1 Model

This document is based on Geers [1].

Key tensors

- *Finger tensor* (also known the name *left Cauchy-Green deformation*)

$$\boldsymbol{b} \equiv \boldsymbol{F} \cdot \boldsymbol{F}^T \tag{1}$$

- *Left stretch tensor*:

$$\boldsymbol{v} \equiv \sqrt{\boldsymbol{b}} \tag{2}$$

- *Hencky's logarithmic strain tensor*:

$$\boldsymbol{\varepsilon} \equiv \ln \boldsymbol{v} = \tfrac{1}{2} \ln \boldsymbol{b} \tag{3}$$

- *Kirchhoff stress tensor* $\boldsymbol{\tau}$, which is related to the *Cauchy stress tensor* $\boldsymbol{\sigma}$ as follows:

$$\boldsymbol{\tau} \equiv J \boldsymbol{\sigma} \tag{4}$$

  where J is the volume change ratio

$$J \equiv \det \boldsymbol{F} \tag{5}$$

Model

This model relies on splitting the deformation gradient $\boldsymbol{F}$ in an elastic part, $\boldsymbol{F}_\mathrm{e}$, and a plastic part, $\boldsymbol{F}_\mathrm{p}$, as follows:

$$\boldsymbol{F} \equiv \boldsymbol{F}_\mathrm{e} \cdot \boldsymbol{F}_\mathrm{p} \tag{6}$$

The model is fully defined in the deformed configuration. To that end it is defined in terms of the logarithmic strain. The Kirchhoff stress $\boldsymbol{\tau}$ is thereby related to the logarithmic elastic strain $\boldsymbol{\varepsilon}_\mathrm{e} \equiv \frac{1}{2} \ln \boldsymbol{b}_\mathrm{e}$ in the usual way:

$$\boldsymbol{\tau} \equiv K \operatorname{tr}(\boldsymbol{\varepsilon}_\mathrm{e}) \, \boldsymbol{I} + 2G \boldsymbol{\varepsilon}_\mathrm{e}^\mathrm{d} \tag{7}$$

where K is the bulk modulus and G is the shear modulus, $\boldsymbol{I}$ is the second order unit tensor, and $\boldsymbol{\varepsilon}_\mathrm{e}^\mathrm{d}$ is the deviator of the elastic strain tensor. Equivalently this can be expressed in terms of a fourth order material stiffness

$$\boldsymbol{\tau} \equiv \mathbb{C} : \boldsymbol{\varepsilon}_\mathrm{e}, \quad \mathbb{C}_\mathrm{e} = K \boldsymbol{I} \otimes \boldsymbol{I} + 2G \mathbb{I}^\mathrm{d} \tag{8}$$

with $\mathbb{I}^\mathrm{d} = \mathbb{I} - \frac{1}{3} \boldsymbol{I} \otimes \boldsymbol{I}$ the deviator projection. The elastic domain is bounded by a yield function, that, following standard J_2 plasticity, reads

$$\Phi(\boldsymbol{\tau}, \varepsilon_\mathrm{p}) \equiv \tau_\mathrm{eq} - \tau_\mathrm{y}(\varepsilon_\mathrm{p}) \leq 0 \tag{9}$$

where

$$\tau_\mathrm{eq} \equiv \sqrt{\boldsymbol{\tau}^\mathrm{d} : \boldsymbol{\tau}^\mathrm{d}} \tag{10}$$

is the equivalent stress; τ_y is the equivalent yield stress that may, in the case of hardening, be a function of the plastic strain; ε_p is the effective plastic strain which depends on the entire strain history:

$$\varepsilon_\mathrm{p} = \int_0^t \dot{\gamma} \; \mathrm{d}t' \tag{11}$$

whereby the plastic strain rate, $\dot{\gamma}$, is by construction non-negative. Also note that it is defined in terms of some pseudo-time, as there is no rate-sensitivity. In line with J_2-plasticity, normality is used to determined the direction of plastic flow. This corresponds to the following associative flow rule:

$$\dot{\boldsymbol{\varepsilon}}_\mathrm{p} = \dot{\gamma} \boldsymbol{N} \tag{12}$$

where $\boldsymbol{N}$ is the Prandtl–Reuss flow vector, which is defined through normality:

$$\boldsymbol{N} = \frac{\partial \Phi}{\partial \boldsymbol{\tau}} = \frac{3}{2} \frac{\boldsymbol{\tau}^\mathrm{d}}{\tau_\mathrm{eq}} \tag{13}$$

Trial state
The path-dependent model is solved by discretising in pseudo-time. Accordingly, the deformation is applied in small steps, transforming Eq. (11) in

$$\varepsilon_\mathrm{p} = \sum \Delta\gamma^t \tag{14}$$

Solving proceeds through the formulation of a trial state, whereby the full deformation increment is assumed elastic. From this potentially unphysical state one looks for the increment in plastic strain that gives rise to a physically admissible state.

