Linear elasticity

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1 Constitutive model

The stress, $\boldsymbol{\sigma}$, is set by to the strain, $\boldsymbol{\varepsilon}$, through the following linear relation:

$$\boldsymbol{\sigma} \equiv K \operatorname{tr}(\boldsymbol{\varepsilon}) + 2G \boldsymbol{\varepsilon}_\mathrm{d} \equiv \mathbb{C} : \boldsymbol{\varepsilon} \tag{1}$$

wherein $\mathbb{C}$ is the elastic stiffness, which reads:

$$\begin{aligned} \mathbb{C} &\equiv K \boldsymbol{I} \otimes \boldsymbol{I} + 2G (\mathbb{I}_\mathrm{s} - \tfrac{1}{3} \boldsymbol{I} \otimes \boldsymbol{I}) &(2)\\ &= K \boldsymbol{I} \otimes \boldsymbol{I} + 2G\, \mathbb{I}_\mathrm{d} &(3) \end{aligned}$$

with K and G the bulk and shear modulus respectively. See Appendix A for nomenclature, including definitions of the unit tensors.

2 Consistency check

To check if the derived tangent $\mathbb{C}$ a *consistency check* can be performed. A (random) perturbation $\delta\boldsymbol{\varepsilon}$ is applied. The residual is compared to that predicted by the tangent. For the general case of linearisation, the following holds:

$$\boldsymbol{\sigma}\big(\boldsymbol{\varepsilon}_\star + \delta\boldsymbol{\varepsilon}\big) = \boldsymbol{\sigma}\big(\boldsymbol{\varepsilon}_\star\big) + \mathbb{C}\big(\boldsymbol{\varepsilon}_\star\big) : \delta\boldsymbol{\varepsilon} + \mathcal{O}(\delta\boldsymbol{\varepsilon}^2) \tag{4}$$

or

$$\underbrace{\boldsymbol{\sigma}\big(\boldsymbol{\varepsilon}_\star + \delta\boldsymbol{\varepsilon}\big) - \boldsymbol{\sigma}\big(\boldsymbol{\varepsilon}_\star\big)}_{\delta\boldsymbol{\sigma}} - \mathbb{C}\big(\boldsymbol{\varepsilon}_\star\big) : \delta\boldsymbol{\varepsilon} = \mathcal{O}(\delta\boldsymbol{\varepsilon}^2) \tag{5}$$

This allows the introduction of a relative error

$$\eta = \Big|\Big| \delta\boldsymbol{\sigma} - \mathbb{C}(\boldsymbol{\varepsilon}_\star) : \delta\boldsymbol{\varepsilon} \Big|\Big| \Big/ \Big|\Big| \delta\boldsymbol{\sigma} \Big|\Big| \tag{6}$$

This *truncation error* thus scales as $\eta \sim ||\delta\boldsymbol{\varepsilon}||^2$ as depicted in Figure 1. As soon as the error becomes sufficiently small the numerical *rounding error* becomes more dominant, the scaling thereof is also included in Figure 1.

Because this model is linear there is no truncation error, the measurement of η and a function of $||\delta\boldsymbol{\varepsilon}||$ thus only displays a rounding error, as depicted in Fig. 2.

A Nomenclature

Tensor products

- Dyadic tensor product

$$\begin{aligned} \mathbb{C} &= \boldsymbol{A} \otimes \boldsymbol{B} &(7)\\ C_{ijkl} &= A_{ij}\, B_{kl} &(8) \end{aligned}$$

- Double tensor contraction

$$\begin{aligned} C &= \boldsymbol{A} : \boldsymbol{B} &(9)\\ &= A_{ij}\, B_{ji} &(10) \end{aligned}$$

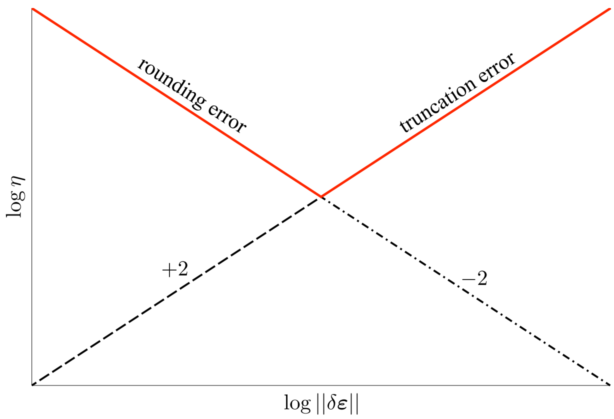

Figure 1. Expected behaviour of the consistency check, see Heath [1, p. 9].

