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ECE 601 - Dr. Gray

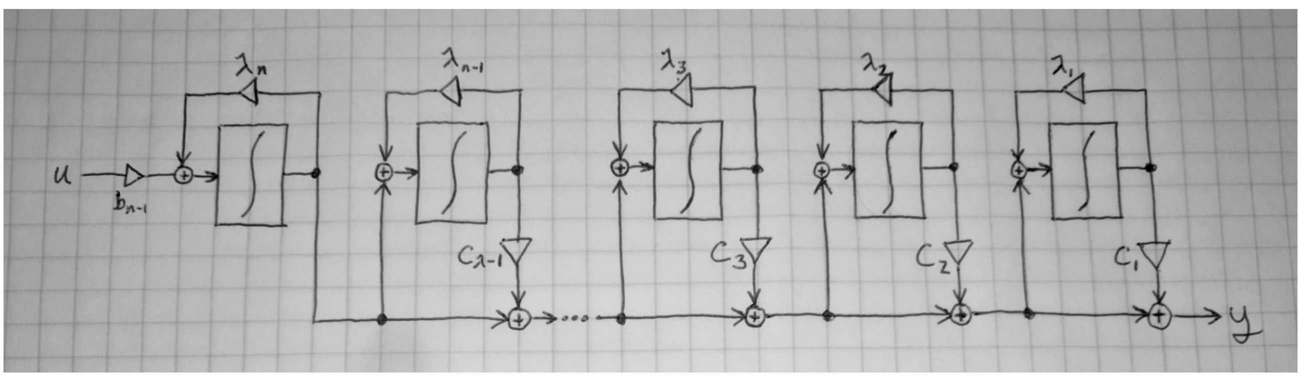
Homework #4

Due 9/28/2017

1. Kailath 2.2-6. *Cascade Form*

Draw a block diagram corresponding to the realization

$$A = \begin{bmatrix} \lambda_1 & c_2 & c_3 & & c_{n-1} & 1 \\ & \lambda_2 & c_3 & \cdots & c_{n-1} & 1 \\ & & \lambda_3 & & c_{n-1} & 1 \\ & & & \ddots & \vdots & \\ & \bigcirc & & & \lambda_{n-1} & 1 \\ & & & & & \lambda_n \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ b_{n-1} \end{bmatrix}, \quad c = \begin{bmatrix} c_1 & c_2 & c_3 & \cdots & c_{n-1} & 1 \end{bmatrix}$$



Update: $c_{\lambda-1}$ should read c_{n-1}.

2. Kailath 2.2-8. *Interconnections of Subsystems*

Write state equations for two realizations $\{A_i, b_i, c_i\}$ connected in:

a) Series

$$\begin{aligned} u_1 &= u \\ \dot{x}_1 &= A_1 x_1 + b_1 u_1 = A_1 x_1 + b_1 u \\ u_2 &= y_1 = c_1 x_1 \\ \dot{x}_2 &= A_2 x_2 + b_2 u_2 = b_2 c_1 x_1 + A_2 x_2 \\ y &= y_2 = c_2 x_2 \end{aligned}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_1 & \bigcirc \\ b_2 c_1 & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ \bigcirc \end{bmatrix} u, \quad y = \begin{bmatrix} \bigcirc & c_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

b) Parallel

$$\begin{aligned} u_1 &= u_2 = u \\ y &= y_1 + y_2 = c_1 x_1 + c_2 x_2 \end{aligned}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_1 & \bigcirc \\ \bigcirc & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u, \quad y = \begin{bmatrix} c_1 & c_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

c) Feedback, with $\{A_1, b_1, c_1\}$ in the forward loop and $\{A_2, b_2, c_2\}$ in the feedback loop

$$\begin{aligned}
u_1 &= u + y_2 = u + c_2 x_2 \\
\dot{x}_1 &= A_1 x_1 + b_1 u_1 = A_1 x_1 + b_1 c_2 x_2 + b_1 u \\
u_2 &= y_1 = y = c_1 x_1 \\
\dot{x}_2 &= A_2 x_2 + b_2 u_2 = b_2 c_1 x_1 + A_2 x_2
\end{aligned}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_1 & b_1 c_2 \\ b_2 c_1 & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ \bigcirc \end{bmatrix} u, \quad y = \begin{bmatrix} c_1 & \bigcirc \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

3. *Properties of Condition Numbers*

Let $A, B \in \mathbb{R}^{n \times n}$ be arbitrary matrices. (Assume A is invertible when necessary.) For each statement below, either prove its validity in general or provide a specific counterexample to disprove it.

