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ECE 601 - Dr. Gray

Homework #1

Due 9/7/2017

1. Attitude Stabilizer for Microsatellite

$$\begin{aligned}
\dot{x}_1 &= \frac{I_2 - I_3}{I_1} x_2 x_3 + b_1 u_1 \\
\dot{x}_2 &= \frac{I_3 - I_1}{I_2} x_3 x_1 + b_2 u_2 \\
\dot{x}_3 &= \frac{I_1 - I_2}{I_3} x_1 x_2 + b_3 u_3,
\end{aligned}$$

This is a nonlinear third order system, but stability is unclear initially. We can easily linearize by looking at level sets, i.e. locking one axis (at a time). Fixing x_3 requires the conditions:

$$\begin{aligned}
\dot{x}_3 = 0 &= \frac{I_1 - I_2}{I_3} x_1 x_2 + b_3 u_3 \\
u_3 &= \frac{I_2 - I_1}{I_3 b_3} x_1 x_2
\end{aligned}$$

Which seems well defined and bounded, provided x_1 and x_2 stay so.

Substituting the constant value of $x_3 = k_{x_3}$, we obtain a 2nd order state space model:

$$\dot{x} = \begin{bmatrix} 0 & a_1 \\ a_2 & 0 \end{bmatrix} x + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u, \begin{cases} a_1 = \frac{I_2 - I_3}{I_1} k_{x_3} \\ a_2 = \frac{I_3 - I_1}{I_2} k_{x_3} \end{cases}$$

This gives more insight into the stability of the system. Assuming zero initial conditions:

$$\begin{aligned}
sX_1 = a_1 X_2 + b_1 U_1 &\rightarrow (sX_1 - b_1 U_1)/a_1 &= X_2 \\
sX_2 = a_2 X_1 + b_2 U_2 &\rightarrow s(sX_1 - b_1 U_1)/a_1 &= a_2 X_1 + b_2 U_2 \\
& \quad X_1 &= \frac{b_1 U_1 + a_1 b_2 U_2}{s^2 - a_1 a_2}
\end{aligned}$$

With poles at $\pm k_{x_3} \sqrt{\frac{(I_2 - I_3)(I_3 - I_1)}{I_1 I_2}}$, we can see that x_1 is unstable or marginally stable, respectively, when $\frac{(I_2 - I_3)(I_3 - I_1)}{I_1 I_2}$ is positive or negative. By symmetry, the system will behave similarly on the other axes.

2. Examples of Vector Spaces

(a) The set of all n × ℓ matrices with real components, $\Re^{n\times\ell}$

Vector addition: $x + y = \{\{x_{ij} + y_{ij}\}_{i=1}^{n}\}_{j=1}^{\ell}$

Commutativity

$$\begin{aligned} x + y &= \{\{x_{ij} + y_{ij}\}_{i=1}^{n}\}_{j=1}^{\ell} \\ &= \{\{y_{ij} + x_{ij}\}_{i=1}^{n}\}_{j=1}^{\ell} \\ &= y + x \end{aligned}$$

Associativity

$$\begin{aligned} x + (y + z) &= \{\{x_{ij} + (y_{ij} + z_{ij})\}_{i=1}^{n}\}_{j=1}^{\ell} \\ &= \{\{(x_{ij} + y_{ij}) + z_{ij}\}_{i=1}^{n}\}_{j=1}^{\ell} \\ &= (x + y) + z \end{aligned}$$

Zero: $\underline{0} = \{\{0\}_{i=1}^{n}\}_{j=1}^{\ell}$

$$\begin{aligned} \underline{0} + x &= \{\{0 + x_{ij}\}_{i=1}^{n}\}_{j=1}^{\ell} \\ &= \{\{x_{ij}\}_{i=1}^{n}\}_{j=1}^{\ell} \\ &= x \end{aligned}$$

