Tim DeChant

ECE 601 - Dr. Gray

Homework #3

Due 9/21/2017

1. Condition Numbers and Error Gains: Consider the linear system of equations $Ax = b$, where

$$A = \begin{bmatrix} 2 & -2 \\ -2 & 1 \end{bmatrix}.$$

(a) Compute the condition number $c(A)$ of A.

$$\begin{aligned} det(A - \lambda I) = 0 = & \quad (2-\lambda)(1-\lambda) - 4 \\ = & \quad \lambda^2 - 3\lambda - 2 \end{aligned}$$

$$\lambda = \frac{3+\sqrt{17}}{2} \qquad \text{or} \qquad \lambda = \frac{3-\sqrt{17}}{2}$$

$$A = A^T \rightarrow c(A) = \frac{|\lambda_{max}|}{|\lambda_{min}|} = \frac{3+\sqrt{17}}{\sqrt{17}-3} = \frac{13+3\sqrt{17}}{4} \approx 6.342$$

In [1]:
```
A = [2 -2; -2 1];
lambda = abs(eig(A));
c = max(lambda)/min(lambda)
```

```
c =

    6.3423
```

(b) Determine specific vectors b and Δb so that the following equations are satisfied:

$$\frac{||\Delta x||}{||x||} = c(A)\frac{||\Delta b||}{||b||}$$

Here b and Δb will be the products of the max/min eigenvalue/eigenvector pairs ($\lambda_{max} x_{max}, \lambda_{min} x_{min}$) for A:

In [2]:
```
[v,d] = eig(A);
l=diag(d);
l_abs=abs(l);
[l_min,i_min] = min(abs(l));
[l_max,i_max] = max(abs(l));
x       = v(:,[i_max]);
delta_x = v(:,[i_min]);
b       = l(i_max)*x
delta_b = l(i_min)*delta_x
error = norm(delta_x)/norm(x) - c * norm(delta_b)/norm(b)
```

```
b =

  -2.8072
   2.1918


delta_b =

   0.3456
   0.4426


error =

  -2.2204e-16
```

$$\frac{||\Delta x||}{||x||} = \frac{1}{c(A)} \frac{||\Delta b||}{||b||}$$

Here the minimum eigenvector is used for x and the maximum for Δx, so b and Δb are also swapped:

In [3]:
```
x       = v(:,[i_min]);
delta_x = v(:,[i_max]);
b       = l(i_min)*x
delta_b = l(i_max)*delta_x
error = norm(delta_x)/norm(x) - 1/c * norm(delta_b)/norm(b)
```

```
b =

   0.3456
   0.4426


delta_b =

  -2.8072
   2.1918


error =

     0
```

2. Spectral Versus Singular Value Decomposition: Consider a linear operator A with matrix representation

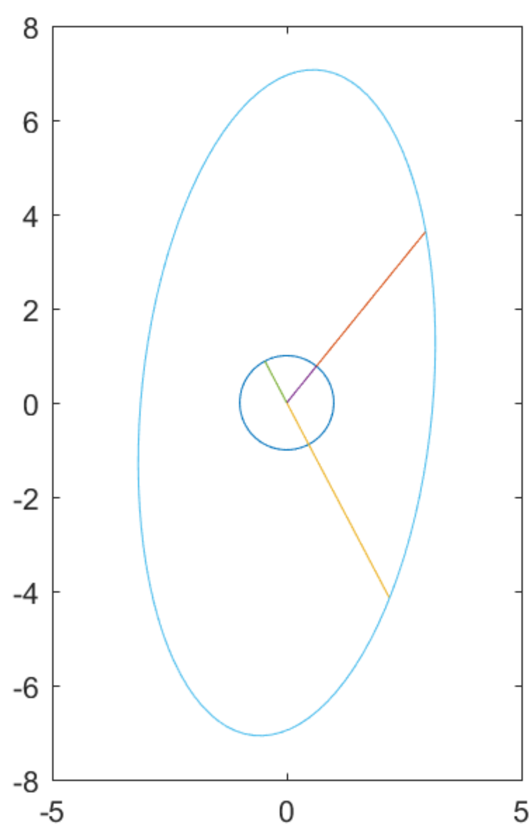
$$A = \begin{bmatrix} 1 & 3 \\ 7 & -1 \end{bmatrix}.$$

(a) Compute the spectral decomposition and singular value decomposition of A.

(b) Sketch $\mathcal{A}(S^1)$, the image of the unit circle under A.

