Isotropic elasto-plasticity

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Abstract

History dependent elasto-plastic material. This corresponds to a non-linear relation between the Cauchy stress, $\boldsymbol{\sigma}$, and the linear strain increment, ε_Δ, depending on the equivalent plastic strain, ε_p. I.e.

$$\boldsymbol{\sigma} = f(\varepsilon_\Delta, \varepsilon_\mathrm{p})$$

The plasticity follows power-law hardening

$$\sigma_\mathrm{y}(\varepsilon_\mathrm{p}) = \sigma_{\mathrm{y}0} + \varepsilon_\mathrm{p}^n$$

The model is implemented in 3-D, hence it can directly be used for either 3-D or 2-D plane strain problems.

1 Constitutive model

The model consists of the following ingredients:

(i) The strain, $\boldsymbol{\varepsilon}$, is additively split in an elastic part, $\boldsymbol{\varepsilon}_\mathrm{e}$, and a plastic plastic, $\boldsymbol{\varepsilon}_\mathrm{p}$. I.e.

$$\boldsymbol{\varepsilon} \equiv \boldsymbol{\varepsilon}_\mathrm{e} + \boldsymbol{\varepsilon}_\mathrm{p} \tag{1}$$

(ii) The stress, $\boldsymbol{\sigma}$, is set by to the elastic strain, $\boldsymbol{\varepsilon}_\mathrm{e}$, through the following linear relation:

$$\boldsymbol{\sigma} \equiv \mathbb{C}_\mathrm{e} : \boldsymbol{\varepsilon}_\mathrm{e} \tag{2}$$

wherein $\mathbb{C}_\mathrm{e}$ is the elastic stiffness, which reads:

$$\mathbb{C}_\mathrm{e} \equiv K \boldsymbol{I} \otimes \boldsymbol{I} + 2G(\mathbb{I}_\mathrm{s} - \tfrac{1}{3}\boldsymbol{I} \otimes \boldsymbol{I}) \tag{3}$$

$$= K \boldsymbol{I} \otimes \boldsymbol{I} + 2G\, \mathbb{I}_\mathrm{d} \tag{4}$$

with K and G the bulk and shear modulus respectively. See Appendix A for nomenclature, including definitions of the unit tensors.

(iii) The elastic domain is bounded by the following yield function

$$\Phi(\boldsymbol{\sigma}, \varepsilon_\mathrm{p}) \equiv \sigma_\mathrm{eq} - \sigma_\mathrm{y}(\varepsilon_\mathrm{p}) \leq 0 \tag{5}$$

where σ_eq the equivalent stress (see Appendix B), and σ_y the yield stress which is a non-linear function of the equivalent plastic strain, ε_p.

(iv) To determine the direction of plastic flow, normality is assumed. This corresponds to the following associative flow rule:

$$\dot{\boldsymbol{\varepsilon}}_\mathrm{p} \equiv \dot{\gamma} \boldsymbol{N} \tag{6}$$

where $\dot{\gamma}$ is the plastic multiplier, and $\boldsymbol{N}$ is the Prandtl–Reuss flow vector, which is defined through normality:

$$\boldsymbol{N} \equiv \frac{\partial \Phi}{\partial \boldsymbol{\sigma}} = \sqrt{\frac{3}{2}}\, \frac{\boldsymbol{\sigma}_\mathrm{d}}{||\boldsymbol{\sigma}_\mathrm{d}||} = \frac{3}{2} \frac{\boldsymbol{\sigma}_\mathrm{d}}{\sigma_\mathrm{eq}} \tag{7}$$

(v) Finally, associative hardening is assumed:

$$\dot{\varepsilon}_\mathrm{p} \equiv \sqrt{\tfrac{2}{3}}\, ||\dot{\boldsymbol{\varepsilon}}_\mathrm{p}|| = \dot{\gamma} \tag{8}$$

The equivalent plastic strain then reads

$$\varepsilon_\mathrm{p} = \int\limits_0^t \dot{\varepsilon}_\mathrm{p}\; \mathrm{d}\tau \tag{9}$$

For a more detailed description the reader is referred to de Souza Neto et al. [1, p. 216-234].

2 Numerical implementation: implicit

For the numerical implementation, first of all a numerical time integration scheme has to be selected. Here, an implicit time discretization is used, which has the favorable property of being unconditionally stable. The numerical implementation of it is done by the commonly used return-map algorithm, in which an increment in strain is first assumed fully elastic (elastic predictor). Then, if needed, a return-map is utilized to return to a physically admissible state (plastic corrector).