Inverse : $-x = \{\{-x_{ij}\}_{i=1}^{n}\}_{j=1}^{\ell}$

$$\begin{aligned} -x + x &= \{\{-x_{ij} + x_{ij}\}_{i=1}^{n}\}_{j=1}^{\ell} \\ &= \{\{0\}_{i=1}^{n}\}_{j=1}^{\ell} \\ &= \underline{0} \end{aligned}$$

Scalar Multiplication: $c \cdot x = \{\{c \cdot x_{ij}\}_{i=1}^{n}\}_{j=1}^{\ell}$

Commutativity

$$\begin{aligned} c \cdot x &= \{\{c \cdot x_{ij}\}_{i=1}^{n}\}_{j=1}^{\ell} \\ &= \{\{x_{ij} \cdot c\}_{i=1}^{n}\}_{j=1}^{\ell} \\ &= x \cdot c \end{aligned}$$

Associativity

$$
\begin{aligned}
c_1 \cdot (c_2 \cdot x) &= \{\{c_1 \cdot (c_2 \cdot x_{ij})\}_{i=1}^{n}\}_{j=1}^{\ell} \\
&= \{\{(c_1 \cdot c_2) \cdot x_{ij}\}_{i=1}^{n}\}_{j=1}^{\ell} \\
&= (c_1 \cdot c_2)\cdot
\end{aligned}
$$

Distribution to scalars

$$
\begin{aligned}
(c_1 + c_2) \cdot x &= \{\{(c_1 + c_2) \cdot x_{ij})\}_{i=1}^{n}\}_{j=1}^{\ell} \\
&= \{\{c_1 \cdot x_{ij} + c_2 \cdot x_{ij})\}_{i=1}^{n}\}_{j=1}^{\ell} \\
&= \{\{c_1 \cdot x_{ij})\}_{i=1}^{n}\}_{j=1}^{\ell} + \{\{c_2 \cdot x_{ij})\}_{i=1}^{n}\}_{j=1}^{\ell} \\
&= c_1 \cdot x + c_2 \cdot x
\end{aligned}
$$

Distribution of scalar

$$
\begin{aligned}
c \cdot (x + y) &= c \cdot \{\{(x_{ij} + y_{ij})\}_{i=1}^{n}\}_{j=1}^{\ell} \\
&= \{\{c \cdot (x_{ij} + y_{ij})\}_{i=1}^{n}\}_{j=1}^{\ell} \\
&= \{\{c \cdot x_{ij} + c \cdot y_{ij}\}_{i=1}^{n}\}_{j=1}^{\ell} \\
&= \{\{c \cdot x_{ij})\}_{i=1}^{n}\}_{j=1}^{\ell} + \{\{c \cdot y_{ij})\}_{i=1}^{n}\}_{j=1}^{\ell} \\
&= c \cdot x + c \cdot y
\end{aligned}
$$

Scalar One: $1 \in \mathfrak{R}$

$$
\begin{aligned}
1 \cdot x &= \{\{1 \cdot x_{ij}\}_{i=1}^{n}\}_{j=1}^{\ell} \\
&= \{\{x_{ij}\}_{i=1}^{n}\}_{j=1}^{\ell} \\
&= x
\end{aligned}
$$

(b) The set of all invertible matrices in $\mathfrak{R}^{n\times n}$, $GL_n(\mathfrak{R}^{n\times n})$.

This is a special case of $\mathfrak{R}^{n\times\ell}$, so all the above will apply.

(c) The set of all functions defined on the interval [0, 1] which have a well defined Laplace transform.