In particular, the incremental deformation tensor

$$\boldsymbol{f} = \boldsymbol{F}^{t+\Delta t} \cdot \left[\boldsymbol{F}^t\right]^{-1} \tag{15}$$

is used to define the trial state

$$\begin{cases} {}^\star\boldsymbol{b}_\mathrm{e} = \boldsymbol{f} \cdot \boldsymbol{b}_\mathrm{e}^t \cdot \boldsymbol{f}^\mathsf{T} \\ {}^\star\varepsilon_\mathrm{p} = \varepsilon_\mathrm{p}^t \end{cases} \tag{16}$$

which gives rise to some ${}^\star\boldsymbol{\tau}$ and corresponding ${}^\star\tau_\mathrm{eq}$ and ${}^\star\boldsymbol{N}$. The trial value for the yield function follows as:

$${}^\star\Phi = {}^\star\tau_\mathrm{eq} - \tau_\mathrm{y}({}^\star\varepsilon_\mathrm{p}) \tag{17}$$

For any ${}^\star\Phi \leq 0$ the trial state is simply the physical admissible state. Otherwise one has to look for a $\Delta\gamma$ such that $\Phi(\boldsymbol{\tau}, \varepsilon_\mathrm{p}) = 0$.

Return mapping
Given the plastic strain update $\Delta\gamma$ (whose value will be specified below), following Eq. (12) the actual elastic strain

$$\boldsymbol{\varepsilon}_\mathrm{e} = {}^\star\boldsymbol{\varepsilon}_\mathrm{e} - \Delta\gamma\, {}^\star\boldsymbol{N} \tag{18}$$

(here and below the superscript $t + \Delta t$ have been dropped, which is trusted not to give confusion). Consequently the stress reads

$$\boldsymbol{\tau} = {}^\star\boldsymbol{\tau} - 2G\Delta\gamma\, {}^\star\boldsymbol{N} \tag{19}$$

which, by construction, only affects the deviatoric part of $\boldsymbol{\tau}$. The stress deviator can be alternatively expressed as

$$\boldsymbol{\tau}^\mathrm{d} = {}^\star\boldsymbol{\tau}^\mathrm{d} - 3G\Delta\gamma \frac{{}^\star\boldsymbol{\tau}^\mathrm{d}}{{}^\star\tau_\mathrm{eq}} = \left(1 - \frac{3G\Delta\gamma}{{}^\star\tau_\mathrm{eq}}\right) {}^\star\boldsymbol{\tau}^\mathrm{d} \tag{20}$$

from which the following expression for the equivalent stress can be expressed:

$$\tau_\mathrm{eq} = {}^\star\tau_\mathrm{eq} - 3G\Delta\gamma \tag{21}$$

It can now be realised that the trial and updated deviatoric stresses are *co-linear*:

$$\frac{\boldsymbol{\tau}_\mathrm{d}}{\tau_\mathrm{eq}} = \frac{{}^\star\boldsymbol{\tau}_\mathrm{d}}{{}^\star\tau_\mathrm{eq}}, \quad \boldsymbol{N} = {}^\star\boldsymbol{N} \tag{22}$$

To summarise, the return-map involves:

$$\begin{cases} \boldsymbol{\tau} = {}^\star\boldsymbol{\tau} - 2G\Delta\gamma\boldsymbol{N} \\ \tau_\mathrm{eq} = {}^\star\tau_\mathrm{eq} - 3G\Delta\gamma \\ \varepsilon_\mathrm{p} = {}^\star\varepsilon_\mathrm{p} + \Delta\gamma \end{cases} \tag{23}$$

The plastic strain update $\Delta\gamma$ follows from enforcing the yield function:

$$\Phi(\boldsymbol{\tau}, \varepsilon_\mathrm{p}) = {}^\star\tau_\mathrm{eq} - 3G\Delta\gamma - \tau_\mathrm{y}({}^\star\varepsilon_\mathrm{p} + \Delta\gamma) = 0 \tag{24}$$