2.1 Elastic predictor

Given an increment in strain

$$\boldsymbol{\varepsilon}_\Delta = \boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^{(t)} \tag{10}$$

and the state variables, evaluate the *elastic trial state*:

$$^\star\boldsymbol{\varepsilon}_\mathrm{e} = \boldsymbol{\varepsilon}_\mathrm{e}^{(t)} + \boldsymbol{\varepsilon}_\Delta \tag{11}$$

$$^\star\varepsilon_\mathrm{p} = \varepsilon_\mathrm{p}^{(t)} \tag{12}$$

The corresponding trial stress is computed by

$$^\star\boldsymbol{\sigma} = \mathbb{C}_\mathrm{e} : {^\star\boldsymbol{\varepsilon}_\mathrm{e}} \tag{13}$$

Finally the trial value of the yield function follows as

$$^\star\Phi = \Phi({^\star\boldsymbol{\sigma}}, {^\star\varepsilon_\mathrm{p}}) = {^\star\sigma_\mathrm{eq}} - \sigma_\mathrm{y}({^\star\varepsilon_\mathrm{p}}) \tag{14}$$

2.2 Trial state: elastic

If the trial state is within the (current) yield surface, i.e. when

$$^\star\Phi \leq 0 \tag{15}$$

the trial state coincides with the actual state, and:

$$\boldsymbol{\varepsilon}_\mathrm{e} = {^\star\boldsymbol{\varepsilon}_\mathrm{e}} = \boldsymbol{\varepsilon}_\mathrm{e}^{(t)} + \boldsymbol{\varepsilon}_\Delta \tag{16}$$

$$\varepsilon_\mathrm{p} = {^\star\varepsilon_\mathrm{p}} = \varepsilon_\mathrm{p}^{(t)} \tag{17}$$

$$\boldsymbol{\sigma} = {^\star\boldsymbol{\sigma}} \tag{18}$$

Otherwise a return-map is needed (see below).

2.3 Trial state elasto-plastic: return-map

If the trial state is outside the (current) yield surface, i.e. when

$$^\star\Phi > 0 \tag{19}$$

plastic flow occurs in the increment. A return-map is needed to return to the admissible state. The admissible state has to satisfy the following system of equations:

$$\begin{cases} \boldsymbol{\varepsilon}_\mathrm{e} & = {^\star\boldsymbol{\varepsilon}_\mathrm{e}} - \Delta\gamma\, \boldsymbol{N} \\ \varepsilon_\mathrm{p} & = \varepsilon_\mathrm{p}^{(t)} + \Delta\gamma \\ \Phi & = \sigma_\mathrm{eq} - \sigma_\mathrm{y}\big(\varepsilon_\mathrm{p}^{(t)} + \Delta\gamma\big) = 0 \end{cases} \tag{20}$$

2.3.1 Scalar equation return-map

This can be reduced by using that

$$\boldsymbol{\sigma}_\mathrm{d} = {}^\star\boldsymbol{\sigma}_\mathrm{d} - 2G\Delta\gamma\ \boldsymbol{N} \tag{21}$$

$$= {}^\star\boldsymbol{\sigma}_\mathrm{d} - 3G\Delta\gamma\ \frac{\boldsymbol{\sigma}_\mathrm{d}}{\sigma_\mathrm{eq}} \tag{22}$$

From this is follows that:

$$\frac{\boldsymbol{\sigma}_\mathrm{d}}{\sigma_\mathrm{eq}} = \frac{{}^\star\boldsymbol{\sigma}_\mathrm{d}}{{}^\star\sigma_\mathrm{eq}} \qquad \text{or} \qquad \boldsymbol{N} = {}^\star\boldsymbol{N} \tag{23}$$

And thus

$$\boldsymbol{\sigma}_\mathrm{d} = \left(1 - \frac{3G\Delta\gamma}{{}^\star\sigma_\mathrm{eq}}\right) {}^\star\boldsymbol{\sigma}_\mathrm{d} \tag{24}$$

whereby the equivalent stress trivially follows as

$$\sigma_\mathrm{eq} = {}^\star\sigma_\mathrm{eq} - 3G\Delta\gamma \tag{25}$$

In stead of the system return the plastic multiplier $\Delta\gamma$ can directly be found by enforcing the yield surface

$$\Phi = {}^\star\sigma_\mathrm{eq} - 3G\Delta\gamma - \sigma_\mathrm{y}\big(\varepsilon_\mathrm{p}^{(t)} + \Delta\gamma\big) = 0 \tag{26}$$

This (non-linear) equation has to be solved for the unknown plastic multiplier $\Delta\gamma$.