Vector addition: $(f + g)(x) = f(x) + g(x), \forall x \in [0, 1]$

Commutativity

$$
\begin{aligned}
(f + g)(x) &= f(x) + g(x) \\
&= g(x) + f(x) \\
&= (g + f)(x)
\end{aligned}
$$

Associativity

$$\begin{aligned}(f + (g + h))(x) &= f(x) + (g(x) + h(x)) \\ &= (f(x) + g(x)) + h(x) \\ &= ((f + g) + h)(x)\end{aligned}$$

Zero: $\underline{0}(x) = 0, \forall x \in [0, 1]$

$$\begin{aligned}(\underline{0} + f)(x) &= 0 + f(x) \\ &= f(x)\end{aligned}$$

Inverse : $(-f)(x) = -f(x)$

$$\begin{aligned}(-f + f)(x) &= -f(x) + f(x) \\ &= 0 \\ &= \underline{0}(x)\end{aligned}$$

Scalar Multiplication: $(c \cdot f)x = c \cdot f(x), \forall c \in \Re, x \in [0, 1]$

Commutativity

$$\begin{aligned}(c \cdot f)(x) &= c \cdot f(x) \\ &= f(x) \cdot c \\ &= (f \cdot c)(x)\end{aligned}$$

Associativity

$$\begin{aligned}(c_1 \cdot (c_2 \cdot f))(x) &= c_1 \cdot (c_2 \cdot f(x)) \\ &= (c_1 \cdot c_2) \cdot f(x) \\ &= ((c_1 \cdot c_2) \cdot f)(x)\end{aligned}$$

Distribution to scalars

$$\begin{aligned}((c_1 + c_2) \cdot f)(x) &= (c_1 + c_2) \cdot f(x) \\ &= c_1 \cdot f(x) + c_2 \cdot f(x) \\ &= (c_1 \cdot f + c_2 \cdot f)(x)\end{aligned}$$

Distribution of scalar

$$\begin{aligned}(c \cdot (f + g))(x) &= c \cdot (f(x) + g(x)) \\ &= c \cdot f(x) + c \cdot g(x) \\ &= (c \cdot f + c \cdot g)(x)\end{aligned}$$

Scalar One: $1 \in \Re$

$$\begin{aligned}(1 \cdot f)(x) &= 1 \cdot f(x) \\ &= f(x)\end{aligned}$$

(d) The set of all infinite sequences of real numbers having zeros in the

odd positions, that is, sequences of the form
$v = (0, v_1, 0, v_2, 0, \ldots)$

Vector addition:

$$
\begin{aligned}
u, v \in V, u + v &= (0+0, u_1 + v_1, 0+0, u_2 + v_2, 0+0, \ldots) \\
&= (0, u_1 + v_1, 0, u_2 + v_2, 0, \ldots)
\end{aligned}
$$

Commutativity

$$
\begin{aligned}
u + v &= (0, u_1 + v_1, 0, u_2 + v_2, 0, \ldots) \\
&= (0, v_1 + u_1, 0, v_2 + u_2, 0, \ldots) \\
&= v + u
\end{aligned}
$$

Associativity

$$
\begin{aligned}
u + (v + w) &= (0, u_1 + (v_1 + w_1), 0, u_2 + (v_2 + w_2), 0, \ldots) \\
&= (0, (u_1 + v_1) + w_1, 0, (u_2 + v_2) + w_2, 0, \ldots) \\
&= (u + v) + w
\end{aligned}
$$

Zero: $\underline{0} = (0, 0, 0, 0, 0, \ldots)$

$$
\begin{aligned}
\underline{0} + v &= (0, 0 + v_1, 0, 0 + v_2, 0, \ldots) \\
&= (0, v_1, 0, v_2, 0, \ldots) \\
&= v
\end{aligned}
$$

Inverse : $-v = (-0, -v_1, -0, -v_2, -0, \ldots) = (0, -v_1, 0, -v_2, 0, \ldots)$

$$
\begin{aligned}
-v + v &= (0, -v_1 + v_1, 0, -v_2 + v_2, 0, \ldots) \\
&= (0, 0, 0, 0, 0, \ldots) \\
&= \underline{0}(x)
\end{aligned}
$$

Scalar Multiplication:
$$
\begin{aligned}
c \cdot v &= (c \cdot 0, c \cdot v_1, c \cdot 0, c \cdot v_2, c \cdot 0, \ldots) \\
&= (0, c \cdot v_1, 0, c \cdot v_2, 0, \ldots)
\end{aligned}
$$

