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ECE 601 - Dr. Gray

Homework #5

Due 10/19/2017

1. Kailath 2.2-22. *A Constant-Resistance Network*

a. Show that a realization for the circuit shown in the figure [$u(\cdot)$ is a current and $y(\cdot)$ is a voltage] can be written as:

$$\dot{x}(t) = \begin{bmatrix} -2R/L & 1/L \\ -1/C & 0 \end{bmatrix} x(t) + \begin{bmatrix} R/L \\ 1/C \end{bmatrix} u(t)$$
$$y(t) = \begin{bmatrix} -R & 1 \end{bmatrix} x(t) + Ru(t)$$

if we choose $x_1(t) = i_L(t), x_2(t) = V_C(t)$.

C Leg: $\begin{aligned} y &= V_C + Ri_C \\ &= V_C + R(u - i_L) \\ &= V_C + -Ri_L + Ru \end{aligned}$, **L Leg**: $\begin{aligned} y &= L\frac{d}{dt}i_L + Ri_L \\ L\frac{d}{dt}i_L + Ri_L &= V_C + -Ri_L + Ru \\ \frac{d}{dt}i_L &= V_C/L + -2R/Li_L + R/Lu \end{aligned}$, **Loops**:

$$\begin{aligned} i_c &= u - i_L \\ C\frac{d}{dt}V_C &= -i_L + u \\ \frac{d}{dt}V_C &= -1/Ci_L + 1/Cu \end{aligned}$$

b. Show that the transfer function is given by:

$$H(s) = \frac{Y(s)}{U(s)} = \frac{Rs^2 + [(1/C) + (R^2/L)]s + (R/LC)}{s^2 + (2R/L)s + (1/LC)}$$

Note that when $R^2 = L/C$ the transfer function is a constant, $H(s) = R$, for all values of s. This is known as a *constant-resistance network*.

$$H(s) = \frac{Y(s)}{H(s)} = c(sI - A)^{-1}b + d$$

$$= \begin{bmatrix} -R & 1 \end{bmatrix} \begin{bmatrix} s + 2R/L & -1/L \\ 1/C & s \end{bmatrix}^{-1} \begin{bmatrix} R/L \\ 1/C \end{bmatrix} + R$$

$$= \frac{\begin{bmatrix} -R & 1 \end{bmatrix} \begin{bmatrix} s & 1/L \\ -1/C & s + 2R/L \end{bmatrix} \begin{bmatrix} R/L \\ 1/C \end{bmatrix}}{det(sI - A)} + R$$

$$= \frac{\begin{bmatrix} -R & 1 \end{bmatrix} \begin{bmatrix} (R/L)s + 1/LC \\ -R/LC + (1/C)s + 2R/LC \end{bmatrix}}{det(sI - A)} + R$$

$$= \frac{-(R^2/L)s - R/LC + (1/C)s + R/LC}{s^2 + (2R/L)s + 1/LC} + \frac{Rs^2 + (2R^2/L)s + R/LC}{s^2 + (2R/L)s + 1/LC}$$

$$= \frac{Rs^2 + (R^2/L)s + (1/C)s + R/LC}{s^2 + (2R/L)s + 1/LC}$$

2. Kailath 2.3-3.

Consider the system illustrated in the figure.

a. Give a state-variable realization of this system.

Realize block $\frac{s+1}{s(s+3)} = \frac{s+1}{s^2+3s+0}$ as controller form: $(A_1, b_1, c_1) = \left(\begin{bmatrix} -3 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \end{bmatrix} \right)$

Realize (negative) feedback block $\frac{k}{s+a}$ as controller form: $(A_2, b_2, c_2) = (-a, 1, k)$

Feedback realization combines blocks:

$$(A, b, c) = \left(\begin{bmatrix} A_1 & -b_1 c_2 \\ b_2 c_1 & A_2 \end{bmatrix}, \begin{bmatrix} b_1 \\ \bigcirc \end{bmatrix}, \begin{bmatrix} c_1 & \bigcirc \end{bmatrix} \right) = \left(\begin{bmatrix} -3 & 0 & -k \\ 1 & 0 & 0 \\ 1 & 1 & -a \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \right)$$

b. Is there any choice of parameters k and/or a for which this realization loses either controllability or obervability or both?

Controllability Matrix:

$$\mathcal{C} = \begin{bmatrix} b & Ab & A^2b \end{bmatrix} = \begin{bmatrix} 1 & -3 & 9 - k \\ 0 & 1 & -3 \\ 0 & 1 & -2 - a \end{bmatrix}$$

$$det(\mathcal{C}) = 1 - a$$

So the system is not controllable when $a = 1$.

