Tim DeChant

ECE 601 - Dr. Gray

Homework #6

Due 10/26/2017

1. Kailath 2.3-28. *Time-Variant Models*

Time-variant coefficients can be handled in descrete time much more easily than in continuous time. Thus suppose

$$x(i+1) = A(i)x(i) + B(i)u(i), \quad i = 0, 1, \dots$$

a. Show that

$$x(k) = \Phi(k,j)x(j) + \sum_{i=j}^{k-1} \Phi(k, i+1)B(i)u(i)$$

where

$$\Phi(k,j) \triangleq A(k-1)\dots A(j), \quad \Phi(i,i) = I$$

This follows from the recursive nature of $x(\cdot)$; recursing from k down to some $j < k$:

$$\begin{aligned}
x(k) &= A(k-1)\Bigg(A(k-2)\Big(\dots A(j+1)\big(A(j)x(j) + B(j)u(j)\big) + B(j+1)u(j+1)\dots\Big) \\
&+ B(k-2)u(k-2)\Bigg) + B(k-1)u(k-1) \\
&= A(k-1)A(k-2)\dots A(j+1)A(j)x(j) \\
&+ A(k-1)A(k-2)\dots A(j+1)B(j)u(j) \\
&+ \dots \\
&+ A(k-1)B(k-2)u(k-2) \\
&+ B(k-1)u(k-1) \\
&= \Phi(k,j)x(j) + \sum_{i=j}^{k-1} \Phi(k, i+1)B(i)u(i)
\end{aligned}$$

b. Is it always true that $\Phi(i,j)\Phi(j,k) = \Phi(i,k)$, **all** $i, j, k \geq 0$ **?**

The recursion in $\Phi(a,b)$ only makes sense for $a > b$, with $a = b$ only covered by the explicit case. We can easily handle the cases $i = j$ or $j = k$, since the corresponding Φ would be I.

Otherwise, we can see:

$$\begin{aligned}\Phi(i,j)\Phi(j,k) &= \left(A(i-1)\dots A(j)\right)\left(A(j-1)\dots A(k)\right) \\ &= A(i-1)\dots A(k) \qquad , \quad i > j > k \\ &= \Phi(i,k)\end{aligned}$$

2. Kailath 2.3-29. *An Alternative ["$u(k+1)$"] Model*

In some problems, it turns out to be more natural to consider state-space models of the form

$$\begin{aligned}\xi(k+1) &= F\xi(k) + gu(k+1), \qquad k > 0 \\ y(k) &= h\xi(k), \qquad \xi(0) = \xi_0\end{aligned}$$

a. Show that the transfer function is

$$\mathcal{H}(z) = hz(zI - F)^{-1}g$$

Note the extra z in the numerator corresponding to the "advanced" input $u(k+1)$.

$$\begin{aligned}\mathscr{L}\{\xi(k+1)\} = z(\Xi - \xi_0) &= F\Xi + gz(U - u_0) \\ (zI - F)\Xi &= gzU + z(\xi_0 - gu_0) = gzU, \quad \text{since } \xi(0) = F\xi(-1) + gu(0) = gu(0) \\ \Xi &= (zI - F)^{-1}gzU\end{aligned}$$

$$\begin{aligned}\mathscr{L}\{y(k)\} = Y &= h\Xi \\ &= h(zI - F)^{-1}gzU\end{aligned}$$

$$\mathcal{H} = \frac{Y}{U} = hz(zI - F)^{-1}g$$

b. Prove that an arbitrary initial state ξ_0 can be transferred to some other arbitrary state in a finite time if and only if the matrix $\begin{bmatrix} g & Fg & \cdots & F^{n-1}g \end{bmatrix}$ is nonsingular.

Since this is the controllability matrix of the equivalent system with (relatively) delayed input, this is really asking to prove that the controllability criteria are the same. This is apparent, because the input required to build an arbitrary state in one would effect the same state in the other if properly delayed/advanced.

c. Consider how to translate results for the "$u(k)$" model to the present model. *Hint:* Define an augmented "state" $[x'(k), u(k)]'$.

Results for the $u(k)$ model would save the u in the state, then add this version into the output equation:

$$\begin{bmatrix} x(k+1) \\ u(k+1) \end{bmatrix} = \begin{bmatrix} F & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x(k) \\ u(k) \end{bmatrix}$$
$$y = \begin{bmatrix} h & g \end{bmatrix} \begin{bmatrix} x(k) \\ u(k) \end{bmatrix}$$