Commutativity

$$
\begin{aligned}
c \cdot v &= (0, c \cdot v_1, 0, c \cdot v_2, 0, \ldots) \\
&= (0, v_1 \cdot c, 0, v_2 \cdot c, 0, \ldots) \\
&= v \cdot c
\end{aligned}
$$

Associativity

$$
\begin{aligned}
c_1 \cdot (c_2 \cdot v) &= (0, c_1 \cdot (c_2 \cdot v_1), 0, c_1 \cdot (c_2 \cdot v_2), 0, \ldots) \\
&= (0, (c_1 \cdot c_2) \cdot v_1, 0, (c_1 \cdot c_2) \cdot v_2, 0, \ldots) \\
&= (c_1 \cdot c_2) \cdot v
\end{aligned}
$$

Distribution to scalars

$$\begin{aligned}(c_1 + c_2) \cdot v &= (0, (c_1 + c_2) \cdot v_1, 0, (c_1 + c_2) \cdot v_2, 0, \ldots) \\ &= (0, c_1 \cdot v_1 + c_2 \cdot v_1, 0, c_1 \cdot v_2 + c_2 \cdot v_2, 0, \ldots) \\ &= (0, c_1 \cdot v_1, 0, c_1 \cdot v_2, 0, \ldots) + (0, c_2 \cdot v_1, 0, c_2 \cdot v_2, 0, \ldots) \\ &= c_1 \cdot v + c_2 \cdot v\end{aligned}$$

Distribution of scalar

$$\begin{aligned}c \cdot (u + v) &= (0, c \cdot (u_1 + v_1), 0, c \cdot (u_2 + v_2), 0, \ldots) \\ &= (0, c \cdot u_1 + c \cdot v_1, 0, c \cdot u_2 + c \cdot v_2, 0, \ldots) \\ &= (0, c \cdot u_1, 0, c \cdot u_2, 0, \ldots) + (0, c \cdot v_1, 0, c \cdot v_2, 0, \ldots) \\ &= c \cdot u + c \cdot v\end{aligned}$$

Scalar One: $1 \in \Re$

$$\begin{aligned}1 \cdot v &= (0, 1 \cdot v_1, 0, 1 \cdot v_2, , \ldots) \\ &= (0, v_1, 0, v_2, 0, \ldots) \\ &= v\end{aligned}$$

3. MatLab Introduction

In [1]:
```
# Bring in some support
from numpy import triu
from scipy.linalg import toeplitz, inv
from util import LaTeX_matrix
```

In [2]:
```
toe = triu(toeplitz(range(1,11)))
LaTeX_matrix(toe)
```

Out[2]:

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 0 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 0 & 0 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 0 & 0 & 0 & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

In [3]:
```
itoe = inv(toe)
LaTeX_matrix(itoe)
```

Out[3]:

$$\begin{bmatrix} 1.0 & -2.0 & 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & -0.0 \\ 0.0 & 1.0 & -2.0 & 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & -0.0 \\ 0.0 & 0.0 & 1.0 & -2.0 & 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & -0.0 \\ 0.0 & 0.0 & 0.0 & 1.0 & -2.0 & 1.0 & 0.0 & 0.0 & 0.0 & -0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 1.0 & -2.0 & 1.0 & 0.0 & 0.0 & -0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 1.0 & -2.0 & 1.0 & 0.0 & -0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 1.0 & -2.0 & 1.0 & -0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 1.0 & -2.0 & 1.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 1.0 & -2.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 1.0 \end{bmatrix}$$

This looks reasonable. It is similarly an upper triangular matrix, which is required; the upper triangle looks sparse, but the three diagonals are sufficient to filter the steady gradient to zero. As a test, evaluating the upper right-hand corner

$$\begin{bmatrix} 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} 10 & 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \end{bmatrix}'$$

yields zero as it should.