result of Bob's measurement. This is sometimes known as the assumption of *locality*.

If you want more thorough discussion about these assumptions take a look the section 2.6 of *Quantum computation and quantum information* (Nielsen and Chuang).

$\square$