(a) $c(A) \geq 1$

$$\sigma_1 \geq \sigma_p \rightarrow c(A) = \frac{\sigma_1}{\sigma_p} \geq 1$$

(b) If $A^{-1} = A^T$ then $c(A) = 1$

$$\begin{aligned}
\sigma_i &= \lambda_i^{1/2}(AA^T) = \lambda_i^{1/2}(I) \\
\sigma_i &= 1, \quad \forall i \\
c(A) &= \frac{\sigma_1}{\sigma_p} = 1
\end{aligned}$$

(c) If $c(A) = 1$ then $A^{-1} = A^T$

False. The condition number does not account for scaling:

In [1]:
```
A = 3*eye(3);
c = cond(A)
A_inv = inv(A)
A_trans = A'
```

```
c =

     1


A_inv =

    0.3333         0         0
         0    0.3333         0
         0         0    0.3333


A_trans =

     3     0     0
     0     3     0
     0     0     3
```

(d) $c(A^T) = c(A)$

Since singular values are eigenvalues of both AA^T and A^TA, this quickly follows:

$$\begin{array}{cccc} \sigma_i(A) = & \lambda_i^{1/2}(AA^T) = & \lambda_i^{1/2}(A^TA) = & \sigma_i(A^T) \\ c(A) = & \dfrac{\sigma_1(A)}{\sigma_p(A)} = & \dfrac{\sigma_1(A^T)}{\sigma_p(A^T)} = & c(A^T) \end{array}$$

(e) $c(A^{-1}) = (c(A))^{-1}$

Since $c(A^{-1}) \geq 1$, this is impossible for $c(A) > 1$:

In [2]:
```
A = [1 2; 3 4];
c_inv = cond(inv(A))
inv_c = 1/cond(A)
```

```
c_inv =

   14.9330


inv_c =

    0.0670
```

(f) $c(\alpha A) = \alpha c(A), \forall \alpha \in \mathbb{R}$

This will never be true for negative α, since $c(\alpha A) \geq 1$ **-and-** $c(A) \geq 1$.

In [3]:
```
c_neg = cond(-A)
neg_c = -cond(A)
```

```
c_neg =

   14.9330


neg_c =

  -14.9330
```

(g) $c(A + B) \leq c(A) + c(B)$

False. Summing matrices can easily result in something singular, or at least sensitive.

In [4]:
```
B = [4 3; 2 1.00001];
c_sum = cond(A+B)
sum_c = cond(A)+cond(B)
```

```
c_sum =

   2.0000e+06


sum_c =

   29.8664
```

(h) $c(AB) \leq c(A)c(B)$

$$
\begin{aligned}
c(AB) &= \|AB\| \|(AB)^{-1}\| \\
&= \|AB\| \|B^{-1}A^{-1}\|
\end{aligned}
$$

$$\|AB\| \leq \|A\| \|B\|$$

$$
\begin{aligned}
c(AB) \leq & \quad \|A\| \|B\| \|B^{-1}\| \|A^{-1}\| \\
c(AB) \leq & \quad (\|A\| \|A^{-1}\|)(\|B\| \|B^{-1}\|) \\
c(AB) \leq & \quad c(A)c(B)
\end{aligned}
$$

4. *Singular Value Decomposition*

Consider a linear operator $\mathcal{A}$ represented by the matrix

$$A = \begin{bmatrix} -1 & 2 & 3 \\ 1 & 0 & -1 \\ 1 & 2 & 0 \end{bmatrix}.$$

(a) Compute the singular value decomposition of A.

In [5]:
```
A = [-1 2 3; 1 0 -1; 1 2 0];
[Q1,S,Q2] = svd(A)
```

```
Q1 =

   -0.9379    0.1096   -0.3291
    0.2540   -0.4290   -0.8669
   -0.2362   -0.8966    0.3745


S =

    3.9798         0         0
         0    2.2609         0
         0         0    0.2223


Q2 =

    0.2402   -0.6348   -0.7344
   -0.5900   -0.6962    0.4088
   -0.7708    0.3351   -0.5417
```

(b) Are singular values invariant under a similarity transformation? Explain.

No, generally. A similarity transformation may change the shape or scale of the operation, which would alter the singular values.

In [6]:
```
T = [1 2 3; 4 5 6; 7 8 10];
A_ = T*A*inv(T)
S_ = svd(A_)
```

```
A_ =

   -7.0000   22.0000  -11.0000
  -11.0000   50.0000  -26.0000
  -17.0000   84.0000  -44.0000


S_ =

  115.0022
    2.5461
    0.0068
```

(c) If your answer to (b) is yes, give another representation of the operator having the same singular values. If the answer is no, can you provide another type of transformation under which the singular values are always preserved?

If we limit the transform to pure rotations, we can preserve the singular values; for example, we could apply the scale-less components of the SVD $T = \overline{Q_1}\,\overline{\Sigma}\,\overline{Q_2}^T$ to create SV invariant transform $\overline{A} = \overline{Q_1} A \overline{Q_2}^T$:

In [7]:
```
[Q1_,ST,Q2_] = svd(T);
A_ = Q1_*A*Q2_'
S_ = svd(A_)
```

```
A_ =

   -1.0181    0.5855   -1.7172
    1.9652    1.4414   -2.0080
   -0.5653    1.2843   -2.1751


S_ =

    3.9798
    2.2609
    0.2223
```