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ECE 601 - Dr. Gray

Homework #2

Due 9/14/2017

1. Equilibria and Linearization

$$\dot{x} = F(x, u), \quad x(0) \ given$$
$$y = G(x, y).$$

(a) Computing *equilibrium state* and *equilibrium input*

For fixed x, we know that $\dot{x} = F(x) = \underline{0}$. If F is linear, we will have a single linear relation $x_e = F_0(u_e)$; generally though, each root of F will yield its own such linear relation. Therefore pairs of (x_e, u_e) are not unique.

(b) Deriving explicit formula for linear model about an equilibrium

$$\begin{aligned} \dot{x}_i &= F_i(x, u) \\ \frac{d}{dt}(x_{ei} + \Delta x_i) &= F_i(x_e + \Delta x, u_e + \Delta u) \\ \cancel{\dot{x}_{ei}} + \frac{d}{dt}\Delta x_i &\approx \cancel{F_i(x_e, u_e)} + \left.\frac{\delta F_i}{\delta x}\right|_{x_e} \Delta x + \left.\frac{\delta F_i}{\delta u}\right|_{u_e} \Delta u \end{aligned}$$

$$\frac{d}{dt}\Delta x = \begin{bmatrix} \left.\frac{\delta F_1}{\delta x}\right|_{x_e} \\ \vdots \\ \left.\frac{\delta F_n}{\delta x}\right|_{x_e} \end{bmatrix} \Delta x + \begin{bmatrix} \left.\frac{\delta F_1}{\delta u}\right|_{u_e} \\ \vdots \\ \left.\frac{\delta F_n}{\delta u}\right|_{u_e} \end{bmatrix} \Delta u$$

$$\begin{aligned} y_i &= H_i(x, u) \\ y_{ei} + \Delta y_i &= H_i(x_e + \Delta x, u_e + \Delta u) \\ \cancel{y_{ei}} + \Delta y_i &\approx \cancel{H_i(x_e, u_e)} + \left.\frac{\delta H_i}{\delta x}\right|_{x_e} \Delta x + \left.\frac{\delta H_i}{\delta u}\right|_{u_e} \Delta u \end{aligned}$$

$$\Delta y = \begin{bmatrix} \left.\frac{\delta H_1}{\delta x}\right|_{x_e} \\ \vdots \\ \left.\frac{\delta H_n}{\delta x}\right|_{x_e} \end{bmatrix} \Delta x + \begin{bmatrix} \left.\frac{\delta H_1}{\delta u}\right|_{u_e} \\ \vdots \\ \left.\frac{\delta H_n}{\delta u}\right|_{u_e} \end{bmatrix} \Delta u$$

$$(A, B, C, D) = \left(\begin{bmatrix} \frac{\delta F_1}{\delta x_1} & \frac{\delta F_1}{\delta x_2} & \cdots & \frac{\delta F_1}{\delta x_n} \\ \frac{\delta F_2}{\delta x_1} & \frac{\delta F_2}{\delta x_2} & \cdots & \frac{\delta F_2}{\delta x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\delta F_n}{\delta x_1} & \frac{\delta F_n}{\delta x_2} & \cdots & \frac{\delta F_n}{\delta x_n} \end{bmatrix}, \begin{bmatrix} \frac{\delta F_1}{\delta u_1} & \frac{\delta F_1}{\delta u_2} & \cdots & \frac{\delta F_1}{\delta u_n} \\ \frac{\delta F_2}{\delta u_1} & \frac{\delta F_2}{\delta u_2} & \cdots & \frac{\delta F_2}{\delta u_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\delta F_n}{\delta u_1} & \frac{\delta F_n}{\delta u_2} & \cdots & \frac{\delta F_n}{\delta u_n} \end{bmatrix}, \begin{bmatrix} \frac{\delta H_1}{\delta x_1} & \frac{\delta H_1}{\delta x_2} & \cdots & \frac{\delta H_1}{\delta x_n} \\ \frac{\delta H_2}{\delta x_1} & \frac{\delta H_2}{\delta x_2} & \cdots & \frac{\delta H_2}{\delta x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\delta H_n}{\delta x_1} & \frac{\delta H_n}{\delta x_2} & \cdots & \frac{\delta H_n}{\delta x_n} \end{bmatrix}, \begin{bmatrix} \frac{\delta H_1}{\delta u_1} & \frac{\delta H_1}{\delta u_2} & \cdots & \frac{\delta H_1}{\delta u_n} \\ \frac{\delta H_2}{\delta u_1} & \frac{\delta H_2}{\delta u_2} & \cdots & \frac{\delta H_2}{\delta u_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\delta H_n}{\delta u_1} & \frac{\delta H_n}{\delta u_2} & \cdots & \frac{\delta H_n}{\delta u_n} \end{bmatrix} \right)$$