2.3.2 Linear hardening

Linear hardening reads

$$\sigma_\mathrm{y} = \sigma_\mathrm{y0} + H\varepsilon_\mathrm{p} \tag{27}$$

In this case (26) can be solved analytically. The solution reads

$$\Delta\gamma = \frac{{}^\star\Phi}{3G + H} \tag{28}$$

2.3.3 Non-linear hardening

(1) Initial guess:

$$\Delta\gamma := 0 \tag{29}$$

and evaluate

$$\tilde{\Phi} := {}^\star\Phi \tag{30}$$

(2) Perform Newton-Raphson iteration:

- Hardening slope

$$H := \left.\frac{\mathrm{d}\sigma_\mathrm{y}}{\mathrm{d}\varepsilon_\mathrm{p}}\right|_{\varepsilon_\mathrm{p}^{(t)}+\Delta\gamma} \tag{31}$$

- Residual derivative:

$$d := \frac{\mathrm{d}\tilde{\Phi}}{\mathrm{d}\Delta\gamma} = -3G - H \tag{32}$$

- Update guess for the plastic multiplier:

$$\Delta\gamma := \Delta\gamma - \frac{\tilde{\Phi}}{d} \tag{33}$$

(3) Check for convergence:

$$\tilde{\Phi} := {}^\star\sigma_\mathrm{eq} - 3G\Delta\gamma - \sigma_\mathrm{y}(\varepsilon_\mathrm{p}^{(t)} + \Delta\gamma) \tag{34}$$

Stop if:

$$\left|\tilde{\Phi}\right| \leq \epsilon_\mathrm{tol} \tag{35}$$

Otherwise continue with (2)

2.3.4 Trial state update

Finally, the trial state is updated:

- The updated stress tensor

$$\boldsymbol{\sigma} = \sigma_\mathrm{m}\boldsymbol{I} + \boldsymbol{\sigma}_\mathrm{d} \tag{36}$$

with

$$\sigma_\mathrm{m} = {}^\star\sigma_\mathrm{m} \tag{37}$$

$$\boldsymbol{\sigma}_\mathrm{d} = \left(1 - \frac{3G\Delta\gamma}{{}^\star\sigma_\mathrm{eq}}\right){}^\star\boldsymbol{\sigma}_\mathrm{d} \tag{38}$$

- The updated elastic strain tensor

$$\boldsymbol{\varepsilon}_\mathrm{e} = \frac{1}{2G}\boldsymbol{\sigma}_\mathrm{d} + \tfrac{1}{3}\,\mathrm{tr}({}^\star\boldsymbol{\varepsilon})\boldsymbol{I} \tag{39}$$

- The updated equivalent plastic strain:

$$\varepsilon_\mathrm{p} = \varepsilon_\mathrm{p}^{(t)} + \Delta\gamma \tag{40}$$

3 Consistent tangent

To derive the consistent tangent, the first step is to combine the above to an explicit relation between the (actual) stress $\boldsymbol{\sigma}$ and the trial elastic strain ${}^\star\boldsymbol{\varepsilon}_\mathrm{e}$.

- If elastic:

$$\boldsymbol{\sigma} = {}^\star\boldsymbol{\sigma} = \mathbb{C}_\mathrm{e} : {}^\star\boldsymbol{\varepsilon}_\mathrm{e} \tag{41}$$

The tangent then trivially follows:

$$\mathbb{C}_\mathrm{ep} = \frac{\partial\boldsymbol{\sigma}}{\partial\boldsymbol{\varepsilon}} = \frac{\partial\,\boldsymbol{\sigma}}{\partial\,{}^\star\boldsymbol{\varepsilon}_\mathrm{e}} = \mathbb{C}_\mathrm{e} \tag{42}$$