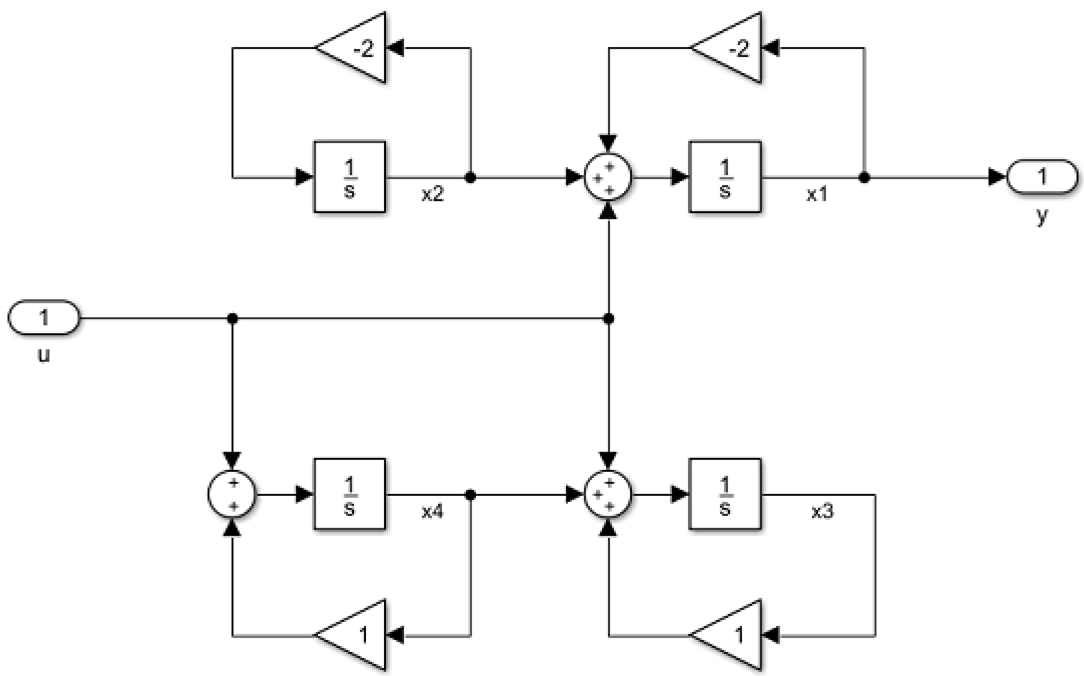
3. Kailath 2.4-1.

Consider the realization with

$$A = \text{block diag}\left\{ \begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right\}$$
$$b' = \begin{bmatrix} 1 & 0 & 1 & 1 \end{bmatrix}, \quad c = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$$

Draw a block diagram of the realization and check its controllability in as many ways as you can.

(Also comment on observability.)



By inspection, this is neither controllable or inspectable. Component x_2 only depends on itself, but is used by $x_1 = y$; so neither the state nor its output is controllable. Similarly, x_3 is only used by itself, with no path to the output; so it is not observable.

In [11]:
```
A = blkdiag([-2 1; 0 -2],[1 1;0 1]);
b = [1;0;1;1];
c = [1 0 0 0];
Con = [b A*b A^2*b A^3*b];
controllable = det(Con)
Obs = [c; c*A; c*A^2; c*A^3];
Observable = det(Obs)
```

```
controllable =

     0


Observable =

     0
```

4. Kailath 2.4-3. *(DeBra)*

An inverted pendulum, of mass m, is hinged at A. A gyro with spin angular momentum, h, is attached to the pendulum but is free to rotate about the pendulum axis (angle ϕ) as shown in the figure. A control torque, Q, can be applied to the gyro from the pendulum. The equations of motion are

$$I\ddot{\theta} = mgl\theta - h\dot{\phi}$$

and

$$J\ddot{\phi} = h\dot{\theta} + Q$$

where $I =$ the moment of inertia of the pendulum plus gyro about A, $J =$ the moment of inertia of the gyro about axis AC, and $C =$ the mass center of the pendulum plus gyro.

(**Remark:** The given equations of motion have already been linearized for you about the equilibrium position $\theta = 0$.)

a. Compute the transfer function from $u(\cdot)$ to $\phi(\cdot)$ and $u(\cdot)$ to $\theta(\cdot)$.

$$\dot{x} = \begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \\ \dot{\phi} \\ \ddot{\phi} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ mgl/I & 0 & 0 & -h/I \\ 0 & 0 & 0 & 1 \\ 0 & h/J & 0 & 0 \end{bmatrix} \begin{bmatrix} \theta \\ \dot{\theta} \\ \phi \\ \dot{\phi} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/J \end{bmatrix} Q$$

$$y_1 = \theta = c_1 x = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} x$$