Observability Matrix:

$$\mathcal{O} = \begin{bmatrix} c \\ cA \\ cA^2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ -2 & 0 & -k \\ 6-k & -k & 2k+ak \end{bmatrix}$$

$$\begin{aligned} det(\mathcal{O}) &= -k^2 + (-4k - 2ak) - (-6k + k^2) \\ &= -2k^2 + 2k - 2ak \\ &= -2k(k - 1 + a) \end{aligned}$$

So the system is not observable when $k = 0$ or $k = 1 - a$.

3. Kailath 2.3-12.

For the constant resistance networks of Exercises 2.2.22 ~~and 2.2.23~~, determine what relations between R, L, and C are required to make them uncontrollable and/or unobservable.

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} -2R/L & 1/L \\ -1/C & 0 \end{bmatrix} x(t) + \begin{bmatrix} R/L \\ 1/C \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} -R & 1 \end{bmatrix} x(t) + Ru(t) \end{aligned}$$

Controllability Matrix:

$$\mathcal{C} = \begin{bmatrix} b & Ab \end{bmatrix} = \begin{bmatrix} \frac{R}{L} & \frac{-2R^2}{L^2} + \frac{1}{LC} \\ \frac{1}{C} & \frac{-R}{LC} \end{bmatrix}$$

$$\begin{aligned} det(\mathcal{C}) &= \frac{-R^2}{L^2C} - \left(\frac{-2R^2}{L^2C} + \frac{1}{LC^2}\right) \\ &= \frac{R^2}{L^2C} - \frac{1}{LC^2} \\ &= \frac{R^2C - L}{L^2C^2} \end{aligned}$$

So the system is not controllable when $L = R^2C$.

Observability Matrix:

$$\mathcal{O} = \begin{bmatrix} c \\ cA \end{bmatrix} = \begin{bmatrix} -R & 1 \\ \frac{2R^2}{L} - \frac{1}{C} & \frac{-R}{L} \end{bmatrix}$$

$$\begin{aligned} det(\mathcal{O}) &= \frac{R^2}{L} - \left(\frac{2R^2}{L} - \frac{1}{C}\right) \\ &= \frac{1}{C} - \frac{R^2}{L} \\ &= \frac{L - R^2C}{LC} \end{aligned}$$

So the system is not observable when $L = R^2 C$.

4. Kailath 2.3-26. *The Fibonacci Sequence*

The Fibonacci sequence $\{0, 1, 1, 2, 3, 5, 8, 13, \cdots\}$is generated by the equation

$$\begin{aligned} y_k &= y_{k-1} + y_{k-2}, \quad k \geq 2 \\ y_0 &= 0, \quad y_1 = 1 \end{aligned}$$

This is equivalent to DT observability form :

$$\left.\begin{aligned} x_{k+1} &= \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} x_k + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} x_k \\ y_k &= \begin{bmatrix} 1 & 0 \end{bmatrix} x_k \end{aligned}\right\}, \quad x_0 = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

a. Show that we can write

$$y_n = \frac{1}{\sqrt{5}}(\lambda_-^n - \lambda_+^n), \quad \lambda_\pm = \frac{1 \pm \sqrt{5}}{2}$$

Here $\lambda_\pm$ are just the eigen values of $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$:

$$\begin{aligned} det\left(\begin{bmatrix} -\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix}\right) &= 0 \\ \lambda^2 - \lambda - 1 &= 0 \\ \lambda &= \frac{1 \pm \sqrt{5}}{2} \end{aligned}$$

So the relation must be based on eigenvectors:

$$v_+ = \begin{bmatrix} -a_- \lambda_- \\ a_- \end{bmatrix} \qquad v_- = \begin{bmatrix} -a_+ \lambda_+ \\ a_+ \end{bmatrix}, \qquad a_\pm = \frac{1}{\sqrt{\lambda_\pm^2 + 1}} = \frac{1}{\sqrt{\frac{1 \pm 2\sqrt{5}+5}{4} + 1}} = \frac{}{\sqrt{}}$$

Showing x_n as a combination $x_k = c_{k+} v_+ + c_{k-} v_-$:

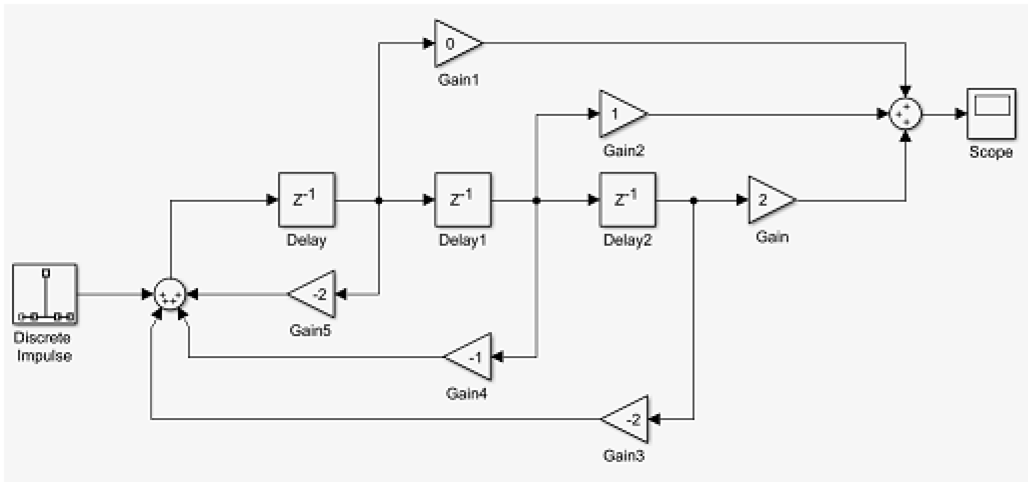
$$\begin{aligned} x_{k+n} = A^n x_k &= A^n c_{k+} v_+ + A^n c_{k-} v_- \\ &= c_{k+} \lambda_+^n v_+ + c_{k-} \lambda_-^n v_- \end{aligned}$$

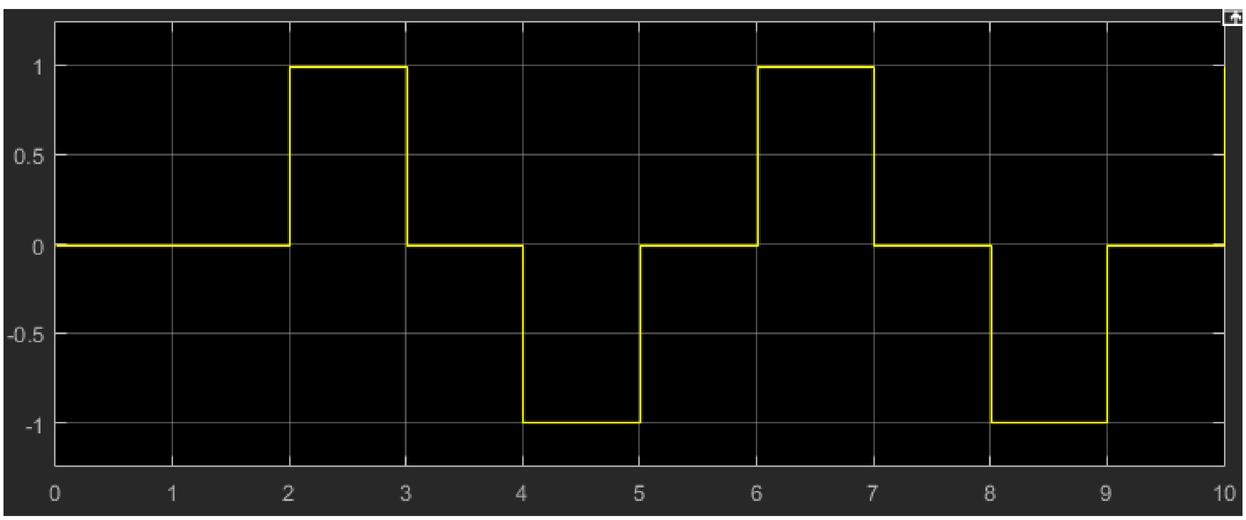
$$\begin{aligned} y_{k+n} &= \begin{bmatrix} 1 & 0 \end{bmatrix} (c_{k+} \lambda_+^n v_+ + c_{k-} \lambda_-^n v_-) \\ y_n = y_{0+n} &= \begin{bmatrix} 1 & 0 \end{bmatrix} (c_{0+} \lambda_+^n v_+ + c_{0-} \lambda_-^n v_-) \end{aligned}$$

b. Show that

$$\lim_{n\to\infty} \frac{\ln y_n}{n} = \ln \frac{\sqrt{5}+1}{2}$$

5. *Simulating a Discrete-time Controller Form Realization*





This looks correct based on the Markov parameters: $H(z) = z^{-2} - z^{-4} + z^{-6} \ldots$

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