- If plastic:

$$\begin{aligned}
\boldsymbol{\sigma} &= \sigma_\mathrm{m}\boldsymbol{I} + \boldsymbol{\sigma}_\mathrm{d} && (43)\\
&= {}^\star\sigma_\mathrm{m}\boldsymbol{I} + \left(1 - \frac{3G\Delta\gamma}{{}^\star\sigma_\mathrm{eq}}\right){}^\star\boldsymbol{\sigma}_\mathrm{d} && (44)\\
&= {}^\star\sigma_\mathrm{m}\boldsymbol{I} + \left(1 - \frac{3G\Delta\gamma}{{}^\star\sigma_\mathrm{eq}}\right)2G\,{}^\star\boldsymbol{\varepsilon}_\mathrm{e}^\mathrm{d} && (45)\\
&= \left[\mathbb{C}_\mathrm{e} - \frac{6G^2\Delta\gamma}{{}^\star\sigma_\mathrm{eq}}\,\mathbb{I}_\mathrm{d}\right] : {}^\star\boldsymbol{\varepsilon}_\mathrm{e} && (46)
\end{aligned}$$

The tangent then follows from

$$\begin{aligned}
\mathbb{C}_\mathrm{ep} &= \frac{\partial\boldsymbol{\sigma}}{\partial\boldsymbol{\varepsilon}} = \frac{\partial\,\boldsymbol{\sigma}}{\partial\,{}^\star\boldsymbol{\varepsilon}_\mathrm{e}} && (47)\\
&= \mathbb{C}_\mathrm{e} - \frac{6G^2\Delta\gamma}{{}^\star\sigma_\mathrm{eq}}\mathbb{I}_\mathrm{d} + 4G^2\left[\frac{\Delta\gamma}{{}^\star\sigma_\mathrm{eq}} - \frac{1}{3G+H}\right]{}^\star\boldsymbol{N}\otimes{}^\star\boldsymbol{N} && (48)
\end{aligned}$$

3.1 Derivation

$$\frac{\partial \boldsymbol{\sigma}}{\partial {}^\star\boldsymbol{\varepsilon}_\mathrm{e}} = \mathbb{C}^e - \frac{6G^2\Delta\gamma}{{}^\star\sigma_\mathrm{eq}}\mathbb{I}^d - \frac{6G^2}{{}^\star\sigma_\mathrm{eq}}\left(\frac{\partial\,\Delta\gamma}{\partial {}^\star\boldsymbol{\varepsilon}_\mathrm{e}}\right)\otimes {}^\star\boldsymbol{\varepsilon}_\mathrm{e}^d + \frac{6G^2\Delta\gamma}{{}^\star\sigma_\mathrm{eq}^2}\left(\frac{\partial {}^\star\sigma_\mathrm{eq}}{\partial {}^\star\boldsymbol{\varepsilon}_\mathrm{e}}\right)\otimes {}^\star\boldsymbol{\varepsilon}_\mathrm{e}^d \tag{49}$$

Apply the following:

- for the equivalent stress

$$\frac{\partial \sigma_\mathrm{eq}}{\partial \boldsymbol{\sigma}} = \frac{\partial \sigma_\mathrm{eq}}{\partial \boldsymbol{\sigma}^d} = \frac{\partial}{\partial \boldsymbol{\sigma}^d}\left(\sqrt{\tfrac{3}{2}\boldsymbol{\sigma}^d : \boldsymbol{\sigma}^d}\right) = \frac{1}{2\sigma_\mathrm{eq}}\frac{\partial}{\partial \boldsymbol{\sigma}^d}\left(\tfrac{3}{2}\boldsymbol{\sigma}^d : \boldsymbol{\sigma}^d\right) = \frac{3}{2}\frac{\boldsymbol{\sigma}^d}{\sigma_\mathrm{eq}} = \boldsymbol{N} = {}^\star\boldsymbol{N} \tag{50}$$

hence:

$$\frac{\partial {}^\star\sigma_\mathrm{eq}}{\partial {}^\star\boldsymbol{\varepsilon}_\mathrm{e}} = 2G\,{}^\star\boldsymbol{N} \tag{51}$$

- for the plastic multiplier

$$\frac{\partial\,\Delta\gamma}{\partial {}^\star\boldsymbol{\varepsilon}_\mathrm{e}} = \left(\frac{\partial\,\Delta\gamma}{\partial\,\Phi}\right)\left(\frac{\partial\,\Phi}{\partial\,{}^\star\sigma_\mathrm{eq}}\right)\left(\frac{\partial\,{}^\star\sigma_\mathrm{eq}}{\partial {}^\star\boldsymbol{\varepsilon}_\mathrm{e}}\right) = \frac{2G}{3G+H}\,{}^\star\boldsymbol{N} \tag{52}$$

A Nomenclature

Tensor products

- Dyadic tensor product

$$\mathbb{C} = \boldsymbol{A}\otimes\boldsymbol{B} \tag{53}$$
$$C_{ijkl} = A_{ij}\,B_{kl} \tag{54}$$