Linear hardening
Linear hardening corresponds to the following evolution of yield stress:

$$\tau_{\text{y}}(\varepsilon_{\text{p}}) = \tau_{\text{y0}} + H\varepsilon_{\text{p}} \tag{25}$$

In this case, enforcing the yield function (as in Eq. (24)) corresponds to:

$$^{\star}\Phi - 3G\Delta\gamma + H\Delta\gamma = 0 \tag{26}$$

Solving for $\Delta\gamma$ results in

$$\Delta\gamma = \frac{^{\star}\Phi}{H + 3G} \tag{27}$$

> Note that although the model is defined in terms of the Kirchhoff stress $\boldsymbol{\tau}$, the implementation outputs the Cauchy stress $\boldsymbol{\sigma} = \boldsymbol{\tau}/J$ for convenience in use with Finite Elements. See below.

2 Tangent

Basic definition
The consistent tangent operator, defined in the current configuration, is of the form

$$\delta\boldsymbol{\tau} = (\mathbb{K}_{\vec{x}})_{(i)} : \boldsymbol{L}_{\delta}^{T} = (\mathbb{K}_{\vec{x}})_{(i)} : \boldsymbol{F}_{(i)}^{-T} \cdot \delta\boldsymbol{F}^{T} \tag{28}$$

where

$$\boldsymbol{L}_{\delta} \equiv \delta\boldsymbol{F} \cdot \boldsymbol{F}_{(i)}^{-1} = \left(\boldsymbol{F}_{(i)}^{-T} \cdot \delta\boldsymbol{F}^{T}\right)^{T} \tag{29}$$

Finite element formulation

$$\int\limits_{\Omega} (\vec{\nabla}\varphi)^{T} : (\mathbb{K}_{\vec{x}})_{(i)} : \boldsymbol{L}_{\delta}^{T} \frac{1}{J_{(i)}} d\Omega = -\int\limits_{\Omega} (\vec{\nabla}\varphi)^{T} : \boldsymbol{\tau}_{(i)} \frac{1}{J_{(i)}} d\Omega \tag{30}$$

> It should now be obvious why the implementation outputs the Cauchy stress $\boldsymbol{\sigma} = \boldsymbol{\tau}/J$ and $(\mathbb{K}_{\vec{x}})_{(i)}/J_{(i)}$.

Details

$$(\mathbb{K}_{\vec{x}})_{(i)} = \mathbb{K}_{\text{geo}} + \mathbb{K}_{\text{mat}} \tag{31}$$

where

$$\mathbb{K}_{\text{geo}} = -\mathbb{I}^{\text{RT}} \cdot \boldsymbol{\tau} \tag{32}$$

and

$$\mathbb{K}_{\text{mat}} = \mathbb{C} : \frac{\partial^{\star}\boldsymbol{\varepsilon}_{\text{e}}}{\partial \ln {^{\star}\boldsymbol{b}_{\text{e}}}} : \frac{\partial \ln {^{\star}\boldsymbol{b}_{\text{e}}}}{\partial^{\star}\boldsymbol{b}_{\text{e}}} : \frac{\partial^{\star}\boldsymbol{b}_{\text{e}}}{\partial \boldsymbol{L}_{\delta}^{T}} \tag{33}$$

where

- The derivative of the constitutive response:

$$\mathbb{C} = \begin{cases} \mathbb{C}_{\text{e}} & \text{if } \phi \leq 0 \\ \mathbb{C}_{\text{ep}} & \text{otherwise} \end{cases} \tag{34}$$

where $\mathbb{C}_\text{e}$ is defined in Eq. (8), and

$$2\,\mathbb{C}_\text{ep} = [a_0 \boldsymbol{I} \otimes \boldsymbol{I} + (1-3a_0)\mathbb{I}^s - 2(a_0 - a_1){}^\star\boldsymbol{N} \otimes {}^\star\boldsymbol{N}] : \mathbb{C}_\text{e} \tag{35}$$

$$= \left(\frac{K}{2} - \frac{G}{3} + a_0 G\right) \boldsymbol{I} \otimes \boldsymbol{I} + (1-3a_0)G\mathbb{I}^s + 2(a_0 - a_1){}^\star\boldsymbol{N} \otimes {}^\star\boldsymbol{N} \tag{36}$$

where

$$a_0 = \frac{\Delta\gamma G}{{}^\star\tau_\text{eq}} \qquad a_1 = \frac{G}{\frac{\partial h}{\partial \varepsilon_\text{p}} + 3G} \tag{37}$$

- The derivative of Hencky's strain with respect to the logarithm of the Finger tensor is simply:

$$\frac{\partial {}^\star\boldsymbol{\varepsilon}_\text{e}}{\partial \ln {}^\star\boldsymbol{b}_\text{e}} = \tfrac{1}{2}\,\mathbb{I} \tag{38}$$

(which is commonly absorbed by the previous term).