(c) Describe equilibria for each system

(i) $\dot{x} = sin(x) + cos(2u), \;\; y = x^2/2$

$$\begin{aligned} sin(x_e) &= -cos(2u_e) \\ &= cos(2u_e - \pi) \\ &= sin(2u_e - \pi + \pi/2) \end{aligned}$$

$$\begin{aligned} x_e &= 2u_e + (2k - 1/2)\pi, \;\; k \in \mathbb{Z} \\ u_e &= x_e/2 - (k - 1/4)\pi \\ y_e &= x_e^2/2 \end{aligned}$$

Any initial state can be a *equilibrium state*, provided the *equilibrium input* is appropriately scaled and shifted; in that case the state feedback and input terms cancel out, leaving the overall state unchanged.

(ii) $\dot{x}_1 = x_1 x_2 + u, \;\; \dot{x}_2 = 1 - x_1 x_2, \quad y = x_1^2 + x_2^2$

$$\begin{aligned} u_e &= -x_{e1}x_{e2} \quad and \quad 1 &&= x_{e1}x_{e2} \\ u_e &= -1 \qquad\;\;\; and \quad x_{e2} &&= 1/x_{e1} \end{aligned}$$

$$y_e = x_{e1}^2 + x_{e1}^{-2}$$

Provided the states of both stages are reciprocals of one another, they can be held in equilibrium by an input of -1.

(iii) $\dot{x} = \begin{bmatrix} a_1 x_2 x_3 & + b_1 u_1 \\ a_2 x_3 x_1 & + b_2 u_2 \\ a_3 x_1 x_2 & + b_3 u_3 \end{bmatrix}, \begin{cases} a_1 & = \frac{I_2 - I_3}{I_1} \\ a_2 & = \frac{I_3 - I_1}{I_2} \\ a_3 & = \frac{I_1 - I_2}{I_3} \end{cases}$

$$\begin{aligned} 0 &= a_1 x_{e2} x_{e3} + b_1 u_{e1} \quad and \quad 0 &= a_2 x_{e3} x_{e1} + b_2 u_{e2} \quad and \quad 0 &= a_3 x_{e1} x_{e2} + b_3 u_{e3} \\ u_{e1} &= \frac{-a_1}{b_1} x_{e2} x_{e3} \quad and \quad u_{e2} &= \frac{-a_2}{b_2} x_{e3} x_{e1} \quad and \quad u_{e3} &= \frac{-a_3}{b_3} x_{e1} x_{e2} \end{aligned}$$

Any initial condition can be held as *equilibrium state*, provided that the momentum wheel for each axis maintains a torque proportional (with specified factors) to the product of the angular velocities of the other two axes.

(d) Determine linear state space model for each system

(i) $\dot{x} = sin(x) + cos(2u), \quad y = x^2/2$

$$\begin{aligned} \frac{d}{dt}\Delta x &= A\Delta x + B\Delta u \\ &= A(x - x_e) + B(u - u_e) \end{aligned}$$

$$\begin{aligned} \Delta y &= C\Delta x + D\Delta u \\ &= C(x - x_e) + D(u - u_e) \end{aligned}$$

$$\begin{aligned} (A, B, C, D) &= \Bigg(\left[\tfrac{\delta}{\delta x}(sin(x) + cos(2u)) \right]_{x_e}, \left[\tfrac{\delta}{\delta u}(sin(x) + cos(2u)) \right]_{u_e}, \\ &\quad \left[\tfrac{\delta}{\delta x}(x^2/2) \right]_{x_e}, \left[\tfrac{\delta}{\delta u}(x^2/2) \right]_{u_e} \Bigg) \\ &= \Big(\left[cos(x_e) \right], \left[2sin(2u_e) \right], \left[x_e \right], \left[0 \right] \Big) \end{aligned}$$