- Double tensor contraction

$$C = \boldsymbol{A} : \boldsymbol{B} \tag{55}$$
$$= A_{ij}\,B_{ji} \tag{56}$$

Tensor decomposition

- Deviatoric part $\boldsymbol{A}_\mathrm{d}$ of an arbitrary tensor $\boldsymbol{A}$:

$$\mathrm{tr}\left(\boldsymbol{A}_\mathrm{d}\right) \equiv 0 \tag{57}$$

and thus

$$\boldsymbol{A}_\mathrm{d} = \boldsymbol{A} - \tfrac{1}{3}\mathrm{tr}\left(\boldsymbol{A}\right) \tag{58}$$

Fourth order unit tensors

- Unit tensor:

$$\boldsymbol{A} \equiv \mathbb{I} : \boldsymbol{A} \tag{59}$$

and thus

$$\mathbb{I} = \delta_{il}\delta jk \tag{60}$$

- Right-transposition tensor:

$$\boldsymbol{A}^T \equiv \mathbb{I}^{RT} : \boldsymbol{A} = \boldsymbol{A} : \mathbb{I}^{RT} \tag{61}$$

and thus

$$\mathbb{I}^{RT} = \delta_{ik}\delta_{jl} \tag{62}$$

- Symmetrisation tensor:

$$\mathrm{sym}\,(\boldsymbol{A}) \equiv \mathbb{I}_{\mathrm{s}} : \boldsymbol{A} \tag{63}$$

whereby

$$\mathbb{I}_{\mathrm{s}} = \tfrac{1}{2}\left(\mathbb{I} + \mathbb{I}^{RT}\right) \tag{64}$$

This follows from the following derivation:

$$\begin{aligned} \mathrm{sym}\,(\boldsymbol{A}) &= \tfrac{1}{2}\left(\boldsymbol{A} + \boldsymbol{A}^{T}\right) && (65)\\ &= \tfrac{1}{2}\left(\mathbb{I} : \boldsymbol{A} + \mathbb{I}^{RT} : \boldsymbol{A}\right) && (66)\\ &= \tfrac{1}{2}\left(\mathbb{I} + \mathbb{I}^{RT}\right) : \boldsymbol{A} && (67)\\ &= \mathbb{I}_{\mathrm{s}} : \boldsymbol{A} && (68) \end{aligned}$$

- Deviatoric and symmetric projection tensor

$$\mathrm{dev}\,(\mathrm{sym}\,(\boldsymbol{A})) \equiv \mathbb{I}_{\mathrm{d}} : \boldsymbol{A} \tag{69}$$

from which it follows that:

$$\mathbb{I}_{\mathrm{d}} = \mathbb{I}_{\mathrm{s}} - \tfrac{1}{3}\boldsymbol{I} \otimes \boldsymbol{I} \tag{70}$$

B Stress measures

- Mean stress

$$\sigma_{\mathrm{m}} = \tfrac{1}{3}\,\mathrm{tr}(\boldsymbol{\sigma}) = \tfrac{1}{3}\,\boldsymbol{\sigma} : \boldsymbol{I} \tag{71}$$

- Stress deviator

$$\boldsymbol{\sigma}_{\mathrm{d}} = \boldsymbol{\sigma} - \sigma_{\mathrm{m}}\,\boldsymbol{I} = \mathbb{I}_{\mathrm{d}} : \boldsymbol{\sigma} \tag{72}$$

- Von Mises equivalent stress

$$\sigma_{\mathrm{eq}} = \sqrt{\tfrac{3}{2}\,\boldsymbol{\sigma}_{\mathrm{d}} : \boldsymbol{\sigma}_{\mathrm{d}}} = \sqrt{3 J_2(\boldsymbol{\sigma})} \tag{73}$$

where the second-stress invariant

$$J_2 = \tfrac{1}{2}\,||\,\boldsymbol{\sigma}_{\mathrm{d}}\,||^2 = \tfrac{1}{2}\,\boldsymbol{\sigma}_{\mathrm{d}} : \boldsymbol{\sigma}_{\mathrm{d}} \tag{74}$$

References

[1] E.A. de Souza Neto, D. Perić, and D.R.J. Owen. *Computational Methods for Plasticity*. John Wiley & Sons, Ltd, 2008. ISBN 9780470694626. doi: 10.1002/9780470694626.