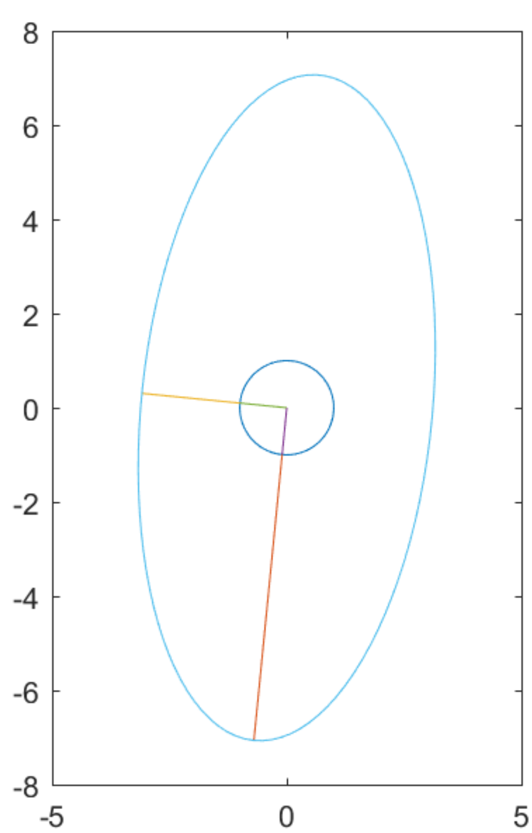
In [4]:
```
imatlab_export_fig('print-png');
A = [1 3; 7 -1];
[v d] = eig(A)
lambda=diag(d);
x1=[[0;0] v(:,[1])];
x2=[[0;0] v(:,[2])];
b1=[[0;0] lambda([1])*v(:,[1])];
b2=[[0;0] lambda([2])*v(:,[2])];
t = linspace(0,2*pi);
x = [ cos(t); sin(t) ];
b = A*x;
plot(x([1],:),x([2],:));
hold on
plot(b1([1],:),b1([2],:));
plot(b2([1],:),b2([2],:));
plot(x1([1],:),x1([2],:));
plot(x2([1],:),x2([2],:));
plot(b([1],:),b([2],:));
hold off
daspect([1 1 1])
```

```
v =

    0.6308   -0.4664
    0.7760    0.8846


d =

    4.6904         0
         0   -4.6904
```



In [5]:
```
imatlab_export_fig('print-png');
A = [1 3; 7 -1];
[u s v] = svd(A)
lambda=diag(s);
x1=[[0;0] u(:,[1])];
x2=[[0;0] u(:,[2])];
b1=[[0;0] lambda([1])*u(:,[1])];
b2=[[0;0] lambda([2])*u(:,[2])];
t = linspace(0,2*pi);
x = [ cos(t); sin(t) ];
b = A*x;
plot(x([1],:),x([2],:));
hold on
plot(b1([1],:),b1([2],:));
plot(b2([1],:),b2([2],:));
plot(x1([1],:),x1([2],:));
plot(x2([1],:),x2([2],:));
plot(b([1],:),b([2],:));
hold off
daspect([1 1 1])
```

```
u =

   -0.0985   -0.9951
   -0.9951    0.0985


s =

    7.0990         0
         0    3.0990


v =

   -0.9951   -0.0985
    0.0985   -0.9951
```



(c) Compare your answers to parts (a) and (b) and draw a conclusion.

The components of the SVD represent the shape of the transform much better than those of the spectral decomposition.

3 Numerically Solving Hilbert Systems: A Hilbert system is a set of linear equations $Hx = b$, where H is the Hilbert matrix (this means: $H_{ij} = 1/(i + j - 1) \forall i, j \geq 1$). For Hilbert matrices of dimension n = 10, 15, 20, try the following experiment:

In [6]:
```
eps
```

```
ans =

   2.2204e-16
```

(a) Compute and save H^{-1}. (Use the MatLab command inv for matrix inversion.)

(b) Set $b = [1, 2, \ldots, n]^T$

(c) Compute $x = H^{-1}b$

(d) Compute $bb := Hx$.

In [7]:
```
n = 10;
H = hilb(n);
iH = inv(H);
b = linspace(1, n, n).'
x = iH*b
bb = H*x
```

```
b =

     1
     2
     3
     4
     5
     6
     7
     8
     9
    10


x =

   1.0e+08 *

   -0.0000
    0.0010
   -0.0233
    0.2330
   -1.2106
    3.5940
   -6.3222
    6.5103
   -3.6227
    0.8405


bb =

    1.0001
    2.0001
    3.0001
    4.0001
    5.0001
    6.0001
    7.0001
    8.0001
    9.0001
   10.0001
```

In [8]:
```
n = 15;
H = hilb(n);
iH = inv(H);
b = linspace(1, n, n).'
x = iH*b
bb = H*x
```

```
Warning: Matrix is close to singular or badly scaled. Results may be inaccurat
e. RCOND =  8.269626e-19.

b =

     1
     2
     3
     4
     5
     6
     7
     8
     9
    10
    11
    12
    13
    14
    15


x =

   1.0e+10 *

    0.0000
   -0.0000
    0.0009
   -0.0129
    0.0917
   -0.3449
    0.5842
    0.2451
   -2.8337
    4.3313
   -0.4098
   -6.4435
    8.4204
   -4.5973
    0.9684


bb =

   10.5181
    4.5068
    3.2783
    3.1009
    3.4141
    4.0027
    4.7570
    5.6148
    6.5392
    7.5070
    8.5037
    9.5196
   10.5482
   11.5853
   12.6278
```

In [14]:
```
n = 20;
H = hilb(n);
iH = inv(H);
b = linspace(1, n, n).'
x = iH*b
bb = H*x
```

```
Warning: Matrix is close to singular or badly scaled. Results may be inaccurat
e. RCOND =  9.542396e-20.

b =

     1
     2
     3
     4
     5
     6
     7
     8
     9
    10
    11
    12
    13
    14
    15
    16
    17
    18
    19
    20


x =

   1.0e+12 *

    0.0000
   -0.0000
    0.0001
   -0.0019
    0.0168
   -0.0873
    0.2707
   -0.4806
    0.3946
   -0.0036
    0.1047
   -0.9998
    1.4091
   -0.7873
    0.4672
   -0.5454
   -0.1398
    0.9149
   -0.7075
    0.1750


bb =

   49.5140
   25.0212
   22.5622
   20.6523
   19.3404
   18.6339
   18.3976
   18.4965
   18.8279
   19.3199
   19.9233
```

20.6050
21.3421
22.1193
22.9255
23.7534
24.5977
25.4545
26.3211
27.1955