(ii) $\dot{x}_1 = x_1 x_2 + u, \;\; \dot{x}_2 = 1 - x_1 x_2, \quad y = x_1^2 + x_2^2$

$$\begin{aligned}\frac{d}{dt}\Delta x &= A\Delta x + B\Delta u \\ &= A(x - x_e) + B(u - u_e)\end{aligned}$$

$$\begin{aligned}\Delta y &= C\Delta x + D\Delta u \\ &= C(x - x_e) + D(u - u_e)\end{aligned}$$

$$\begin{aligned}(A, B, C, D) &= \left(\begin{bmatrix} \frac{\delta}{\delta x_1}(x_1 x_2 + u) & \frac{\delta}{\delta x_2}(x_1 x_2 + u) \\ \frac{\delta}{\delta x_1}(1 - x_1 x_2) & \frac{\delta}{\delta x_2}(1 - x_1 x_2) \end{bmatrix}_{x_e}, \qquad \begin{bmatrix} \frac{\delta}{\delta u}(x_1 x_2 + u) \\ \frac{\delta}{\delta u}(1 - x_1 x_2) \end{bmatrix}_{u_e}, \right. \\ &\qquad \left. \begin{bmatrix} \frac{\delta}{\delta x_1}(x_1^2 + x_2^2) & \frac{\delta}{\delta x_2}(x_1^2 + x_2^2) \end{bmatrix}_{x_e}, \qquad \begin{bmatrix} \frac{\delta}{\delta u}(x_1^2 + x_2^2) \end{bmatrix}_{u_e} \right) \\ &= \left(\begin{bmatrix} x_{2e} & x_{1e} \\ -x_{2e} & -x_{1e} \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2x_{1e} & 2x_{2e} \end{bmatrix}, \begin{bmatrix} 0 \end{bmatrix} \right)\end{aligned}$$

(iii) $\dot{x} = \begin{bmatrix} a_1 x_2 x_3 + b_1 u_1 \\ a_2 x_3 x_1 + b_2 u_2 \\ a_3 x_1 x_2 + b_3 u_3 \end{bmatrix}, \begin{cases} a_1 = \frac{I_2 - I_3}{I_1} \\ a_2 = \frac{I_3 - I_1}{I_2} \\ a_3 = \frac{I_1 - I_2}{I_3} \end{cases}$

$$\begin{aligned}\frac{d}{dt}\Delta x &= A\Delta x + B\Delta u \\ &= A(x - x_e) + B(u - u_e)\end{aligned}$$

$$\begin{aligned}(A, B) &= \left(\begin{bmatrix} \frac{\delta}{\delta x_1}(a_1 x_2 x_3 + b_1 u_1) & \frac{\delta}{\delta x_2}(a_1 x_2 x_3 + b_1 u_1) & \frac{\delta}{\delta x_3}(a_1 x_2 x_3 + b_1 u_1) \\ \frac{\delta}{\delta x_1}(a_2 x_3 x_1 + b_2 u_2) & \frac{\delta}{\delta x_2}(a_2 x_3 x_1 + b_2 u_2) & \frac{\delta}{\delta x_3}(a_2 x_3 x_1 + b_2 u_2) \\ \frac{\delta}{\delta x_1}(a_3 x_1 x_2 + b_3 u_3) & \frac{\delta}{\delta x_2}(a_3 x_1 x_2 + b_3 u_3) & \frac{\delta}{\delta x_3}(a_3 x_1 x_2 + b_3 u_3) \end{bmatrix}_{x_e}, \right. \\ &\qquad \left. \begin{bmatrix} \frac{\delta}{\delta u_1}(a_1 x_2 x_3 + b_1 u_1) & \frac{\delta}{\delta u_2}(a_1 x_2 x_3 + b_1 u_1) & \frac{\delta}{\delta u_3}(a_1 x_2 x_3 + b_1 u_1) \\ \frac{\delta}{\delta u_1}(a_2 x_3 x_1 + b_2 u_2) & \frac{\delta}{\delta u_2}(a_2 x_3 x_1 + b_2 u_2) & \frac{\delta}{\delta u_3}(a_2 x_3 x_1 + b_2 u_2) \\ \frac{\delta}{\delta u_1}(a_3 x_1 x_2 + b_3 u_3) & \frac{\delta}{\delta u_2}(a_3 x_1 x_2 + b_3 u_3) & \frac{\delta}{\delta u_3}(a_3 x_1 x_2 + b_3 u_3) \end{bmatrix}_{u_e} \right) \\ &= \left(\begin{bmatrix} 0 & a_1 x_{3e} & a_1 x_{2e} \\ a_2 x_{3e} & 0 & a_2 x_{1e} \\ a_3 x_{2e} & a_3 x_{1e} & 0 \end{bmatrix}, \begin{bmatrix} b_1 & 0 & 0 \\ 0 & b_2 & 0 \\ 0 & 0 & b_3 \end{bmatrix} \right)\end{aligned}$$