$$y_2 = \phi = c_2 x = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} x$$

$$
\begin{aligned}
H_\theta &= c_1(sI - A)^{-1}b \\
&= c_1 \begin{bmatrix} s & -1 & 0 & 0 \\ -mgl/I & s & 0 & h/I \\ 0 & 0 & s & -1 \\ 0 & -h/J & 0 & s \end{bmatrix}^{-1} b \\
&= \frac{c_1 \begin{bmatrix} \cdot & \cdot & \cdot & -det\left(\begin{bmatrix} -1 & 0 & 0 \\ s & 0 & h/I \\ 0 & s & -1 \end{bmatrix}\right) \\ \cdot & \cdot & \cdot & det\left(\begin{bmatrix} s & 0 & 0 \\ -mgl/I & 0 & h/I \\ 0 & s & -1 \end{bmatrix}\right) \\ \cdot & \cdot & \cdot & -det\left(\begin{bmatrix} s & -1 & 0 \\ -mgl/I & s & h/I \\ 0 & 0 & -1 \end{bmatrix}\right) \\ \cdot & \cdot & \cdot & det\left(\begin{bmatrix} s & -1 & 0 \\ -mgl/I & s & 0 \\ 0 & 0 & s \end{bmatrix}\right) \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/J \end{bmatrix}}{det(A)} \\
&= \frac{c_1 \begin{bmatrix} -h/Is \\ -h/Is^2 \\ s^2 - mgl/I \\ s^3 - mgl/Is \end{bmatrix} [1/J]}{-h/J(-h/Is^2) + s(s^3 - mgl/Is)} = \frac{c_1 \begin{bmatrix} -h/(IJ)s \\ -h/(IJ)s^2 \\ 1/Js^2 - mgl/(IJ) \\ 1/Js^3 - mgl/(IJ)s \end{bmatrix}}{s^4 + (h^2/(IJ) - mgl/I)s^2} = \frac{-h/(IJ)s}{s^4 + (h^2/(IJ) - mgl/I)s^2}
\end{aligned}
$$

$$
H_\phi = c_2(sI - A)^{-1}b = \frac{1/Js^2 - mgl/(IJ)}{s^4 + (h^2/(IJ) - mgl/I)s^2}
$$

b. Show that the system is controllable by Q, observable with ϕ and unobservable with θ.

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ mgl/I & 0 & 0 & -h/I \\ 0 & 0 & 0 & 1 \\ 0 & h/J & 0 & 0 \end{bmatrix}$$

$$\mathcal{C} = \begin{bmatrix} b & Ab & A^2b & A^3b \end{bmatrix} = \begin{bmatrix} 0 & 0 & -h/(IJ) & 0 \\ 0 & -h/(IJ) & 0 & (-hmglJ + h^3)/(I^2J^2) \\ 0 & 1/J & 0 & -h^2/(IJ^2) \\ 1/J & 0 & -h^2/(IJ^2) & 0 \end{bmatrix}$$

$$\begin{aligned} det(\mathcal{C}) &= \frac{-1}{J}\left(\frac{1}{J}\frac{-h}{IJ}\frac{-hmglJ + h^3}{I^2J^2} + \frac{-h^2}{IJ^2}\frac{-h^2}{I^2J^2}\right) \\ &= \frac{h^2mglJ - h^4}{I^3J^5} + \frac{h^4}{I^3J^5} = \frac{h^2mglJ}{I^3J^5} \quad \text{[physically, this must be non-zero]} \end{aligned}$$

$$\begin{aligned} \mathcal{O}_\theta &= \begin{bmatrix} c_1 \\ c_1A \\ c_1A^2 \\ c_1A^3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ mgl/I & 0 & 0 & -h/I \\ 0 & mgl/I - h^2/(IJ) & 0 & 0 \end{bmatrix} \\ det(\mathcal{O}_\phi) &= 0 \end{aligned}$$

$$\begin{aligned} \mathcal{O}_\phi &= \begin{bmatrix} c_2 \\ c_2A \\ c_2A^2 \\ c_2A^3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & h/J & 0 & 0 \\ hmgl/(IJ) & 0 & 0 & -h^2/(IJ) \end{bmatrix} \\ det(\mathcal{O}_\phi) &= -h^2mgl/(IJ^2) \quad \text{[physically, this must be non-zero]} \end{aligned}$$

c. Show that the system is always unstable.

With modes at $\{0, 0, \pm\sqrt{mgl/I - h^2/(IJ)}\}$, the system will have a mode in the right-half if $mgl/I > h^2/(IJ)$; otherwise all modes will lie on the imaginary axis.

5. *A Certain Commutativity Property*

Let $A \in \mathbb{R}^{n\times n}$ be an arbitrary matrix and $b(s)$ be any degree n polynomial. Extend the domain of $b(s)$ to $\mathbb{R}^{n\times n}$ in the usual way.

(a) Show that in general $\lambda(b(A)) = b(\lambda(A))$, where $\lambda(\cdot)$ denotes any eigenvalue of its matrix argument.

Looking at corresponding eigenvector v:

$$\begin{aligned} b(\lambda(A))v &= (b_1\lambda^{n-1} + b_2\lambda^{n-2} + b_3\lambda^{n-3} + \cdots + b_n I)v \\ &= b_1\lambda^{n-1}v + b_2\lambda^{n-2}v + b_3\lambda^{n-3}v + \cdots + b_n Iv \\ &= b_1 A^{n-1}v + b_2 A^{n-2}v + b_3 A^{n-3}v + \cdots + b_n Iv \\ &= (b_1 A^{n-1} + b_2 A^{n-2} + b_3 A^{n-3} + \cdots + b_n I)v \\ &= \lambda(b(A))v \end{aligned}$$

(b) How does the result in part (a) simplify if $b(s)$ is the characteristic polynomial of A?

Then $b(s)$ could be directly factored to get the eigenvalues.