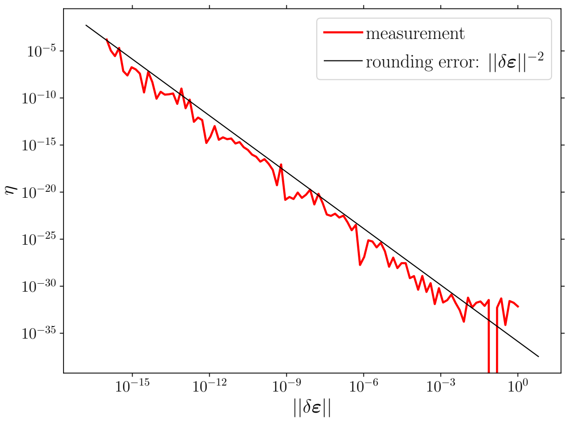

Figure 2. Measured consistency check, cf. Fig. 1.

Tensor decomposition

- Deviatoric part $\boldsymbol{A}_{\mathrm{d}}$ of an arbitrary tensor $\boldsymbol{A}$:

$$\operatorname{tr}\left(\boldsymbol{A}_{\mathrm{d}}\right) \equiv 0 \tag{11}$$

  and thus

$$\boldsymbol{A}_{\mathrm{d}} = \boldsymbol{A} - \frac{1}{3}\operatorname{tr}\left(\boldsymbol{A}\right) \tag{12}$$

Fourth order unit tensors

- Unit tensor:

$$\boldsymbol{A} \equiv \mathbb{I} : \boldsymbol{A} \tag{13}$$

  and thus

$$\mathbb{I} = \delta_{il}\delta jk \tag{14}$$

- Right-transposition tensor:

$$\boldsymbol{A}^{T} \equiv \mathbb{I}^{RT} : \boldsymbol{A} = \boldsymbol{A} : \mathbb{I}^{RT} \tag{15}$$

  and thus

$$\mathbb{I}^{RT} = \delta_{ik}\delta_{jl} \tag{16}$$

- Symmetrisation tensor:

$$\operatorname{sym}\left(\boldsymbol{A}\right) \equiv \mathbb{I}_{\mathrm{s}} : \boldsymbol{A} \tag{17}$$

  whereby

$$\mathbb{I}_{\mathrm{s}} = \frac{1}{2}\left(\mathbb{I} + \mathbb{I}^{RT}\right) \tag{18}$$

  This follows from the following derivation:

$$\begin{aligned} \operatorname{sym}\left(\boldsymbol{A}\right) &= \frac{1}{2}\left(\boldsymbol{A} + \boldsymbol{A}^{T}\right) && (19)\\ &= \frac{1}{2}\left(\mathbb{I} : \boldsymbol{A} + \mathbb{I}^{RT} : \boldsymbol{A}\right) && (20)\\ &= \frac{1}{2}\left(\mathbb{I} + \mathbb{I}^{RT}\right) : \boldsymbol{A} && (21)\\ &= \mathbb{I}_{\mathrm{s}} : \boldsymbol{A} && (22) \end{aligned}$$

- Deviatoric and symmetric projection tensor

$$\operatorname{dev}\left(\operatorname{sym}\left(\boldsymbol{A}\right)\right) \equiv \mathbb{I}_{\mathrm{d}} : \boldsymbol{A} \tag{23}$$

  from which it follows that:

$$\mathbb{I}_{\mathrm{d}} = \mathbb{I}_{\mathrm{s}} - \frac{1}{3}\boldsymbol{I} \otimes \boldsymbol{I} \tag{24}$$

References

[1] M.T. Heath. *Scientific computing.* 2002.