$$\begin{aligned} 0 &= a_1 x_{e2} x_{e3} + b_1 u_{e1} \quad \textit{and} \quad 0 = a_2 x_{e3} x_{e1} + b_2 u_{e2} \quad \textit{and} \quad 0 = a_3 x_{e1} x_{e2} + b_3 u_{e3} \\ u_{e1} &= \frac{-a_1}{b_1} x_{e2} x_{e3} \quad \textit{and} \quad u_{e2} = \frac{-a_2}{b_2} x_{e3} x_{e1} \quad \textit{and} \quad u_{e3} = \frac{-a_3}{b_3} x_{e1} x_{e2}\end{aligned}$$

2 Two Operators on the Vector Space $\mathbb{R}^{n \times n}$

$$\mathcal{S} : \mathbb{R}^{n \times n} \to \mathbb{R}^{n \times n} \quad : M \mapsto \frac{M + M^T}{2}$$
$$\mathcal{A} : \mathbb{R}^{n \times n} \to \mathbb{R}^{n \times n} \quad : M \mapsto \frac{M - M^T}{2}$$

(a) Are these operations linear?

Yes, we can show this by superposition:

$$\mathcal{S} : \frac{(c_1 m_{ij_1} + c_2 m_{ij_2}) + (c_1 m_{ji_1} + c_2 m_{ji_2})}{2} = c_1 \frac{m_{ij_1} + m_{ji_1}}{2} + c_2 \frac{m_{ij_2} + m_{ji_2}}{2}$$
$$\mathcal{A} : \frac{(c_1 m_{ij_1} + c_2 m_{ij_2}) - (c_1 m_{ji_1} + c_2 m_{ji_2})}{2} = c_1 \frac{m_{ij_1} - m_{ji_1}}{2} + c_2 \frac{m_{ij_2} - m_{ji_2}}{2}$$

(b) Determine the null space and range space of each operator.

Null Space

$N(\mathcal{S})$ is the set of *skew symmetric* matrices: $m_{ij} = -m_{ji}$. (Note: this implies that diagonal is zeros.)

$N(\mathcal{A})$ is the set of all symmetric matrices: $m_{ij} = m_{ji}$.

Range Space

$R(\mathcal{S})$ is the set of all symmetric matrices.

$R(\mathcal{A})$ is the set of all symmetric matrices with zeroes on the diagonal:
$i = j \to m_{ij} = m_{ji} \to \bar{m}_{ij} = \frac{m_{ij} - m_{ji}}{2} = 0.$

(c) What are the eigenvectors and eigenvalues of $\mathcal{S}$ and $\mathcal{A}$?

$\mathcal{S}$ has no effect on symmetric matrices: $m_{ij} = m_{ji} = \frac{m_{ij} + m_{ji}}{2}$. Thus the eigenvectors of $\mathcal{S}$ compose the basis of symmetric matrices:

$$\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \right)$$

The eigenvalues are all one.

$\mathcal{A}$ has no eigenvectors: all outputs are symmetric, but symmetric matrices compose the null space.