- The derivative of the logarithm of the Finger tensor reads:

$$\frac{\partial \ln {}^\star\boldsymbol{b}_\text{e}}{\partial {}^\star\boldsymbol{b}_\text{e}} = \sum_{n=1}^{3}\sum_{m=1}^{3} g(\lambda_n, \lambda_m)\vec{v}_n \otimes \vec{v}_m \otimes \vec{v}_n \otimes \vec{v}_m \tag{39}$$

where

$$g(\lambda_n, \lambda_m) = \begin{cases} \dfrac{\ln\lambda_m - \ln\lambda_n}{\lambda_m - \lambda_n} & \text{if } \lambda_m \neq \lambda_n \\ \dfrac{1}{\lambda_n} & \text{otherwise} \end{cases} \tag{40}$$

- And finally:

$$\delta {}^\star\boldsymbol{b}_\text{e} = \frac{\partial {}^\star\boldsymbol{b}_\text{e}}{\partial \boldsymbol{L}_\delta^T} : \boldsymbol{L}_\delta^T = 2\big(\mathbb{I}^s \cdot {}^\star\boldsymbol{b}_\text{e}\big) : \boldsymbol{L}_\delta^T \tag{41}$$

Consistency check
The pull-back of the Kirchhoff stress to the first Piola-Kirchhoff stress reads

$$\boldsymbol{P}_{(i)} = \boldsymbol{\tau}_{(i)} \cdot \boldsymbol{F}_{(i)}^{-T} \tag{42}$$

This expression can be used to pull-back the consistent tangent operator, which in the form of Eq. (28) reads:

$$\delta\boldsymbol{\tau} = (\mathbb{K}_{\vec{x}})_{(i)} : \boldsymbol{F}_{(i)}^{-T} \cdot \delta\boldsymbol{F}^T \tag{43}$$

The subscript $\vec{x}$ indicates that the tangent is defined in the last-known iterative configuration. In terms of a variation in the first Piola-Kirchhoff stress, as in Eq. (43), can be rewritten as

$$\delta\boldsymbol{P}^T = \big[\boldsymbol{F}_{(i)}^{-1} \cdot (\mathbb{K}_{\vec{x}})_{(i)} \cdot \boldsymbol{F}_{(i)}^{-T}\big] : \delta\boldsymbol{F}^T \tag{44}$$

Note that this pull-back does not introduce additional approximations. We now have a tangent of the form:

$$\delta\boldsymbol{P}^T = \mathbb{K}_{(i)} : \delta\boldsymbol{F}^T \tag{45}$$

This form can be conveniently used to perform a consistency check, as is done in Fig. 1.

References

[1] M.G.D. Geers. Finite strain logarithmic hyperelasto-plasticity with softening: a strongly non-local implicit gradient framework. *Comput. Methods Appl. Mech. Eng.*, 193 (30-32):3377–3401, 2004. doi: 10.1016/j.cma.2003.07.014.

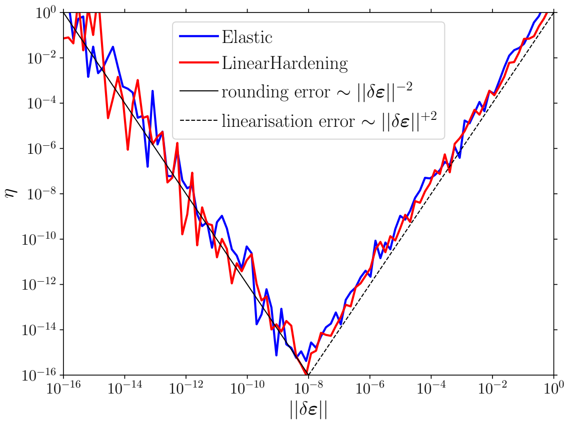

Figure 1. Result of the consistency check (performed in the reference configuration, as in Eq. (45))